Unimodular Transformations and Canonical Forms for Singular Systems

Dimitris Vafiadis * **Nicos Karcanias** *

* *Systems and Control Research Centre*
City University, London EC1V 0HB, England

Abstract: The relationship between the unimodular matrices relating coprime and column reduced matrix fraction descriptions (MFD) of a nonproper transfer function, and the restricted system equivalence (r.s.e.) transformations relating the corresponding generalised state space realisations is considered. It is shown that the r.s.e and unimodular transformations can be directly obtained from each other by inspection. The r.s.e. transformations leading to the canonical form are derived from the unimodular transformations leading to the echelon canonical form of the composite matrix of the MFD of the system.

Keywords: singular systems, canonical forms, matrix fraction description .

1. INTRODUCTION

Canonical forms is a very old and well known topic in the study of linear systems Dickinson et al. (1974), Popov (1972), Kailath (1980). In the case of state space systems description, the canonical forms are usually considered in the context of similarity transformations, while in the case of singular (or descriptor) systems case the relevant transformation is the restricted system equivalence (r.s.e. or coordinate) transformation. In both cases the canonical form is directly related to a specific external representation of the system, the matrix fraction description (MFD) of the transfer function and more specifically, the echelon form of the composite matrix consisting of the numerator and denominator of the MFD of the transfer function. Canonical forms of state space systems are obtained directly from the echelon form of the denominator matrix of a coprime and column reduced MFD of the system transfer function and result in a canonical pair (A, B) of the state and input matrix of the system. In the case of singular systems the whole composite matrix (numerator and denominator) is necessary for the derivation of the canonical form, which includes the whole quadruple (E, A, B, C) of the matrices describing a singular system Vafiadis and Karcanias (1997, 1995); Lebret and Loiseau (1994).

It is well known from the literature that all polynomial minimal bases of a rational vector space Wolovich (1974); Forney (1975) are related by structured unimodular matrices Karcanias (2013). Therefore all coprime and column reduced (i.e. minimal) composite matrices corresponding to the MFDs of the transfer function of the system are related by structured unimodular transformations. In (Vafiadis and Karcanias (1997)) it was shown that the canonical form of the system is readily obtained from the echelon form of the MFD via an appropriate realisation procedure and in (Vafiadis and Karcanias (1995)) the procedure for obtaining the canonical form by applying a series of elementary strict system equivalence transformations was described.

In the present paper a method for the derivation of the r.s.e. transformations leading to the canonical form is proposed. It is based on the unimodular transformation matrix relating the composite matrix of the MFD of the original system to its echelon canonical form. It is shown that the r.s.e. transformations can be obtained from the unimodular transformations by inspection and vice versa.

The development of the paper reveals the duality between transformations in the frequency domain (unimodular transformations) and the time domain (coordinate transformations), in the sense that for every unimodular transformation on the composite matrix of a given MFD a unique r.s.e. transformation can be derived on the corresponding generalised state space (g.s.s.) realisation such that the resulting generalised state space system is a direct realisation of the transformed composite matrix.

2. PRELIMINARIES AND PROBLEM STATEMENT

Let $H(s) \in \mathbb{R}^{m\times\ell}(s)$ be nonproper transfer function and consider two different coprime and column reduced MFDs of $H(s)$:

$$H(s) = N_1(s)D_1^{-1}(s) = N_2(s)D_2^{-1}(s) \tag{1}$$

Let the composite matrices of the above MFDs be $T_j(s) = \left[N_j^T(s), D_j^T(s)\right]^T$, $j = 1, 2$. It is assumed that $T_1(s)$ and $T_2(s)$ are ordered minimal bases of the rational vector spaces spanned by their columns (Forney (1975)), i.e. they have no finite Smith zeros and are column reduced. Then, there exists unimodular matrix $U(s)$ such that

$$T_2(s) = T_1(s)U(s) \tag{2}$$

The generalised state–space realisations based on the above MFDs, denoted by $\mathscr{S}_{(E_j,A_j,B_j,C_j)}$, $j = 1, 2$ are as follows:

$$E_j\dot{x}(t) = A_jx(t) + B_ju(t),\ y(t) = C_jx(t) \tag{3}$$

where $E \in \mathbb{R}^{n\times n}$, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times \ell}$ $C \in \mathbb{R}^{m\times n}$. The system matrices $R_j(s)$ of the above systems are (Rosenbrock (1974))

$$R_j(s) = \begin{bmatrix} sE_j - A_i & -B_j \\ C_j & 0 \end{bmatrix} \qquad (4)$$

