A Bayesian High-Frequency Estimator of the Multivariate Covariance of Noisy and Asynchronous Returns

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Abstract

A multivariate positive definite estimator of the covariance matrix of noisy and asynchronously observed asset returns is proposed. We adopt a Bayesian Dynamic Linear Model which allows us to interprete microstructure noise as measurement errors, and the asynchronous trading as missing observations in an otherwise synchronous series. These missing observations are treated as any other parameter of the problem as typically done in a Bayesian framework. We use an augmented Gibbs algorithm and thus sample the covariance matrix, the observational error variance matrix, the latent process and the missing observations of the noisy process from their full conditional distributions. Convergence issues and robustness of the Gibbs sampler are discussed. A simulation study compares our Bayesian estimator with recently proposed pair-wise QMLE-type and Multivariate Realized Kernel estimators, under different liquidity and microstructure noise conditions. The results suggest that our estimator is superior in terms of RMSE in both a two- and ten-dimensional settings, especially with dispersed and high missing percentages and with high noise. This suggests that our Bayesian estimator is more robust in severe conditions, such as portfolios of assets with heterogeneous liquidity profiles, or particularly illiquid, or when there is a high level of microstructure noise in the market.

Keywords: Asinchronicity; Data Augmentation; Gibbs Sampler; Missing Observations; Realized Covariance.

1 Introduction

Available intra-day prices can be used to improve the estimation of the covariance among several financial assets, so that even the covariation of asset prices within the day can be included in the inferential process. The two main concerns when dealing with several time series of Ultra-High Frequency (UHF) prices is that they are observed at different trading times and with microstructure noise. The first problem is known as asynchronicity of UHF asset prices and its effect on the estimation of the covariance was first identified by Epps (1979), who found that the correlation is biased toward zero as the sampling frequency increases. The Realized Covariance estimator proposed by Hayashi and Yoshida (2005) (HY) is unaffected by this asynchronicity problem. The second feature of UHF asset prices (and in general of asset prices) is that they present a component due to the microstructure noise[1]. A consistent QMLE-type estimator of the high-frequency covariance of two assets observed asynchronously with microstructure noise was introduced by Ait-Sahalia et al. (2010), and this will be our first benchmark.

When instead of the covariance between two assets we consider the covariance matrix of several assets, all estimators mentioned above, that successfully deal with the asynchronicity and noise in the bivariate case, do not guarantee a positive semi-definite estimator in the multivariate setting. To our knowledge, the only work in the literature proposing a multivariate covariance estimator that preserves positivity is the Multivariate Realized Kernel (MRK) of Barndorff-Nielsen et al. (2011), and this will provide our second benchmark. They suggest to synchronize the high frequency prices using a Refresh Time Scheme combined with a multivariate realized kernel to provide a consistent and positive semi-definite estimator of

[1]In the literature, there are several proposed corrections to HY that make it robust to this noise. For example, Voev and Lunde (2007), Bibinger (2011), based on a multiscale subsampling correction of HY that improves the convergence properties of the estimator proposed by Palandri (2006) and the Two-Scales Realized Covariance (TSRC) estimator of Zhang (2011).

the covariance matrix. A drawback of their methodology is that the synchronization of the time series with the Refresh Time Scheme can cause a large loss of information if the involved assets are traded with very different liquidity. Furthermore, they need to tune a bandwidth parameter. Finally, their results are valid only asymptotically, when the mesh of trading interval converges to zero and the number of observations goes to infinity: their simulation study (limited to the bivariate case) shows that there is still an important bias due to the finiteness of the sample and coarse trading intervals.

We are motivated by the need of an unbiased, positive-semidefinite estimator of the multivariate covariance matrix of asynchronous and noisy UHF asset prices. We cast the problem into a Bayesian framework and consider the asynchronous times series as synchronous series with missing observations, that are treated as any other parameter of the problem as typically done in a Bayesian framework. The flexibility of the dynamic linear model we adopt allows us to easily treat the true latent price process, not affected by microstructure noise, as additional parameters to be estimated. Since the joint posterior distribution of the parameters and the missing values is not standard, we use Markov chain Monte Carlo algorithms to obtain samples from it. In particular, the posterior covariance matrix is sampled through a Gibbs sampler from an Inverse Wishart distribution which naturally preserves its positivity.

We present our methodology in Section 2: in 2.1 we introduce the case of asynchronicity without noise, which is extended in 2.2 to noisy observations and propose the augmented Gibbs sampler. In Section 3 several Monte Carlo simulation experiments are performed to compare our Bayesian estimators with the AFX estimator of Ait-Sahalia et al. (2010) and the MRK estimator of Barndorff-Nielsen et al. (2011) in a bivariate (3.1) and multivariate (3.2) setting. The results favour our estimator, particularly for a high number of assets and dispersed missing probabilities. We check convergence properties of the Gibbs sampler and its robustness to strong microstructure noise and high missings percentages in Sections 4.1 and 4.2, respectively. Section 5 concludes.

2 Methodology

2.1 Asynchronicity without noise

We start by considering the model with asynchronous prices that are observed at different times within the day, but without being contaminated by microstructure noise. The simplified model is

$$d\mathbf{X}_t \;\; = \mu(\mathbf{X}_t, \theta)dt + \;\; \sqrt{\Sigma(\mathbf{X}_t, \theta)}d\mathbf{W}_t \tag{1}$$

where, for some compact $\Theta \subseteq \mathbb{R}^k$, θ the unknown parameter vector, $\mu : \mathbb{R}^d \times \Theta \to \mathbb{R}^d$, $\Sigma : \mathbb{R}^d \times \Theta \to \mathbb{R}^{d \times d,+}$, $\mathbf{X}_t$ is a d-dimensional log-price diffusion process, $\mathbf{W}_t$ is a d-dimensional Brownian motion and $\mathbb{R}^{d \times d,+} := \{M \in \mathbb{R}^{d \times d} : M > 0, \text{ symmetric}\}$, that is the space of square, $d \times d$ positive definite symmetric matrices. We assume that the drift and the diffusion functions satisfy the Lipschitz condition

$$||\mu(x, \theta) - \mu(y, \theta)|| + ||\sqrt{\Sigma(x, \theta)} - \sqrt{\Sigma(y, \theta)}|| \leq C||x - y|| \tag{2}$$

for some positive constant C and with $|| \cdot ||$ indicating the Euclidean norm. We need this assumption to ensure the existence of a strong (unique), square integrable solution $\mathbf{X}_t$ to (1) (Oksendal (2002)).

