Unimodular Equivalence and Similarity for Linear Systems

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The problem of finding the mapping between unimoduar transformations relating two minimal matrix fraction descriptions (MFDs) of a transfer function, and the similarity transformations relating the respective minimal state–space representations is considered. It is shown that the problem is equivalent to finding the relation of MFDs of the input–state transfer functions of the two systems. This relation turns out to be an equivalence relation involving the unimodular and the similarity matrices relating the MFDs and the state–space systems respectively. A canonical form for MFDs under this equivalence relation is obtained and it is shown that it leads to a canonical state–space representation, via a realisation procedure.

Keywords: linear systems, unimodular transformations, similarity, matrix fraction description.

1. Introduction

The relationship between matrix fraction descriptions (MFD) and state space representations in linear systems theory is a thoroughly discussed topic, see for instance Dickinson et al. (1974), Rosenbrock (1970), Kailath (1980) etc. MFDs of the input – output transfer (i.o.t.f) functions have their corresponding state – space description which is called realisation. A basic fact in the study of state space realisations is that minimal realisations of a given i.o.t.f. are related by similarity transformations and conversely, all similarity equivalent state space descriptions of a system give rise to the same i.o.t.f. On the other hand, a minimal (coprime and column reduced) MFD representation of a t.f. is not unique. All minimal MFDs are related to each other by unimodular transformations Popov (1969), Forney (1975) i.e. the corresponding composite matrices consisting of the "numerator" and the "denominator" of the MFD can be obtained from each other by multiplication by a unimodular matrix.

The aim of the present paper is the investigation of the relation between the aforementioned similarity and unimodular transformations. The formulation of such a problem can be roughly summarised as follows: Given two minimal MFD representations of a transfer function, find the similarity transformation relating the corresponding realisations in terms of the unimodular matrix relating the MFDs, or conversely, given two minimal state–space systems, find the appropriate corresponding minimal MFDs and the unimodular transformation relating them in terms of the similarity transformation matrix. In this way the duality between the two types of minimal system descriptions (minimal MFD and minimal state space realisation of a given transfer function) is completed by the duality of the transformations in the frequency domain (MFD) and time domain (state–space) respectively.

The paper is organised as follows: In section 2 it is shown that that the relationship between the unimodular and similarity equivalence can be obtained by considering only the input–to–state equations either in the state–space description, or in the MFD description of the system i.e. by considering the input – state equations, and the corresponding input – state transfer function (i.s.t.f.) and its minimal MFDs. The i.o.t.f can be readily obtained from the i.s.t.f. by multiplication with the output matrix, in both state–space and MFD descriptions. Furthermore it is shown that, when similarity is applied, the composite matrix of the i.s.t.f. MFD of the transformed system multiplied by a unimodular matrix is equal to the composite matrix of the original system pre – multiplied by a constant matrix. The latter defines an equivalence relation between MFDs of i.s.t.f. of systems related by similarity. The above unimodular and the constant matrices can be obtained from each other by inspection. The mapping so defined is shown to be an isomorphism.

Section 3 provides a realisation method of the i.s.t.f. into state equations, starting from a coprime and column reduced MFD. The results of this section are used in section 4 where, first, a canonical form for MFDs if i.s.t.f. of systems with state space descriptions related by similarity. Then it is shown that application of the realisation procedure of section 3 to the canonical MFD leads to a canonical state space description.

2. Similarity and unimodular transformations

In this section the problem statement is given, the unimodular – similarity relationship is established and its properties are further investigated. Consider the state - space system denoted by (A, B, C) and described by the equation

$$\dot{x}(t) = Ax(t) + Bu(t),\ y = Cx(t) \tag{1}$$

where $x \in \mathbb{X} \approx \mathbb{R}^n$, $u \in \mathbb{U} \approx \mathbb{R}^\ell$, $(A, B, C) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times \ell} \times \mathbb{R}^{m\times n}$ and $n \geq \ell$. Without loss of generality it is assumed that rank $B = \ell$. System (1) is assumed to be minimal, i.e. $[sI - A, -B]$ and $[sI - A^T, -C^T]^T$ do not have finite Smith zeros Kailath (1980). The controllability indices of a pair (A, B) are given by the right Kronecker indices Gantmacher (1959) of $[sI - A, -B]$ Kailath (1980). Therefore the assumption rank $B = \ell$ means that the system has no controllability indices of value zero.

Before continuing we recall some definitions of notions related to MFDs.

Definition 2.1: *Consider the rational matrix $H(s)$. The factorisation of $H(s)$ in the form $H(s) = N(s)D^{-1}(s)$, where $N(s)$ and $D(s)$ are polynomial matrices, is called (right)* **Matrix Fraction Description** *of $H(s)$ and is denoted by $(N(s), D(s))$. The matrix $T(s) = [N^T(s), D^T(s)]^T$ is called the* **composite matrix** *of the MFD. If $T(s)$ has no Smith zeros and is column reduced Kailath (1980), then the MFD is called* **minimal**. □

It is important to note that if $H(s)$ is the i.o.t.f. of system (1), it is strictly proper and the i–th column degree of $N(s)$ are strictly less than the i–th column degree of $D(s)$ Kailath (1980).