Write $N_j(s)$ and $D_j(s)$ in (1) as

$$N_j(s) = \tilde{C}_j S(s),\ D_j(s) = [\underline{d}_{j1}(s), \cdots, \underline{d}_{j\ell}(s)] \qquad (5)$$

where $S(s) = \text{bl–diag}\{[1, s, \cdots, s^{r_i-1}]^T\},\ i = 1, \cdots, \ell$ and r_i the reachability indices (r.i.) of the triple (E_j, A_j, B_j), $j = 1, 2$ and $\underline{d}_{ji}(s) = \underline{k}^i_{jr_\ell} s^{r_i} + \underline{\lambda}^i_{jr_i} s^{r_i-1} + \cdots + \underline{\lambda}^i_{j1}, i = 1, \cdots, \ell$

Consider the special form of system matrices (4)

$$R_j(s) = \left[\begin{array}{c|c} L(s) & 0 \\ sK^j - \Lambda^j & -I \\ \hline \tilde{C}_j & 0 \end{array}\right] \qquad (6)$$

where

$$L(s) = \text{block} - \text{diag}\{\ldots, L_{r_i-1}(s), \ldots\} \qquad (7)$$

$$r_1 \le r_2 \le \cdots \le r_\ell$$

$$L_{r_i-1}(s) = s\,[I_{r_i-1}, 0] - [0, I_{r_i-1}] \qquad (8)$$

$$sK^j - \Lambda^j = [sK^j_1 - \Lambda^j_1, \ldots, sK^j_\ell - \Lambda^j_\ell] \qquad (9)$$

and

$$K^j_i = \left[\, 0_{\ell\times(r_i-1)},\ \underline{k}^i_{jr_\ell} \,\right] \Lambda^j_i = [-\underline{\lambda}^j_1, \cdots, -\underline{\lambda}^j_{r_i}] \qquad (10)$$

corresponding to realisations of $H(s)$. It is straightforward that they are obtained by inspection from the numerator and denominator of the corresponding MFDs $T_1(s)$ and $T_2(s)$. It can be shown that the above realisations are minimal, as long as $T_1(s)$ and $T_2(s)$ are column reduced and have no Smith zeros, and are related by the restricted system equivalence (r.s.e.) transformations.

$$E_2 = PA_1Q, A_2 = PA_1Q, B_2 = PB, C_2 = C_1Q \qquad (11)$$

where P, Q nonsingular constant matrices. The following example clarifies the matrices defined above

Example 1. Let the given MFD be

$$T(s) = \begin{bmatrix} N(s) \\ D(s) \end{bmatrix} = \left[\begin{array}{ccc} 2s+1 & 4+s & -s^2+2s+3 \\ 2s-1s & 4+s & 2\,s^2+2s \\ \hline 2 & s^2+1 & 4 \\ s+3 & s-1 & 4 \\ s-1 & s+2 & -2s+2 \end{array}\right]$$

We have $r_1 = 2, r_2 = 2$ and $r_3 = 3$ (for the relation of the reachability indices with the column degrees of the composite matrix $T(s)$ see Remark 4 of the next section). Then

$$S^T(s) = \left[\begin{array}{cc:cc:ccc} 1 & s & 0 & 0 & 0 & 0 & 0 \\ \hdashline 0 & 0 & 1 & s & 0 & 0 & 0 \\ \hdashline 0 & 0 & 0 & 0 & 1 & s & s^2 \end{array}\right]$$

$$L(s) = \left[\begin{array}{cc:cc:ccc} s & -1 & 0 & 0 & 0 & 0 & 0 \\ \hdashline 0 & 0 & s & -1 & 0 & 0 & 0 \\ \hdashline 0 & 0 & 0 & 0 & s & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & s & -1 \end{array}\right]$$

$$sK - \Lambda = \left[\begin{array}{cc:cc:ccc} 2 & 0 & 1 & s & 4 & 0 & 0 \\ 3 & 1 & -1 & 1 & 4 & 0 & 0 \\ -1 & 1 & 2 & 1 & 2 & -2 & 0 \end{array}\right]$$

$$\tilde{C} = \begin{bmatrix} 1 & 2 & 4 & 1 & 3 & 2 & -1 \\ -1 & 2 & 4 & 1 & 0 & 2 & 2 \end{bmatrix}$$

The problem considered in the paper is the following: Given the unimodular matrix $U(s)$ in (2) relating the minimal MFDs of the transfer function, find the matrices of the r.s.e. transformations in (11) relating the corresponding realisations and conversely, given the r.s.e. transformations relating the state space descriptions find the unimodular transformations relating the composite matrices of the MFDs. Also investigate the relationship between the canonical form of singular state descriptions under r.s.e. and the canonical MFDs of the system transfer function.

