ESTIMATION AND TESTING OF LATENT FACTORS IN TERM STRUCTURE OF INTEREST RATES

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To my god parents.

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ABSTRACT

Factor analysis has contributed imperatively towards solving the dimensionality problem and identifying the underlying factor structure governing term structure of interest rates. The estimated latent factors are known as level, slope, and curvature. These factors can be estimated using Principal Component Analysis (PCA) or Nelson-Siegel (1987) framework as reparameterized by Diebold and Li (2006). The two statistical methods have been shown to produce the same three factors.

The thesis contributes towards testing of level, slope, and curvature factors extracted using the statistical models. We investigate the issues of stability in the eigenspace variables governing level, slope, and curvature. We develop a stability testing procedure to examine the presence of significant structural changes in the latent factors estimated using PCA. Bootstrapped critival values have been employed in order to draw inferences. Monte Carlo evidence suggests good finite sample size and power properties of the tests. Empirical test results on zero coupon bond yield curves show significant structural changes in factors.

Further, we propose some extensions to estimating level, slope, and curvature factors for term structures where the interest rate maturities are coupled together into dependence clusters. In this, we extend the Nelson-Siegel (1987) framework to the case of modelling yield curves with correlation clusters. We identify the short maturity and long maturity clusters governing the term structure and propose a block dynamic representation to model the factors. We find that the proposed model generated superior forecasts than the benchmark model proposed by Diebold and Li (2006).

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DECLARATION

INTRODUCTION

Factor analysis has been widely used in areas of finance where large number of variables is known to be driven by a few common sources of variations. In the literature of term structures of interest rates, it is well noted that around 98% of variations in bond yield term structures can be explained by just three factors. These effects were first established by Steeley (1990), Litterman and Scheinkman (1991) among others.

Since shocks affecting interest rates disseminate contemporaneously across units, factor models have been employed to understand the common factors of risk in a term structure and how they load onto the interest rates. In using panel factor models for interest rates, since the risk factors are not observable and latent, estimation of these factors are commonly done using principal component analysis where the factors are a function of the eigenvectors. The loadings of these factors can be then estimated using ordinary least squares (OLS). Bai (2003), Kao et al. (2006) among others show that in the case of approximate factor models where the idiosyncratic errors are allowed to weekly correlate, OLS estimation of factor loadings is inconsistent when the number of cross-sectional units is not large. This issue is of paramount importance for empirical applications in finance where we commonly come across this situation. The inconsistency arises due to the fact that there is limited information available to estimate the factors when the number of cross sectional units is fixed and not large. However, in the case of using Principal Component Analysis (PCA) for estimating the factor loadings (that are functions of eigenvalues and eigenvectors) from a stationary term structure panel, we show consistency of the estimated factor loadings.

In PCA framework, there is no guarantee that factors are the same across time. Since the factors are defined as factor score times the term structure panel, changes in the factor scores would cause the estimated factors to vary over time. In investigating changes in term structure factors, we therefore can simply determine the constancy of the factor scores which are defined in terms of the eigenvalues and eigenvectors. Perignon and Villa (2006) and Bliss

(1997) document the time variations in these common risk factors driving interest rates but do not infer anything about structural changes that might have been caused due to abrupt time variations or regime changes.

Stability in the factors is crucial for modelling and forecasting of the yield curve. Authors usually rely on split-sample analysis in understanding whether the factors have been stable through time. In this dissertation we develop a statistical testing procedure in order to formally evaluate the stability in the factor structure of level, slope, and curvature factors governing the yield curve. We estimate the factors using PCA, and investigate stability of the eigenspace variables governing the factors. The results indicate that the PCA technique, based on the eigen decomposition of the correlation or covariance matrix, produce factors that can be subject to structural changes observed within the term structure. Several validations to these results, both empirical and theoretical (via simulations), have been conducted to robustify the findings.

In extracting the factors via PCA, we have assumed the following. First, the contemporaneous dependence structure between interest rate maturities is similar. In this case, we assume that the interest rate maturities are (linearly) correlated in the same way and that PCA would detect the principal factors by the rotation of the principal axes in the order to capture the direction explaining the maximum variability in the whole term structure. However, the estimation of the factors from a correlation matrix with multiple correlation clusters would fail to discover the true factors governing the entire term structure. Presence of multiple data clusters within the term structure would mean that PCA factors would be representative to part of the term structure and not to the entire maturity domain of the term structure. Second, the factors are static. This assumption is restrictive since the term structure is observed to change shapes and evolve over time. Authors have found that a dynamic structure to the factors better explain the movements in the term structure. We extend the estimation of term structure models to the case of "Block Dynamic" latent factor models. Particularly, we draw from the dynamic Nelson-Siegel representation of yield curve factors by Diebold and Li (2006) and extend the framework to the case of term structures with multiple data clusters. We find that accounting for the multiple maturity clusters increase forecastability of the term structure.

Object of research

Zero coupon rates are one of the fundamental building blocks in fixed income markets. They are widely used in many applications from financial engineering to forecasting business cycles. Modelling the zero coupon bond yield term structures have therefore been of central interest of fixed income researchers and central bankers. Since the interest rate maturities within the zeros term structure have a strong dependence structure (usually measured using correlations or covariances), a few common factors are known to sufficiently explain most of the dynamics underlying the term structure panel. These latent common factors have been estimated using various factor analysis representations. The three principal factors governing the bond yield term structures have been commonly called level, slope, and curvature. The shape of the factor loadings associated with these three factors are flat, sloped, and curved respectively. Hence the factors are called level, slope, and curvature. Since the factors can be decomposed from a covariance matrix, they inherently capture the risk exposures due to movements in the yield curve. The level factor explains linear shifts in the yield curve, and slope and curvature factors explain the non-linear shifts observed in the yield curve. We refer to Litterman and Scheinkman (1991) and Diebold and Li (2006) that advocates the level, slope, and curvature explanations to term structure factors.

These factors can be extracted parametrically using the Nelson-Siegel model or non-parametrically using PCA. For the Nelson-Siegel model, the loadings are predetermined by the process governing the forward rates. Nelson-Siegel (1987) recommends the functional form for the loadings that produce the level, slope, and curvature.

The object of research in this dissertation is estimating and testing the term structure latent factors level, slope, and curvature commonly estimated using function-based statistical models such as PCA and Nelson-Siegel factor models. The empirical study is restricted to 'pure' discount bonds.

Objective of research

Modelling of yield curves with factor models would mean describing the relationship between interest rates in a term structure and its factors, capturing the cross-sectional dependencies between rates. A stable relationship between the term structure and its factors (measured by the factor loadings) are assumed in practice. Further, the dependence structure (measured by quantities such as covariance or correlation) between rates is assumed to be similar.

The objective of research in this dissertation is broadly two-fold. First, to propose a formal testing procedure in order to test for stability in the factor structure of the yield curves. In this we evaluate the stability of the eigensystem while estimating the PCA factors. Second, to allow for estimation of separate factors in a term structure panel that is characterized by more than one dependence structures (or correlation clusters). In this we model the factors governing the two clusters separately in a Nelson-Siegel dynamic framework and show that allowing for such block dynamic estimation increase forecastability of the term structure. Thus we propose a block dynamic Nelson-Siegel model for term structures with more than one dependence clusters.

Dissertation structure

The dissertation is comprised of four main chapters outlined below.

Chapter one describes the various term structure theories and frameworks fundamental to modelling interest rates. In this, we outline the classical theories that provide an explanation to the contemporaneous movements across rates that cause systematic shifts in the yield curve. This clearly influences the yield curve shapes (normal, inverted, humped) observed in practice. The chapter concentrates on discussing commonly used statistical factor models for yield curves such as the principal components model and the Nelson-Siegel type models. These models generate latent factors economically interpreted as level, slope, and curvature.

Chapter two develops a testing procedure for evaluating the stability in eigenvalues, eigenvectors, and factor loadings of level, slope, and curvature factors of the US zero coupon yield term structure. In this, we formulate a series of hypotheses, develop a test statistic, and conduct statistical tests for stability. For the US zero coupon bond yield term structure, we find that the variance process (measured by eigenvalues) of level, slope, and curvature factors were indeed unstable during the sample period considered. We find the factor structure of level to be unstable over the sample period considered. Slope and curvature factor structures are however found to be stable. We corroborate the literature that variances (volatility) explained by the level, slope, and curvature factors are unstable over time. We find evidence of the presence of common economic shocks affecting the level and slope factors, unlike slope and curvature factors that responded differently to economic shocks and were unaffected by any common instabilities.

Chapter three provides a robustification to the stability test and empirical conclusions drawn in chapter two. First, we provide a simulations based evidence that provides the validity of the use of bootstrapped critical values for the testing procedure implemented in chapter one. Second, we conduct stability tests on Fama-Bliss and Federal Reserve zero coupon bond yield term structures with daily and monthly frequencies and compare the results drawn in chapter two. The Monte Carlo simulation results show that the bootstrap procedure well approximates the finite sample distribution of the test statistics with good size and power properties. Conducting the stability tests on four term structures with different frequencies, we find that overall eigensystem of three factors are unstable, all the eigensystem variables governing the level factor is unstable, the long rates governing the slope factor are stable over time, and there is evidence of common points of instability among the three factors.

Chapter four introduces a block dynamic for structure to the factors governing term structures with maturity clusters. In this, we generalize the dynamic representation proposed by Diebold and Li (2006) for constructing yield curve forecasts of the Nelson-Siegel factors and relax the assumption of common factor dynamics among maturity clusters within a term structure. The new framework is formulated in a state space system and estimated using a

Kalman filter. The optimization implements the Marquardt and BHHH algorithms written in Matlab 7. Due to space limitations, the complete listings of the source code are not included in the dissertation. However, careful documentation of the algorithms is presented in the relevant chapters. Application of the new Nelson-Siegel block dynamic factor model on the term structure of daily zero coupon bond yields with maturity clusters show better out-of-sample forecasting performance than the benchmark model proposed by Diebold and Li (2006).

Chapter 1

Review of Literature

1.1 Introduction

Movements in the bond yields can be characterized by defining the dynamics of the term structure of interest rates. The markets are complete with no arbitrage opportunities if the long term yields are the risk adjusted expectations of the average future short term yields. This implies commonality across the movements in the cross-section of the yield term structure. Modelling the evolution of these common factors can directly relate to modelling the entire yield term structure. This approach allows for obvious computational advantages when dealing with a large panel framework. The term structure literature has relied heavily on the data decomposition techniques in the factor analysis literature in order to remove the dimensionality issue faced in modelling interest rates panel data structures.

Empirical evidence confirms that single factor (short rate) specifications are insufficient to capture all the dynamics of yields across maturities. By assuming that yield dynamics are driven by the same risk factor, single factor models impose the restriction that yields are locally perfectly correlated. However, empirical results show that yield correlations are different from unity - though highly correlated at similar or closer time to maturities. Further, the correlations are considerably reduced if the yields are in different segments of the yield curve. Also, literature concludes that the volatility dynamics of the yields have an inconsistent structure, which is neither constant nor affine in the short rate, as assumed for simplicity (e.g. Chapman and Pearson (2001)). This highlights the importance of multifactor term structure models.

Multifactor models accommodate for the unspanned factors that can explain the evolution in the term structure of interest rates better. Here the yields are functions of all these state factors that are unobserved. Studies have used two, three, or four leading factors for explaining the dynamics in the yield term structure. Leading papers show that three principal factors are sufficient to explain most (about 99%) of the variations in interest rates. Though it is difficult to interpret what these latent factors correspond to, there is a common consensus that the three leading factors correspond to level, steepness, and curvature of the term structure (Litterman and Scheinkman (1991)). Some authors relate the factors to macroeconomic variables and monetary policy shocks (e.g. Ang and Piazzesi (2003), Ang et al. (2005)). Dewachter and Lyrio (2003) suggest that the level factor corresponds to the long-run inflation expectation, the slope factor corresponds to the business cycle, and the curvature factor corresponds to the monetary policies.

There is no clear idea of how many factors should be ideally considered in modelling the term structure of interest rates. Including different number of factors would generate different conclusions. Also, incorporating a large number of factors would generate a non-parsimonious representation of our data. This leads us to concerns regarding the number of optimal factors to be included, the dynamics of the underlying factors, what factor estimation methods should be employed when factors are unobserved, distributions of the latent factors, and so on. The factor analysis literature in macroeconomics and panel econometrics contribute towards some of the issues. A series of papers by Bai and Ng (2002, 2006, 2007), Bai (2003, 2004), Bovin and Ng (2005, 2006), Stock and Watson (2005), Kao et al. (2006) provide us with the asymptotic results in panel framework that can be employed into the term structure literature in finance. We provide some of the key results in section 1.2 which can be implemented in the term structure literature.

In the following sections we introduce issues that are of interests to the development of the next three chapters of the dissertation. We organize the remaining sections as follows: Section 1.2 and 1.3 reviews the fundamentals in modelling yield curves and introduces two important methods of yield factor estimations namely, principal component analysis and exponential components models. Section 1.4 reviews some major contributions in panel factor models literature in macroeconomics and econometrics and relates it to the literature on term structure of interest rates.

1.2 Fundamentals in modelling term structure of interest rates

Term structure of interest rates has been known to be forward looking and therefore well-suited in understanding the market expectations of the future. An explanation of the term structure would enable extract the relationships between the interest rate maturities and study their dynamics over time. Several theories have been proposed that explain the driving forces to change in relationships observed between interest rates. They provide a theoretical link between short term and long term interest rates and present an explanation to the different shapes of the yield curve.

One of the simplest theories is the expectations theory which conjectures that the long maturity interest rates reflect the expected future short term rates. This would mean that a yield curve with a positive slope imply the investors view that short rates would be higher in the future. On the other hand, an inverted yield curve implies that investors anticipate decrease in future spot rates. An alternative view is the market segmentation theory that assumes no relationship between short and long interest rate maturities. The theory conjectures that demand and supply forces in each market determine their rates and very little influence of neighbouring maturities affects the rates. In this, the investors are invariably located at the same point on the yield curve. Along this theory, the preferred habitat theory combines it with investor preferences and states that investors would switch between yield curve segments based on changes in term premiums. The third and the most appealing theory is the liquidity preference theory that coincides with the expectations hypothesis but also places weight on risk preferences of the investors. In this, the long rates are equal to the sum of average future short rates and a liquidity premium. The theory produces an upward sloping yield curve and consistent with empirical evidence that more often yield curves slope upwards.

A model for the term structure is required to explain the relationship between the interest rate maturities over time. Theoretical frameworks proposed aim to satisfy the empirical characteristics of evolution of interest rates observed in the market. However, not all models capture all the characteristics of the term structure. Financial researchers mainly use two approaches to modelling the dynamics of interest rates. The first type is a general equilibrium framework where the bond yields are modelled based on the dynamics of the short rates. In this, the implicit assumption is that bond prices and yields are determined by the market's assessment on the evolution of the short term interest rates. A general equilibrium framework describing the instantaneous short rates is given by the stochastic process

$$dr_t = \kappa\left(\theta - r_t\right) dt + \sigma r_t^{\gamma} dB_t \tag{1.1}$$

where κ measures the speed of mean reversion and affects the shape of the yield curve, θ is the long run mean (equilibrium level) to which the short rate mean reverts, and σ is the instantaneous volatility of the short rate. The parameterization for the instantaneous drift in the model explains the main feature of mean reversion in short rates. The framework includes the models of Vasicek (1977); Dothan (1978); Cox, Ingersoll, and Ross (CIR) (1985); Brennan and Schwartz (1979); Longstaff and Schwartz (1992) among others.

Since fluctuations in the interest rates vary in different parts of the yield curve, a richer framework with several stochastic factors in addition to the short rates are necessary in order to capture interdependence across all maturities. Litterman and Scheinkman (1991) show that additional to the short rates, there are two other factors significant in explaining the dynamics in medium rates and the long rates. The first factor is the slope oscillation component that captures the differential effect in the short end and the long end of the curve, leading to relative steepening or flattening at different ends. The second factor is the curvature component that captures the differential effect on all three segments of the curve (short, medium, and long rate), leading to a "humped" shape to the yield curve. A technique through which these features may be determined is principal component analysis.

The equilibrium models are generally adopted by traders who trade the yield curve. These models can be used to spot possible mispricing in bond portfolios. The traders pick a one or multi factor equilibrium model and then calibrate the model using time series data. Then each day they reconsider the parameters (inputs) in order to model the current term structure. The difference between the yield curves given by the model and that existing in the market on that day is the potential arbitrage opportunity.

The main criticism of the equilibrium models is that they do not automatically fit the current market term structure. In the equilibrium approach, we assume a functional form for the drift in the diffusion process of the short rate equation. Then interest rate securities such as zero coupon bonds are priced using this assumed drift. In this approach, we find that bond prices given by the model are usually inconsistent with prices observed in the market. Therefore the parameters are reconsidered every day in order to fit the current yield curve. However the fit is never perfect.

The drawbacks of equilibrium models are overcome by modelling interest rates using a different approach of non-arbitrage where assets in the economy are assumed to be arbitrage free. In this approach, interest rate models are designed to be exactly consistent with the current market term structure. Here the current yield curve is observed in the market and the dynamics of this yield curve is then modelled with the constraint that the changes in yield curves would not produce any arbitrage situations. In this approach we fit the drift to the observable market prices of the securities of zero coupon bond prices and not the other way round.

These models are generally preferred by interest rate option traders. The traders would use the market yield curve and solve for the model with non-arbitrage conditions. Even with the non-arbitrage models, an options trader is required to reconsider the model each day in order to match the implied volatility given by the market. Clearly, an equilibrium model cannot be used when trading derivatives such as options on bonds. This is because the underlying bond price has to match the market term structure so that the trader can simultaneously hedge his options position by using the underlying.

A common non-arbitrage model is by Heath, Jarrow, and Morton (HJM) (1992) that develops a process for the instantaneous forward rates and uses the non-arbitrage relationship implied by the bond prices in order to determine the drift and diffusion parameters of the rates. The drift term can be expressed in terms of the forward rates volatility processes and captures the full dynamics of the entire forward rate curve.

Using a exponential factor loading specification for the forward rate curve, Nelson and Siegel (1987) developed a yield factor model that produced three factors that could be interpreted as level, slope, and curvature as in Litterman and Scheinkman (1991). Diebold and Li (2006) show that the Nelson-Siegel model fit the yield curve quite well and also produce accurate forecasts as compared to other benchmarks.

1.3 Modelling zero coupon yield curve using factor models

Consider the factor representation for the term structure

$$Y_t = \alpha + \gamma F_t + \varepsilon_t \tag{1.2}$$

where Y_t is the $N \times 1$ vector of yield maturities observed at time t, F_t denotes the $k \times 1$ vector of yield curve factors with the factor loadings matrix γ which is of dimension $N \times k$. ε_t is the idiosyncratic errors of the system.

In an attempt to parsimoniously model the behaviour of the yield curves over time, factor models (such as the above) have been widely implemented in literature. Several authors have suggested different parameterizations and econometric techniques in order to implement this general framework in the case of yield curves.

We document below two statistical framework that have been prominently applied to yield curve modelling, namely principal component models and exponential component models.

1.3.1 Principal components modelling

In the above factor model (equation 1.2), factors driving the yield cuves are latent and therefore needs to be estimated. One means of estimating these latent factors is using Principal Component Analysis (PCA) where the estimated principal components are the latent factors and the factor loadings are given as by the scaled eigenvectors. PCA is a statistical data exploration technique that has been employed to provide insights to interest rate movements and to summarize the significant shifts in the yield curve by means of a few factors. This data decomposition techniques have been first advocated for yield curve modelling by Steeley (1990) and Litterman and Scheinkman (1991) who found that the primary sources of interest rate risks summarized by a covariance structure can be decomposed into first few principal components. The first three principal components have been named level, slope, and curvature based on its economic behaviour and has been shown to capture around 95-98% of variations observed in yield curves. The first principal level factor is the most significant one, explaining all the level shifts or parallel movements noticed along yield curve maturities. The slope and curvature factors measure the non-parallel movements along the yield curve. Though the non-parallel risks explain a small amount of the overall variations observed, the factors have been shown to be significant for the purpose of risk management.