Consider now two minimal systems (A_1, B_1, C_1) and (A_2, B_2, C_2) of type (1), giving rise to the same transfer function $G(s)$. Then, it is well known that they are related by similarity transformations i.e. there exists invertible matrix $Q \in \mathbb{R}^{n\times n}$ such that

$$A_2 = Q^{-1}A_1Q,\ B_2 = Q^{-1}B_1,\ C_2 = C_1Q \tag{2}$$

Let $H(s)$ be the i.o.t.f. of the above systems. Then

$$H(s) = C_1(sI - A_1)^{-1}B_1 = C_2(sI - A_2)^{-1}B_2 \tag{3}$$

The i.s.t.f. of (A_1, B_1, C_1) and (A_2, B_2, C_2) are $G_1(s) = (sI - A_1)^{-1}B_1$ and $G_2(s) = (sI - A_2)^{-1}B_2$ respectively. Clearly,

$$G_1(s) = QG_2(s) \tag{4}$$

Let $(N_1(s), D_1(s))$ and $(N_2(s), D_2(s))$ be minimal MFDs of $G_1(s)$ and $G_2(s)$ respectively. The MFDs $(C_1N_1(s), D_1(s))$ and $(C_2N_2(s), D_2(s))$ of $H(s)$ are also minimal and their composite matrices are related as follows Kailath (1980):

$$\begin{bmatrix} C_1N_1(s) \\ D_1(s) \end{bmatrix} = \begin{bmatrix} C_2N_2(s) \\ D_2(s) \end{bmatrix} \cdot U(s) \tag{5}$$

where $U(s)$ is a unimodular polynomial matrix.

We may now give the statement of the problem of relating the unimodular and similarity transformations applied on the MFD and state–space descriptions of system (1) respectively: Given two minimal representations of (1) related by Q as in (2), find $U(s)$ satisfying (5) and conversely, given (5) and $U(s)$ find Q satisfying (2).

Since $C_2 = C_1Q$, we have from (5)

$$C_1[N_1(s) - QN_2(s)U(s)] = 0 \tag{6}$$

$$D_1 = D_2(s)U(s) \tag{7}$$

Note that (6) holds true only if

$$N_1(s) - QN_2(s)U(s) = 0 \tag{8}$$

because otherwise (4) would not be satisfied. We may thus state the following:

Proposition 2.1: *Let (A_1, B_1, C_1), (A_2, B_2, C_2) be minimal state – space realisations of the transfer function $G(s)$. If the MFDs $(C_1N_1(s), D_1(s))$ and $(C_2N_2(s), D_2(s))$ are two minimal MFDs of $G(s)$, Q is the similarity transformation relating the realisations and $U(s)$ the unimodular matrix relating the MFDs as in (5) then,*

$$\begin{bmatrix} N_1(s) \\ D_1(s) \end{bmatrix} = \begin{bmatrix} QN_2(s) \\ D_2(s) \end{bmatrix} \cdot U(s) \tag{9}$$

□

The above equation provides a relation between the composite matrices of the MFDs of the i.s.t.f. of systems (A_1, B_1, C_1) and (A_2, B_2, C_2), involving $U(s)$ and Q.

Let

$$\hat{Q} = \begin{bmatrix} Q & 0 \\ 0 & I_\ell \end{bmatrix}, \tag{10}$$

and

$$T_1(s) = \left[N_1^T(s)\ D_1^T(s)\right]^T, T_2(s) = \left[N_2^T(s)\ D_2^T(s)\right]^T \tag{11}$$

Then (9) can be written as

$$T_1(s) = \hat{Q}T_2(s) \cdot U(s) \tag{12}$$

Without loss of generality, we may assume that $T_1(s)$ and $T_2(s)$ are column ordered i.e. their columns are arranged in ascending order. It is easy to verify the following:

Proposition 2.2: *The relation between two coprime and column reduced composite matrices $T_1(s)$ and $T_2(s)$ defined by (12) , with $\hat{Q}$ as in (10) and $U(s)$ unimodular, is an equivalence relation.* □

In section 4 a canonical form of a composite matrix $T(s)$ under the above equivalence relation is derived. This canonical form combined with the realisation procedure presented in section 3 lead to a canonical form of the state equations under similarity.

In the rest of this section relationship (12) is investigated, it is shown that there is an isomorphism between $U(s)$ and Q and the explicit form of the mapping between them is derived.

Definition 2.2: *The integers as $\sigma_1, \cdots, \sigma_\ell$ are the* **ordered** *controllability indices of (A, B).* □

For simplicity, σ_i above will be referred to as controllability indices.

Lemma 2.1: *Let $(N(s), D(s))$ be a minimal MFD of the i.s.t.f. $(sI - A)^{-1}B$ or equivalently*

$$[sI - A, -B]\begin{bmatrix} N(s) \\ D(s) \end{bmatrix} = 0 \tag{13}$$

and write $N(s) = C_0 S(s)$ where

$$S(s) = block - diag\{\cdots, [1, s, \cdots, s^{\sigma_i - 1}]^T, \cdots\}, i = 1, \cdots, \ell \tag{14}$$

Then C_0 is invertible.

Proof. Equation (13) is equivalent to Karcanias (1979)

$$\begin{bmatrix} B^\perp \\ B^\dagger \end{bmatrix}\begin{bmatrix} sI - A & -B \end{bmatrix}\begin{bmatrix} N(s) \\ D(s) \end{bmatrix} = 0 \tag{15}$$

where $B^\perp$ is a basis matrix for the left null space of B and $B^\dagger$ is a left inverse of B. In Karcanias (1990) it has been shown that when the system (1) is controllable, then the pencil $sB^\perp - B^\perp A$ has only column minimal indices (c.m.i.) Gantmacher (1959) ε_i, $\varepsilon_i = \sigma_i - 1$. Thus, (15) may be further transformed by using strict equivalence transformations Gantmacher (1959), Kailath (1980) of the following type

$$\begin{bmatrix} R_1 & 0 \\ 0 & I \end{bmatrix}\begin{bmatrix} sB^\perp - B^\perp A & 0 \\ sB^\dagger - B^\dagger A & -I_\ell \end{bmatrix}\begin{bmatrix} R_2 & 0 \\ 0 & I \end{bmatrix}\begin{bmatrix} R_2^{-1}N(s) \\ D(s) \end{bmatrix} = 0, \tag{16}$$

to the form

$$\begin{bmatrix} L(s) & 0 \\ sB^\dagger - B^\dagger A & -I_\ell \end{bmatrix}\begin{bmatrix} R_2^{-1}C_0 S(s) \\ D(s) \end{bmatrix} = 0 \tag{17}$$