3. UNIMODULAR TRANSFORMATIONS AND THE STABILIZER OF $L(s)$

The stabilizer of $L(s)$ is the set of s.e. transformations leaving the pencil $L(s)$ unaltered (Vafiadis and Karcanias (1995)) and will be used for the development of the paper. In this section the connection of the stabilizer transformations to a class unimodular transformations of polynomial matrices is established.

Definition 2. The stabilizer of $L(s)$ is defined as the set of all pairs $(\hat{P}, \hat{Q})$ of invertible matrices such that $\hat{P}L(s)\hat{Q} = L(s)$ and is denoted by $Stab(L(s))$. □

Lemma 3. (Vafiadis and Karcanias (1995)) Let $\hat{P}$, Q' be such that $\hat{P}L(s) = L(s)Q'$. Then $\hat{P}$, Q' are upper block triangular matrices with blocks

$$\hat{P}_{ij} = \begin{bmatrix} \lambda_{ij0} \cdots \lambda_{ij(r_j-r_i)} & & \\ \ddots & & \ddots \\ & \lambda_{ij0} & \cdots \lambda_{ij(r_j-r_i)} \end{bmatrix}_{(r_i-1)\times(r_j-1)} \qquad (12)$$

$$Q'_{ij} = \begin{bmatrix} \lambda_{ij0} \cdots \lambda_{ij(r_j-r_i)} & & \\ \ddots & & \ddots \\ & \lambda_{ij0} & \cdots \lambda_{ij(r_j-r_i)} \end{bmatrix}_{r_i\times r_j} \qquad (13)$$

when $r_j \ge r_i$ and $\hat{P}_{ij} = 0$, $Q'_{ij} = 0$ if $r_j < r_i$ □

Matrix $S(s)$ defined in (5) is a basis matrix for $\mathscr{N}_r\{L(s)\}$, where $\mathscr{N}_r\{\bullet\}$ denotes the right null space. Then $L(s)S(s) = 0$ and, if $(\hat{P}, Q) \in Stab(L(s)$, we have

$$\hat{P}L(s)QS(s) = 0 \qquad (14)$$

which means that

$$QS(s) \in \mathscr{N}_r\{L(s)\} = \langle S(s)\rangle_{\mathbb{R}(s)} \qquad (15)$$

where $\langle\bullet\rangle_{\mathbb{R}(s)}$ denotes the column span over $\mathbb{R}(s)$. Thus, there exists invertible polynomial matrix $V(s)$ such that

$$QS(s) = S(s)V(s) \qquad (16)$$

By equating the coefficients of like powers of s above, we obtain Q and $V(s)$:

Q is an upper block triangular matrix with blocks $Q_{ij} \in \mathbb{R}^{r_i\times r_j}$ of the following Toeplitz form:

$$Q_{ij} = \begin{cases} \begin{bmatrix} v_{ij0} & \cdots & v_{ij(r_j-r_i)} & \cdots & 0 \\ & \ddots & & \ddots & \\ 0 & \cdots & v_{ij0} & \cdots & v_{ij(r_j-r_i)} \end{bmatrix}, & r_i \leq r_j \\ 0_{r_i \times r_j}, & r_i > r_j \end{cases} \tag{17}$$

and $V(s)$ is a polynomial matrix with entries $v_{ij}(s)$:

$$v_{ij}(s) = \begin{cases} s^{r_j-r_i} v_{ij(r_j-r_i)} + \cdots + v_{ij0}, & r_i \leq r_j \\ 0 & r_i > r_j \end{cases} \tag{18}$$

