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Perturbative computation of string one-loop corrections to Wilson loop minimal surfaces in $AdS_5 \times S^5$

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Abstract

We revisit the computation of the 1-loop string correction to the "latitude" minimal surface in $AdS_5 \times S^5$ representing 1/4 BPS Wilson loop in planar $\mathcal{N}$=4 SYM theory previously addressed in arXiv:1512.00841 and arXiv:1601.04708. We resolve the problem of matching with the subleading term in the strong coupling expansion of the exact gauge theory result (derived previously from localization) using a different method to compute determinants of 2d string fluctuation operators. We apply perturbation theory in a small parameter (angle of the latitude) corresponding to an expansion near the AdS_2 minimal surface representing 1/2 BPS circular Wilson loop. This allows us to compute the corrections to the heat kernels and zeta-functions of the operators in terms of the known heat kernels on AdS_2. We apply the same method also to two other examples of Wilson loop surfaces: generalized cusp and k-wound circle.

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1 Introduction

The expectation value of a Wilson loop (WL) operator in planar $\mathcal{N} = 4$ super Yang-Mills theory is conjected to be given, at strong coupling, by the $AdS_5 \times S^5$ superstring path integral with appropriate boundary conditions [1–3]. The computation of the leading strong-coupling correction to the classical area term given by the logarithm of the 1-loop string partition function was addressed in [4–6] and, in general, is technically challenging.

Simplest examples correspond to supersymmetric Wilson loops, e.g., 1/2 BPS circular loop [6–8], 1/4 BPS family of "latitudes" [9–14], the k-wound circle case (dual to WL in k-fundamental representation) [7, 15], etc. Even in the circular WL case the first string correction appears to disagree with the subleading term in the strong coupling expansion of the gauge-theory result [16–22].

To avoid the subtle issue of the overall normalization of the string path integral one may consider the computation of the *ratio* of partition functions for minimal surfaces of the same (disc) topology. Then the universal UV divergences and possible string tension factors associated with the Killing vector volume [6, 18] that are independent of local world-sheet geometry should cancel out and the result should be a well-defined function of the non-trivial WL (i.e. world-surface) parameters. This strategy was followed in [13], where the one-loop determinants for fluctuations about the classical string solutions corresponding to a generic 1/4 BPS "latitude" WL of [9–11] were evaluated with the Gel'fand-Yaglom (GY) method. The same

result for the string partition function was obtained in [14], with a slightly different application of the GY method.[1] Still, the resulting string prediction was found to be in disagreement with the exact gauge theory result obtained by the localization method [20, 21].

In this paper we will reconsider the computation in [13, 14] using a different approach to evaluation of the fluctuation determinants. We shall use the perturbation theory in a small parameter α, such that for $\alpha = 0$ the world-surface becomes the same as the circular WL surface, i.e. is equivalent to the Euclidean AdS_2. Then the leading correction in α can be found by the perturbative expansion of the heat kernels (see, e.g., [23, 24]) using that for $\alpha = 0$, i.e. in the AdS_2 case, the heat kernels for the bosonic and fermionic operators are known explicitly [25–28]. This will allow us to find the leading-order correction to the string partition function for the near-AdS_2 geometry corresponding to the latitude in $S^2 \subset S^5$ parametrized by a small angle θ_0. Since for $\theta_0 = 0$ it reduces to the AdS_2 (circular WL) geometry, here the small expansion parameter may be chosen as $\alpha = \theta_0^2$.

Remarkably, we will be able to reproduce the first non-trivial term in the small-θ_0 expansion of the exact gauge-theory result [20, 21] for the latitude WL expectation value $Z = \langle \mathcal{W}(\lambda, \theta_0) \rangle$ in the strong-coupling ($\lambda \gg 1$) limit. Explicitly, the gauge-theory prediction for the string "effective action" $\Gamma = -\log Z$ is

$$\Gamma(\lambda, \theta_0) - \Gamma(\lambda, 0) = \sqrt{\lambda}\,(1 - \cos\theta_0) + \tfrac{3}{2}\log\cos\theta_0 + \mathcal{O}(\lambda^{-1/2}) \ , \tag{1.1}$$

and we will reproduce precisely the leading small-θ_0 term in the $O(\lambda^0)$ part of (1.1), i.e. $\frac{3}{2}\log\cos\theta_0 = -\frac{3}{4}\theta_0^2 + O(\theta_0^4)$, from the one-loop string-theory computation (see (3.2),(3.45)).

A possible reason why the two previous attempts in [13] and [14] failed to find the agreement with the gauge theory result may be related to some subtleties in their application of the GY method to computation of functional determinants.[2] Compared to the heat-kernel approach, here the spectral problem is treated (after Fourier-transforming in τ) as effectively a one-dimensional operator problem; one also uses a zeta-function-like regularization in σ world-sheet direction and a cutoff regularization of the sum over the Fourier modes in τ-direction. This method also requires considering ratios of determinants for differential operators with the same principal symbol, which in turns implies a functional rescaling by a conformal factor.[3] Together with a possible regularization ambiguity in the sum over modes mentioned above, what may account for the disagreement is the fictitious boundary (a cut at the origin of the disk) introduced in [7, 13, 14] to allow for the calculation of determinants on a compact interval (see also [37, 38]). It would be interesting to perform an explicit comparison of the two computations eliminating the need for this regulator, which does not appear in the heat

[1] In [14], the fermionic contribution was found starting with the Dirac-like first-order operator rather than its square, as in [13]. Using a particular organization of the determinant ratios, ref. [14] computed the analytic expression for the resulting string 1-loop correction (while the analysis in [13] was partially numerical). Ref. [14] presented also a detailed study of the supermultiplet structure of the fluctuations.

[2] This method was originally suggested in [29] and later improved in [30–35]; for a review see, for example, [34, 36], or Appendix B of [13].

[3] One may quantify (see, e.g., Appendix A of [6]) how such conformal rescaling of the operators affects the finite part of the regularized determinants. However, a simple check for the ratio of two bosonic operators in [13, 14] reveals that adding this contribution does not explain the discrepancy with the result obtained here.

kernel approach.[4]

Below we will also test our perturbative approach based on constructing heat kernels for 2d fluctuation operators in an expansion in a small parameter on two other examples. The first will be the near-BPS limit of the generalized cusp of [40], corresponding to the the strong coupling expansion of the "Bremsstrahlung function" of $\mathcal{N}=4$ SYM theory, derived exactly using supersymmetric localization in [41]. In this case the GY method applied to the computation of the string 1-loop correction reproduced [40] the gauge-theory result.[5] Our perturbative computation will also be consistent with this matching.

Another example will be the 1-loop partition function for the surface ending on the k-wound circle that should be representing the k-fundamental circular Wilson loop [3, 46]. Here the gauge theory result is a generalization of the $k=1$ circular WL case [9, 20], see (3.107). The string one-loop computation was previously discussed in [7] (using the GY method and again introducing an unphysical cutoff) and in [15] (using heat kernel construction on a cone of AdS_2 with angular deficit $2\pi(1-k)$). Both approaches failed to find an agreement with gauge theory. We will use an expansion about the $k=1$ case, i.e. set the small parameter to be $\alpha=k-1$. Our result (3.105) for the coefficient of the $O(k-1)$ term in the 1-loop correction will differ from the gauge theory one just by an extra γ-term (the Euler-Mascheroni constant). We will suggest that this disagreement is due to a regularization ambiguity related to the fact that the expansion near the regular $k=1$ (i.e. AdS_2) surface appears to be problematic due to a conical singularity appearing for $k\neq 1$.

We will start in Section 2 with the description of the perturbative procedure for computing the heat kernel in a small-parameter expansion. In Section 3 we will apply this method the 1-loop string computations of the leading corrections to the three WL surfaces mentioned above. We will collect useful formulae and details of the calculations in Appendices A and B.

2 Perturbative expansion of heat kernel and determinant of an elliptic operator

To prepare for the computation of leading string 1-loop corrections to Wilson loop expectation values in expansion in some small parameter α here we shall present the general relations for the perturbative expansion of the heat kernel and determinant of a differential operator parametrized by α.

Let $\mathcal{O}$ be a second order elliptic operator defined on (sections of a bundle over) a d-dimensional Riemannian manifold $\mathcal{M}$ with metric g_{ij}. The standard expression for the loga-

[4] A more general application of the GY method [39] suggests that in the case of a non-compact interval one may try to proceed by selecting suitably "well-behaved" eigenfunctions of the auxiliary initial value problem.

[5] Here the application of the GY method does not require an unphysical regulator and thus the agreement could be expected. The GY procedure is known also to reproduce the predictions of integrability on gauge-theory side in other non-trivial fluctuation problems [42–45].

rithm of its determinant defined using zeta-function regularization is (see, e.g., [47, 48])

$$\log \mathrm{Det}_{\mathcal{M}} \mathcal{O} = -\zeta_{\mathcal{O}}^{'}(0) \ , \tag{2.1}$$

$$\zeta_{\mathcal{O}}(s) = \frac{1}{\Gamma(s)} \int_0^\infty dt\, t^{s-1} K_{\mathcal{O}}(t) \ , \qquad K_{\mathcal{O}}(t) = \int d^d x \sqrt{g}\ \mathrm{tr}\, K_{\mathcal{O}}(x,x;t) \ , \tag{2.2}$$

$$(\partial_t + \mathcal{O}_x) K_{\mathcal{O}}(x,x';t) = 0 \ , \qquad K_{\mathcal{O}}(x,x';0) = \frac{1}{\sqrt{g}} \delta^{(d)}(x-x')\, \mathbb{I} \ . \tag{2.3}$$

Here tr and the unit operator $\mathbb{I}$ correspond to the internal indices in the (vector or spinor) bundle.

Suppose the metric g_{ij} on $\mathcal{M}$ as well as $\mathcal{O}$ depend on some parameter α, such that for $\alpha = 0$, corresponding to $\bar{\mathcal{M}}$ with metric $\bar{g}_{ij}$, the spectral problem can be solved exactly. Then we can compute $K_{\mathcal{O}}$ and $\mathrm{Det}_{\mathcal{M}} \mathcal{O}$ in perturbation theory in α. Namely, let us set

$$\begin{aligned} g_{ij} &= \bar{g}_{ij} + \alpha\ \tilde{g}_{ij} + O\left(\alpha^2\right) \ , \\ \mathcal{O} &= \bar{\mathcal{O}} + \alpha\, \tilde{\mathcal{O}} + O\left(\alpha^2\right) \ , \\ K_{\mathcal{O}}(x,x';t) &= \bar{K}_{\mathcal{O}}(x,x';t) + \alpha\ \tilde{K}_{\mathcal{O}}(x,x';t) + O\left(\alpha^2\right) \ , \end{aligned} \tag{2.4}$$

where $\bar{K}_{\mathcal{O}}$ is the heat kernel corresponding to $\bar{\mathcal{O}}$, i.e.

$$\left(\partial_t + \bar{\mathcal{O}}_x\right) \bar{K}_{\mathcal{O}}(x,x';t) = 0 \ , \qquad \bar{K}_{\mathcal{O}}(x,x';0) = \frac{1}{\sqrt{\bar{g}}} \delta^{(d)}\left(x-x'\right) \mathbb{I} \ . \tag{2.5}$$

Then $\tilde{K}_{\mathcal{O}}$ may be found by solving

$$\left(\partial_t + \bar{\mathcal{O}}_x\right) \tilde{K}_{\mathcal{O}}(x,x';t) + \tilde{\mathcal{O}}_x \bar{K}_{\mathcal{O}}(x,x';t) = 0 \ , \qquad \tilde{K}_{\mathcal{O}}(x,x';0) = -\frac{\tilde{g}}{2\bar{g}^{3/2}} \delta^{(d)}\left(x-x'\right) \mathbb{I} \ . \tag{2.6}$$

The resulting solution is (see Appendix A.1 for details)

$$\begin{aligned} \tilde{K}_{\mathcal{O}}\left(x,x';t\right) \ = \ & -\frac{\tilde{g}}{2\bar{g}^{3/2}} \delta^{(d)}(x-x') \mathbb{I} \\ & + \int_0^t dt' \int d^d x'' \sqrt{\bar{g}}\, \bar{K}_{\mathcal{O}}\left(x,x'';t-t'\right) \bar{\mathcal{O}}_{x''} \left(\frac{\tilde{g}}{2\bar{g}^{3/2}} \delta^{(d)}(x''-x') \right) \\ & - \int_0^t dt' \int d^d x'' \sqrt{\bar{g}} \bar{K}_{\mathcal{O}}\left(x,x'';t-t'\right) \tilde{\mathcal{O}}_{x''} \bar{K}_{\mathcal{O}}\left(x'',x';t'\right) \ . \end{aligned} \tag{2.7}$$

Then the trace $K_{\mathcal{O}}(t)$ in (2.2) takes the form

$$K_{\mathcal{O}}(t) = \bar{K}_{\mathcal{O}}(t) + \alpha\, \tilde{K}_{\mathcal{O}}(t) + O\left(\alpha^2\right) \ , \tag{2.8}$$

$$\tilde{K}_{\mathcal{O}}(t) = -t \int d^d x \sqrt{\bar{g}}\ \ \mathrm{tr}\Big[\tilde{\mathcal{O}}_x\ \bar{K}_{\mathcal{O}}\left(x,x';t\right) \Big]_{x=x'} . \tag{2.9}$$

Thus the perturbative expansion of the determinant of $\mathcal{O}$ in (2.1) becomes

$$\frac{\mathrm{Det}_{\mathcal{M}} \mathcal{O}}{\mathrm{Det}_{\bar{\mathcal{M}}} \bar{\mathcal{O}}} = e^{-\alpha\, \tilde{\zeta}_{\mathcal{O}}'(0) + O(\alpha^2)} \ , \qquad \log \mathrm{Det}_{\mathcal{M}} \mathcal{O} = -\bar{\zeta}_{\mathcal{O}}'(0) - \alpha\, \tilde{\zeta}_{\mathcal{O}}'(0) + O(\alpha^2) \, , \tag{2.10}$$

$$\bar{\zeta}_{\mathcal{O}}(s) = \frac{1}{\Gamma(s)} \int_0^\infty dt\, t^{s-1} \bar{K}_{\mathcal{O}}(t) \ , \qquad \tilde{\zeta}_{\mathcal{O}}(s) = \frac{1}{\Gamma(s)} \int_0^\infty dt\, t^{s-1} \tilde{K}_{\mathcal{O}}(t) \ . \tag{2.11}$$

From (2.8),(2.9) one can find similar perturbative expansions for the coefficients in the small-t expansion of the heat kernel, i.e. for the Seeley coefficients that control the UV divergent part of $\log \mathrm{Det}_{\mathcal{M}} \mathcal{O}$ in, e.g., the proper-time regularization (see, e.g., [48, 49]). As a check of (2.7) we show in Appendix A.2 that the small-t expansion of (2.9) reproduces the results of the standard perturbation theory applied directly to the Seeley coefficients of the scalar Laplace operator on a manifold with no singularities.

In the following section we will consider examples where scalar and spinor operators will be defined on the two-dimensional $\bar{\mathcal{M}}$ which will be real hyperbolic space H^2. In this case the homogeneity of H^2 allows one to construct the relevant heat kernels $\bar{K}_{\mathcal{O}}$ for generic pair of points x, x' [25–28] (see also Appendix B) and thus to compute the first corrections $\tilde{K}_{\mathcal{O}}$ according to (2.7).

3 Perturbative expansion of 1-loop string correction to Wilson loop minimal surfaces

Our aim will be to use the above expressions to develop a perturbative approach to computation of $AdS_5 \times S^5$ superstring partition function Z expanded near a particular minimal surface ending on the AdS boundary that represents the leading strong-coupling correction to the corresponding Wilson loop in gauge theory. In general,

$$Z = \langle W(\lambda, \alpha) \rangle \equiv e^{-\Gamma}\,, \qquad \Gamma = \sqrt{\lambda}\,\Gamma^{(0)}(\alpha) + \Gamma^{(1)}(\alpha) + \mathcal{O}(\lambda^{-1/2})\ . \tag{3.1}$$

Here $\sqrt{\lambda}\Gamma^{(0)}(\alpha)$ is the classical string action ($\frac{\sqrt{\lambda}}{2\pi}$ is the string tension) evaluated on a minimal surface with parameter α and $\Gamma^{(1)}(\alpha)$ is the 1-loop correction expressed in terms of ratios of determinants of 2nd order fluctuation operators [4–6].

While computing these determinants for a generic minimal surface is hard, expanding in some small parameter α (such that for $\alpha = 0$ the surface becomes simple) that can be done in perturbation theory. We shall demonstrate this below in a number of cases:

(i) "latitudes" in $S^2 \subset S^5$ (Section 3.1);

(ii) generalized cusp (Section 3.2);

(iii) k-wound circle (Section 3.3).