We work with the discretized version ($\Delta t = 1$) of (1) with constant diffusion coefficient and zero drift, so that the inference is directely conducted on Σ. Assuming a deterministic drift equal to zero is reasonable since we are dealing with infra-day log-prices. Furthermore, we recognize that the volatility is a time-varying process, but we are interested in the estimation of a constant volatility parameter for each day. For this purpose we rely on the recent result of Xiu (2010) that shows that the QMLE of the volatility of a misspecified model with constant volatility remains consistent and optimal in terms of its rate of convergence under fairly general assumptions. Let us first consider the bivariate case, for $i = 1, 2$

$$X_{i,t} \quad = \quad X_{i,t-1} + \epsilon_{i,t} \quad \epsilon_{i,t} \sim N(0, \sigma_i^2), \tag{3}$$

with $\mathrm{corr}(\epsilon_{1,t}\epsilon_{2,t}) = \rho$, $X_{i,t}$ is the log-price of asset i observed at time t, σ_i is its volatility and ρ is the correlation coefficient.

We define $\Sigma = \begin{bmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix}$, the missing and observed parts of $\mathbf{X}$ respectively as $\mathbf{X}^{miss}$ and $\mathbf{X}^{obs}$ and partition the time interval $[1, \cdots, T]$ as $[\mathbf{t}_i^{miss}, \mathbf{t}_i^{obs}]$ accordingly. The likelihood is

$$L(\Sigma, \mathbf{X}_{1:T}^{miss} | \mathbf{X}_{1:T}^{obs}) \propto \prod_{i=1}^{2} \frac{1}{(1-\rho^2)\sigma_i^2} \exp\left(-\frac{1}{2} \sum_{t \in \mathbf{t}_i^{obs}} \frac{(X_{i,t} - m_{i,t})^2}{(1-\rho^2)\sigma_i^2} \right),$$

where $X_{i,s:t}$ indicates the log price of asset i, from time s to time t, both extremes included, $m_{i,t} := X_{i,t-1} - \rho\frac{\sigma_i}{\sigma_{-i}}(X_{-i,t} - X_{-i,t-1})$ and $-i$ is the other asset. We assume a Jeffrey's uninformative prior for Σ, $p(\Sigma) \sim |\Sigma|^{-3/2}$, but an informative prior that incorporates in the analysis prior knowledge on the problem can easily be adopted. For example, in an empirical Bayesian approach, an Inverse Wishart prior for Σ, with parameters T_0 and S_0 being respectively, the sample size and the sum of squared observed returns, ignoring the missing data, would still retain the conjugacy of the problem.

The asynchronicity (or, in other words, the presence of missing observations) complicates the form of the likelihood, but, once we condition on missing observations, we can still use standard results from multivariate normal theory to derive the full conditional for Σ:

$$p(\Sigma|\mathbf{X}) \propto |\Sigma|^{-(T+3)/2} \exp\left(-\frac{1}{2}\mathrm{tr}(\Sigma^{-1}\sum_{t=1}^{T}\epsilon_t\epsilon_t')\right) \propto IW\left(\sum_{t=1}^{T}(\mathbf{X}_t - \mathbf{X}_{t-1})(\mathbf{X}_t - \mathbf{X}_{t-1})', T\right)$$

and find that $\phi(X_{i,t}^{miss}|\Sigma)$, the full conditionals of $X_{i,t}^{miss}$, $i = 1, 2$, are normal:

$$X_{i,t}^{miss}|X_{i,1:t-1}, X_{-i,1:t}, \Sigma \sim N\left(m_{i,t}, (1-\rho^2)\sigma_i^2\right),$$

where the subscript $-i$ refer to the other asset. The extension of (3) to the multivariate case is straightforward and sampling the covariance matrix from an Inverse Wishart assures that the resulting estimate is positive definite. As mentioned in the introduction, up to our knowledge, this is the first attempt to have a positive definite multivariate estimator, besides **?**, who operate in a frequentist context and pay the price of disregarding many observations, due to the Refresh Time Scheme they adopt. The generic d-dimensional discretized model is, for $i = 1, \cdots, d$:

$$X_{i,t} = X_{i,t-1} + \epsilon_{i,t} \quad \epsilon_{i,t} \sim N(0, \sigma_i^2), \tag{4}$$

with $\mathrm{corr}(\epsilon_{i,t}\epsilon_{j,t}) = \rho_{ij}$. $\mathbf{X}$ and Σ are, respectively, a d x T and a d x d matrix. We write the likelihood as

$$L(\Sigma, \mathbf{X}_{1:T}^{miss}|\mathbf{X}_{1:T}^{obs}) \propto \prod_{i=1}^{d} v_i^{-1} \exp\left(-\frac{1}{2}\sum_{t\in\mathbf{t}_i^{obs}} v_i^{-1}(X_{i,t} - \tilde{m}_{i,t})^2\right),$$

where $v_i := \sigma_i^2 - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{12}'$, Σ_{22} is obtained from Σ by dropping the row and column corresponding to asset i, whilst Σ_{12} is the i-th row of Σ without its i-th element, $\tilde{m}_{i,t} := X_{i,t-1} + \Sigma_{12}\Sigma_{22}^{-1}(\mathbf{X}_{-i,t} - \mathbf{X}_{-i,t-1})$ and $\mathbf{X}_{-i,s:t}$ is the matrix of assets log prices j, $\forall j \neq i$ and from time s to time t. We have suppressed the dependence of Σ_{12} and Σ_{22} on i for notational convenience. The Jeffrey's uninformative prior for Σ is now $p(\Sigma) \sim |\Sigma|^{-(d+1)/2}$ and its full conditional is

$$p(\Sigma|\mathbf{X}) \propto |\Sigma|^{-(T+d+1)/2} \exp\left(-\frac{1}{2}\mathrm{tr}(\Sigma^{-1}\sum_{t=1}^{T}\epsilon_t\epsilon_t')\right) \propto IW\left(\sum_{t=1}^{T}(\mathbf{X}_t - \mathbf{X}_{t-1})(\mathbf{X}_t - \mathbf{X}_{t-1})', T\right)$$

The full conditional of $X_{i,t}^{miss}$, $\phi(X_{i,t}^{miss}|\Sigma)$, is easily derived as:

$$X_{i,t}^{miss}|X_{i,1:t-1}, \mathbf{X}_{-i,1:t}, \Sigma \sim N\left(\tilde{m}_{i,t}, v_i\right).$$

To sample from the posterior distribution of X^{miss} and Σ, we use a Gibbs sampler and iteratively sample $\{\mathbf{X}^{miss}, \Sigma\}$ from their full conditionals. The algorithm at each iteration thus consists of the following two steps:

