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Structure of large spin expansion of anomalous dimensions at strong coupling

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Abstract

The anomalous dimensions of planar $\mathcal{N} = 4$ SYM theory operators like $\text{tr}(\Phi D_+^S \Phi)$ expanded in large spin S have the asymptotics $\gamma = f \ln S + f_c + \frac{1}{S}(f_{11} \ln S + f_{10}) + \dots$, where f (the universal scaling function or cusp anomaly), f_c and f_{mn} are given by power series in the 't Hooft coupling λ . The subleading coefficients appear to be related by the so-called functional relation and parity (reciprocity) property of the function expressing γ in terms of the conformal spin of the collinear group. Here we study the structure of such large spin expansion at strong coupling via AdS/CFT, i.e. by using the dual description in terms of folded spinning string in AdS_5 . The large spin expansion of the classical string energy happens to have exactly the same structure as that of γ in the perturbative gauge theory. Moreover, the functional relation and the reciprocity constraints on the coefficients are also satisfied. We compute the leading string 1-loop corrections to the coefficients f_c , f_{11} , f_{10} and verify the functional/reciprocity relations at subleading $\frac{1}{\sqrt{\lambda}}$ order. This provides a strong indication that these relations hold not only in weak coupling (gauge-theory) but also in strong coupling (string-theory) perturbative expansions.

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1. Introduction and summary

Recent advances in the study of duality between planar $\mathcal{N} = 4$ SYM theory and free $AdS_5 \times S^5$ superstring theory which utilise their integrability property led to important insights into the structure of dependence of anomalous dimensions of gauge-invariant operators on the quantum numbers like spin and on the 't Hooft coupling. While there was a remarkable recent progress in understanding the asymptotic large spin limit in which the compactness of the spatial direction of the world sheet may be ignored [1,2],³ it is important to study corrections to this limit.

Here we shall consider the famous example [6] of folded spinning string in AdS_5 dual to a minimal twist gauge theory operator like $\text{tr}(\Phi D_+^S \Phi)$. Starting with the classical string energy for the solution of [6,7] and expanding it in large semiclassical spin parameter \mathcal{S} one finds (see [8] and below)

$$E = \sqrt{\lambda} \mathcal{E}(\mathcal{S}), \quad \mathcal{S} = \frac{S}{\sqrt{\lambda}}, \tag{1.1}$$

$$\begin{aligned} \mathcal{E}(\mathcal{S})_{\mathcal{S} \gg 1} = & \mathcal{S} + a_0 \ln \mathcal{S} + a_c + \frac{1}{\mathcal{S}}(a_{11} \ln \mathcal{S} + a_{10}) \\ & + \frac{1}{\mathcal{S}^2}(a_{22} \ln^2 \mathcal{S} + a_{21} \ln \mathcal{S} + a_{20}) + \mathcal{O}\left(\frac{\ln^3 \mathcal{S}}{\mathcal{S}^3}\right), \end{aligned} \tag{1.2}$$

with $a_0 = \frac{1}{\pi}$, $a_c = \frac{1}{\pi}(\ln 8\pi - 1)$, etc.⁴ That means that in the semiclassical string theory limit in which one first takes the string tension $\frac{\sqrt{\lambda}}{2\pi}$ to be large for fixed \mathcal{S} and then expands in large \mathcal{S} , the corresponding string energy can be written as ($\sqrt{\lambda} \gg 1$, $\frac{S}{\sqrt{\lambda}} \gg 1$)

$$\begin{aligned} E = & S + f \ln S + f_c + \frac{1}{S}[f_{11} \ln S + f_{10}] \\ & + \frac{1}{S^2}[f_{22} \ln^2 S + f_{21} \ln S + f_{20}] + \mathcal{O}\left(\frac{\ln^3 S}{S^3}\right), \end{aligned} \tag{1.3}$$

where $f = \sqrt{\lambda} a_0 + \dots$, $f_c = \sqrt{\lambda} a_c + \dots$, etc. The subleading coefficients simplify if we absorb the constant f_c into the $\ln S$ term, i.e. if we re-write (1.3) as

$$\begin{aligned} E = & S + f \ln(S/\tilde{f}_c) + \frac{1}{S}[f_{11} \ln(S/\tilde{f}_c) + f'_{10}] \\ & + \frac{1}{S^2}[f_{22} \ln^2(S/\tilde{f}_c) + f'_{21} \ln(S/\tilde{f}_c) + f'_{20}] + \mathcal{O}\left(\frac{\ln^3 S}{S^3}\right), \end{aligned} \tag{1.4}$$

where to leading order in $\frac{1}{\sqrt{\lambda}}$ expansion

$$\begin{aligned} f = \frac{\sqrt{\lambda}}{\pi}, \quad \tilde{f}_c = \frac{e\sqrt{\lambda}}{8\pi}, \quad f_{11} = \frac{\lambda}{2\pi^2}, \quad f'_{10} = 0, \\ f_{22} = -\frac{\lambda^{3/2}}{8\pi^3}, \quad f'_{21} = \frac{5\lambda^{3/2}}{16\pi^3}, \quad f'_{20} = \frac{\lambda^{3/2}}{8\pi^3}. \end{aligned}$$

³ Here we refer to the integral equations that describe the minimal anomalous dimension in the band [3,4]. These equations were obtained from the all-loop Bethe ansatz by taking a special scaling limit [3,5] which describes a condensation of magnons and holes at the origin.

⁴ Note that the small S behaviour of the energy is quite different [9]: $\mathcal{E} = \sqrt{2S}[h_0 + h_1 S + \dots]$.

Following the analysis of quantum corrections to the folded string solution in [10], one may conclude that this structure of the large S expansion is preserved by the $\alpha' \sim \frac{1}{\sqrt{\lambda}}$ corrections, with the coefficients f, f_c, f_{11}, \dots being promoted to power series in $\frac{1}{\sqrt{\lambda}}$, i.e. $f_{mk} \sim \sum_n \frac{b_{mk,n}}{(\sqrt{\lambda})^n}$.

Indeed, as we shall find below, the 1-loop corrections for leading coefficients in (1.4) are⁵

$$f = \frac{\sqrt{\lambda}}{\pi} \left[1 - \frac{3 \ln 2}{\sqrt{\lambda}} + \mathcal{O}\left(\frac{1}{\lambda}\right) \right], \tag{1.5}$$

$$\tilde{f}_c = \frac{e\sqrt{\lambda}}{8\pi} \left[1 + \frac{1}{\sqrt{\lambda}}(3 \ln 2 - c) + \mathcal{O}\left(\frac{1}{\lambda}\right) \right], \tag{1.6}$$

$$f_{11} = \frac{\lambda}{2\pi^2} \left[1 - \frac{6 \ln 2}{\sqrt{\lambda}} + \mathcal{O}\left(\frac{1}{\lambda}\right) \right], \tag{1.7}$$

$$f'_{10} = \frac{\lambda}{2\pi^2} \left[0 - \frac{0}{\sqrt{\lambda}} + \mathcal{O}\left(\frac{1}{\lambda}\right) \right]. \tag{1.8}$$

Here c is a constant that we were not able to determine with the method for evaluation of 1-loop string correction we used below. Equivalently, in (1.3) we get the same f, f_{11} and

$$f_c = f \ln \frac{1}{\tilde{f}_c} = \frac{\sqrt{\lambda}}{\pi} \left[\ln \frac{8\pi}{\sqrt{\lambda}} - 1 + \frac{1}{\sqrt{\lambda}} \left(-3 \ln 2 \ln \frac{8\pi}{\sqrt{\lambda}} + c \right) + \mathcal{O}\left(\frac{1}{\lambda}\right) \right], \tag{1.9}$$

$$f_{10} = f'_{10} + f_{11} \ln \frac{1}{\tilde{f}_c} = f'_{10} + \frac{f_c}{f} f_{11} \tag{1.10}$$

$$= \frac{\lambda}{2\pi^2} \left[\ln \frac{8\pi}{\sqrt{\lambda}} - 1 + \frac{1}{\sqrt{\lambda}} \left(-6 \ln 2 \left[\ln \frac{8\pi}{\sqrt{\lambda}} - \frac{1}{2} \right] + c \right) + \mathcal{O}\left(\frac{1}{\lambda}\right) \right]. \tag{1.11}$$

The (at first surprising) vanishing of the first two terms in f'_{10} in (1.8) is a consequence of the relation $f_{10} - \frac{f_c}{f} f_{11} = 0$ (here verified at first two leading orders, i.e. to order $\mathcal{O}(\lambda^0)$ in f_{10}). In fact, since we also see that in (1.5), (1.7) $f_{11} = \frac{1}{2} f^2$, this relation is equivalent to $f_{10} - \frac{1}{2} f_c f = 0$. As we shall see below, these relations are consequences of the “functional relation” and reciprocity at strong coupling. We thus led to expect that $f'_{10} = 0$ should be true in to *all orders* in the strong coupling expansion, suggesting the advantage of the form of E in (1.4) over (1.3) and the importance of the function \tilde{f}_c .

Reversing the usual logic, we may then conjecture that structurally same large spin expansion should appear also at weak coupling, i.e. in the perturbative expressions for the corresponding gauge theory anomalous dimensions. This is not, a priori, guaranteed since the limit taken on the gauge theory side is different from the above string-theory limit: there one first expands the anomalous dimension in small λ at fixed S and then takes S large in each of the λ^n coefficients. Yet, remarkably, expanding in large S the known 2-, 3- and 4-loop perturbative anomalous dimensions of twist 2 and twist 3 operators in SYM theory one does find [8,11–13, 15–18] the expression of the form (1.3) with the coefficients given by power series in λ , i.e. $f_{mk} \sim \sum_n a_{mk,n} \lambda^n$.⁶

⁵ The leading correction to f was found in [10].

⁶ The 4-loop prediction for twist 2 and 3 anomalous dimension at finite S [17] so far was not based on direct gauge theory computation.

Assuming that the expansion (1.3) or⁷

$$E - S = \sum_{m=0}^{\infty} \frac{e_m(\lambda, \ln S)}{S^m}, \quad e_m(\lambda, \ln S) = \sum_k f_{mk}(\lambda) \ln^k S \quad (1.12)$$

applies for any λ , and given the important role of the universal scaling function or cusp anomalous dimension $f(\lambda)$ [19], one may raise the question about the interpretation of other “interpolating” functions $f_{mk}(\lambda)$ in (1.3).

From the gauge theory point of view, the function $f(\lambda)$ appears in the asymptotics of anomalous dimensions of gauge invariant operators as well as (for twist 2) in the IR asymptotics of gluon scattering amplitudes related to UV cusp anomaly of light-like Wilson loops (for a review and references see, e.g., [20]). On string side that corresponds, respectively, to the closed string [6] and the open string [21,22] sectors. They are connected in the strict large S limit since then the ends of the folded spinning string reach the boundary of AdS_5 and thus the associated world surface has a Wilson line interpretation [23]. This open string sector interpretation should not be expected to apply to other subleading coefficients $f_{mk}(\lambda)$ since for finite S the end points of the folded string no longer touch the boundary (cf., however, [24,25] where the gauge theory interpretation of the constant f_c is discussed).

In fact, many of the f_{mk} coefficients in (1.3), (1.12) are not actually independent, as was first observed at few leading orders in weak coupling expansion and then given a general interpretation in [15,16]. According to [15], these coefficients are constrained by (i) the so-called “functional relation” suggested by the conformal invariance (which relates the leading f_{mm} functions to powers of the scaling function f and thus implies their universality) and also by (ii) the “parity preserving relation” or “reciprocity” [12,15,16,18,26] (which relates some subleading non-universal coefficients, e.g., f_{10} to f_c , f_{32} to f_{21} , etc.).

Our aim here will be to investigate the presence of such relations at strong coupling, i.e. in the semiclassical string theory expansion for the spinning string states, extending earlier observations made in [15].⁸

Before proceeding let us add an important clarification. While for low twists $J = 2, 3$ in weak-coupling expansion the powers of $\ln S$ in e_m in the anomalous dimension (1.12) appear to be positive,⁹ for $J > 3$ one finds also terms with negative powers like $\frac{J^k}{\ln^m S}$. Namely, for $J > 3$ the leading term in $E - S - J$ appears to be $e_0 = k_1(J) + \frac{k_2(J)}{\ln^2 S} + \dots$, where for $k_1 = k_{11}J + k_{10}$, $k_2 = k_{23}J^3 + k_{22}J^2 + k_{21}J + k_{20}$ and k_2 vanishes for $J = 2, 3$.¹⁰ The presence of such terms was first observed in [3] (in the 1-loop approximation in the $sl(2)$ sector) in the limit of large J and S with $j \equiv \frac{J}{\ln S}$ fixed and small, i.e. $e_0 = (k_{11}j + k_{23}j^3 + \dots) \ln S$, and they are likely to be present also for finite J with $S \gg 1$.¹¹ At strong coupling, i.e. in the string-theory semiclassical expansion, one cannot distinguish between finite values of J and $J = 0$; for large

⁷ Here, as in (1.2), (1.3), we suppress the dependence on finite twist J .

⁸ Let us note also that the fine structure of the constant term f_c in (1.3) for “non-minimal” operators and the dual string states was studied in [3,4] and in [27].

⁹ Here we have in mind the *minimal* anomalous dimension in the band [3]; the $\frac{J^3}{\ln^2 S}$ terms appear [28] in non-minimal dimensions even for low twists. We thank G. Korchemsky for this clarification.

¹⁰ We thank A. Rej for a discussion of the structure of these terms at 1-loop order in $sl(2)$ sector (see also [3]).

¹¹ There is a numerical evidence for the presence of $\frac{k_2(J)}{\ln^2 S}$ term for $J \geq 4$ from the analysis of the corresponding 1-loop Baxter equation (A. Rej, private communication). For a general method to derive higher order terms in $1/S$ expansion at fixed J see [29].

J one finds (see [3] and Section 2.2 below) that in a similar limit of large S with $\ell \equiv \frac{J}{\sqrt{\lambda \ln S}}$ fixed $e_0 = (n_1 \ell^2 + n_2 \ell^4 + \dots) \ln S$. Thus the dependence on J and S is different at strong and weak coupling and to relate the two expansions one needs a non-trivial resummation rather than simple interpolation of coefficients in λ [2,40].

Let us now review in more detail what is known at weak coupling (see [15] and references there). The results of explicit higher-loop planar gauge-theory computations of anomalous dimensions $\gamma(S, J, \lambda)$ of operators like $\text{tr}(D_+^S \Phi^J)$ (S is the Lorentz spin and J is the twist) were interpreted in [15] in the following way (see also [16]). Observing that such Wilson-type operators can be classified according to representations of the collinear $SL(2, R)$ subgroup of the $SO(2, 4)$ conformal group [30] which are labeled by the *conformal spin* $s = \frac{1}{2}(S + \Delta)$ one may argue that the anomalous dimension $\gamma = \Delta - S - J$ should be a function of S only through its dependence on the conformal spin s . Since the scaling dimension Δ is¹²

$$\Delta = S + J + \gamma(S, J), \tag{1.13}$$

that then leads to the following “functional relation” for γ

$$\gamma(S, J) = f(s; J) = f\left(S + \frac{1}{2}J + \frac{1}{2}\gamma(S, J); J\right). \tag{1.14}$$

Without further information, this relation is nothing more than a change of variable, since, at least in perturbation theory, it is always possible to compute the function f in terms of the anomalous dimension $\gamma(S, J)$. Nevertheless, the above reasoning suggests that f could be more fundamental than γ .¹³ This is what we shall assume below when referring to the functional relation.

Suppressing the dependence on J in γ and f we may write this functional relation simply as¹⁴

$$\gamma(S) = f\left(S + \frac{1}{2}\gamma(S)\right). \tag{1.15}$$

At weak coupling $\gamma(S) = \sum_{n=1}^{\infty} \gamma_n(S) \lambda^n$; expanding the coefficients γ_n in large S (for fixed J) one finds that for all explicitly known perturbative gauge-theory results one gets the same expansion as in (1.3)

$$\begin{aligned} \gamma(S)_{S \gg 1} = & f \ln S + f_c + \frac{f_{11} \ln S + f_{10}}{S} + \frac{f_{22} \ln^2 S + f_{21} \ln S + f_{20}}{S^2} \\ & + \frac{f_{33} \ln^3 S + f_{32} \ln^2 S + f_{31} \ln S + f_{30}}{S^3} + \mathcal{O}\left(\frac{\ln^4 S}{S^4}\right), \end{aligned} \tag{1.16}$$

where the coefficients f, f_c, f_{11}, \dots are power series in λ . Remarkably, the structure of this expansion turns out to be perfectly consistent with the functional relation (1.15): the function f

¹² Here we assume that Φ in the operator $\text{tr}(D_+^S \Phi^J)$ is a scalar field (as is the case in the $sl(2)$ sector of SYM theory). The relation between the notation used in [15] and ours is: $N \rightarrow S, L \rightarrow J, J \rightarrow C$ and $j \rightarrow s$.

¹³ Apart from conformal invariance, for twist 2 case, this conclusion is also expected on the basis of the QCD origin of the functional relation (1.14) [13]. In that context, the function f for twist 2 operators is closely related to a special reformulation of the parton distribution functions evolution equation which aims at treating symmetrically the space-like channel of deep inelastic scattering and the time-like crossed channel describing e^+e^- annihilation [14]. In particular, the functional relation (1.14) turns out to be predictive within the large S expansion (1.12) because the function f happens to be simpler than γ and, in particular, it does not contain $\ln^m S/S^m$ terms [15,16,26].