In these cases the $\alpha = 0$ limit of the minimal surface will be the Euclidean AdS_2 space or H^2 for which the heat kernels and determinants or relevant operators are known explicitly, i.e. $\Gamma^{(1)}(0) \equiv \bar{\Gamma}^{(1)}$ is known. Our aim will be to find the first correction to $\Gamma^{(1)}(0)$:

$$\Gamma^{(1)}(\alpha) = \bar{\Gamma}^{(1)} + \alpha\,\tilde{\Gamma}^{(1)} + O\left(\alpha^2\right)\ . \tag{3.2}$$

3.1 Latitude Wilson loop

Let us start with a family of 1/4-BPS Wilson loops with the minimal surface of half-sphere topology ending on a unit circle at the boundary of AdS_5 and stretched also along the latitude located at the polar angle θ_0 in a $S^2 \subset S^5$ [9–11]. The minimal surface is embedded into a

subspace $H^3 \times S^2$ of $AdS_5 \times S^5$ with the metric

$$ds^2_{H^3\times S^2} = z^{-2}(dx_1^2 + dx_2^2 + dz^2) + d\theta^2 + \sin^2\theta\, d\phi^2 \tag{3.3}$$

as follows

$$x_1 = \frac{\cos\tau}{\cosh\sigma}\,, \qquad x_2 = \frac{\sin\tau}{\cosh\sigma}\,, \qquad z = \tanh\sigma\,, \tag{3.4}$$

$$\sin\theta = \frac{1}{\cosh(\sigma+\sigma_0)}\,, \qquad \cos\theta = \tanh(\sigma+\sigma_0)\,, \qquad \phi = \tau\,, \tag{3.5}$$

$$\sigma \in [0,\infty)\,, \qquad \tau \in [0, 2\pi)\,, \qquad \tanh\sigma_0 \equiv \cos\theta_0\,. \tag{3.6}$$

The world-sheet boundary at $\sigma = 0$ is located at the boundary of AdS_5, and $\sigma_0 \in [0,\infty)$ related to $\theta_0 \in [0, \frac{\pi}{2}]$ describes a one-parameter family of latitudes on S^5. The maximally supersymmetric (1/2-BPS) case corresponds to $\theta_0 = 0$ or $\sigma_0 = \infty$ when the latitude in S^2 shrinks to a point ($\theta = \theta_0 = 0$) and thus the minimal surface becomes the same as of the circular Wilson loop. In what follows θ_0 will thus play the role of the small expansion parameter α.

The induced world-sheet geometry is that of the 2d Euclidean manifold $\mathcal{M}$ with the metric

$$\begin{aligned} ds^2_{\mathcal{M}} &= \Omega^2(\sigma)\left(d\tau^2 + d\sigma^2\right)\,, \\ \Omega^2(\sigma) &\equiv \frac{1}{\sinh^2\sigma} + \frac{1}{\cosh^2(\sigma+\sigma_0)} = \frac{1}{\sinh^2\sigma} + O\left(\theta_0^2\right)\,, \end{aligned} \tag{3.7}$$

which for $\sigma_0 = \infty$, i.e. $\theta_0 = 0$, becomes the hyperbolic plane H^2. The leading term in (3.1), i.e. the area of this minimal surface, regularized in a standard way by introducing a small cutoff near the boundary of AdS_5, at $z = \epsilon \to 0$, or, equivalently, at $\sigma = \operatorname{arctanh}\epsilon \to \infty$ is then

$$\Gamma^{(0)}(\theta_0) = \frac{1}{2\pi}\int_0^{2\pi} d\tau \int_{\operatorname{arctanh}\epsilon}^{\infty} d\sigma\ \Omega^2(\sigma) = \frac{1}{\epsilon} - \cos\theta_0 \to -\cos\theta_0\,. \tag{3.8}$$

The singular term here is θ_0-independent and thus is the same as in the singular part of the volume of Euclidean AdS_2 space.[6]

Expanding the $AdS_5 \times S^5$ superstring action to second order in the fluctuation fields leads to the following one-loop contribution to (3.1) [12–14] [7]

$$\Gamma^{(1)}(\theta_0) = -\log \frac{\prod_{p_{12},p_{56}=\pm 1} \mathrm{Det}^{2/4}\left[\mathcal{O}^2_{p_{12},p_{56}}(\theta_0)\right]}{\mathrm{Det}^{3/2}\left[\mathcal{O}_1(\theta_0)\right]\mathrm{Det}^{3/2}\left[\mathcal{O}_2(\theta_0)\right]\mathrm{Det}^{1/2}\left[\mathcal{O}_{3+}(\theta_0)\right]\mathrm{Det}^{1/2}\left[\mathcal{O}_{3-}(\theta_0)\right]}\,. \tag{3.9}$$

[6] The linearly divergent part $\frac{1}{\epsilon}$, proportional to the length of the boundary at $z = \epsilon$, may be subtracted by a Legendre transform of the Wilson loop as in [6, 19, 46].

[7] As in earlier discussions [6, 12] it is assumed here that the same boundary conditions are imposed on the operator of the longitudinal bosonic modes and the one of the ghosts associated with the diffeomorphisms gauge-fixing, so that their net contribution to the ratio (3.9) equals to one.

Here the bosonic second-order operators [8]

$$\mathcal{O}_1(\theta_0) \equiv \frac{1}{\Omega^2(\sigma)}\Big(-\partial_\tau^2 - \partial_\sigma^2 + \frac{2}{\sinh^2\sigma}\Big)\,, \quad \mathcal{O}_2\,(\theta_0) \equiv \frac{1}{\Omega^2(\sigma)}\Big(-\partial_\tau^2 - \partial_\sigma^2 - \frac{2}{\cosh^2(\sigma+\sigma_0)}\Big)\,, \tag{3.10}$$

$$\begin{aligned}\mathcal{O}_{3\pm}\,(\theta_0) \equiv \frac{1}{\Omega^2(\sigma)}\Big[&-\partial_\tau^2 - \partial_\sigma^2 \pm 2i\,(\tanh(2\sigma+\sigma_0) - 1)\,\partial_\tau \\ &- 1 - 2\tanh(2\sigma+\sigma_0) + 3\tanh^2(2\sigma+\sigma_0)\Big]\end{aligned} \tag{3.11}$$

act on the world-sheet scalars, and the fermionic first-order operators

$$\begin{aligned}\mathcal{O}_{p_{12},p_{56}}\,(\theta_0) \equiv{}& \frac{i}{\Omega(\sigma)}\Big(\partial_\sigma + \frac{\Omega'(\sigma)}{2\,\Omega(\sigma)}\Big)\sigma_1 + \frac{1}{\Omega(\sigma)}\Big(-i\partial_\tau + \frac{p_{56}}{2}\big[1-\tanh(2\sigma+\sigma_0)\big]\Big)\,\sigma_2 \\ &+ \frac{p_{12}}{\Omega^2(\sigma)\sinh^2\sigma}\sigma_3 - \frac{p_{12}\,p_{56}}{\Omega^2(\sigma)\cosh^2(\sigma+\sigma_0)}\mathbb{I}_2\end{aligned} \tag{3.12}$$

act on two-dimensional spinors and are labeled by $p_{12}, p_{56} = \pm 1$ (σ_i are Pauli matrices).[9] The determinants of these operators have been evaluated exactly (for any θ_0) in [13, 14].

To apply the perturbative approach developed in Section 2, we choose

$$\alpha_{\text{latitude}} \equiv \theta_0^2\,, \tag{3.13}$$

so that the reference manifold $\bar{\mathcal{M}}$ for $\alpha = 0$ is H^2 corresponding to the circular Wilson loop ($\theta_0 = 0$, or $\sigma_0 = \infty$), i.e.

$$ds^2_{\bar{\mathcal{M}}} = \frac{d\tau^2 + d\sigma^2}{\sinh^2\sigma} = d\rho^2 + \sinh^2\rho\, d\tau^2\,, \qquad \sinh\rho \equiv \frac{1}{\sinh\sigma}\,, \tag{3.14}$$

with the S^1 boundary at

$$\rho = \Lambda \to \infty\,, \qquad \Lambda \equiv \operatorname{arccosh}(\epsilon^{-1})\,. \tag{3.15}$$

The string action proportional to the (renormalized) volume of this space is

$$\Gamma^{(0)}(0) = \frac{1}{2\pi}V_{H^2} = \frac{1}{2\pi}\int_0^{2\pi} d\tau \int_0^{\Lambda} d\rho\, \sinh\rho = \frac{1}{\epsilon} - 1 \to -1\,, \tag{3.16}$$

which is the $\theta_0 = 0$ term in (3.8). In the limit $\theta_0 = 0$ the operators (3.10)-(3.12) take the form of the Laplacian (B.8) and the Dirac operator (B.11)

$$\bar{\mathcal{O}}_1 = -\Delta_{\rho,\tau} + 2\,, \qquad \bar{\mathcal{O}}_2 = \bar{\mathcal{O}}_{3\pm} = -\Delta_{\rho,\tau}\,, \qquad \bar{\mathcal{O}}_{p_{12},p_{56}} = -i\not{\nabla}_{\rho,\tau} + p_{12}\,\sigma_3\,. \tag{3.17}$$

The spectrum of physical excitations which contribute to $\Gamma^{(1)}\,(\theta_0 = 0)$ in (3.9), is composed of 3 massive scalars $\big(m^2 = 2\big)$, 5 massless scalars and 8 massive 2d Majorana spinors ($m^2 = 1$)

[8] The operators $\mathcal{O}_{3\pm}(\theta_0)$ in (3.10) of [13] coincide with the ones in (3.11) upon the replacement $-i\partial_\tau \to -i\partial_\tau \pm 1$, which implements the shift explained in Section 4 of [13]. This is equivalent to a choice of the normal bundle gauge connection [12] that is regular everywhere on the world-sheet (see discussion below (4.20) of [14]).

[9] Compared to the notation used in (3.26) of [13], the fermionic determinants are raised in (3.9) to an additional power of two because the irrelevant label p_{89} is suppressed. We also made the replacement $-i\partial_\tau \to -i\partial_\tau + \frac{p_{56}}{2}$ to arrive at (3.12), as motivated in Section 4 of [13].

propagating in H^2 [6, 7]. The regularized determinants were computed in [8] with the heat kernel method using (B.30) and (B.31)

$$\bar{\zeta}'_{\mathcal{O}_1}(0) = -\frac{25}{12} + \frac{3}{2}\log 2\pi - 2\log A\,, \tag{3.18}$$

$$\bar{\zeta}'_{\mathcal{O}_2}(0) = \bar{\zeta}'_{\mathcal{O}_{3\pm}}(0) = -\frac{1}{12} + \frac{1}{2}\log 2\pi - 2\log A\,, \tag{3.19}$$

$$\bar{\zeta}'_{\mathcal{O}^2_{p_{12},p_{56}}}(0) = -\frac{5}{3} + 2\log 2\pi - 4\log A\,, \tag{3.20}$$

where A is the Glaisher constant (see (B.33), (B.34) and (B.39)). As a result, the one-loop correction (3.9) in the circular Wilson loop case is

$$\Gamma^{(1)}(0) = -\frac{3}{2}\bar{\zeta}'_{\mathcal{O}_1}(0) - \frac{3}{2}\bar{\zeta}'_{\mathcal{O}_2}(0) - \frac{1}{2}\bar{\zeta}'_{\mathcal{O}_{3+}}(0) - \frac{1}{2}\bar{\zeta}'_{\mathcal{O}_{3-}}(0) + \frac{1}{2}\sum_{p_{12},p_{56}=\pm 1}\bar{\zeta}'_{\mathcal{O}^2_{p_{12},p_{56}}}(0) = \frac{1}{2}\log 2\pi. \tag{3.21}$$

Expanding (3.7) in small $\alpha = \theta_0^2$ we find that the leading correction to the metric (3.14) in (2.4) is given by

$$\bar{g}_{ij}(\rho,\tau) = \begin{pmatrix} 1 & 0 \\ 0 & \sinh^2\rho \end{pmatrix}, \qquad \tilde{g}_{ij}(\rho,\tau) = \begin{pmatrix} \frac{1}{(1+\cosh\rho)^2} & 0 \\ 0 & \frac{\cosh\rho-1}{\cosh\rho+1} \end{pmatrix}. \tag{3.22}$$

From (3.10)-(3.12) we find that the expansion of the relevant differential operators [10]

$$\mathcal{O}_i(\theta_0) = \bar{\mathcal{O}}_i + \theta_0^2\,\tilde{\mathcal{O}}_i + O(\theta_0^4)\,, \qquad i = 1,\, 2,\, 3+,\, 3-\,, \tag{3.23}$$

$$\mathcal{O}_{p_{12},p_{56}}(\theta_0) = \bar{\mathcal{O}}_{p_{12},p_{56}} + \theta_0^2\,\tilde{\mathcal{O}}_{p_{12},p_{56}} + O(\theta_0^4)\,, \tag{3.24}$$

$$\mathcal{O}^2_{p_{12},p_{56}}(\theta_0) = \bar{\mathcal{O}}^2_{p_{12},p_{56}} + \theta_0^2\left\{\bar{\mathcal{O}}_{p_{12},p_{56}}, \tilde{\mathcal{O}}_{p_{12},p_{56}}\right\} + O(\theta_0^4)\,, \tag{3.25}$$

contains

$$\tilde{\mathcal{O}}_1 = \tilde{\mathcal{O}}_2 = \frac{1}{(1+\cosh\rho)^2}\left(\Delta_{\rho,\tau} - 2\right), \tag{3.26}$$

$$\tilde{\mathcal{O}}_{3\pm} = \frac{1}{(1+\cosh\rho)^2}\left[\Delta_{\rho,\tau} - \frac{\sinh^2\rho}{(1+\cosh\rho)^2}\left(2 \pm i\partial_\tau\right)\right], \tag{3.27}$$

$$\begin{aligned}\tilde{\mathcal{O}}_{p_{12},p_{56}} = {} & \frac{i}{2\,(1+\cosh\rho)^2}\not{\nabla}_{\rho,\tau} - \frac{i\,(1-\cosh\rho)}{2\sinh\rho\,(1+\cosh\rho)^2}\sigma_1 \\ & + \frac{p_{56}\sinh^3\rho}{4\,(1+\cosh\rho)^4}\sigma_2 - \frac{p_{12}}{(1+\cosh\rho)^2}\left(\sigma_3 + p_{56}\,\mathbb{I}_2\right).\end{aligned} \tag{3.28}$$

For the bosonic operator $\mathcal{O}_1(\theta_0)$ in (3.23), substituting (3.26) into (2.9), we obtain

$$\tilde{K}_{\mathcal{O}_1}(t) = -t\int_0^{2\pi} d\tau \int_0^{\Lambda} d\rho \frac{\sinh\rho}{(1+\cosh\rho)^2}\Big[\left(\Delta_{\rho,\tau} - 2\right)\bar{K}_{-\Delta+2}(\rho,\tau,\rho',\tau';t)\Big]_{\rho=\rho',\tau=\tau'}, \tag{3.29}$$

where Λ was defined in (3.15). As $\bar{\mathcal{O}}_1$ in (3.17) is the Laplacian for a scalar field of mass $m^2 = 2$, its heat kernel satisfies

$$\left(\partial_t - \Delta_{\rho,\tau} + 2\right)\bar{K}_{\mathcal{O}_1}(\rho,\tau,\rho',\tau';t) = 0 \tag{3.30}$$

[10] Here by $\{,\}$ we indicate the anticommutator of two (matrix-valued) differential operators.

so that we can trade the Laplacian in (3.29) for the derivative ∂_t, and then take the coincident-point limit, getting

$$\tilde{K}_{\mathcal{O}_1}(t) = -t\int_0^{2\pi} d\tau \int_0^{\Lambda} d\rho \frac{\sinh\rho}{(1+\cosh\rho)^2}\,\partial_t \bar{K}_{\mathcal{O}_1}(\rho,\tau,\rho,\tau;t)\ . \tag{3.31}$$

Here we can send the upper limit to infinity ($\Lambda\to\infty$ corresponds to $\epsilon\to 0$ in (3.15)) and then use the integral representation of the traced heat kernel (B.23) for mass $m^2=2$

$$\tilde{K}_{\mathcal{O}_1}(t) = \frac{t}{2}\int_0^{\infty} dv\, v\tanh(\pi v)\left(v^2+\tfrac{9}{4}\right)\, e^{-t\left(v^2+\frac{9}{4}\right)}\,. \tag{3.32}$$