Lord and Pelsser (2007) investigate whether the systematic shapes (level, slope, and curvature) extracted by the PCA are merely an artefact of principal component decomposition. Some suggest that the shapes hold for any factors extracted from a highly correlated and ordered system such as a bond yield term structure. The paper provides the sufficient conditions under which a term structure factors display the level-slope-curvature effect. They show that the ordered correlation matrix with positive elements display such effects. Others suggest that the effect arised due to the method constructing the curves. Using simulations Lekkos (2000) show that when the forward rates are independent, the correlation matrix governing the zero coupon yields display level, slope, and curvature.

Suppose matix Σ explains the centered interest rate changes (covariances) or the standardized changes (correlations) in interest rates. Using PCA, we estimate the eigenvalues $\lambda_1 \geqq \lambda_2 \geqq ... \geqq \lambda_N$ of the matix Σ satisfying the equality

$$|\Sigma - \Lambda I| = 0 \tag{1.3}$$

where $\Lambda = (\lambda_1, \lambda_2, ..., \lambda_N)'$ and their corresponding vectors $\beta_1, \beta_2, ..., \beta_N$ satisfying the two conditions

$$\Sigma\beta_i = \lambda_i\beta_i \tag{1.4}$$

$$\beta_i'\beta_i = 1 \tag{1.5}$$

The conditions ensure that the characteristic vectors β_i for $i = 1, 2, ..., N$ are orthogonal to each other and are of unit length. The factor loadings matrix γ is by definition the unit length eigenvectors scaled by its singular value (eigenvalues) where

$$\gamma_i = \beta_i\lambda_i^{1/2} \quad \text{for } i = 1, ...N \tag{1.6}$$

Geometrically, PCA is a multivariate procedure in which we rotate the axes in multidimensional space such that maximum variability in term structure data is projected onto its principal axes (or principal components). In other words, we rotate the axes in order to find linear combinations (or principal components) of the original variables that can summarize as much information, in the original variable, as possible. Since the original term structure is raw and noisy, the PCA technique tries to find a linear *basis*[1] that is able to effectively express the original dataset. This first rotation of axes gives us the first basis vector or the first principal component. Since the basis vectors are orthonormal, the PCA restricts the subsequent rotations to directions perpendicular to the previous directions. This means that

[1]A basis is a set of vectors that, in a linear combination, can represent every vector in a given vector space, and such that no element of the set can be represented as a linear combination of the others. In PCA, the Basis vector is extracted using eigen decomposition and are the eigenvectors.

the second basis vector captures the maximum variations in term structure data that is orthonormal to the first; the third basis vector captures the maximum variations orthonormal to the first and the second basis vectors; and so on. This process gives us N basis vectors (principal components). In PCA we impose the linearity assumption in order to filter noise in the term structure data and express the data as a linear combination of its basis vectors.

Alternative to using the PCA in extracting the factors of yield curves, Matzner-Løber and Villa (2004) propose using a functional PCA . This functional analog of the PCA considers extracting the principal components from data that is assumed to be a set of functions or curves. Unlike the PCA, at each step the FPCA estimate orthogonal basis functions (than basis vectors) that accounts for the maximum variations. In this case, the primary aim is to summarize most of the variations in data with fewer basis functions as possible. The paper finds that the resuts from PCA and FPCA are indeed different. FPCA has a distinct advantage that one can also estimate the yields for maturities not observed in the original term structure.

PCA assumes a stable contemporaneous correlation structure over time. In the case of bond yield curves, Periognon and Villa (2006) test the time-varying nature of covariances and advocate for the use of common principal component analysis (CPCA). The changing nature of variances is well supported by empirical evidence. Bliss (1997), Phoa (2000), Chapman and Pearson (2001) show that even though the risk factors of the term structure have remained constant over time, the variances of the factors have been time-varying. CPCA accounts for the time-variation of the eigenvalues with constant eigenvectors and therefore a practical extension of PCA in the field of modelling bond yield curves. Since the methodology is based on maximum likelihood, tests based on the likelihood function can be constructed in order to test several empirical restrictions on yield curves. Implementing this methodology on Fama-Bliss bond yields term structure, Periognon and Villa (2006) show that appointment of new Federal Reserve chairmans played an important role in characterizing the time variation in loadings of the common factors driving interest rates.

1.3.2 Exponential components framework

This class of models provide a functional form for the instantaneous forward rates and model the yield curves under the objective measure. The models belonging to this class are also called the Nelson-Siegel class of models following the seminal contribution by Nelson-Siegel (1987). Central banks and fixed income traders heavily rely on directly the Nelson-Siegel model or modified version of this model. The Bank of International Settlements (2005) technical documentation show that the participating central banks of Belgium, Finland, France, Germany, Italy, Norway, Spain, and Switzerland use the Nelson-Siegel class of models to estimate zero-coupon yield curves. Bernadell et al. (2005) report that the foreign reserve management of the European Central Bank use a regime switching Nelson-Siegel model. Given the simplicity in estimating the parameters using a two step cross-sectional OLS regression (as in Fabozzi et al. (2005), Diebold and Li (2006)) and the time series forecastability properties of the model, as documented by Diebold and Li (2006), these models have been widely implemented in practice. A series of research papers by Diebold and coauthors, and others have shown the usefulness and extension of these models in various settings. In this section, we briefly outline the Nelson-Siegel class of models as introduced in literature.

Consider the price of a zero-coupon bond

$$p(t,T) = e^{-y(t,T)(T-t)} \tag{1.7}$$

where $y(t,T)$ is the yield to maturity of a bond with time to maturity $\tau = T - t$.

From the above, if we know the price of a zero-coupon bond in the market, we can calculate the yield to maturity as

$$y(t,T) = -\frac{\ln p(t,T)}{(T-t)}$$

Since we usually observe coupon bonds traded mainly for long term maturities, we can extract the zero-coupon bond prices from the coupon bearing bonds. A coupon bearing bond can be seen as a sum of constituting coupons, each coupon payment seen as a single zero-coupon bond. Suppose c_τ is the coupon paid at time τ, then the price of a coupon bond can be written as

$$p(t,T) = \sum_{\tau=t}^{T} c_\tau \cdot e^{-y(t,\tau)(\tau-t)} \tag{1.8}$$

where $y(t,\tau)$ is the zero coupon spot rate at time t for a time to maturity of τ periods. The spot rate curve can be determined by calculating $y(t,\tau)$ for different times to maturity $\tau - t$. Given the spot rate curve, the forward rate at time t for investments at time τ for time to maturity $T-\tau$ can be determined by the formula

$$f(t,\tau,T) = \frac{(T-t)\,y(t,T) - (\tau-t)\,y(t,\tau)}{T-\tau} \tag{1.9}$$

The instantaneous forward rate can be obtained for infinitesimal time to maturity

$$\begin{aligned} f(t,\tau) &= \lim_{h\to 0} f(t,\tau,\tau+h) \\ &= y(t,\tau) - (\tau-t)\,\frac{\partial y(t,\tau+h)}{\partial y} \\ &= -\frac{p'(t,\tau)}{p(t,\tau)} \end{aligned}$$

Integrating, we find that the zero coupon rate is an equally weighted average forward rate

$$y(t,T) = \frac{1}{(T-t)} \int_t^T f(t,\tau)d\tau \tag{1.10}$$

Nelson and Siegel (1987) considered the instantaneous forward rate for time to maturity τ to be

$$f(\tau) = \beta_1 + \beta_2\left(e^{-\tau/\lambda}\right) + \beta_3\left(\frac{\tau}{\lambda}e^{-\tau/\lambda}\right) \tag{1.11}$$

where $\beta_0, \beta_1, \beta_2$ are the coefficients and λ is a constant decay parameter. Solving the integral, we get the Nelson-Siegel formulation for the yield curve

$$Y(\tau) = \beta_1 + (\beta_2 + \beta_3)\left(\frac{1 - e^{-\tau/\lambda}}{\tau/\lambda}\right) - \beta_3\left(e^{-\tau/\lambda}\right) \tag{1.12}$$

Nelson and Siegel (1987) show that the assumed exponential polynomial functions are general enough to generate monotonic and humped shaped curves as observed in the market. The parameters of the model have the following explanations: β_1 specifies the long rates to which the forward rate curve horizontally asymptotes, β_2 is the weight attached to the short term component, and β_3 is the weight attached to the medium term component. The parameter λ is a time constant measuring the point of the beginning of the decay.

Svensson (1994) provides a generalization to higher order models including four factors. The Svensson (1994) model incorporated an additional hump shaped parameter with a seperate decay parameter. The forward rate is assumed to take the form

$$f(\tau) = \beta_1 + \beta_2\left(e^{-\tau/\lambda_1}\right) + \beta_3\left(\frac{\tau}{\lambda_1}e^{-\tau/\lambda_1}\right) + \beta_4\left(\frac{\tau}{\lambda_2}e^{-\tau/\lambda_2}\right) \tag{1.13}$$

and the yield curve formulation of the model becomes

$$Y(\tau) = \beta_1 + \beta_2\left(\frac{1 - e^{-\tau/\lambda_1}}{\tau/\lambda_1}\right) + \beta_3\left(\frac{1 - e^{-\tau/\lambda_1}}{\tau/\lambda_1} - e^{-\tau/\lambda_1}\right) + \beta_4\left(\frac{1 - e^{-\tau/\lambda_2}}{\tau/\lambda_2} - e^{-\tau/\lambda_2}\right) \tag{1.14}$$

where the parameter β_4 is the loadings to the additional medium term component introduced in the model. This generalization of the model helps fit the data well as it accounts for term structures with more than one maximum or minimum along the maturity spectrum. Recently, concerns have been raised about multicollinearity problems when $\lambda_1 \approx \lambda_2$. De Pooter (2007) introduces an "Adjusted" Svensson (1994) model that addresses this issue. The paper proposes forward rates taking the form

$$f(\tau) = \beta_1 + \beta_2\left(e^{-\tau/\lambda_1}\right) + \beta_3\left(\frac{\tau}{\lambda_1}e^{-\tau/\lambda_1}\right) + \beta_4\left(e^{-\tau/\lambda_2} + \frac{2\tau}{\lambda_2}e^{-2\tau/\lambda_2} - e^{-2\tau/\lambda_2}\right) \tag{1.15}$$

and the "Adjusted" Svensson (1994) yield curve model is

$$Y(\tau) = \beta_1+\beta_2\left(\frac{1-e^{-\tau/\lambda_1}}{\tau/\lambda_1}\right)+\beta_3\left(\frac{1-e^{-\tau/\lambda_1}}{\tau/\lambda_1} - e^{-\tau/\lambda_1}\right)+\beta_4\left(\frac{1-e^{-\tau/\lambda_2}}{\tau/\lambda_2} - e^{-2\tau/\lambda_2}\right) \quad (1.16)$$

Comparing the fit of the adjusted version of the Svensson model to the original model, the paper finds that the adjusted fourth component is steeper and increases at a faster rate.

Various extensions of these models have been proposed to increase flexibility. Bjork and Christensen (1999) introduced an additional fourth factor to the Nelson-Siegel model forward curve function

$$f(\tau) = \beta_1 + \beta_2\left(e^{-\tau/\lambda}\right) + \beta_3\left(\frac{\tau}{\lambda}e^{-\tau/\lambda}\right) + \beta_4\left(e^{-2\tau/\lambda}\right) \quad (1.17)$$

and the four factor model of the yield curve is

$$Y(\tau) = \beta_1 + \beta_2\left(\frac{1-e^{-\tau/\lambda}}{\tau/\lambda}\right) + \beta_3\left(\frac{1-e^{-\tau/\lambda}}{\tau/\lambda} - e^{-\tau/\lambda}\right) + \beta_4\left(\frac{1-e^{-2\tau/\lambda}}{2\tau/\lambda}\right) \quad (1.18)$$

where the parameter β_4 is the weight to fourth component that is attached to short-term maturities. In this model, the slope factor of the term structure is captured by the weighted sum of β_2 and β_4 coefficients. Bliss (1997) estimated the term structure by relaxing the assumption of constant decay parameter λ for the short and meduim term components. The paper assumes the forward rate structure to be

$$f(\tau) = \beta_1 + \beta_2\left(e^{-\tau/\lambda_1}\right) + \beta_3\left(\frac{\tau}{\lambda_2}e^{-\tau/\lambda_2}\right) \quad (1.19)$$

and the yield curve model is

$$Y(\tau) = \beta_1 + \beta_2\left(\frac{1-e^{-\tau/\lambda_1}}{\tau/\lambda_1}\right) + \beta_3\left(\frac{1-e^{-\tau/\lambda_2}}{\tau/\lambda_2} - e^{-\tau/\lambda_2}\right) \quad (1.20)$$

The yield curves of the model collapses to the original Nelson-Siegel curves when $\lambda_1 = \lambda_2$

Recently, Diebold and Li (2006) modified the Nelson-Siegel (1987) model to provide economic meaning to the three components of the model and provide a dynamized version of the model widely known for forecastability. The reformulation of the Nelson-Siegel model is given by

$$Y_t(\tau) = \beta_{1t} + \beta_{2t}\left(\frac{1-e^{-\tau/\lambda}}{\tau/\lambda}\right) + \beta_{3t}\left(\frac{1-e^{-\tau/\lambda}}{\tau/\lambda} - e^{-\tau/\lambda}\right) \tag{1.21}$$

where β_{1t} is the factor loading associated with the first component which can be interpreted as the level factor, β_{2t} is the factor loading for the second component which captures the slope factor mostly influencing the short term factors, and β_{3t} is the factor loading for the third component associated with the medium rates, interpreted as the curvature factor loadings. The paper proposes a two-step OLS cross-sectional regression procedure to estimate the unknown parameters. Alternatively, one can represent the framework in a state space formulation and use Kalman filtering to estimate the parameters. Koopman, Mallee, and van der Wel (2007) analysed the dynamic Nelson-Siegel model of Diebold and Li (2006) by introducing time varying factor loadings. They introduce a step function and a cubic-spline function to the decay parameter λ and investigate the evolution of the factor loadings. Further, the paper also includes common time-varying volatility factors specified as a spline function. These generalization show significant improvements in model fit and forecastability.

The Dynamized Nelson-Siegel model as proposed by Diebold and Li (2006) has become popular in fitting the cross-section of bond yields and significant in explaining the yield curve dynamics. The dynamic factors extracted from the model corroborate to the interpretation of level, slope, and curvature. Unifying the cross-sectional and time series properties into the model has proved to generate economically significant forecastability both in the short and long horizons as shown by Diebold and Li (2006). Diebold, Ji, and Li (2006) show that there exist no three factor affine term structure model that generates the Nelson-Siegel (1987) three factors. The paper suggest that the unforecastability of affine term structure models documented in literature could not be therefore contributed to the Nelson-Siegel factors. Further, the papers finds that the cross section of yields are well explained by the loadings of the model.

Diebold, Rudebusch, and Aruoba (2006) extend this yield model also to include macroeconomic variables in order to study the dynamic interactions between the macro economy and yield factors. They find evidence of bilateral interactions between macro variables and future movements in yield curve. This flexibility of matching changes in yield curves to exogeneous variables have also lead to studies about cross-country yield curve interactions. Diebold, Li, and Yue (2007) study dynamic properties of global bond yield curves by modelling the cross-country yield curve factors. They extract the global bond yield factors from U.S., Germany, Japan, and U.K. government bond yields, decompose the variations of country yields into global and idiosyncratic components, and study the evolution of the dynamics of global bond yield curve factors. The paper finds existence of economically significant global factors with global level and global slope factors reflecting global inflation and global real activity. In a similar vein; Koivu, Nyholm, and Stromberg (2007) model the yield curves across multiple currency areas in order to capture cross-effects of adverse movements in one currency area affecting the dynamic evolution of other currency areas. The paper defines one currency economy yield curve as the 'cardinal' yield curve and models cross-country interactions as spreads to the cardinal yield curve. For the U.S., German, and Japanese markets; the paper finds that the spreads are well modelled with the Nelson-Siegel model with little estimation errors. The Nelson-Siegel model has also been applied to predicting US NBER recessions from 1973 to 2004 by Nyholm (2007). Since central banks intervene by lowering (or increasing) short rates to stimulate (or dampen) economic growth, the evolution of the slope factor capturing the shape of the yield curve is known as a good predictor of economic activity. The paper implements a regime switching dynamic Nelson-Siegel model where the slope factor exhibit switches between regimes based on the states of macroeconomic variables predicting recession. The findings of the paper provide a new approach to modelling and forecasting recessions.

Given the widespread empirical merits of the dynamic Nelson-Siegel model, various authors have also considered investigating the theoretical consistency of this framework. Since the model does not fall under the affine class of term structure models (Diebold, Ji, and

Li (2004)) and initially shown to unsatisfy the principles of non-arbitrariness (Bjork and Christensen (1999)), concerns have been raised in using this model. Recently, two papers contribute to this research question, Christensen, Diebold, and Rudebusch (2007) and Coroneo, Nyholm, and Vidova-Koleva (2008). The first paper derives an arbitrage correction term that is incorporated into the dynamic Nelson-Siegel model to ensure arbitrage-freeness of the model. The authors find that imposing the non-arbitrage constraints significantly improves the forecastability of the model, supporting the imposition of the restrictions. The predictive gains are more significant in the case of moderate to long term rates and for long forecast horizons. Coroneo, Nyholm, and Vidova-Koleva (2008) confirms the non-arbitrariness of the Nelson-Siegel model using a very different approach. The paper conducts a statistical test for equality between factor loadings of Nelson-Siegel model and factor loadings derived from an non-arbitrage model. The find that factor loadings of the two models are statistically indifferent. Performing an out-of-sample forecasting exercise, the authors show that both the models performed equally well. The results of the paper indicate that non-arbitrariness of the model might be induced by misspecification of the imposed Nelson-Siegel curvature factor loadings structure.

1.4 Panel factor analysis

Factor analysis and principal component analysis have been originally developed in order to capture the main sources of variations and covariations among the N independent random variables in a panel framework. These methods were extended by Geweke (1977) and Brillinger (1964) into dynamic factor models and dynamic principal component analysis respectively, that were able to predict the covariation in economic variables by few underlying latent factors. Although the two methods differed for small N, they gave similar inferences as N increased and was large. Chamberlain and Rothschild (1983) then distinguished the dynamic models into exact and approximate dynamic factor models. In the case of exact dynamic factor models, the idiosyncratic terms are assumed to be mutually uncorrelated whereas the approximate factor models relaxes this restriction and allows for limited correlation among the idiosyncratic terms. Applications in finance particularly favor the approximate factor models where the idiosyncratic terms are weekly correlated and where large number of cross-sectional units can be competently summarized by a few common factors.

Consider the factor representation for N cross-sectional by T time-series term structure panel:

$$Y_t = \gamma' F_t + \varepsilon_t \tag{1.22}$$

where Y_t is an $N \times 1$ vector of cross sectional observations from the panel data structure at time period t, γ is an $r \times N$ matrix of the factor loadings, F_t is the $r \times 1$ vector of common factors for all cross-sectional units at time period t, and ε_t is the $N \times 1$ vector of idiosyncratic components which are allowed to be weakly correlated in an approximate factor model framework. $\gamma' F_t$ are known as the common components. The sources of idiosyncrasies in bond prices include bid-ask effects, tax effects, liquidity, and transaction costs that can be explained by using an approximate factor structure to the model. In the term structure literature, it is reasonable to assume that the errors are independent of the factors, i.e $E\left[\varepsilon_t F_{t-s}'\right] = 0$ for all s. Since yields and the common components are functions of the maturities, empirical evidence suggest high positive cross correlations between error terms of neighbourby maturities (say $\varepsilon_{it}, \varepsilon_{jt}$ for i and j close to each other). Also, yields in levels models are shown to have strong positive time series correlations, that can be removed by taking forward differences (see De Jong (2000)).

Since the right hand side variables are unobservable, one can use principal component analysis in order to estimate the factors and their loadings. In classical factor analysis, where the cross-sectional dimension N is finite, the covariance matrix of the yields, $\Sigma = \Lambda\Lambda' + \Omega$ where Ω is the covariance matrix of the errors ε_t and under some assumptions, we can obtain a root-T consistent estimator of Σ, say $\hat{\Sigma}$ where

$$\hat{\Sigma} = \frac{1}{T-1}\sum_{t=1}^{T}\left(Y_t - \bar{Y}_t\right)\left(Y_t - \bar{Y}_t\right)' \tag{1.23}$$

This forms the basis for using the principal component analysis in order to estimate the factor loadings in the case of N fixed and large T. Bai (2003) show that for panels with N fixed, using OLS one can only estimate the factor loadings (γ) consistently but not the factors (F_t). For large N and T panels, both the factors and their loadings can be estimated consistently.