¹⁴ In the approach of [13,16], which recently received a nice confirmation in [31], this relation follows from a suitable modification of the evolution equations governing the renormalization of the twist operators.

starts with a logarithmic term so that the coefficients of the leading $\frac{\ln^m S}{S^m}$ terms are all determined by the scaling function f [4,15,26]

$$\gamma(S) = f \ln \left(S + \frac{1}{2} f \ln S + \dots \right) + \dots = f \ln S + \frac{f^2 \ln S}{2 S} - \frac{f^3 \ln^2 S}{8 S^2} + \frac{f^4 \ln^3 S}{24 S^3} + \dots \tag{1.17}$$

The universality (i.e. twist and flavor independence) of the scaling function or cusp anomalous dimension f thus implies the universality of all of the coefficients f_{mm} in (1.16) as they are simply proportional to f^{m+1} ,

$$f_{11} = \frac{1}{2} f^2, \quad f_{22} = -\frac{1}{8} f^3, \quad f_{33} = \frac{1}{24} f^4, \quad \dots \tag{1.18}$$

These should be understood as relations between the functions $f_{mm}(\lambda)$ and $f(\lambda)$ defined as power series in λ .

Let us note that anomalous dimensions of operators with twist higher than two occupy a band [3], the lower bound of which is the *minimal* dimension for given S and J . The relation (1.17) is expected to apply for the minimal dimension in the band. Interestingly, as was found at weak coupling, a similar relation also holds for the excited trajectories [4]. This is also what we shall see at strong coupling on the example of the spiky string in Section 2.3 (see (2.32)).

In Appendix F below we will summarize the known perturbative expansions for the minimal anomalous dimensions of twist 2 and twist 3 operators of various flavors, as obtained from the asymptotic Bethe ansatz of [32]. These results are indeed consistent with (1.17), (1.18) and thus with the universality of the f_{mm} coefficients in (1.16).

As for the subleading ($\frac{\ln^k S}{S^m}$, $k < m$) terms in (1.16), their coefficients are, at least partially, controlled by special properties of the function f in the functional relation (1.15). Indeed, it was observed on many examples that the function f should satisfy a “parity preserving relation” or *reciprocity* property.¹⁵ This property implies that the large S expansion of $f(S)$ should run in the inverse *even* powers of the quadratic Casimir of the collinear $SL(2, R)$ group, namely, [15]

$$f(S) = \sum_{n=0}^{\infty} \frac{a_n (\ln C)}{C^{2n}}, \tag{1.19}$$

where C is the “bare” quadratic Casimir defined in terms of the “canonical” value of the conformal spin s_0 as $C^2 \equiv s_0(s_0 - 1)$, $s_0 = \frac{1}{2}(S + \Delta_0) = S + \frac{1}{2}J$, i.e.¹⁶

$$C^2 = \left(S + \frac{1}{2}J \right) \left(S + \frac{1}{2}J - 1 \right). \tag{1.20}$$

The reciprocity property (1.19) of the function f in the relation (1.15) then imposes constraints on some of the coefficients of the subleading terms in the expansion (1.16):

¹⁵ The name reciprocity has its origin in the formulation of this property, for twist 2 case, in terms of the Mellin transform: $F(x) = -x F(1/x)$, where $f(S) = \int_0^1 dx x^{S-1} F(x)$. For twist larger than 2 the “parity preserving relation” was suggested in [15] as a more appropriate name. Here for simplicity we shall not make this distinction and will often refer to reciprocity when implying the “parity invariance” property in (1.19), (1.20) below.

¹⁶ Here we again consider the operator built of scalar fields. For twist 2, i.e. $J = 2$, one then has $C^2 = S(S + 1)$, so that $C = S + \frac{1}{2} + \mathcal{O}(\frac{1}{S})$. For generic flavour one is to replace, in (1.20), $S + \frac{1}{2}J \rightarrow S + \ell J$, where $\ell = \frac{1}{2}, 1, \frac{3}{2}$ for a scalar, spinor or vector cases [30].

$$f_{10} = \frac{1}{2}f(f_c - 1 + J), \quad f_{32} = \frac{1}{16}f[f^3 - 2f^2(f_c - 1 + J) - 16f_{21}], \quad \dots, \quad (1.21)$$

where dots stand for similar expressions for f_{31} , f_{30} , f_{55} , \dots , f_{50} , etc. Again, these equations relate functions $f_{mk}(\lambda)$ defined as power series in λ . For twist $J = 2$ we get simply

$$f_{10} = \frac{1}{2}f(f_c + 1), \quad f_{32} = \frac{1}{16}f[f^3 - 2f^2(f_c + 1) - 16f_{21}], \quad \dots \quad (1.22)$$

These so-called MVV [12] relations were first observed for twist 2 QCD anomalous dimensions up to 3 loops. The large S expansions for the known twist 2 and twist 3 SYM anomalous dimensions that we will present in Appendix F are indeed consistent with these relations, i.e. with the reciprocity property of the function f .¹⁷

It is natural to expect that the functional relation and the reciprocity property should hold also at higher orders in small λ expansion. Since the planar perturbation theory should be convergent, they should then also be visible at strong coupling [15], i.e. in the large spin expansion of the corresponding semiclassical string energies.

One may also wonder if the reciprocity property may apply to higher twist operators above the lower bound of the band [3,4]. If that were the case, it could then be checked also at strong coupling on the example of the spiky string solution of [34].

The agreement in the *structure* of the large S expansion found in perturbative gauge theory and in perturbative string theory is already quite remarkable. This agreement is non-trivial since, as was already mentioned, the gauge-theory and string-theory perturbative expansions are organized differently: the gauge-theory limit is to expand in small λ at fixed S and then expand the λ^n coefficients in large S , while the semiclassical string-theory limit is to expand in large λ with fixed $S = \frac{S}{\sqrt{\lambda}}$ and then expand the $\frac{1}{(\sqrt{\lambda})^n}$ terms in E in large S . Even assuming these limits commute (which so far appears to be verified only for the leading universal $\ln S$ term) the reason for the validity of the functional relation (1.15) and, moreover, of the reciprocity property (1.19) is obscure on the semiclassical string theory side.

The functional relation (1.15) for the anomalous dimensions of Wilson-type operators on the gauge theory side was argued [15] to follow from the invariance under the collinear $SL(2, R)$ subgroup of the conformal $SO(2, 4)$ group. Given that this argument is based on the conformal symmetry, one may think that it should then apply also on the string theory side. However, as we will review in Appendix A, the realization of the conformal group on states represented by classical spinning string solutions in global AdS_5 coordinates is a priori different from the one used on the gauge-theory side (which is based on the collinear subgroup), so that the direct connection is not obvious. The reason for the reciprocity property on the string theory side is even far less clear.

If one identifies the energy E and the spin S of a string rotating in a plane in global AdS_5 with dimension and Lorentz spin of the gauge theory operator like $\text{tr}(D_+^S \Phi^J)$, the functional relation (1.15) would then imply that $\gamma = E - S - J$ should be a function of $s = \frac{1}{2}(E + S)$, i.e.

$$E - S - J = f(E + S, J). \quad (1.23)$$

¹⁷ Three-loop tests of reciprocity for QCD and for the universal twist 2 supermultiplet in $\mathcal{N} = 4$ SYM were discussed in [15,16]. A four-loop test for the twist 3 anomalous dimension in the $sl(2)$ sector was performed in [26]. The case of twist 3 gauge field strength operators was analyzed in [33] (at three loops) and in [18] (at four loops). In the latter paper it was also proved that even the wrapping-affected four loop result for the twist two operators [17] is reciprocity respecting.

As we shall discuss below (extending earlier observations in [8,15]), not only the structure of the large spin expansion on the string theory side happens to be the same as on the gauge theory side but also its coefficients are indeed consistent with the functional relation and the reciprocity for the minimal dimension case represented by the folded spinning string. This will be demonstrated at the classical as well as 1-loop string theory level.

We shall also show that the functional relation but not the reciprocity appears to apply also to the case of the classical spiky string solution.

The rest of this paper is organized as follows.

In Section 2 we shall first consider the large spin expansion of the classical energy of folded spinning string in AdS_5 and show that the large spin expansion has the structure (1.2) and the functional and reciprocity relations between the coefficients are satisfied. We shall then include (in Section 2.2) the dependence on the angular momentum J in S^5 in the “long string” limit ($\mathcal{J} \ll S$). In Section 2.3 we shall study the same large spin expansion for a spiky string in AdS_5 ; in this case we shall find that the reciprocity condition is violated which should be related to the fact that the corresponding operator has higher than minimal dimension for a given spin.

In Section 3 we shall return to the case of the folded spinning string in AdS_5 (i.e. assume that J is negligible compared to S) and compute the 1-loop correction to the energy expanded in large S , determining corrections to several leading coefficients. As result, we shall verify that the string 1-loop corrections preserve the structure (1.3) of the large spin expansion and, moreover, that the reciprocity condition is satisfied beyond the string tree level.

In Appendix A we shall make some comments on relation between different realizations of conformal group. In Appendices B and C we shall review the folded spinning string solution and discuss long-string or large-spin expansions used in the 1-loop computation in Section 3. In Appendix D we shall give details of large spin expansions for (S, J) string considered in Section 2. In Appendix E we shall discuss some consequences of the functional relation and the reciprocity at strong coupling, pointing out a subtlety in the definition of the latter in the semiclassical string expansion. In Appendix F we shall summarize the known weak coupling planar SYM results for the large spin expansion of twist 2 and 3 anomalous dimensions up to 4-loop order in the 't Hooft coupling.

2. Large spin expansion: Classical string theory

2.1. Folded spinning string with $J = 0$

We shall start with a discussion of the limit when the S^5 momentum J of the string state can be ignored, i.e. we shall concentrate only on the AdS_5 spin S dependence of the string energy. This is the limit when the twist of the gauge theory operator is sufficiently small compared to the Lorentz spin.

We review the folded spinning string solution [6] in Appendix B. The integrals of motion are the energy $E = \sqrt{\lambda} \mathcal{E}$ and the spin $S = \sqrt{\lambda} \mathcal{S}$, which can be expressed in terms of the elliptic functions \mathbb{E} and \mathbb{K} of an auxiliary variable η ¹⁸

$$\mathcal{E} - \mathcal{S} = \frac{2}{\pi} \sqrt{\frac{1+\eta}{\eta}} \left[\mathbb{E}\left(-\frac{1}{\eta}\right) \left(\frac{1}{\sqrt{1+\eta}} - 1 \right) + \mathbb{K}\left(-\frac{1}{\eta}\right) \right], \quad (2.1)$$

¹⁸ Here we follow the notation of [35]. Equivalently, one can express the conserved charges in terms of the hypergeometric functions as in Appendix B.

$$\mathcal{S} = \frac{2}{\pi} \sqrt{\frac{1+\eta}{\eta}} \left[\mathbb{E}\left(-\frac{1}{\eta}\right) - \mathbb{K}\left(-\frac{1}{\eta}\right) \right]. \tag{2.2}$$

To find the energy in terms of the spin one is to solve for η . Here we are interested in the large spin expansion which corresponds to the long string limit (when the string ends are close to the boundary of AdS_5). For such long string one has $\eta \rightarrow 0$. Solving (2.2) for small η and substituting it into (2.1) one finds for \mathcal{E} as a function of \mathcal{S}

$$\begin{aligned} \mathcal{E} = \mathcal{S} &+ \frac{\ln \bar{\mathcal{S}} - 1}{\pi} + \frac{\ln \bar{\mathcal{S}} - 1}{2\pi^2 \mathcal{S}} - \frac{2 \ln^2 \bar{\mathcal{S}} - 9 \ln \bar{\mathcal{S}} + 5}{16\pi^3 \mathcal{S}^2} \\ &+ \frac{2 \ln^3 \bar{\mathcal{S}} - 18 \ln^2 \bar{\mathcal{S}} + 33 \ln \bar{\mathcal{S}} - 14}{48\pi^4 \mathcal{S}^3} + \dots, \quad \bar{\mathcal{S}} \equiv 8\pi \mathcal{S}, \end{aligned} \tag{2.3}$$

as was already claimed in (1.2).

The functional relation (1.15), (1.23) implies that $\mathcal{E} - \mathcal{S}$ should be a function of $\mathcal{E} + \mathcal{S}$. It is not immediately obvious from (2.2) (or from the form of the exact solution in global AdS_5 coordinates) why such a relation should be natural for any value of \mathcal{S} . Still, the coefficients of the leading $(\frac{\ln \mathcal{S}}{\mathcal{S}})^m$ terms in (2.3) happen, indeed, to be consistent with such a relation, with the leading term in the function f being simply the logarithm (cf. (1.17))

$$E - S = \frac{\sqrt{\lambda}}{\pi} \ln \left[S + \frac{1}{2} \frac{\sqrt{\lambda}}{\pi} \ln S + \dots \right] + \dots. \tag{2.4}$$

Furthermore, it is possible to verify that the expansion of $E - S$ also satisfies the reciprocity property (1.19), (1.22). The large \mathcal{S} expansion of the function f (its leading term in the strong-coupling limit) is much simpler than that of the anomalous dimension $E - S$ in (2.1) and contains only *even* powers of $C^{-1} \sim \mathcal{S}^{-1}$ (see (1.20))

$$f = \sqrt{\lambda} \tilde{f}, \quad \tilde{f}(\mathcal{S}) = \frac{1}{\pi} \left[\ln \bar{\mathcal{S}} - 1 + \frac{\ln \bar{\mathcal{S}} + 1}{16\pi^2 \mathcal{S}^2} + \mathcal{O}\left(\frac{1}{\mathcal{S}^4}\right) \right] + \mathcal{O}\left(\frac{1}{\sqrt{\lambda}}\right). \tag{2.5}$$

Equivalently, we find that the MVV-like relations (1.22) are satisfied.¹⁹

A more systematic analysis of the reciprocity (parity invariance) property of the function f is possible with the help of an integral representation for it. Using that (1.15) implies $\tilde{f}(\mathcal{S}') = \tilde{\gamma}(\mathcal{S}' - \frac{1}{2}\tilde{f}(\mathcal{S}'))$, where $\mathcal{S}' = \mathcal{S} + \frac{1}{2}\tilde{\gamma}(\mathcal{S})$, $\tilde{\gamma}(\mathcal{S}) = \mathcal{E} - \mathcal{S}$, and renaming $\mathcal{S}' \rightarrow \mathcal{S}$ we have

$$\tilde{f}(\mathcal{S}) = \frac{1}{2\pi i} \oint_{\Gamma} d\omega \tilde{\gamma}(\omega) \frac{1 + \frac{1}{2}\tilde{\gamma}'(\omega)}{\omega - \mathcal{S} + \frac{1}{2}\tilde{\gamma}(\omega)}, \tag{2.6}$$

where the contour Γ encircles the pole of the integrand and prime stands for derivative.²⁰ It is natural to replace the variable ω in (2.6) with the expression (2.2) for the semiclassical spin $\mathcal{S}(\eta)$

¹⁹ The definition of reciprocity condition in string semiclassical expansion is discussed in Appendix E.

²⁰ The expression that multiplies $\tilde{\gamma}$ in the integrand has residue 1, so that the integral is $\tilde{\gamma}$ evaluated at the pole $\omega = \mathcal{S} - \frac{1}{2}\tilde{\gamma}$. Then defining $x = \mathcal{S} - \frac{1}{2}\tilde{f}(\mathcal{S})$ we have $2\mathcal{S} - 2x = \tilde{\gamma}$ which coincides with the equation for the pole with $x = \omega$. Note that assuming f exists, one can formally reconstruct it from γ using [15] $f(\mathcal{S}) = \sum_{k=1}^{\infty} \frac{1}{k!} (-\frac{1}{2}\partial_{\mathcal{S}})^{k-1} [\gamma(\mathcal{S})]^k = \gamma - \frac{1}{4}(\gamma^2)' + \frac{1}{24}(\gamma^3)'' + \dots$. This relation also arises by expanding the denominator in (2.6) in small $\tilde{\gamma}$ and integrating the resulting series.

$$\tilde{f}(\mathcal{S}) = \frac{1}{2\pi i} \oint_{\Gamma} d\eta \tilde{\gamma}(\eta) \frac{\tilde{s}'(\eta)}{\tilde{s}(\eta) - \mathcal{S}}, \tag{2.7}$$

where $\tilde{s}(\eta) \equiv \mathcal{S}(\eta) + \frac{1}{2}\tilde{\gamma}(\eta) = \frac{1}{2}(\mathcal{E} + \mathcal{S})$ is the ‘‘conformal spin’’ expressed in terms of the semiclassical quantities. The integral then gives the function $\tilde{\gamma}$ evaluated at the zero of the denominator; this is the same as the statement that the anomalous dimension as a function of the Lorentz spin is, effectively, a function of the conformal spin \tilde{s} .

To verify the reciprocity property of the function $\tilde{f}(\mathcal{S})$ in (2.7) it is useful to redefine the variable η as²¹ $\eta \rightarrow -1 + 16\eta + \sqrt{1 + 256\eta^2}$ and examine the large \mathcal{S} or small η limit of the expressions. One finds that $\tilde{\gamma}(\eta)$ is a series in even powers of η

$$\tilde{\gamma}(\eta) = -\frac{1 + \ln \eta}{\pi} + \frac{4(\ln \eta + 12)}{\pi} \eta^2 - \frac{6(62 \ln \eta + 777)}{\pi} \eta^4 + \dots, \tag{2.8}$$

while the expression for the conformal spin runs in odd powers of η

$$\tilde{s}(\eta) = \frac{1}{8\pi\eta} + \frac{11 + 2 \ln \eta}{2\pi} \eta - \frac{877 + 92 \ln \eta}{2\pi} \eta^3 + \dots. \tag{2.9}$$

From the equation for the pole of the integrand in (2.7), $\tilde{s} - \mathcal{S} = 0$, one can find the parameter η in terms of the spin \mathcal{S} , concluding that it is given by a power series in odd negative powers of \mathcal{S} . As a result, $\tilde{f}(\mathcal{S})$, which is same as $\tilde{\gamma}(\eta)$ evaluated at the pole, should also run only in even negative powers of \mathcal{S} or $\mathcal{C} = \frac{\mathcal{C}}{\sqrt{\lambda}}$ (cf. (1.20)).

