# A Note on Ex-Ante Stable Lotteries

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**Abstract**

We study ex-ante priority respecting (ex-ante stable) lotteries in the context of object allocation under thick priorities. We show that ex-ante stability as a fairness condition is very demanding: Only few agent-object pairs have a positive probability of being matched in an ex-ante stable assignment. We interpret our result as an impossibility result. With ex-ante stability one cannot go much beyond randomly breaking ties and implementing a (deterministically) stable matching with respect to the broken ties. *JEL-classification:* C78, D47
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## 1 Introduction

A classical matching problem with many real-world applications is the assignment of indivisible objects to agents where objects are rationed according to priorities. In applications, such as the school choice problem (Abdulkadiroglu and Sönmez, 2003), priorities are often thick, i.e. many agents have the same priority to obtain a certain object. Thus, one can sometimes not avoid to treat agents differently ex-post even though they have the same priorities and preferences. However, ex-ante, some form of fairness can be restored by the use of lotteries. This has motivated researchers to study the problem of designing priority respecting lotteries for allocating objects.

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A minimal ex-ante fairness requirement for random assignments under priorities is that the lottery should respect the priorities. One way of formalizing this requirement is the following: An agent $i$ has *ex-ante justified envy* if there is an object $s$ where a lower priority agent $j$ has a positive probability of receiving the object and $i$ would rather have the object $s$ than another object which he receives with positive probability in the lottery under consideration. In this case, it would be natural to eliminate the justified envy, i.e. changing the probability shares such that $i$ has a higher chance of receiving $s$ at the expense of the lower ranked agent $j$. *Ex-ante stability* requires that there is no ex-ante justified envy. In the school choice set-up, ex-ante stability has been introduced by Kesten and Ünver (2015). For the classical marriage model the condition was first considered by Roth et al. (1993). He et al. (2017) define an appealing class of mechanisms that implement ex-ante stable lotteries.

Even though ex-ante stability is, in a sense, a minimal ex-ante fairness requirement, it is demanding. In an environment with strict priorities (no ties) and where each school has one seat to allocate, it follows from an earlier result by Roth et al. (1993) that each student has a positive probability of receiving a seat at, at most, two schools. In other words, an ex-ante stable lottery is almost deterministic. We generalize this result to the more general set-up with quotas and ties. With strict priorities, we show that an ex-ante stable lottery is almost degenerate, since

- each agent has a positive probability at at most *two* distinct objects for receiving a copy of that object.
- For each object all but possibly *one* copy are assigned deterministically. For the one copy that is assigned by a lottery, two agents have a positive probability of receiving it.

With ties in the priorities, ex-ante stability is naturally less demanding. However, ex-ante stability imposes a lot of structure on the lottery. We show that the size of the support of an ex-ante stable lottery (the number of pairs being matched with positive probability) is determined by the number of ties the lottery "uses" (i.e. how many agents who have equal priority at some object are matched with positive probability to that object). More precisely, we show that for each ex-ante stable lottery the size of the support is determined by the size of the "cut-off" priority classes: Here, cut-off priority classes are the lowest priority classes at an object, such that an agent of that priority class gets that object with positive probability.

The proofs in this paper use the graph representation of assignment problems due to Balinski and Ratier (1997). As far as we know, this representation has not been used so far in the study of lotteries. We think that our results demonstrate the usefulness of this particular representation for the study of random assignments with priorities.

# 2 Model

There is a set of $n$ agents $N$ and a set of $m$ object types $M$. A generic agent is denoted by $i$ and a generic object type by $j$. Of each object type $j$, there is a finite number of copies $q_j \in \mathbb{N}$. We assume that there are as many objects as agents, $\sum_{j\in M} q_j = n$.[1] Each agent $i$ has strict preferences $P_i$ over different types of objects. Each object type $j$ has a strict priority ranking $\succ_j$ of agents. Later in Subsections 3.2 we will also consider the case where object types have indifferences in their priorities.