The issue of consistency of the estimated common components clearly depend on the dimensions of the panel under consideration. Connor and Korajczyk (1986) prove the consistency of the latent factors estimated via principle component analysis when the cross-sectional dimension (N) is greater than the time-series dimension (T). However, from standard factor analysis literature (see Anderson (2003)) it is well-known that consistent estimation of latent factors cannot be achieved when either N or T is finite. Bai and Ng (2002) and Bai (2003) developed an inferential theory for large panels where the cross-sectional dependence is explicitly considered in the factor loadings representation. Bai (2003) proves consistency of the estimated factors and the factor loadings, for N and T tending infinity. It also shows consistency of the estimates when T is fixed but under the assumption of (i) asymptotic orthogonality between the estimated factors and the idiosyncratic errors, and (ii) asymptotic homoskedasticity in the idiosyncratic errors, for all T as N tends to infinity. The results suggest that the limiting distributions of the "estimated factors" are asymptotically normal and the rate of convergence of the estimated factors is min $\left\{\sqrt{N}, T\right\}$. When the true factor loadings are all known, then the true factors can be estimated by cross section least squares and the rate of convergence of estimated factors is $\sqrt{N}$. Else if the factor loadings are unknown and estimated, then the rate of convergence of the estimated factors is min $\left\{\sqrt{N}, T\right\}$. In the case of the "estimated factor loadings", the limiting distributions is again asymptotically normal and the convergence rate is min $\left\{N, \sqrt{T}\right\}$. When the true factors are all observable, then the true factor loadings can be estimated by time series regression and the rate of convergence of estimated factor loadings is $\sqrt{T}$. Else if the true factors are unknown and estimated, then the rate of convergence of estimated factor loadings is min $\left\{N, \sqrt{T}\right\}$. A recent paper, Heaton and Solo (2006) evaluated the importance of using large dimensional panels in approximate factor models framework. The paper showed that the consistency and rate of convergence derived for the case of large dimensional panels depend on the rate at which the cross-sectional correlation of the model disturbances grow, as N grows to infinity. The implication of this finding is the empirical application of these asymptotic results for real world situations where we work with panels of finite dimensions.

Since the factors are generally unobserved, they can be estimated from sample covariance matrices using statistical techniques. There exist estimation procedures such as maximum likelihood and principal component analysis in order to estimate the latent factors and their factor loadings. Though the maximum likelihood method is successful to estimate factors in low-dimensional systems, there arise computational complexities in maximizing the likelihood functions over large number of variables as the cross-sectional dimensions increase. Principal component analysis however is easy to compute for higher dimensional panels and therefore have been widely used in the term structure literature. Chamberlain and Rothschild (1983) show that the principal component estimators converge to the maximum likelihood estimators as N increase. Bai and Ng (2006) developed test statistics in order to compare the observed economic variables with estimates of the observed factors and gather inferences on whether the observed variables were in fact the underlying unobserved factors.

Another important question concerns the number of factors in existence. The number of factors governing the term structure of interest rates have been commonly assumed to be three. The three factors namely level, slope, and curvature have been found to significantly explain most of the variations in the yield curve. Therefore we do not estimate the number of factors and assume the number of latent factors to be given and equal to three. Connor and Korajczyk (1993) estimated the number of factors using sequential limit asymptotics whereby N tends to infinity first, followed by T. Cragg and Donald (1997) shows that the BIC information criterion could be used to infer about the rank of a consistently estimated sample covariance matrix for N and T fixed. Stock and Watson (1999) assumes that $\sqrt{N}/T$ goes to infinity and estimates the number of latent factors. Bai and Ng (2002) considers N and T to be infinite and develops a statistical procedure that consistently estimates the number of factors from observed data. The paper allows for hetroskedasticity in both the cross-sectional and time-series dimensions, and weak serial dependence and cross-sectional dependence. The paper proposes that the true number of factors r can be estimated by minimizing one of the following information criteria:

$$IC_1(r) = \ln(V(r,\hat{F}^r)) + r\left(\frac{N+T}{NT}\right)\ln\left(\frac{NT}{N+T}\right); \quad (1.24)$$

$$IC_2(r) = \ln(V(r,\hat{F}^r)) + r\left(\frac{N+T}{NT}\right)\ln C_{NT}^2; \quad (1.25)$$

$$IC_3(r) = \ln(V(r,\hat{F}^r)) + r\left(\frac{\ln C_{NT}^2}{C_{NT}^2}\right). \quad (1.26)$$

where F^r are the r factors, and $V(k,F^r)$ be the overall sum of squared residuals (divided by NT) from time series regression on Y_i for r factors for all i, defined as

$$V(r,F^r) = \min_r (NT)^{-1}\sum_{i=1}^{N}\sum_{t=1}^{T}(Y_{it} - \gamma_i^{r\prime}F_t^r)^2 \quad (1.27)$$

The estimated number of factors for r is obtained from minimizing the information criterion in the range $r = 0,1,...,r_{\max}$ where $r_{\max}$ is some pre-specified upper bound for the number of factors. The convergence rate $C_{NT}^2 = \min\{N,T\}$ in the stationary model framework. For a non-stationary framework, Bai (2004) shows that $C_{NT}^2 = \min\{N,T^2\}$. The finite sample properties for the information criteria show that in small panels IC_3 is less robust than IC_1 and IC_2, however, as N and T exceed 40, the estimates are rather precise with small standard deviations.

Recently Amengual and Watson (2007) proposed an extension of Bai and Ng (2002) by modifying the information criteria for the case of consistently estimating the number of dynamic factors for large N and T approximate factor model framework. Estimating the dynamic factors is important when working with dynamic factor models and care should be taken to distinguish between the *number of significant static factors*, estimated by decomposing the covariance matrix of the original data and the *number of significant dynamic factors*, estimated from the spectral density matrix of the orginal data; governing the underlying dynamic factor model. In the recent paper, Bai and Ng (2007) proposed an ultimate methodology for determining the number of dynamic factors from the estimated number of static factors, without being required to estimate the dynamic factors itself. The static factors were estimated using the Bai and Ng (2002) information criteria.

Chapter 2

TESTING FOR INSTABILITY IN FACTOR STRUCTURE OF YIELD CURVES

ABSTRACT

A widely relied upon but a formally untested consideration is the issue of stability in factors underlying the term structure of interest rates. In testing for stability, practitioners as well as academics have employed ad-hoc techniques such as splitting the sample into a few sub-periods and determining whether the factor loadings have appeared to be similar over all sub-periods. Various authors have found mixed evidence on stability in the factors. In this chapter, we develop a series of hypotheses and statistically evaluate the factor structure stability of the US zero coupon yield term structure. We find that the level, slope, and curvature factors were indeed unstable during the sample period considered. The level instability was caused due to structural changes common to all maturities; the slope instability was caused due to structural changes affecting the short rates; and the curvature instability was caused due to structural changes affecting the long rates. We find evidence of the presence of common economic shocks affecting the level and slope factors, unlike slope and curvature factors that responded differently to economic shocks and were not affected by any common instability.

2.1 Introduction

Statistical models using factor decomposition techniques such as principal component analysis (PCA), where the yield curve dynamics can be summarized by a few estimated principal factors have been highly favored in extracting the yield curve factors. Instabilities present within the factor structure of yield curves have commonly been assumed to be nil. This inherently implies that the latent factors, generally extracted using PCA rotations, are robust to structural changes. Some authors informally test for factor structure instabilities using graphical methods. The term structure literature that uses statistical factor models have all relied upon graphical methods for analysing the stability of the factors. The standard procedure implemented in this regard has been to divide the sample data into sub-periods and to identify the factor loading for the corresponding sub-periods. If the explanatory power of the factor loadings appeared to be similar over all sub-periods, then the factors were said to be stable over time. There have been no other formal tests conducted in this respect except a recent paper by Audrino et al. (2005) that concluded instability in the filtered innovations of the principal factors governing the US Discount bond yields. The instability detected in the paper could not however be interpreted directly as instability in the level, slope, or curvature factors.

This chapter develops a formal stability test on all the eigenspace variables associated with the level, slope, and curvature factors of the US zero coupon yield term structure. In particular, we consider instabilities in factors that are associated with instabilities in eigenvalues, instabilities in eigenvectors, and instabilities in the factor loadings governing the system. These eigenspace variables are estimated using PCA. We formalize a series of hypotheses in order to test for instabilities present in the eigenspace variables of the factors. To anticipate the results, we find that the eigenvalues (volatility) of the level, slope, and curvature factors were unstable over the sample period considered. The level instability was caused due to structural changes common to all interest rate maturities; the slope instability was caused due to structural changes affecting the short rates; and the curvature instability was caused due to structural changes affecting the long rates. We find evidence of the presence of common economic shocks affecting the level and slope factors, unlike slope and curvature factors that responded differently to economic shocks and were not affected by any common instability.

The remainder of this chapter is structured as follows. Section 2.2 provides an account of the instability in yield curves documented in literature. We motivate the formal testing of instability in the yield factor structures by examining the evolution of eigenspace variables (eigenvalues, eigenvectors, and factor loadings) graphically. In Section 2.3 we present the factor analysis framework for the term structure level, slope and curvature factors, estimated using the principal component analysis. We provide the asymptotic properties of the estimated eigenspace variables for the three factors, which is applied into developing the stability testing procedure. In Section 2.4 we formulate six hypotheses for statistically evaluating the stability in the eigenspace variables governing the level, slope, and curvature factors of the yield curves and device the test statistics for evaluating each hypothesis. Section 2.5 describes the dataset used and presents the results of the testing procedure developed in previous sections. Section 2.6 concludes with the summary and findings of the study. The relevant proofs are presented in the Appendix of the chapter.

2.2 Yield curve dynamics instability

Modelling the dynamics of interest rates is vital in trading fixed income securities that are sensitive to movements in interest rates. The concern is to fit the interest rates data within a framework (model) that is able to capture the future evolution of the term structure of interest rates. This is important for valuation of securities such as interest rate derivatives. Also, given that we understand the process that governs the interest movements, we are able to analyse and alter the risk exposures at a given point of time. Bliss and Smith (1997) argue that model selection and stability of the parameters underlying the process are closely related. The paper illustrates by critically examining the findings of Chan et al. (1992) and show that the unaccounted structural break, caused due to Fed change in the monetary policy, has indeed affected the conclusions drawn.

Structural changes have also been modelled by allowing for regime switches in interest rates. Following the seminal work of Hamilton (1989) that introduced modelling the short rates using a regime switching process, authors such as Lewis (1991), Evans and Lewis (1995), Garcia and Perron (1996), Gray (1996), and Ang and Bekaert (2002) have studied regime switches in interest rate models. Empirical evidence suggest that not only the short rates but also the whole term structure of interest rates might experience shifts in regimes caused due to business cycle expansions and contractions, changes in monetary policies and regime changes in economic variables such as consumption and inflation. Bansal and Zhou (2002) show that term structure models incorporating regime shifts provide considerable improvements over multifactor CIR and affine models. They develop a model allowing for regime switches in both the state vector and the risk premium and show that the model accommodates for the conditional joint dynamics (the conditional distribution) of short and long yields.

The presence of instabilities in the short and long term yields can also seep into the factor structures governing these yields. One of the earlier work in factor analysis of term structure of interest rates is the Nelson-Siegel (1987) model. This parsimonious representation is very popular among practitioners for calibrating the yield curve. Since the model is linear in coefficients, they are estimated using ordinary least squares. The coefficients of the yield curves were interpreted to be level, slope, and curvature. Various other authors have found the same statistical interpretation to the coefficients estimated via statistical techniques such as the principal component analysis and factor analysis. Litterman and Scheinkman (1991) show that the three principal factors, explaining around 99 percent of the changes in treasury bond yields, could be interpreted to be the level (or parallel movement component), slope (or slope oscillation component), and curvature component. The level factor or the parallel movement component alone was the most important factor that accounted for an average of 89 percent of the variations observed in the yield changes data.

Given the widespread use of factor analysis for term structure of interest rates, there arises a need to evaluate the factor structure stability of interest rates. Many authors have assumed that the principal factors driving the evolution of interest rates are stable or robust through time. Some use ad-hoc methods to investigate factor stability. For instance, Bliss (1997) divided the sample period January 1970 – December 1995 into three sub-periods of arbitrary lengths and investigated the change in the factor loadings. Since the factor loadings patterns in the different sub-periods seemed similar in the case of all three factors, the factors were concluded to be stable. However, the factor volatilities were found to fluctuate over the sub-periods considered. In the forecasting setting using the Nelson-Siegel model, Diebold and Li (2006) found similar results with stable factors and time-varying factor volatilities. Since the parameters were stable over time, the proposed model produced much accurate forecasts at both the short and long horizons than other standard forecasting benchmarks. Since empirical results noted the time-varying nature of volatility associated with the factors, Perignon and Villa (2006) accounted for a time-varying covariance matrix when estimating the factor structure of interest rates. Using the U.S. term structure data between January 1960 and December 1999, Perignon and Villa observed that the factor structure (factor loadings) remained constant across sub-periods considered but the volatility (eigenvalues) of the factors varied through time. Reisman and Zohar (2004) use the yield to maturity data of US discount bonds from 1982. They found that the first two principal components were quite stable; the third component was marginally stable; and the fourth component was unstable. Fabozzi et al. (2005) used the Nelson-Siegel (1987) model to parameterize twelve monthly yields term structure data from June 7, 1994 to September 5, 2003. They plot the factor loadings from the model, and observed that the level and slope coefficients of the model seemed stable, while the curvature coefficient showed instability. Chantziara and Skiadopoulos (2005) evaluated stability in the principal factors of the term structure of petroleum futures by performing the principal component analysis (PCA) individually on two sub-periods before and after May 1997, the cut-off date being identified as the beginning of the Asian crisis. Since the PCA results for the two sub-periods were not different from the results obtained for the whole sample, the paper concluded stability in the factor structure over the whole sample period.

As it appears empirically, the stability analyses on factors were carried out by graphically plotting the factor loadings and by weighing the similarity in results over time. The standard procedure implemented in this regard was to divide the data into sub-periods and to identify the factor loading for the corresponding periods. If the explanatory power of the factor loadings appeared to be similar over all periods, then the factors were concluded to be stable over time. There were no other formal tests conducted in this regard. The first formal test (to the best of our knowledge) in evaluating stability of factors governing interest rates was introduced in Audrino et al. (2005) that considered a three-factor model with conditional hetroskedastic factors. The paper found contradicting conclusions that the factor loadings of the US discount bond yields were in fact unstable over the period January 1986 to May 1995. The paper used independent filtered innovations in order to find the principal factors for the different sub-periods considered and then using a regression framework on the filtered innovations, tested the hypothesis that the regression coefficients (factor loadings) in the different sub-periods are indeed equal. Since the authors constructed factors on the filtered innovations, the instability detected could not be interpreted as instability of the level, slope, or curvature factors.

The main contribution of this chapter is to introduce a testing framework that would enable us to formally investigate the instability present in the factor structure of level, slope, and curvature of the yield curves.

2.2.1 A first examination of factor structure instability

As evident in the term structure literature, the instability risks present in the yield curves can persist also within its factor structures. As a first examination to this argument, we carry out some graphical analyses for the term structure of US zero coupon bond yields between Jan

1999 and May 2006 obtained from Datastream[1]. First, we arbitrarily split the seven and half year's bond yield data into three approximately equal, two and half year subsample periods; Jan '99 - June '01, July '01 - Dec '03, and Jan '04 - May '06 and graphically investigate whether the eigensystem has remained stable over the three subperiods.

We perform the principal component analysis on the 5 year holding period returns data for the three subsamples, in order to extract the level, slope, and curvature factors that drive the evolution of change in interest rates data. We concur with Litterman and Scheinkman (1991) when we consider the first three principal factors in explaining the evolution of term structure of interest rates. In order to extract the three principal factors using PCA, we perform the following steps:

1. Form the covariance matrix from the change in yields panel data for the three subsample periods considered.

2. Compute the eigenvalues and the corresponding eigenvectors from the covariance matrix for each period.

The eigenvectors are the principal components and the eigenvalues present the explanatory power of the corresponding eigenvectors.

Second, we graphically investigate instability along the short end, medium term, and long end of the yield curve separately over the three subsample periods. For this, we draw the direction of the principal axes (which are the eigenvectors), along with the scatter plot of the original yield changes data for the three subsample periods. In order to visualize the direction of the eigenvectors, we have to limit our analysis to two dimensions. We use the three month and six month rate as a proxy for the short end of the curve; the five year and seven year rate as a proxy for the medium term of the curve; and the ten and twelve year rate as a proxy for the long end of the curve.

[1]The information on the data sources are detailed in section 2.5.1, where we undertake formal empirical tests on the issue of factor structure stability.

Third, in order to examine the evolution of the entire eigenspace, we conduct recursive PCA by expanding the estimation window at every run by including one new observation and then record the evolution of the eigenvalues, eigenvectors and the factor loadings. We undertake two recursive schemes, namely the Forward Recursive Scheme (FRS) and Backward Recursive Scheme (BRS). The two schemes allow us to evaluate stability in an informal way. The FRS allows us to visually gauge the impact of adding one extra observation at each recursion and the BRS allows us to visually gauge the impact of removing one observation at each recursion. The instability can be seen as the abrupt increase in variability at a point in time in the case of the FRS and a reduction in variability at a point in time in the case of the BRS. This FRS and BRS patterns can also be used to check if there are more than one change affecting the variability in the recursion.

Figures *2.1* to *2.7* present the results towards the preliminary study of the issue of instability. Figure *2.1* shows the evolution of three principal factors determined over the three subsamples. We observe that considering the first three principal components would be sufficient in explaining the dynamics of the term structure and therefore we can say that though the three factors vary in detail, the term structure responsiveness to these factors has remained stable over time. This stability result concurs with that recorded by Bliss (1997), Perignon and Villa (2006), and others. However, the bar charts show that the level risks, captured by the first principal component, was the highest in the third subsample period; the slope risks, explained by the second principal component, was the lowest in the third subsample period, and the curvature risks, explained by the third principal component, was the highest in the second subsample period. This means that the shocks to the term structure varied during the subperiods considered, though we are unable to (at the point) make any statistical inference of instability.

[Insert Figures 2.1 to 2.4 here]

Figures *2.2*, *2.3*, and *2.4* plot the short run, medium term, and long run principal axes (also called principal components or directional vectors) for the three subsample period considered.

The two directional vectors are orthogonal to each other by construction. The plot shows how well the principal axes explain the variability in yields. The table below the plot records the eigenvalues (volatility), eigenvectors, and the percentage of variances explained by the two principal components. For the case of short rates, if we compare the direction of the principal axes across the three subsample period, we find that the first principal axis differ across the three subsamples and by the orthogonality condition, so does the second principal axis. Further, we observe that the sample data for the short rates are dispersed distinctly across the three subsample periods. This means there exists different volatility patterns during the three subperiods and support the argument for allowing distinct time-varying covariance matrices. Therefore considering a constant covariance matrix decomposition of principal components may induce instability in the components. For the case of medium term and long term rates governing the yield curves (*2.3* and *2.4* respectively), we find that the two eigenvalues have similar directional vectors for the three subsample periods, with around 99% explanatory power of the variances.

Further consider the recursive plots of the eigenvalues, eigenvectors, and factor loadings (Figures *2.5*, *2.6*, and *2.7*). The plots obtained from the recursion clearly show endurance of instability in the eigensystem. In the case of eigenvalues governing the factors (Figure *2.5*), we can clearly see that the dynamics have not remained the same over time even though the percentage variation explained by the eigenvalues have remained the same. The eigenvalues for the level and curvature factors seems to have one prominent change but the eigenvalues governing the slope seems to have more than one abrupt changes. Looking at the recursion patterns for eigenvectors (Figure *2.6*), the level and curvature eigenvectors show two prominent patterns and the slope eigenvector shows three prominent patterns suggesting possible structural changes in the eigenvectors. In the case of factor loadings (Figure *2.7*), The FRS suggest one possible pattern change in the case of level, and two pattern changes in the case of slope and curvature. However, if we also consider the BRS, we can see there exist one possible intermittent blip in all the three level, slope, and curvature factor loadings. The observations of pattern changes surely corroborate the time-varying nature of the eigensystem, which may have caused possible structural breaks in the series. In order to formally conclude instability in the data, we would further require a formal test that would evaluate the significance of the observed blips in data.