Equation (16) can be written as

$$Q'S(s) = S(s)W(s) \tag{19}$$

where $Q' = Q^{-1}$ and $W(s) = V^{-1}(s)$. Then $W(s)$ and Q' are block matrices with

$$Q'_{ij} = \begin{cases} \begin{bmatrix} w_{ij0} & \cdots & w_{ij(r_j-r_i)} & \cdots & 0 \\ & \ddots & & \ddots & \\ 0 & \cdots & w_{ij0} & \cdots & w_{ij(r_j-r_i)} \end{bmatrix}, & r_i \leq r_j \\ 0_{r_i \times r_j}, & r_i > r_j \end{cases} \tag{20}$$

and $W(s)$ is a polynomial matrix with entries $w_{ij}(s)$:

$$w_{ij}(s) = \begin{cases} s^{r_j-r_i} w_{ij(r_j-r_i)} + \cdots + w_{ij0}, & r_i \leq r_j \\ 0 & r_i > r_j \end{cases} \tag{21}$$

Note that $V(s)$ and $W(s)$ (and consequently Q and Q^{-1}) above, have the same structure. This is expected, since the structure of matrices $V(s)$ and $W(s)$ is that of a unimodular matrix relating two ordered minimal bases of the same rational vector space (Karcanias (2013)). Such a matrix is called **structured unimodular** and has the same structure to its inverse.

Remark 4. The assumption that the matrices $T_1(s)$ and $T_2(s)$ are ordered is necessary, in order to have $U(s)$ in the triangular form and all the related constant matrices in the upper block triangular form. For the case of strictly proper systems the column degrees of the matrices $T_1(s)$ and $T_2(s)$ (the controllability indices (c.i.) of the system) coincide with the r.i., and the ordering of the columns is based on the c.i. For singular systems we have two types of c.i., the proper and the nonproper (Karcanias (2013); Karcanias and Eliopoulou (1990); Malabre et al. (1990)). If the polynomials of the i–th column of a minimal composite matrix with degree equal to the degree of the column appear only in the corresponding denominator column, the i–th c.i. is called *proper* and the value of the i–th c.i. is equal to the value of the i–th r.i. If the column degree appears in the numerator then the "plus one" property holds (Malabre et al. (1990); Karcanias (2013)) i.e. the corresponding value of the he i–th r.i. is the degree of the corresponding column plus one, the case of *nonproper* c.i. In order to have $U(s)$ and matrices $\hat{P}$, Q, Q' in the forms shown by the equations (12) – (21) the column ordering of $T_1(s)$ and $T_2(s)$ must be considered with respect to r.i. in the same way the blocks in the diagonal of $L(s)$ in (7) are ordered. □

The following example clarifies the above Remark.

Example 5. Consider the composite matrix

$$T_1(s) = \begin{bmatrix} N_1(s) \\ D_1(s) \end{bmatrix} = \begin{bmatrix} s+1 & s & -s^2 \\ s & s & 2s^2 \\ \hline 1 & s^2 & 1 \\ s & s & s-2 \\ s-2 & s-1 & -2s \end{bmatrix}$$

$T_1(s)$ above is a minimal basis, since it has no Smith zeros and is column reduced. The controllability indices are the column degrees of the above i.e. $c_1 = 1$, $c_2 = c_3 = 2$. The c.i. c_1 and c_3 are nonproper, since the column degrees of columns 1 and 3 occur in $N_1(s)$. The r.i. are $r_1 = r_2 = 2$ and $r_3 = 3$. $T_1(s)$ is column ordered. A unimodularly equivalent to $T_1(s)$ is

$$T_2(s) = \begin{bmatrix} s+1 & -\frac{1}{3} & -1 \\ s & s^2 - \frac{s}{3} & 0 \\ \hline 1 & \frac{s}{3} & s^2 - 1 \\ s & \frac{1}{3}s^2 - \frac{2}{3} & 0 \\ s-2 & \frac{1}{3}s^2 - 5s + \frac{2}{3} & 1 \end{bmatrix}$$

The converting unimodular matrix is

$$U(s) = \begin{bmatrix} 1 & s/3 - 1/3 & -1 \\ 0 & 0 & 1 \\ 0 & 1/3 & 0 \end{bmatrix}$$

which, clearly, is not of the form (18) due to the element $s/3 - 1/3$ (the degree of that element according to (18) should be 0 because $r_2 = r_1$). If $T_2(s)$ is ordered with respect to the r.i., we take

$$T'_2(s) = \begin{bmatrix} s+1 & -1 & -\frac{1}{3} \\ s & 0 & s^2 - \frac{s}{3} \\ \hline 1 & s^2 - 1 & \frac{s}{3} \\ s & 0 & \frac{1}{3}s^2 - \frac{2}{3} \\ s-2 & 1 & \frac{1}{3}s^2 - 5s + \frac{2}{3} \end{bmatrix}$$