1. *Draw a covariance matrix Σ from its full conditional, that is an Inverse Wishart distribution, $IW(S(\boldsymbol{X}^{miss}), T)$, with $S(\boldsymbol{X}^{miss}) = \sum_{t=1}^{T}(\boldsymbol{X}_t - \boldsymbol{X}_{t-1})(\boldsymbol{X}_t - \boldsymbol{X}_{t-1})'$. S is expressed as a function of $\boldsymbol{X}^{miss}$ to highlight the dependence on the imputed missing log prices.*

2. *Impute the missing observations, $\boldsymbol{X}^{miss}$, by drawing from $\phi(X_{i,t}^{miss}|\Sigma)$, $\forall t \in \boldsymbol{t}^{miss}$, $i = 1, \cdots, d$.*

2.2 Asynchronicity and noise

A step toward a more realistic model is made by introducing the microstructure noise, so that our model becomes, for $t = 1, \cdots, T$

$$\mathbf{Y}_t = \mathbf{X}_t + \sqrt{\Omega}d\mathbf{B}_t \tag{5}$$

$$d\mathbf{X}_t = \mu(\mathbf{X}_t, \theta)dt + \sqrt{\Sigma(\mathbf{X}_t, \theta)}d\mathbf{W}_t, \tag{6}$$

where μ, Σ, $\mathbf{X}_t$ and $\mathbf{W}_t$ are defined as in (1), Ω is the constant variance of the microstructure noise term, Y_t is a d-dimensional observed (with noise) log-price diffusion process, $\mathbf{B}_t$ is a d-

dimensional Brownian motion, $\mathbf{B}_t \perp \mathbf{W}_t$. Assume that the Lipschitz condition (2) is satisfied. The discretized version of (5) and (6) with constant diffusion coefficient is

$$\mathbf{Y}_t = \mathbf{X}_t + \eta_t \quad , \quad \eta_t \sim N(\mathbf{0}, \Omega), \tag{7}$$

$$\mathbf{X}_t = \mathbf{X}_{t-1} + \epsilon_t \quad , \quad \epsilon_t \sim N(\mathbf{0}, \Sigma), \tag{8}$$

where $\mathbf{Y}_t$ is the observed log-price, $\mathbf{X}_t$ is considered as the true latent log-price process, η_t is the microstructure noise and Ω its covariance matrix, ϵ is the true latent return with covariance Σ. η_t and ϵ_t are assumed independent.

This is a linear state space model, consisting in the observation equation (7) and the state equation (8). In this particular form, is also known in the literature as local level model, or random walk plus noise model or steady forecasting model, extensively covered in Harvey (1989) and in West and Harrison (1997). Despite its simplicity, the local level model can be used to analyse real data sets in various settings and scenarios, as it has been pointed out by many authors, see e.g. Durbin (2004) or Triantafyllopoulos (2010). The model can also be viewed as a particular case of a Dynamic Linear Model (West and Harrison (1997)), DLM in short, characterized in its general form by $\{A, C, R, Q\}$, respectively the observation matrix, transition matrix, observation error variance matrix and transition error variance matrix, possibly time-varying. Then, our model is a time-invariant DLM with matrices $\{I_d, I_d, \Omega, \Sigma\}$.

Under this model, the observed log-return follows an MA(1) process (Ait-Sahalia et al. (2010)), and we can still obtain the likelihood in a product form by noting that $\{\epsilon_t | \mathbf{Y}_t, \mathbf{X}_{t-1}, \Omega\}_{t=1}^T$ are i.n.i.d (independent but not identically distributed) Gaussian random vectors with zero means and covariance matrix $\tilde{\Sigma}_t = V_t^{t-1} + \Omega$. V_t^{t-1} is the variance of the prediction error obtained through the Kalman filter. Hence, the likelihood is

$$L(\Sigma, \Omega, \mathbf{Y}_{1:T}^{miss}, \mathbf{X}_{1:T}|\mathbf{Y}_{1:T}^{obs}) \propto \prod_{i=1}^{d} \prod_{t \in \mathbf{t}_i^{obs}} \tilde{v}_{i,t}^{-1} \exp\left(-\frac{1}{2}\tilde{v}_{i,t}^{-1}(X_{i,t} - \tilde{m}_{i,t})^2\right),$$

with $\tilde{v}_{i,t} := \tilde{\sigma}_{i,t}^2 - \tilde{\Sigma}_{12,t}\tilde{\Sigma}_{22,t}^{-1}\tilde{\Sigma}_{12,t}'$ and $\mathbf{Y}^{miss}$ and $\mathbf{Y}^{obs}$ are the missing and observed parts of $\mathbf{Y}$ corresponding to the partition $[\mathbf{t}^{miss}, \mathbf{t}^{obs}]$. If we assume uninformative Jeffrey's priors for Σ and Ω, the complete date joint posterior density of Σ and Ω is proportional to

$$(|\Omega||\Sigma|)^{-\frac{d+1}{2}} \prod_{t=1}^{T} N(\mathbf{Y}_t; \mathbf{X}_t, \Omega) N(\mathbf{X}_t; \mathbf{X}_{t-1}, \Sigma).$$

The Gibbs sampler can be used also in this more complicated setting since the full conditionals are available in standard form. In our simulation approach, we augment the data twice, by considering both the missing observations and the latent process as additional parameters. Therefore, we implement the algorithm by iteratively sampling $\{\mathbf{X}, \Sigma, \Omega, \mathbf{Y}^{miss}\}$ from their full conditional densities. The full conditionals of Σ and Ω are still proportional to Inverse Wishart: $p(\Sigma|\Omega, \mathbf{Y}_{1:t}, \mathbf{X}_{1:t}) \propto IW(SS_\Sigma, T)$and $p(\Omega|\Sigma, \mathbf{Y}_{1:t}, \mathbf{X}_{1:t}) \propto IW(SS_\Omega, T)$, with $SS_\Sigma = \sum_{t=1}^{T}(\mathbf{X}_t - \mathbf{X}_{t-1})(\mathbf{X}_t - \mathbf{X}_{t-1})'$and $SS_\Omega = \sum_{t=1}^{T}(\mathbf{Y}_t - \mathbf{X}_t)(\mathbf{Y}_t - \mathbf{X}_t)'$.