[Insert Figures 2.5 to 2.7 here]

Summarizing the preliminary results, we have used various ad hoc graphical techniques in order to infer about the term structure stability. By evaluating the three arbitrarily split subsamples, we find that the shocks contributing to the level, slope, and curvature instability risks have varied during the three subsamples. Also, we find that the directional axes of the short end interest rates have varied over time. The forward and backward recursive plots give us an idea about the evolution of the eigenspace variables and we find indications of instability in them. In Section 2.4 we develop a formal statistical test for stability and empirically evaluate a series of hypotheses in order to infer about the instability risks associated with the level, slope, and curvature factors.

2.3 Framework and estimation of eigenspace variables

In this chapter, we use the classical principal component analysis framework, which incorporates the approximate structure in the cross sectional correlation among units. The factor structure is considered static, as we do not assume any dynamic evolution for the factors. Consider the stationary representation for term structure of interest rates with cross-sectional (N) and time-series (T) dimension and with (r) factors:

$$Y_t = \gamma' F_t + \varepsilon_t \qquad t = 1, 2, ..., T \tag{2.1}$$

Let $Y_t = (Y_{1t}, ..., Y_{Nt})'$ be the term structure panel with Y_t being an $N \times 1$ vector of cross sectional observations from the panel data structure at time period t, γ is an $r \times N$ matrix of the factor loadings, F_t is the $r \times 1$ vector of common factors for all cross-sectional units at time period t, and $\varepsilon_t = (\varepsilon_{1t}, ..., \varepsilon_{Nt})'$ is the $N \times 1$ vector of idiosyncratic disturbances. In the term structure literature, the number of common factors that are sufficient in order to explain the dynamics of interest rates are commonly established to be equal to three. Therefore we consider the case of $r = 3$. The factor loadings matrix loads the factors on to the variables, explaining the correlation between the factors and the variables. The factor loadings (γ) can be computed as the unit length eigenvectors matrix multiplied by its singular value, which is the square-root of eigenvalues. Thus γ characterizes the unit length eigenvectors in its true size and encompasses in them the information of direction as well as magnitude.

The loadings underlying the factor structure of Y, by definition is a function of eigenvalues and eigenvectors. In order to estimate the loadings, we use the principal component analysis (PCA) that undertakes the eigen decomposition of the covariance matrix Σ of Y. When Σ is unknown, we estimate the sample variance covariance matrix whose elements at position i, j is given as

$$\left[\hat{\Sigma}\right]_{i,j} = \frac{1}{T-1} \sum_{t=1}^{T} \left(y_{it} - \mu_{y_i}\right)\left(y_{jt} - \mu_{y_j}\right) \qquad i, j = 1, ..., N \tag{2.2}$$

where $(y_{i1}, ..., y_{iT})$ for $i = 1, ..., N$ are each independent and identically distributed.

In Principal Component Analysis we estimate the eigenvalues $\lambda_1 \geqq \lambda_2 \geqq ... \geqq \lambda_N$ of the matix Σ satisfying the equality

$$|\Sigma - \Lambda I| = 0 \tag{2.3}$$

where $\Lambda = (\lambda_1, \lambda_2, ..., \lambda_N)'$ and their corresponding vectors $\beta_1, \beta_2, ..., \beta_N$ satisfying the two conditions

$$\Sigma\beta_i = \lambda_i\beta_i \tag{2.4}$$

$$\beta_i'\beta_i = 1 \tag{2.5}$$

The conditions ensure that the characteristic vectors β_i for $i = 1, 2, ..., N$ are orthogonal to each other and are of unit length.

The estimated vectors $\beta_1, \beta_2, ., \beta_i, .., \beta_N$ are such that the vector $\beta_i' Y$ is the directional vector that captures the maximum variability in Y. Therefore the estimation of β_i can be seen as solution to the optimization problem

$$\begin{aligned} & max\ E(\beta_i' Y\ Y' \beta_i) \\ = & \ max\ \beta_i' \Sigma \beta_i \end{aligned} \tag{2.6}$$

subject to the conditions $\beta_i' \beta_i = 1$ and $\beta_i Y' \perp \beta_j Y'$ for $i < j$. The orthogonality condition between the characteristic vectors means that

$$0 = E\left[\left(\beta_j' Y'\right)\left(\beta_i' Y'\right)'\right] = E\left(\beta_j' Y' Y \beta_i\right) = \beta_j' \Sigma \beta_i \tag{2.7}$$

The Lagrangian equation to be maximized is therefore

$$L_j = \beta_j' \Sigma \beta_j - \xi(\beta_j' \beta_j - 1) - 2 \sum_{i=1}^{j-1} \phi_i \beta_j' \Sigma \beta_i \tag{2.8}$$

where ξ and $\phi = \left(\phi_1, ..., \phi_{j-1}\right)$ are the Lagrange multipliers and $j = 1, 2, ..., N$. The solution to this optimization problem satisfies the equation (2.4) and (2.3) and therefore the eigenvalues λ_i summarize the amount of variability captured by the corresponding eigenvector β_i.

The factor variables can be consistently estimated in an approximate factor model framework, only in the case of large N and large T (see Bai (2003)). For term structure of interest rates, since the number of maturities (N) is small and fixed, the variables estimated in a factor model framework (such as the one proposed by Bai (2003)) cannot be consistently estimated. This is mainly because the factors themselves are unobserved and cannot be estimated consistently for the case of N fixed.

We estimate the eigenspace variables from the term structure indirectly, without observing the factors themselves. Estimating the term structure model using PCA provides us with consistent estimates for the factor loadings since the factors loadings are defined in terms of the eigenvalues and eigenvectors obtained via eigen decomposition that are consistent for panels with large number of observations. In the following subsection, we derive the limiting distributions of the eigenvalues and eigenvectors of a covariance matrix, which is Wishart distributed and combine the results in order to get the limiting distributions of the factor loadings for the case of large time dimension interest rate panels. The limiting distributions of the eigenspace variables are help us construct the asymptotic test statistics for evaluating the issue of eigensystem instability.

2.3.1 Asymptotic properties of the eigenspace variables

In this section, we systematize the inferential theory for the eigenvalues, eigenvectors, and the factor loadings that are estimated using the classical principal component analysis. Let $Z = \left(z_1', ..., z_T'\right)$ be $N \times T$ matrix such that $ZZ' = (T-1)\,\hat{\Sigma}$ in equation (2.2). Therefore

$$\hat{\Sigma} = \frac{1}{T-1}\sum_{t=1}^{T} z_t z_t' \qquad (2.9)$$

where $z_t = (y_t - \bar{y})$ is the demeaned vector and $z_t \sim N_N(0, \Sigma)$.

Definition. *($N-$variate Wishart distribution) Let $x_1, ..., x_k$ be k-independent $N-$vectors. Suppose each $x_i \sim N_N(0, \Sigma)$. Let $U = x_1 x_1' + x_2 x_2' + ... + x_k x_k'$. Then U is said to have a $N-$variate Wishart Distribution with k degrees of freedom and covariance matrix Σ. That is,*

$$U \sim W_N(\Sigma, k)$$

According to the above definition, $\hat{\Sigma}(T-1) = \sum_{t=1}^{T} z_t z_t' = \sum_{t=1}^{T} y_{it} \cdot y_{jt} \sim W_N(\Sigma, T-1)$. Therefore

$$\hat{\Sigma} \sim W_N((T-1)^{-1}\Sigma, T-1) \tag{2.10}$$

The density function of matrix $\hat{\Sigma}$ is

$$f\left(\hat{\Sigma}\right) = \frac{\left((T-1)^{-N}\left|\hat{\Sigma}\right|\right)^{\frac{1}{2}(T-N-2)} e^{-\frac{1}{2(T-1)}tr\left(\hat{\Sigma}\Sigma^{-1}\right)}}{2^{\frac{1}{2}N(T-1)}\pi^{\frac{1}{4}N(N-1)}\left|\Sigma\right|^{\frac{1}{2}(T-1)}\prod_{i=1}^{N}\Gamma\left[\frac{1}{2}(T-i)\right]} \tag{2.11}$$

where $\Gamma(.)$ is the gamma function.

The following theorem provides the rate of convergence and the limiting distribution of the eigenvalues and eigenvectors decomposed from a covariance matrix $\hat{\Sigma}$.

Theorem. *(Limiting distribution of eigenvalues and eigenvectors) Let $y_1, ..., y_T$ be independently distributed, each being an $N-$vector of $N_N(0, \Sigma)$. Define $\Lambda = (\lambda_1, \lambda_2, ..., \lambda_N)'$ a $N \times 1$ vector of independent eigenvalues and $\beta = (\beta_1, \beta_2, ..., \beta_N)$ a $N \times N$ matrix of orthogonal eigenvectors. The sample covariance matrix $\hat{\Sigma}$ is such that $\hat{\Sigma} \sim W_N((T-1)^{-1}\Sigma, T-1)$. Then as $T \to \infty$,*

$$\left(\hat{\Lambda} - \Lambda\right) = O_p(T^{-1/2}) \tag{2.12}$$

$$\left(\hat{\beta} - \beta\right) = O_p(T^{-1/2}) \tag{2.13}$$

where the sequence $\left(\hat{\Lambda} - \Lambda\right)$ and $\left(\hat{\beta} - \beta\right)$ are independent to each other. The limiting distribution is given by

$$\sqrt{T}\left(\hat{\Lambda} - \Lambda\right) \xrightarrow{d} N(0, \Upsilon) \tag{2.14}$$

where $\Upsilon = diag\left(2\lambda_1^2, 2\lambda_2^2, ..., 2\lambda_N^2\right)$ *and*

$$\sqrt{T}\left(\hat{\beta} - \beta\right) \xrightarrow{d} N(0, \Theta) \tag{2.15}$$

where $\Theta = \sum_{i=1}^{N}\sum_{j=1}^{N}(U_{ij} \otimes \Theta_{ij})$ *with*

$$\Theta_{ij} = \begin{cases} \lambda_i \sum\limits_{\substack{k=1 \\ k\neq i}}^{N} \frac{\lambda_k}{(\lambda_i - \lambda_k)^2}\beta_k\beta_k' & \text{for } i = j \\ -\frac{\lambda_i\lambda_j}{(\lambda_i-\lambda_j)^2}\beta_j\beta_i' & \text{for } i \neq j \end{cases} \tag{2.16}$$

and U_{ij} *is an* $N \times N$ *matrix that has 1 in the* ij^{th} *position and 0's elsewhere.*

Proof: see Appendix 2.7.1

Corollary. *(Limiting distribution of factor loadings) Consider* $\lambda_1 \geqslant \lambda_2 \geqslant ... \geqslant \lambda_N$ *and* $\beta_1, \beta_2, ..., \beta_N$ *as the first N ordered eigenvalues and their corresponding eigenvectors of* Σ *respectively. Define* $\beta_i\lambda_i^{1/2} = \gamma_i$ *as the* i^{th} *factor loading vector where* $\gamma_i = (\gamma_{i1}, \gamma_{i2}, .., \gamma_{iN})'$ *and* $\gamma = (\gamma_1, ..., \gamma_N)$. *From the theorem above, since* $\hat{\lambda}_i - \lambda_i$ *is independent of* $\hat{\beta}_i - \beta_i$ *we can show that*

$$(\hat{\gamma} - \gamma) = O_p(T^{-1/2}) \tag{2.17}$$

$$\sqrt{T}(\hat{\gamma} - \gamma) \xrightarrow{d} N(0, \Psi) \tag{2.18}$$

where $\Psi = \sum_{i=1}^{N}\sum_{j=1}^{N}(U_{ij} \otimes \Psi_{ij})$ *where* U_{ij} *is an* $N \times N$ *matrix that has 1 in the* ij^{th} *position and 0's elsewhere. The asymptotic covariance matrix*

$$\Psi_{ij} = \begin{cases} \lambda_i\Theta_{ij} + \frac{1}{2}\lambda_i\beta_j\beta_i' & \text{for } i = j \\ (\lambda_i\lambda_j)^{1/2}\Theta_{ij} & \text{for } i \neq j \end{cases} \tag{2.19}$$

Proof: see Appendix 2.7.2

The above corollary gives us the rate of convergence and the limiting distribution of the factor loadings matrix, $\hat{\gamma}_i$. Let $\hat{\Psi}$ be the estimated covariance matrix of Ψ. As $T \longrightarrow \infty$, $\hat{\Psi} \xrightarrow{p} \Psi$. $\hat{\Psi}$ is consistent since it is a continuous function of the estimated eigenvalues and eigenvectors that are consistent. The corollary states the consistency of the estimated covariance matrix for the factor loadings, as a consequence of the continuous mapping theorem.

In the case of estimating factor loadings in stationary approximate factor models using regression methods, it is well known that since the factors (F_t) are unknown and usually estimated via principal component analysis, the factor loadings estimated in regression are in fact inconsistent for the case of fixed number of cross sectional units (N). The inconsistency is due to the fact that there is limited information available to estimate the factors when the number of cross sectional units is fixed and not large. A recent paper Heaton and Solo (2006) evaluated the importance of using large N dimensional panels when using PCA to estimate approximate factor models. The paper showed that the consistency and rate of convergence derived for this case depend on the rate at which the cross sectional correlations of the model disturbances grow as N grows to infinity (that is, the rate at which the maximum eigenvalues of covariance matrix of the error terms grow with N). This means that for panels where there is an increase in the cross-correlation of the disturbances with growing N, we would need N really large. The implication of this finding is that the issue of inconsistency in finite dimensional panels is dependent on the panel data under consideration and have to be explored case by case. This conclusion pertains in the case of term structure of interest rates where the cross sectional interest rate maturities are finite. Since the inconsistency of the estimated factor loadings in a regression framework is induced by inconsistency in estimation of the unobserved factors in the case of fixed N dimensional panels, we do not estimate the factor loadings in a regression framework but simply as an eigen decomposition problem. The factor loadings are defined in terms of the eigenvalues and eigenvectors, which are consistent for panels with fixed N and for large number of observations (T).

2.4 Testing for instability in the eigensystem

In this section, we formulate a series of hypotheses that will enable us to evaluate stability among the eigenspace variables of the yield curves. Since we are primarily concerned with the level, slope, and curvature factors governing the yield curves, we investigate stability in the eigenspace variables of the first three principal factors.

We examine instability by testing the null hypothesis of no change point against the alternative of at least one change point happening at the unknown time, τ. We define τ as a fraction of the sample space T such that $\tau = [T\epsilon]$ where $\epsilon = [0, 1]$. We define the eigenvalues, eigenvectors, and the factor loadings for the sample split around the unknown time point τ as

$$\Lambda = \begin{cases} \Lambda^a & for\ t = 1, ..., \tau \\ \Lambda^b & for\ t = \tau + 1, ..., T \end{cases} \quad for\ some\ \tau$$

$$\beta_i = \begin{cases} \beta_i^a & for\ t = 1, ..., \tau \\ \beta_i^b & for\ t = \tau + 1, ..., T \end{cases} \quad for\ some\ \tau$$

$$\gamma_i = \begin{cases} \gamma_i^a & for\ t = 1, ..., \tau \\ \gamma_i^b & for\ t = \tau + 1, ..., T \end{cases} \quad for\ some\ \tau$$

We test the following hypotheses in order to gather inference on instability in the underlying eigensystem of the yield curves:

I. $H_0 : \Lambda^a = \Lambda^b$

$H_1 : \Lambda^a \neq \Lambda^b$

II. $H_0 : \lambda_i^a = \lambda_i^b$

$H_1 : \lambda_i^a \neq \lambda_i^b \quad$ for $i = 1, 2, 3$

III. $H_0 : \beta_i^a = \beta_i^b$

$H_1 : \beta_i^a \neq \beta_i^b$ for $i = 1, 2, 3$

IV. $H_0 : \beta_{ip}^a = \beta_{ip}^b$

$H_1 : \beta_{ip}^a \neq \beta_{ip}^b$ for $i = 1, 2, 3$, and $p = 1, 2, ..., N$

V. $H_0 : \gamma_i^a = \gamma_i^b$

$H_1 : \gamma_i^a \neq \gamma_i^b$ for $i = 1, 2, 3$

VI. $H_0 : \gamma_i^a = \gamma_i^b$ and $\gamma_j^a = \gamma_j^b$ for $i, j = 1, 2, 3$ and $i \neq j$

$H_1 : \gamma_i^a \neq \gamma_i^b$ or $\gamma_j^a \neq \gamma_j^b$

Remarks:

- In testing the series of hypotheses formulated above, we aim to study the economic shocks causing structural changes and their impact on the eigensystem of the yield curves. Since the risks associated with the yield curves are summarized within its eigensystem, we are able to envisage which economic shocks have caused what kind of risks.

- Hypothesis I tests for the stability of the overall eigensystem by testing the restriction on $\Lambda = (\lambda_1, \lambda_2, \lambda_3)'$. The result from this test would help us conclude whether there persist structural changes in the eigensystem of the yield curves.

- A natural extension to this would be asking the question "What kind of structural changes have occurred?". Hypotheses II, III, and V tests for instabilities in the individual level, slope, and curvature factor magnitude, direction, and loadings respectively. The result from these tests would help us conclude whether the instability has been induced by level breaks, slope breaks, or rather curvature breaks. The corollary to this test would be to understand the risks associated with level, slope, and curvature shocks.

- Hypothesis IV relates to testing for instability in each factor, and understanding which interest rates have experienced structural changes, causing the instability.
- Hypothesis VI, unlike the previous ones, tests for common structural changes in factors. Since the level, slope, and curvature factors are correlated, the test tries to capture change points in one factor that might ripple into the other factors causing common change points in all factors.

In what follows, we develop the stability test statistics for evaluating the six hypotheses formulated above. Define $\Pi = (\Lambda, \beta, \gamma)$ as the parameters. Let $\hat{\Pi}^a$ and $\hat{\Pi}^b$ be the consistent estimators of Π^a and Π^b. The limiting distribution of Π for the restricted sample space before the break and after the break, given the change point τ is

$$\sqrt{T}\left(\hat{\Pi}^a - \Pi\right) = \sqrt{T}\begin{pmatrix} \hat{\Lambda}^a - \Lambda \\ \hat{\beta}^a - \beta \\ \hat{\gamma}^a - \gamma \end{pmatrix} \xrightarrow{d} \begin{pmatrix} N\left(0, \bar{\Upsilon}^a\right) \\ N\left(0, \bar{\Theta}^a\right) \\ N\left(0, \bar{\Psi}^a\right) \end{pmatrix}, \tag{2.20}$$

and

$$\sqrt{T}\left(\hat{\Pi}^b - \Pi\right) = \sqrt{T}\begin{pmatrix} \hat{\Lambda}^b - \Lambda \\ \hat{\beta}^b - \beta \\ \hat{\gamma}^b - \gamma \end{pmatrix} \xrightarrow{d} \begin{pmatrix} N\left(0, \bar{\Upsilon}^b\right) \\ N\left(0, \bar{\Theta}^b\right) \\ N\left(0, \bar{\Psi}^b\right) \end{pmatrix} \tag{2.21}$$

where the superscript a and b denote estimation from restricted sample before and after the break respectively. The associated covariance weighting structure $\bar{\Upsilon}^a = \frac{\Upsilon^a}{\epsilon}, \bar{\Upsilon}^b = \frac{\Upsilon^b}{1-\epsilon}, \bar{\Theta}^a = \frac{\Theta^a}{\epsilon}, \bar{\Theta}^b = \frac{\Theta^b}{1-\epsilon}, \bar{\Psi}^a = \frac{\Psi^a}{\epsilon}$ and $\bar{\Psi}^b = \frac{\Psi^b}{(1-\epsilon)}$. We have derived the limiting distribution of these eigenspace variables in previous section.