and the corresponding unimodular matrix leading to the echelon form is

$$U'(s) = \begin{bmatrix} 1 & -1 & s/3 - 1/3 \\ 0 & 1 & 0 \\ 0 & 0 & 1/3 \end{bmatrix}$$

which now conforms to (18). This discrepancy between the two unimodular matrices is due to the Forney's definition of *pivot indices* (p.i.) (Forney (1975)), which dictates that the p.i. corresponding to columns of equal degrees are increasingly ordered, and the fact that proper p.i. (p.i. corresponding to columns with proper c.i.) are by definition greater than the nonproper p.i. since they appear in the numerator $D(s)$ only.

4. R.S.E. EQUIVALENCE AND UNIMODULAR TRANSFORMATIONS

In this section the relationship of unimodular transformations relating minimal MFDs of a nonproper transfer function and the r.s.e. transformations relating the corresponding generalised state space realisations of the form (6) is considered. It is shown that the r.s.e. transformations are obtained directly from the unimodular transformations and vice – versa.

Proposition 6. Let $R_1(s)$ and $R_2(s)$ be two system matrices of the form (6) of systems related by (11). Then

$$P = \begin{bmatrix} \hat{P} & 0 \\ 0 & I_\ell \end{bmatrix}, \; Q = \hat{Q}^{-1} \tag{22}$$

where $(\hat{P}, Q) \in \mathrm{Stab}(L(s))$.

Proof. Let

$$P = \begin{bmatrix} \hat{P} & P_2 \\ P_3 & P_4 \end{bmatrix}$$

Then, from (11) it follows that

$$\begin{bmatrix} \hat{P} & P_2 \\ P_3 & P_4 \end{bmatrix} \begin{bmatrix} L(s) & 0 \\ sK^1 - \Lambda^1 & -I \end{bmatrix} \begin{bmatrix} Q & 0 \\ 0 & I \end{bmatrix} = \begin{bmatrix} L(s) & 0 \\ sK^2 - \Lambda^2 & -I \end{bmatrix}$$

The above may be expanded to the following equations:

$$\hat{P}L(s)Q + P_2(sK^1 - \Lambda^1)Q = L(s) \tag{23}$$
$$P_2 = 0 \tag{24}$$
$$P_3 L(s)Q + P_4(sK^1 - \Lambda^1)Q = sK^2 - \Lambda^2 \tag{25}$$
$$P_4 = I \tag{26}$$

Equations (23) and (24) yield that $(\hat{P}, Q) \in Stab(L(s))$.

It remains to show that $P_3 = 0$. From (6) it follows that

$$(sK^1 - \Lambda^1 1)S(s) = D_1(s) \tag{27}$$

then from (2)

$$(sK^1 - \Lambda^1)S(s)U(s) = D_1(s)U(s) = D_2(s) \tag{28}$$

Matrix $U(s)$) is structured unimodular (see. Remark 4), because $T_1(s)$ and $T_2(s)$ are ordered minimal bases and has the form (18). Then, since $(\hat{P}, Q) \in Stab(L(s))$, we have (see (16)) $QS(s) = S(s)U(s)$ and (28) becomes

$$(sK^1 - \Lambda^1)QS(s) = D_2(s) \tag{29}$$

thus

$$(sK^1 - \Lambda^1)Q = sK^2 - \Lambda^2 \tag{30}$$

and from (25), (26) it follows that $P_3 = 0$. □

Consider now the two realisations of $H(s)$ in the form (6) and the corresponding MFDs related as in (2). Then

$$\left[\begin{array}{c|c} L(s) & 0 \\ sK^1 - \Lambda^1 & -I \\ \hline \tilde{C}_1 & 0 \end{array}\right] \begin{bmatrix} S(s) \\ D_1(s) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ N_1(s) \end{bmatrix} \tag{31}$$

From $U(s)$ in (2) we obtain matrices Q and P as follows: Q is obtained from $U(s)$ by equation (17). Then Q^{-1} is taken either by direct inversion of Q or from $U^{-1}(s)$ and (20) and P from Q^{-1} by using (12) and (13). Then

$$\left[\begin{array}{c|c} PL(s)Q & 0 \\ sK^1 - \Lambda^1 & -I \\ \hline \tilde{C}_1 & 0 \end{array}\right] \begin{bmatrix} Q^{-1}S(s) \\ D_1(s) \end{bmatrix} U(s) = \begin{bmatrix} 0 \\ 0 \\ N_1(s) \end{bmatrix} U(s) \tag{32}$$

or

$$\left[\begin{array}{c|c} L(s) & 0 \\ sK^2 - \Lambda^2 & -I \\ \hline \tilde{C}_2 & 0 \end{array}\right] \begin{bmatrix} S(s) \\ D_2(s) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ N_2(s) \end{bmatrix} \tag{33}$$