The above discussion has a straightforward generalization to the multifolded spinning string case. The leading terms in the large \mathcal{S} expansion of the energy of a string with m folds are (see Appendix D)

$$\mathcal{E} - \mathcal{S} = \frac{m}{\pi} \left[\ln \bar{\mathcal{S}} - 1 + \frac{4}{\bar{\mathcal{S}}} (\ln \bar{\mathcal{S}} - 1) - \frac{4}{\bar{\mathcal{S}}^2} (\ln^2 \bar{\mathcal{S}} - 9 \ln \bar{\mathcal{S}} + 5) + \dots \right], \quad \bar{\mathcal{S}} \equiv \frac{8\pi}{m} \mathcal{S}. \tag{2.10}$$

In this case it is possible to show again that the large \mathcal{S} expansion is consistent with the reciprocity property.

2.2. Folded spinning string with $J \neq 0$

Let us now consider the case when the S^5 angular momentum of the string is not negligible compared to \mathcal{S} , i.e. when the string state is dual to an operator with large spin \mathcal{S} and large twist \mathcal{J} . The corresponding charges are the energy $E = \sqrt{\lambda} \mathcal{E}$ and the two angular momenta $S = \sqrt{\lambda} \mathcal{S}$ and $J = \sqrt{\lambda} \mathcal{J}$ [10,35]:

$$\mathcal{E} = \kappa + \frac{\kappa}{\omega} \mathcal{S}, \quad \frac{\omega^2 - \mathcal{J}^2}{\kappa^2 - \mathcal{J}^2} \equiv 1 + \eta, \tag{2.11}$$

$$\sqrt{\kappa^2 - \mathcal{J}^2} = \frac{2}{\pi\sqrt{\eta}} \mathbb{K}\left(-\frac{1}{\eta}\right), \quad \mathcal{S} = \frac{2\pi\sqrt{\eta}\omega}{\sqrt{\kappa^2 - \mathcal{J}^2}} \left[\mathbb{E}\left(-\frac{1}{\eta}\right) - \mathbb{K}\left(-\frac{1}{\eta}\right) \right]. \tag{2.12}$$

Here κ and ω (or η) are parameters of the classical solution which should we eliminated to find \mathcal{E} as a function of \mathcal{S} and \mathcal{J} .

²¹ This choice is not unique. An analogous transformation was used in [15].

We will be interested in large \mathcal{S} expansion with $\mathcal{S} \gg \mathcal{J}$ since only in this case the expansions like (1.16), (1.19), i.e. going in the inverse powers of \mathcal{S} with the coefficients being polynomials in $\ln \mathcal{S}$, will apply (see also [10,15]).

In the large $\mathcal{S} \gg \mathcal{J}$ or *long string* limit, when $\eta \ll 1$, one should distinguish between “small” or “large” \mathcal{J} cases [10,36]. In the “slow long string” approximation (corresponding to taking \mathcal{S} to be large with $\ell \equiv \frac{\mathcal{J}}{\ln \mathcal{S}}$ fixed and then expanding in powers of ℓ) the leading terms in the semiclassical energy read (cf. (2.3))

$$\begin{aligned} \mathcal{E} - \mathcal{S} - \mathcal{J} &\approx \frac{1}{\pi} (\ln \bar{\mathcal{S}} - 1) + \frac{\pi \mathcal{J}^2}{2 \ln \bar{\mathcal{S}}} - \frac{\pi^3 \mathcal{J}^4}{8 \ln^3 \bar{\mathcal{S}}} \left(1 - \frac{1}{\ln \bar{\mathcal{S}}} \right) + \dots \\ &+ \frac{4}{\bar{\mathcal{S}}} \left[\frac{1}{\pi} (\ln \bar{\mathcal{S}} - 1) + \frac{\pi \mathcal{J}^2}{2 \ln^2 \bar{\mathcal{S}}} - \frac{3\pi^3 \mathcal{J}^4}{4 \ln^4 \bar{\mathcal{S}}} \left(1 - \frac{2}{3 \ln \bar{\mathcal{S}}} \right) + \dots \right] \\ &- \frac{4}{\bar{\mathcal{S}}^2} \left[\frac{1}{\pi} (2 \ln^2 \bar{\mathcal{S}} - 9 \ln \bar{\mathcal{S}} + 5) \right. \\ &\left. + \pi \mathcal{J}^2 \left(1 + \frac{3}{2 \ln \bar{\mathcal{S}}} - \frac{1}{\ln^2 \bar{\mathcal{S}}} - \frac{2}{\ln^3 \bar{\mathcal{S}}} \right) + \dots \right], \end{aligned} \quad (2.13)$$

where $\bar{\mathcal{S}} \equiv 8\pi \mathcal{S}$, and dots stand for higher order corrections depending on \mathcal{J} .²²

In the case of “fast long string”, when $\ln \mathcal{S} \ll \mathcal{J} \ll \mathcal{S}$, the corrections to the energy read

$$\begin{aligned} \mathcal{E} - \mathcal{S} - \mathcal{J} &\approx \frac{1}{\pi^2 \mathcal{J}} \left[\frac{1}{2} \ln^2 \hat{\mathcal{S}} - \ln \hat{\mathcal{S}} + \frac{4 \ln \hat{\mathcal{S}}}{\hat{\mathcal{S}}} \right. \\ &\left. + \frac{4}{\hat{\mathcal{S}}^2} \left(-2 \ln \hat{\mathcal{S}} + 1 + \frac{3}{\ln \hat{\mathcal{S}}} + \frac{2}{\ln^2 \hat{\mathcal{S}}} + \dots \right) + \dots \right] \\ &+ \frac{1}{\pi^4 \mathcal{J}^3} \left[-\frac{\ln^4 \hat{\mathcal{S}}}{8} - \frac{2}{\hat{\mathcal{S}}} \left(3 \ln^2 \hat{\mathcal{S}} + \ln \hat{\mathcal{S}} + 1 + \frac{1}{\ln \hat{\mathcal{S}}} + \frac{1}{\ln^2 \hat{\mathcal{S}}} + \dots \right) \right. \\ &\left. - \frac{2}{\hat{\mathcal{S}}^2} \left(2 \ln^3 \hat{\mathcal{S}} - 19 \ln^2 \hat{\mathcal{S}} + 11 \ln \hat{\mathcal{S}} + 13 + \frac{13}{\ln \hat{\mathcal{S}}} + \frac{11}{\ln^2 \hat{\mathcal{S}}} + \dots \right) + \dots \right], \end{aligned} \quad (2.14)$$

where $\hat{\mathcal{S}} \equiv \frac{8\mathcal{S}}{\mathcal{J}} = \frac{8\mathcal{S}}{\mathcal{J}} \gg 1$. Dots in the square brackets indicate corrections in $1/\hat{\mathcal{S}}$, corrections in $1/\ln \hat{\mathcal{S}}$ can be added in the round brackets and terms like $\ln(\ln \hat{\mathcal{S}})$ have been neglected.

The leading terms here can be summed up as [3]

$$\mathcal{E} - \mathcal{S} = \sqrt{\mathcal{J}^2 + \frac{1}{\pi^2} \ln^2 \frac{8\mathcal{S}}{\mathcal{J}}} + \dots, \quad (2.15)$$

where $\frac{\ln \mathcal{S}}{\mathcal{J}} \ll 1$ plays the role of an expansion parameter.

Notice that in contrast to the slow long string case where the expansion (2.13) has the same structure as in (1.12), in the fast long string case (2.14) we get higher powers of $\ln \mathcal{S}$ not suppressed by \mathcal{S} , and so this case (cf. also its discussion in Appendix D) is somewhat outside our main theme here.

To study the properties of the subleading corrections, one may again make use of the integral representation for the functional relation as in (2.6). The discussion will apply to both the “slow”

²² Note that the leading terms in expression of the previous subsection (2.3) dominate in the limit when $\frac{\mathcal{J}^2}{\ln \mathcal{S}} \ll \frac{\ln \mathcal{S}}{\mathcal{S}}$.

and the “fast” long string limits. Here the “conformal spin” is $\tilde{s} = \frac{1}{2}(\mathcal{S} + \mathcal{E}) = \mathcal{S} + \frac{1}{2}\mathcal{J} + \frac{1}{2}\tilde{\gamma}$, while the “semiclassical” value of the Casimir operator in (1.20) is $\mathcal{C} = \frac{\mathcal{C}}{\sqrt{\lambda}} \approx \mathcal{S} + \frac{1}{2}\mathcal{J}$. Then the integral in (2.7) can be written as

$$\tilde{f}(\mathcal{C}) = \frac{1}{2\pi i} \oint_{\Gamma} d\eta \tilde{\gamma}(\eta) \frac{\tilde{s}'(\eta)}{\tilde{s}(\eta) - \mathcal{C}}, \quad \tilde{s}(\eta) = \mathcal{S}(\eta) + \frac{1}{2}\tilde{\gamma}(\eta). \tag{2.16}$$

After a redefinition of η one can then show that the expansion of f in large \mathcal{C} runs only in even negative powers of \mathcal{C} . Some details are given in Appendix D. In the kinematic region of “fast” long strings, with $1 \ll \ln \mathcal{S} \ll \mathcal{J} \ll \mathcal{S}$, this parity invariance property was already demonstrated in a closely related way in [15].

2.3. Large spin expansion of energy of a spiky string in AdS₅

Let us now consider the spiky spinning string in AdS₅ [34], and find corrections to the leading $\ln S$ term in its large spin expansion.

The integrals of motion here are the difference between the position of the spike and of the middle of the valley between the two spikes, the spin and the energy [34]²³

$$\Delta\theta = \frac{\pi}{n} = \frac{\sinh 2\rho_0}{\sqrt{2} \sinh \rho_1} \frac{1}{\sqrt{u_1 + u_0}} \left[\Pi\left(\frac{\pi}{2}, \frac{u_1 - u_0}{u_1 - 1}, p\right) - \Pi\left(\frac{\pi}{2}, \frac{u_1 - u_0}{u_1 + 1}, p\right) \right], \tag{2.17}$$

$$\mathcal{S} = \frac{n \cosh \rho_1}{\sqrt{2} \pi \sqrt{u_1 + u_0}} \left[-(1 + u_0)\mathbb{K}(p) + (u_1 + u_0)\mathbb{E}(p) - \frac{u_0^2 - 1}{u_1 + 1} \Pi\left(\frac{\pi}{2}, \frac{u_1 - u_0}{u_1 + 1}, p\right) \right], \tag{2.18}$$

$$\mathcal{E} - \omega\mathcal{S} = \frac{n\sqrt{u_1 + u_0}}{\sqrt{2} \pi \sinh \rho_1} [\mathbb{K}(p) - \mathbb{E}(p)], \tag{2.19}$$

where n is the number of the spikes and

$$u_0 = \cosh 2\rho_0, \quad u_1 = \cosh 2\rho_1, \quad \omega = \coth \rho_1, \quad p = \sqrt{\frac{u_1 - u_0}{u_1 + u_0}}. \tag{2.20}$$

The string is rigidly rotating with the radial coordinate being $\rho = \rho(\sigma)$, with ρ_0 and ρ_1 as its minimal and maximal values (positions of the bottom of the valley between the spikes and the spikes themselves). ρ_0 and ρ_1 are related by the condition (2.17). Solving for the remaining free parameter gives $\mathcal{E} = \mathcal{E}(\mathcal{S}, n)$.

The large spin limit corresponds to $\rho_1 \rightarrow \infty$, i.e. to the case when the ends of the spikes approach the boundary of AdS₅. Let us set

$$y = e^{-2\rho_1}, \tag{2.21}$$

and expand in $y \rightarrow 0$. Then, at leading order, $\Delta\theta = \frac{\pi}{n} = \arcsin \frac{1}{u_0} + \mathcal{O}(y)$ implies $u_0 = \cosh 2\rho_0 = \csc \frac{\pi}{n}$ and

²³ In the case of the multiply folded string with n spikes multiplying formulas one should multiply (2.19) by the number m of the folds, and use that in this case $\Delta\theta = \frac{\pi}{nm}$. As a result, one is simply to substitute $n \rightarrow nm$.

$$\mathcal{E} - \mathcal{S} = -\frac{n}{2\pi} \ln y + \mathcal{O}(y), \quad \mathcal{S} = \frac{n}{4\pi} \frac{1}{y} + \mathcal{O}(\ln y), \tag{2.22}$$

i.e.

$$\mathcal{E} - \mathcal{S} = \frac{n}{2\pi} \ln \frac{16\pi \mathcal{S}}{n} + \dots \tag{2.23}$$

This is the result already found in [34], which reduces to the case of the folded string when $n = 2$.²⁴ Expanding further near $y \simeq 0$ one gets

$$\mathcal{S} = \frac{n}{4\pi} \left(\frac{1}{y} + \ln y + 1 - 2\sqrt{u_0^2 - 1} \arccos \sqrt{\frac{u_0 + 1}{2u_0}} + \ln \frac{u_0}{4} \right) + \dots, \tag{2.24}$$

$$\Delta\theta = \frac{\pi}{n} = \arcsin \frac{1}{u_0} + y \left(2 \arcsin \sqrt{\frac{u_0 + 1}{2u_0}} - \pi \right) + \dots, \tag{2.25}$$

where the second equation can be used to fix u_0 in terms of y and the number of spikes n . Eliminating then y in favor of \mathcal{S} , we have from (2.19)

$$\begin{aligned} \mathcal{E} - \mathcal{S} = \frac{n}{2\pi} \left[\ln \bar{\mathcal{S}} + p_1 + \frac{4}{\bar{\mathcal{S}}} (\ln \bar{\mathcal{S}} + p_2) - \frac{4}{\bar{\mathcal{S}}^2} (2 \ln^2 \bar{\mathcal{S}} + p_3 \ln \bar{\mathcal{S}} + p_4) \right. \\ \left. + \frac{32}{3\bar{\mathcal{S}}^3} (2 \ln^3 \bar{\mathcal{S}} + p_5 \ln^2 \bar{\mathcal{S}} + p_6 \ln \bar{\mathcal{S}} + p_7) + \dots \right], \end{aligned} \tag{2.26}$$

where

$$\bar{\mathcal{S}} = \frac{16\pi}{n} \mathcal{S}, \tag{2.27}$$

$$p_1 = -1 + \ln \sin \frac{\pi}{n}, \quad p_2 = -1 + \ln \sin \frac{\pi}{n} + \frac{\pi(n-2)}{2n} \cot \frac{\pi}{n}, \tag{2.28}$$

$$p_3 = -10 + \frac{2\pi(n-2)}{n} \cot \frac{\pi}{n} - 2 \cot^2 \frac{\pi}{n} - 4 \ln \csc \frac{\pi}{n} + \csc^2 \frac{\pi}{n},$$

$$\begin{aligned} p_4 = 6 - \csc^2 \frac{\pi}{n} + \frac{\pi^2(n-2)^2}{2n^2} - \frac{4\pi(n-2)}{n} \cot \frac{\pi}{n} + \cot^2 \frac{\pi}{n} \left[\frac{\pi^2(n-2)^2}{n^2} + 1 \right] \\ + \ln \csc \frac{\pi}{n} \left[2 \cot^2 \frac{\pi}{n} - \frac{2\pi(n-2)}{n} \cot \frac{\pi}{n} - \csc^2 \frac{\pi}{n} + 2 \ln \csc \frac{\pi}{n} + 10 \right], \end{aligned} \tag{2.29}$$

$$p_5 = -18 + \mathcal{O}(n-2), \quad p_6 = 33 + \mathcal{O}(n-2), \quad p_7 = -14 + \mathcal{O}(n-2). \tag{2.30}$$

It is easy to check that (2.26) coincides with the energy (2.3) for the folded string in AdS_5 when $n = 2$.²⁵

Retaining in (2.26) only the dominant contributions at each order of the above expansion we obtain

$$\mathcal{E} - \mathcal{S} = \frac{n}{2\pi} \ln \mathcal{S} + \frac{n^2}{8\pi^2 \mathcal{S}} \ln \mathcal{S} - \frac{n^3}{64\pi^3 \mathcal{S}^2} \ln^2 \mathcal{S} + \frac{n^4}{384\pi^4 \mathcal{S}^3} \ln^3 \mathcal{S} + \dots \tag{2.31}$$

This may be rewritten as

²⁴ For $n = 2$ we have $\Delta\theta = \frac{\pi}{2}$ (i.e. the angle between spikes is π), and thus $\rho_0 = 0$ or $u_0 = 1$.

²⁵ Note also that the form of p_1 is consistent with the interpretation of the subleading term in the energy in [27].

$$E - S = \frac{\sqrt{\lambda} n}{2\pi} \ln \left[S + \frac{1}{2} \frac{\sqrt{\lambda} n}{2\pi} \ln S \right] + \dots, \quad (2.32)$$

implying that the functional relation is satisfied (cf. (2.4)).

However, the reciprocity property is not respected in this case. Indeed, the analog of the function $\tilde{f}(S)$ in (2.5) has the following expansion

$$\tilde{f}(S) = \frac{n}{2\pi} \left[\ln \bar{S} + q_1 + \frac{q_2}{\bar{S}} + \frac{1}{\bar{S}^2} (q_3 \ln \bar{S} + q_4) + \frac{1}{\bar{S}^3} (q_5 \ln \bar{S} + q_6) \dots \right] + \dots, \quad (2.33)$$

where

$$q_1 = -1 + \ln \sin \frac{\pi}{n}, \quad q_2 = \frac{2\pi(n-2)}{n} \cot \frac{\pi}{n}, \quad q_3 = 4 \csc^2 \frac{\pi}{n}, \quad (2.34)$$

$$q_4 = 4 + 2\pi^2 \left(\frac{n-2}{n} \right)^2 \left(1 - 2 \csc^2 \frac{\pi}{n} \right) + 4 \ln \sin \frac{\pi}{n} \csc^2 \frac{\pi}{n}, \quad (2.35)$$

$$q_5 = \mathcal{O}(n-2), \quad q_6 = \mathcal{O}(n-2), \quad (2.36)$$

where q_5, q_6 are non-zero for $n \neq 2$. The expansion (2.33), even if considerably simpler compared to the energy (2.26), is not parity invariant under $S \rightarrow -S$. It is interesting though that higher powers of $\ln S$ appear to cancel in the subleading terms in (2.33).²⁶ The parity invariance is restored in the case of the folded string when $n = 2$, where indeed (2.33) coincides with (2.5).