In testing the hypothesis I, we construct the Wald test statistic under the null hypothesis of no structural change in Λ against the alternative of at least one structural change in Λ can be constructed as under:

$$\begin{aligned} W_I(\tau) &= \left(\left(\hat{\Lambda}^a - \Lambda\right) - \left(\hat{\Lambda}^b - \Lambda\right)\right)' \left[(\bar{\Upsilon}^a) + (\bar{\Upsilon}^b)\right]^{-1} \left(\left(\hat{\Lambda}^a - \Lambda\right) - \left(\hat{\Lambda}^b - \Lambda\right)\right) \\ &\xrightarrow{d} Z' \left[(\bar{\Upsilon}^a) + (\bar{\Upsilon}^b)\right]^{-1} Z \end{aligned} \tag{2.22}$$

where $Z \sim N\left(0, \bar{\Upsilon}^a + \bar{\Upsilon}^b\right)$.

Define $\hat{\Upsilon} = \bar{\Upsilon}^a + \bar{\Upsilon}^b$ where $\hat{\Upsilon}$ is positive definite. Using Cholesky decomposition, we have $\hat{\Upsilon} = LL'$ and $\hat{\Upsilon}^{-1} = L^{-1}L^{-1\prime}$ where L is a lower triangular matrix with strictly positive diagonal entries. Premultiplying Z by the inverse of L,

$$\begin{aligned} L^{-1}Z &\sim N\left(0, L^{-1}\hat{\Upsilon}L^{-1\prime}\right) \\ &= N\left(0, L^{-1}LL'L^{-1\prime}\right) \\ &= N\left(0, I_r\right) \end{aligned}$$

Therefore using this result, we can show asymptotically

$$W_I(\tau) \xrightarrow{d} Z'\hat{\Upsilon}^{-1}Z = Z'L^{-1\prime}L^{-1}Z = Q(\tau) \tag{2.23}$$

where for a given $\tau = [T\epsilon], Q(\tau) \sim \chi^2(q)$ with the degrees of freedom q corresponding to the number of restrictions being tested. Thus the distribution of our test statistic under the null is asymptotically pivotal.

Since the eigenvectors and the factor loadings are also asymptotically Normal, we can test all the other five hypotheses using the statistic $W(\tau)$, which when normalized with their respective asymptotic variances, can again be shown to converge to a chi-squared as above. The form of the Wald statistics corresponding to the five hypotheses are given below.

$$W_{II}(i,\tau) = \frac{\left(\hat{\lambda}_i^a - \hat{\lambda}_i^b\right)^2}{\left[\left(2\left(\hat{\lambda}_i^a\right)^2/\epsilon\right) + \left(2\left(\hat{\lambda}_i^b\right)^2/(1-\epsilon)\right)\right]} \tag{2.24}$$

$$W_{III}(i,\tau) = \left(\hat{\beta}_i^a - \hat{\beta}_i^b\right)' \left[\bar{\Theta}_{ii}^a + \bar{\Theta}_{ii}^b\right]^{-1} \left(\hat{\beta}_i^a - \hat{\beta}_i^b\right) \tag{2.25}$$

$$W_{IV}(i,p,\tau) = \frac{\left(\hat{\beta}_{ip}^a - \hat{\beta}_{ip}^b\right)^2}{\left[\left(\bar{\Theta}_{ii,pp}^a\right) + \left(\bar{\Theta}_{ii,pp}^b\right)\right]} \text{ where } \bar{\Theta}_{ii,pp}^s \text{ is } pp^{th} \text{ position of matrix } \bar{\Theta}_{ii}^s, s = (a,b) \tag{2.26}$$

$$W_V(i,\tau) = \left(\hat{\gamma}_i^a - \hat{\gamma}_i^b\right)' \left[\bar{\Psi}_{ii}^a + \bar{\Psi}_{ii}^b\right]^{-1} \left(\hat{\gamma}_i^a - \hat{\gamma}_i^b\right) \tag{2.27}$$

$$W_{VI}(i,j,\tau) = \begin{pmatrix} \hat{\gamma}_i^a - \hat{\gamma}_i^b \\ \hat{\gamma}_j^a - \hat{\gamma}_j^b \end{pmatrix}' \begin{bmatrix} (\bar{\Psi}_{ii}^a + \bar{\Psi}_{ii}^b) & (\bar{\Psi}_{ij}^a + \bar{\Psi}_{ij}^b) \\ (\bar{\Psi}_{ij}^a + \bar{\Psi}_{ij}^b) & (\bar{\Psi}_{jj}^a + \bar{\Psi}_{jj}^b) \end{bmatrix}^{-1} \begin{pmatrix} \hat{\gamma}_i^a - \hat{\gamma}_i^b \\ \hat{\gamma}_j^a - \hat{\gamma}_j^b \end{pmatrix} \tag{2.28}$$

Note that the Wald test derived in the above framework are equivalent to the F test or Chow type tests. One could also use the Langrange Multiplier (LM) or the Likelihood Ratio (LR) tests in order to test the linear restrictions. It is possible to show that Wald, LM and LR type tests have the same asymptotic distributions. When the date of the structural change is unknown but known to fall within a finite range, Andrews (1993) and Andrews and Ploberger (1994) introduced the “*Sup*”, “*Exp*”, and “*Avg*” tests for the Wald, LM and LR test statistics and derived its asymptotic distributions. If we define the Wald test statistic as

W for a break occurring at time τ, then

$$SupW = \max_{t_1<\tau<t_2} W \tag{2.29}$$

$$AvgW = \frac{1}{t_2 - t_1 + 1}\sum_{\tau=t_1}^{t_2} W \tag{2.30}$$

$$ExpW = \ln\left[\frac{1}{t_2 - t_1 + 1}\sum_{\tau=t_1}^{t_2} \exp\left(\frac{1}{2}W\right)\right] \tag{2.31}$$

where the breakpoint τ lies between t_1 and t_2 such that $t_1 = [T\epsilon_1]$, $t_2 = [T\epsilon_2]$, $t_1 \neq t_2, \epsilon_2 = 1 - \epsilon_1$,and t_1 is bounded away from zero and t_2 is bounded away from boundary, T. This condition is required since the proposed test statistic is unbounded in limit at the boundaries. Andrews (1993) suggested the restricted interval $t_1 = 0.15T$ and $t_2 = 0.85T$ such that ϵ_1 and ϵ_2 lies in the interval $[0.15, 0.85]$ and T denotes the number of observations in the sample. Therefore the test statistics does not capture the breakpoints occurring at the end of sample.

Under the null of no structural change, according to the continuous mapping theorem the asymptotic distributions of the test statistics converge as follows:

$$SupW \xrightarrow{d} \max_{\epsilon_1<\epsilon<\epsilon_2} Q(\epsilon) \tag{2.32}$$

$$AvgW \xrightarrow{d} \int_{\epsilon_1}^{\epsilon_2} Q(\epsilon)d\epsilon \tag{2.33}$$

$$ExpW \xrightarrow{d} \ln\left[\int_{\epsilon_1}^{\epsilon_2} \exp\left(\frac{1}{2}Q(\epsilon)\right) d\epsilon\right] \tag{2.34}$$

where if we know the break point fraction $\epsilon, Q(\epsilon)$ will be $\chi^2(q)$ with the degrees of freedom q corresponding to the number of restrictions being tested. In the GMM estimation framework, Andrews (1993) and Andrews and Ploberger (1994) provide the critical values for *Sup*, *Avg*, and *Exp* of the Wald test statistic using Monte Carlo simulation. The Monte Carlo critical values provided by Andrews and Ploberger break down under various circumstances as documented by Diebold and Chen (1996), Hansen (2000), and O'Reilly and Whelan (2005).

O'Reilly and Whelan (2005) proposed a wild bootstrap approach for generating critical values for the *Sup* statistic in dynamic time series models.

In providing inference on the eigensystem stability, we rely upon the test statistic distribution obtained using the bootstrap methodology. In this, we bootstrap the space vector of N maturities by resampling across time.

[Insert Table 2.1 here]

It is well established that the bootstrap procedures provide much accurate and reliable inferences than asymptotics based inferences. Andrews (1993) and Andrews and Ploberger (1994) provide asymptotic critical values for the *Sup*, *Avg*, and *Exp* of optimal tests based on a regression type framework. We know that the principal component analysis provides a different solution than the least squares solution, in which the least squares problem minimizes the vertical distance between the points and the principal component analysis problem minimizes the orthogonal distances between the points. The following diagram tries to explain this notion:

[*Insert Figure 2.8 here*]

Since the estimation framework developed in this chapter is based on orthogonal rotation of axes and different to the regression type framework in Andrews' papers, we use the bootstrapped critical values rather than asymptotic critical values provided by Andrews. Since the Wald test statistic is asymptotically pivotal, the asymptotic distribution of the test statistics does not depend on a particular data generating process under the null. Therefore bootstrap distribution can consistently estimate the asymptotic distribution of the test statistics and provide more reliable inference than asymptotically based inferences by removing the finite sample biases. Davidson and MacKinnon (1999) find that for asymptotically pivotal test statistics using critical values from the bootstrap will produce smaller size distortions (reduced by an order of $T^{-1/2}$) than when using the critical values obtained from the first order asymptotics. Using the bootstrapped critical values, one may be able to mimic the skewness and kurtosis of the empirical distribution that is not captured by the first order limiting distribution.

In order to construct the bootstrap distribution of the test statistics, we undertake the following steps:

1. For a given value of ϵ, the break fraction and α, the significance level, randomly draw the vector of maturities from the $T \times N$ term structure data in order to construct the $T \times N$ bootstrapped data.
2. Then construct the covariance matrix for the bootstrapped data and conduct the principal component analysis in order to estimate the eigenspace variables $\hat{\Lambda}, \hat{\beta}, \hat{\gamma}$.
3. Compute the Wald statistics $W_k(., \tau)$ for $k = 1, 2, ..., 6$ and calculate the *Sup*, *Avg*, and *Exp* of the Wald statistics.
4. Repeat steps 1 through 3 for BR number of bootstrap replications.
5. Determine the $\alpha\%$ bootstrap critical value for *Sup*, *Avg*, and *Exp* of the various Wald statistics.

A set of Monte Carlo experiments are conducted in order to study the finite sample properties of the tests proposed. The simulation results with 5000 Monte Carlo runs and 500 bootstraps show good size (at 5% significance level) and power properties for varied cross sectional dimension panels ($N = 5, 10, 20$), with three possible change points ($\tau = T/3, T/2, 2T/3$), and three possible break sizes generated from different intervals of the uniform distribution. Overall, we find that the empirical size of the bootstrap tests is very close to the nominal size in the case of *Sup*, *Avg*, and *Exp* of almost all of the six Wald type test statistics. In the case of testing the curvature factors, we find under-sizing for small N. However, we see substantial size improvements as N increase. In terms of power performance, we find that the test statistics $W_I(\cdot), W_{II}(\cdot), W_{III}(\cdot), W_V(\cdot)$ and $W_{VI}(\cdot)$ show power essentially close to one. The test statistic $W_{IV}(\cdot)$ however show low power in evaluating the curvature factor for small N and small structural change magnitudes. There is however power gain as the magnitude of structural change increase. The full set of results is presented in chapter three.

2.5 Empirical results

2.5.1 Data

We use the term structure of US zero coupon bond yields obtained from Datastream. The term structure of zeros are extremely useful in fixed income applications such as pricing bonds, swaps, and other fixed income derivatives; financial engineering the interest rates exposures; obtaining the forward rate curves, par yield curves; and so on. Thus understanding the stability issues of the factors governing the zero coupon yield term structure will provide an explanation to possible problems encountered in statistical modelling techniques such as principal component analysis and factor analysis commonly used in literature. Table *2.2*, summarizes the datasets used in the previous studies that directly or in passing evaluates term structure stability.

[Insert Table 2.2 here]

[Insert Figure 2.9 here]

The term structure of US zero coupon bond yields from Datastream include 21 maturities of 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 24, 36, 48, 60, 72, 84, 96, 108, 120, 132 and 144 months. Figure *2.9* plots the evolution of the bond yield curve over the sample period considered. The sample period extends from 11 Jan 1999 to 31 May 2006, with daily frequency (1927 observations).

The data period covers both the period of downturn (during the technology stock boom in 2001) and recent upswings, where the risk aversion of investors is high causing gains in the bond markets. The bond yields data for maturities less than three months were filtered out in order to reduce the market microstructure effects and avoid liquidity issues. On the same note, we use the five day change in yields or the five day holding period returns in order to perform the eigen decomposition on its covariance structure as recommended by Lardic et al. (2003) and as commonly used in factor analysis literature of term structure of interest rates. Working with changes rather than levels allow us to reduce the high autocorrelation and work in a stationary framework.

2.5.2 Stability testing results

Table *2.3* records the results from implementing the *Sup*, *Avg*, and *Exp* test statistics for the six hypotheses formulated above along with p-values in italics. The tests are evaluated for significant structural changes within the restricted sample period $[0.15T, 0.85T]$. We avoid the boundaries since the test statistic will produce unstable results at the boundaries as documented by Andrews (1993). We test the linear restrictions of equality in eigenspace variables for a given change point occurring at time τ using the Wald test. In practice since we do not know this change point τ, we calculate the weighted statistics *Sup*, *Avg*, and *Exp* for all possible change points within the restricted sample period. The conclusions are drawn only when the results from all the three weighted statistics *Sup*, *Avg*, and *Exp* are in agreement with each other.

[Insert Table 2.3 here]

Investigating stability in the overall eigensystem

Evaluating the weighted test statistics for $W_I(\tau)$, we reject the null in favor of the alternative that $\Lambda^a \neq \Lambda^b$. Thus the *Sup*, *Avg*, and *Exp* test statistics of $W_I(\tau)$ infer that significant changes persist in the eigensystem of the yield curves. Instability in the vector of eigenvalues would mean structural instability in the variance process governing the factors. The instability detected has also been concluded in literature by Bliss (1997), Andrino et al. (2005) among others.

It is worth mentioning that the conclusions on instability in the factors governing the volatility are indeed different to the conclusions drawn in this chapter where we evaluated the volatility governing the factors. The distinction lies within the fact that the information extracted (using eigen decomposition) from the covariance matrix of the yields are different than the information summarized in the covariance matrix of unobserved volatility. In regard to the latter, Perignon and Villa (2006) document the time-varying nature of the volatility governing the factors and Bliss (1997) reported instability present in the factor volatility structures using graphical methods.

Investigating stability in eigensystem of the level factor

Evaluating the weighted test statistics for $W_{II}(1,\tau)$, $W_{III}(1,\tau)$, and $W_V(1,\tau)$ we reject the null in favor of the alternative that $\lambda_1^a \neq \lambda_1^b, \beta_1^a \neq \beta_1^b$, and $\gamma_1^a \neq \gamma_1^b$ respectively. Thus according to all the three weighted measures (*Sup*, *Avg*, and *Exp*) for the various hypotheses, we can conclude that all the three eigenspace variables (eigenvalues, eigenvectors, and factor loadings) governing the level factor has had statistically significant structural changes inducing instability. The result differs to the graphical inferences gathered by several authors such as Reisman and Zohar (2004) and Fabozzi et al. (2005) who have drawn stability conclusions for the level factor of discount bond yields and swap rates respectively.

In order to gauge which interest rate maturities have contributed to the structural instability in the level factor, we evaluated the weighted test statistics for $W_{IV}(1,\tau)$. According to all the three weighted measures (*Sup*, *Avg*, and *Exp*) we can conclude that the structural instability was common and evident in all the 21 interest rate maturities governing the level factor. This means that the structural change in the level factor has been caused by economic shocks that eminently influenced the whole yield curve (short end as well as the long end maturities).

Investigating stability in eigensystem of the slope factor

In the case of the slope factor, we find that the eigenvalues or volatility governing the factor have incurred structural changes. Using all the three weighted statistics for $W_{II}(2, \tau)$, we reject the null in favor of the alternative of $\lambda_2^a \neq \lambda_2^b$. However, by evaluating $W_{III}(2, \tau)$ and $W_V(2, \tau)$ we find that the eigenvectors and the factor loadings governing the slope factor have remained stable over time. By evaluating the weighted test statistic of $W_{IV}(2, \tau)$ for the slope factor, we can find that the short term interest rates (3 months - 10 months) governing the factor were unstable whereas the medium - long term interest rates (2 years - 12 years) governing the factor were tested to be stable over time. The test results for the slope factor do not concur with Reisman and Zohar (2004) that document stability of the slope factor.

In light of the results, we can infer that the instability of the slope factor lie in the volatility governing the factor structure and since the eigenvalues and eigenvectors have a one to one correspondence with each other, we can infer that the instability in the volatility of the slope factor is caused due to economic shocks causing structural changes within the short term interest rates.

Investigating stability in eigensystem of the curvature factor

In the case of testing for instability in the eigenspace variables of the curvature factor, we find similar results to that of the slope factor. Using the *Sup*, *Avg*, and *Exp* for $W_{II}(3, \tau)$, $W_{III}(3, \tau)$, and $W_V(3, \tau)$ we find that the curvature factor eigenvalue (volatility) has been subject to statistically significant structural changes but the corresponding eigenvectors and factor loadings have remained stable through time. By evaluating stability in the interest rates governing the curvature factor (using $W_{IV}(3, \tau)$), we find that the medium and long term rates (2 years - 12 years) have contributed to the structural change in the volatility of the curvature factor. Unlike the slope factor, we find that the short and the medium term (3 months - 1 year) interest rates were stable through time. In literature, Reisman and Zohar (2004) document marginal stability of the curvature factors using graphical tools. Again using visual aid, Fabozzi et al. (2005) find that the curvature factors were unstable for the swap rates.

Thus we can conclude that, as in the case of the slope factor structure, the volatility governing the curvature factor has incurred statistically significant structural changes. Evaluating the interest rates governing the eigenvectors, we find instability present in the medium and long end of the yield curves, unlike the case of the slope factor. Since the instability in the slope and curvature factors have been caused due to economic shocks influencing different ends of the yield curve, we can say that the slope factors are sensitive to movements and shocks affecting in the short rates and the curvature factors are sensitive to movements and shocks affecting in the long rates.

Investigating common instability in factor loadings

Since we have found that the eigenspace variables for the level, slope, and curvature factors have incurred instability and since the three factors are correlated with each other, the economic shocks affecting one factor could also have affected the other. Therefore we investigated the presence of common structural changes due to common shocks in factors. By evaluating the weighted test statistics of $W_{VI}(1,2,\tau)$ we do not reject the null of presence of common structural changes in level and slope factor loadings. Thus we can conclude that there exist statistically significant change points common to the level and slope factors. Combining this result with the instability conclusions found for the level and slope eigenvectors, we can identify the common sources of instability within the level and slope factors as the economic shocks that have caused structural changes in the short term interest rates (3 months - 10 months). For testing the common instabilities in level and curvature factor loadings, we cannot infer the presence of common structural changes since the *Sup*, *Avg*, and *Exp* for $W_{VI}(1,3,\tau)$ provide variant conclusions. In the case of evaluating common instabilities present in the slope and curvature factor loadings, we reject the null of the test statistics $W_{VI}(2,3,\tau)$ in favor of the alternative of no common structural changes between the slope and curvature factor loadings. Thus we can conclude that the slope and curvature factors behave dissimilarly to economic shocks that may cause structural instabilities in them. This result corroborates with the above findings that the slope and curvature factors are sensitive to economic shocks influencing different ends of the yield curve.

2.6 Conclusion

This chapter explores the important question of whether the yield curve factor structure is stable through time. Several authors have either assumed stability or relied upon graphical analysis to make inferences. We propose a formal testing procedure and evaluate its asymptotic properties. We formulate six hypotheses for statistically evaluating the stability in the eigenspace variables (eigenvalues, eigenvectors, and factor loadings) governing the level, slope, and curvature factors of the yield curves. We then formally test for stability of the US zero coupon bond yield factor structures between January 1999 and May 2006.

We find that the overall variance process governing the first three factors of the yield curves were unstable over time. Previous literature documents the time-varying volatility of yield factors; see for example, Perignon and Villa (2006). Our results corroborate that but further, we also find abrupt fluctuations (instabilities) present in the factors, captured by the Wald-type testing procedure. We find that even if the volatility (eigenvalues) of factors were unstable, the linear relationship (factor loadings) of slope and curvature were stable.