To evaluate $\tilde{\zeta}_{\mathcal{O}_1}(s)$ one proceeds as in Appendix B.1, interchanging the integration over the spectral parameter v and the proper time t in the definition (2.11) of the zeta-function, and writing $\tanh(\pi v)=1-2/(e^{2\pi v}+1)$ to get

$$\tilde{\zeta}_{\mathcal{O}_1}(s) = \int_0^{\infty} dv \frac{sv}{2\left(v^2+\frac{9}{4}\right)^s} - \int_0^{\infty} dv \frac{sv}{\left(e^{2\pi v}+1\right)\left(v^2+\frac{9}{4}\right)^s}\ . \tag{3.33}$$

As the first integral above converges only for $\mathrm{Re}\, s>1$, one can first integrate over v assuming this is true and then analytically continue to all values of s

$$\tilde{\zeta}_{\mathcal{O}_1}(s) = \tfrac{s}{4(s-1)}\left(\tfrac{9}{4}\right)^{1-s} - s\int_0^{\infty} dv \frac{v}{\left(e^{2\pi v}+1\right)\left(v^2+\frac{9}{4}\right)^s}\,, \tag{3.34}$$

and one obtains

$$\tilde{\zeta}'_{\mathcal{O}_1}(0) = -\tfrac{7}{12}\,. \tag{3.35}$$

The same steps may be followed for $\mathcal{O}_2(\theta_0)$, for which one gets

$$\tilde{K}_{\mathcal{O}_2}(t) = \frac{t}{2}\int_0^{\infty} dv\, v\tanh(\pi v)\left(v^2+\tfrac{9}{4}\right)\, e^{-t\left(v^2+\frac{1}{4}\right)}\ , \tag{3.36}$$

$$\begin{aligned}\tilde{\zeta}_{\mathcal{O}_2}(s) &= \textstyle\int_0^{\infty} dv \frac{sv}{\left(v^2+\frac{1}{4}\right)^s}\left(\frac{1}{2}+\frac{1}{v^2+\frac{1}{4}}\right) - \int_0^{\infty} dv \frac{sv}{\left(e^{2\pi v}+1\right)\left(v^2+\frac{1}{4}\right)^s}\left(1+\frac{2}{v^2+\frac{1}{4}}\right)\\ &= \tfrac{s}{4(s-1)}\left(\tfrac{1}{4}\right)^{1-s} + \tfrac{1}{2}\left(\tfrac{1}{4}\right)^{-s} - s\textstyle\int_0^{\infty} dv \frac{v}{\left(e^{2\pi v}+1\right)\left(v^2+\frac{1}{4}\right)^s} - 2s\int_0^{\infty} dv \frac{v}{\left(e^{2\pi v}+1\right)\left(v^2+\frac{1}{4}\right)^{s+1}}\ ,\end{aligned}$$

$$\tilde{\zeta}'_{\mathcal{O}_2}(0) = -\tfrac{1}{12}+\gamma\,. \tag{3.37}$$

Here we used (B.36) and γ is the Euler-Mascheroni constant.

The operators $\mathcal{O}_{3+}(\theta_0)$ and $\mathcal{O}_{3-}(\theta_0)$ coincide for $\theta_0=0$ in (3.17) and therefore the derivatives ∂_τ in (3.27) cancel each other in the sum [11]

$$\begin{aligned}&\tilde{K}_{\mathcal{O}_{3+}}(t) + \tilde{K}_{\mathcal{O}_{3-}}(t)\\ &= -2t\int_0^{2\pi} d\tau\int_0^{\infty} d\rho \frac{\sinh\rho}{(1+\cosh\rho)^2}\left[\left(\Delta_{\rho,\tau} - 2\frac{\sinh^2\rho}{(1+\cosh\rho)^2}\right)\bar{K}_{-\Delta}(\rho,\tau,\rho',\tau';t)\right]_{\rho=\rho',\tau=\tau'}\\ &= t\int_0^{\infty} dv\, v\tanh(\pi v)\left(v^2+\tfrac{5}{4}\right)\, e^{-t\left(v^2+\frac{1}{4}\right)}\,.\end{aligned} \tag{3.38}$$

[11] The derivatives come with opposite signs in (3.27) as the fields acted upon by (3.11) in the fluctuation Lagrangian [12–14] are a complex scalar and its complex conjugate, coupled to a $U(1)$ connection with opposite charges [14].

Then for the combined zeta-functions one obtains

$$\tilde\zeta_{\mathcal{O}_{3+}}(s)+\tilde\zeta_{\mathcal{O}_{3-}}(s)=\int_0^\infty dv\tfrac{sv}{(v^2+\frac14)^s}(1+\tfrac{1}{v^2+\frac14})+\int_0^\infty dv\tfrac{-2sv}{(e^{2\pi v}+1)(v^2+\frac14)^s}(1+\tfrac{1}{v^2+\frac14}) \tag{3.39}$$
$$=\tfrac{s}{2(s-1)}\left(\tfrac14\right)^{1-s}+\tfrac12\left(\tfrac14\right)^{-s}-2s\int_0^\infty dv\tfrac{v}{(e^{2\pi v}+1)\left(v^2+\frac14\right)^s}-2s\int_0^\infty dv\tfrac{v}{(e^{2\pi v}+1)\left(v^2+\frac14\right)^{s+1}}\,,$$
$$\tilde\zeta'_{\mathcal{O}_{3+}}(0)+\tilde\zeta'_{\mathcal{O}_{3-}}(0)=-\tfrac16+\gamma\,, \tag{3.40}$$

where we used (B.36). In the fermionic case the relevant operator is the *square* of $\mathcal{O}_{p_{12},p_{56}}(\theta_0)$, a positive-definite operator with a well-defined θ_0-expansion of its heat kernel defined in (3.24)

$$\tilde K_{\mathcal{O}^2_{p_{12},p_{56}}}(t)=-t\int_0^{2\pi}d\tau\int_0^\infty d\rho\,\sinh\rho\,\mathrm{tr}\Big[\Big\{\bar{\mathcal{O}}^{\rho,\tau}_{p_{12},p_{56}},\tilde{\mathcal{O}}^{\rho,\tau}_{p_{12},p_{56}}\Big\}\bar K_{-\nabla^2+1}(\rho,\tau,\rho',\tau';t)\Big]_{\rho=\rho',\tau=\tau'}$$
$$=t\int_0^\infty dv\,v\coth(\pi v)\left(v^2+2\right)e^{-t\left(v^2+1\right)}\,. \tag{3.41}$$

Here one has to work with the full heat kernel (B.13) for $m^2=1$ and the rest of the computation is essentially unchanged, giving

$$\tilde\zeta_{\mathcal{O}^2_{p_{12},p_{56}}}(s)=\int_0^\infty dv\tfrac{sv}{(v^2+1)^s}(1+\tfrac{1}{v^2+1})+\int_0^\infty dv\tfrac{2sv}{(e^{2\pi v}-1)(v^2+1)^s}(1+\tfrac{1}{v^2+1})$$
$$=\tfrac{s}{2(s-1)}+\tfrac12+2s\int_0^\infty dv\tfrac{v}{(e^{2\pi v}-1)(v^2+1)^s}+2s\int_0^\infty dv\tfrac{v}{(e^{2\pi v}-1)(v^2+1)^{s+1}}\,, \tag{3.42}$$
$$\tilde\zeta'_{\mathcal{O}^2_{p_{12},p_{56}}}(0)=-\tfrac{11}{12}+\gamma\,. \tag{3.43}$$

where we split $\coth(\pi v)=1+2/(e^{2\pi v}-1)$ and the last relation follows from (B.41).

We can now sum over the bosonic and fermionic contributions to get

$$\Gamma^{(1)}(\theta_0)-\Gamma^{(1)}(0)=\theta_0^2\,\tilde\Gamma^{(1)}+O(\theta_0^4)\,, \tag{3.44}$$
$$\tilde\Gamma^{(1)}=-\tfrac32\tilde\zeta'_{\mathcal{O}_1}(0)-\tfrac32\tilde\zeta'_{\mathcal{O}_2}(0)-\tfrac12\tilde\zeta'_{\mathcal{O}_{3+}}(0)-\tfrac12\tilde\zeta'_{\mathcal{O}_{3-}}(0)+\tfrac12\textstyle\sum_{p_{12},p_{56}=\pm1}\tilde\zeta'_{\mathcal{O}^2_{p_{12},p_{56}}}(0)$$
$$=-\tfrac34\,. \tag{3.45}$$

Remarkably, we thus find the agreement with the strong-coupling expansion of the exact gauge-theory result (1.1), expanded also in small θ_0.

Let us note that to the same result (3.45) can be found by reversing the order of taking the derivative in the zeta-function variable s and summing over the scalar and spinor fields. The expressions for zeta-functions in (3.33)–(3.42) above are written as $\tilde\zeta_{\mathcal{O}}(s)\equiv\tilde\zeta^{(\mathrm{power})}_{\mathcal{O}}(s)+\tilde\zeta^{(\mathrm{exp})}_{\mathcal{O}}(s)$, where $\tilde\zeta^{(\mathrm{power})}_{\mathcal{O}}(s)$ includes the 1 from the expansion of the hyperbolic functions and is defined for $\mathrm{Re}\,s>1$, and $\tilde\zeta^{(\mathrm{exp})}_{\mathcal{O}}(s)$ is well-defined for s close to 0. The analytic continuation of each $\tilde\zeta^{(\mathrm{power})}_{\mathcal{O}}(s)$ is not necessary if one considers, before taking the derivative, the sum of all (perturbed) zeta-functions. It can be easily checked that the sum of "power" contributions

$$\tfrac32\tilde\zeta^{(\mathrm{power})}_{\mathcal{O}_1}(s)+\tfrac32\tilde\zeta^{(\mathrm{power})}_{\mathcal{O}_2}(s)+\tfrac12\tilde\zeta^{(\mathrm{power})}_{\mathcal{O}_3}(s)-\tfrac12\textstyle\sum_{p_{12},p_{56}=\pm1}\tilde\zeta^{(\mathrm{power})}_{\mathcal{O}^2_{p_{12},p_{56}}}(s) \tag{3.46}$$
$$=\int_0^\infty dv\big[\tfrac{3sv}{4\left(v^2+\frac94\right)^s}+\tfrac{3sv}{2\left(v^2+\frac14\right)^s}(\tfrac12+\tfrac{1}{v^2+\frac14})+\tfrac{sv}{2\left(v^2+\frac14\right)^s}(1+\tfrac{1}{v^2+\frac14})-\tfrac{2sv}{(v^2+1)^s}(1+\tfrac{1}{v^2+1})\big]$$

is well defined for $\mathrm{Re}\,s>s_0$ for a certain negative s_0. One may then first take s-derivative of the *integrands* in

$$\tilde\zeta_{\mathrm{tot}}(s)=\tfrac32\tilde\zeta_{\mathcal{O}_1}(s)+\tfrac32\tilde\zeta_{\mathcal{O}_2}(s)+\tfrac12\tilde\zeta_{\mathcal{O}_{3+}}(s)+\tfrac12\tilde\zeta_{\mathcal{O}_{3-}}(s)-\tfrac12\textstyle\sum_{p_{12},p_{56}=\pm1}\tilde\zeta_{\mathcal{O}^2_{p_{12},p_{56}}}(s)\,, \tag{3.47}$$

set $s=0$ and then integrate over v. It is easy to check that this leads again to (3.45).

One may track down the origin of such regular behavior for the full sum (3.47) by studying the small-t expansion of the leading correction terms in heat kernels in (3.32), (3.36), (3.38), (3.41). For that one may isolate the exponentials of t and integrate the rest [12],

$$\begin{aligned}
\tilde{K}_{\mathcal{O}_1}(t) &= \tfrac{t}{2}\int_0^\infty dv\ (v-\tfrac{2v}{e^{2\pi v}+1})(v^2+\tfrac{9}{4})e^{-t(v^2+\frac{9}{4})} \\
&= \tfrac{4+9t}{16t}e^{-9t/4} - t\int_0^\infty dv \tfrac{v}{e^{2\pi v}+1}(v^2+\tfrac{9}{4})e^{-t(v^2+\frac{9}{4})} = \tfrac{1}{4t}+O(t)\,, \\
\tilde{K}_{\mathcal{O}_2}(t) &= \tfrac{1}{4t}+\tfrac{1}{2}+O(t)\,, \qquad \tilde{K}_{\mathcal{O}_{3+}}(t)+\tilde{K}_{\mathcal{O}_{3-}}(t) = \tfrac{1}{2t}+\tfrac{1}{2}+O(t)\,, \\
\tilde{K}_{\mathcal{O}^2_{p_{12},p_{56}}}(t) &= \tfrac{1}{2t}+\tfrac{1}{2}+O(t)\,.
\end{aligned} \tag{3.48}$$

Then considering the zeta-function

$$\tilde{\zeta}_{\mathcal{O}}(s) = \frac{1}{\Gamma(s)}\int_0^1 dt\, t^{s-1}\tilde{K}_{\mathcal{O}}(t) + \frac{1}{\Gamma(s)}\int_1^\infty dt\, t^{s-1}\tilde{K}_{\mathcal{O}}(t) \tag{3.49}$$

one finds that the second integral here is finite for $s=0$ while the first one is singular due to the asymptotics in (3.48) [13]. This explains the need to analytically extend zeta-functions to $s=0$ before computing their derivatives.

The $t\to 0$ singularities cancel in the sum of heat traces, due to the special spectrum of scalar and spinor fields and the values of their masses

$$\tfrac{3}{2}\tilde{K}_{\mathcal{O}_1}(t)+\tfrac{3}{2}\tilde{K}_{\mathcal{O}_2}(t)+\tfrac{1}{2}\tilde{K}_{\mathcal{O}_{3+}}(t)+\tfrac{1}{2}\tilde{K}_{\mathcal{O}_{3-}}(t)-\tfrac{1}{2}\sum_{p_{12},p_{56}=\pm 1}\tilde{K}_{\mathcal{O}^2_{p_{12},p_{56}}}(t) = \frac{0}{t}+0+O(t)\,. \tag{3.50}$$

Thus, in the θ_0^2 term in the total zeta-function (3.47) no analytic continuation to $s=0$ is necessary. This regularity of the leading correction (3.50) to the sum of traces of heat kernels or, equivalently, the UV finiteness of the θ_0^2 term (and, in fact, higher terms) in the expansion of the logarithm of the string 1-loop partition function has a simple explanation. The logarithmic UV divergences (determined by the Seeley coefficient a_2 of the t^0 part in the small-t expansion of heat kernel) in 2d are proportional, for smooth manifolds, to the Euler number which is the same for both the minimal surface (3.7) and its $\theta_0=0$ limit (3.14), both having the same topology (see also [13]).[14] These divergences thus cancel in the ratio of the partition functions of the latitude and the circle minimal surfaces, i.e. in $\Gamma(\theta_0)-\Gamma(0)$.

3.2 Cusped Wilson loop

Next, let us consider the string world-sheet ending on a pair of oppositely oriented ("antiparallel") lines in $\mathbb{R}\times S^3\subset AdS_5$, separated by a geometric angle $\pi-\phi$ along a great circle of

[12] Equivalently, as explained in Appendix A.2, one could use (A.19).

[13] More generally, since the operators (3.10)–(3.11) and the square of (3.12) have positive eigenvalues, the Mellin transform of their heat kernel traces (2.11) is convergent at the upper limit of the integral and singularities originate only from $t=0$ (cf. [47,48]).

[14] The part of the 1-loop superstring partition function on the disc given by the ratio of determinants as in (3.9) is known to contain a universal logarithmic UV divergence which is cancelled in the total partition function against the cutoff dependent factors in the conformal Killing vector measure included [6].