This breakdown of parity invariance for a string with $n > 2$ spikes is not totally surprising, as such spiky string should correspond to an operator with *non-minimal* anomalous dimension for a given spin, while the reciprocity was checked at weak coupling only for the minimal anomalous dimensions. Indeed, anomalous dimensions of operators of twist higher than two with trajectories close to the upper boundary of the band also do not respect the reciprocity as was seen recently in the twist three case at weak coupling in [4].

It is interesting that our strong-coupling result (2.32), (2.33) has close similarity with weak-coupling one found for $n = 3$ in [4]: the functional relation (2.32) is still satisfied, and the parity invariance is broken at level $1/S$. Interestingly, the $1/S$ coefficient $\sim nq_2$ in (2.33) (cf. (2.27)) is proportional, for $n = 3$, to $\sqrt{3}$, which is, surprisingly, the same factor appearing also in the corresponding expression at weak coupling [4].²⁷ In general, this coefficient should be a function of λ interpolating from weak to strong coupling but its dependence on n might be the same for any λ .

3. Large spin expansion of folded string energy: 1-loop order

Let us now go back to the folded spinning string case of Section 2.1 and compute the leading 1-loop corrections to its energy (2.3) in the large S expansion. We shall follow the general approach for computation of quantum string corrections developed in [10] where the 1-loop shift of the $\ln S$ term was found.²⁸ We shall find the 1-loop corrections to the subleading terms in (2.3)

²⁶ This feature of the \tilde{f} -function is in a marked contrast with the anomalous dimension, whose large S expansion includes growing powers of $\ln S$ in the coefficients of $1/S^n$ terms. This reduction of singularity of the large S expansion of \tilde{f} was observed also at weak coupling [4,18,26].

²⁷ We thank G. Korchemsky for this observation.

²⁸ The 2-loop correction to the scaling function was found in [38,39]; a generalization to non-zero J was considered in [36,40] (parallel results from the Bethe ansatz were found in [41,42]).

by applying a perturbative procedure similar to the one used in [9] in the small spin expansion case.

Our aim will be to verify that: (i) the structure of the large spin expansion (2.3) remains the same also with the 1-loop corrections included, and (ii) the constraints on the coefficients imposed by the functional relation and the reciprocity remain to be satisfied at the 1-loop order.

The fluctuation action in the conformal gauge expanded to quadratic order in fluctuations near the folded spinning string solution $\tilde{S} = -\frac{\sqrt{\lambda}}{4\pi} \int d\tau \int_0^{2\pi} d\sigma \tilde{L}$ has the following bosonic part (see [10] and Appendix B)

$$\begin{aligned} \tilde{L}_B = & -\partial_a \tilde{t} \partial^a \tilde{t} - \mu_{\tilde{t}}^2 \tilde{t}^2 + \partial_a \tilde{\phi} \partial^a \tilde{\phi} + \mu_{\tilde{\phi}}^2 \tilde{\phi}^2 \\ & + 4\tilde{\rho}(\kappa \sinh \rho \partial_0 \tilde{t} - w \cosh \rho \partial_0 \tilde{\phi}) + \partial_a \tilde{\rho} \partial^a \tilde{\rho} + \mu_{\tilde{\rho}}^2 \tilde{\rho}^2 \\ & + \partial_a \beta_u \partial^a \beta_u + \mu_{\tilde{\beta}}^2 \beta_u^2 + \partial_a \varphi \partial^a \varphi + \partial_a \zeta_s \partial^a \zeta_s, \end{aligned} \tag{3.1}$$

where

$$\mu_{\tilde{t}}^2 = 2\rho'^2 - \kappa^2, \quad \mu_{\tilde{\phi}}^2 = 2\rho'^2 - w^2, \quad \mu_{\tilde{\rho}}^2 = 2\rho'^2 - w^2 - \kappa^2, \quad \mu_{\tilde{\beta}}^2 = 2\rho'^2. \tag{3.2}$$

Here β_u ($u = 1, 2$) are the two AdS_5 fluctuations transverse to the AdS_3 subspace in which the string is moving, while φ, ζ_s ($s = 1, 2, 3, 4$) are fluctuations in S^5 . The fermionic part of the quadratic fluctuation Lagrangian can be put into the form [10]

$$\tilde{L}_F = 2i(\bar{\Psi} \gamma^a \partial_a \Psi - \mu_F \bar{\Psi} \Gamma_{234} \Psi), \quad \mu_F = \rho', \tag{3.3}$$

and can be interpreted as describing a system of $4 + 4$ 2d Majorana fermions with σ -dependent mass ρ' . As explained in [9], after squaring the corresponding “Dirac” operator, the fermionic contribution to the 1-loop partition function can be represented as

$$-\frac{1}{2}(4 \ln \det \Delta_{F+} + 4 \ln \det \Delta_{F-}), \quad \Delta_{F\pm} \equiv -\partial^a \partial_a \pm \rho'' + \rho'^2. \tag{3.4}$$

In the leading-order computation in the long-string limit that we are going to discuss below the term $\pm \rho''$ in the effective fermionic mass squared term in (3.4) can be ignored: as follows from (B.12), $\rho'' = O(\eta)$ and since according to (3.4) half of fermions has $+\rho''$ and half $-\rho''$ in their mass term, the leading $O(\eta)$ contribution to the partition function can come only from the ρ'^2 term in $\Delta_{F\pm}$. We shall assume this when writing the fermionic contribution below.

Switching to Euclidean signature ($\tau \rightarrow i\tau$), the 1-loop correction to the energy can be found from the 2d effective action

$$E_1 = \frac{\Gamma_1}{\kappa T}, \quad \mathcal{T} \equiv \int d\tau \rightarrow \infty. \tag{3.5}$$

Since the spinning string solution is stationary, both the bosonic and the fermionic fluctuation Lagrangians do not depend on τ ; thus, as in [9], we may compute the relevant 2d functional determinants by reducing them to 1d functional determinants using

$$\det[-\partial_1^2 - \partial_0^2 + m^2] = \mathcal{T} \int \frac{d\omega}{2\pi} \det[-\partial_1^2 + \omega^2 + m^2], \tag{3.6}$$

where m^2 is a generic mass term which may depend on σ .

Given that $\rho(\sigma)$ is a complicated function (see (B.4)), we are unable to determine the fluctuation spectrum exactly, and, as in [9], we will resort to perturbation theory in $\frac{1}{\mathcal{T}}$ or in parameter

η determining the maximal string length (see Appendices B and C). In (3.5) we have from (B.6)

$$\kappa = \kappa_0 - \frac{\eta}{4\pi}(\pi\kappa_0 - 2) + \mathcal{O}(\eta^2), \quad \kappa_0 \equiv \frac{1}{\pi} \ln \frac{16}{\eta}. \quad (3.7)$$

Γ_1 will also be expected to have expansion in powers of $\eta \sim \frac{1}{S}$ (see (B.9)) with the coefficients counting powers of $\ln \eta$.

To proceed, we need to expand the fluctuation Lagrangian in small η corresponding to large S . Some relations needed below can be found in Appendix B. Let us first perform (as in [36]) the following rotation $(\tilde{t}, \tilde{\phi}) \rightarrow (\xi, \chi)$:

$$\xi = -\tilde{t} \sinh \rho + \tilde{\phi} \cosh \rho, \quad \chi = -\tilde{\phi} \sinh \rho + \tilde{t} \cosh \rho. \quad (3.8)$$

Then the fluctuation Lagrangian takes the form

$$\begin{aligned} \tilde{L}_B = & -\partial_a \chi \partial^a \chi + (\mu_\phi^2 \sinh^2 \rho - \mu_t^2 \cosh^2 \rho + \rho'^2) \chi^2 + \partial_a \xi \partial^a \xi \\ & + (\mu_\phi^2 \cosh^2 \rho - \mu_t^2 \sinh^2 \rho - \rho'^2) \xi^2 + 4\tilde{\rho}(\kappa \sinh^2 \rho - w \cosh^2 \rho) \partial_0 \xi \\ & + \partial_a \tilde{\rho} \partial^a \tilde{\rho} + \mu_\rho^2 \tilde{\rho}^2 + 2\rho'(\chi \xi' - \xi \chi') + \chi \xi (\mu_\phi^2 - \mu_t^2) \sinh 2\rho \\ & + 2\tilde{\rho} \dot{\chi} (\kappa - w) \sinh 2\rho + \partial_a \beta_u \partial^a \beta_u + \mu_\beta^2 \beta_u^2 + \partial_a \varphi \partial^a \varphi + \partial_a \zeta_s \partial^a \zeta_s. \end{aligned} \quad (3.9)$$

The reason for this rotation is that in the subsequent small η expansion the bosonic fluctuation Lagrangian at order $\mathcal{O}(\eta^0)$ will become σ -independent, i.e. will have constant coefficients as at the leading order in long-string expansion considered in [10,36].²⁹

Expanding the solution for $\rho(\sigma)$ and the parameters κ and w in small η (see Appendix B), the bosonic fluctuation Lagrangian becomes $\tilde{L}_B = \tilde{L}_0 + \eta \tilde{L}_1 + \dots$, where

$$\begin{aligned} \tilde{L}_0 = & -\partial_a \chi \partial^a \chi + \partial_a \xi \partial^a \xi + 2\kappa_0 \chi \xi' - 2\kappa_0 \chi' \xi - 4\kappa_0 \tilde{\rho} \dot{\xi} \\ & + \partial_a \tilde{\rho} \partial^a \tilde{\rho} + \partial_a \beta_u \partial^a \beta_u + 2\kappa_0^2 \beta_u^2 + \partial_a \varphi \partial^a \varphi + \partial_a \zeta_s \partial^a \zeta_s, \end{aligned} \quad (3.10)$$

and

$$\begin{aligned} \tilde{L}_1 = & -\kappa_0^2 \cosh(2\kappa_0 \sigma) \xi^2 - \kappa_0^2 \cosh(2\kappa_0 \sigma) \tilde{\rho}^2 - \kappa_0^2 \sinh(2\kappa_0 \sigma) \xi \chi \\ & - \frac{\kappa_0}{\pi} [\kappa_0 \pi \cosh(2\kappa_0 \sigma) - 2] \beta_u^2 + (\chi \xi' - \xi \chi') \left[\frac{1}{\pi} - \frac{\kappa_0}{2} \cosh(2\kappa_0 \sigma) \right] \\ & - \tilde{\rho} \dot{\chi} \kappa_0 \sinh(2\kappa_0 \sigma) - \tilde{\rho} \dot{\xi} \left[\frac{2}{\pi} + \kappa_0 \cosh(2\kappa_0 \sigma) \right]. \end{aligned} \quad (3.11)$$

As already mentioned, the 1-loop effective action can be expressed in terms of 1d functional determinants (with $\partial_0 \rightarrow i\omega$, see (3.6)). We shall denote the quadratic fluctuation operator in the coupled $(\chi, \xi, \tilde{\rho})$ sector as Q_ω . Since \tilde{L}_0 has constant coefficients, the leading part of the fluctuation operator coming from \tilde{L}_0 can be written as

$$Q_\omega^{(0)} = \begin{pmatrix} -(-\partial_1^2 + \omega^2) & 2\kappa_0 \partial_1 & 0 \\ -2\kappa_0 \partial_1 & -\partial_1^2 + \omega^2 & -2\omega \kappa_0 \\ 0 & 2\omega \kappa_0 & -\partial_1^2 + \omega^2 \end{pmatrix}. \quad (3.12)$$

The 1-loop correction to the effective action is then

²⁹ As discussed below, we shall ignore the contribution of the turning points at $\sigma = \frac{\pi}{2}$ and $\frac{3\pi}{2}$ and will treat the fluctuation problem separately on each “quarter-string” interval.

$$\Gamma_1 = \frac{\mathcal{T}}{4\pi} \int_{-\infty}^{\infty} d\omega \left[-8 \ln \frac{\det[-\partial_1^2 + \omega^2 + \rho'^2]}{\det[-\partial_1^2 + \omega^2 + \kappa_0^2]} + 2 \ln \frac{\det[-\partial_1^2 + \omega^2 + 2\rho'^2]}{\det[-\partial_1^2 + \omega^2 + 2\kappa_0^2]} \right. \\ \left. - \ln \frac{\det^8[-\partial_1^2 + \omega^2 + \kappa_0^2]}{\det^2[-\partial_1^2 + \omega^2 + 2\kappa_0^2] \det^6[-\partial_1^2 + \omega^2]} + \ln \frac{\det Q_\omega}{\det Q_\omega^{(0)}} - \ln \frac{\det P_\omega}{\det Q_\omega^{(0)}} \right], \quad (3.13)$$

where

$$P_\omega = \begin{pmatrix} -(-\partial_1^2 + \omega^2) & 0 & 0 \\ 0 & -\partial_1^2 + \omega^2 & 0 \\ 0 & 0 & -\partial_1^2 + \omega^2 \end{pmatrix}. \quad (3.14)$$

Also, $Q_\omega = Q_\omega^{(0)} + \eta Q_\omega^{(1)} + \dots$, where $Q_\omega^{(1)}$ is the next to leading order coupled operator from (3.11)

$$Q_\omega^{(1)} = \begin{pmatrix} 0 & Q_{12} & Q_{13} \\ Q_{21} & -\kappa_0^2 \cosh(2\kappa_0\sigma) & Q_{23} \\ -Q_{13} & -Q_{23} & -\kappa_0^2 \cosh(2\kappa_0\sigma) \end{pmatrix}. \quad (3.15)$$

Here

$$Q_{12} = -\frac{\kappa_0^2}{2} \sinh(2\kappa_0\sigma) + in \left[\frac{1}{\pi} - \frac{\kappa_0}{2} \cosh(2\kappa_0\sigma) \right], \quad (3.16)$$

$$Q_{21} = -\frac{\kappa_0^2}{2} \sinh(2\kappa_0\sigma) - in \left[\frac{1}{\pi} - \frac{\kappa_0}{2} \cosh(2\kappa_0\sigma) \right], \quad (3.17)$$

$$Q_{13} = -\frac{\kappa_0\omega}{2} \sinh(2\kappa_0\sigma), \quad Q_{23} = -\omega \left[\frac{1}{\pi} + \frac{\kappa_0}{2} \cosh(2\kappa_0\sigma) \right] \quad (3.18)$$

and we performed the Fourier transform in σ , i.e. replaced $\partial_1 \rightarrow in$, $n = 0, \pm 1, \dots$, as appropriate for fluctuation fields which are 2π periodic in σ .

Our aim will be to determine the 1-loop correction to string energy to order η by computing

$$\Gamma_1 = \Gamma_1^{(0)} + \Gamma_1^{(1)} + \mathcal{O}(\eta^2), \quad \Gamma_1^{(1)} = \mathcal{O}(\eta). \quad (3.19)$$

As in [9] the first, second and fourth terms in (3.13) can be computed to order $\mathcal{O}(\eta)$ using that

$$\ln \frac{\det[O^{(0)} + \eta O^{(1)}]}{\det O^{(0)}} = \eta \text{Tr}[(O^{(0)})^{-1} O^{(1)}] + \mathcal{O}(\eta^2). \quad (3.20)$$

While in [9] a similar contribution to the effective action happened to vanish since it was proportional to the sum of squares of fluctuation masses,³⁰ here this leading term is no longer zero as in the present case the expansion is around a non-trivial string background with different propagators for different string fluctuations.

In (3.11) we used the expansion (B.11) of the solution $\rho(\sigma)$ in small η . As discussed in Appendices B and C, this expansion breaks down at the turning points where subleading terms are of the same order as the leading term. As in the computation of the leading order in [10], here we shall assume that one can ignore the contributions from the turning points. The classical folded string solution is built out of four parts making up the closed string (e.g., the expansion

³⁰ The mass sum rule implies the 1-loop UV finiteness of the superstring; it was proven in general for any string solution in [43].

(B.11) used in (3.11) is defined for $0 \leq \sigma \leq \frac{\pi}{2}$.³¹ The closed string fluctuations by definition must be periodic in $0 \leq \sigma \leq 2\pi$.

We shall assume that we can treat the problem “piece-wise” also at the fluctuation level. Direct implementation of this may effectively bring back the turning-point contributions. We shall parametrize our current lack of control of such terms by including the possible contribution with an arbitrary coefficient in the final result.

We shall split the integral over σ as follows

$$\int_0^{2\pi} \frac{d\sigma}{2\pi} \rightarrow \frac{1}{2\pi} \left[\int_0^{\frac{\pi}{2}} d\sigma + \int_{\frac{\pi}{2}}^{\pi} d\sigma + \int_{\pi}^{\frac{3\pi}{2}} d\sigma + \int_{\frac{3\pi}{2}}^{2\pi} d\sigma \right]. \tag{3.21}$$

Considering the first interval $(0, \frac{\pi}{2})$, the order η contribution of the decoupled boson β_u in (3.1) and (3.10), (3.11) can be obtained as

$$\begin{aligned} & \left(\ln \frac{\det[-\partial_1^2 + \omega^2 + 2\rho'^2]}{\det[-\partial_1^2 + \omega^2 + 2\kappa_0^2]} \right)^{(1)} \\ &= -\frac{\eta\kappa_0}{\pi} \sum_{n=-\infty}^{\infty} \frac{1}{n^2 + \omega^2 + 2\kappa_0^2} \int_0^{\frac{\pi}{2}} \frac{d\sigma}{2\pi} [\pi\kappa_0 \cosh(2\kappa_0\sigma) - 2] \\ &= -\frac{\eta\kappa_0}{4\pi} \sum_{n=-\infty}^{\infty} \frac{\sinh(\pi\kappa_0) - 2}{n^2 + \omega^2 + 2\kappa_0^2}. \end{aligned} \tag{3.22}$$

Similarly, for the fermionic contribution (the first term in (3.13)) we get

$$\begin{aligned} & \left(\ln \frac{\det[-\partial_1^2 + \omega^2 + \rho'^2]}{\det[-\partial_1^2 + \omega^2 + \kappa_0^2]} \right)^{(1)} = -\frac{\eta\kappa_0}{2\pi} \sum_{n=-\infty}^{\infty} \frac{1}{n^2 + \omega^2 + \kappa_0^2} \int_0^{\frac{\pi}{2}} \frac{d\sigma}{2\pi} [\pi\kappa_0 \cosh(2\kappa_0\sigma) - 2] \\ &= -\frac{\eta\kappa_0}{8\pi} \sum_{n=-\infty}^{\infty} \frac{\sinh(\pi\kappa_0) - 2}{n^2 + \omega^2 + \kappa_0^2}. \end{aligned} \tag{3.23}$$

For the coupled part one finds

$$\begin{aligned} & \left(\ln \frac{\det Q_\omega}{\det Q_\omega^{(0)}} \right)^{(1)} = \eta \int_0^{\frac{\pi}{2}} \frac{d\sigma}{2\pi} \text{Tr}[(Q_\omega^{(0)})^{-1} Q_\omega^{(1)}] \\ &= \frac{\eta\kappa_0}{\pi} \sum_{n=-\infty}^{\infty} \frac{(n^2 + \omega^2)^2 - n^2(n^2 + \omega^2 + \kappa_0^2) \sinh(\pi\kappa_0)}{(n^2 + \omega^2)^2 (n^2 + \omega^2 + 4\kappa_0^2)}. \end{aligned} \tag{3.24}$$

The contributions of the other three intervals of σ are the same.