To summarize the results; for the level factor, we find structural instability in all the eigenspace variables. Structural changes affecting all the interest rate maturities in the term structure panel fostered instability in the factor structure as well as the volatility explained by the factor. In the case of the slope and curvature factors, we find that the variances accounted by the factors incur structural instabilities. However, we find the eigenvectors and loadings have remained stable through time. Therefore we can conclude that the slope and curvature factor structures have remained stable; though the volatility associated with the factors are unstable over time. The instability in the volatility of the slope factor is caused by instability affecting only the short term maturities (3 months - 1 year) whereas in the case of the curvature factor, the instability in the volatility of the factor is caused by instability affecting only the medium and long term rates (2 years - 12 years). In investigating the presence of common structural changes in factors, we find statistically significant breaks common to level and slope factors and no statistically significant common breaks in the slope and curvature factors.

In concluding instability in factor structure, we allude to the limitations of the implemented test procedure, which is that the test is non-constructive. The implemented test procedure tests the null hypothesis of no change point against the alternative hypothesis of more than one change points. In this, the conclusion of instability would provide us with little information about the date of the break, and the number of breaks. However, in constructing several hypotheses on stability in interest rate maturities governing the factors, we have gathered inference on the source of the instability. Though the graphical techniques motivate towards structural changes, the statistical test introduced in this chapter provide inference on the statistical significance of those changes observed. For further work, one can conduct several break tests on the factors using multiple break techniques proposed by Bai and Perron (1998).

While this chapter evaluated the stability of PCA factors - level, slope, and curvature, for future work, one can also study the stability issues in factors estimated using the popular function-based Nelson-Siegel factor model as parameterized by Diebold and Li (2006).

2.7 Appendix

2.7.1 Proof to Theorem

The results mentioned in this theorem have been proved almost simultaneously by Girshick (1939), Hsu (1939), Fisher (1939), Roy (1939), Mood (1951), Anderson (1963) and widely presented in advanced multivariate statistics books. For the proof, we refer the reader to any of the above papers or book by Anderson (2003, pp.546). Here we would sketch the proof to equation (2.12) and (2.14) alone.

To prove equation (2.12) and equation (2.14):

Consider the covariance matrix $\hat{\Sigma}$ such that $(T-1)\hat{\Sigma} \sim W_N(\Sigma, T-1)$ where $\hat{\Sigma} = \frac{1}{T-1}\sum_{t=1}^{T} z_t' z_t$ and $z_t \sim N_N(0, \Sigma)$. Multiplying by $\beta_i'\beta_i$ where $\beta_i'\beta_i = I$, we define

$$(T-1)\hat{V}_i \equiv (T-1)\beta_i'\hat{\Sigma}\beta_i \sim W_1(\lambda_i, T-1) \text{ for } i = 1, 2, ..., r$$

Define $\hat{V}_i^* = (T-1)\,\hat{V}_i$ so that $\hat{V}_i^* \sim \lambda_i \cdot \chi^2\,(T-1)$. Therefore the moments $E\left(\hat{V}_i\right) = \lambda_i$ and $Var\left(\hat{V}_i\right) = 2\lambda_i^2$.

Since $z_t's$ are iid, we can represent

$$\hat{V}_i^* = \sum_{t=1}^{T} \hat{V}_{it}^* \quad \text{where } \hat{V}_{it}^* \sim W_1(\lambda_i, 1)$$

As $T \to \infty$, using the multivariate central limit theorem,

$$\frac{(T-1)}{\sqrt{T}}\left(\hat{V}_i - \frac{T}{T-1}\lambda_i\right) \xrightarrow{d} N\left(0, 2\lambda_i^2\right)$$
$$\Rightarrow \sqrt{T}\cdot\frac{(T-1)}{T}\left(\hat{V}_i - \frac{T}{T-1}\lambda_i\right) \xrightarrow{d} N\left(0, 2\lambda_i^2\right)$$

Since $\frac{T-1}{T} \to 1$ for $T \to \infty$, we can get $O_p(T^{-1/2})$ order of convergence

$$\sqrt{T}\left(\hat{V}_i - \lambda_i\right) \xrightarrow{d} N\left(0, 2\lambda_i^2\right)$$

Here $\hat{V}_i = \beta_i' \hat{\Sigma} \beta_i$ is the eigenvalue estimated from the sample covariance matrix $\hat{\Sigma}$. This proves equation (2.12) and (2.14).

2.7.2 Proof to Corollary

We know from the theorem that as $T \to \infty$,

$$\sqrt{T}\left(\hat{\lambda}_i - \lambda_i\right) \xrightarrow{d} N(0, 2\lambda_i^2)$$

and

$$\sqrt{T}\left(\hat{\beta}_i - \beta_i\right) \xrightarrow{d} N(0, \Theta_{ii})$$
$$\sqrt{T}\left(\left(\hat{\beta}_i - \beta_i\right)\left(\hat{\beta}_j - \beta_j\right)\right) \xrightarrow{d} N(0, \Theta_{ij})$$

where

$$\Theta_{ij} = \begin{cases} \lambda_i \sum_{\substack{k=1 \\ k \neq i}}^{N} \frac{\lambda_k}{(\lambda_i - \lambda_k)^2} \beta_k \beta_k' & \text{for } i = j \\ -\frac{\lambda_i \lambda_j}{(\lambda_i - \lambda_j)^2} \beta_j \beta_i' & \text{for } i \neq j \end{cases}$$

We define the error in estimation of the eigenvalues $\left(\hat{\lambda}_i - \lambda_i\right)$ as ε_{λ_i} and the error in estimation of the eigenvectors $\left(\hat{\beta}_i - \beta_i\right)$ as ε_{β_i}. Note that $E\left(\varepsilon_{\lambda_i} \varepsilon_{\beta_i}\right) = 0$.

$\hat{\lambda}_i^{1/2} = (\lambda_i + \varepsilon_{\lambda_i})^{1/2} = \lambda_i^{1/2}\left(1 + \frac{\varepsilon_{\lambda_i}}{\lambda_i}\right)^{1/2}$. Using Taylor expansion up to the first order, $\hat{\lambda}_i^{1/2} = \lambda_i^{1/2}\left(1 + \frac{1}{2}\frac{\varepsilon_{\lambda_i}}{\lambda_i}\right) + o_p(1)$. Therefore we can write $\hat{\lambda}_i^{1/2} - \lambda_i^{1/2} = \frac{1}{2}\frac{\varepsilon_{\lambda_i}}{\lambda_i^{1/2}}$. Since we know the limiting distribution of the ε_{λ_i}, we have

$$\sqrt{T}\left(\hat{\lambda}_i^{1/2} - \lambda_i^{1/2}\right) \xrightarrow{d} N\left(0, \frac{1}{2}\lambda_i\right) \tag{2.35}$$

We define $\hat{\lambda}_i^{1/2} - \lambda_i^{1/2} \equiv \tilde{\varepsilon}_{\lambda_i}$. Therefore we can write

$$\begin{aligned} \hat{\lambda}_i^{1/2}\hat{\beta}_i &= \left(\lambda_i^{1/2} + \tilde{\varepsilon}_{\lambda_i}\right)\left(\beta_i + \varepsilon_{\beta_i}\right) \\ &= \lambda_i^{1/2}\beta_i + \lambda_i^{1/2}\varepsilon_{\beta_i} + \beta_i\tilde{\varepsilon}_{\lambda_i} + \tilde{\varepsilon}_{\lambda_i}\varepsilon_{\beta_i} \end{aligned} \tag{2.36}$$

Therefore

$$\hat{\lambda}_i^{1/2}\hat{\beta}_i - \lambda_i^{1/2}\beta_i = \lambda_i^{1/2}\varepsilon_{\beta_i} + \beta_i\tilde{\varepsilon}_{\lambda_i} + \tilde{\varepsilon}_{\lambda_i}\varepsilon_{\beta_i}$$

We know, $\varepsilon_{\beta_i} = O_p(T^{-1/2}), \tilde{\varepsilon}_{\lambda_i} = O_p(T^{-1/2})$ and $\tilde{\varepsilon}_{\lambda_i}\varepsilon_{\beta_i} = O_p(T^{-1})$. Therefore $\hat{\lambda}_i^{1/2}\hat{\beta}_i - \lambda_i^{1/2}\beta_i = O_p\left(T^{-1/2}\right)$. This proves equation (2.17).

From the above, $\sqrt{T}\left(\lambda_i^{1/2}\varepsilon_{\beta_i}\right) \xrightarrow{d} N(0, \lambda_i\Theta_{ii})$ and $\sqrt{T}\left(\beta_i\tilde{\varepsilon}_{\lambda_i}\right) \xrightarrow{d} N\left(0, \frac{1}{2}\lambda_i\beta_i\beta_i'\right)$

Define $\underset{r\times 1}{X} = \lambda_i^{1/2}\varepsilon_{\beta_i}$ and $\underset{r\times 1}{Y} = \beta_i\tilde{\varepsilon}_{\lambda_i}$. From the above, we know that $X + Y$ have a limiting distribution that is Normal. The first moment of the distribution is $E(X+Y) = 0$. The second moment (variance) can be calculated as follows:

$$E\left((X+Y)(X+Y)'\right) = E\left[X'X\right] + E\left[Y'Y\right] + E\left[X'Y\right] + E\left[Y'X\right]$$

We know $E\left[X'X\right] = \lambda_i\Theta_{ii}$, $E\left[Y'Y\right] = \frac{1}{2}\lambda_i\beta_i\beta_i', E\left[X'Y\right] = E\left[Y'X\right] = E\left[\lambda_i^{1/2}\varepsilon_{\beta_i}\tilde{\varepsilon}_{\lambda_i}\beta_i'\right] = \lambda_i^{1/2}E\left[\varepsilon_{\beta_i}\tilde{\varepsilon}_{\lambda_i}\right]\beta_i' = 0$. Therefore, $E\left[\left(\left(\lambda_i^{1/2}\varepsilon_{\beta_i}\right) + \left(\beta_i\tilde{\varepsilon}_{\lambda_i}\right)\right)\left(\left(\lambda_i^{1/2}\varepsilon_{\beta_i}\right) + \left(\beta_i\tilde{\varepsilon}_{\lambda_i}\right)\right)'\right] = \lambda_i\Theta_{ii} + \frac{1}{2}\lambda_i\beta_i\beta_i'$

Therefore

$$\sqrt{T}\left(\hat{\lambda}_i^{1/2}\hat{\beta}_i - \lambda_i^{1/2}\beta_i\right) \overset{D}{=} N\left(0, \lambda_i\Theta_{ii} + \frac{1}{2}\lambda_i\beta_i\beta_i'\right) + Q + o_p(1)$$

where $\frac{1}{\sqrt{T}}\left(\bar{\varepsilon}_{\lambda_i}\varepsilon_{\beta_i}\right) \xrightarrow{d} Q$ where Q is a distribution of the product of two mean zero independent normal variates. As $T \to \infty$, the effect of Q is $O_p(T^{-1/2})$ is negligible and therefore

$$\sqrt{T}\left(\hat{\lambda}_i^{1/2}\hat{\beta}_i - \lambda_i^{1/2}\beta_i\right) \xrightarrow{d} N\left(0, \lambda_i\Theta_{ii} + \frac{1}{2}\lambda_i\beta_i\beta_i'\right)$$

This proves equation (2.18).

The asymptotic covariance matrix for $(\hat{\gamma}_i - \gamma_i)\left(\hat{\gamma}_j - \gamma_j\right)$, $i \neq j$:

$$\begin{aligned}
& Cov\left((\hat{\gamma}_i - \gamma_i), (\hat{\gamma}_j - \gamma_j)\right) \\
= \; & E\left[\hat{\gamma}_i\left(\hat{\gamma}_j - \gamma_j\right) - \gamma_i\left(\hat{\gamma}_j - \gamma_j\right)\right] \\
= \; & E\left[\hat{\lambda}_i^{1/2}\hat{\beta}_i\left(\hat{\lambda}_j^{1/2}\hat{\beta}_j - \lambda_j^{1/2}\beta_j\right)\right] - E\left[\lambda_i^{1/2}\beta_i\left(\hat{\lambda}_j^{1/2}\hat{\beta}_j - \lambda_j^{1/2}\beta_j\right)\right] \\
= \; & I - II
\end{aligned}$$

Substituting for the estimators of $\hat{\lambda}_l^{1/2}\hat{\beta}_l$ for $l = i, j$, we solve the two parts below:

I :

$$\begin{aligned}
& E\left[\hat{\lambda}_i^{1/2}\hat{\beta}_i\left(\hat{\lambda}_j^{1/2}\hat{\beta}_j - \lambda_j^{1/2}\beta_j\right)\right] \\
= \; & E\left[\lambda_i^{1/2}\beta_i + \lambda_i^{1/2}\varepsilon_{\beta_i} + \beta_i\bar{\varepsilon}_{\lambda_i} + \bar{\varepsilon}_{\lambda_i}\varepsilon_{\beta_i}\left(\lambda_j^{1/2}\beta_j + \lambda_j^{1/2}\varepsilon_{\beta_j} + \beta_j\bar{\varepsilon}_{\lambda_j} + \bar{\varepsilon}_{\lambda_j}\varepsilon_{\beta_j} - \lambda_j^{1/2}\beta_j\right)\right] \\
= \; & E\left[\lambda_i^{1/2}\lambda_j^{1/2}\varepsilon_{\beta_i}\varepsilon_{\beta_j}\right]
\end{aligned}$$

II :

$$\begin{aligned}
& E\left[\lambda_i^{1/2}\beta_i\left(\hat{\lambda}_j^{1/2}\hat{\beta}_j - \lambda_j^{1/2}\beta_j\right)\right] \\
= \; & E\left[\lambda_i^{1/2}\beta_i\left(\lambda_j^{1/2}\beta_j + \lambda_j^{1/2}\varepsilon_{\beta_j} + \beta_j\bar{\varepsilon}_{\lambda_j} + \bar{\varepsilon}_{\lambda_j}\varepsilon_{\beta_j} - \lambda_j^{1/2}\beta_j\right)\right] \\
= \; & 0
\end{aligned}$$

Therefore

$$
\begin{aligned}
Cov\left((\hat{\gamma}_i - \gamma_i), (\hat{\gamma}_j - \gamma_j)\right) &= E\left[\lambda_i^{1/2}\lambda_j^{1/2}\varepsilon_{\beta_i}\varepsilon_{\beta_j}\right] \\
&= \lambda_i^{1/2}\lambda_j^{1/2}\Theta_{ij}
\end{aligned}
$$

TABLE 2.1. Bootstrapped Critical values. The table reports the critical values of the bootstrapped distributions of *Sup*, *Avg*, and *Exp* of the test statistics $W(\tau) = (W_I(\tau), W_{II}(i,\tau), W_{III}(i,\tau), W_{IV}(i,\tau), W_V(i,\tau), W_{VI}(i,\tau))$ associated with the six hypotheses formulated in equations 2.23 - 2.28. The critical values correspond to testing the null hypotheses of stability in the eigenspace variables against the alternative of the presence of atleast one point of instability in the eigenspace variables for parameters $\epsilon = 0.15$ and significance level, $\alpha = 0.05$.

Testing overall system			
	Sup	*Avg*	*Exp*
$W_I(\tau)$	0.10647	0.016155	-0.6376
Testing the level factor			
	Sup	*Avg*	*Exp*
$W_{II}(1,\tau)$	0.027513	0.004888	-0.6485
$W_{III}(1,\tau)$	0.17855	0.041762	-0.61261
$W_V(1,\tau)$	0.16094	0.039824	-0.61471
Testing the slope factor			
	Sup	*Avg*	*Exp*
$W_{II}(2,\tau)$	0.044315	0.006072	-0.64735
$W_{III}(2,\tau)$	69573	1494.9	628.23
$W_V(2,\tau)$	1.999	0.85389	0.1815
Testing the curvature factor			
	Sup	*Avg*	*Exp*
$W_{II}(3,\tau)$	0.067158	0.008053	-0.64545
$W_{III}(3,\tau)$	38165	1341.9	660.17
$W_V(3,\tau)$	1.9993	0.79494	0.1429
Testing the common factors			
	Sup	*Avg*	*Exp*
$W_{VI}(1,2,\tau)$	2.2677	0.92023	0.24683
$W_{VI}(1,3,\tau)$	2.0918	0.81725	0.16398
$W_{VI}(2,3,\tau)$	5.5412	1.6858	1.2748

Testing the IRs governing the level factor			
	Sup	*Avg*	*Exp*
$W_{IV}(1,1,\tau)$	0.036288	0.00635	-0.64709
$W_{IV}(1,2,\tau)$	0.029168	0.005343	-0.64805
$W_{IV}(1,3,\tau)$	0.027792	0.005012	-0.64838
$W_{IV}(1,4,\tau)$	0.04486	0.008422	-0.64509
$W_{IV}(1,5,\tau)$	0.057391	0.010442	-0.64315
$W_{IV}(1,6,\tau)$	0.065338	0.011304	-0.64231
$W_{IV}(1,7,\tau)$	0.063509	0.010929	-0.64268
$W_{IV}(1,8,\tau)$	0.056745	0.009598	-0.64395
$W_{IV}(1,9,\tau)$	0.05125	0.008461	-0.64505
$W_{IV}(1,10,\tau)$	0.047699	0.00756	-0.64592
$W_{IV}(1,11,\tau)$	0.07329	0.011633	-0.642
$W_{IV}(1,12,\tau)$	0.062959	0.010263	-0.64332
$W_{IV}(1,13,\tau)$	0.0588	0.009607	-0.64396
$W_{IV}(1,14,\tau)$	0.054342	0.009161	-0.64437
$W_{IV}(1,15,\tau)$	0.052668	0.008919	-0.64462
$W_{IV}(1,16,\tau)$	0.049518	0.008384	-0.64513
$W_{IV}(1,17,\tau)$	0.046074	0.007915	-0.64558
$W_{IV}(1,18,\tau)$	0.044357	0.007526	-0.64596
$W_{IV}(1,19,\tau)$	0.040922	0.007315	-0.64615
$W_{IV}(1,20,\tau)$	0.040231	0.00722	-0.64625
$W_{IV}(1,21,\tau)$	0.039394	0.007005	-0.64646

Testing the IRs governing the slope factor

	Sup	*Avg*	*Exp*
$W_{IV}(2,1,\tau)$	0.051577	0.006352	-0.64708
$W_{IV}(2,2,\tau)$	0.12512	0.02597	-0.62804
$W_{IV}(2,3,\tau)$	0.077594	0.013121	-0.64057
$W_{IV}(2,4,\tau)$	0.32226	0.083682	-0.57185
$W_{IV}(2,5,\tau)$	0.61571	0.17536	-0.48234
$W_{IV}(2,6,\tau)$	0.85115	0.25188	-0.40749
$W_{IV}(2,7,\tau)$	0.87334	0.27685	-0.38139
$W_{IV}(2,8,\tau)$	0.82397	0.26617	-0.3939
$W_{IV}(2,9,\tau)$	0.72726	0.2428	-0.41755
$W_{IV}(2,10,\tau)$	0.63881	0.21369	-0.44538
$W_{IV}(2,11,\tau)$	1.9012	0.66653	0.008678
$W_{IV}(2,12,\tau)$	3.6191	1.2345	0.64373
$W_{IV}(2,13,\tau)$	6.5092	2.0605	1.6257
$W_{IV}(2,14,\tau)$	7.6617	2.3896	2.0536
$W_{IV}(2,15,\tau)$	8.122	2.5765	2.2694
$W_{IV}(2,16,\tau)$	7.2262	2.4504	2.0487
$W_{IV}(2,17,\tau)$	5.9639	2.0917	1.6156
$W_{IV}(2,18,\tau)$	4.9678	1.7805	1.2349
$W_{IV}(2,19,\tau)$	4.0719	1.4843	0.88861
$W_{IV}(2,20,\tau)$	3.7643	1.3731	0.77263
$W_{IV}(2,21,\tau)$	3.427	1.249	0.63609