where $R_1 \in \mathbb{R}^{(n-\ell)\times(n-\ell)}$, $det(R) \neq 0$, $R_2 \in \mathbb{R}^{n\times n}$, $det(R_2) \neq 0$, $L(s)$ is in Kronecker canonical form Gantmacher (1959), Kailath (1980) i.e. $L(s) = \text{block} - \text{diag}\{\ldots, L_{\varepsilon_i}(s), \ldots\}$, $\varepsilon_1 \leq \varepsilon_2 \leq \cdots \leq \varepsilon_\ell$, $L_{\varepsilon_i}(s) = s\,[I_{\varepsilon_i}|0_{\varepsilon_i\times 1}] - [0_{\varepsilon_i\times 1}|I_{\varepsilon_i}]$, $\varepsilon_i = \sigma_i - 1$. Then

$$L(s)R_2^{-1}C_0 S(s) = 0 \tag{18}$$

The above means that the columns of $R_2^{-1}C_0 S(s)$ lie in the right null space of $L(s)$. A basis matrix of the right null space of $L(s)$ is $S(s)$. Thus $R_2^{-1}C_0 S(s) = S(s)\Xi(s), \Xi(s) \in \mathbb{R}^{n\times n}(s)$

Matrix $N(s)$ has full column rank because B, and consequently $G(s)$, has full column rank. Then, since $N(s) = C_0 S(s)$ and Q is invertible, it follows that $R_2^{-1}C_0 S(s) = S(s)\Xi(s)$ has rank ℓ, which in turn means that $det(\Xi(s)) \neq 0$ and therefore $rank(R_2^{-1}C_0 S(s)) = \ell$.

Let now $det(C_0) = 0$. Then, rank $rank(R_2^{-1}C_0) < \ell$, therefore there exists nonzero constant vector $z^T \in \mathbb{R}^n$ such that $z^T R_2^{-1}C_0$. which, in turn, means that $z^T R_2^{-1}C_0 S(s) = z^T S(s)\Xi(s) = 0$.

The latter means that z^T lies in the left null space of $S(s)$. The result follows from the fact that the left null space of $S(s)$ has as minimal basis matrix the matrix $L(s)$ and no constant vector exists in the row span of $L(s)$.

□

The following definition is essential for the deployment of the rest of the paper

Definition 2.3: *The matrices K and $U(s)$ are defined as follows: K is a block matrix with blocks $K_{ij} \in \mathbb{R}^{\sigma_i \times \sigma_j}$ of the following Toeplitz form:*

$$K_{ij} = \begin{cases} \begin{bmatrix} k_{ij0} & \cdots & k_{ij(\sigma_j-\sigma_i)} & \cdots & 0 \\ & \ddots & & \ddots & \\ 0 & \cdots & k_{ij0} & \cdots & k_{ij(\sigma_j-\sigma_i)} \end{bmatrix}, & \sigma_i \leq \sigma_j \\ 0_{\sigma_i \times \sigma_j}, & \sigma_i > \sigma_j \end{cases} \tag{19}$$

and $U(s)$ is a polynomial matrix with entries $u_{ij}(s)$:

$$u_{ij}(s) = \begin{cases} s^{\sigma_j-\sigma_i} k_{ij(\sigma_j-\sigma_i)} + \cdots + k_{ij0}, & \sigma_i \leq \sigma_j \\ 0 & \sigma_i > \sigma_j \end{cases} \tag{20}$$

□

It is easy to verify that

$$KS(s) = S(s)U(s) \tag{21}$$

Consider the partitioning of the set of columns of $S(s)$ into subsets of columns with equal column degrees. Recall that the degree of the $i-th$ column of $S(s)$ is equal to $\sigma_i - 1$. If the number of distinct values of the controllability indices is φ, then the set of integers $\{1, \cdots, \ell\}$ is partitioned into φ subsets of cardinalities $c_1, \cdots, c_\varphi$. Each subset corresponds to a group of controllability indices of the same value. Define $\xi_1 = 1$ and $\xi_i = \sum_{j=1}^{i-1} c_j + 1$. Matrix K can be considered as block diagonal with diagonal blocks defined as follows: The diagonal blocks of K consist of φ matrices $\hat{K}_i$, $i = 1, \cdots, \varphi$ of the following block matrix form

$$\hat{K}_i = \begin{bmatrix} K_{\xi_i\xi_i} & \cdots & K_{\xi_i,(\xi_i+c_i-1)} \\ \vdots & & \vdots \\ K_{(\xi_i+c_i-1),i} & \cdots & K_{(\xi_i+c_i-1),(\xi_i+c_i-1)} \end{bmatrix} \tag{22}$$

Note that when the controllability index σ_i is distinct then $\hat{K}_i = K_{\xi_i\xi_i}$. From (20) we have

$$\begin{aligned} det(U(s)) = \prod_{i=1}^{\varphi} \begin{vmatrix} k_{\xi_i\xi_i0} & \cdots & k_{\xi_i(\xi_i+c_i-1)0} \\ \vdots & & \vdots \\ k_{(\xi_i+c_i-1)\xi_i0} & \cdots & k_{(\xi_i+c_i-1)(\xi_i+c_i-1)0} \end{vmatrix} \\ \triangleq \prod_{i=1}^{\varphi} \Delta_i \end{aligned} \tag{23}$$

Proposition 2.3: *The following hold true*

(i) Matrix $U(s)$ in (21) is unimodular with $det(U(s)) = \prod_{i=1}^{\varphi} \Delta_i$
(ii) $det(K) = \prod_{i=1}^{\varphi} \Delta_i^{\sigma_i}$