The latter equation and Proposition 6 mean that matrices P and Q in (22) are the r.s.e. transformations relating systems $\mathscr{S}_{(E_j,A_j,B_j,C_j)}$, $j = 1, 2$. We have thus established the following:

Theorem 7. Consider two coprime and column reduced MFDs of a given nonproper transfer function $H(s)$. Let the composite matrices be related by a unimodular transformation $U(s)$ as in (2). If $\mathscr{S}_{(E_j,A_j,B_j,C_j)}$, $j = 1, 2$ are the corresponding realisations of the form (6) then

i) $U(s)$ is structured unimodular of the form (18)
ii) The r.s.e. transformations P and Q relating the systems $\mathscr{S}_{(E_j,A_j,B_j,C_j)}$, $j = 1, 2$ are obtained by $U(s)$ as described by (17) and (12) (13) and (22).

□

Remark 8. The above theorem can be stated the reverse way, i.e. starting from the r.s.e. transformations relating two systems of the form (6), we can obtain the unimodular matrix $U(s)$ according to (17) and (12) (13) and (22).

The meaning of the above result is that there is a mapping between unimodular transformations of the input–output description and r.s.e. transformations in the state–space description.

Theorem 9. Let $\mathscr{Q}_r$ be the set of matrices of the form (17) corresponding to the set of integers r_i (reachability indices) and $\mathscr{U}_\ell$ the set of unimodular matrices of the form (18). The map

$$\mathscr{F} : \mathscr{U}_\ell \longrightarrow \mathscr{Q}_r \tag{34}$$

defined by (17) –(18) is an isomorphism.

Proof: We have to prove that $\mathscr{F}$ is bijective i.e. it is (i) injective and (ii) surjective.

(i) Let $U_1(s)$ and $U_2(s) \in \mathscr{U}_\ell$. Then $F(U_1(s)) = Q_1$ and $F(U_2(s)) = Q_2$ with $Q_1, K_2 \in \mathscr{Q}_r$. If $Q_1 = Q_2$ it readily follows that $U_1(s) = U_2(s)$.

(ii) Given matrix $M \in \mathscr{Q}_r$, there always exists a polynomial matrix $U(s) \in \mathscr{U}_\ell$ such that $F(U(s)) = Q_1$ as it is clearly derived from equations (17) –(18). □

This relationship allows the derivation of transformations leading to canonical forms of the generalised state equations from transformations leading to canonical forms of polynomial descriptions of the system as it is discussed in the following section.

5. DERIVATION OF THE CANONICAL FORM

The problem of Popov type canonical forms under r.s.e. transformations was considered in Vafiadis and Karcanias (1997, 1995) where the sequence of the elementary coordinate (r.s.e.) transformations leading to the canonical quadruple (E, A, B, C) of a generalised state space system was described in detail. In this section the results of the previous section are used for the derivation of the coordinate transformations directly from the transformations relating minimal MFDs.

Definition 10. The MFD $N(s)D^{-1}(s)$ of a transfer function is called *canonical MFD* if the matrix $T(s) = \left[N^T(s), D^T(s)\right]^T$ has no Smith zeros, is column reduced and is in the echelon canonical form for polynomial matrices (Forney (1975)). □

It is known that all minimal generalised state–space realisations of a nonproper transfer function belong to the same r.s.e. class, i.e. they are r.s. equivalent to each other. The canonical element is given by the following result.

Theorem 11. (Vafiadis and Karcanias (1997)) The canonical form of the singular system $\mathscr{S}_{(E,A,B,C)}$ under r.s.e. transformations is the minimal realisation of the type (6) obtained by the canonical MFD of the transfer function. □

The canonical MDF is the one derived from the transformation to the echelon canonical form.