Testing the IRs governing the curvature factor

	Sup	*Avg*	*Exp*
$W_{IV}(3,1,\tau)$	4.2162	1.3185	0.77694
$W_{IV}(3,2,\tau)$	6.636	2.0783	1.7184
$W_{IV}(3,3,\tau)$	5.3056	1.6169	1.1638
$W_{IV}(3,4,\tau)$	2.8565	0.89162	0.28807
$W_{IV}(3,5,\tau)$	1.1404	0.34882	-0.30065
$W_{IV}(3,6,\tau)$	0.1145	0.028903	-0.6253
$W_{IV}(3,7,\tau)$	0.29948	0.085398	-0.56978
$W_{IV}(3,8,\tau)$	1.6242	0.47963	-0.16645
$W_{IV}(3,9,\tau)$	2.2641	0.74217	0.1056
$W_{IV}(3,10,\tau)$	2.1419	0.70569	0.06553
$W_{IV}(3,11,\tau)$	0.064511	0.016448	-0.63737
$W_{IV}(3,12,\tau)$	0.026064	0.005589	-0.64782
$W_{IV}(3,13,\tau)$	0.016456	0.003439	-0.64989
$W_{IV}(3,14,\tau)$	0.056857	0.014074	-0.63962
$W_{IV}(3,15,\tau)$	0.091825	0.023438	-0.63058
$W_{IV}(3,16,\tau)$	0.10196	0.02641	-0.62765
$W_{IV}(3,17,\tau)$	0.09857	0.025447	-0.62864
$W_{IV}(3,18,\tau)$	0.094809	0.024353	-0.62966
$W_{IV}(3,19,\tau)$	0.090232	0.022286	-0.63162
$W_{IV}(3,20,\tau)$	0.09193	0.02293	-0.63106
$W_{IV}(3,21,\tau)$	0.091584	0.023557	-0.63049

Authors	*Term Structure (rates considered)*	*Source*	*Frequency*	*Period*
Bliss (1997)	Fama-Bliss Discount bond yields (0.25, 0.5, 1, 2, 3, 5, 7, 10, 15, and 20 yrs)	CRSP	Monthly	Jan 1970 - Dec 1995
Diebold and Li (2006)	Govt bonds (3, 6, 9, 12, 15, 18, 21, 24, 30, 36, 48, 60, 72, 84, 96, 108, and 120 months)	CRSP	Monthly	Jan 1985 - Dec 2000
Perignon and Villa (2006)	Fama-Bliss Discount Bond Yields (0.25, 0.5, 1, 2, 6, 12 yrs)	CRSP	Monthly	Jan 1960 - Dec 1999
Reisman and Zohar (2004)	Discount bond yields (0.25, 0.5, 1, 2, 3, 5, 7, and 10 yrs)	Fed	Monthly & weekly	1982 - 2003
Fabozzi et al (2005)	Swap rates (0.25, 0.5, 1-5, 7, 10, 15, 20, 30 yrs)	-	Monthly	June 1994 - Sept 2003
Chantziara and Skiadopoulos (2005)	4 Petroleum NIMEX & IPE Futures with 9 maturities	WTI, Bloomberg	Daily	Jan 1993 - Dec 2003
Audrino et al (2005)	Discount bond yields (1,...,30 yrs)	J.P. Morgan	Daily	Jan 1986 - May 1995

TABLE 2.2. List of datasets used in studies that evaluated the issue of PCA factor's stability

TABLE 2.3. Testing results for US zero coupon bond yields. The table reports the *Sup*, *Avg*, and *Exp* values for the test statistics $W_I(\tau), W_{II}(i,\tau), W_{III}(i,\tau), W_{IV}(i,\tau), W_V(i,\tau)$, and $W_{VI}(i,\tau)$ associated with the six hypotheses formulated in equations 2.23 - 2.28. The p-values are reported in italics.

Testing overall system

	Sup	*Avg*	*Exp*
$W_I(\tau)$	0.96701	0.26218	-0.38973
	0.000	*0.000*	*0.000*

Testing the level factor

	Sup	*Avg*	*Exp*
$W_{II}(1,\tau)$	0.24475	0.062922	-0.59174
	0.000	*0.000*	*0.000*
$W_{III}(1,\tau)$	0.87365	0.22575	-0.43265
	0.000	*0.000*	*0.000*
$W_V(1,\tau)$	0.70327	0.24609	-0.41435
	0.000	*0.000*	*0.000*

Testing the slope factor

	Sup	*Avg*	*Exp*
$W_{II}(2,\tau)$	0.34214	0.090797	-0.5643
	0.000	*0.000*	*0.000*
$W_{III}(2,\tau)$	249.64	10.664	118.11
	0.817	*0.793*	*0.817*
$W_V(2,\tau)$	1.9815	0.77043	0.11505
	0.466	*0.156*	*0.180*

Testing the curvature factor

	Sup	*Avg*	*Exp*
$W_{II}(3,\tau)$	0.38012	0.10846	-0.54758
	0.000	*0.000*	*0.000*
$W_{III}(3,\tau)$	335.49	5.6174	160.9
	0.948	*0.963*	*0.948*
$W_V(3,\tau)$	1.9899	0.44182	-0.20286
	0.541	*0.478*	*0.572*

Testing the common factors

	Sup	*Avg*	*Exp*
$W_{VI}(1,2,\tau)$	3.4721	1.2964	0.62638
	0.000	*0.000*	*0.000*
$W_{VI}(1,3,\tau)$	2.393	0.70712	0.063935
	0.000	*0.150*	*0.185*
$W_{VI}(2,3,\tau)$	5.4103	1.4711	0.83286
	0.119	*0.149*	*0.293*

Testing the IRs governing the level factor

	Sup	*Avg*	*Exp*
$W_{IV}(1,1,\tau)$	0.084479	0.008332	-0.64515
	0.001	*0.028*	*0.028*
$W_{IV}(1,2,\tau)$	0.13845	0.019008	-0.63475
	0.000	*0.000*	*0.000*
$W_{IV}(1,3,\tau)$	0.087745	0.011164	-0.64241
	0.000	*0.002*	*0.002*
$W_{IV}(1,4,\tau)$	0.1796	0.031018	-0.62314
	0.000	*0.000*	*0.000*
$W_{IV}(1,5,\tau)$	0.23842	0.058936	-0.59595
	0.000	*0.000*	*0.000*
$W_{IV}(1,6,\tau)$	0.39888	0.087758	-0.56738
	0.000	*0.000*	*0.000*
$W_{IV}(1,7,\tau)$	0.52171	0.105	-0.54994
	0.000	*0.000*	*0.000*
$W_{IV}(1,8,\tau)$	0.53798	0.11548	-0.53989
	0.000	*0.000*	*0.000*
$W_{IV}(1,9,\tau)$	0.48691	0.1161	-0.53983
	0.000	*0.000*	*0.000*
$W_{IV}(1,10,\tau)$	0.42221	0.11239	-0.54391
	0.000	*0.000*	*0.000*
$W_{IV}(1,11,\tau)$	0.5422	0.078327	-0.57431
	0.000	*0.000*	*0.000*
$W_{IV}(1,12,\tau)$	0.5252	0.090457	-0.56329
	0.000	*0.000*	*0.000*
$W_{IV}(1,13,\tau)$	0.49403	0.09326	-0.56105
	0.000	*0.000*	*0.000*
$W_{IV}(1,14,\tau)$	0.41073	0.08511	-0.56987
	0.000	*0.000*	*0.000*
$W_{IV}(1,15,\tau)$	0.3606	0.08916	-0.5662
	0.000	*0.000*	*0.000*
$W_{IV}(1,16,\tau)$	0.3292	0.086432	-0.56912
	0.000	*0.000*	*0.000*
$W_{IV}(1,17,\tau)$	0.30412	0.080209	-0.57535
	0.000	*0.000*	*0.000*
$W_{IV}(1,18,\tau)$	0.27069	0.074814	-0.5807
	0.000	*0.000*	*0.000*
$W_{IV}(1,19,\tau)$	0.24245	0.06894	-0.58645
	0.000	*0.000*	*0.000*
$W_{IV}(1,20,\tau)$	0.2327	0.064932	-0.59037
	0.000	*0.000*	*0.000*
$W_{IV}(1,21,\tau)$	0.22566	0.060909	-0.5943
	0.000	*0.000*	*0.000*

Testing the IRs governing the slope factor

	Sup	*Avg*	*Exp*
$W_{IV}(2,1,\tau)$	0.18589	0.033578	-0.62067
	0.000	*0.000*	*0.000*
$W_{IV}(2,2,\tau)$	0.33459	0.060201	-0.59436
	0.000	*0.000*	*0.000*
$W_{IV}(2,3,\tau)$	0.27183	0.059894	-0.59495
	0.000	*0.000*	*0.000*
$W_{IV}(2,4,\tau)$	0.79596	0.17493	-0.47831
	0.000	*0.000*	*0.000*
$W_{IV}(2,5,\tau)$	1.3499	0.27752	-0.36647
	0.000	*0.000*	*0.000*
$W_{IV}(2,6,\tau)$	1.6302	0.32541	-0.31064
	0.000	*0.002*	*0.001*
$W_{IV}(2,7,\tau)$	1.5131	0.29581	-0.34281
	0.000	*0.033*	*0.013*
$W_{IV}(2,8,\tau)$	1.5473	0.27801	-0.36074
	0.000	*0.036*	*0.021*
$W_{IV}(2,9,\tau)$	1.3915	0.23797	-0.40494
	0.000	*0.053*	*0.032*
$W_{IV}(2,10,\tau)$	1.1574	0.19343	-0.45422
	0.000	*0.077*	*0.060*
$W_{IV}(2,11,\tau)$	1.5084	0.27769	-0.37404
	0.278	*0.465*	*0.490*
$W_{IV}(2,12,\tau)$	2.5622	0.48654	-0.12241
	0.394	*0.478*	*0.514*
$W_{IV}(2,13,\tau)$	4.7042	0.87814	0.46817
	0.283	*0.443*	*0.471*
$W_{IV}(2,14,\tau)$	5.4103	1.0661	0.79033
	0.285	*0.431*	*0.426*
$W_{IV}(2,15,\tau)$	5.7059	1.2922	1.0658
	0.304	*0.384*	*0.375*
$W_{IV}(2,16,\tau)$	5.9104	1.265	0.99521
	0.201	*0.379*	*0.370*
$W_{IV}(2,17,\tau)$	5.421	1.1148	0.7536
	0.128	*0.384*	*0.379*
$W_{IV}(2,18,\tau)$	4.7358	0.95788	0.51804
	0.087	*0.397*	*0.389*
$W_{IV}(2,19,\tau)$	4.0563	0.81108	0.30857
	0.053	*0.402*	*0.397*
$W_{IV}(2,20,\tau)$	3.6786	0.75652	0.23197
	0.070	*0.400*	*0.402*
$W_{IV}(2,21,\tau)$	3.2564	0.69068	0.14226
	0.107	*0.402*	*0.406*

Testing the IRs governing the curvature factor

	Sup	*Avg*	*Exp*
$W_{IV}(3,1,\tau)$	3.2792	0.092784	-0.50577
	0.352	*0.938*	*0.940*
$W_{IV}(3,2,\tau)$	4.8078	0.1234	-0.36822
	0.595	*0.942*	*0.940*
$W_{IV}(3,3,\tau)$	3.8986	0.13472	-0.44289
	0.468	*0.930*	*0.937*
$W_{IV}(3,4,\tau)$	2.5426	0.10303	-0.52279
	0.177	*0.922*	*0.920*
$W_{IV}(3,5,\tau)$	1.0262	0.067277	-0.58338
	0.141	*0.858*	*0.863*
$W_{IV}(3,6,\tau)$	0.15693	0.019444	-0.63427
	0.004	*0.165*	*0.163*
$W_{IV}(3,7,\tau)$	0.31938	0.009214	-0.64412
	0.029	*0.933*	*0.933*
$W_{IV}(3,8,\tau)$	1.3524	0.025804	-0.62289
	0.185	*0.949*	*0.948*
$W_{IV}(3,9,\tau)$	2.0552	0.044823	-0.59434
	0.155	*0.946*	*0.945*
$W_{IV}(3,10,\tau)$	2.0115	0.053352	-0.5844
	0.124	*0.940*	*0.939*
$W_{IV}(3,11,\tau)$	0.23199	0.06061	-0.59451
	0.000	*0.000*	*0.000*
$W_{IV}(3,12,\tau)$	0.50206	0.058314	-0.59496
	0.000	*0.000*	*0.000*
$W_{IV}(3,13,\tau)$	0.53585	0.046009	-0.60558
	0.000	*0.000*	*0.000*
$W_{IV}(3,14,\tau)$	0.077917	0.011581	-0.64201
	0.004	*0.102*	*0.102*
$W_{IV}(3,15,\tau)$	0.20318	0.038946	-0.61529
	0.000	*0.004*	*0.004*
$W_{IV}(3,16,\tau)$	0.39983	0.069968	-0.58369
	0.000	*0.000*	*0.000*
$W_{IV}(3,17,\tau)$	0.52036	0.089416	-0.56384
	0.000	*0.000*	*0.000*
$W_{IV}(3,18,\tau)$	0.56755	0.097685	-0.55579
	0.000	*0.000*	*0.000*
$W_{IV}(3,19,\tau)$	0.58345	0.10417	-0.54988
	0.000	*0.000*	*0.000*
$W_{IV}(3,20,\tau)$	0.55312	0.10467	-0.54966
	0.000	*0.000*	*0.000*
$W_{IV}(3,21,\tau)$	0.51483	0.10014	-0.55433
	0.000	*0.000*	*0.000*

FIGURE 2.1. Evolution of the three principal factors across three subsamples. The line charts plot the first three principal components and the column charts show the percentage variations explained by those factors for the three subsample periods: Jan 1999 - Jun 2001, Jul 2001 - Dec 2003, and Jan 2004 - May 2006.

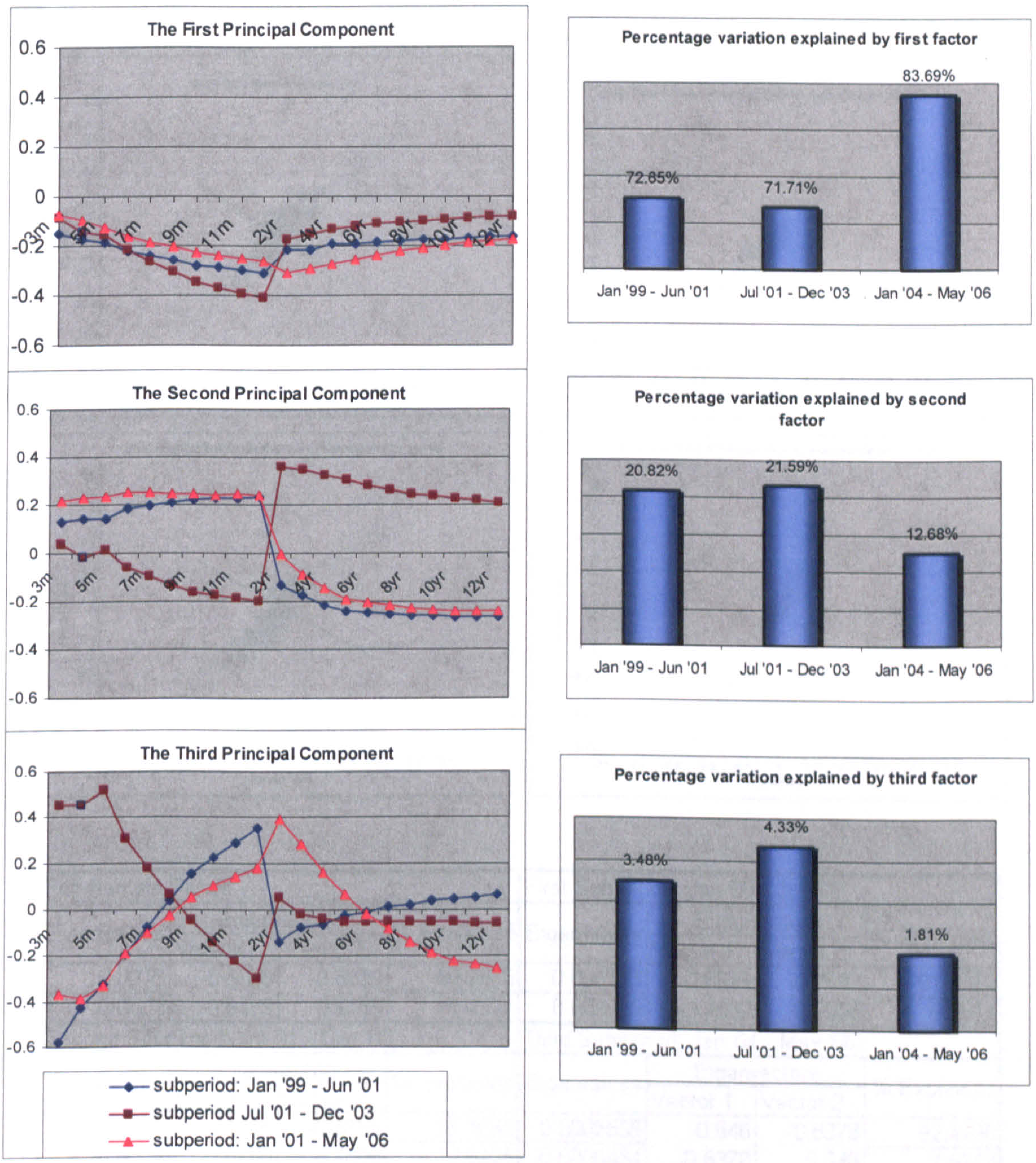
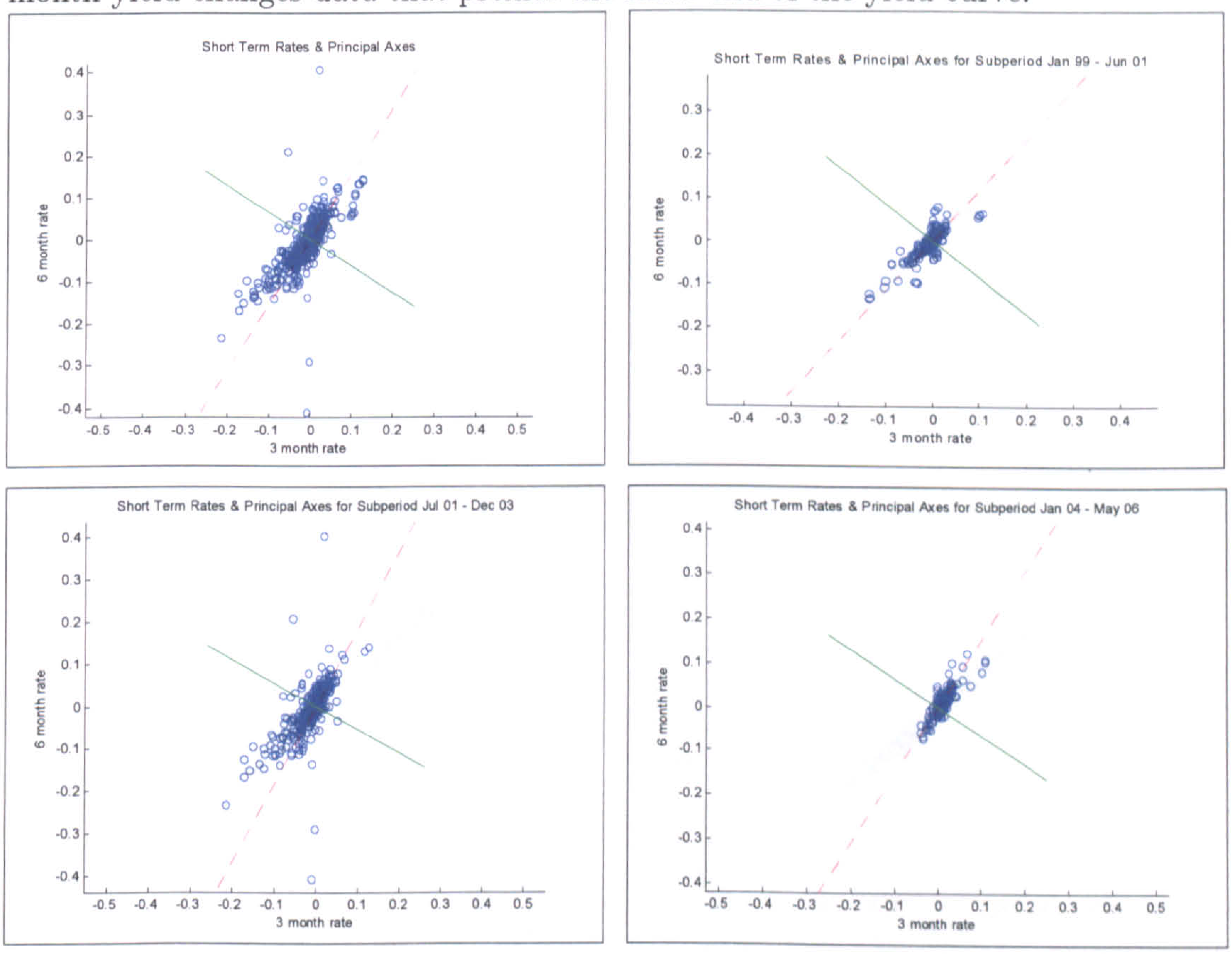
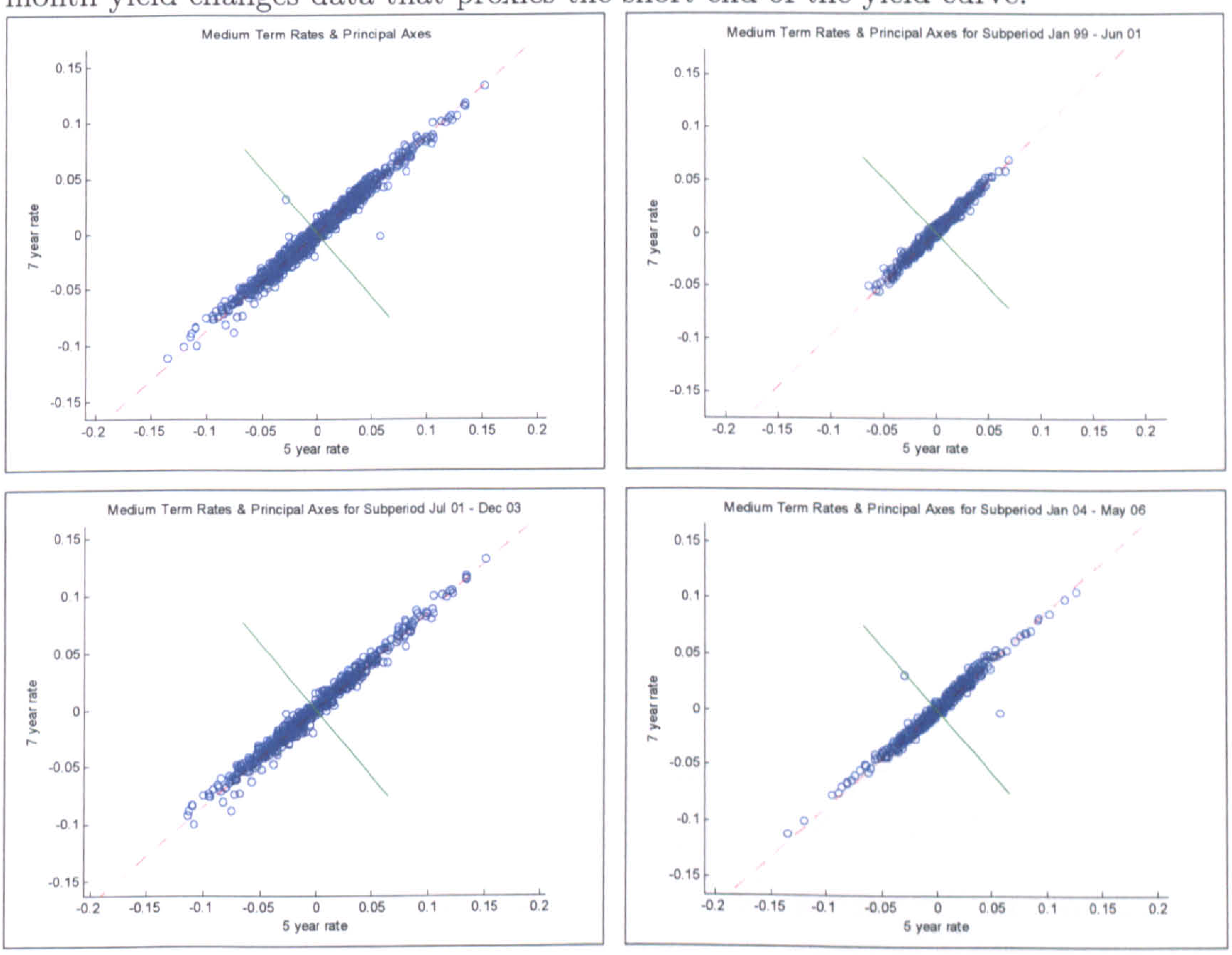