Proof. Result (i) is obvious, since the determinants in (23) are constants. Note that when $c_i = 1$ then $\Delta_i = k_{\xi_i \xi_i 0}$. In order to prove (ii), consider the determinant of matrix K in (19). It is equal to the product of the determinants of the diagonal blocks of K. The number of the diagonal blocks is φ and each one of them has dimensions equal to $\sigma_i c_i \times \sigma_i c_i$. Blocks corresponding to $c_i > 1$, form a $c_i \times c_i$ block matrix having as blocks diagonal matrices of dimension σ_i, with k_{ij1} on the diagonal. It can be shown (by using permutations on the rows and columns) that the determinant of such a block matrix is equal to $\Delta_i^{\sigma_i}$ where Δ_i is defined in (23) and the result follows. □

Remark 2.1: *Matrix $U(s)$ of (20) is a unimodular which relates two ordered minimal bases of the same rational vector space (the space spanned by their columns) Wolovich (1974), Forney (1975) and is called* **structured unimodular**. □

Remark 2.2: *Equation (21) can be written as $KS(s)V(s) = S(s)$ where $V(s) = U^{-1}(s)$ is also structured unimodular. Then the pair $(K, U(s))$ is the* **stabilizer** *(i.e. the subgroup of the transformation group that leaves $S(s)$ unaltered) MacLane & Birkhoff (1967) of $S(s)$ with respect to the transformation group defined by pre–multiplication by a constant invertible matrix and post–multiplication by a unimodular matrix.* □

In the following example the structure of K and $U(s)$ is clarified.

Example 1: *Consider the case where $\sigma_1 = 2$, $\sigma_2 = \sigma_3 = 3$, $\sigma_4 = 4$. Then $\varphi = 3$, $c_1 = 1$, $c_2 = 2$ $c_3 = 1$ and $\xi_1 = 1$, $\xi_2 = 2$, $\xi_3 = 4$*

$$S^T(s) =, \left[\begin{array}{c|c|c|c} 1\ s & & & \\ \hline & 1\ s\ s^2 & & \\ \hline & & 1\ s\ s^2 & \\ \hline & & & 1\ s\ s^2\ s^3 \end{array}\right] \tag{24}$$

$$K = \begin{bmatrix} K_{11} & K_{12} & K_{13} & K_{14} \\ 0_{2\times 2} & K_{22} & K_{23} & K_{24} \\ 0_{3\times 2} & K_{32} & K_{33} & K_{34} \\ 0_{4\times 2} & 0_{3\times 3} & 0_{3\times 3} & K_{44} \end{bmatrix} \tag{25}$$

where

$$K_{11} = \begin{bmatrix} k_{1,1,0} & 0 \\ 0 & k_{1,1,0} \end{bmatrix}, K_{12} = \begin{bmatrix} k_{1,2,0} & k_{1,2,1} & 0 \\ 0 & k_{1,2,0} & k_{1,2,1} \end{bmatrix}$$
$$K_{13} = \begin{bmatrix} k_{1,3,0} & k_{1,3,1} & 0 \\ 0 & k_{1,3,0} & k_{1,3,1} \end{bmatrix}$$
$$K_{14} = \begin{bmatrix} k_{1,4,0} & k_{1,4,1} & k_{1,4,2} & 0 \\ 0 & k_{1,4,0} & k_{1,4,1} & k_{1,4,2} \end{bmatrix}$$
$$K_{22} = diag(k_{2,2,0}, k_{2,2,0}, k_{2,2,0})$$
$$K_{23} = diag(k_{2,3,0}, k_{2,3,0}, k_{2,3,0})$$

$$K_{24} = \begin{bmatrix} k_{2,4,0} & k_{2,4,1} & 0 & 0 \\ 0 & k_{2,4,0} & k_{2,4,1} & 0 \\ 0 & 0 & k_{2,4,0} & k_{2,4,1} \end{bmatrix}$$
$$K_{32} = diag(k_{3,2,0}, k_{3,2,0}, k_{3,2,0})$$
$$K_{33} = diag(k_{3,3,0}, k_{3,3,0}, k_{3,3,0})$$
$$K_{34} = \begin{bmatrix} k_{3,4,0} & k_{3,4,1} & 0 & 0 \\ 0 & k_{3,4,0} & k_{3,4,1} & 0 \\ 0 & 0 & k_{3,4,0} & k_{3,4,1} \end{bmatrix}$$
$$K_{44} = diag(k_{4,4,0}, k_{4,4,0}, k_{4,4,0}, k_{4,4,0})$$

Notice the diagonal blocks according to (22)

$$\hat{K}_1 = K_{11}, \hat{K}_3 = K_{44}, \hat{K}_2 = \begin{bmatrix} K_{22} & K_{23} \\ K_{32} & K_{33} \end{bmatrix} \tag{26}$$

of K, corresponding to the group of the two equal controllability indices σ_2 and σ_3. It is easy to verify that $\hat{K}_2$ can be transformed by column and row permutations only, into the following matrix

$$K'_2 = \left[\begin{array}{cc|cc|cc} k_{2,2,0} & k_{2,3,0} & 0 & 0 & 0 & 0 \\ k_{3,2,0} & k_{3,3,0} & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & k_{2,2,0} & k_{2,3,0} & 0 & 0 \\ 0 & 0 & k_{3,2,0} & k_{3,3,0} & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & k_{2,2,0} & k_{2,3,0} \\ 0 & 0 & 0 & 0 & k_{3,2,0} & k_{3,3,0} \end{array}\right]$$