Collecting the above results we observe that the final expression for the order η term in the effective action $\Gamma_1^{(1)}$ is UV finite. Moreover, the part that does not contain the $\sinh(\pi\kappa_0)$ factor is IR finite, i.e. the non-trivial potentially IR divergent contributions of the two unphysical

³¹ Note that for the second and the fourth σ intervals where ρ decreases we need to use the minus sign in (B.2).

AdS_5 massless modes (χ, ξ) (time-like and longitudinal) that appear in the coupled part of the fluctuation Lagrangian cancel.³²

Explicitly, integrating first over ω we obtain³³ the order η contribution to the 1-loop effective action (3.19) coming from (3.20) as

$$\Gamma_1^{(1)} = -\frac{\mathcal{T}\eta}{4\pi} \sum_{n=-\infty}^{\infty} [A_n + C_n \sinh(\pi\kappa_0)], \quad (3.25)$$

where

$$A_n = \frac{8\kappa_0}{\sqrt{n^2 + \kappa_0^2}} - \frac{4\kappa_0}{\sqrt{n^2 + 2\kappa_0^2}} - \frac{4\kappa_0}{\sqrt{n^2 + 4\kappa_0^2}}, \quad (3.26)$$

$$C_n = \frac{\kappa_0}{2n} + \frac{3n}{4\kappa_0} - \frac{4\kappa_0}{\sqrt{n^2 + \kappa_0^2}} + \frac{2\kappa_0}{\sqrt{n^2 + 2\kappa_0^2}} - \frac{3n^2}{4\kappa_0\sqrt{n^2 + 4\kappa_0^2}}. \quad (3.27)$$

The coefficient of the part proportional to $\sinh(\pi\kappa_0)$ given by $\sum_n C_n$ is UV finite but formally has an IR singular contribution.³⁴ This term should be an artifact of our computational procedure related to the problem with expansion in η in (B.11) near the turning points (see Appendices B and C). Insisting on omitting the turning point contributions means that we should drop this IR singular $\sim \sinh(\pi\kappa_0)$ term, and this is what we will do below.³⁵ We believe that in a more systematic treatment that consistently treats the turning point contributions such terms will be automatically absent (equivalently, in our present form of the expansion, such terms should resum away, see also the discussion in Appendix C).

Computing the remaining $\sum_n A_n$ contribution in (3.25), (3.26) using the Euler–MacLaurin formula

$$\sum_{n=1}^{\infty} f(n) = \int_1^{\infty} dn f(n) + \frac{f(1) + f(\infty)}{2} + \sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)!} [f^{(2k-1)}(\infty) - f^{(2k-1)}(1)], \quad (3.28)$$

we can extract its large κ_0 behaviour

$$\sum_{n=-\infty}^{\infty} A_n = 12\kappa_0 \ln 2 + \mathcal{O}(e^{-\kappa_0}). \quad (3.29)$$

The contribution of this term in $\Gamma_1^{(1)}$ in (3.25) to the energy (3.5) is then (using (B.9))

$$E_1^{(1)} = -\frac{3 \ln 2}{\pi} \frac{\kappa_0}{\kappa} \eta. \quad (3.30)$$

³² Their flat-space contribution is cancelled against the conformal gauge ghost contribution.

³³ Let us mention that if we perform the sum over n first, then using the residue theorem in the integral over ω we arrive back at the same sum as below.

³⁴ The IR singular contribution goes away if one separates it before doing the integral over ω . Then we get for the large κ_0 behaviour of $\sum_n C_n$: $2 \sum_{n=1}^{\infty} C_n = \kappa_0 (\ln \kappa_0 - 6 \ln 2 - \frac{3}{2} + \gamma_E) + \mathcal{O}((\kappa_0)^0)$.

³⁵ To justify the expansion in (B.11) we need to omit the turning point contribution. That can be done by shifting the upper limit of the integration over σ in (3.22)–(3.24) to $\frac{\pi}{2} - \epsilon$, $\epsilon \rightarrow +0$. Then the coefficient of C_n will become $\sinh[(\pi - 2\epsilon)\kappa_0] \sim \eta^{-1 + \frac{2}{\pi}\epsilon}$ and thus is subleading compared to the contribution of order $\mathcal{O}(\eta^0)$.

Let us now include the $\mathcal{O}(\eta^0)$ contribution to Γ_1 (3.19) coming from the third and fifth terms in (3.13). Since $Q_\omega^{(0)}$ in (3.12) has no σ dependence, its functional determinant

$$\det Q_\omega^{(0)} = -\det^2(-\partial_1^2 + \omega^2) \det(-\partial_1^2 + \omega^2 + 4\kappa_0^2) \tag{3.31}$$

can be easily computed as a product over integer n of a matrix determinant (after $\partial_1 \rightarrow in$). Since $\det P_\omega = -\det^3(-\partial_1^2 + \omega^2)$ we may write the relevant contribution from (3.13) as

$$\Gamma_1^{(0)} = -\frac{\mathcal{T}}{4\pi} \int_{-\infty}^{\infty} d\omega \ln \frac{\det^8(-\partial_1^2 + \omega^2 + \kappa_0^2)}{\det^2(-\partial_1^2 + \omega^2 + 2\kappa_0^2) \det^5(-\partial_1^2 + \omega^2) \det(-\partial_1^2 + \omega^2 + 4\kappa_0^2)}. \tag{3.32}$$

Since $\ln \det(-\partial_1^2 + \omega^2 + \kappa^2) = \sum_{n=-\infty}^{\infty} \ln(n^2 + \omega^2 + \kappa^2)$, doing the integral over ω we finally obtain the 1-loop correction to the string energy to order $\mathcal{O}(\eta)$ as

$$E_1^{(0)} = \frac{1}{2\kappa} \sum_{n=-\infty}^{\infty} [2\sqrt{n^2 + 2\kappa_0^2} + \sqrt{n^2 + 4\kappa_0^2} + 5\sqrt{n^2} - 8\sqrt{n^2 + \kappa_0^2}]. \tag{3.33}$$

This turns out to be a direct generalization of the leading-order result of [10] where κ should be replaced by κ_0 in the fluctuation mass terms (but not in the overall $\frac{1}{\kappa}$ factor due to $t = \kappa\tau$).

Using again the Euler–MacLaurin formula to transform the sum into an integral we find³⁶

$$E_1^{(0)} = \frac{1}{\kappa} \left[-3 \ln 2\kappa_0^2 - \frac{5}{12} + \mathcal{O}(e^{-\kappa_0}) \right]. \tag{3.34}$$

This is, of course, in agreement with the result of [10] and also, for the subleading term, with Ref. [44].³⁷

As we shall argue in Appendix C, an additional contribution that may come from near turning point regions can be parametrized as follows:

$$\Gamma_1^{(2)} = \frac{c}{\pi} \kappa_0 \mathcal{T}, \quad E_1^{(2)} = \frac{c}{\pi} \frac{\kappa_0}{\kappa}, \tag{3.35}$$

where c is an undetermined constant (we included factor of π for convenience).

Inverting the relation between \mathcal{S} and η in (B.9) to order $\mathcal{O}(\eta)$ we get

$$\eta = \frac{2}{\pi \mathcal{S}} - \frac{\ln(8\pi \mathcal{S}) - 3}{\pi^2 \mathcal{S}^2} + \dots, \tag{3.36}$$

which, plugged into (3.7), gives³⁸

$$\begin{aligned} \kappa &= \frac{\ln(8\pi \mathcal{S})}{\pi} - \frac{1}{2\pi^2 \mathcal{S}} + \dots, & \kappa_0 &= \frac{\ln(8\pi \mathcal{S})}{\pi} + \frac{\ln(8\pi \mathcal{S}) - 3}{2\pi^2 \mathcal{S}} + \dots, \\ \frac{\kappa_0}{\kappa} &= 1 + \frac{1}{2\pi \mathcal{S}} + \dots. \end{aligned} \tag{3.37}$$

³⁶ Note that the sum of (3.26) is minus the derivative over κ_0 of the sum in (3.33), which explains why the corresponding coefficients are closely related.

³⁷ Ref. [44] considered, following [10], the formal sum (3.32) with $\kappa_0 \rightarrow \kappa$ and with the $n = 0$ term omitted (this term was omitted in [10] since there it was subleading in the infinite κ limit). As a result, the expression in [44] contained an extra (minus “zero mode”) term $3 - \sqrt{2}$.

³⁸ Note that in the expression for the energy in (3.5) we need to keep κ to order $\mathcal{O}(\eta)$ to get the correction in E_1 also to order $\eta \sim \frac{1}{\mathcal{S}}$.

This means that the dominant term in (3.34) is the $-3 \ln 2 \frac{\kappa_0^2}{\kappa}$ one: the other terms $-\frac{5}{12\kappa} \sim \frac{1}{\ln \mathcal{S}}$ and $\frac{1}{\kappa} e^{-\kappa_0} \sim \frac{1}{\ln \mathcal{S}^{1/\pi}}$ will be subleading and should be ignored in the approximation we considered above where we dropped terms of higher order in η at earlier stages.³⁹

Thus we find that the 1-loop correction to the folded string energy to order $\mathcal{O}(\frac{\ln^2 \mathcal{S}}{\mathcal{S}^2})$ can be written in the same form as the classical energy (2.3), i.e. as in (1.2), (1.3)

$$E_1 = b_0 \ln \mathcal{S} + b_c + \frac{b_{11} \ln \mathcal{S} + b_{10}}{\mathcal{S}} + \mathcal{O}\left(\frac{\ln^2 \mathcal{S}}{\mathcal{S}^2}\right). \tag{3.38}$$

The contribution from $E_1^{(0)}$ in (3.34) to the 1-loop coefficients is

$$\begin{aligned} b_0^{(0)} &= -\frac{3 \ln 2}{\pi}, & b_c^{(0)} &= -\frac{3 \ln 2}{\pi} \ln 8\pi, \\ b_{11}^{(0)} &= -\frac{3 \ln 2}{\pi^2}, & b_{10}^{(0)} &= -\frac{3 \ln 2}{\pi^2} \left(\ln 8\pi - \frac{5}{2} \right), \end{aligned} \tag{3.39}$$

while $E_1^{(1)}$ in (3.30) and $E_1^{(2)}$ in (3.35) contribute as

$$b_{10}^{(1)} = -\frac{6 \ln 2}{\pi^2}, \quad b_c^{(2)} = \frac{1}{\pi} c, \quad b_{10}^{(2)} = \frac{1}{2\pi^2} c. \tag{3.40}$$

Thus finally we obtain for the full 1-loop coefficients

$$b_0 = -\frac{3 \ln 2}{\pi}, \quad b_c = \frac{1}{\pi} (-3 \ln 2 \ln 8\pi + c), \tag{3.41}$$

$$b_{11} = -\frac{3 \ln 2}{\pi^2}, \quad b_{10} = \frac{1}{2\pi^2} \left[-6 \ln 2 \left(\ln 8\pi - \frac{1}{2} \right) + c \right]. \tag{3.42}$$

The functional and reciprocity relations in (1.18), (1.22) at strong coupling are (see discussion in Appendix E)

$$\tilde{f}_{11} = \frac{1}{2} \tilde{f}^2, \quad \tilde{f}_{10} = \frac{1}{2} \tilde{f} \tilde{f}_c, \quad \tilde{f} \equiv \frac{f}{\sqrt{\lambda}}, \quad \tilde{f}_c \equiv \frac{f_c}{\sqrt{\lambda}}, \quad \tilde{f}_{1k} \equiv \frac{f_{1k}}{\lambda}, \tag{3.43}$$

$$\tilde{f} = a_0 + \frac{b_0}{\sqrt{\lambda}} + \dots, \quad \tilde{f}_c = a_c + \frac{b_c}{\sqrt{\lambda}} + \dots, \quad \tilde{f}_{1k} = a_{1k} + \frac{b_{1k}}{\sqrt{\lambda}} + \dots. \tag{3.44}$$

They imply that the coefficients in (3.38) should obey (see (E.6), (E.7))

$$b_{11} = a_0 b_0, \quad b_{10} = \frac{1}{2} (a_0 b_c + b_0 a_c). \tag{3.45}$$

Recalling the values of the leading coefficients at the classical level in (2.3)

$$a_0 = \frac{1}{\pi}, \quad a_c = \frac{1}{\pi} (\ln 8\pi - 1), \tag{3.46}$$

we see that the relations (3.45) are indeed satisfied by the expressions in (3.41), i.e. the functional and the reciprocity relations appear to apply also including string 1-loop corrections. Note that this is true for any value of the undetermined coefficient c .

Needless to say, it would be interesting to generalize the 1-loop computation of this section and the check of reciprocity to the case of non-zero J and to attempt to relate the strong-coupling version of the reciprocity discussed in Appendix E to its weak-coupling finite twist one in (1.21).

³⁹ The role of these subleading terms and their possible resummation remains to be studied in more detail.

Note added

As we have learned (N. Gromov, private communication) an independent way of evaluating the 1-loop correction to the folded string energy based on the algebraic curve approach to extracting fluctuation frequencies [53] leads to the value $c = 6 \ln 2 + \pi$ for the undetermined coefficient in (1.6), (1.9), (1.11), (3.35), (3.41), (3.42). This c contribution changes the 1-loop coefficient in (1.6) from $3 \ln 2$ to $-3 \ln 2 - \pi$.

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Appendix A. Comments on conformal algebra realizations

Starting with a conformal theory in $R^{1,3}$ with the standard $SO(2, 4)$ conformal group generators P_m, M_{mn}, K_m, D ($m, n = 0, 1, 2, 3$) one may define the collinear $SL(2, R)$ subgroup as generated by the following light-cone components [30]:

$$L_+ \equiv -i P_+, \quad L_- \equiv \frac{i}{2} K_-, \quad L_0 \equiv \frac{i}{2} (D + M_{-+}), \tag{A.1}$$

$$[L_0, L_\pm] = \pm L_\pm, \quad [L_+, L_-] = -2L_0. \tag{A.2}$$

If the eigenvalue of D is dimension Δ and the eigenvalue of M_{-+} – the collinear projection of the Lorentz spin S , then the eigenvalue of L_0 is the conformal spin $s = \frac{1}{2}(\Delta + S)$. The corresponding quadratic Casimir operator is $C^2 = s(s - 1)$.

At the same time, in the $R^{2,4}$ embedding representation of the global AdS_5 space the generators Σ_{MN} ($M, N = 0, 1, 2, 3, 4, 5$ with the signature $- + + + -$) of $SO(2, 4)$ linearly realised on the embedding coordinates Y^M can be related to the standard boundary conformal group generators as (see, e.g., [45])

$$\Sigma_{mn} = M_{mn}, \quad \Sigma_{m4} = \frac{1}{2}(K_m - P_m), \quad \Sigma_{m5} = \frac{1}{2}(K_m + P_m), \quad \Sigma_{54} = D. \tag{A.3}$$

Then the standard spin is $S = \Sigma_{12} = M_{12}$, the conformal spin is $S' = \Sigma_{34} = \frac{1}{2}(K_3 - P_3)$, and the conformal energy is the rotation generator in the 05 plane, i.e. the global AdS_5 energy, $E = \Sigma_{05} = \frac{1}{2}(K_0 + P_0)$.

In general, the energy E of a string state in global AdS_5 space with boundary $R \times S^3$ should be equal to the energy of the corresponding SYM state on $R \times S^3$. Through radial quantization (and analytic continuation) this state may be associated to a local operator in $R^{1,3}$ that creates it. The AdS_5 energy $E = \Sigma_{05}$ or conformal Hamiltonian generates an $SO(2)$ subgroup while the

dilatation operator $D = \Sigma_{54}$ generates an $SO(1, 1)$ subgroup of $SO(2, 4)$.⁴⁰ After the Euclidean continuation of the embedding coordinate $Y_0 \rightarrow iY_{0E}$ (to allow for the mapping from $R \times S^3$ to R^4) one may exchange Y_{0E} with Y_4 which exchanges the generator $\Sigma_{54} = D$ with $E = \Sigma_{05} = \frac{1}{2}(P_0 + K_0)$.

To relate the $SO(1, 2)$ subgroup of $SO(2, 2)$ which is a symmetry of global AdS_3 subspace of AdS_5 where the folded spinning string is moving to the collinear $SL(2, R)$ subgroup classifying the operators like $\text{tr}(\Phi D_+^S \Phi)$ one is also to perform an additional analytic continuation that interchanges the Euclidean (12) plane with the hyperbolic (+−) plane. Since different choices are formally related via $SO(2, 4)$ transformations and a re-identification of the generators one may expect that the two representations should be equivalent.⁴¹

Still, the representations of $SO(2, 2)$ or string states in AdS_3 are naturally labeled by (E, S) , and the relation to $SO(1, 2)$ labels $\frac{1}{2}(E + S)$ does not appear to be natural, unless one is interested in the large spin limit (see in this connection [47]). That relation may possibly be made more explicit by choosing a different set of coordinates in global AdS_5 in which the boundary is not $R \times S^3$ but $AdS_3 \times S^1$ (see [48] where such coordinates in the boundary theory were used to explain the leading $E \sim \ln S$ behaviour).