S^3 (that can be mapped to a cusp on the plane) and with an internal (R-symmetry) angle θ. The classical solution was written in [11] in terms of Jacobi elliptic functions [15]. Here we will consider only the case of vanishing θ [43,46]. Then the angular opening ϕ and the parameters b, p, q of the classical solution in Appendix B of [40] can be expressed in terms of just one independent parameter $k \in [0\,, \frac{1}{\sqrt{2}})$

$$b = \tfrac{\sqrt{1-2k^2}}{k}\,, \quad p^2 = \tfrac{b^4}{1+b^2}\,, \quad q = 0\,, \quad \phi = \pi - \tfrac{2p^2}{b\sqrt{b^4+p^2}}\Big[\Pi\big(\tfrac{b^4}{b^4+p^2}|k^2\big) - \mathbb{K}\big(k^2\big)\Big] \tag{3.51}$$

and the classical surface $\mathcal{M}$ lies entirely inside an AdS_3 subspace of AdS_5 with the metric $ds^2_{AdS_3} = -\cosh^2\rho\, dt^2 + d\rho^2 + \sinh^2\rho\, d\varphi^2$. After $t \to it$, the induced world-sheet metric is Euclidean

$$ds^2_{\mathcal{M}} = \frac{1-k^2}{\mathrm{cn}^2\left(\sigma|k^2\right)}\left(d\sigma^2 + d\tau^2\right)\,, \qquad -\mathbb{K}\left(k^2\right) < \sigma < \mathbb{K}\left(k^2\right)\,, \qquad \tau \in \mathbb{R}\,, \tag{3.52}$$

where σ, τ are related to ρ, t by

$$\cosh\rho = \frac{\sqrt{1+b^2}}{b\,\mathrm{cn}\left(\sigma|k^2\right)}\,, \qquad t = \frac{b\,p}{\sqrt{b^4+p^2}}\,\tau\,. \tag{3.53}$$

Introducing large cutoffs $0 \le \rho \le \rho_0, \;\; 0 < t \le T$ translates into

$$\sigma \in (-\sigma_0,\, \sigma_0)\,, \quad \tau \in [0, \mathcal{T}]\,, \qquad \sigma_0 \equiv \mathrm{cn}^{-1}\Big(\frac{\sqrt{1+b^2}}{b\cosh\rho_0}|k^2\Big)\,, \quad \mathcal{T} \equiv \frac{\sqrt{b^4+p^2}}{b\,p}\,T\,. \tag{3.54}$$

The classical string action (the first term in (3.1)) proportional to the regularized area of the surface is given, after the subtraction of the divergence due to the two boundary lines at $\rho = \rho_0 \to \infty$, in terms of elliptic integrals [40]

$$\begin{aligned}
\Gamma^{(0)}(k) &= \frac{1}{2\pi}\int_0^{\mathcal{T}} d\tau \int_{-\sigma_0}^{\sigma_0} d\sigma \frac{1-k^2}{\mathrm{cn}^2(\sigma|k^2)} \\
&= \tfrac{T}{2\pi}\Big[e^{\rho_0} + \tfrac{2\sqrt{b^4+p^2}}{b\,p}\Big(\tfrac{(b^2+1)p^2}{b^4+p^2}\mathbb{K}(k^2) - \mathbb{E}(k^2)\Big) + O(e^{-\rho_0})\Big] \\
&\to \frac{T}{2\pi}\frac{2\sqrt{b^4+p^2}}{b\,p}\Big[\frac{(b^2+1)p^2}{b^4+p^2}\mathbb{K}(k^2) - \mathbb{E}(k^2)\Big]\,.
\end{aligned} \tag{3.55}$$

The one-loop effective action reads formally (cf. (3.9)) [40,43]

$$\Gamma^{(1)}\left(k\right) = -\log\frac{\mathrm{Det}^{8/4}[\mathcal{O}_F^2(k)]}{\mathrm{Det}^{5/2}[\mathcal{O}_0(k)]\,\mathrm{Det}^{2/2}[\mathcal{O}_1(k)]\,\mathrm{Det}^{1/2}[\mathcal{O}_2(k)]} \tag{3.56}$$

with the bosonic and the fermionic fluctuation operators given by

$$\mathcal{O}_0\left(k\right) \equiv \frac{\mathrm{cn}^2\left(\sigma|k^2\right)}{1-k^2}\left(-\partial_\sigma^2 - \partial_\tau^2\right)\,, \qquad \mathcal{O}_1\left(k\right) \equiv \mathcal{O}_0\left(k\right) + 2\,, \tag{3.57}$$

$$\mathcal{O}_2\left(k\right) \equiv \mathcal{O}_0\left(k\right) + 2 - 2\frac{k^2\mathrm{cn}^4\left(\sigma|k^2\right)}{1-k^2}\,, \tag{3.58}$$

$$\mathcal{O}_F\left(k\right) \equiv -i\frac{\mathrm{cn}\left(\sigma|k^2\right)}{\sqrt{1-k^2}}\sigma_1\Big(\partial_\sigma + \frac{\mathrm{sn}\left(\sigma|k^2\right)\mathrm{dn}\left(\sigma|k^2\right)}{2\mathrm{cn}\left(\sigma|k^2\right)}\Big) - i\frac{\mathrm{cn}\left(\sigma|k^2\right)}{\sqrt{1-k^2}}\sigma_2\partial_\tau + \sigma_3\,. \tag{3.59}$$

[15] We adhere to the notation in Appendix F of [40]: $\mathrm{sn}, \mathrm{cn}, \mathrm{dn}$ are the three basic Jacobi elliptic functions, $\mathbb{K}$ is the complete elliptic integral of the first kind and Π is the complete elliptic integral of the third kind.

The limiting case of $k = 0$ ($\phi = 0$) corresponds to a surface $\bar{\mathcal{M}}$ stretching between a pair of lines that are antipodal in $\mathbb{R} \times S^3$ [16] at the AdS boundary, a configuration for which the corresponding Wilson loop is a 1/2 BPS protected observable with the expectation value equal to one [50]. Thus the natural choice for the expansion parameter α is

$$\alpha_{\rm cusp} \equiv k^2 \,. \tag{3.60}$$

In this case the world-sheet cutoffs in (3.54) also depend on α and that may be confusing. A sensible expansion would require introducing new world-sheet coordinates r, w with the range independent of k. For the world-sheet time, one can simply choose it to be the AdS time

$$w \equiv t = \frac{b\,p}{\sqrt{b^4 + p^2}}\,\tau \,, \qquad\qquad w \in [0,\, T] \,, \tag{3.61}$$

while finding a suitable spatial world-sheet coordinate appears to be more problematic [17]. A good candidate [51] in the $\rho_0 \to \infty$ limit is

$$r \equiv \frac{\pi\sigma}{2\mathbb{K}(k^2)} \,, \qquad\qquad r \in \Big(-\frac{\pi}{2},\, \frac{\pi}{2}\Big) \,, \tag{3.62}$$

as in this limit $\sigma_0 = \mathbb{K}(k^2)$, see (3.54). At finite and large ρ_0, however, the maximum value of $|r|$ is $\pi\sigma_0/(2\mathbb{K}(k^2)) = \pi/2 + O(e^{-\rho_0})$ and k reappears in the exponentially suppressed terms. We will later take into account that the integrals over r may generate such k-dependent contributions (see footnote 23).

In the limiting case $k = 0$ eqs. (3.53) and (3.61)-(3.62) simplify to $\sinh\rho = |\tan\sigma| = |\tan r|$ and $t = \tau = w$, the cutoffs (3.54) become $\sigma_0 = \arctan(\sinh\rho_0)$ and $\mathcal{T} = T$, and (3.52) reduces to that of the infinite-strip parametrization or H^2 that we will call $\hat{H}^2$ (with boundary $\mathbb{R}$ instead of S^1)

$$ds^2_{\bar{\mathcal{M}}} = \frac{1}{\cos^2 r}(dr^2 + dw^2) \,. \tag{3.63}$$

In this case the regularized volume or the value of string action vanishes (cf. (3.55))

$$\Gamma^{(0)}(0) = \frac{1}{2\pi}V_{\hat{H}^2} = \frac{1}{2\pi}\int_0^T dw \int_{-\arctan(\sinh\rho_0)}^{\arctan(\sinh\rho_0)} \frac{dr}{\cos^2 r} = \frac{T}{2\pi}\big[e^{\rho_0} + O(e^{-\rho_0})\big] \;\to\; 0 \,, \tag{3.64}$$

in agreement with the $k = 0$ limit of (3.55). For $k = 0$ the operators (3.57)-(3.59) become those of the straight line Wilson loop [5, 6]

$$\bar{\mathcal{O}}_0 = -\Delta_{r,w} \,, \qquad\qquad \bar{\mathcal{O}}_1 = \bar{\mathcal{O}}_2 = -\Delta_{r,w} + 2 \,, \qquad\qquad \bar{\mathcal{O}}_F = -i\not{\nabla}_{r,w} + \sigma_3 \,, \tag{3.65}$$

[16] Considering the theory in $\mathbb{R}^4$, related to the theory in $\mathbb{R} \times S^3$ by the stereographic projection, this is the infinite straight line.

[17] For instance, we discard ρ because its minimum value $\operatorname{arccosh}(\sqrt{1+b^2}/b)$ is a function of k, and the relation (3.53) between σ and ρ is not one-to-one. Another possibility is $r' \equiv \pi\sigma/(2\sigma_0)$ which varies in the constant interval $(-\pi/2, \pi/2)$, however this choice would introduce the cutoff ρ_0 via σ_0 into (3.52) and (3.57)-(3.59) once the change of coordinates is made. This implies that the metric at $k = 0$ is still dependent on one parameter and cannot have the geometry of H^2. The perturbative analysis for small k would be then problematic, as the procedure relies on the knowledge of the heat kernels at $k = 0$, which in this case one would still need to evaluate.

with the Laplacian given in (B.9) and the Dirac operator in (B.12). Here the multiplicities and the masses coincide with those in the spectrum (3.17) corresponding to a circular Wilson loop in $\mathbb{R}^4$. The zeta-functions (B.30) of these operators are proportional to the volume $V_{\hat{H}^2}$ whose renormalized value is zero (3.64) and thus we get [8, 18] [18]

$$\Gamma^{(1)}(0) = -\tfrac{5}{2}\bar{\zeta}'_{\mathcal{O}_0}(0) - \tfrac{2}{2}\bar{\zeta}'_{\mathcal{O}_1}(0) - \tfrac{1}{2}\bar{\zeta}'_{\mathcal{O}_2}(0) + \tfrac{8}{4}\bar{\zeta}'_{\mathcal{O}_F^2}(0) = 0\,. \tag{3.66}$$

For small values of $\alpha_{\text{cusp}} = k^2$ the angle in (3.51) is also small, $\phi = \pi k + O(k^3)$. In this near-BPS limit we get, expanding the elliptic integral in the metric (3.52) (cf. (2.4))

$$\bar{g}_{ij}(r,w) = \frac{1}{\cos^2 r}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \tilde{g}_{ij}(r,w) = \begin{pmatrix} -\frac{1}{2} & 0 \\ 0 & \frac{3}{2\cos^2 r} - \frac{1}{2} \end{pmatrix}. \tag{3.67}$$

The order k^2 terms in the operators (3.57)–(3.59) are found to be barred operators given by (3.65) and their perturbations by

$$\tilde{\mathcal{O}}_0 = \tilde{\mathcal{O}}_1 = \tfrac{\cos^2 r}{2}\Big[-\cos^2 r\,\partial_r^2 + \big(2+\sin^2 r\big)\,\partial_w^2\Big]\,, \tag{3.68}$$

$$\tilde{\mathcal{O}}_2 = \tfrac{\cos^2 r}{2}\Big[-\cos^2 r\,\partial_r^2 + \big(2+\sin^2 r\big)\,\partial_w^2 - 4\cos^2 r\Big]\,, \tag{3.69}$$

$$\tilde{\mathcal{O}}_F = -\tfrac{i\cos^3 r}{4}\sigma_1\partial_r - \tfrac{i(\cos 3r - 9\cos r)}{16}\sigma_2\partial_w - \tfrac{3i\sin r\cos^2 r}{8}\sigma_1\,. \tag{3.70}$$

As in (3.25), we will actually be using the expansion of the square of the fermionic operator:

$$\mathcal{O}_F^2(k) = \bar{\mathcal{O}}_F^2 + k^2\,\{\bar{\mathcal{O}}_F, \tilde{\mathcal{O}}_F\} + O(k^4)\,. \tag{3.71}$$

For each operator in (3.56) we will repeat a procedure similar to that explained between (3.29)–(3.35), with two differences. Since we rescaled the world-sheet coordinates (3.61)–(3.62) differently, none of the operators (3.68)–(3.70) can be written in terms of the Laplacian (B.9) or the Dirac operator (B.12). Therefore we will use the full heat kernels (B.10) and (B.13) instead of their simpler expressions at coincident points. Also, in the integrals over the (regularized) world-sheet, the domain of integration of r depends on the perturbative parameter k, and divergences appear if the radial cutoff $\rho_0 \to \infty$ is removed at fixed k. By analogy with (3.64), we shall assume that a sensible regularization at small k consists in doing the integrals for finite ρ_0, expanding in $\rho_0 \to \infty$ and dropping all positive powers of e^{ρ_0}. It is easy to check that since negative powers of k^2 are absent, in what is left we can simply take the limit $k \to 0$ (see also footnote 23). Applying this to the bosonic operator $\mathcal{O}_0 = \bar{\mathcal{O}}_0 + k^2\tilde{\mathcal{O}}_0 + \ldots$

[18] The minimal surface (3.4)-(3.6) bounded by a circle and the one ending on a straigh line or two antiparallel lines have the same local geometry of H^2, as they are mapped to each other through an isometry of AdS_5. The difference in the values of their regularized volumes (3.16) and (3.64) is a regularization effect due to the different global properties of the two spaces – different topology of the boundary (see [8, 15, 18] for a discussion of this point).

we find for the correction to its heat kernel (see (2.9))

$$\begin{aligned}\tilde{K}_{\mathcal{O}_0}(t) &= -t\int_{-\frac{\pi\sigma_0}{2\mathbb{K}(k^2)}}^{\frac{\pi\sigma_0}{2\mathbb{K}(k^2)}} dr\int_0^T \frac{dw}{\cos^2 r}\Big[\tilde{\mathcal{O}}_0\bar{K}_{-\Delta}(r,w,r',w';t)\Big]_{r=r',w=w'}\\ &= \tfrac{1}{8\pi}\Big[\big(3e^{\rho_0}-2\pi+O(e^{-\rho_0})\big)+O(k^2)\Big]t\,T\int_0^\infty dv\,v\tanh\pi v\ (v^2+\tfrac14)\ e^{-t\left(v^2+\frac14\right)}\\ &\to -\frac{t\,T}{4}\int_0^\infty dv\,v\tanh\pi v\ (v^2+\tfrac14)\ e^{-t(v^2+\frac14)}\,, \end{aligned}\tag{3.72}$$

where we used that $\pi\sigma_0/(2\mathbb{K}(k^2)) = \arctan(\sinh\rho_0)+O(k^2)$ after taking the large-ρ_0 limit. The corresponding zeta-function is

$$\begin{aligned}\tilde{\zeta}_{\mathcal{O}_0}(s) &= -\frac{s\,T}{4}\int_0^\infty dv\frac{1}{\big(v^2+\frac14\big)^s}\Big(v-\frac{2v}{e^{2\pi v}+1}\Big)\\ &= -\frac{s\,T}{8\big(s-1\big)}\big(\tfrac14\big)^{1-s}+\frac{s\,T}{2}\int_0^\infty dv\frac{v}{\big(e^{2\pi v}+1\big)\big(v^2+\frac14\big)^s}\,,\end{aligned}\tag{3.73}$$

$$\tilde{\zeta}'_{\mathcal{O}_0}(0) = \tfrac{1}{24}T\ .\tag{3.74}$$

Similarly, for the remaining bosonic and fermionic operators one gets

$$\tilde{K}_{\mathcal{O}_1}(t) = -t\int_{-\frac{\pi\sigma_0}{2\mathbb{K}(k^2)}}^{\frac{\pi\sigma_0}{2\mathbb{K}(k^2)}} dr\int_0^T \frac{dw}{\cos^2 r}\Big[\tilde{\mathcal{O}}_1\bar{K}_{-\Delta+2}(r,w,r',w';t)\Big]_{r=r',w=w'}\tag{3.75}$$

$$= -\frac{t\,T}{4}\int_0^\infty dv\,v\tanh\pi v\ \big(v^2+\tfrac14\big)\ e^{-t(v^2+\frac94)}\ ,$$

$$\begin{aligned}\tilde{\zeta}_{\mathcal{O}_1}(s) &= -\frac{s\,T}{4}\int_0^\infty dv\frac{1}{\big(v^2+\frac94\big)^s}\big(1-\frac{2}{v^2+\frac94}\big)\big(v-\frac{2v}{e^{2\pi v}+1}\big)\\ &= -\frac{s\,T}{8\big(s-1\big)}\big(\tfrac94\big)^{1-s}+\frac{T}{4}\big(\tfrac94\big)^{-s}+\frac{s\,T}{2}\int_0^\infty dv\frac{v}{\big(e^{2\pi v}+1\big)\big(v^2+\frac94\big)^s}\\ &\quad -s\,T\int_0^\infty dv\frac{v}{\big(e^{2\pi v}+1\big)\big(v^2+\frac94\big)^{s+1}}\ ,\qquad \tilde{\zeta}'_{\mathcal{O}_1}(0) = \big(-\tfrac{5}{24}+\tfrac12\gamma\big)T\,,\end{aligned}\tag{3.76, 3.77}$$