To sample the missing observations, we partition $\mathbf{Y}$ in $[\mathbf{Y}^{miss}, \mathbf{Y}^{obs}]$ and the full conditional density of the missing observations are still available in a standard form since they are obtained as the conditional normal density that result from the joint density of observed and missing log-prices $\mathbf{Y}_t|\mathbf{Y}_{1:t-1}, \mathbf{X}, \Sigma, \Omega \propto N(\mathbf{X}_t, \Omega)$, that is $Y_{i,t}^{miss}|Y_{i,1:t-1}, \mathbf{Y}_{-i,1:t}, \mathbf{X}, \Sigma, \Omega$, distributed as $N\left(\mathbf{X}_{i,t} + \Omega_{12}\Omega_{22}^{-1}(\mathbf{Y}_{-i,t} - \mathbf{X}_{-i,t}), \omega_i^2 - \Omega_{12}\Omega_{22}^{-1}\Omega_{12}'\right)$, where $\mathbf{Y}_{-i,s:t}$ is the matrix of log-prices for assets j, $\forall j \neq i$, from time s to time t and, similarly, $\mathbf{X}_{-i,s:t}$ is the matrix of latent log-prices sampled with the FFBS algorithm for assets j, $\forall j \neq i$ and from time s to time t. Ω_{22} is obtained from Ω by dropping the row and column corresponding to asset i, and Ω_{12} is the i-th row of Ω without its i-th element.

Finally, we extract the latent log-priceby using the FFBS (Forward Filtering Backward Simulation) algorithm, a Kalman smoother in which the smoothing recursions are replaced by simulations of the latent process. Following Fruwirth-Schnatter (1994), we can write the distribution of $\mathbf{X}|\mathbf{Y}, \Sigma, \Omega$ as

$$p(\mathbf{X}|\mathbf{Y}, \Sigma, \Omega) = \prod_{t=1}^{T} p(\mathbf{X}_t|\mathbf{X}_{t+1:T}, \mathbf{Y}) \tag{9}$$

where the last factor in the product is simply $p(\mathbf{X}_T|\mathbf{Y})$, i.e., the filtering distribution of $\mathbf{X}_T$, which is $N(\mathbf{X}_t^t, V_t^t)$, with $\mathbf{X}_t^t$ the filtered latent log-price and V_t^t its covariance matrix. In order to obtain a draw from the distribution on the left-hand side, one can start by drawing $\mathbf{X}_T$ from $N(\mathbf{X}_t^t, V_t^t)$ and then, for $t = T-1, T-2, \cdots, 1$, recursively draw $\mathbf{X}_t$ from $p(\mathbf{X}_t|\mathbf{X}_{t+1:T}, \mathbf{Y})$. It can be shown that $p(\mathbf{X}_t|\mathbf{X}_{t+1:T}, \mathbf{Y}) = p(\mathbf{X}_t|\mathbf{X}_{t+1}, \mathbf{Y}_{1:t})$ and this distribution is $N(\mathbf{X}_t^t + V_t^t(V_{t+1}^t)^{-1}(\mathbf{X}_{t+1} - \mathbf{X}_{t+1}^t), V_t^t - V_t^t(V_{t+1}^t)^{-1}V_t^t)$, where $\mathbf{X}_{t+1}^t$ is the predicted latent log-price.

Summarizing, the implemented Gibbs sampler executes the following steps at each iteration:

1. *Draw the covariance matrix Σ from its full conditional, that is an Inverse Wishart distribution $IW(SS_\Sigma, T)$, with $SS_\Sigma = \sum_{t=1}^{T}(\boldsymbol{X}_t - \boldsymbol{X}_{t-1})(\boldsymbol{X}_t - \boldsymbol{X}_{t-1})'$.*

2. *Draw the covariance matrix Ω from its full conditional, that is an Inverse Wishart distribution $IW(SS_\Omega, T)$, with $SS_\Omega = \sum_{t=1}^{T}(\boldsymbol{Y}_t - \boldsymbol{X}_t)(\boldsymbol{Y}_t - \boldsymbol{X}_t)'$.*

3. *Impute, for $i = 1, \cdots, d$ and $t \in \boldsymbol{t}_i^{miss}$, the missing observations $\boldsymbol{Y}^{miss}$ by drawing from $N\left(\boldsymbol{X}_{i,t} + \Omega_{12}\Omega_{22}^{-1}(\boldsymbol{Y}_{-i,t} - \boldsymbol{X}_{-i,t}), \omega_i^2 - \Omega_{12}\Omega_{22}^{-1}\Omega_{12}'\right)$, where the dependence on i of Ω_{12} and Ω_{22} has been suppressed to simplify notation.*

4. *Apply the FFBS algorithm to the DLM $\{I_d, I_d, \Omega, \Sigma\}$ to extract the latent process $\boldsymbol{X}$ from*

its full conditional $\prod_{t=1}^{T} N(m_t, W_t)$*, where we have defined* $m_t \equiv \boldsymbol{X}_t^t + V_t^t (V_{t+1}^t)^{-1} (\boldsymbol{X}_{t+1} - \boldsymbol{X}_{t+1}^t)$ *and* $W_t \equiv V_t^t - V_t^t (V_{t+1}^t)^{-1} V_t^t$.

3 Simulation Study

In this section we compare the performance of our Gibbs estimator with other estimators available in the literature. The first alternative is proposed by Ait-Sahalia et al. (2010) (AFX), who estimate the covariance as a function of variances after sinchronizing the asset returns. They use the Refresh Time Scheme introduced by **?**, which consists in aligning the returns on an irregular time grid by selecting those ticks at which all the assets have been traded at least once in the interval. This scheme includes the largest amount of data among all the Generalized Synchronization Schemes as defined in Ait-Sahalia et al. (2010), but the loss of information still strongly depends on the presence of illiquid assets since several observations for the more liquid assets are neglected within each grid interval. After the sinchronization, they estimate the covariance by applying the QMLE estimator suggested in Ait-Sahalia et al. (2005) to the identity $\mathrm{Cov}(X_1, X_2) = \frac{1}{4}(\mathrm{Var}(X_1 + X_2) - \mathrm{Var}(X_1 - X_2))$, valid for any random variables X_1 and X_2.

The second estimator we include in the comparison study is the Multivariate Realized Kernel of Barndorff-Nielsen et al. (2011) (MRK). MRK synchronizes the high frequency prices using a Refresh Time Scheme combined with a multivariate realized kernel to provide a consistent and positive semi-definite estimator of the covariance matrix. In particular, for its implementation, we choose a jittering parameter equal to 2 and a Parzen kernel function as weigth function for the realized autocovariances. The bandwidth of each series is computed using the true Σ and Ω and the multivariate bandwidth is chosen to be the average of the single bandwidths. Note that we set the bandwidth to its optimal value by using the true Σ and Ω (that cannot be observed in reality) and thus we favour the MRK methodology. For more details, we refer the reader to the original paper. To our knowledge, this is the only estimator that guarantees that the estimated multivariate covariance matrix is positive semidefinite. This property is essential in many applications and, to enforce it, when using Ait-Sahalia et al. (2010) in a multivariate setting, it is suggested to project the resulting estimated matrix into the space of positive semi-definite matrices, by minimizing some notion of distance between the two matrices. The main problem with this projection is that we can lose the financial interpretation of the covariance between the assets since, as noted in Frigessi et al. (2010), some entries of the covariance matrix, upon projection, can dramatically change.