A **deterministic assignment** is a mapping $\mu : N \to M$ such that for each $j \in M$ we have $|\mu^{-1}(j)| = q_j$. A **random assignment** is a probability distribution over deterministic assignments. By the Birkhoff-von Neumann Theorem, each random assignment corresponds to a bi-stochastic matrix and, vice versa, each such matrix corresponds to a random assignment (see Kojima and Manea (2010) for a proof in the set-up that we consider). Thus each random assignment is represented by a matrix $\Pi = (\pi_{ij}) \in \mathbb{R}^{N\times M}$ such that

$$0 \leq \pi_{ij} \leq 1, \quad \sum_{j\in M} \pi_{ij} = 1, \quad \sum_{i\in N} \pi_{ij} = q_j,$$

where $\pi_{ij}$ is the probability that agent $i$ is matched to an object of type $j$. The **support** of $\Pi$ is the set of all non-zero entries of the matrix $\Pi$, i.e.

$$\text{supp}(\Pi) := \{ij \in N \times M : \pi_{ij} \neq 0\}.$$

We say that agent $i$ is **fractionally matched** to object type $j$ if there is a positive probability of the pair being matched but they are not matched

[1]Our results can be generalized to the case where the number of objects and agents differ by adding dummy agents and objects. See Aziz and Klaus (2017), for the details of this construction.

for sure, i.e. $0 < \pi_{ij} < 1$. A random assignment represented by the matrix $\Pi = (\pi_{ij})$ is **ex-ante blocked** by agent $i$ and object type $j$ if there is some agent $i' \neq i$ with $\pi_{i'j} > 0$ and $i \succ_j i'$ and some object type $j'$ with $\pi_{ij'} > 0$ and $j \, P_i \, j'$. A random assignment is **ex-ante stable** if it is not blocked by any agent-object type pair.[2]

## 2.1 Graph representation

Next, we introduce the graph representation of Balinski and Ratier (1997). In the following, a directed graph $\Gamma$ is a pair $(V(\Gamma), E(\Gamma))$, where $V(\Gamma)$ is a finite set of vertices and $E(\Gamma)$ is a set of ordered pairs of vertices called arcs. For a random assignment $\Pi$, we construct a directed graph $\Gamma(\Pi)$ as follows: The vertices are the agent-object type pairs,

$$V = N \times M.$$

There are two kind of arcs. A **horizontal arc** connects two vertices $ij$ and $ij'$ that contain the same agent. A **vertical arc** connects two vertices $ij$ and $i'j$ that contain the same object type. The direction of the arc is determined by the preferences respectively priorities. A horizontal arc points to the more preferred object type according to the agent's preferences. A vertical arc points to the agent with higher priority in the object type's priority. Moreover we only consider those arcs which origin in a pair $ij$ with $\pi_{ij} > 0$. Thus

$$E(\Pi) := \{(ij, i'j') \in V^2 : \pi_{ij} > 0, (i = i', j' \, P_i \, j \text{ or } j = j', i' \succ_j i)\}.$$

Immediately from the definition of ex-ante stability we obtain the following necessary and sufficient condition for ex-ante stability (see Figure 1).

**Lemma 1.** *If $\Pi = (\pi_{ij})$ is ex-ante stable, then there cannot exist both a horizontal arc $(ij', ij)$ and a vertical arc $(i'j, ij)$ in $\Gamma(\Pi)$ pointing to $ij$.*

[2]For deterministic assignments, ex-ante stability is equivalent to the usual notion of a stable matching. In particular, ex-ante stable assignments always exists, since stable matchings always exist.

| $P_1$ | $P_2$ | $P_3$ | $\succeq_1$ | $\succeq_2$ | $\succeq_3$ |
|---|---|---|---|---|---|
| 2 | 1 | 3 | 2 | 1 | 1 |
| 1 | 2 | 2 | 1 | 2 | 2 |
| 3 | 3 | 1 | 3 | 3 | 3 |

$$\begin{pmatrix} \frac{1}{3} & 0 & \frac{2}{3} \\ 0 & \frac{2}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} & 0 \end{pmatrix}$$

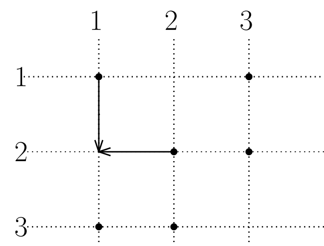


Figure 1: The matrix represents a random assignment. Preferences and priorities are as in the table in the middle. The random assignment has several blocking pairs, for example the pair (21): Since $\pi_{11} > 0$ and $\pi_{22} > 0$, agent 2 and object type 1 ex-ante block the random assignment. In the corresponding directed graph, there is a horizontal arc from $(2, 2)$ to $(2, 1)$ and a vertical arc from $(1, 1)$ to $(2, 1)$.