Let us mention also that the relation (1.23) or $E - S = f(E + S)$ is reminiscent of a light-cone gauge expression, where f would be a light-cone Hamiltonian (cf. [47,49,50]).

Appendix B. Review of folded string solution with $J = 0$

In this appendix we review the folded spinning string solution in AdS_3 [6,7] and consider its large spin expansion (see also [10]).

The solution is given by

$$\begin{aligned}
 t &= \kappa \tau, & \phi &= w \tau, & \rho &= \rho(\sigma), \\
 ds^2 &= -\cosh^2 \rho dt^2 + d\rho^2 + \sinh^2 \rho d\phi^2,
 \end{aligned}
 \tag{B.1}$$

where $\rho(\sigma)$ satisfies

$$\rho' = \pm \kappa \sqrt{1 - \eta \sinh^2 \rho}.
 \tag{B.2}$$

Here ρ varies from 0 to its maximal value ρ_0 related to the parameter η by

$$\coth^2 \rho_0 = \frac{w^2}{\kappa^2} \equiv 1 + \eta.
 \tag{B.3}$$

The solution in the interval $0 \leq \sigma \leq \frac{\pi}{2}$ with the initial condition $\rho(0) = 0$ is

$$\sinh \rho = \frac{1}{\sqrt{\eta}} \text{sn} \left[\kappa \sqrt{\eta} \sigma, -\frac{1}{\eta} \right], \quad 0 \leq \sigma \leq \frac{\pi}{2}.
 \tag{B.4}$$

⁴⁰ Their eigenvalues happen to be the same since the two representations (the unitary one classified by $SO(4) \times SO(2)$ and the one classified by $SO(4) \times SO(1, 1)$) are related by a global $SO(2, 4)$ similarity transformation (see, e.g., [46]).

⁴¹ The formal relation can be achieved by a continuation to Euclid: by replacing null direction like $x_0 + x_3$ with a complex one $x_1 + ix_2$, i.e. replacing the operator $\text{tr}(\Phi D_+^S \Phi)$ with $\text{tr}(\Phi D_*^S \Phi)$, where $D_* = D_1 + iD_2$.

The condition satisfied at the turning point ρ_0 at $\sigma = \frac{\pi}{2}$ is $\rho'(\frac{\pi}{2}) = 0$. To construct the full (2π periodic) folded closed string solution one should glue together four such functions on $\frac{\pi}{2}$ intervals to cover the full $0 \leq \sigma \leq 2\pi$ interval; e.g., for $\frac{\pi}{2} < \sigma < \pi$ we have

$$\sinh \rho = \frac{1}{\sqrt{\eta}} \operatorname{sn} \left[\kappa \sqrt{\eta} (\pi - \sigma), -\frac{1}{\eta} \right], \quad \frac{\pi}{2} \leq \sigma \leq \pi. \tag{B.5}$$

The expressions for the parameter κ , the energy and the spin in terms of η are

$$\begin{aligned} \kappa &= \frac{1}{\sqrt{\eta}} {}_2F_1 \left(\frac{1}{2}, \frac{1}{2}; 1; -\frac{1}{\eta} \right), & \mathcal{E} &= \frac{1}{\sqrt{\eta}} {}_2F_1 \left(-\frac{1}{2}, \frac{1}{2}; 1; -\frac{1}{\eta} \right), \\ \mathcal{S} &= \frac{\sqrt{1+\eta}}{2\eta\sqrt{\eta}} {}_2F_1 \left(\frac{1}{2}, \frac{3}{2}; 2; -\frac{1}{\eta} \right). \end{aligned} \tag{B.6}$$

In this paper we are interested in the large spin or long string limit, i.e. small η expansion. Since in Section 3 we compute 1-loop correction only to order $\frac{1}{\mathcal{S}}$ we will only need expansions to order $\mathcal{O}(\eta)$. Expanding κ , \mathcal{E} , \mathcal{S} in small η we obtain⁴²

$$\kappa = \kappa_0 - \frac{\eta}{4\pi} (\pi \kappa_0 - 2) + \mathcal{O}(\eta^2), \quad \kappa_0 \equiv \frac{1}{\pi} \ln \frac{16}{\eta}, \tag{B.7}$$

$$\mathcal{E} = \frac{2}{\pi \eta} + \frac{\pi \kappa_0 + 1}{2\pi} - \frac{\eta}{32\pi} (2\pi \kappa_0 - 3) + \mathcal{O}(\eta^2), \tag{B.8}$$

$$\mathcal{S} = \frac{2}{\pi \eta} - \frac{\pi \kappa_0 - 3}{2\pi} - \frac{\eta}{32\pi} (2\pi \kappa_0 + 13) + \mathcal{O}(\eta^2). \tag{B.9}$$

Expanding the solution (B.4) in small η we obtain, for $0 < \sigma < \frac{\pi}{2}$,

$$\sinh \rho = \sinh(\kappa \sigma) - \frac{\eta}{8} [\sinh(2\kappa \sigma) - 2\kappa \sigma] \cosh(\kappa \sigma) + \mathcal{O}(\eta^2), \tag{B.10}$$

or, using (B.7),

$$\sinh \rho = \sinh(\kappa_0 \sigma) - \frac{\eta}{8} \left[\sinh(2\kappa_0 \sigma) - \frac{4}{\pi} \sigma \right] \cosh(\kappa_0 \sigma) + \mathcal{O}(\eta^2). \tag{B.11}$$

To leading order when $\eta \rightarrow 0$, $\kappa_0 \rightarrow \infty$ the string touches the boundary of AdS_5 ($\rho_0 = \infty$) and the solution can be approximated (away from the turning points) by simply $\rho = \kappa_0 \sigma$. This limiting case proved to be a useful framework for computing 1- and 2-loop string corrections [10,36,38,39]. At the next order in small η expansion the “ends” (turning points) of the string are close to the boundary but no longer touch it.

We should add a word of caution about the use of the formal expansion in (B.10) or (B.11). Notice that it goes in powers of η with coefficients containing $\ln \eta$ and is not, strictly speaking, valid close enough to the turning points. Indeed, for $\sigma = \frac{\pi}{2}$ we have $\sinh(2\kappa_0 \sigma) = \sinh(\pi \kappa_0) \approx \frac{1}{2} e^{\pi \kappa_0} \sim \eta^{-1}$ and similarly $\cosh(\kappa_0 \sigma) = \sinh(\frac{\pi}{2} \kappa_0) \sim \eta^{-1/2}$. Hence at the turning point the order η term in (B.11) goes actually as $\eta^{-1/2}$, i.e. is of the same order as the leading term $\sinh(\kappa_0 \sigma)$. If σ is slightly away from the turning point the subleading terms are smaller than the leading

⁴² Let us note that these expansions were found using pre-Mathematica 6 versions of Mathematica (Mathematica 6 apparently has some bug leading to inconsistent expansions for some elliptic and hypergeometric functions).

term but then the expansion and the contributions to the energy need to be resummed. The resummation at the level of the string profile $\rho(\sigma)$ is completely equivalent to its expansion near the turning point $\sigma = \frac{\pi}{2}$, as will be discussed in [Appendix C](#).

In [Section 3](#) we ignored the regions near the turning points and thus uses the formal expansion [\(B.11\)](#) to compute it. Similar assumption was made in [\[10\]](#) in the computation of the 1-loop shift of the coefficient of the $\ln \mathcal{S}$ term in the energy (where it was indeed justified). A reason behind this assumption is that the masses of string fluctuations in [\(3.2\)](#) depend on ρ'^2 which is small near the turning points where $\rho' = 0$. Thus the correction to the leading result of [\[10\]](#) should mainly come from the “internal” parts of the σ -interval, where the expansion [\(B.11\)](#) is justified. We shall comment more on this point in [Appendix C](#). In [Section 3](#) we included a possible term which we may thus miss with an arbitrary coefficient yet to be determined.

For the computation of the 1-loop correction in [Section 3](#) we will need the following expansions

$$\rho'^2 = \kappa_0^2 - \frac{\eta}{2\pi} \kappa_0 [\pi \kappa_0 \cosh(2\kappa_0 \sigma) - 2] + \mathcal{O}(\eta^2), \quad (\text{B.12})$$

$$\begin{aligned} \kappa \sinh \rho &= \kappa_0 \sinh(\kappa_0 \sigma) + \frac{\eta}{8\pi} \{4\kappa_0 \sigma \cosh(\kappa_0 \sigma) \\ &\quad - [\pi \kappa_0 \cosh(2\kappa_0 \sigma) + 3\pi \kappa_0 - 4] \sinh(\kappa_0 \sigma)\} + \mathcal{O}(\eta^2), \end{aligned} \quad (\text{B.13})$$

$$\begin{aligned} w \cosh \rho &= \kappa_0 \cosh(\kappa_0 \sigma) + \frac{\eta}{8\pi} \{4\kappa_0 \sigma \sinh(\kappa_0 \sigma) \\ &\quad - [\pi \kappa_0 \cosh(2\kappa_0 \sigma) - 3\pi \kappa_0 - 4] \cosh(\kappa_0 \sigma)\} + \mathcal{O}(\eta^2). \end{aligned} \quad (\text{B.14})$$

Notice again that these expansions are formally invalid at the turning point (but are justified away from it). This is evident, e.g., in [\(B.12\)](#) where the leading term does not vanish at the turning point where one should have $\rho' = 0$. What happens is that at $\sigma = \frac{\pi}{2}$ the leading term κ_0^2 gets cancelled against the sum of subleading terms which all are of the same order, i.e. are proportional to κ_0^2 (see [Appendix C](#)).

The masses appearing in the fluctuation Lagrangian in [Section 3](#) are then expanded as follows

$$\mu_t^2 = \kappa_0^2 - \frac{\eta \kappa_0}{\pi} \left[\pi \kappa_0 \cosh(2\kappa_0 \sigma) - \frac{1}{2} \pi \kappa_0 - 1 \right] + \mathcal{O}(\eta^2), \quad (\text{B.15})$$

$$\mu_\phi^2 = \kappa_0^2 - \frac{\eta \kappa_0}{\pi} \left[\pi \kappa_0 \cosh(2\kappa_0 \sigma) + \frac{1}{2} \pi \kappa_0 - 1 \right] + \mathcal{O}(\eta^2), \quad (\text{B.16})$$

$$\mu_\rho^2 = -\eta \kappa_0^2 \cosh(2\kappa_0 \sigma) + \mathcal{O}(\eta^2). \quad (\text{B.17})$$

Appendix C. Resummation of “long string” expansion near the turning points

Continuing the discussion of the previous [Appendix B](#) here we will show that it is possible to resum systematically the terms $\sim e^{2n\kappa_0 \sigma}$ appearing in the formal η expansion of $\rho(\sigma)$ in [\(B.10\)](#). These terms are potentially dangerous since they scale at the turning point $\sigma = \pi/2$ as $e^{n\pi \kappa_0} = (16/\eta)^n$ and spoil the perturbative expansion.

In the first “quarter-string” interval $0 \leq \sigma \leq \frac{\pi}{2}$, the function $\rho(\sigma)$ obeys the differential equation [\(B.2\)](#) with plus sign and $\rho(0) = 0$. Its formal expansion in powers of η (treating κ_0 in [\(B.7\)](#) as a constant parameter) reads (cf. [\(B.10\)](#))

$$\rho(\sigma) = \kappa_0 \sigma + \left[\frac{\sigma}{2\pi} - \frac{1}{8} \sinh(2\kappa_0 \sigma) \right] \eta$$

$$\begin{aligned}
 & + \left[-\frac{\cosh(2\kappa_0\sigma)\sigma}{8\pi} - \frac{13\sigma}{64\pi} + \frac{1}{16} \sinh(2\kappa_0\sigma) + \frac{1}{256} \sinh(4\kappa_0\sigma) \right] \eta^2 \\
 & + \left[-\frac{\sinh(2\kappa_0\sigma)\sigma^2}{16\pi^2} + \frac{29 \cosh(2\kappa_0\sigma)\sigma}{256\pi} + \frac{\cosh(4\kappa_0\sigma)\sigma}{128\pi} + \frac{23\sigma}{192\pi} \right. \\
 & \left. - \frac{1}{128} \cosh(2\kappa_0\sigma) \sinh(2\kappa_0\sigma) - \frac{\cosh(4\kappa_0\sigma) \sinh(2\kappa_0\sigma)}{3072} - \frac{125 \sinh(2\sigma\kappa_0)}{3072} \right] \eta^3 \\
 & + \mathcal{O}(\eta^4).
 \end{aligned} \tag{C.1}$$

Since here, in fact, $\kappa_0 = -\frac{1}{\pi} \ln \frac{\eta}{16}$ the hyperbolic functions potentially reduce the true order in the η expansion. The dangerous terms can be easily identified by setting

$$t = e^{\kappa_0\sigma}, \quad r(t) \equiv \rho \left(\frac{\ln t}{\kappa_0} \right), \tag{C.2}$$

and neglecting exponentially suppressed terms in the above expansion. The next-to-leading (NLO) result which is correct at $\mathcal{O}(\eta)$ can be written

$$\begin{aligned}
 r_{\text{NLO}}(t) = \ln t & - \frac{\eta t^2}{16} + \frac{\eta^2 t^4}{512} - \frac{\eta^3 t^6}{12288} + \frac{\eta^4 t^8}{262144} + \dots \\
 & + \left[\frac{\eta t^2}{32} - \frac{\eta^2 t^4}{512} + \frac{\eta^3 t^6}{8192} + \dots \right. \\
 & \left. + \frac{\ln t}{2\pi\kappa_0} \left(1 - \frac{\eta t^2}{8} + \frac{\eta^2 t^4}{128} - \frac{\eta^3 t^6}{2048} + \dots \right) \right] \eta + \dots.
 \end{aligned} \tag{C.3}$$

All terms $(\eta t^2)^k$ are $\mathcal{O}(1)$ at $\sigma = \frac{\pi}{2}$ and the above infinite series need resummation. This can be accomplished as follows. Introducing $h(t) \equiv r(t) - \ln t$ we get

$$1 + th'(t) = \frac{\kappa}{\kappa_0} \sqrt{1 - \eta \left(\frac{te^h - t^{-1}e^{-h}}{2} \right)^2}, \quad h(t) = r(t) - \ln t. \tag{C.4}$$

Taking the large t limit we arrive at the following equation for the leading order term h_{LO}

$$1 + th'_{\text{LO}}(t) \simeq \sqrt{1 - \frac{1}{4} \eta t^2 e^{2h_{\text{LO}}(t)}}, \tag{C.5}$$

which can be integrated and gives

$$h_{\text{LO}}(t) = -\ln \left(1 + \frac{\eta t^2}{16} \right). \tag{C.6}$$

As a check we can reexpand and find indeed

$$h_{\text{LO}}(t) = -\frac{\eta t^2}{16} + \frac{\eta^2 t^4}{512} - \frac{\eta^3 t^6}{12288} + \frac{\eta^4 t^8}{262144} - \frac{\eta^5 t^{10}}{5242880} + \mathcal{O}(t^{11}), \tag{C.7}$$

which are the leading terms in $r(t)$.

The NLO approximation $h_{\text{NLO}}(t)$ is simply obtained by including an extra piece in the square root and taking into account that the ratio κ/κ_0 has a non-trivial expansion in η ,

$$1 + th'_{\text{NLO}}(t) = \frac{\kappa}{\kappa_0} \sqrt{1 - \eta \frac{t^2}{4} e^{2h_{\text{NLO}}(t)} + \frac{\eta}{2}}. \tag{C.8}$$

Integrating this equation, substituting the necessary terms in the expansion of κ/κ_0 , and neglecting all NNLO terms we then get

$$h_{\text{NLO}}(t) = -\ln\left(1 + \frac{\eta t^2}{16}\right) + \eta \frac{\ln t}{2\pi\kappa_0} + \frac{\eta^2 t^2}{32} \frac{1}{1 + \frac{\eta t^2}{16}} \left(1 - \frac{2\ln t}{\pi\kappa_0}\right). \quad (\text{C.9})$$

The expansion of (C.9) in powers of η gives

$$\begin{aligned} h_{\text{NLO}}(t) &= \left(\frac{\ln t}{2\pi\kappa_0} - \frac{t^2}{16}\right)\eta + \left(\frac{t^4}{512} - \frac{t^2 \ln t}{16\pi\kappa_0} + \frac{t^2}{32}\right)\eta^2 \\ &+ \left(-\frac{t^6}{12288} + \frac{t^4 \ln t}{256\pi\kappa_0} - \frac{t^4}{512}\right)\eta^3 + \left(\frac{t^8}{262144} - \frac{t^6 \ln t}{4096\pi\kappa_0} + \frac{t^6}{8192}\right)\eta^4 \\ &+ \left(-\frac{t^{10}}{5242880} + \frac{t^8 \ln t}{65536\pi\kappa_0} - \frac{t^8}{131072}\right)\eta^5 + \dots, \end{aligned} \quad (\text{C.10})$$

which agrees indeed with the expansion of r_{NLO} in (C.3).