The number of row permutations performed for the conversion of $\hat{K}_2$ into K'_2 is equal to the number of column permutations, thus the total number of permutations is even, which means that $det(\hat{K}_2) = det(K'_2) = \Delta_2^3$, where Δ_2 is defined in (23). The determinant of K is $\Delta_1^2 \cdot \Delta_2^3 \cdot \Delta_3^4$. The corresponding matrix $U(s)$ has the block form

$$U(s) = \begin{bmatrix} U_{11}(s) & U_{12}(s) & U_{13}(s) \\ 0_{2\times 1} & U_{22}(s) & U_{23}(s) \\ 0_{1\times 1} & 0_{1\times 2}(s) & U_{33}(s) \end{bmatrix} \tag{27}$$

where

$$U_{11}(s) = k_{1,1,0}$$
$$U_{12}(s) = [\, sk_{1,2,1} + k_{1,2,0} \;\; sk_{1,3,1} + k_{1,3,0} \,]$$
$$U_{13}(s) = s^2 k_{1,4,2} + sk_{1,4,1} + k_{1,4,0}$$
$$U_{22}(s) = \begin{bmatrix} k_{2,2,0} & k_{2,3,0} \\ k_{3,2,0} & k_{3,3,0} \end{bmatrix}$$
$$U_{23}(s) = \begin{bmatrix} sk_{2,4,1} + k_{2,4,0} \\ sk_{3,4,1} + k_{3,4,0} \end{bmatrix}$$
$$U_{33}(s) = k_{4,4,0}$$

Finally,

$$det(U(s)) = k_{1,1,0} \cdot \left| \begin{bmatrix} k_{2,2,0} & k_{2,3,0} \\ k_{3,2,0} & k_{3,3,0} \end{bmatrix} \right| \cdot k_{4,4,0}$$

Proposition 2.4: *Let K and Λ be two matrices of the form (19). Then the product $K\Lambda$ is in the form (19) i.e. the set of matrices corresponding to the same set of controllability indices, is closed under matrix multiplication.*

Proof: Let the K, Λ be partitioned into blocks as in (19) and denote by P their product i.e.

$$P_{ij} = \sum_{\mu=1}^{\ell} K_{i\mu}\Lambda_{\mu j} \tag{28}$$

and consider the row vector obtained by multiplying the first row of $K_{i\mu}$ by $\Lambda_{\mu j}$. The last $\sigma_i - 1$ entries of this vector are zero due to the "band" structure of $K_{i\mu}$ and $\Lambda_{\mu j}$. Denote this vector by $[\hat{t}^{\mu}_{ij}, 0_{\sigma_i-1}]$ where $\hat{t}^{\mu}_{ij} = [t^{\mu}_{ij0}, \cdots, t^{\mu}_{ij(\sigma_j-\sigma_i)}]$. The rest of the rows $2, \cdots \sigma_i$ of $K_{i\mu} \cdot \Lambda_{\mu j}$ are obtained by "shifting" $\hat{t}^{\mu}_{ij}$ to the right, one position per row. Then

$$K_{i\mu} \cdot \Lambda_{\mu j} = \begin{bmatrix} \hat{t}^{\mu}_{ij} & 0 & \cdots & 0 \\ & \ddots & \ddots & \\ 0 & \cdots & 0 & \hat{t}^{\mu}_{ij} \end{bmatrix}_{\sigma_i \times \sigma_j}$$

and from (28)

$$P_{ij} = \begin{bmatrix} \hat{t}_{ij} & 0 & \cdots & 0 \\ & \ddots & \ddots & \\ 0 & \cdots & 0 & \hat{t}_{ij} \end{bmatrix}_{\sigma_i \times \sigma_j}$$

where $\hat{t}_{ij} = \sum_{\mu=1}^{\ell} \hat{t}^{\mu}_{ij}$, which proves the result. □

Proposition 2.5: *The inverse of matrix K in (19) has the same form with K.*

Proof: Only the case of distinct controllability indices is going to be considered, for the sake of simplicity. Then $\hat{K}_i = K_{ii}$ (see (22)). In order to prove the Proposition we must show that, given an invertible K of the form (19), the equation

$$K \cdot \Lambda = I \tag{29}$$

has a solution with respect to Λ, where Λ has the same block structure (including the dimensions of blocks) with K. Similarly to Proposition 2.4, let $P = K\Lambda = I$ and assume that Λ is upper diagonal with block dimensions as in (19). Then

$$P_{ij} = \sum_{\mu=1}^{\ell} K_{i\mu}\Lambda_{\mu j} \tag{30}$$

The block triangular structure of K and Λ imply that P is also block triangular with the same block dimensions. For (29) to hold true we must have

$$P_{ii} = I_{\sigma_i}, P_{ij} = 0_{\sigma_i \times \sigma_j}, i \neq j, i, j \in 1, \cdots, \ell \tag{31}$$

The above equations imply that

$$P_{ii} = \Lambda_{ii} \cdot K^{-1}_{ii} = I_{\sigma_i} \ \text{ or } \ \Lambda_{ii} = K^{-1}_{ii} \tag{32}$$

By equating the non-diagonal blocks $P_{ij}, i \neq j$ to zero we can see that (29) is solvable with respect to Λ as follows: Start from the equation $P_{\ell-1,\ell} = K_{\ell-1,\ell-1}\Lambda_{\ell-1,\ell} + K_{\ell-1,\ell}\Lambda_{\ell,\ell} = 0$ or

$$\Lambda_{\ell-1,\ell} = -K^{-1}_{\ell-1,\ell-1}K_{\ell-1,\ell}K^{-1}_{\ell,\ell} \tag{33}$$

The next equation considered is $P_{\ell-2,\ell} = 0$ or

$$K_{\ell-2,\ell-2}\Lambda_{\ell-2,\ell} + K_{\ell-2,\ell-1}\Lambda_{\ell-1,\ell} + K_{\ell-2,\ell}\Lambda_{\ell,\ell} = 0 \text{ or}$$

$$\Lambda_{\ell-2,\ell} = -K^{-1}_{\ell-2,\ell-2}\left(K_{\ell-2,\ell-1}K^{-1}_{\ell-1,\ell-1}K_{\ell-1,\ell} - K_{\ell-2,\ell}\right)K^{-1}_{\ell,\ell} \tag{34}$$

Proceeding this way, on the ℓ–th block–column of $P(s)$ we obtain $\Lambda_{\ell-2,\ell}, \cdots, \Lambda_{1,\ell}$. Next we apply the same procedure to the block – columns $\ell-1, \cdots, 1$ and obtain the inverse of K. All blocks of Λ are given by expressions involving products of blocks of K in the fashion of Proposition 2.4 an therefore, their (Toeplitz) structure is that of equation (19). □

Based on the above results it is easy to show the following:

Proposition 2.6: *The set of matrices with the structure defined in equation (19) for a given set of controllability indices $\sigma_1, \cdots, \sigma_\ell$, endowed with the operations of matrix addition and multiplication, is a ring.* □

The detailed relation between the similarity transformations and the corresponding unimodular matrices as it is defined in (12) is given by the following result.