# 3 Results

## 3.1 Strict Priorities

We are ready to state and prove the main results for the case with strict priorities. First we show that if $\Pi$ represents an ex-ante stable random assignment, then it has small support.

**Proposition 1.** *If priorities are strict, then for each ex-ante stable random assignment* $\Pi$ *we have*

$$|supp(\Pi)| \leq n + m.$$

*Proof.* We prove the proposition by a double counting argument. Let $U \subseteq V$ be the set of vertices $ij$ that have an incoming horizontal arc in $\Gamma(\Pi)$ and positive probability $\pi_{ij} > 0$. For each $i \in N$, let $M_i(\Pi) \subseteq M$ be the set of object types $j$ such that $ij$ has an incoming horizontal arc and $\pi_{ij} > 0$. By definition, we have $|U| = \sum_{i \in N} |M_i(\Pi)|$. Let $i \in N$. Either $i$ is deterministically matched or he is fractionally matched to multiple object types. In the first case, we have $M_i(\Pi) = \emptyset$. In the second case, let $j \in M_i(\Pi)$ be the least preferred object type (according to $i$'s preferences) among the object types that are fractionally matched to $i$ under $\Pi$. Since $j$ is $i$'s least preferred object type to which he is matched, there is for each such object type $j' \neq j$ a horizontal arc pointing from $ij$ to $ij'$. Thus, in either case, $|\text{supp}(\Pi_i)| - 1 = |M_i(\Pi)|$ where $\text{supp}(\Pi_i)$ is the support of the $i$-row of $\Pi$.

Summing over $N$ we obtain

$$\text{supp}(\Pi) - n \leq \sum_{i \in N} |M_i(\Pi)| = |U|. \tag{1}$$

Next we bound $|U|$ from above. Let $j \in M$ and $i, i' \in N$. Suppose $\pi_{ij} > 0$, $\pi_{i'j} > 0$ and furthermore that there is a horizontal arc pointing to $ij$ and another horizontal arc pointing to $i'j$. If there were a vertical arc pointing from $ij$ to $i'j$, we would have a contradiction to Lemma 1 and vice versa if there were a vertical arc pointing from $i'j$ to $ij$, we would also have a contradiction to Lemma 1. Thus for each $j$ there is at most one agent $i$ such that $\pi_{ij} > 0$ and $ij$ has an incoming horizontal arc. Thus $|U| \leq m$. Combining this inequality with Inequality 1, we obtain the desired result. $\square$

The opposite direction is not necessarily true. There can exists random assignment that satisfy the bound on the support, but are not ex-ante stable. The random assignment in Figure 2 is an example of such a random assignment.

It follows from the bound on the support that ex-ante stable random assignments under strict priorities are almost degenerate.

**Corollary 1.** *If priorities are strict, then for each ex-ante stable random assignment the following holds:*

1. *For each agent $i$ there are at most two object types that are fractionally matched to $i$.*

2. *For each object type $j$ there are at most two agents that are fractionally matched to $j$.*

*Proof.* Suppose there is an ex-ante stable $\Pi$ such that some agent $i'$ is fractionally matched to at least three object types. Consider a minimal bistochastic sub-matrix $\Pi' \subseteq \Pi$ containing $i'$, i.e. a minimal (in terms of number of rows and columns) matrix $(\pi_{ij})_{(i,j) \in N' \times M'}$ with $i' \in N' \subseteq N$ and $M' \subseteq M$ such that

1. $\sum_{j \in M'} \pi_{ij} = 1$ for any $i \in N'$,

2. $q'_j := \sum_{i \in N'} \pi_{ij} \in \mathbb{N}$ for any $j \in M'$,

3. $\sum_{j \in M'} q'_j = |N'|$.

$$
\begin{pmatrix}
\frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 \\
0 & \frac{2}{3} & \frac{1}{3} & 0 & 0 & 0 \\
\frac{2}{3} & 0 & \frac{1}{3} & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\
0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
$$

Figure 2: A bi-stochastic matrix representing a random assignment. The object types corresponding to the second and to the third column have two copies. The other object types have one copy. The agent $i'$ corresponding to the first row is fractionally matched to three object types. The minimal bi-stochastic matrix $\Pi'$ containing $i'$ is the $3 \times 3$-matrix shaded in red. Note that $|\text{supp}(\Pi')| = 7 > 3 + 3 = |N'| + |M'|$. Thus $\Pi'$ is not ex-ante stable and therefore $\Pi$ is not ex-ante stable.