Here, we are going to consider a slightly different definition of the echelon form than the one given in (Forney (1975)), in order to be compliant with Remark 4 above. In what follows, echelon form of minimal basis polynomial matrix, is considered the usual echelon form defined by Forney with the difference that the column ordering is done on the basis of reachability indices and not the column degrees. This means that in the case where one proper and one nonproper c.i. of the system have the same value, the column corresponding to the proper c.i. goes first, since the corresponding r.i. is less than the r.i. which corresponds to the nonproper c.i.

Then the results of the previous section can be used for the derivation of the r.s.e. transformations leading a system of the form (6) to the canonical form.

Theorem 12. The r.s.e. transformations leading a system of the form (6) to the canonical form are obtained from the unimodular matrix $U(s)$ transforming the composite matrix

$$T(s) = \begin{bmatrix} N(s) \\ D(s) \end{bmatrix} = \begin{bmatrix} CS(s) \\ (sK - \Lambda)S(s) \end{bmatrix}$$

to the echelon canonical form according to Theorem 7. □

Example 13. Consider the transfer function $H(s)$

$$H(s) = \begin{bmatrix} -\frac{1}{2}\,\frac{9\,s^3-23\,s^2-33\,s+35}{s^4+4\,s^3-6\,s^2+18\,s-5} & -\frac{1}{2}\,\frac{s^5-7\,s^4-2\,s^2+19\,s-23}{s^4+4\,s^3-6\,s^2+18\,s-5} & \frac{1}{2}\,\frac{s^5+s^4-2\,s^3-2\,s^2+25\,s+9}{s^4+4\,s^3-6\,s^2+18\,s-5} \\ 3\,\frac{2\,s^3+7\,s^2+2\,s-5}{s^4+4\,s^3-6\,s^2+18\,s-5} & \frac{s^5+2\,s^4-5\,s^3-3\,s^2-14\,s+13}{s^4+4\,s^3-6\,s^2+18\,s-5} & -\frac{s^5+4\,s^4-2\,s^3+8\,s^2-15\,s-4}{s^4+4\,s^3-6\,s^2+18\,s-5} \end{bmatrix} \quad (35)$$

Two minimal MFDs of the above are

$$T_1(s) = \left[\begin{array}{ccc} 1+2\,s & 4+s & -s^2+2\,s+3 \\ -1+2\,s & 4+s & 2\,s^2+2\,s \\ \hline 2 & s^2+1 & 4 \\ s+3 & s-1 & 4 \\ s-1 & s+2 & -2\,s+2 \end{array}\right]$$

$$T_2(s) = \left[\begin{array}{ccc} s+\frac{1}{2} & 7/2 & \frac{7}{12} \\ s-\frac{1}{2} & \frac{9}{2} & s^2-\frac{s}{3}+\frac{5}{12} \\ \hline 1 & s^2 & \frac{s}{3}+\frac{1}{2} \\ \frac{s}{2}+\frac{3}{2} & \frac{s}{2}-\frac{5}{2} & \frac{1}{6}\,s^2+\frac{s}{12}+\frac{1}{12} \\ \frac{s}{2}-\frac{1}{2} & \frac{s}{2}+\frac{5}{2} & \frac{1}{6}\,s^2-\frac{5}{4}\,s+\frac{13}{12} \end{array}\right]$$

For the above systems we have realisations of the form (6) with

$$sK^1 - \Lambda^1 = \begin{bmatrix} 2 & 0 & 1 & s & 4 & 0 & 0 \\ 3 & 1 & -1 & 1 & 4 & 0 & 0 \\ -1 & 1 & 2 & 1 & 2 & -2 & 0 \end{bmatrix}$$

$$C_1 = \begin{bmatrix} 1 & 2 & 4 & 1 & 3 & 2 & -1 \\ -1 & 2 & 4 & 1 & 0 & 2 & 2 \end{bmatrix}$$

$$sK^2 - \Lambda^2 = \begin{bmatrix} 1 & 0 & 0 & s & 1/2 & 1/3 & 0 \\ \frac{3}{2} & \frac{1}{2} & -\frac{5}{2} & \frac{1}{2} & \frac{1}{12} & \frac{1}{12} & \frac{1}{6} \\ -\frac{1}{2} & \frac{1}{2} & \frac{5}{2} & \frac{1}{2} & \frac{13}{12} & -\frac{5}{4} & \frac{1}{6} \end{bmatrix}$$

$$C_2 = \begin{bmatrix} \frac{1}{2} & 1 & \frac{7}{2} & 0 & \frac{7}{12} & 0 & 0 \\ -\frac{1}{2} & 1 & \frac{9}{2} & 0 & \frac{5}{12} & -\frac{1}{3} & 1 \end{bmatrix}$$