Theorem 2.1: *Consider two systems with corresponding i.s.t.f. coprime and column reduced composite matrices related by (12), i.e. $T_1(s) = \hat{Q}T_2(s)\cdot U(s)$. Let $N_1(s) = C_{0_1}S(s)$ and $N_2(s) = C_{0_2}S(s)$ be the numerator matrices of $T_1(s)$ and $T_2(s)$ as in (11). Then, (i) $U(s)$ is unimodular matrix of the form (20) and (ii) $Q^{-1} = C_{0_2}KC^{-1}_{0_1}$ where K is derived from $U(s)$ by (19).*

Proof: Result (i) readily follows from the fact that both composite matrices in (12), i.e. $T_1(s)$ and $\hat{Q}T_2(s)$ are coprime and column reduced, i.e. are minimal bases of the same vector space. Additionally, they are considered as column ordered (see (12)). Thus, according to Remark 2.1, $U(s)$ if of the form (20). (ii)From Lemma 2.1 we have that C_{0_1} and C_{0_2} are invertible. From (12) we take

$$C_{0_1}S(s) = QC_{0_2}S(s)U(s) \tag{35}$$

or

$$C^{-1}_{0_2}Q^{-1}C_{0_1}S(s) = S(s)U(s) \tag{36}$$

The above equation is actually equation (21) with $K = C^{-1}_{0_2}Q^{-1}C_{0_1}$. □

Theorem 2.2: *Let $\mathscr{K}_\sigma$ be the set of matrices of the form (19) corresponding to the set σ of integers (controllability indices) and $\mathscr{U}_\sigma$ the set of unimodular matrices of the form (20). The map*

$$\mathscr{F} : \mathscr{U}_\sigma \longrightarrow \mathscr{K}_\sigma \tag{37}$$

defined by (19) –(20) is an isomorphism.

Proof: We have to prove that $\mathscr{F}$ is bijective i.e. it is (i) injective and (ii) surjective.

(i) Let $U_1(s)$ and $U_2(s) \in \mathscr{U}_\sigma$. Then $F(U_1(s)) = K_1$ and $F(U_2(s)) = K_2$ with $K_1, K_2 \in \mathscr{K}_\sigma$. If $K_1 = K_2$ it readily follows that $U_1(s) = U_2(s)$.

(ii) Given matrix $M \in \mathscr{K}_\sigma$, there always exists a polynomial matrix $U(s) \in \mathscr{U}_\sigma$ such that $F(U(s)) = K_1$ as it is clearly derived from equations (19) – (20). □

An immediate consequence of the above two theorems (since $Q^{-1} = C_2 K C_1^{-1}$) is the following:

Corollary 2.1: *The relationship between matrices Q and $U(s)$ in (36) is an isomorphism.* □

3. Controller type realisation of the input – state transfer function

In realisation theory the transfer function considered is the input – output transfer function of the system which is invariant under similarity transformations. Here, the realisation is obtained with regard to the input – state transfer function. As it was pointed out in Section 2 (see (4)), the transfer functions of similar systems are equal modulo pre – multiplication with the similarity matrix. The controllability pencil of a controllable system is the canonical (in echelon form, see Forney (1975)) basis matrix of the left null space of the composite matrix of the MFD of the i.s.t.f. Thus it is uniquely defined, which is equivalent to the statement that the realisation of an i.s.t.f. in state – space form, is uniquely defined. Next, we proceed to the realisation of a given minimal MFD $(N(s), D(s))$ of the i.s.t.f. in the form of a pair (A, B). This realisation is obtained by inspection from the MFD.

The construction is the one proposed in Forney (1975) with the difference that there is no output matrix C in the realisation or, equivalently, C is the identity matrix.

Write the numerator of the MFD as $N(s) = C_0 S(s)$ and the denominator as (see Kailath (1980))

$$D(s) = D_{hc} \cdot diag(s^{\sigma_1}, \cdots, s^{\sigma_\ell}) + D_{lc} \cdot S(s) \tag{38}$$

and consider the equation

$$[sM - H, -I] \begin{bmatrix} C_0 S(s) \\ D(s) \end{bmatrix} = 0 \text{ or} \tag{39}$$

$$\left[s\hat{M} - \hat{H}, -D_{hc}^{-1} \right] \hat{T}(s) = 0 \tag{40}$$

where $\hat{M} = D_{hc}^{-1} M C_0$ and $\hat{H} = D_{hc}^{-1} H C_0$ and $\hat{T}(s) = [S^T(s) \; D^T(s)]^T$. For (39) to hold true, $\hat{M}$ and $\hat{H}$ must be such that

$$(s\hat{M} - \hat{H}) S(s) = diag(s^{\sigma_1}, \cdots, s^{\sigma_\ell}) + D_{hc}^{-1} \cdot D_{lc} \cdot S(s) \tag{41}$$