By minimality, $\Pi'$ contains only fractionally matched agents (see Figure 2 for an example). Thus every agent in $\Pi'$ is fractionally matched to two or more object types. Moreover, $i'$ is fractionally matched to at least three object types. Thus $|\text{supp}(\Pi')| \geq 3 + 2 \cdot (|N'| - 1) > 2 \cdot |N'| \geq |N'| + |M'|$ and, by Proposition 1, $\Pi'$ is not ex-ante stable. But each blocking pair of $\Pi'$ is also a blocking pair of $\Pi$. Therefore, $\Pi$ is not ex-ante stable, contradicting our assumption. A symmetric argument shows the second part of the corollary. $\square$

## 3.2 Thick Priorities

Now we consider the more general case where priorities can be weak. For each object type $j$ we have a weak (reflexive, complete and transitive) priority order $\succeq_j$ of the agents. We let $i \sim_j i'$ if and only if $i \succeq_j i'$ and $i' \succeq_j i$. We let $i \succ_j i'$ if and only if $i \succeq_j i'$ but not $i' \succeq_j i$. The priorities $\succeq_j$ of an object type $j$ partition $N$ in equivalence classes of equal priority agents, i.e. in equivalence classes with respect to $\sim_j$. We call these equivalence classes

**priority classes** and denote them by $N_j^1, N_j^2 \dots, N_j^{\ell_j}$ with indices increasing with priority. Thus for $a > b$, $i \in N_j^a$ and $i' \in N_j^b$ we have $i \succ_j i'$. We use the notation $i \succ_j N_j^k$ to indicate that $i$ has higher priority at $j$ than the agents in the priority class $N_j^k$. The definition of ex-ante stability remains the same as before, in particular, the object type $j$ in the blocking pair must strictly prioritize $i$ over $i'$ in order to ex-ante block.

For each random assignment $\Pi$ and priority profile $\succeq = (\succeq_j)_{j \in M}$, we define **priority cut-offs** $c(\Pi) = (c_j(\Pi))_{j \in M}$ by

$$c_j(\Pi) := \min\{c \in \{1, \dots, \ell_j\} : \exists i \in N_j^c \text{ with } \pi_{ij} > 0\},$$

i.e. $c_j(\Pi)$ is the lowest priority of an agent that is matched to $j$ under $\Pi$. We define **cut-off priority classes** $N(\Pi) = (N_j(\Pi))_{j \in M}$ by $N_j(\Pi) := N_j^{c_j(\Pi)}$ and define for each agent $i \in N$ the set $M_i(\Pi) := \{j \in M : i \in N_j(\Pi)\}$ of **cut-off object types**.

We now generalize Proposition 1 to the case with thick priority classes.

**Theorem 1.** *If* $\Pi$ *is ex-ante stable then*

$$|supp(\Pi)| \leq n + \sum_{j \in M} |N_j(\Pi)|.$$

*Proof.* Again we use the graph representation as introduced in Section 2. We model indifferences in priorities by undirected edges, i.e. unordered pairs of vertices. Now there are two kind of vertical edges: Vertical arcs pointing from a vertex $ij$ to $i'j$ such that $i' \succ_j i$ and **neutral** vertical edges connecting vertices $ij$ and $i'j$ such that $i \sim_j i'$. Neutral edges do not have a direction. Note that Lemma 1 remains to hold.