$$\tilde{K}_{\mathcal{O}_2}(t) = -t\int_{-\frac{\pi\sigma_0}{2\mathbb{K}(k^2)}}^{\frac{\pi\sigma_0}{2\mathbb{K}(k^2)}} dr\int_0^T \frac{dw}{\cos^2 r}\Big[\tilde{\mathcal{O}}_2\bar{K}_{-\Delta+2}(r,w,r',w';t)\Big]_{r=r',w=w'}\tag{3.78}$$

$$= \frac{t\,T}{4}\int_0^\infty dv\,v\tanh\pi v\Big[-\big(v^2+\frac14\big)+2\Big]e^{-t\left(v^2+\frac94\right)}\ ,$$

$$\begin{aligned}\tilde{\zeta}_{\mathcal{O}_2}\big(s\big) &= \frac{s\,T}{4}\int_0^\infty dv\frac{1}{\big(v^2+\frac94\big)^s}\big(-1+\frac{4}{v^2+\frac94}\big)\big(v-\frac{2v}{e^{2\pi v}+1}\big)\\ &= -\frac{s\,T}{8\big(s-1\big)}\big(\tfrac94\big)^{1-s}+\frac{T}{2}\big(\tfrac94\big)^{-s}+\frac{s\,T}{2}\int_0^\infty dv\frac{v}{\big(e^{2\pi v}+1\big)\big(v^2+\frac94\big)^s}\\ &\quad -2s\,T\int_0^\infty dv\frac{v}{\big(e^{2\pi v}+1\big)\big(v^2+\frac94\big)^{s+1}}\ ,\qquad \tilde{\zeta}'_{\mathcal{O}_2}(0) = \big(-\tfrac{17}{24}+\gamma\big)T\ ,\end{aligned}\tag{3.79, 3.80}$$

$$\tilde{K}_{\mathcal{O}_F}(t) = -t\int_{-\frac{\pi\sigma_0}{2\mathbb{K}(k^2)}}^{\frac{\pi\sigma_0}{2\mathbb{K}(k^2)}} dr\int_0^T \frac{dw}{\cos^2 r}\mathrm{tr}\Big[\Big\{\bar{\mathcal{O}}_F^{r,w},\tilde{\mathcal{O}}_F^{r,w}\Big\}\bar{K}_{-\not{\nabla}^2+1}(r,w,r',w';t)\Big]_{r=r',w=w'}$$

$$= \frac{t\,T}{2}\int_0^\infty dv\,v\coth\pi v\Big[1-\big(v^2+1\big)\Big]e^{-t\left(v^2+1\right)}\ ,\tag{3.81}$$

$$\tilde{\zeta}_{\mathcal{O}_F^2}(s) = \frac{s\,T}{2}\int_0^\infty dv \frac{1}{\big(v^2+1\big)^s}\Big(\frac{1}{v^2+1}-1\Big)\Big(v+\frac{2v}{e^{2\pi v}-1}\Big) \tag{3.82}$$

$$= -\frac{s\,T}{4\big(s-1\big)} + \frac{T}{4} - \int_0^\infty dv \frac{s\,T v}{\big(e^{2\pi v}-1\big)\big(v^2+1\big)^s} + \int_0^\infty dv \frac{s\,T v}{\big(e^{2\pi v}-1\big)\big(v^2+1\big)^{s+1}},$$

$$\tilde{\zeta}'_{\mathcal{O}_F^2}(0) = \big(-\tfrac{1}{24}+\tfrac{\gamma}{2}\big)T\ , \tag{3.83}$$

where (B.37) was used to compute (3.77) and (3.80), and (B.41) – to find (3.83).

The resulting one-loop effective action is then

$$\Gamma^{(1)}(k) - \Gamma^{(1)}(0) = k^2\Big(-\tfrac{5}{2}\tilde{\zeta}'_{\mathcal{O}_0}(0) - \tfrac{2}{2}\tilde{\zeta}'_{\mathcal{O}_1}(0) - \tfrac{1}{2}\tilde{\zeta}'_{\mathcal{O}_2}(0) + \tfrac{8}{4}\tilde{\zeta}'_{\mathcal{O}_F^2}(0)\Big) + O(k^4)$$

$$= \tfrac{3}{8}Tk^2 + O(k^4) \equiv \tfrac{3}{8\,\pi^2}T\phi^2 + O(\phi^4)\ , \tag{3.84}$$

where in the last line we substituted the expansion of (3.51). This reproduces, as it should, the result of [40] for the so-called Bremsstrahlung function [41].

As in the case of the latitude Wilson loop, it is not difficult to check that considering the sum of perturbed contributions to the zeta-functions

$$\tilde{\zeta}_{\rm tot}(s) = \tfrac{5}{2}\tilde{\zeta}_{\mathcal{O}_0}(s) + \tfrac{2}{2}\tilde{\zeta}_{\mathcal{O}_1}(s) + \tfrac{1}{2}\tilde{\zeta}_{\mathcal{O}_2}(s) - \tfrac{8}{4}\tilde{\zeta}_{\mathcal{O}_F^2}(s) \tag{3.85}$$

eliminates the need of an analytical continuation in s: setting $s=0$ in the total integrand and then performing the integration gives (3.84). This is again consistent with the fact that the trace of the full heat kernel, which equals to the sum of (3.72), (3.75), (3.78) and (3.81), vanishes for small t

$$\tilde{K}_{\rm tot}(t) = \tfrac{5}{2}\tilde{K}_{\mathcal{O}_0}(t) + \tfrac{2}{2}\tilde{K}_{\mathcal{O}_1}(t) + \tfrac{1}{2}\tilde{K}_{\mathcal{O}_2}(t) - \tfrac{8}{4}\tilde{K}_{\mathcal{O}_F}(t) = \frac{0}{t} + 0 + O(t)\ , \tag{3.86}$$

which implies that (3.85) does not develop any singularity in $s=0$.

3.3 k-wound circular Wilson loop

Our next example is the minimal surface generalizing the circular Wilson loop one (given by the $\theta_0=0$ limit of (3.4)-(3.6)) to the case of an arbitrary integer winding number k along the circle. The string theory solution should be representing, at strong coupling, the gauge-theory circular Wilson loop in the k-fundamental representation.

This classical solution can be found simply by the replacements $\sigma \to k\sigma$ and $\tau \to k\tau$ in (3.14) [7,9], so that the induced metric becomes

$$ds^2 = \Omega^2(\sigma)\left(d\sigma^2+d\tau^2\right)\ , \qquad \Omega(\sigma) = \frac{k}{\sinh(k\sigma)}\ , \qquad \sigma\in[0,\infty)\,, \qquad \tau\in[0,2\pi)\,. \tag{3.87}$$

The corresponding geometry is a cone of AdS_2 with negative angular deficit $\delta = 2\pi(1-k)$. Given a singular nature of this geometry one may wonder if a perturbation theory near $k=1$ limit is meaningful. We will first proceed formally and then comment on possible issues at the end of this section.

The relation $z=\tanh(k\sigma)$ from (3.4) implies that the world-sheet coordinate σ is to be cut off at $k^{-1}\text{arctanh}\,\epsilon$ in order keep the same physical cutoff at $z=\epsilon$ for any value of k. Then the classical string action is [7] (cf. (3.16))

$$\Gamma^{(0)}(k)=\frac{1}{2\pi}\int_0^{2\pi}d\tau\int_{k^{-1}\text{arctanh}\,\epsilon}^{\infty}d\sigma\ \frac{k^2}{\sinh^2(k\sigma)}=\frac{k}{\epsilon}-k\ \to -k\,. \tag{3.88}$$

One may define the new coordinate ρ that ranges in the same interval $[\text{arccosh}(\epsilon^{-1}),\infty)$ for any k by

$$\sinh\rho\equiv(\sinh(k\sigma))^{-1}\ . \tag{3.89}$$

The one-loop correction in (3.1) is [7, 15]

$$\Gamma^{(1)}(k)=-\log\frac{\text{Det}^{5/2}[\mathcal{O}_0(k)]\,\text{Det}^{3/2}[\mathcal{O}_1(k)]}{\text{Det}^{8/4}[\mathcal{O}_F(k)]}\ , \tag{3.90}$$

where

$$\mathcal{O}_0\,(k)\equiv\frac{\sinh^2(k\sigma)}{k^2}\left(-\partial_\tau^2-\partial_\sigma^2\right)\ ,\qquad \mathcal{O}_1\,(k)\equiv\mathcal{O}_0\,(k)+2\,, \tag{3.91}$$

$$\mathcal{O}_F\,(k)\equiv\tfrac{i}{k}\sinh(k\sigma)\sigma_1\left[\partial_\sigma-\tfrac{k}{2}\coth(k\sigma)\right]-\tfrac{i}{k}\sinh(k\sigma)\,\sigma_2\partial_\tau+\sigma_3\,. \tag{3.92}$$

For $k=1$ the corresponding world-sheet surface (3.87) becomes that of $\bar{\mathcal{M}}=H^2$, i.e. (3.14), with the boundary S^1 at $\rho=\text{arccosh}(\epsilon^{-1})$ and the regularized area in (3.16). The spectrum of excitations then coincides with (3.17)

$$\bar{\mathcal{O}}_0=-\Delta_{\rho,\tau}\,,\qquad \bar{\mathcal{O}}_1=-\Delta_{\rho,\tau}+2\,,\qquad \bar{\mathcal{O}}_F=-i\not{\nabla}_{\rho,\tau}+\sigma_3 \tag{3.93}$$

so that the 1-loop correction in (3.1) is also the same as in (3.21)

$$\Gamma^{(1)}\,(k=1)=-\tfrac{5}{2}\bar{\zeta}'_{\mathcal{O}_0}(0)-\tfrac{3}{2}\bar{\zeta}'_{\mathcal{O}_1}(0)+\tfrac{8}{4}\bar{\zeta}'_{\mathcal{O}_F^2}(0)=\tfrac{1}{2}\log 2\pi\ . \tag{3.94}$$

For $k=2,3,...$ the space (3.87) is a cone of H^2 with a conical singularity at $\rho=0$. We shall formally treat k as a real number and expand in $k-1$, i.e. define the small parameter α as

$$\alpha_{\text{k}-\text{circle}}\equiv k-1\,. \tag{3.95}$$

The small-α expansion of the metric (3.87) yields the leading and subleading terms as

$$\bar{g}_{ij}(\rho,\tau)=\begin{pmatrix}1 & 0\\ 0 & \sinh^2\rho\end{pmatrix}\ ,\qquad \tilde{g}_{ij}(\rho,\tau)=\begin{pmatrix}0 & 0\\ 0 & 2\sinh^2\rho\end{pmatrix}\ . \tag{3.96}$$

For the leading-order corrections in the operators (3.91),(3.92) we find

$$\tilde{\mathcal{O}}_0=\tilde{\mathcal{O}}_1=\frac{2}{\sinh^2\rho}\partial_\tau^2\,,\qquad \tilde{\mathcal{O}}_F=\frac{i}{\sinh\rho}\sigma_2\partial_\tau\,. \tag{3.97}$$

In the perturbative expansion in $k-1$ of the heat kernels and zeta-functions the integrals will contain similar $\frac{1}{\epsilon}$ divergences as in the volume (3.88). As in Section 3.2, we will first compute the integrals at finite cutoff, then take the limit $\epsilon\to 0$ in the result and finally drop terms with

negative powers of ϵ. Using this regularization prescription we find (here $\Lambda = \text{arccosh}(\epsilon^{-1})$ as in (3.15))

$$\begin{aligned}
\tilde{K}_{\mathcal{O}_0}(t) &= -t\int_0^\Lambda d\rho\int_0^{2\pi} d\tau \frac{2}{\sinh\rho}\Big[\partial_\tau^2 \bar{K}_{-\Delta}(\rho,\tau,\rho',\tau';t)\Big]_{\rho=\rho',\tau=\tau'} \\
&= -t\int_0^\infty dv\, v\tanh(\pi v)\left(v^2+\tfrac{1}{4}\right) e^{-t\left(v^2+\frac{1}{4}\right)} , \qquad (3.98)\\
\tilde{\zeta}_{\mathcal{O}_0}(s) &= \textstyle\int_0^\infty dv \frac{-sv}{\left(v^2+\frac{1}{4}\right)^s} + \int_0^\infty dv \frac{2sv}{(e^{2\pi v}+1)\left(v^2+\frac{1}{4}\right)^s} \\
&= -\frac{s}{2(s-1)}\left(\tfrac{1}{4}\right)^{1-s} + 2s\textstyle\int_0^\infty dv \frac{v}{(e^{2\pi v}+1)\left(v^2+\frac{1}{4}\right)^s} , \qquad \tilde{\zeta}'_{\mathcal{O}_0}(0) = \tfrac{1}{6} , \qquad (3.99)\\
\tilde{K}_{\mathcal{O}_1}(t) &= -t\int_0^\Lambda d\rho\int_0^{2\pi} d\tau \frac{2}{\sinh\rho}\Big[\partial_\tau^2 \bar{K}_{-\Delta+2}(\rho,\tau,\rho',\tau';t)\Big]_{\rho=\rho',\tau=\tau'} \\
&= -t\int_0^\infty dv\, v\tanh(\pi v)\left(v^2+\tfrac{1}{4}\right) e^{-t\left(v^2+\frac{9}{4}\right)} , \qquad (3.100)\\
\tilde{\zeta}_{\mathcal{O}_1}(s) &= \textstyle\int_0^\infty dv \frac{-sv}{\left(v^2+\frac{9}{4}\right)^s}\left(1-\frac{2}{v^2+\frac{9}{4}}\right) + \int_0^\infty dv \frac{2sv}{(e^{2\pi v}+1)\left(v^2+\frac{9}{4}\right)^s}\left(1-\frac{2}{v^2+\frac{9}{4}}\right) \\
&= -\frac{s}{2(s-1)}\left(\tfrac{9}{4}\right)^{1-s} + \left(\tfrac{9}{4}\right)^{-s} + 2s\textstyle\int_0^\infty dv \frac{v}{(e^{2\pi v}+1)\left(v^2+\frac{9}{4}\right)^s} \\
&\quad - 4s\textstyle\int_0^\infty dv \frac{v}{(e^{2\pi v}+1)\left(v^2+\frac{9}{4}\right)^{s+1}} , \qquad \tilde{\zeta}'_{\mathcal{O}_1}(0) = -\tfrac{5}{6}+2\gamma , \qquad (3.101)\\
\tilde{K}_{\mathcal{O}_F^2}(t) &= -\int_0^\Lambda d\rho\int_0^{2\pi} d\tau \sinh\rho\,\mathrm{tr}\Big[\left\{\bar{\mathcal{O}}_F^{\rho,\tau},\tilde{\mathcal{O}}_F^{\rho,\tau}\right\}\bar{K}_{-\nabla^2+1}(\rho,\tau,\rho',\tau';t)\Big]_{\rho=\rho',\tau=\tau'} \\
&= -t\int_0^\infty dv\, v\coth(\pi v)\left(2v^2+1\right) e^{-t\left(v^2+1\right)} , \qquad (3.102)\\
\tilde{\zeta}_{\mathcal{O}_F^2}(s) &= \textstyle\int_0^\infty dv\, \frac{-sv}{(v^2+1)^s}\left(2-\frac{1}{v^2+1}\right) + \int_0^\infty dv\, \frac{-2sv}{(e^{2\pi v}-1)(v^2+1)^s}\left(2-\frac{1}{v^2+1}\right) \\
&= -\frac{s}{s-1}+\frac{1}{2}-4s\textstyle\int_0^\infty dv\, \frac{v}{(e^{2\pi v}-1)(v^2+1)^s} + 2s\int_0^\infty dv\, \frac{v}{(e^{2\pi v}-1)(v^2+1)^{s+1}} , \qquad (3.103)\\
\tilde{\zeta}'_{\mathcal{O}_F^2}(s) &= \tfrac{1}{3}+\gamma , \qquad (3.104)
\end{aligned}$$

where we used (B.37) and (B.41). Combining these results, the one-loop effective action reads

$$\begin{aligned}
&\Gamma^{(1)}(k) - \Gamma^{(1)}(k=1) = c_1(k-1) + O\big((k-1)^2\big) , \qquad (3.105)\\
&c_1 = -\tfrac{5}{2}\tilde{\zeta}'_{\mathcal{O}_0}(0) - \tfrac{3}{2}\tilde{\zeta}'_{\mathcal{O}_1}(0) + \tfrac{8}{4}\tilde{\zeta}'_{\mathcal{O}_F^2}(0) = \tfrac{3}{2}-\gamma . \qquad (3.106)
\end{aligned}$$

At the same time, the strong-coupling expansion of the gauge theory prediction for the expectation value $\langle \mathcal{W}(\lambda,k)\rangle = e^{-\Gamma(\lambda,k)}$ of k-fundamental circular loop normalized to the $k=1$ value is (cf. (1.1)) [9, 20]

$$\Gamma(\lambda,k) - \Gamma(\lambda,k=1) = \sqrt{\lambda}\,(1-k) + \tfrac{3}{2}\log k + \mathcal{O}(\lambda^{-1/2}) . \qquad (3.107)$$

Our string theory result (3.106) thus coincides with the $k\to 1$ expansion of the $\log k$ term in (3.107) just up to an extra γ (the Euler-Mascheroni constant) term in c_1.