We further report the results of the estimator proposed by Hayashi and Yoshida (2005) (HY), that is the cross-product of all returns with at least a partial overlapping. HY is robust to the asynchronicity but not to the microstructure noise. In our simulation study, we pretend to observe the latent log-price process, in order to apply this estimator to the latent log-prices free of noise, so that this becomes, in our setting, a benchmark unattainable in real applications.

The data generating process is the stochastic volatility model of Heston (1993), for $i = 1, \cdots, d$ and $t = 1, \cdots, T$:

$$\begin{aligned} dX_{i,t} &= \sigma_{i,t} dW_{i,t}, \\ d\sigma^2_{i,t} &= k_i(\bar{\sigma}^2_i - \sigma^2_{i,t}) + s_i \sigma_{i,t} dB_{i,t}, \end{aligned}$$

where $E(dW_{i,t}dW_{j,t}) = \rho_{ij}dt$, $E(dW_{i,t}dB_{j,t}) = \delta_{ij}\pi_i dt$. We use an Euler discretization scheme to generate the data. The first observation for the variance process is drawn from a Gamma distribution $\Gamma(2k_i\bar{\sigma}_i^2/s_i^2, s_i^2/2k_i)$ centered in the mean variance. All the codes have been written in Matlab 7.11.0 (R2010b) and run (possibly in parallel) with Intel(R) Xeon(R) CPU X7460 @ 2.66 GHz.

3.1 Bivariate case

To replicate the same setting adopted in Ait-Sahalia et al. (2010), we first run a bivariate simulation study with: sample size of $T = 10000$, 1/3 and 1/2 of the observations removed completely at random from the whole generated sample, for the first and second asset, respectively, and two starting log prices of log(100) and log(40). The true mean covariance matrix of the returns and the covariance matrix of the microstructure noise are, respectively,

$$\Sigma = \begin{bmatrix} 0.16 & 0.06 \\ 0.06 & 0.09 \end{bmatrix} \quad , \quad \Omega = \begin{bmatrix} 0.08 & 0 \\ 0 & 0.04 \end{bmatrix}.$$

The other parameters $\{k_i, s_i, \pi_i\}$ of the Heston model are $\{6, 0.5, -0.6\}$ and $\{4, 0.3, -0.75\}$ for asset 1 and 2 respectively. For comparison purposes, all parameter values are chosen as in Ait-Sahalia et al. (2010). For each compared estimator we generate $M = 100$ matrices of prices and we run our Gibbs sampler for each generated sample of prices for 5000 steps, after 5000 initial iterations of burn-in. The starting point (of the MCMC simulation) of the missing values is the local mean ignoring the missing data up to 10 ticks before and after the missing trade. To speed up the convergence of the Markov chain, we need to be careful about the initial values for the covariance parameters: We initialize the sampler from the pairwise Hayashi and Yoshida (2005) estimate of the covariance of the partially observed noisy returns series for the off-diagonal terms. As for the variances, we use the Two Time Scale Estimator of Zhang et al. (2005). Then, we interpret the return variability not captured by the Two Time Scale Estimator as explained by the microstructure noise. We stress the fact that different starting values affect the convergence speed of the Markov chain, but not the validity of the results, since the chain is independent on the chosen starting values, once it has reached stationarity. We report the results of this first simulation in Table 1, together with the HY estimator proposed by Hayashi and Yoshida (2005), applied to the latent log-prices free of noise, as unattainable benchmark.

On average, the Gibbs estimator performs better than all other methods in terms of RMSE and bias. The MRK, the only direct competitor in the multivariate setting, has an important bias and is more volatile. This could be quite relevant in empirical studies if we also consider that the results for MRK are obtained with optimal bandwidth parameters computed from the true unobserved Ω and Σ.

3.2 Multivariate case

For the multivariate simulation analysis, we estimate the covariance matrix for a portfolio of 10 assets. Our Gibbs estimator is naturally extended to the 10-dimensional case, as the MRK estimator. If the covariance matrix obtained with the pointwise AFX procedure is not positive semi-definite, we project it onto the space of positive semi-definite matrices by minimizing the Frobenius distance. This projection is done following Higham (2002) algorithm, slightly modified to avoid singular matrices. We also estimate with HY each off-diagonal term of the covariance matrix for comparison purposes. The true data generating process variance matrices are Σ, reported in Table 2, and $\Omega = \text{diag}([0.08, 0.04, 0.02, 0.05, 0.1, 0.06, 0.1, 0.15, 0.15, 0.08])$. The simulation is initialized from $P_0 = log([100, 40, 60, 80, 40, 20, 90, 30, 50, 60])$, with probabilities of missing observations for each series equal to $\{1/2, 1/3, 1/2, 1/4, 1/4, 1/3, 1/5, 1/4, 1/3, 1/4\}$. We note that, as the dimension of the portfolio increases, it becomes more and more difficult to obtain ,through pairwise inferences, an estimated covariance matrix that retains positivity, and the projection onto the space of positive semi-definite matrices can severely distort the estimation.

We iterate a random scan Gibbs sampler for 5000 (burn-in) plus 5000 steps and summarize the results in Figure 1a that shows the RMSE of the estimators. We need to project the pairwise AFX onto the space of positive semidefinite matrices for 27 cases out of 100 Monte Carlo simulations. The superiority of the Bayesian estimator relative to MRK is clear, since the latter is both more biased and volatile. As synthetic measure of the performance we choose the Frobenius norm of the matrix difference Δ between the estimated and the true (used to generate the data) covariance matrix. In particular, we define our norm as $||\Delta_{i,j}|| = ||(\hat{\Sigma}_i^j - \Sigma)||_F$, where $||\cdot||_F$ is the traditional Frobenius norm, and $\hat{\Sigma}_i^j$ is the j-th covariance matrix estimated with methodology i, with $i \in \{Gibbs, AFX, MRK\}$ and $j = 1, \ldots, M$. We further define $\hat{E}[||\Delta_i||] := \frac{1}{M}\sum_{j=1}^{M} ||\Delta_{i,j}||$ and $\hat{\sigma}[||\Delta_i||] := \left(\frac{1}{M-1}\sum_{j=1}^{M}(||\Delta_{i,j}|| - \hat{E}[||\Delta_i||])^2\right)^{1/2}$ as the estimated expected value and standard deviation of the Frobenius distance for methodology i. The relative values are reported in Table 3 and they suggest that the results favour our multivariate Bayesian estimator, with MRK performing worst. In Figure 2a we compare graphically the kernel density of the Frobenius distances for the three methodologies and again we observe that the Gibbs estimator has the best performance, followed by AFX and MRK.