Again we use a double counting argument. As before, let $U \subseteq V$ be the set of vertices $ij$ that have an incoming horizontal arc and positive probability $\pi_{ij} > 0$. The same argument as in the proof of Proposition 1, shows that

$$|\text{supp}(\Pi)| - n \leq |U|. \tag{2}$$

Next we bound $|U|$ from above in terms of the sizes of cut-off classes. For each $j \in M$, let $\tilde{N}_j(\Pi) \subseteq N$ be the set of agents $i$ such that $ij$ has an incoming horizontal arc and $\pi_{ij} > 0$. By definition, we have $|U| = \sum_{j \in M} |\tilde{N}_j(\Pi)|$. Let $j \in M$ and suppose $i, i' \in \tilde{N}_j(\Pi)$. If there were a vertical arc pointing from $ij$ to $i'j$ we would have a contradiction to Lemma 1 and vice versa if there were a vertical arc pointing from $i'j$ to $ij$ we would also have a contradiction to Lemma 1. Thus, in this case $ij$ and $i'j$ are connected by a neutral edge. Thus, each $i \in \tilde{N}_j(\Pi)$ is in the same indifference class at $j$. Moreover, by Lemma 1, for each $i \in \tilde{N}_j(\Pi)$ there is no vertical arc from a $i'j$ with $\pi_{i'j} > 0$ pointing to $ij$. Therefore $\tilde{N}_j(\Pi) \subseteq N_j(\Pi)$. Thus for each $j \in M$ we have $|\tilde{N}_j(\Pi)| \leq |N_j(\Pi)|$. Summing over $M$ we obtain

$$|U| = \sum_{j \in M} |\tilde{N}_j(\Pi)| \leq \sum_{j \in M} |N_j(\Pi)|.$$

Combining this inequality with Inequality 2, we obtain the theorem. $\square$

Note that Theorem 1 generalizes Proposition 1. If the profile $\succeq$ is strict then for each $j \in M$ we have $|N_j(\Pi)| = 1$. Therefore, the second term on the right hand side of the inequality is $m$. The following example illustrates Theorem 1.

*Example* 1. Consider five agents, five object types, each with a single copy ($q_j = 1$ for each object type), and the following preferences and priorities.

| $P_1$ | $P_2$ | $P_3$ | $P_4$ | $P_5$ | $\succeq_1$ | $\succeq_2$ | $\succeq_3$ | $\succeq_4$ | $\succeq_5$ |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 2 | 5 | $1,2,3$ | 5 | $3,5$ | 4 | 1 |
| 4 | 2 | 2 | 5 | 3 | $\vdots$ | $2,3,4$ | $\vdots$ | 1 | 2 |
| 5 | 5 | 3 | 4 | 2 | | 1 | | $\vdots$ | 3 |
| 2 | 3 | 5 | 3 | 1 | | | | | $4,5$ |
| 3 | 4 | 4 | 1 | 4 | | | | | |

In Figure 3, we consider a random assignment that is ex-ante stable for the above preferences and priorities. In this random assignment, the upper bound on the size of the support holds with equality.

$$\begin{pmatrix} \frac{1}{3} & 0 & 0 & \frac{1}{2} & \frac{1}{6} \\ \frac{1}{3} & \frac{1}{3} & 0 & 0 & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{6} & \frac{1}{3} & 0 & \frac{1}{6} \\ 0 & \frac{1}{3} & 0 & \frac{1}{2} & \frac{1}{6} \\ 0 & \frac{1}{6} & \frac{2}{3} & 0 & \frac{1}{6} \end{pmatrix}$$

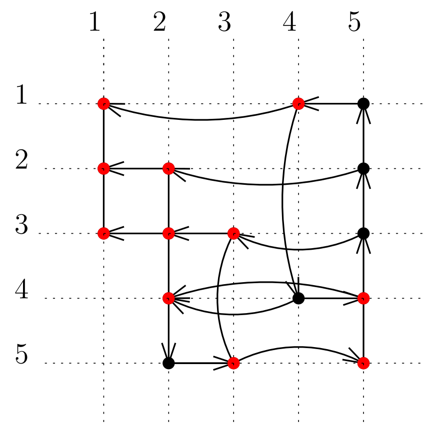


Figure 3: The red vertices have an incoming horizontal arc. In this example: $\sum_{j \in M} |N_j(\Pi)| = 3 + 3 + 2 + 1 + 2 = 11$. Thus, the right-hand side of the inequality in the theorem is $5 + 11 = 16$. In this example the bound is sharp and the inequality holds with equality: $|\text{supp}(\Pi)| = 16$.

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