In terms of the string profile $\rho(\sigma) = r(e^{\kappa_0\sigma}) = h(e^{\kappa_0\sigma}) - \kappa_0\sigma$ this result can be written as

$$\begin{aligned} \rho_{\text{NLO}}(\sigma) &= \kappa_0\sigma - \ln\left[1 + \left(\frac{\eta}{16}\right)^{1-2\sigma/\pi}\right] \\ &+ \eta\left[\frac{\sigma}{2\pi} + \frac{1}{2} \frac{\left(\frac{\eta}{16}\right)^{1-2\sigma/\pi}}{1 + \left(\frac{\eta}{16}\right)^{1-2\sigma/\pi}} \left(1 - \frac{2\sigma}{\pi}\right)\right]. \end{aligned} \quad (\text{C.11})$$

This expression resums at NLO order the contributions near the boundary point $\sigma = \frac{\pi}{2}$. This expression is not, of course, expected to be correct near $\sigma = 0$, but it must reproduce the exact value of $\rho(\frac{\pi}{2})$ with order $\mathcal{O}(\eta)$ included. This is true since

$$\rho_{\text{NLO}}\left(\frac{\pi}{2}\right) = \left(\ln 2 - \frac{1}{2} \ln \eta\right) + \frac{\eta}{4} + \mathcal{O}(\eta^2) \quad (\text{C.12})$$

is in agreement with the exact value of $\rho(\frac{\pi}{2})$ which is

$$\rho\left(\frac{\pi}{2}\right) = \operatorname{arcsinh} \frac{1}{\sqrt{\eta}} = \left(\ln 2 - \frac{1}{2} \ln \eta\right) + \frac{\eta}{4} - \frac{3\eta^2}{32} + \frac{5\eta^3}{96} + \mathcal{O}(\eta^4). \quad (\text{C.13})$$

Also, as an additional check, we immediately reproduce that $\rho'_{\text{NLO}}(\frac{\pi}{2}) = 0$.

In order to obtain the resummation in a systematic way and to show that it comes from the behaviour of the string profile around the turning point we can work out the expansion of the differential equation for $\rho(\sigma)$ around $\sigma = \frac{\pi}{2}$. To this aim, let us define

$$x = \kappa\left(\frac{\pi}{2} - \sigma\right), \quad \hat{\rho}(x) \equiv \rho\left(\frac{\pi}{2} - \frac{x}{\kappa}\right), \quad (\text{C.14})$$

and solve the corresponding equation (cf. (B.2))

$$\hat{\rho}'(x) = -\sqrt{1 - \eta \sinh^2 \hat{\rho}(x)}, \quad \hat{\rho}(0) = \operatorname{arcsinh} \frac{1}{\sqrt{\eta}}, \quad (\text{C.15})$$

perturbatively in η , i.e.

$$\hat{\rho}(x) = \operatorname{arcsinh} \frac{1}{\sqrt{\eta}} + \ln \operatorname{sech} x - \frac{\eta}{4} x \tanh x + \frac{\eta^2}{128} (-1 + \cosh(2x) - 4x^2 \operatorname{sech}^2 x + 10x \tanh x) + \mathcal{O}(\eta^3). \quad (\text{C.16})$$

Here the value $\hat{\rho}(0)$ was left unexpanded. Expanding it consistently we get the final result

$$\hat{\rho}(x) = \left(\frac{\pi \kappa_0}{2} + \ln \frac{\operatorname{sech} x}{2} \right) + \left(\frac{1}{4} - \frac{1}{4} x \tanh x \right) \eta + \left(-\frac{1}{32} x^2 \operatorname{sech}^2 x + \frac{1}{128} \cosh(2x) + \frac{5}{64} x \tanh x - \frac{13}{128} \right) \eta^2 + \mathcal{O}(\eta^3). \quad (\text{C.17})$$

If we now use the definition of $x = \kappa(\frac{\pi}{2} - \sigma)$ in this expression, we get precisely the NLO resummation in Eq. (C.11) plus a new η^2 term which is beyond the order of accuracy of (C.11).

One may wonder if this systematic resummation of $\rho(\sigma)$ can be used to resum the associated contributions in the 1-loop correction to string energy discussed in Section 3. This is not, however, immediately clear. Plugging the expansion of ρ around $\sigma = \frac{\pi}{2}$ in the Q_ω operator in (3.13) and denoting the resulting terms with label “fold” to indicate the expansion point, we find

$$Q_\omega = Q_{\omega, \text{fold}}^{(0)} + \eta Q_{\omega, \text{fold}}^{(1)} + \dots, \quad (\text{C.18})$$

where

$$Q_{\omega, \text{fold}}^{(0)} = \begin{pmatrix} -n^2 - \omega^2 & -V_1 & -V_2 \\ V_1 & n^2 + \omega^2 - \frac{2\kappa_0}{\omega} V_2 & -2\omega\kappa_0 - V_2 \\ V_2 & 2\omega\kappa_0 + V_2 & n^2 + \omega^2 - \frac{2\kappa_0}{\omega} V_2 \end{pmatrix}, \quad (\text{C.19})$$

$$z \equiv \frac{\pi}{2} - \sigma, \quad V_1 = \frac{\kappa_0^2}{\cosh^2(\kappa_0 z)} + 2in\kappa_0 \tanh(\kappa_0 z), \quad V_2 = \frac{\omega\kappa_0}{\cosh^2(\kappa_0 z)}. \quad (\text{C.20})$$

In the large κ_0 limit, we can make the following replacements ($\int_0^\infty dz \delta_+(z) = 1$)

$$\kappa_0 \operatorname{sech}^2(\kappa_0 z) \rightarrow \delta_+(z), \quad \kappa_0 \tanh(\kappa_0 z) \rightarrow \kappa_0 - \ln(2)\delta_+(z). \quad (\text{C.21})$$

After this substitution we can write

$$Q_{\omega, \text{fold}}^{(0)} = Q_\omega^{(0)} + Q_{\omega, \text{fold}}^{(0)'} \delta_+(z), \quad (\text{C.22})$$

where $Q_\omega^{(0)}$ is the same operator (3.12) we found in Section 3 in the expansion valid near $\sigma = 0$, while the new piece is

$$Q_{\omega, \text{fold}}^{(0)'} = \begin{pmatrix} 0 & 2in \ln 2 - \kappa_0 & -\omega \\ -2in \ln 2 - \kappa_0 & -2\kappa_0 & -\omega \\ \omega & \omega & -2\kappa_0 \end{pmatrix}. \quad (\text{C.23})$$

This is a $\mathcal{O}(\kappa_0)$ perturbation over $Q_\omega^{(0)}$ whose matrix elements are $\mathcal{O}(\kappa_0^2)$ (since $n, \omega \sim \kappa_0$ in the combined sum and integral like in (3.24)). Unfortunately, higher order terms coming from this term can be estimated to have the same order of magnitude and thus must be resummed. In principle, the contribution from $Q_{\omega, \text{fold}}^{(0)}$ must be treated exactly and separately, a task which we leave for the future.

Still, it is encouraging to note that a possible non-zero contribution from the near-turning-point region is expected to change the one-loop energy by a term proportional to

$$\frac{\kappa_0}{\kappa} = 1 + \left(\frac{1}{4} - \frac{1}{2\pi\kappa_0} \right) \eta + \dots = 1 + \frac{1}{2\pi\mathcal{S}} + \dots \tag{C.24}$$

This means that the induced modification of the coefficients in appearing in (3.38) must obey

$$\delta b_0 = \delta b_{11} = 0, \quad \delta b_{10} = \frac{1}{2\pi} \delta b_c. \tag{C.25}$$

Remarkably, this is precisely what is required by the reciprocity conditions in (3.45), (3.46).

Appendix D. Details of large spin expansion for folded (\mathcal{S} , \mathcal{J}) spinning string

In this section we collect some details on large spin expansions used in Section 2.2.

In the “slow long strings” regime ($\mathcal{S} \gg 1$, $\mathcal{J} \ll \mathcal{S}$), the small η expansions for the “anomalous” part of the energy and the conformal spin read⁴³

$$\begin{aligned} \tilde{\gamma}_{\mathcal{J} \ll 1} &= \kappa + \frac{\kappa}{\omega} \mathcal{S} - \mathcal{S} - \mathcal{J} \approx \left[-\frac{1 + \ln \eta}{\pi} + \frac{4(\ln \eta + 12)}{\pi} \eta^2 + \mathcal{O}(\eta^4) \right] - \mathcal{J} \\ &\quad + \pi \mathcal{J}^2 \left[\frac{(1 - \ln \eta)}{2 \ln^2 \eta} - \eta^2 \left(\frac{10}{\ln \eta} + \frac{20}{\ln^2 \eta} - \frac{44}{\ln^3 \eta} \right) + \mathcal{O}(\eta^4) \right] + \dots, \\ \tilde{s}_{\mathcal{J} \ll 1} &= \mathcal{S} + \frac{1}{2} \mathcal{J} + \frac{1}{2} \tilde{\gamma} \approx \left[\frac{1}{8\pi\eta} + \eta \frac{2 \ln \eta + 11}{2\pi} + \mathcal{O}(\eta^3) \right] \\ &\quad + \pi \mathcal{J}^2 \left[\frac{1}{16\eta \ln^2 \eta} - \eta \left(\frac{3}{2 \ln \eta} - \frac{13}{4 \ln^2 \eta} - \frac{11}{2 \ln^3 \eta} \right) + \mathcal{O}(\eta^3) \right] + \dots. \end{aligned} \tag{D.1}$$

For the “fast long strings” ($\mathcal{S} \gg 1$, with $\ln \mathcal{S} \ll \mathcal{J} \ll \mathcal{S}$) one finds

$$\begin{aligned} \tilde{\gamma}_{\ln \mathcal{S} \ll \mathcal{J} \ll \mathcal{S}} &\approx \frac{1}{\pi^2 \mathcal{J}} \left[\ln \eta \left(1 + \frac{1}{2} \ln \eta \right) + 44\eta^2 \left(\frac{1}{11} \ln^2 \eta - \ln \eta - 1 \right) + \mathcal{O}(\eta^4) \right] \\ &\quad + \frac{1}{\pi^4 \mathcal{J}^3} \left[-\frac{1}{8} (\ln^4 \eta + 4 \ln^3 \eta) + 2\eta^2 (-5 \ln^4 \eta + 5 \ln^3 \eta + 33 \ln^2 \eta) \right. \\ &\quad \left. + \mathcal{O}(\eta^4) \right] + \dots, \\ \tilde{s}_{\ln \mathcal{S} \ll \mathcal{J} \ll \mathcal{S}} &\approx \mathcal{J} \left[-\frac{1}{8\eta \ln \eta} + \eta \left(1 - \frac{11 + 12 \ln \eta}{2 \ln^2 \eta} \right) + \mathcal{O}(\eta^3) \right] \\ &\quad + \frac{1}{\pi^2 \mathcal{J}^2} \left[-\frac{\ln \eta}{16\eta} - \eta \left(\frac{3 \ln^2 \eta}{2} + 4 \ln \eta - \frac{11}{4} \right) + \mathcal{O}(\eta^3) \right] + \dots. \end{aligned} \tag{D.2}$$

Since the function $\tilde{f} = \frac{f}{\sqrt{\lambda}}$ in (1.14) coincides with the anomalous dimension evaluated at zero of the denominator in (2.16) in both cases we get an equation expressing the parameter η in terms of only the odd powers of the Casimir \mathcal{C} . From the power series expressions for $\tilde{\gamma}$, even in \mathcal{S} , it follows that the function \tilde{f} has expansion in even negative powers of the semiclassical Casimir \mathcal{C} .

Explicitly, the first few corrections read, for slow long strings

⁴³ The expansions are obtained from (2.11) and (2.12) after the redefinition $\eta \rightarrow -1 + 16\eta + \sqrt{1 + 256\eta^2}$.

$$\begin{aligned} \tilde{f} \approx & \left[\frac{\ln 8\pi\mathcal{C} - 1}{\pi} + \frac{\ln 8\pi\mathcal{C} + 1}{16\pi^3\mathcal{C}^2} + \mathcal{O}(1/\mathcal{C}^4) \right] - \mathcal{J} \\ & + \pi\mathcal{J}^2 \left[\frac{1}{2\ln 8\pi\mathcal{C}} - \frac{3}{32\pi^2\mathcal{C}^2 \ln 8\pi\mathcal{C}} + \mathcal{O}(1/\mathcal{C}^4) \right] + \dots, \end{aligned} \tag{D.3}$$

where $\mathcal{C} = \mathcal{S} + \frac{1}{2}\mathcal{J}$ and dots indicate corrections in \mathcal{J} . For fast long strings,

$$\begin{aligned} \tilde{f} \approx & \frac{1}{\pi^2\mathcal{J}} \left[\frac{\ln^2 \hat{\mathcal{C}}}{2} - \ln \hat{\mathcal{C}} + \frac{1}{16\hat{\mathcal{C}}^2} \left(4\ln \hat{\mathcal{C}} + 3 + \frac{3}{\ln \hat{\mathcal{C}}} + \frac{7}{\ln^2 \hat{\mathcal{C}}} + \dots \right) + \mathcal{O}(1/\hat{\mathcal{C}}^4) \right] \\ & - \frac{1}{\pi^4\mathcal{J}^3} \left[\frac{\ln^4 \hat{\mathcal{C}}}{8} + \frac{1}{32\hat{\mathcal{C}}^2} \left(4\ln^3 \hat{\mathcal{C}} + 5\ln^2 \hat{\mathcal{C}} + 9\ln \hat{\mathcal{C}} \right. \right. \\ & \left. \left. + 16 + \frac{24}{\ln \hat{\mathcal{C}}} + \frac{34}{\ln^2 \hat{\mathcal{C}}} + \dots \right) + \mathcal{O}(1/\hat{\mathcal{C}}^4) \right] + \dots, \end{aligned} \tag{D.4}$$

where $\hat{\mathcal{C}} = \frac{\mathcal{C}}{\mathcal{J}} = \frac{\mathcal{S}}{\mathcal{J}} + \frac{1}{2}$ and dots inside round brackets indicate corrections in $1/\ln \hat{\mathcal{C}}$. As was already noted in Section 2.2, the expansion in the case of the fast long strings is not of the same type as in (1.12) and (1.19) assumed in the main part of this paper.

Let us mention also that in the case of the m -folded string the interval $0 \leq \sigma < 2\pi$ is split into $4m$ segments: for $0 < \sigma < \frac{\pi}{2m}$ the function $\rho(\sigma)$ increases reaching its maximal value ρ_0 , then decreases to zero for $\frac{\pi}{2m} \leq \sigma \leq \frac{\pi}{m}$, etc. This implies the condition

$$2\pi = \int_0^{2\pi} d\sigma = 4m \int_0^{\rho_0} \frac{d\rho}{\sqrt{(\kappa^2 - \mathcal{J}^2) \cosh^2 \rho - (\omega^2 - \mathcal{J}^2) \sinh^2 \rho}}, \tag{D.5}$$

which leads to a factor of m in front of the relevant expressions for \mathcal{E} , \mathcal{S} , $\sqrt{\kappa^2 - \mathcal{J}^2}$. The large spin expansion is then similar to the $m = 1$ case. Once \mathcal{E} is expressed in terms of \mathcal{S} and \mathcal{J} , the parameter m enters only in combination with the string tension $\frac{\sqrt{\lambda}}{2\pi}$.

Appendix E. Higher order relations from reciprocity at strong coupling

The evidence for the functional relation and reciprocity (1.14), (1.19) at weak coupling suggests that the corresponding constraints should hold also in strong-coupling expansion. As we have seen, the large spin expansion of anomalous dimensions at strong coupling appears to have the same structure as at weak coupling (1.16) where now

$$f \equiv \sqrt{\lambda} \tilde{f}, \quad \tilde{f} = a_0 + \frac{b_0}{\sqrt{\lambda}} + \frac{c_0}{(\sqrt{\lambda})^2} + \dots, \tag{E.1}$$

$$f_c \equiv \sqrt{\lambda} \tilde{f}_c, \quad \tilde{f}_c = a_c + \frac{b_c}{\sqrt{\lambda}} + \frac{c_c}{(\sqrt{\lambda})^2} + \dots, \tag{E.2}$$

$$f_{mk} \equiv (\sqrt{\lambda})^{m+1} \tilde{f}_{mk}, \quad \tilde{f}_{mk} = a_{mk} + \frac{b_{mk}}{\sqrt{\lambda}} + \frac{c_{mk}}{(\sqrt{\lambda})^2} + \dots. \tag{E.3}$$

Assuming the functional relation or (1.17), one is then able to compute the coefficients f_{mm} of $\frac{\ln^m \mathcal{S}}{\mathcal{S}^m}$ in terms of the strong-coupling expansion coefficients in the scaling function f . The latter are known up to 2-loop order directly from the string-theory computations [10,39]

$$a_0 = \frac{1}{\pi}, \quad b_0 = -\frac{1}{\pi} 3 \ln 2, \quad c_0 = -\frac{1}{\pi} \mathbf{K}, \quad \dots \tag{E.4}$$

and also to a high (in principle, arbitrarily high) order from the analytic strong coupling solution [37] of the BES [1] equation for the function f . This means that f_{mm} are then effectively determined if the functional relation applies.

Assuming the validity of the reciprocity condition (1.19) should lead to additional constraints on the subleading coefficients like (1.22) which here should be understood in terms of power series in $\frac{1}{\sqrt{\lambda}}$. As a result, one should find non-trivial relations between strong-coupling expansion coefficients in (E.2) and (E.3).

There is, however, a subtlety in formulating the reciprocity condition in the context of large spin expansion at strong coupling as defined by string semiclassical perturbation theory where all non-zero charges are automatically large at large λ . For example, the case of finite twist $J = 2, 3, \dots$ cannot be distinguished from the formal case of $J = 0$. It is usually assumed that the folded string in AdS_5 with zero angular momentum in S^5 describes an operator of small twist, but that can be $J = 2$ or $J = 3$, etc. To establish a relation to the definition of reciprocity in weakly coupled gauge theory expansion with finite twist one would need to consider the case of semiclassical (S, J) string and then resum the series for its energy (both in J and in $\sqrt{\lambda}$) so that the limit of finite J would make sense.