We believe that the performance of the Bayesian estimator should improve relative to that of AFX when we use more dispersed probabilities of observing missing values, that is assets with very different liquidity, since in this case our naturally multivariate Bayesian estimator catches information that the pairwise AFX cannot use. To validate this expected result, we repeat the simulations for 10 assets, holding everything as in the previous simulation setting, but with more dispersed missing probabilities $\{0, 0.5, 0.8, 0.9, 0.25, 0, 0.5, 0.8, 0.9, 0.25\}$. The results in Figure 1b confirm our intuition: now the pairwise nature of the AFX methodology affects more severely the estimation and we indeed need to project the covariance matrix estimated by AFX 73% of the times. The relative norms are summarized in Table 3 and in Figure 2b the relative kernel densities estimates are plotted. The explanation we give is that information contained in the co-movement of more liquid assets is incorporated in the estimation of covariances of less liquid asset only with a multivariate approach. The improvement of the resulting estimates can be quite important in portfolios containing assets with very different liquidity profiles.

4 MCMC convergence issues and robustness

Our model with noise is a non stationary latent Gaussian Bayesian model with Gaussian response variables and we use an MCMC approach for inference. It is well known that MCMC tends to exhibit poor performance when applied to these models. The first reason is that the different points of the latent process $\mathbf{X}$ are strongly dependent on each other. Second, the latent process and Σ are also strongly dependent, especially in large sample settings as ours. This is known in the literature as *Roberts-Stramer critique* (Roberts and Stramer (2001)) and is formalized by noting that

$$\operatorname*{plim}_{\Delta t \to 0} \sum_{t=1}^{T} (\mathbf{X}_t - \mathbf{X}_{t-1})'(\mathbf{X}_t - \mathbf{X}_{t-1}) = \Sigma. \tag{10}$$

This asymptotic relationship between Σ and $\mathbf{X}$ causes, in the limit, the Markov Chain to be reducible, that is unable to escape from the current value.

A common approach to overcome the strong posterior dependence *within* the latent process, is to sample the whole process $\mathbf{X}$ jointly. This is what we do in (9) by using the FFBS algorithm of Fruwirth-Schnatter (1994) and the simulation study results suggest that our Gibbs estimator is not severely affected by the Roberts-Stramer reducibility problem. This happens thanks to the double augmentation scheme, since the sampling of the missing values of the noisy observed process adds a disturbance element to the sampling of the latent process, breaking the deterministic relation between Σ and $\mathbf{X}$. Our doubly augmented algorithm naturally complements the traditional Gibbs sampler that does not sample $\mathbf{Y}^{miss}$: the convergence speed of the traditional Gibbs sampler increases when the signal-to-noise ratio of the problem is higher (Roberts and Sahu (1997)), whilst in our augmented Gibbs sampler, as the observational noise increases, sampling of $\mathbf{X}$ will be more "disturbed" by sampling of $\mathbf{Y}^{miss}$, improving the performance of the Bayesian estimator.

We investigate the robustness of our methodology in two ways: we increase the noise, everything else being fixed, and look at simulation results to validate the hypothesis that with higher noise, the Gibbs sampler works relatively better. Then, to test the robustness to a finer grid, that is to a higher number of points between observations in the latent process, we increase the missing percentage, holding everything else constant.

With an increase in noise, $\mathbf{Y}^{miss}$ introduces an higher "disturbance" to the deterministic relation (10) between the latent process and the covariance matrix. Since the reducibility problem is attenuated, it is natural to expect that the Bayesian estimation improves relative to the alternative methodologies. In Figure 1c we report the results for the different methodologies compared, with noise variance $\Omega_1 = \Omega + 0.35$, and we indeed note that there is an improvement in favor of the Bayesian approach. We project AFX 29% of the times to obtain a positive definite covariance matrix. The estimated expected values and standard deviations of the Frobenius distances are reported in Table 3, Figure 1c compares the RMSEs and components of the methodologies and Figure 2c plots the Frobenius distance kernel densities.

The increase in missing percentage has the effect of increasing the number of points of the latent process to be estimated, severing the reducibility problem. Still, with missing probabilities equal to the missing percentages of the standard multivariate case plus 0.35, the simulation study shows superior results for the Bayesian estimator as reported in Figures 1d, 2d and in Table 3. AFX is projected 60% of the times to guarantee the positivity of the estimated covariance matrix. We thus conclude that the other estimation procedures deteriorate more rapidly than our as the conditions become more severe.

Finally, we add simultaneously more variance noise and more missings and we report the results in Figures 1e, 2e and in Table 3, confirming the higher robustness of the Bayesian methodology to more extreme market conditions. AFX has been projected 59% of the times.

For clarification, we stress the difference between the two Frobenius distances so far mentioned in the text. The Frobenius used in the projection of AFX, is the distance between the (non necessarily positive) covariance matrix estimated with AFX methodology and the closest positive definite matrix. Its purpose is to minimize the impact of the projection. On the other hand, the Frobenius shown in the Figures, is the distance between the covariance matrix estimated by the generic methodology and the true covariance matrix. Its purpose is to compare the performances of the estimators. It is reasonable to expect that the projection negatively affects the performance of the AFX estimator, but results not reported (available upon request) show that the impact of the projection is negligible.

5 Conclusions

In this paper we study the problem of estimation of the multivariate covariance matrix of noisy and asynchronous observations. The Dynamic Linear Model is the setting chosen to deal with presence of noise in the data, and we treat the asynchronous time series as synchronous series with missing observations. The Bayesian approach allows us to deal with missing observations (asynchronicity) by treating them as additional parameters of the problem. An augmented Gibbs algorithm is implemented to sample the covariance matrix, the observational error variance matrix, the latent process and the missing observations of the noisy process. Our MCMC estimator is positive definite by construction and we compare it with the bivariate estimator of Ait-Sahalia et al. (2010) (AFX) and the Multivariate Realized Kernel (MRK) of Barndorff-Nielsen et al. (2011). A simulation study suggests that our estimator is superior in terms of RMSE in a two- and ten-dimensional setting, especially with dispersed and high missing percentages and with high noise. This suggests that MRK and AFX perform worse than our Bayesian estimator in severe conditions, as with portfolios of assets with heterogeneous liquidity profiles, or particularly illiquid, or when there is a high level of microstructure noise in the market. We naturally overcome the Roberts-Stramer reducibility critique (Roberts and Stramer (2001)) without losing the conjugacy of the problem, by sampling the missing observations of the noisy process. As possible extension, our methodology could be applied to factors of much larger portfolios. Furthermore, we could extend the simulation algorithm to adaptive MCMC samplers (Haario et al. (2001)) or to Particle MCMC methods (Andrieu et al. (2010)) to face the problem of lost conjugacy of the full conditionals in case of non-linear and non-normal measurement and transition equations.