The r.i. of the system are $r_1 = 2$, $r_2 = r_3 = 3$ and the c.i. are $c_1 = 1$, $c_2 = c_3 = 2$. The c.i. c_1, c_3 are nonproper. The unimodular matrix of equation (2) relating the two MFDs is

$$U(s) = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{s}{6}-\frac{5}{12} \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix}$$

then Q (see (17))

$$Q = \left[\begin{array}{cc:cc:ccc} 1/2 & 0 & -1/2 & 0 & -\frac{5}{12} & 1/6 & 0 \\ 0 & 1/2 & 0 & -1/2 & 0 & -\frac{5}{12} & 1/6 \\ \hdashline 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ \hdashline 0 & 0 & 0 & 0 & 1/3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1/3 \end{array}\right]$$

Notice the dimensions of the blocks of Q above which are derived from the reachability indices of the system and the correspondence of the coefficients of the polynomial entries of $U(s)$ and the values of the elements of Q.

$$U^{-1}(s) = \begin{bmatrix} 2 & 1 & -s+5/2 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

Note that

$$(sK^1 - \Lambda^1)Q = sK^2 - \Lambda^2$$

Matrix P_1 is obtained from $U^{-1}(s)$

$$P_1 = \left[\begin{array}{c|c|cc} 2 & 1 & 5/2 & -1 \\ \hline 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{array}\right]$$

It can be verified that $P_1 L(s) Q = L(s)$. The composite matrix $T_2(s)$ is obtained from the echelon form of $T_1(s)$ in the sense of Forney, with the last columns permuted, in order to have the columns ordered according to the reachability indices (see Remark 4), therefore the resulting system described by C_2, $sK^2 - \Lambda^2$ (see (6)) is the canonical form.

6. CONCLUSIONS

In the present paper the problem of deriving the coordinate transformations relating minimal generalised state space realisations of a given nonproper transfer function from the unimodular transformations relating the corresponding polynomial (MFD) descriptions of the system was considered. It was shown that there is a correspondence between the transformations in the frequency and time domain. This correspondence is actually an isomorphism. The transformation matrices can be obtained by inspection from each other. Based on this correspondence, the coordinate transformation resulting in the canonical form was derived from the unimodular transformation yielding the echelon canonical form of the corresponding matrix fraction description.

REFERENCES

Dickinson, B.W., Kailath, T., and Morf, M. (1974). Canonical matrix fraction and state-space descriptions for deterministic and stochastic linear systems. *IEEE Transactions on Automatic Control*, 19.

Forney, G.D. (1975). Minimal bases of rational vector spaces, with applications to multivariable linear systems. *SIAM J. Control*, 13, 493–520.

Kailath, T. (1980). *Linear Systems.* Prentice Hall, Englewood Cliffs.

Karcanias, N. and Eliopoulou, H. (1990). A classification of minimal bases for singular systems. In M.A. Kaashoek (ed.), *Progress in Systems and control Theory, Proceedings of 1989 MTNS Conference*, 255–262. Birkhauser, Amsterdam.

Karcanias, N. (2013). Geometric and algebraic properties of minimal bases of singular systems. *Int. J. Control*, 86(11), 1924–1945.

Lebret, G. and Loiseau, J.J. (1994). Proportional and proportional-derivative canonical forms for descriptor systems with outputs. *Automatica*, 30(5), 847–864.

Malabre, M., Kucera, V., and Zagalak, P. (1990). Reachability and controllability indices for linear descriptor systems. *Syst. Contr. Letters*, 1, 119–123.

Popov, V.M. (1972). Invariant description of linear time–invariant controllable systems. *SIAM J. Control*, 13, 252–264.

Rosenbrock, H. (1974). Structural Properties of Linear Dynamical Systems. *Int. J. Contr.*, 20(2), 191–202.

Vafiadis, D. and Karcanias, N. (1995). Canonical forms for singular systems with outputs under restricted system equivalence. In *Proceed. of IFAC Conference on Systems Structure and Control*, 324–329.

Vafiadis, D. and Karcanias, N. (1997). Canonical forms for descriptor systems under restricted system equivalence. *Automatica*, 33(5), 955–958.

Wolovich, W.A. (1974). *Linear multivariable systems*, volume 11 of *Applied Math. Sciences.* Springer.