Our value for $c_1 = \frac{3}{2}-\gamma \approx 0.923$ may be compared to the results of the two previous string theory computations of $\Gamma^{(1)}(k)$ in [7] and in [15]. The 1-loop correction in [7] was

$$\Gamma^{(1)}_{\rm KT}(k) = \tfrac{1}{2}\ln(2\pi) + (2k+\tfrac{1}{2})\ln k - \ln\Gamma(k+1) , \qquad (3.108)$$

so that $(c_1)_{\rm KT} = \frac{3}{2} + \gamma \approx 2.077$, which, surprisingly, differs from (3.106) just by the sign of the γ term. This suggests that the presence of this extra γ term in both approaches is a regularization artifact (see also below). The result of [15] was given by

$$\Gamma^{(1)}_{\rm BT}(k) = \tfrac{1}{2} k \log(2\pi) + I(k)\ , \tag{3.109}$$
$$I(k) = -\tfrac{1}{4}\int_0^\infty \tfrac{dy}{y\sinh y}\Big[(5e^{-y}+3e^{-3y})\big(\coth\tfrac{y}{k} - k\coth y\big) + 16e^{-2y}\big(\tfrac{1}{\sinh\frac{y}{k}} - \tfrac{k}{\sinh y}\big)\Big]\ ,$$

so that $(c_1)_{\rm BT} = \frac{1}{2}\ln(2\pi) + I'(1) \approx 0.9189 + 0.3161 = 1.235$ which is closer but still different from the gauge-theory prediction $c_1 = 1.5$ in (3.106).

From a technical point of view, the presence of the extra γ term in (3.106) can be traced back to the dependence on a regularization used to define the $s = 0$ limit in the zeta-functions, which, in contrast to the examples in the previous two subsections, does not cancel out in the sum of leading-order corrections to the zeta-functions. Splitting the power and exponential terms in the integrands in (3.98)–(3.102) (cf. (3.46)), i.e. $\tilde\zeta_{\mathcal O} \equiv \tilde\zeta^{(\rm power)}_{\mathcal O}(s) + \tilde\zeta^{(\rm exp)}_{\mathcal O}(s)$, we find

$$\begin{aligned}&\tfrac{5}{2}\tilde\zeta^{(\rm power)}_{\mathcal O_0}(s) + \tfrac{3}{2}\tilde\zeta^{(\rm power)}_{\mathcal O_1}(s) - \tfrac{8}{4}\tilde\zeta^{(\rm power)}_{\mathcal O_F^2}(s) \\ &\quad = \int_0^\infty dv\Big[-\tfrac{5sv}{s\left(v^2+\frac14\right)^s} - \tfrac{3sv}{2\left(v^2+\frac94\right)^s}\Big(1-\tfrac{2}{v^2+\frac94}\Big) + \tfrac{2sv}{(v^2+1)^s}\Big(2-\tfrac{1}{v^2+1}\Big)\Big]\ ,\end{aligned} \tag{3.110}$$

which is divergent for $s \to 0$. Proceeding without performing an analytical continuation in s gives

$$\begin{aligned}&\tfrac{d}{ds}\Big(-\tfrac{5}{2}\tilde\zeta_{\mathcal O_0}(s) - \tfrac{3}{2}\tilde\zeta_{\mathcal O_1}(s) + \tfrac{8}{4}\tilde\zeta_{\mathcal O_F^2}(s)\Big)_{s=0} \\ &\quad = \int_0^\infty dv\, \tfrac{2v(2v^2-3)}{4v^4+13v^2+9} + \tfrac{d}{ds}\Big(-\tfrac{5}{2}\tilde\zeta^{(\rm exp)}_{\mathcal O_0}(s) - \tfrac{3}{2}\tilde\zeta^{(\rm exp)}_{\mathcal O_1}(s) + \tfrac{8}{4}\tilde\zeta^{(\rm exp)}_{\mathcal O_F^2}(s)\Big)_{s=0}\ ,\end{aligned} \tag{3.111}$$

where the integral diverges logarithmically for large v. This reflects the presence of t^0 term in the small-t expansion of the leading $k-1$ correction to the heat kernel (cf. (3.50),(3.86))

$$\tilde K_{\rm tot}(t) = \tfrac{5}{2}\tilde K_{\mathcal O_0}(t) + \tfrac{3}{2}\tilde K_{\mathcal O_1}(t) - \tfrac{8}{4}\tilde K_{\mathcal O_F^2}(t) = \tfrac{0}{t} + \tfrac{1}{2} + O(t) \tag{3.112}$$

where we used that according to (3.98), (3.100), (3.102),

$$\tilde K_{\mathcal O_0}(t) = -\tfrac{1}{2t} + O(t)\,, \qquad \tilde K_{\mathcal O_1}(t) = -\tfrac{1}{2t} + 1 + O(t)\,, \qquad \tilde K_{\mathcal O_F^2}(t) = -\tfrac{1}{t} + \tfrac{1}{2} + O(t)\,. \tag{3.113}$$

It is interesting to note that (3.113) matches the small-t asymptotics of the heat kernels on the cone of H^2 found in [15] when expanded for $k \to 1$ [19]

$$[\tilde K_{\mathcal O_0}(t)]_{\rm BT} = \bar K_{-\Delta}(t) + \frac{e^{-\frac14 t}}{\sqrt{4\pi t}}\int_0^\infty dy\, e^{-\frac{y^2}{t}}\,\frac{y-\sinh y\cosh y}{\sinh^3 y} = -\tfrac{1}{2t} + O(t)\,, \tag{3.114}$$

$$[\tilde K_{\mathcal O_1}(t)]_{\rm BT} = \bar K_{-\Delta+2}(t) + \frac{e^{-\frac94 t}}{\sqrt{4\pi t}}\int_0^\infty dy\, e^{-\frac{y^2}{t}}\,\frac{y-\sinh y\cosh y}{\sinh^3 y} = -\tfrac{1}{2t} + 1 + O(t)\,, \tag{3.115}$$

$$[\tilde K_{\mathcal O_F^2}(t)]_{\rm BT} = \bar K_{-\nabla^2+1}(t) - \frac{4e^{-t}}{\sqrt{4\pi t}}\int_0^\infty dy\, e^{-\frac{y^2}{t}}\,\frac{y\cosh y-\sinh y}{\sinh^3 y} = -\tfrac{1}{t} + \tfrac{1}{2} + O(t)\ . \tag{3.116}$$

[19] Here we give the terms proportional to $k-1$ in the expansion of eqs. (2.19) (with $m^2 = 0, 2$) and (3.17) (with $m^2 = 1$) in [15].

Here we used the expansions (B.25)-(B.26) and performed the change of variable $y \to \sqrt{t}\, y$.

The non-vanishing t^0 term in the $O(k-1)$ correction to the total heat kernel $\tilde{K}_{\rm tot}$ in (3.112) implies the presence of k-dependent logarithmic UV divergence in the logarithm of the one-loop string partition function (implicit also in [15]). The presence of this k-dependent UV divergence appears to be in contradiction with the fact that the Euler number of the cone of AdS_2 is the same as of the disc ($\chi = 1$) for any k which suggests that the UV divergence should actually cancel in the ratio of $k \neq 1$ and $k = 1$ partition functions (as in the latitude example of Section 3.1). In general, it is known that conical singularities produce extra contributions to the heat coefficient a_2 [47, 48] (cf. (A.14)). What happens is that the regular k-dependent bulk contribution to the Euler number is cancelled against the k-dependent tip contribution.[20]

One may then suspect that our perturbative approach to computation of heat kernels may be missing some subtleties of the heat asymptotics around the tip of the cone.[21] Namely, it may be missing the singular tip of the cone contribution to a_2 so that instead of being proportional to the full (k-independent) Euler number equal to 1 it appears to be given just by the regular bulk contribution $\chi_v^{\rm reg} = k$ (we drop the $\frac{1}{\epsilon}$ part in $\chi_v^{\rm reg}$ in footnote 20 as our usual IR regularization prescription). Explicitly, one may then interpret (3.112) as the $O(k-1)$ term in the total heat kernel where the t^0 term is given by $\chi_v^{\rm reg}$, i.e.

$$K_{\rm tot}^{\rm reg}(t) = \tfrac{0}{t} + \tfrac{k}{2}\, t^0 + O(t) = \tfrac{0}{t} + \Big[\, \tfrac{1}{2} + \tfrac{1}{2}(k-1) \Big]\, t^0 + O(t)\ , \tag{3.117}$$

Then the effect of proper accounting for the tip contribution should be, in particular, the replacement of the $\frac{k}{2}\, t^0$ term in (3.117) by $\frac{1}{2}\, t^0$ and thus the cancellation of the $\frac{1}{2}$ term in (3.112).

This suggests that the presence of the extra γ term in (3.106) (which represents the difference with the gauge theory result) may be an artifact of the superficial presence of k-dependent UV divergences before the tip contribution is taken into account. We leave a careful resolution of this issue for the future.

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[20] The Euler number is given by the sum of the volume and boundary contributions, $\chi = \chi_v + \chi_b$. The volume part of the Euler number $\chi_v = \frac{1}{4\pi}\int d^2x\sqrt{g}R$ contains the "regular" and "singular" (tip) contributions: $\chi_v = \chi_v^{\rm reg} + \chi_v^{\rm tip}$. The regular part of the curvature of the metric (3.87) is $R_{\rm reg} = -2\Omega^{-2}\partial_\sigma^2 \log\Omega = -2$, while the tip part is $R_{\rm tip} = 4\,\pi\,(1-k)\,\delta^{(2)}(x)$, so that $\chi_v^{\rm reg} = \frac{1}{4\pi}\int_0^{2\pi} d\tau \int_{k^{-1}{\rm arctanh}\,\epsilon}^{\infty} d\sigma\, \Omega^2\,(\sigma)\, R_{\rm reg} = -\frac{k}{\epsilon} + k$ and $\chi_v^{\rm tip} = 1-k$. Thus $\chi_v = 1 - \frac{k}{\epsilon}$. The geodesic curvature of the boundary at $\bar\sigma = k^{-1}{\rm arctanh}\,\epsilon$ is $\kappa_g = \partial_\sigma \Omega^{-1}(\sigma)|_{\sigma=\bar\sigma} = \frac{1}{\epsilon}$, so that the boundary part of the Euler number is $\chi_b = \frac{1}{2\pi}\int ds\,\kappa_g \equiv \frac{1}{2\pi}\int_0^{2\pi} d\tau\,\Omega(\bar\sigma)\,\kappa_g = \frac{k}{\epsilon}$. As a result, the total Euler character $\chi = \chi_v + \chi_b = 1$ is finite and does not depend on k, i.e. is the same as of a disc.

[21] See, for example [52] and references therein. We thank D. Seminara for a discussion on this point. Note also that our test for the scalar Laplacian in Appendix A.2 applied only to smooth manifolds.

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A Perturbation theory for heat kernel and Seeley coefficients

In this Appendix we collect some details on the derivation of (2.7),(2.9) and show, as non-trivial consistency check, that our perturbative expansion reproduces the standard perturbation theory applied directly to the Seeley coefficients of the scalar Laplacian operator.

A.1 Perturbation theory for heat kernel

To obtain the first correction $\tilde{K}(x,x';t)$ to the heat kernel in (2.4), we solve equation (2.6) using the standard method of variation of constants. We start with the ansatz

$$\begin{aligned}\bar{K}_{\mathcal{O}}(x,x';t) &\equiv -\frac{\tilde{g}(x)}{2\bar{g}^{3/2}(x)}\delta^{(d)}(x-x')\mathbb{I} + \int d^dx''\sqrt{\bar{g}(x'')}\bar{K}_{\mathcal{O}}(x,x'';t)C_{\mathcal{O}}(x'',x';t)\,, \\ &\lim_{t\to 0^+} C_{\mathcal{O}}(x,x';t) = 0\,,\end{aligned} \tag{A.1}$$

which guarantees that the initial condition in (2.6) is satisfied, and solve for $C_{\mathcal{O}}(x'',x';t)$

$$\int d^dx''\sqrt{g(x'')}\bar{K}_{\mathcal{O}}(x,x'';t)\;\partial_t C_{\mathcal{O}}(x'',x';t) = \bar{\mathcal{O}}_x\Big(\frac{\tilde{g}(x)}{\bar{g}^{3/2}(x)}\delta^{(d)}(x-x')\Big) - \tilde{\mathcal{O}}_x\bar{K}_{\mathcal{O}}(x,x';t)\,. \tag{A.2}$$

We now multiply both sides by $\sqrt{\bar{g}(x)}\bar{K}_{\mathcal{O}}(x''',x;-t)$ [22] and integrate over x, using the composition law

$$\int d^dx'\sqrt{\bar{g}(x')}\bar{K}_{\mathcal{O}}(x,x';t)\bar{K}_{\mathcal{O}}(x',x'';t') = \bar{K}_{\mathcal{O}}(x,x'';t+t')\,, \qquad t,\,t'>0\,. \tag{A.3}$$

With the initial condition in (2.5), we then obtain

$$\begin{aligned}\partial_t C_{\mathcal{O}}(x''',x';t) = &\int d^dx\sqrt{\bar{g}(x)}\bar{K}_{\mathcal{O}}(x''',x;-t)\bar{\mathcal{O}}_x\left(\frac{\tilde{g}(x)}{\bar{g}^{3/2}(x)}\delta^{(d)}(x-x')\right) \\ &- \int d^dx\sqrt{\bar{g}(x)}\bar{K}_{\mathcal{O}}(x''',x;-t)\tilde{\mathcal{O}}_x\bar{K}_{\mathcal{O}}(x,x';t)\,.\end{aligned} \tag{A.4}$$

Relabeling $x'''\to x''$, $x\to x'''$, $t\to t'$ it is straightforward to integrate over the proper time to get

$$\begin{aligned}C_{\mathcal{O}}(x'',x';t) = &\int_0^t dt'\int d^dx'''\,\sqrt{\bar{g}(x''')}\bar{K}_{\mathcal{O}}(x'',x''';-t')\bar{\mathcal{O}}_{x'''}\left(\frac{\tilde{g}(x''')}{2\bar{g}^{3/2}(x''')}\delta^{(d)}(x'-x''')\right) \\ &- \int_0^t dt'\int d^dx'''\sqrt{\bar{g}(x''')}\bar{K}_{\mathcal{O}}(x'',x''';-t')\tilde{\mathcal{O}}_{x'''}\bar{K}_{\mathcal{O}}(x''',x';t')\,.\end{aligned} \tag{A.5}$$

[22] This step is not fully rigorous because the identity (A.3) holds only for positive values of the proper times. In fact, the inverse heat kernel $\bar{K}^{-1}_{\mathcal{O}}(x''',x;t) = \bar{K}_{\mathcal{O}}(x''',x;-t)$ is not guaranteed to be a well-defined operator when it acts on arbitrary functions $f(x)$ taking values in a vector bundle. However, this potentially problematic operator will not enter the final formula (2.7), which indeed contains heat kernels with only positive arguments t' and $t-t'$. We could alternatively start with (2.7) and check that it is a solution of (2.5) without the need of inverting heat kernels. A similar discussion is found in Chapter 14 of [24].

Substituting in (A.1) this leads to the explicit integral form (2.7), with a few more steps that employ (2.5) and (A.3).