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6 Tables and Figures

Table 1: Simulation results: bivariate case. $M = 100$ Monte Carlo estimates for each compared estimator are computed and the mean is reported. The Gibbs sampler runs for 5000 iterations, plus 5000 of burn-in. The RMSE is reported in parenthesis. The HY estimator is the ideal benchmark computed on the unobserved latent process.

	σ_1^2	σ_2^2	σ_{12}
Gibbs	0.1604	0.0901	0.0600
	(0.0052)	(0.0029)	(0.0030)
AFX	0.1611	0.0908	0.0607
	(0.0078)	(0.0032)	(0.0041)
MRK	0.1652	0.0929	0.0614
	(0.0106)	(0.0054)	(0.0051)
HY (latent)			0.0599
			(0.0015)
True value	0.16	0.09	0.06

Table 2: 10-dimensional state covariance matrix used to generate simulated data.

$$
\Sigma = \begin{bmatrix}
0.16 & 0.06 & -0.01 & 0.10 & 0.12 & 0.04 & 0.06 & 0.09 & 0.09 & 0.04 \\
0.06 & 0.09 & 0.04 & 0.05 & 0.05 & 0.02 & 0.04 & 0.03 & 0.05 & 0.10 \\
-0.01 & 0.04 & 0.12 & 0.03 & 0.08 & 0.05 & 0.09 & 0.05 & 0.11 & 0.04 \\
0.10 & 0.05 & 0.03 & 0.11 & 0.11 & 0.04 & 0.06 & 0.11 & 0.13 & 0.04 \\
0.12 & 0.05 & 0.08 & 0.11 & 0.25 & 0.06 & 0.09 & 0.07 & 0.16 & 0.03 \\
0.04 & 0.02 & 0.05 & 0.04 & 0.06 & 0.16 & 0.10 & 0.07 & 0.14 & 0.06 \\
0.06 & 0.04 & 0.09 & 0.06 & 0.09 & 0.10 & 0.18 & 0.07 & 0.16 & 0.05 \\
0.09 & 0.03 & 0.05 & 0.11 & 0.07 & 0.07 & 0.07 & 0.32 & 0.11 & 0.05 \\
0.09 & 0.05 & 0.11 & 0.13 & 0.16 & 0.14 & 0.16 & 0.11 & 0.28 & 0.05 \\
0.04 & 0.10 & 0.04 & 0.04 & 0.03 & 0.06 & 0.05 & 0.05 & 0.05 & 0.24
\end{bmatrix}
$$

,

Table 3: Estimated expected values and standard deviations of the Frobenius distances between the estimated and the true covariance matrix. $\hat{E}[||\Delta_i||] := \frac{1}{M}\sum_{j=1}^{M}||\Delta_{i,j}||$ and $\hat{\sigma}[||\Delta_i||] := \left(\frac{1}{M-1}\sum_{j=1}^{M}(||\Delta_{i,j}|| - \hat{E}[||\Delta_i||])^2\right)^{1/2}$, with $M = 100$ and 10 assets. Define missing probabilities $v = \{1/2, 1/3, 1/2, 1/4, 1/4, 1/3, 1/5, 1/4, 1/3, 1/4\}$ and noise matrix $\Omega = \text{diag}([0.08, 0.04, 0.02, 0.05, 0.1, 0.06, 0.1, 0.15, 0.15, 0.08])$

(a) *Standard*: missing probabilities v and noise matrix Ω.

(b) *Dispersed missings*: more dispersed missing probabilities $\{0, 0.5, 0.8, 0.9, 0.25, 0, 0.5, 0.8, 0.9, 0.25\}$ and noise matrix Ω.

(c) *High noise*: missing probabilities v and noise matrix $\Omega + 0.35$.

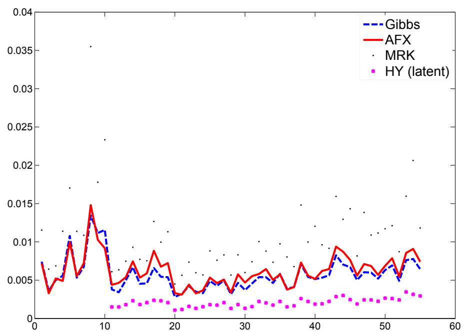
(d) *High missings*: missing probabilities $v + 0.35$ and noise matrix Ω.

(e) *High noise and missings*: missing probabilities $v + 0.35$ and noise matrix $\Omega + 0.35$.

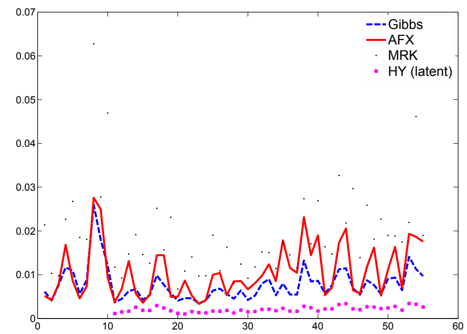
	Gibbs	AFX	MRK
(a) *Standard*			
$\hat{E}[\|\|\Delta_i\|\|]$	0.0560	0.0622	0.1063
$\hat{\sigma}[\|\|\Delta_i\|\|]$	0.0142	0.0137	0.0314
(b) *Dispersed missings*			
$\hat{E}[\|\|\Delta_i\|\|]$	0.0790	0.1164	0.1930
$\hat{\sigma}[\|\|\Delta_i\|\|]$	0.0184	0.0259	0.0624
(c) *High noise*			
$\hat{E}[\|\|\Delta_i\|\|]$	0.0641	0.0759	0.1669
$\hat{\sigma}[\|\|\Delta_i\|\|]$	0.0180	0.0165	0.0454
(d) *High missings*			
$\hat{E}[\|\|\Delta_i\|\|]$	0.0877	0.0929	0.1626
$\hat{\sigma}[\|\|\Delta_i\|\|]$	0.0184	0.0170	0.0446
(e) *High noise and missings*			
$\hat{E}[\|\|\Delta_i\|\|]$	0.0811	0.1121	0.2531
$\hat{\sigma}[\|\|\Delta_i\|\|]$	0.0179	0.0235	0.0852