Let $s\hat{M} - \hat{H}$ be partitioned according to the partitioning of $S(s)$ as follows

$$s\hat{M} - \hat{H} = [s\hat{M}_1 - \hat{H}_1, \ldots, s\hat{M}_\ell - \hat{H}_\ell] \tag{42}$$

where $\hat{M}_i \in \mathbb{R}^{\ell \times \sigma_i}$ $\hat{H}_i \in \mathbb{R}^{\ell \times \sigma_i}$, $i = 1, \cdots, \ell$. Then from (41) it follows

$$\hat{M}_i = [\, 0_{\ell \times (\sigma_i - 1)}, \; \underline{e}_i \,] \text{ and } \hat{H} = -D_{hc}^{-1} \cdot D_{lc} \tag{43}$$

where $\underline{e}_i$ are the vectors of the orthonormal basis of $\mathbb{R}^\ell$. Denote by $\mu_i(s)$ the i–th row of $s\hat{M} - \hat{H}$, by $\hat{L}_i(s)$ the matrix formed by the rows $\sum_{j=1}^{i-1}(\sigma_j) + 1, \cdots, \sum_{j=1}^{i}(\sigma_j)$ of $L(s)$ in (17) and by $\hat{d}_i$ the i–th row of D_{hc}^{-1}. Then the matrix

$$\left[\begin{array}{c} \hat{L}_1(s) \\ \mu_1(s) \\ \hline \vdots \\ \hline \hat{L}_\ell(s) \\ \mu_\ell(s) \end{array} \right] \tag{44}$$

is of the form $sI - \hat{A}$. If $\hat{B}$ is defined as

$$\hat{B} = \begin{bmatrix} 0_{(\sigma_1-1)\times\ell} \\ \hat{d}_1 \\ \hline \vdots \\ \hline 0_{(\sigma_\ell-1)\times\ell} \\ \hat{d}_\ell \end{bmatrix} \tag{45}$$

we have that

$$\left[sI - \hat{A}, -\hat{B}\right] \hat{T}(s) = 0 \tag{46}$$

The dimensions of $\hat{L}_i(s)$ and the blocks which form $\hat{B}$ are derived from the controllability indices (column degrees) of the MFD while the coefficients of the nontrivial rows of $\hat{A}$ and are $\hat{B}$ are taken by inspection from $D(s)$.

From the above equations (43)–(46) it is clear that the realisation $(\hat{A}, \hat{B})$ is derived directly from the MFD of the input-state transfer function. The pair $(\hat{A}, \hat{B})$ is a realisation of the given input state transfer function pre–multiplied by C_0, therefore it is related to the realisation (A, B) sought, by the similarity transformation C_0^{-1} (note that from Lemma 2.1 we have that C_0 is invertible) as follows:

$$A = C_0 \hat{A} C_0^{-1}, \; B = C_0 \hat{B} \tag{47}$$

4. Realisations and canonical forms

In the theory of canonical forms under similarity, the canonical representation is related to the denominator matrix of the echelon canonical form of the composite matrix of the MFD of the i.s.t.f. Kailath (1980), Popov (1969), Forney (1975). The strict properness of a state space system (A, B) ensures that when $T(s)$ is in echelon form, $D(s)$ is also in echelon form and vice versa. Then the so called Popov parameters Kailath (1980), Popov (1972) , of the canonical representation, are directly related to the echelon form of $D(s)$. Actually they are the coefficients of the polynomial entries of the latter polynomial matrix. This is the *controllable* canonical form Kailath (1980) and it can be obtained by any of the i.s.t.f. corresponding to the orbit of the systems related by similarity. Each one member of the equivalence class of the system (A, B) has its uniquely defined corresponding minimal composite matrix $T(s)$ in echelon form, since the composite matrix is a basis matrix of the right null space of the controllability pencil (recall that $[sI - A, -B]T(s) = 0$, $T(s)$ is a minimal basis and $[sI - A, -B]$ has full row rank). All the echelon composite matrices share the same denominator and differ on the numerators as shown below:

Proposition 4.1: *Consider two pairs (A_1, B_1) and (A_2, B_2) related by similarity transformation Q and let*

$$[sI - A_1, -B_1]\, T_1(s) = 0, \quad [sI - A_2, -B_2]\, T_2(s) = 0 \tag{48}$$

where

$$T_1(s) = \begin{bmatrix} N_1(s) \\ D_1(s) \end{bmatrix}, \quad T_2(s) = \begin{bmatrix} N_2(s) \\ D_2(s) \end{bmatrix} \tag{49}$$

and $T_1(s)$, $T_2(s)$ coprime and column reduced. Write

$$T_{1ech}(s) = T_1(s)U_1(s) = \begin{bmatrix} N_{1ech}(s) \\ D_{1ech}(s) \end{bmatrix} \tag{50}$$

$$T_{2ech}(s) = T_2(s)U_2(s) = \begin{bmatrix} N_{2ech}(s) \\ D_{2ech}(s) \end{bmatrix} \tag{51}$$

where $U_1(s)$ and $U_2(s)$ unimodular matrices, and $T_{1ech}(s)$, $T_{2ech}(s)$ are in echelon form Forney (1975), then $D_{1ech}(s)$ is in echelon form, $D_{1ech}(s) = D_{2ech}(s)$ and $N_{2ech}(s) = QN_{1ech}(s)$.

Proof: Let $A_2 = Q^{-1}A_1Q, \quad B_2 = Q^{-1}B_1$. Then the second of (48) yields

$$Q^{-1}\left[sI - A_1, -B_1\right] \begin{bmatrix} Q & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} N_2(s) \\ D_2(s) \end{bmatrix} U_2(s) = 0 \tag{52}$$

or

$$[sI - A_1, -B_1] \begin{bmatrix} QN_{2ech}(s) \\ D_{2ech}(s) \end{bmatrix} = 0 \tag{53}$$

The column degrees of $N_{2ech}(s)$ are strictly less than the corresponding column degrees of $D_{2ech}(s)$ because the system is strictly proper. This means that the matrix

$$\begin{bmatrix} QN_{2ech}(s) \\ D_{2ech}(s) \end{bmatrix} \tag{54}$$

is in echelon form so $D_{2ech}(s)$ is also in echelon form. Furthermore, matrix in (54) is equal to $T_{1ech}(s)$ which yields that $D_{2ech}(s) = D_{1ech}(s)$ and $N_{2ech}(s) = QN_{1ech}(s)$. □

Between the composite matrices corresponding to the equivalence class defined by a system (A, B) we distinguish the following

$$T_c(s) = \begin{bmatrix} S(s) \\ D_{ech}(s) \end{bmatrix} \tag{55}$$

In the rest of the paper $T_c(s)$ will be referred to as the *canonical composite matrix* of the system (A, B). Obviously, $T_c(s)$ is the canonical composite matrix of all systems in the orbit of (A, B).