Next, the order α correction $\tilde{K}_{\mathcal{O}}(t)$ in (2.8) receives contributions[23] from both the α-correction to volume factor $\sqrt{g(x)}$ (cf. (2.4)) and from the α-correction to the heat kernel in (2.7), i.e.

$$\tilde{K}_{\mathcal{O}}(t) = \int d^d x \frac{\tilde{g}(x)}{2\sqrt{\bar{g}(x)}} \,\mathrm{tr}\bar{K}_{\mathcal{O}}(x,x;t) + \int d^d x \sqrt{\bar{g}(x)}\,\mathrm{tr}\tilde{K}_{\mathcal{O}}(x,x;t)\,. \tag{A.6}$$

Plugging here the diagonal element $x = x'$ of (2.7), we get

$$\begin{aligned}\tilde{K}_{\mathcal{O}}(t) &= \int d^d x \frac{\tilde{g}(x)}{2\sqrt{\bar{g}(x)}} \,\mathrm{tr}\bar{K}_{\mathcal{O}}(x,x;t) - \delta^{(d)}(0)\int d^d x \frac{\tilde{g}(x)}{2\bar{g}(x)}\,\mathrm{tr}\mathbb{I} \\ &- \int_0^t dt' \int d^d x \sqrt{\bar{g}(x)} \int d^d x'' \sqrt{\bar{g}(x'')}\,\mathrm{tr}\Big[\bar{K}_{\mathcal{O}}(x,x'';t-t')\tilde{\mathcal{O}}_{x''}\bar{K}_{\mathcal{O}}(x'',x;t')\Big] \\ &+ \int_0^t dt' \int d^d x \sqrt{\bar{g}(x)} \int d^d x'' \sqrt{\bar{g}(x'')}\,\mathrm{tr}\Big[\bar{K}_{\mathcal{O}}(x,x'';t-t')\bar{\mathcal{O}}_{x''}\left(\frac{\tilde{g}(x'')}{2\bar{g}(x'')^{3/2}}\delta^{(d)}(x-x'')\right)\Big]\,.\end{aligned} \tag{A.7}$$

This expression is potentially affected by two types of divergences. The first one is the $\delta^{(d)}(0)$, short-distance divergence originating from (2.7); it will eventually cancel against the delta-function in the last integrand. The second is a possible infrared divergence that may appear if $\mathcal{M}$ and $\bar{\mathcal{M}}$ are non-compact. In the applications to string theory in Section 3 the volume divergences will be regulated by a cutoff and then subtracted through the renormalization prescription suggested in similar calculations in [8, 15].

To bring the integral (A.7) into a more convenient form, we shall assume that $\bar{\mathcal{O}}$ is a self-adjoint operator on a vector bundle of the manifold $\bar{\mathcal{M}}$ [24]

$$\int d^d x \sqrt{\bar{g}(x)}\, f^\dagger(x)\bar{\mathcal{O}}_x h(x) = \int d^d x \sqrt{\bar{g}(x)}\,(\bar{\mathcal{O}}_x f(x))^\dagger h(x)\,. \tag{A.8}$$

Combining this with (2.5) and setting $t'' = t - t'$, we rewrite the last term in (A.7) as

$$\begin{aligned}&-\int_0^t dt'' \int d^d x \frac{\tilde{g}(x)}{2\sqrt{\bar{g}(x)}}\,\mathrm{tr}\big(\partial_{t''}\bar{K}_{\mathcal{O}}(x,x;t'')\big) \\ &\quad = -\int d^d x \frac{\tilde{g}(x)}{2\sqrt{\bar{g}(x)}}\,\mathrm{tr}\bar{K}_{\mathcal{O}}(x,x;t) + \delta^{(d)}(0)\int d^d x \frac{\tilde{g}(x)}{2\sqrt{\bar{g}(x)}}\mathrm{tr}\mathbb{I}\,,\end{aligned} \tag{A.9}$$

which simplifies (A.7) to

$$\tilde{K}_{\mathcal{O}}(t) = -\int_0^t dt' \int d^d x \sqrt{\bar{g}(x)} \int d^d x'' \sqrt{\bar{g}(x'')}\,\mathrm{tr}\Big[\bar{K}_{\mathcal{O}}(x,x'';t-t')\tilde{\mathcal{O}}_{x''}\bar{K}_{\mathcal{O}}(x'',x;t')\Big]\,. \tag{A.10}$$

[23] There is no correction due the integration over the x^i because they range in a subset of $\mathbb{R}^n$ that is the same for $\mathcal{M}$ and $\bar{\mathcal{M}}$. Although one may argue that the expansion (2.4) needs to assume that the range of coordinates should not depend on α, the analysis in Section 3.2 shows that one may allow their domain to change infinitesimally when expanding in small α. This weaker condition on the choice of coordinates should be valid as long as the change in the integration domain in the final formula (2.9) produces only small additional terms, proportional to positive powers of α, that are eventually neglected in (2.8) at linear order in α.

[24] A natural inner product is defined as $(f,h) \equiv \int d^d x \sqrt{g(x)}\, f^\dagger(x) h(x)\,.$

We can now use the cyclicity of the trace to write

$$\tilde{K}_{\mathcal{O}}(t) = -\int_0^t dt' \int d^d x \sqrt{\bar{g}(x)} \int d^d x'' \sqrt{\bar{g}(x'')}\, \mathrm{tr}\Big[\tilde{\mathcal{O}}_{x''}\Big(\bar{K}_{\mathcal{O}}(x'', x; t')\bar{K}_{\mathcal{O}}(x, x'; t - t')\Big)\Big]_{x'=x''}\,, \tag{A.11}$$

where it is understood that the limit $x' \to x''$ is taken after $\mathcal{O}_{x''}$ has acted on the argument in round brackets. Making use of (A.3) we then get the compact expressions in (2.9).

A.2 Perturbative expansion of Seeley coefficients of scalar Laplacian

Consider the scalar Laplacian on a compact non-singular space $\mathcal{M}$ with metric g_{ij}

$$\mathcal{O} = -\frac{1}{\sqrt{g}}\partial_i\left(\sqrt{g}g^{ij}\partial_j\right) + E\,. \tag{A.12}$$

Under the standard conditions the corresponding heat kernel may be expanded as [48, 49]

$$K_{\mathcal{O}}(x, x; t) \simeq \frac{1}{(4\pi)^{d/2}}\sum_{k=0}^{\infty} t^{(k-d)/2}\, b_{k/2}(x)\,, \qquad K_{\mathcal{O}}(t) \simeq \sum_{k=0}^{\infty} t^{(k-d)/2}\, a_k\,, \tag{A.13}$$

where $a_k \equiv (4\pi)^{-d/2}\int d^d x\sqrt{g(x)}\, b_k(x)$. Then the UV divergences of $\log \mathrm{Det}\mathcal{O}$ may be expressed in terms of the Seeley coefficients a_k with $k \le d$. As in Section 3 we are interested in the case of $d = 2$, here we shall concentrate only on the leading a_k. For compact manifolds without boundary, odd Seeley coefficients a_{2l+1} vanish and the first non-trivial ones read [48, 49][25]

$$a_0 = \frac{1}{(4\pi)^{d/2}}\int d^2 x\sqrt{g}\,, \qquad\qquad a_2 = \frac{1}{(4\pi)^{d/2}}\int d^2 x\sqrt{g}\,\Big(\frac{R}{6} - E\Big)\,. \tag{A.14}$$

Consider now two conformally equivalent metrics g_{ij} and $\bar{g}_{ij}$, $\;g_{ij} = e^{2\,\alpha\,\Omega(x)}\bar{g}_{ij}$, with α being a small parameter. Setting $E = \bar{E} + \alpha\,\tilde{E} + O(\alpha^2)$ and using (2.4) we get

$$\begin{aligned}\mathcal{O} &= \bar{\mathcal{O}} + \alpha\;\tilde{\mathcal{O}} + O\left(\alpha^2\right) \qquad\qquad (A.15)\\ &= \Big[-\frac{1}{\sqrt{\bar{g}}}\partial_i(\sqrt{\bar{g}}\bar{g}^{ij}\partial_j) + \bar{E}\Big] + \alpha\Big[-(d-2)\,\bar{g}^{ij}\partial_i\Omega\partial_j + \frac{2\Omega}{\sqrt{\bar{g}}}\partial_i\left(\sqrt{\bar{g}}\bar{g}^{ij}\partial_j\right) + \tilde{E}\Big] + O\left(\alpha^2\right)\,.\end{aligned}$$

Using also the expansion for the scalar curvature[26]

$$R = \bar{R} + \alpha\tilde{R} + O(\alpha^2) = \bar{R} - 2\alpha\Big[\Omega\bar{R} + \frac{d-1}{\sqrt{\bar{g}}}\partial_i\left(\sqrt{\bar{g}}\bar{g}^{ij}\partial_j\Omega\right)\Big] + O(\alpha^2)\,, \tag{A.16}$$

[25] The manifolds discussed in Section 3 are *not* compact. We regularize integrations over infinite regions by introducing a cutoff, i.e. a boundary at a finite distance. This renders the integrals defining the Seeley coefficients a_0 and a_2 finite, and suggests that we should also consider the boundary term $a_1 = -\frac{1}{4}\left(\frac{1}{4\pi}\right)^{\frac{d-1}{2}}\int_{\partial\mathcal{M}}\sqrt{g}$ proportional to the length of the boundary. However, if we assume that all the IR divergences are completely subtracted, that implies that the renormalized value of a_1 is effectively zero and we can restrict consideration to a_0 and the (the volume part of) a_2 (cf. also [6, 12, 53]).

[26] Under a conformal rescaling of the metric, $\bar{R} \to R = e^{-2\alpha\Omega}\Big[\bar{R} - \frac{2\alpha(d-1)}{\sqrt{\bar{g}}}\partial_i\left(\sqrt{\bar{g}}\bar{g}^{ij}\partial_j\Omega\right) - \alpha^2(d-1)(d-2)\bar{g}^{ij}\partial_i\Omega\partial_j\Omega\Big]$.

one obtains to linear order in α the following expansion for the relevant Seeley coefficients,

$$a_k = \bar{a}_k + \alpha\, \tilde{a}_k + O(\alpha^2)\,, \tag{A.17}$$

$$\bar{a}_0 = \frac{1}{(4\pi)^{d/2}} \int d^d x \sqrt{\bar{g}}\,, \qquad \bar{a}_2 = \frac{1}{(4\pi)^{d/2}} \int d^d x \sqrt{\bar{g}} \Big(\frac{1}{6} \bar{R} - \bar{E} \Big)\,, \tag{A.18}$$

$$\tilde{a}_0 = \frac{d}{(4\pi)^{d/2}} \int d^d x \sqrt{\bar{g}}\, \Omega\,, \qquad \tilde{a}_2 = \frac{1}{(4\pi)^{d/2}} \int d^d x \sqrt{\bar{g}} \Big(\frac{d-2}{6}\, \Omega\, \bar{R} - d\, \Omega\, \bar{E} - \tilde{E} \Big). \tag{A.19}$$

As a consistency check of the perturbative approach developed in Section 2 let us show that $\tilde{a}_0$ and $\tilde{a}_2$ in (A.19) are reproduced from the small-t expansion of the heat kernel trace in (2.8),(2.9). Using (A.15) we get

$$\begin{aligned} \tilde{K}_{\mathcal{O}}(t) = t \int d^d x \Big\{ & (d-2) \sqrt{\bar{g}(x)} \bar{g}^{ij}(x) \partial_i^x \Omega(x) \left(\partial_j^x \bar{K}_{\mathcal{O}}(x, x'; t) \right)_{x=x'} \\ & -2\, \Omega(x) \Big[\partial_i^x \left(\sqrt{\bar{g}(x)} \bar{g}^{ij}(x) \partial_j^x \bar{K}_{\mathcal{O}}(x, x'; t) \right) \Big]_{x=x'} - \sqrt{\bar{g}(x)} \tilde{E}(x) \bar{K}_{\mathcal{O}}(x, x; t) \Big\}\,. \end{aligned} \tag{A.20}$$

Integrating by parts in the first term using that the unperturbed Laplacian satisfies (2.5) gives

$$\begin{aligned} \tilde{K}_{\mathcal{O}}(t) = t \int d^d x \Big[& -(d-2) \partial_j^x \left(\sqrt{\bar{g}(x)} \bar{g}^{ij}(x) \partial_i^x \Omega(x) \right) \bar{K}_{\mathcal{O}}(x, x; t) \\ & - 2 \sqrt{\bar{g}(x)}\, \Omega(x)\, \partial_t \bar{K}_{\mathcal{O}}(x, x; t) - \sqrt{\bar{g}(x)} \left(2 \Omega(x) \bar{E}(x) + \tilde{E}(x) \right) \bar{K}_{\mathcal{O}}(x, x; t) \Big]\,. \end{aligned} \tag{A.21}$$

Expanding in $t \to 0^+$ and using (A.13) we get

$$\begin{aligned} K_{\mathcal{O}}(t) = \bar{K}_{\mathcal{O}}(t) + \frac{\alpha}{(4\pi)^{d/2}} \Big[& d\, t^{-d/2} \int d^d x \sqrt{\bar{g}(x)}\, \Omega(x) \\ & + t^{(2-d)/2} \int d^d x \sqrt{\bar{g}(x)} \Big(\tfrac{d-2}{6} \Omega(x) \bar{R}(x) - d\, \Omega(x) \bar{E}(x) - \tilde{E}(x) \Big) + O(t^{(3-d)/2}) \Big] + O(\alpha^2)\,. \end{aligned} \tag{A.22}$$

Reading off the values of the first corrections $\tilde{a}_0, \tilde{a}_2$ one finds that they match the ones in (A.19).

B Heat kernels and zeta-functions for operators on H^2

In this Appendix we will review the known expressions for heat kernels of Laplace and Dirac operators on the Euclidean AdS_2 or 2d hyberbolic space H^2 with the metric

$$ds^2 = d\rho^2 + \sinh^2 \rho\, d\tau^2\,, \qquad \rho > 0\,, \qquad \tau \in [0, 2\pi)\,, \tag{B.1}$$

where τ parametrizes the S^1 boundary at $\rho = \infty$. The geodesic distance $d(x, x')$ between two points $x = (\rho, \tau)$ and $x' = (\rho', \tau')$ is

$$\cosh d(x, x') = \cosh \rho \cosh \rho' - \sinh \rho \sinh \rho' \cos(\tau - \tau')\,. \tag{B.2}$$

We will also considered the "infinite-strip" parametrization $x = (r, w)$ of AdS_2 that we call $\hat{H}^2$, which has the real line instead of S^1 as its boundary

$$ds^2 = \frac{1}{\cos^2 r} (dr^2 + dw^2)\,, \qquad r \in \left(-\tfrac{\pi}{2}, \tfrac{\pi}{2} \right)\,, \qquad w \in \mathbb{R}\,, \tag{B.3}$$

with geodesic distance function

$$\cosh d(x,x') = -\tan r \tan r' + \frac{\cosh(w-w')}{\cos r \cos r'} \ . \tag{B.4}$$

The change of coordinates between the two systems is

$$\cosh\rho = \tfrac{\cosh w}{\cos r}\,, \qquad \sin\tau = \tfrac{\sin r}{\sqrt{\sin^2 r + \sinh^2 w}}\,, \qquad \cos\tau = \tfrac{\sinh w}{\sqrt{\sin^2 r + \sinh^2 w}}\,, \tag{B.5}$$

$$\tan r = \sinh\rho \sin\tau\,, \qquad \tanh w = \tanh\rho\cos\tau\,. \tag{B.6}$$

We shall consider a Laplace type operator acting on function in a vector bundle $-\frac{1}{\sqrt{g}}(\partial_i + A_i)\left(\sqrt{g}g^{ij}(\partial_j + A_j)\right) + E$ and also a Dirac type acting on two-dimensional spinors $-i\not{\nabla} + V \equiv ie_a{}^i\Gamma^a\nabla_i + V$, where the spinor derivative is $\nabla_i \equiv \partial_i + \frac{1}{4}\omega^{ab}{}_c e^c{}_i \Gamma_{ab}$.[27] $e^a{}_i$ is the zweibein, $\omega^{ab}{}_c$ is the spin connection and Γ_a are hermitian $SO(2)$ Dirac matrices

$$\Gamma_1 = \sigma_1\,, \qquad \Gamma_2 = \sigma_2\,, \qquad \Gamma_3 = -i\,\Gamma^1\Gamma^2 = \sigma_3\,, \qquad \{\Gamma_a\,,\Gamma_b\} = 2\delta_{ab}\mathbb{I}_2\,, \tag{B.7}$$

The explicit expressions for the scalar Laplacian in the two coordinates (B.1) and (B.3) are

$$\Delta \equiv \frac{1}{\sqrt{g}}\partial_i\left(\sqrt{g}g^{ij}\partial_j\right)\ , \qquad \Delta_{\rho,\tau} = \partial_\rho^2 + \coth\rho\,\partial_\rho + \sinh^{-2}\rho\,\partial_\tau^2\,, \tag{B.8}$$

$$\Delta_{r,w} = \cos^2 r\left(\partial_r^2 + \partial_w^2\right)\ . \tag{B.9}$$

The operator $-\Delta$ is hermitian with a continuous spectrum of positive eigenvalues $\lambda \in (\frac{1}{4}, \infty]$. The corresponding heat kernel for the massive operator $-\Delta + m^2$ is [25, 26, 54–56]

$$K_{-\Delta+m^2}(x,x';t) = \tfrac{1}{2\pi}\int_0^\infty dv\, v\tanh(\pi v) P_{-\frac{1}{2}+iv}(\cosh d(x,x'))\ e^{-t(v^2+\frac{1}{4}+m^2)}\,, \tag{B.10}$$

where the Legendre function is indexed by $v \equiv \sqrt{\lambda - \frac{1}{4}} > 0$ and the geodesic distance is given by (B.2) and (B.4) in the coordinate sets (B.1) and (B.3) respectively.