Here we shall assume that in checking the reciprocity (1.19) at subleading order in strong coupling at $J = 0$ one may simply take the Casimir C in (1.19) as $C = \sqrt{\lambda} \mathcal{C}$, $\mathcal{C} = \mathcal{S}$, and ignore the shifts in brackets in (1.21) or (1.22), i.e. getting

$$\tilde{f}_{10} = \frac{1}{2} \tilde{f} \tilde{f}_c, \quad \tilde{f}_{32} = \frac{1}{16} f (\tilde{f}^3 - 2\tilde{f}^2 \tilde{f}_c - 16\tilde{f}_{21}), \quad \dots \quad (\text{E.5})$$

Multiplying the series in (E.1)–(E.3) we then find that some of the 1-loop coefficients can be expressed in terms of the tree-level coefficients and the coefficients in f . Explicitly,

$$b_{11} = a_0 b_0, \quad b_{10} = \frac{1}{2} (a_0 b_c + a_c b_0), \quad b_{22} = -\frac{3}{8} a_0^2 b_0, \quad (\text{E.6})$$

$$b_{33} = \frac{1}{6} a_0^3 b_0, \quad b_{32} = \frac{1}{8} a_0^3 (2b_0 - b_c) - a_{21} b_0 - \frac{3}{8} a_0^2 a_c b_0 - a_0 b_{21}, \quad \dots \quad (\text{E.7})$$

We have verified the validity of these relations for b_{11} and b_{10} in Section 3.

Appendix F. Large S expansions for twist 2 and twist 3 anomalous dimensions at weak coupling

Here we shall collect the coefficients of large spin expansion of anomalous dimensions for planar SYM operators of twist 2 and 3, up to four loops in the gauge coupling and up to $1/S^3$ order.⁴⁴ They are derived from the closed expressions in terms of the harmonic sums that were obtained (mainly exploiting the maximum transcendentality principle and the asymptotic Bethe ansatz), respectively, in [17] (at four loops) for the twist two scalar sector, in [17,51] for the twist three scalar sector and in [33] (at three loops) and [18] (at four loops) for the “gauge” sector.

All these expansions were proven to satisfy the reciprocity property, for a review see [18]. The expansions are indeed of the generic form (1.16) where the coefficients satisfy the relations (1.21) once J (and the flavor index ℓ , see footnote 11 in the introduction) are fixed accordingly.

⁴⁴ In the case of twist 3 operators, the anomalous dimensions we will consider here are the *minimal* in the band.

The coefficients of the leading $\ln^m S/S^m$ terms below are manifestly universal in twist and flavor.⁴⁵ As far as these leading terms are concerned, there is no need to explicitly write down the results for the twist two gaugino and gauge sectors, and the twist three gaugino sector. Indeed, the closed formulas for their anomalous dimensions can be deduced from the one for the twist two scalar case by just shifting the argument of the harmonic sums⁴⁶ but such shifts do not affect the coefficients of the leading $\ln^m S/S^m$ terms. It is worth stressing again that this universality, a well-known feature of the leading $\ln S$ coefficient (or cusp anomaly), is a non-trivial consequence of the functional relation (1.14), as was noticed in [33] and emphasized in [4].

At weak coupling it is useful to rewrite (1.16) as

$$\begin{aligned} \gamma(S)_{S \gg 1} = & f \ln \bar{S} + \bar{f}_c + \frac{f_{11} \ln \bar{S} + \bar{f}_{10}}{S} + \frac{f_{22} \ln^2 \bar{S} + \bar{f}_{21} \ln \bar{S} + \bar{f}_{20}}{S^2} \\ & + \frac{f_{33} \ln^3 \bar{S} + \bar{f}_{32} \ln^2 \bar{S} + \bar{f}_{31} \ln \bar{S} + \bar{f}_{30}}{S^3} + \mathcal{O}\left(\frac{\ln^4 \bar{S}}{S^4}\right), \end{aligned} \quad (\text{F.1})$$

where $\bar{S} = e^{\gamma_E} S$ and the coefficients will be power series in $\hat{\lambda} = \frac{\lambda}{16\pi^2}$. Then one finds:

Twist two scalar sector:

$$\begin{aligned} f &= 8\hat{\lambda} - \frac{8\pi^2}{3}\hat{\lambda}^2 + \frac{88\pi^4}{45}\hat{\lambda}^3 - \left(\frac{584\pi^6}{315} + 64\zeta_3^2\right)\hat{\lambda}^4, \\ \bar{f}_c &= -24\zeta_3\hat{\lambda}^2 + \left(\frac{16}{3}\pi^2\zeta_3 + 160\zeta_5\right)\hat{\lambda}^3 + \left(-\frac{56}{15}\pi^4\zeta_3 - \frac{80}{3}\pi^2\zeta_5 - 1400\zeta_7\right)\hat{\lambda}^4 \\ f_{11} &= 32\hat{\lambda}^2 - \frac{64\pi^2}{3}\hat{\lambda}^3 + \frac{96\pi^4}{5}\hat{\lambda}^4, \\ \bar{f}_{10} &= 4\hat{\lambda} - \frac{4\pi^2}{3}\hat{\lambda}^2 + \left(\frac{44\pi^4}{45} - 96\zeta_3\right)\hat{\lambda}^3 + \left(-\frac{292\pi^6}{315} + \frac{160}{3}\pi^2\zeta_3 - 32\zeta_3^2 + 640\zeta_5\right)\hat{\lambda}^4, \\ f_{22} &= -64\hat{\lambda}^3 + (64\pi^2 - 128\zeta_3)\hat{\lambda}^4, \\ \bar{f}_{21} &= -16\hat{\lambda}^2 + \left(128 + \frac{16\pi^2}{3}\right)\hat{\lambda}^3 + \left(-128\pi^2 - \frac{32\pi^4}{15} + 448\zeta_3\right)\hat{\lambda}^4, \\ \bar{f}_{20} &= -\frac{2}{3}\hat{\lambda} + \left(24 + \frac{2\pi^2}{9}\right)\hat{\lambda}^2 - \left(\frac{32\pi^2}{3} + \frac{22\pi^4}{135} - 48\zeta_3\right)\hat{\lambda}^3 \\ &+ \left(\frac{136\pi^4}{15} + \frac{146\pi^6}{945} - 384\zeta_3 - \frac{32}{3}\pi^2\zeta_3 + \frac{16\zeta_3^2}{3} - 320\zeta_5\right)\hat{\lambda}^4 \\ f_{33} &= \frac{512}{3}\hat{\lambda}^4, \end{aligned}$$

⁴⁵ The only exception being the four loop coefficient of the term $\ln^2 S/S^2$ in the case of twist two scalar operators. However, it seems reasonable to relate this exception to the wrapping-induced breakdown of the Bethe equations at four loops for twist two operators.

⁴⁶ It is well known that in $\mathcal{N} = 4$ SYM all twist two operators belong to the same supermultiplet, and their anomalous dimension is expressed in terms of a universal function with shifted arguments $\gamma_{J=2}^\varphi(S) = \gamma_{\text{univ}}(S)$, $\gamma_{J=2}^\psi(S) = \gamma_{\text{univ}}(S+1)$, $\gamma_{J=2}^A(S) = \gamma_{\text{univ}}(S+2)$. In [52] it was proved that the anomalous dimension for twist three operators built out of gauginos is related to the one of the twist two universal supermultiplet as $\gamma_{J=3}^\psi(S) = \gamma_{J=2}^\varphi(S+2)$.

$$\begin{aligned}
\bar{f}_{32} &= 64\hat{\lambda}^3 + \left(-768 - \frac{64\pi^2}{3} + 128\zeta_3\right)\hat{\lambda}^4, \\
\bar{f}_{31} &= \frac{16}{3}\hat{\lambda}^2 + \left(-256 + \frac{16\pi^2}{9}\right)\hat{\lambda}^3 + \left(512 + \frac{512\pi^2}{3} - \frac{64\pi^4}{15} - 576\zeta_3\right)\hat{\lambda}^4, \\
\bar{f}_{30} &= -\frac{56}{3}\hat{\lambda}^2 + \left(96 + \frac{40\pi^2}{9} - 16\zeta_3\right)\hat{\lambda}^3 \\
&\quad - \left(\frac{224\pi^2}{3} + \frac{32\pi^4}{15} - 800\zeta_3 + \frac{64}{9}\pi^2\zeta_3 - \frac{320\zeta_5}{3}\right)\hat{\lambda}^4.
\end{aligned} \tag{F.2}$$

Twist three scalar sector:

$$\begin{aligned}
f &= 8\hat{\lambda} - \frac{8\pi^2}{3}\hat{\lambda}^2 + \frac{88\pi^4}{45}\hat{\lambda}^3 - \left(\frac{584\pi^6}{315} + 64\zeta_3^2\right)\hat{\lambda}^4, \\
\bar{f}_c &= -8\ln 2\hat{\lambda} + \left(\frac{8}{3}\pi^2\ln 2 - 8\zeta_3\right)\hat{\lambda}^2 + \left(-\frac{88}{45}\pi^4\ln 2 + \frac{8}{3}\pi^2\zeta_3 - 8\zeta_5\right)\hat{\lambda}^3 \\
&\quad + \frac{8}{315}(73\pi^6\ln 2 - 84\pi^4\zeta_3 + 2520\ln 2\zeta_3^2 + 105\pi^2\zeta_5 + 17325\zeta_7)\hat{\lambda}^4, \\
f_{11} &= 32\hat{\lambda}^2 - \frac{64\pi^2}{3}\hat{\lambda}^3 + \frac{96\pi^4}{5}\hat{\lambda}^4, \\
\bar{f}_{10} &= 8\hat{\lambda} + \left(-\frac{8\pi^2}{3} - 32\ln 2\right)\hat{\lambda}^2 + \left(\frac{88\pi^4}{45} + \frac{64}{3}\pi^2\ln 2 - 32\zeta_3\right)\hat{\lambda}^3 \\
&\quad - \frac{8}{315}(73\pi^6 + 756\pi^4\ln 2 - 840\pi^2\zeta_3 + 2520\zeta_3^2 + 1260\zeta_5)\hat{\lambda}^4, \\
f_{22} &= -64\hat{\lambda}^3 + 64\pi^2\hat{\lambda}^4, \\
\bar{f}_{21} &= -32\hat{\lambda}^2 + \left(128 + \frac{64\pi^2}{3} + 128\ln 2\right)\hat{\lambda}^3 \\
&\quad + \left(-256 - 128\pi^2 - \frac{96\pi^4}{5} - 128\pi^2\ln 2 + 256\zeta_3\right)\hat{\lambda}^4, \\
\bar{f}_{20} &= -\frac{8\lambda}{3} + \left(48 + \frac{8\pi^2}{9} + 32\ln 2\right)\hat{\lambda}^2 \\
&\quad + \left(32 - \frac{80\pi^2}{3} - \frac{88\pi^4}{135} - 128\ln 2 - \frac{64}{3}\pi^2\ln 2 - 64\ln^2 2 + 32\zeta_3\right)\hat{\lambda}^3 \\
&\quad + \left(-512 - \frac{32\pi^2}{3} + \frac{352\pi^4}{15} + \frac{584\pi^6}{945} + 256\ln 2 + 128\pi^2\ln 2 + \frac{96}{5}\pi^4\ln 2\right. \\
&\quad \left.+ 64\pi^2\ln^2 2 - 128\zeta_3 - \frac{64}{3}\pi^2\zeta_3 - 256\ln 2\zeta_3 + \frac{64\zeta_3^2}{3} + 32\zeta_5\right)\hat{\lambda}^4, \\
f_{33} &= \frac{512}{3}\hat{\lambda}^4, \\
\bar{f}_{32} &= 128\hat{\lambda}^3 - (768 + 128\pi^2 + 512\ln 2)\hat{\lambda}^4,
\end{aligned}$$

$$\begin{aligned}
\bar{f}_{31} &= \frac{64}{3}\hat{\lambda}^2 + \left(-512 - \frac{128\pi^2}{9} - 256\ln 2\right)\hat{\lambda}^3 \\
&\quad + \left(768 + \frac{1408\pi^2}{3} + \frac{64\pi^4}{5} + 1536\ln 2 + 256\pi^2\ln 2 + 512\ln^2 2 - 512\zeta_3\right)\hat{\lambda}^4, \\
\bar{f}_{30} &= -\left(\frac{224}{3} + \frac{64\ln 2}{3}\right)\hat{\lambda}^2 \\
&\quad + \left(128 + \frac{352\pi^2}{9} + 512\ln 2 + \frac{128}{9}\pi^2\ln 2 + 128\ln^2 2 - \frac{64\zeta_3}{3}\right)\hat{\lambda}^3 \\
&\quad + \left(896 - \frac{448\pi^2}{3} - \frac{512\pi^4}{15} - 768\ln 2 - \frac{1408}{3}\pi^2\ln 2 - \frac{64}{5}\pi^4\ln 2 - 768\ln^2 2 \right. \\
&\quad \left. - 128\pi^2\ln^2 2 - \frac{512\ln^3 2}{3} + 640\zeta_3 + \frac{128}{9}\pi^2\zeta_3 + 512\ln 2\zeta_3 - \frac{64\zeta_5}{3}\right)\hat{\lambda}^4. \quad (\text{F.3})
\end{aligned}$$

Twist three “gauge” sector:

$$\begin{aligned}
f &= 8\hat{\lambda} - \frac{8\pi^2}{3}\hat{\lambda}^2 + \frac{88\pi^4}{45}\hat{\lambda}^3 - \left(\frac{584\pi^6}{315} + 64\zeta_3^2\right)\hat{\lambda}^4, \\
\bar{f}_c &= 8(1 - \ln 2)\hat{\lambda} + \frac{8}{3}(-12 - \pi^2 + \pi^2\ln 2 - 3\zeta_3)\hat{\lambda}^2 - \frac{8}{45}(-1440 - 60\pi^2 - 11\pi^4 \\
&\quad + 11\pi^4\ln 2 - 15\pi^2\zeta_3 + 45\zeta_5)\hat{\lambda}^3 + \frac{8}{315}(-100800 - 3360\pi^2 - 336\pi^4 \\
&\quad - 73\pi^6 + 73\pi^6\ln 2 - 84\pi^4\zeta_3 - 2520\zeta_3^2 + 2520\ln 2\zeta_3^2 + 105\pi^2\zeta_5 + 17325\zeta_7)\hat{\lambda}^4, \\
f_{11} &= 32\hat{\lambda}^2 - \frac{64\pi^2}{3}\hat{\lambda}^3 + \frac{96\pi^4}{5}\hat{\lambda}^4, \\
\bar{f}_{10} &= 32\hat{\lambda} + \left(32 - \frac{32\pi^2}{3} - 32\ln 2\right)\hat{\lambda}^2 + \frac{32}{45}(-180 - 30\pi^2 + 11\pi^4 \\
&\quad + 30\pi^2\ln 2 - 45\zeta_3)\hat{\lambda}^3 - \frac{32}{315}(-10080 - 840\pi^2 - 189\pi^4 + 73\pi^6 \\
&\quad + 189\pi^4\ln 2 - 210\pi^2\zeta_3 + 2520\zeta_3^2 + 315\zeta_5)\hat{\lambda}^4, \\
f_{22} &= -64\hat{\lambda}^3 + 64\pi^2\hat{\lambda}^4, \\
\bar{f}_{21} &= -128\hat{\lambda}^2 + \left(\frac{256\pi^2}{3} + 128\ln 2\right)\hat{\lambda}^3 + \left(256 - \frac{384\pi^4}{5} - 128\pi^2\ln 2\right)\hat{\lambda}^4, \\
\bar{f}_{20} &= -\frac{200}{3}\hat{\lambda} + \left(16 + \frac{200\pi^2}{9} + 128\ln 2\right)\hat{\lambda}^2 + \left(480 - \frac{16\pi^2}{3} - \frac{440\pi^4}{27} - \frac{256}{3}\pi^2\ln 2 \right. \\
&\quad \left. - 64\ln^2 2 + 128\zeta_3\right)\hat{\lambda}^3 + \left(-2816 - \frac{1120\pi^2}{3} + \frac{64\pi^4}{15} + \frac{2920\pi^6}{189} - 256\ln 2 \right. \\
&\quad \left. + \frac{384}{5}\pi^4\ln 2 + 64\pi^2\ln^2 2 - 128\zeta_3 - \frac{256}{3}\pi^2\zeta_3 + \frac{1600\zeta_3^2}{3} + 128\zeta_5\right)\hat{\lambda}^4, \\
f_{33} &= \frac{512}{3}\hat{\lambda}^4, \\
\bar{f}_{32} &= 512\hat{\lambda}^3 + (-256 - 512\pi^2 - 512\ln 2)\hat{\lambda}^4,
\end{aligned}$$

$$\begin{aligned}
\bar{f}_{31} &= \frac{1600}{3}\hat{\lambda}^2 + \left(-640 - \frac{3200\pi^2}{9} - 1024\ln 2\right)\hat{\lambda}^3 \\
&\quad + \left(-1792 + \frac{1792\pi^2}{3} + 320\pi^4 + 512\ln 2 + 1024\pi^2\ln 2 + 512\ln^2 2\right)\hat{\lambda}^4, \\
\bar{f}_{30} &= 192\hat{\lambda} + \left(-\frac{1120}{3} - 64\pi^2 - \frac{1600\ln 2}{3}\right)\hat{\lambda}^2 + \left(-\frac{5824}{3} + \frac{1856\pi^2}{9} + \frac{704\pi^4}{15}\right. \\
&\quad + 640\ln 2 + \frac{3200}{9}\pi^2\ln 2 + 512\ln^2 2 - \frac{1600\zeta_3}{3}\left.\right)\hat{\lambda}^3 + \left(\frac{25984}{3} + \frac{15488\pi^2}{9}\right. \\
&\quad - \frac{544\pi^4}{3} - \frac{4672\pi^6}{105} + 1792\ln 2 - \frac{1792}{3}\pi^2\ln 2 - 320\pi^4\ln 2 - 256\ln^2 2 \\
&\quad \left. - 512\pi^2\ln^2 2 - \frac{512\ln^3 2}{3} + 1152\zeta_3 + \frac{3200}{9}\pi^2\zeta_3 - 1536\zeta_3^2 - \frac{1600\zeta_5}{3}\right)\hat{\lambda}^4. \quad (\text{F.4})
\end{aligned}$$

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