Remark 4.1: *The term "canonical" above, is justified as follows: Relation (12) is an equivalence relation for the coprime and column reduced composite matrices of the state – space systems belonging to the same orbit with respect to similarity. More specifically, we may define the following transformation on this set of composite matrices*

$$\hat{Q}^{-1}T_1(s)V(s) = T_2(s) \tag{56}$$

where $V(s) = U^{-1}(s)$ and $U(s)$ is defined in (12). Then it is quite straightforward that $T_c(s)$ is the canonical form of all composite matrices of this orbit, under this transformation. □

Consider now the realisation of $T_c(s)$ of section 3. For this realisation we have that $C_0 = I$, therefore $A = \hat{A}$ and $B = \hat{B}$ (see (47)). We have the following

Theorem 4.1: *The realisation $(\hat{A}, \hat{B})$ of $T_c(s)$ is canonical.*

Proof: If we apply the similarity transformation C_0^{-1} (the coefficient matrix of the numerator of the echelon form of the echelon composite matrix of (A, B)) to system (A, B) we end up with $(\hat{A}, \hat{B})$ which is the same for all pairs (A, B) in the same orbit, since $\hat{A}$ and $\hat{B}$ are uniquely defined from the echelon form of the denominator matrix, which is common to all systems in the orbit. The continuous invariants are obtained by the coefficients of the denominator. The discrete invariants are the controllability indices which are equal to the column degrees of $T_c(s)$. □

Given a system (A, B), the similarity transformation leading to the canonical form can be found as follows: Find a coprime an column reduced MFD $(N(s), D(s))$ of the i.s.t.f. of the original system. Then find the echelon form $D_{ech}(s)$ (see Forney (1975)) of $D(s)$.Then the matrix $U^{-1}(s)$ obtained as the solution of $D_{ech}(s) = D(s)U^{-1}(s)$ is the unimoduar matrix transforming the original denominator to the echelon form. Then from (19) find K by inspection. The similarity transformation is given by $Q^{-1} = KC_{0_1}^{-1}$ as stated in Theorem 2.1.

Example 2:

Consider the system MFD of the i.s.t.f. $S(s)D^{-1}(s)$ with

$$S(s) = \begin{bmatrix} 1 & s & & & \\ & & 1 & s & s^2 \end{bmatrix}^T, \quad D(s) = \begin{bmatrix} s^2 & s^3 + s^2 + 1 \\ 2 & s^3 - 2\,s + 2 \end{bmatrix} \tag{57}$$

Here we have $C_{0_1} = I_5$. The corresponding controllability pencil, obtained by using the realisation method of section 3, is

$$[\, sI - A, -B\,] = \left[\begin{array}{cc:ccc|cc} s & -1 & 0 & 0 & 0 & 0 & 0 \\ 2 & s & 3 & -2 & 1 & -1 & -1 \\ \hdashline 0 & 0 & s & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & s & -1 & 0 & 0 \\ 2 & 0 & 2 & -2 & s & 0 & -1 \end{array}\right] \tag{58}$$

The echelon form of the denominator $D(s)$ of the original system and the corresponding $U(s)$ are

$$D_{ech}(s) = \begin{bmatrix} s^2 & 1 \\ 2 & s^3 \end{bmatrix}, \quad U(s) = \begin{bmatrix} 1 & -s+1 \\ 0 & 1 \end{bmatrix} \tag{59}$$

The canonical composite matrix is that of (55) and has corresponding controllability pencil

$$[\, sI - A_c, -B_c\,] = \left[\begin{array}{cc:ccc|cc} s & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & s & 1 & 0 & 0 & -1 & 0 \\ \hdashline 0 & 0 & s & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & s & -1 & 0 & 0 \\ 2 & 0 & 0 & 0 & s & 0 & -1 \end{array}\right] \tag{60}$$

Matrix K is obtained from $U(s)$ according to (19) and (20)

$$K = \left[\begin{array}{cc|ccc} 1 & 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & 1 & -1 \\ \hline 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right] \tag{61}$$

and the similarity transformation matrix is $Q = K$ since $Q^{-1} = KC_{0_1}^{-1}$ and $C_{0_1} = I_5$;

5. Conclusions

In the present paper two minimal types of representations (coprime and column reduced MFDs and minimal state– space) were considered. It was shown that unimodular transformations in the frequency domain (composite matrix of the MFD of input – state transfer function) can be directly mapped to similarity transformations in the time domain (state – space). This mapping was shown to be an isomorphism. Thus, the well known duality of MFD and state–space representations was also established for the transformations on the frequency and time domain respectively. The transformation matrices can be obtained by inspection from each other. In this context, new canonical forms were proposed for both MFD composite matrix and the corresponding state space realisation. The results about canonical forms are based on the fact that the denominator of the echelon form of different input state transfer functions of systems related by similarity, is invariant.

Some extensions of the present results could be the subject of further research: The case of non-proper transfer functions and the corresponding singular system representations can be considered. However in that case the echelon form of the denominator of thr i.s.t.f. is not invariant and the corresponding coordinate transformations for the state equations are the so called restricted system equivalence transformations. The challenge in this case is to define the canonical MFD of the input state transfer function in a way similar to the present paper for state–space systems (see equation (55)).

The results of the paper could also be extended to the case of implicit descriptor systems where the transfer equivalence is replaced by external equivalence and the transfer function by the autoregressive equations relating the external variables of the system.

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