Figure 1: Simulated Root Mean Squared Errors. $M = 100$ Monte Carlo estimates for each estimator are computed. The Gibbs sampler runs for 5000 iterations, plus 5000 of burn-in. The HY estimator is computed on the unobserved latent process. The x-axis is the index for the true 55 parameters of the covariance matrix, starting from the 10 variances. Define missing probabilities $v = \{1/2, 1/3, 1/2, 1/4, 1/4, 1/3, 1/5, 1/4, 1/3, 1/4\}$ and noise matrix $\Omega = \mathrm{diag}([0.08, 0.04, 0.02, 0.05, 0.1, 0.06, 0.1, 0.15, 0.15, 0.08])$
(a) *Standard*: missing probabilities v and noise matrix Ω.
(b) *Dispersed missings*: more dispersed missing probabilities $\{0, 0.5, 0.8, 0.9, 0.25, 0, 0.5, 0.8, 0.9, 0.25\}$ and noise matrix Ω.
(c) *High noise*: missing probabilities v and noise matrix $\Omega + 0.35$.
(d) *High missings*: missing probabilities $v + 0.35$ and noise matrix Ω.
(e) *High noise and missings*: missing probabilities $v + 0.35$ and noise matrix $\Omega + 0.35$.

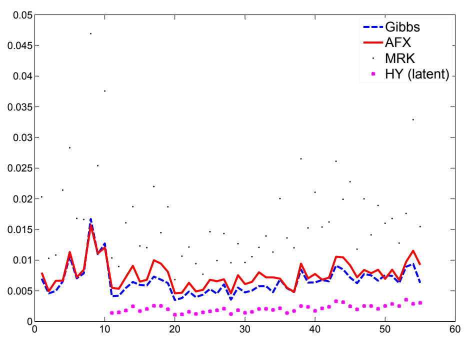
(a) *Standard*



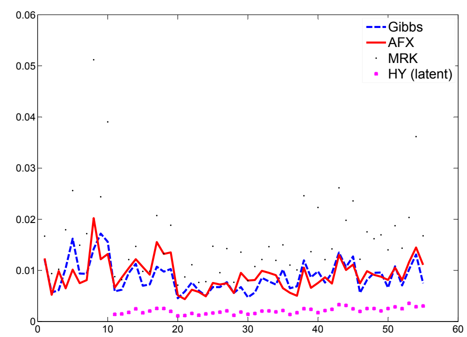
(b) *Dispersed missings*



(c) *High noise*



(d) *High missings*



(e) *High noise and missings*

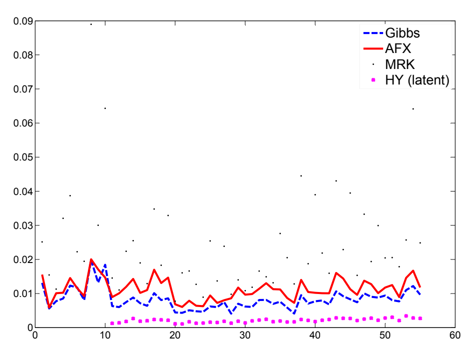
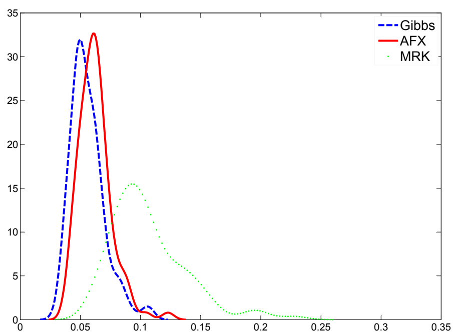
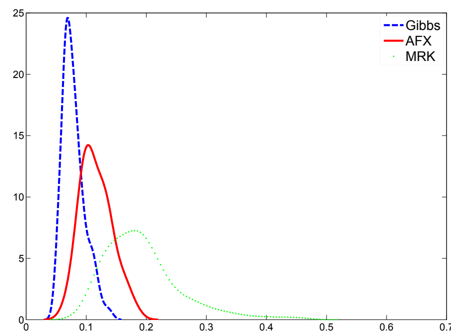


Figure 2: Kernel density estimates of the Frobenius distances $||\Delta_{i,j}|| = ||(\hat{\Sigma}_i^j - \Sigma)||_F$, $\hat{\Sigma}_i^j$ is the j-th covariance matrix estimated with methodology i, $i \in \{gibbs, afx, mrk\}$ and $j = 1, \dots, M = 100$ and Σ is the true covariance matrix. Define missing probabilities $v = \{1/2, 1/3, 1/2, 1/4, 1/4, 1/3, 1/5, 1/4, 1/3, 1/4\}$ and noise matrix $\Omega = \mathrm{diag}([0.08, 0.04, 0.02, 0.05, 0.1, 0.06, 0.1, 0.15, 0.15, 0.08])$
(a) *Standard*: missing probabilities v and noise matrix Ω.
(b) *Dispersed missings*: more dispersed missing probabilities $\{0, 0.5, 0.8, 0.9, 0.25, 0, 0.5, 0.8, 0.9, 0.25\}$ and noise matrix Ω.
(c) *High noise*: missing probabilities v and noise matrix $\Omega + 0.35$.
(d) *High missings*: missing probabilities $v + 0.35$ and noise matrix Ω.
(e) *High noise and missings*: missing probabilities $v + 0.35$ and noise matrix $\Omega + 0.35$.

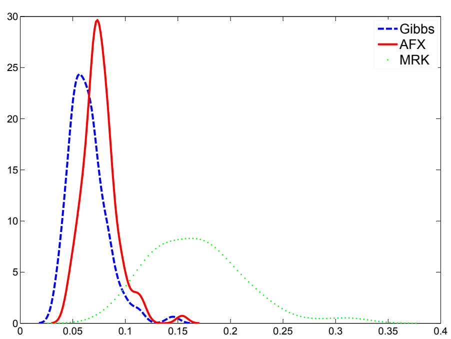
(a) *Standard*



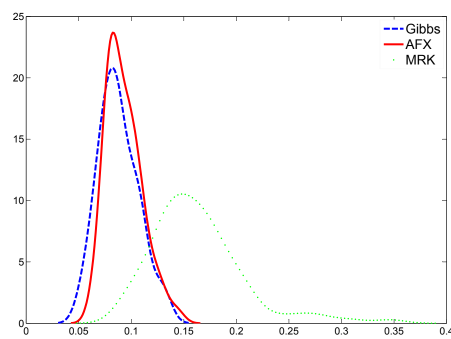
(b) *Dispersed missings*



(c) *High noise*



(d) *High missings*



(e) *High noise and missings*