The Dirac operator $-i\not{\nabla}$ has the following explicit form

$$-i\not{\nabla}_{\rho,\tau} = -i\Gamma^1\left(\partial_\rho + \tfrac{1}{2}\coth\rho\right) - i\sinh^{-1}\rho\ \Gamma^2\partial_\tau\,, \tag{B.11}$$

$$-i\not{\nabla}_{r,w} = -i\Gamma^1\left(\cos r\partial_r + \tfrac{1}{2}\sin r\right) - i\cos r\,\Gamma^2\partial_w \tag{B.12}$$

in the two coordinate sets (B.1) and (B.3). The spinor heat kernel for the Dirac operator with a constant chiral mass term $-i\not{\nabla}_{\rho,\tau} + m\,\Gamma^3$ (with $m \in \mathbb{R}$) that satisfies the heat equation for $-\not{\nabla}^2 + m^2$ can be written in a coordinate-independent form as the product of the parallel spinor propagator $U(x,x')$ and a scalar function of the geodesic distance $d(x,x')$ between the two points x, x' [28]

$$K_{-\not{\nabla}^2+m^2}(x,x';t) = \tfrac{1}{2\pi}U(x,x')\int_0^\infty dv\, v\,\coth\pi v\,\cosh\left(\tfrac{1}{2}d(x,x')\right) \tag{B.13}$$
$$\times\ {}_2F_1\Big(iv+1,\ -iv+1,\ 1,\ \tfrac{1}{2} - \tfrac{1}{2}\cosh(d(x,x'))\Big)e^{-t\,(v^2+m^2)}\,.$$

[27] The coordinate indices are $i,j,\ldots = 1,2$, the indices of the local orthonormal frame are $a,b,\ldots = 1,2$ and $\alpha,\beta,\ldots = 1,2$ are the indices of the spinor bundle over $\mathcal{M}$ (we follow mainly the conventions of Appendix A of [13]).

The unitary 2×2 matrix $U(x,x')$ is the regular solution of the parallel transport equation [27]

$$n^i(u)\nabla_i U(x(0),x(u))=0, \qquad\qquad U\left(x(0),x(0)\right)=\mathbb{I}_2\,, \tag{B.14}$$

where $n_i(u)=\partial_i d(x(0),x(u))$ is the unit vector tangent to the shortest geodesic $x(u)$ between $x(0)=x'$ and $x(1)=x$. The explicit expression of $U(\rho,\tau,\rho',\tau')$ for the metric (B.1) in the matrix representation (B.7) reads [15]

$$U(\rho,\tau,\rho',\tau') = \mathbb{I}_2\cos\theta(\rho,\tau,\rho',\tau')+i\Gamma_3\sin\theta(\rho,\tau,\rho',\tau')\,, \tag{B.15}$$

$$\theta(\rho,\tau,\rho',\tau')\equiv \arctan\left(\tfrac{\cosh(\frac{\rho+\rho'}{2})\tan(\frac{\tau-\tau'}{2})}{\cosh(\frac{\rho-\rho'}{2})}\right). \tag{B.16}$$

The expression in (B.13) is the solution of the heat equation

$$\begin{aligned}&(\partial_t-\nabla^2_{\rho,\tau}+m^2)K_{-\nabla^2+m^2}(\rho,\tau,\rho',\tau';t)=0\,,\\ &\lim_{t\to 0^+}K_{-\nabla^2+m^2}(\rho,\tau,\rho',\tau';t)=\frac{\delta(\rho-\rho')\delta(\tau-\tau')}{\sinh\rho}\,\mathbb{I}_2\,.\end{aligned} \tag{B.17}$$

To change the coordinates to the infinite-strip parametrization (B.3) through (B.6) recall that spinors and the parallel spinor propagator are scalars under the diffeomorphisms, while under local rotations of the orthonormal frame they transform as

$$\begin{aligned}&\psi(x)\to\psi(\hat{x})=S(\hat{x})\psi(x(\hat{x}))\,, \qquad U(x,x')\to U(\hat{x},\hat{x}')=S(\hat{x})U(x(\hat{x}),x'(\hat{x}'))S^\dagger(\hat{x}')\,,\\ &S(\hat{x})\,\Gamma^{\hat{a}}\,S^\dagger(\hat{x})=\Lambda^{\hat{a}}{}_b(\hat{x})\,\Gamma^b\,, \qquad S(\hat{x})\in Spin(2)\,, \qquad \Lambda(\hat{x})\in SO(2)\,.\end{aligned} \tag{B.18}$$

Here $x=(\rho,\tau)$ and $\hat{x}=(r,w)$ represent one point and $x'=(\rho',\tau')$ and $\hat{x}'=(r',w')$ another point. The tangent frame rotation $\Lambda(\hat{x})$, satisfying $e^{\hat{a}}{}_{\hat{i}}(\hat{x})=\Lambda^{\hat{a}}{}_b(\hat{x})\,\frac{\partial x^j(\hat{x})}{\partial\hat{x}^{\hat{i}}}\,e^b{}_j(x(\hat{x}))$, reads

$$\Lambda^{\hat{a}}{}_b(\hat{x})\equiv\begin{pmatrix}\cos\delta(\hat{x}) & \sin\delta(\hat{x})\\ -\sin\delta(\hat{x}) & \cos\delta(\hat{x})\end{pmatrix}, \quad \sin\delta(\hat{x})=\tfrac{\cos r\sinh w}{\sqrt{\sin^2 r+\sinh^2 w}}\,, \quad \cos\delta(\hat{x})=\tfrac{\sin r\cosh w}{\sqrt{\sin^2 r+\sinh^2 w}}\,, \tag{B.19}$$

and the associated unitary rotation on the spinor indices is

$$S(\hat{x})=\cos\left(\tfrac{\delta(\hat{x})}{2}\right)\mathbb{I}_2+i\Gamma^3\sin\left(\tfrac{\delta(\hat{x})}{2}\right). \tag{B.20}$$

The parallel spinor propagator in the infinite-strip coordinates (B.3) is thus explicitly

$$\begin{aligned}U(r,w,r',w') = \mathbb{I}_2\cos\left(\theta(\rho,\tau,\rho',\tau')+\tfrac{1}{2}\delta(r,w)-\tfrac{1}{2}\delta(r',w')\right)\\ +\,i\,\Gamma_3\sin\left(\theta(\rho,\tau,\rho',\tau')+\tfrac{1}{2}\delta(r,w)-\tfrac{1}{2}\delta(r',w')\right).\end{aligned} \tag{B.21}$$

In these coordinates the spinor heat kernel $K_{-\nabla^2+m^2}(r,w,r',w';t)$ is given by (B.13) with $d(x,x')$ in (B.4) and U in (B.21) satisfies

$$\begin{aligned}&\left(\partial_t-\nabla^2_{r,w}+m^2\right)K_{-\nabla^2+m^2}(r,w,r',w';t)=0\,,\\ &\lim_{t\to 0^+}K_{-\nabla^2+m^2}(r,w,r',w';t)=\cos r\,\delta(r-r')\delta(w-w')\,\mathbb{I}_2\,.\end{aligned} \tag{B.22}$$

B.1 Zeta-functions of the Laplace and Dirac operator

The finite parts of the determinants of the massive Laplace and Dirac operator in H^2 are given by the derivative of the corresponding spectral zeta-function which itself can be expressed in terms of the functional trace of the heat kernels (B.10) and (B.13) (see also Appendix B of [8] and [57]). The integrated heat kernel for the massive Laplace operator $-\Delta + m^2$ is [25, 26]

$$K_{-\Delta+m^2}(t) = \frac{V_{H^2}}{2\pi}\int_0^\infty dv\, v \tanh(\pi v)\, e^{-t\left(v^2+\frac{1}{4}+m^2\right)}\,, \tag{B.23}$$

and for the square of the massive Dirac operator $-i\not{\nabla} + m\Gamma^3$ is [27, 28]

$$K_{-\not{\nabla}^2+m^2}(t) = \frac{V_{H^2}}{\pi}\int_0^\infty dv\, v \coth(\pi v)\, e^{-t\left(v^2+m^2\right)}\,. \tag{B.24}$$

The Seeley coefficients can be read off from the small-t expansions

$$\begin{aligned}\bar{K}_{-\Delta+m^2}(t) &= \frac{V_{H^2}}{2\pi}\left[\frac{e^{-\left(\frac{1}{4}+m^2\right)t}}{2t} - \int_0^\infty dv\,\frac{2v}{e^{2\pi v}+1}e^{-t\left(v^2+\frac{1}{4}+m^2\right)}\right] \qquad \text{(B.25)}\\ &= \frac{V_{H^2}}{4\pi}\left[\frac{1}{t} - \left(\frac{1}{3}+m^2\right) + O(t)\right]\\ \bar{K}_{-\not{\nabla}^2+m^2}(t) &= \frac{V_{H^2}}{\pi}\left[\frac{e^{-m^2t}}{2t} + \int_0^\infty dv\frac{2v}{e^{2\pi v}-1}e^{-t\left(v^2+m^2\right)}\right] \qquad \text{(B.26)}\\ &= \frac{V_{H^2}}{4\pi}\left[\frac{2}{t} + \left(\frac{1}{3}-2m^2\right) + O(t)\right]\end{aligned}$$

by replacing $\tanh(\pi v) = 1 - 2/(e^{2\pi v}+1)$ and $\coth(\pi v) = 1 + 2/(e^{2\pi v}-1)$, and they agree with the general results in [49]. The zeta-function for the massive Laplace operator is

$$\zeta_{-\Delta+m^2}(s) = \frac{V_{H^2}}{2\pi}\int_0^\infty dv \frac{v\tanh\pi v}{\left(v^2+m^2+\frac{1}{4}\right)^s}\,. \tag{B.27}$$

This expression is valid for $\operatorname{Re} s > 1$. For the analytic continuation to a neighbourhood of $s = 0$, we first use $\tanh(\pi v) = 1 - 2/(e^{2\pi v}+1)$ so that

$$\zeta_{-\Delta+m^2}(s) = \frac{V_{H^2}}{2\pi}\left[\int_0^\infty dv \frac{v}{\left(v^2+m^2+\frac{1}{4}\right)^s} - \int_0^\infty dv \frac{2v}{\left(e^{2\pi v}+1\right)\left(v^2+m^2+\frac{1}{4}\right)^s}\right], \tag{B.28}$$

where the second integral is exponentially convergent for large v at $s = 0$. The analytic continuation of the first integral can be easily found giving

$$\zeta_{-\Delta+m^2}(s) = \frac{V_{H^2}}{2\pi}\left[\frac{\left(m^2+\frac{1}{4}\right)^{1-s}}{2(s-1)} - 2\int_0^\infty dv \frac{v}{\left(e^{2\pi v}+1\right)\left(v^2+m^2+\frac{1}{4}\right)^s}\right]. \tag{B.29}$$

Then taking the derivative with respect to s and using the integral in (B.32), we obtain

$$\zeta'_{-\Delta+m^2}(0) = \frac{V_{H^2}}{2\pi}\left[\frac{1+\log 2}{12} - \log A + \int_0^{m^2+1/4} dx\,\psi\left(\sqrt{x}+\frac{1}{2}\right)\right]. \tag{B.30}$$

Similarly, for (B.24) using $\coth(\pi v) = 1 + 2/(e^{2\pi v} - 1)$ and (B.38) we get [28]

$$\begin{aligned}
\zeta_{-\nabla^2+m^2}(s) &= \frac{V_{H^2}}{\pi}\Big[\frac{(m^2)^{1-s}}{2(s-1)} + 2\int_0^\infty dv \frac{v}{(e^{2\pi v}-1)(v^2+m^2)^s}\Big], \\
\zeta'_{-\nabla^2+m^2}(0) &= \frac{V_{H^2}}{\pi}\Big[-\frac{1}{6} + 2\log A + \sqrt{m^2} + \int_0^{m^2} dx\, \psi\big(\sqrt{x}\big)\Big].
\end{aligned} \tag{B.31}$$

As for any homogeneous space, for which the heat kernel $K_{\mathcal{O}}(x,x;t)$ is independent of the point x, the integrated heat kernels above are all proportional to the volume of H^2. The latter has to replaced by its renormalized value, as discussed in Section 3.

B.2 Useful integrals

Here we collect some integrals useful for the computation of the regularized determinants of the Laplace and Dirac operators in H^2 (see also [6, 8, 57]). Below, c is some non-negative constant, $A \approx 1.282$ is the Glaisher constant, $\gamma \approx 0.577$ is the Euler-Mascheroni constant and $\psi(x) \equiv \frac{d}{dx}\log\Gamma(x)$ is the digamma function:

$$\int_0^\infty dv \frac{v\log\left(v^2+c\right)}{e^{2\pi v}+1} = \tfrac{c}{4}(1-\log c) + \tfrac{1+\log 2}{24} - \tfrac{\log A}{2} + \tfrac{1}{2}\int_0^c dx\, \psi(\sqrt{x}+\tfrac{1}{2}) \tag{B.32}$$

$$\int_0^\infty dv \frac{v\log\left(v^2+\frac{1}{4}\right)}{e^{2\pi v}+1} = \tfrac{5}{48} - \tfrac{\log 2}{8} + \log A - \tfrac{\log\pi}{4} \tag{B.33}$$

$$\int_0^\infty dv \frac{v\log\left(v^2+\frac{9}{4}\right)}{e^{2\pi v}+1} = \tfrac{77}{48} + \tfrac{3\log 2}{8} - \tfrac{9\log 3}{8} + \log A - \tfrac{3\log\pi}{4} \tag{B.34}$$

$$\int_0^\infty dv \frac{v}{(e^{2\pi v}+1)(v^2+c)} = -\tfrac{\log c}{4} + \tfrac{1}{2}\psi\left(\sqrt{c}+\tfrac{1}{2}\right) \tag{B.35}$$

$$\int_0^\infty dv \frac{v}{(e^{2\pi v}+1)\left(v^2+\frac{1}{4}\right)} = \tfrac{\log 2}{2} - \tfrac{\gamma}{2} \tag{B.36}$$

$$\int_0^\infty dv \frac{v}{(e^{2\pi v}+1)\left(v^2+\frac{9}{4}\right)} = -\tfrac{\log\frac{3}{2}}{2} + \tfrac{1}{2} - \tfrac{\gamma}{2} \tag{B.37}$$

$$\int_0^\infty dv \frac{v\log\left(v^2+c\right)}{e^{2\pi v}-1} = \tfrac{c}{4}(\log c - 1) + \tfrac{1}{12} - \log A - \tfrac{\sqrt{c}}{2} - \tfrac{1}{2}\int_0^c dx\, \psi\left(\sqrt{x}\right) \tag{B.38}$$

$$\int_0^\infty dv \frac{v\log\left(v^2+1\right)}{e^{2\pi v}-1} = -\tfrac{2}{3} + \tfrac{\log 2}{2} - \log A + \tfrac{\log\pi}{2} \tag{B.39}$$

$$\int_0^\infty dv \frac{v}{(e^{2\pi v}-1)(v^2+c)} = \tfrac{\log c}{4} - \tfrac{1}{4\sqrt{c}} - \tfrac{1}{2}\psi\left(\sqrt{c}\right) \tag{B.40}$$

$$\int_0^\infty dv \frac{v}{(e^{2\pi v}-1)(v^2+1)} = -\tfrac{1}{4} + \tfrac{\gamma}{2} \tag{B.41}$$

[28] Compared to [8], here we do not include the minus sign of fermionic statistics of the spinor fields in the definition of the zeta-function, but we account for it in the sum over the scalar and spinor contributions to the one-loop effective actions (3.21), (3.66) and (3.94). We also recall that the spinor heat kernel in Appendix B of [8] and [15] is for Majorana fermions, so the integrated heat kernel and zeta-function include an extra factor of 1/2 with respect to the expressions (B.24) and (B.31) for Dirac spinors derived from the heat kernel in [27, 28].

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