FIGURE 2.2. Short term rates and principal axes for the whole sample period and the three subsample periods. The dashed line depicts the first principal axis and the continuous line depicts the second principal axis of the short rates considered for the whole sample period and the three subsample periods: Jan 1999 - Jun 2001, Jul 2001 - Dec 2003, and Jan 2004 - May 2006.. The two orthogonal axes are fitted onto the scatter plot of the three month and six month yield changes data that proxies the short end of the yield curve.



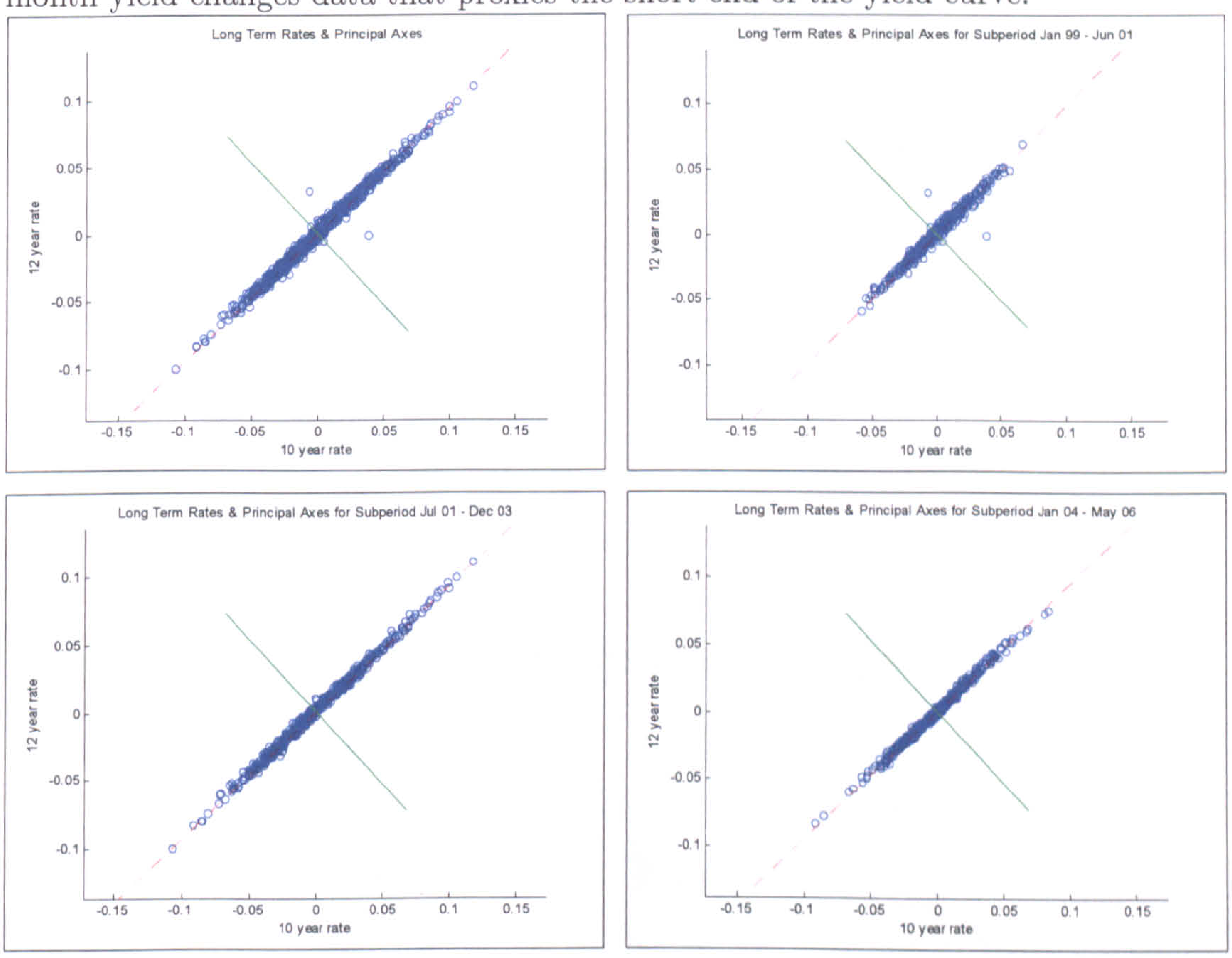
Full Sample Period				First Subperiod: Jan '99 - Jun '01			
Eigenvalues	Eigenvectors		% Explained	Eigenvalues	Eigenvectors		% Explained
	Vector 1	Vector 2			Vector 1	Vector 2	
0.0015	-0.5365	-0.8439	88.5676	0.000723	-0.6509	-0.7592	91.7486
0.0002	-0.8439	0.5365	11.4324	0.000065	-0.7592	0.6509	8.2514
Second Subperiod: Jul '01 - Dec '03				**Third Subperiod: Jan '04 - May '06**			
Eigenvalues	Eigenvectors		% Explained	Eigenvalues	Eigenvectors		% Explained
	Vector 1	Vector 2			Vector 1	Vector 2	
0.003	-0.4808	-0.8768	87.3595	0.0005558	-0.546	-0.8378	92.4486
0.0004	-0.8768	0.4808	12.6405	0.0000454	-0.8378	0.546	7.5514

FIGURE 2.3. Medium term rates and principal axes for the whole sample period and the three subsample periods. The dashed line depicts the first principal axis and the continuous line depicts the second principal axis of the short rates considered for the whole sample period and the three subsample periods: Jan 1999 - Jun 2001, Jul 2001 - Dec 2003, and Jan 2004 - May 2006.. The two orthogonal axes are fitted onto the scatter plot of the three month and six month yield changes data that proxies the short end of the yield curve.



Full Sample Period				First Subperiod: Jan '99 - Jun '01			
Eigenvalues	Eigenvectors		% Explained	Eigenvalues	Eigenvectors		% Explained
	Vector 1	Vector 2			Vector 1	Vector 2	
0.0018	-0.7523	-0.6589	99.3849	0.0007643	-0.7193	-0.6947	99.3881
0	-0.6589	0.7523	0.6151	0.0000047	-0.6947	0.7193	0.6119
Second Subperiod: Jul '01 - Dec '03				Third Subperiod: Jan '04 - May '06			
Eigenvalues	Eigenvectors		% Explained	Eigenvalues	Eigenvectors		% Explained
	Vector 1	Vector 2			Vector 1	Vector 2	
0.0031	-0.7609	-0.6489	99.5701	0.0014	-0.7491	-0.6625	99.1222
0	-0.6489	0.7609	0.4299	0	-0.6625	0.7491	0.8778

FIGURE 2.4. Long term rates and principal axes for the whole sample period and the three subsample periods. The dashed line depicts the first principal axis and the continuous line depicts the second principal axis of the short rates considered for the whole sample period and the three subsample periods: Jan 1999 - Jun 2001, Jul 2001 - Dec 2003, and Jan 2004 - May 2006.. The two orthogonal axes are fitted onto the scatter plot of the three month and six month yield changes data that proxies the short end of the yield curve.



Full Sample Period				First Subperiod: Jan '99 - Jun '01			
Eigenvalues	Eigenvectors		% Explained	Eigenvalues	Eigenvectors		% Explained
	Vector 1	Vector 2			Vector 1	Vector 2	
0.0012	-0.7265	-0.6871	99.7203	0.0007137	-0.713	-0.7011	99.2305
0	-0.6871	0.7265	0.2797	0.0000055	-0.7011	0.713	0.7695
Second Subperiod: Jul '01 - Dec '03				**Third Subperiod: Jan '04 - May '06**			
Eigenvalues	Eigenvectors		% Explained	Eigenvalues	Eigenvectors		% Explained
	Vector 1	Vector 2			Vector 1	Vector 2	
0.0019	-0.7315	-0.6818	99.8754	0.0009108	-0.7259	-0.6878	99.8185
0	-0.6818	0.7315	0.1246	0.0000017	-0.6878	0.7259	0.1815

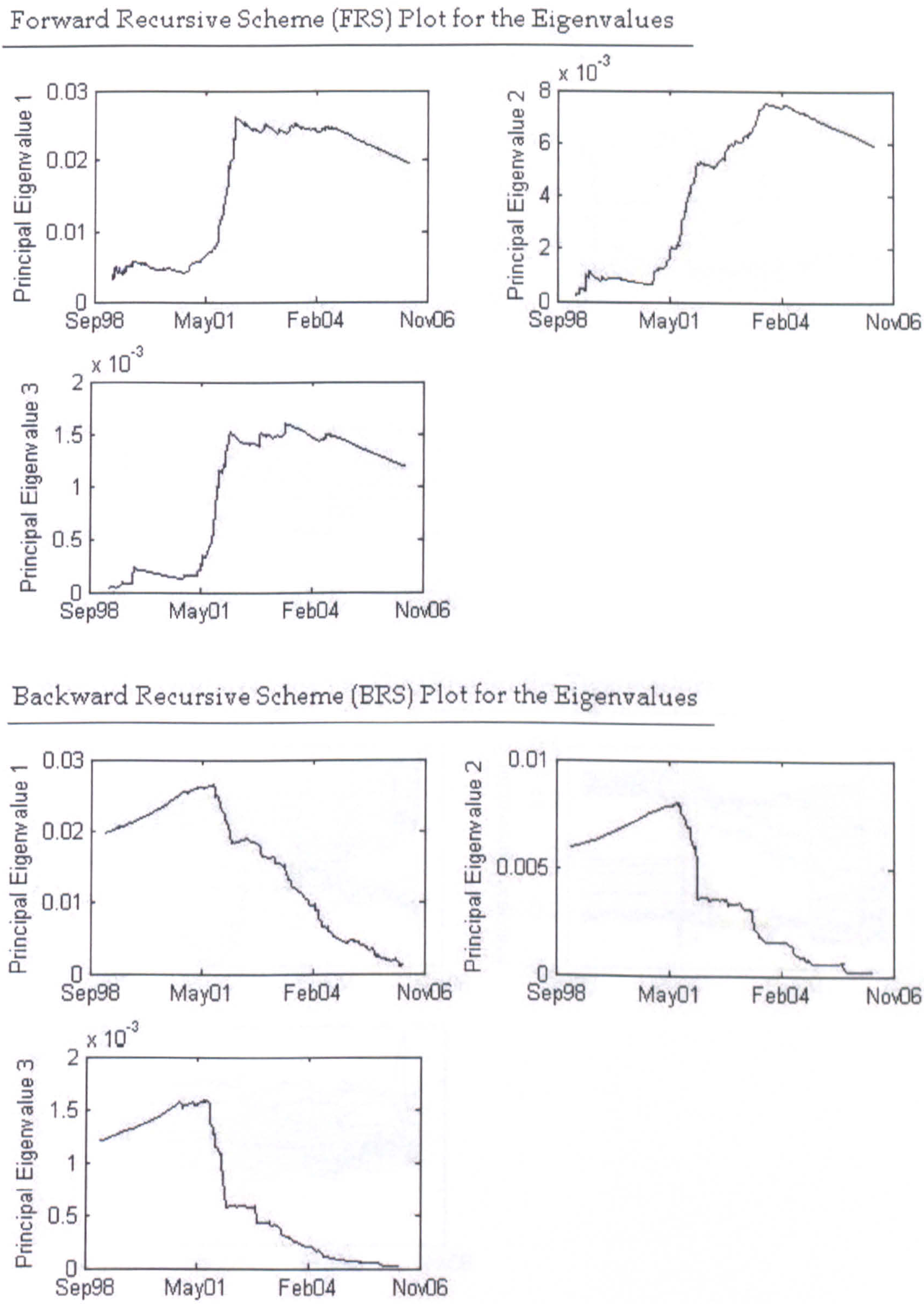


FIGURE 2.5. Forward recursive scheme (FRS) and backward recursive scheme (BRS) of the eigenvalues

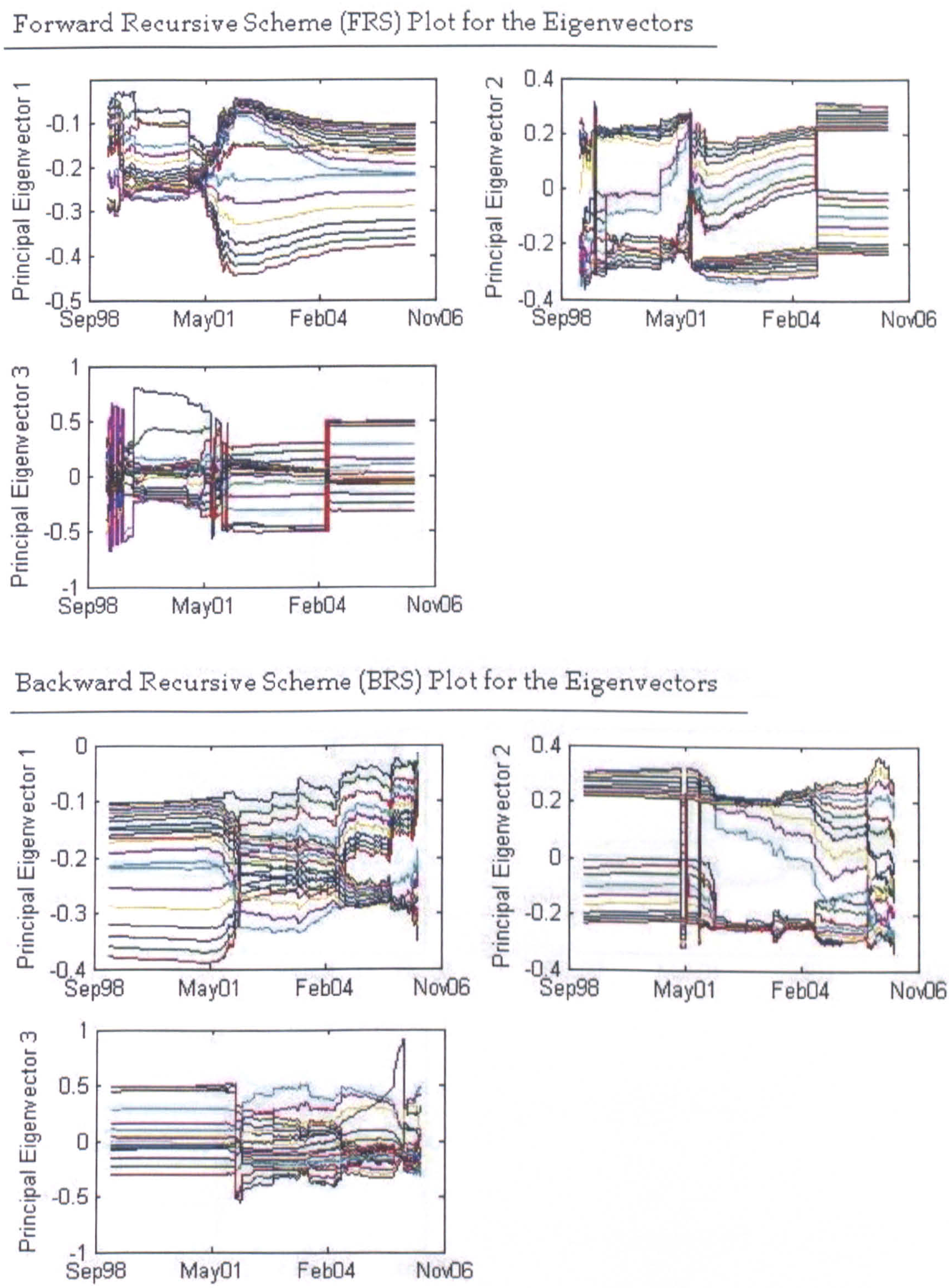


FIGURE 2.6. Forward recursive scheme (FRS) and backward recursive scheme (BRS) of the eigenvectors

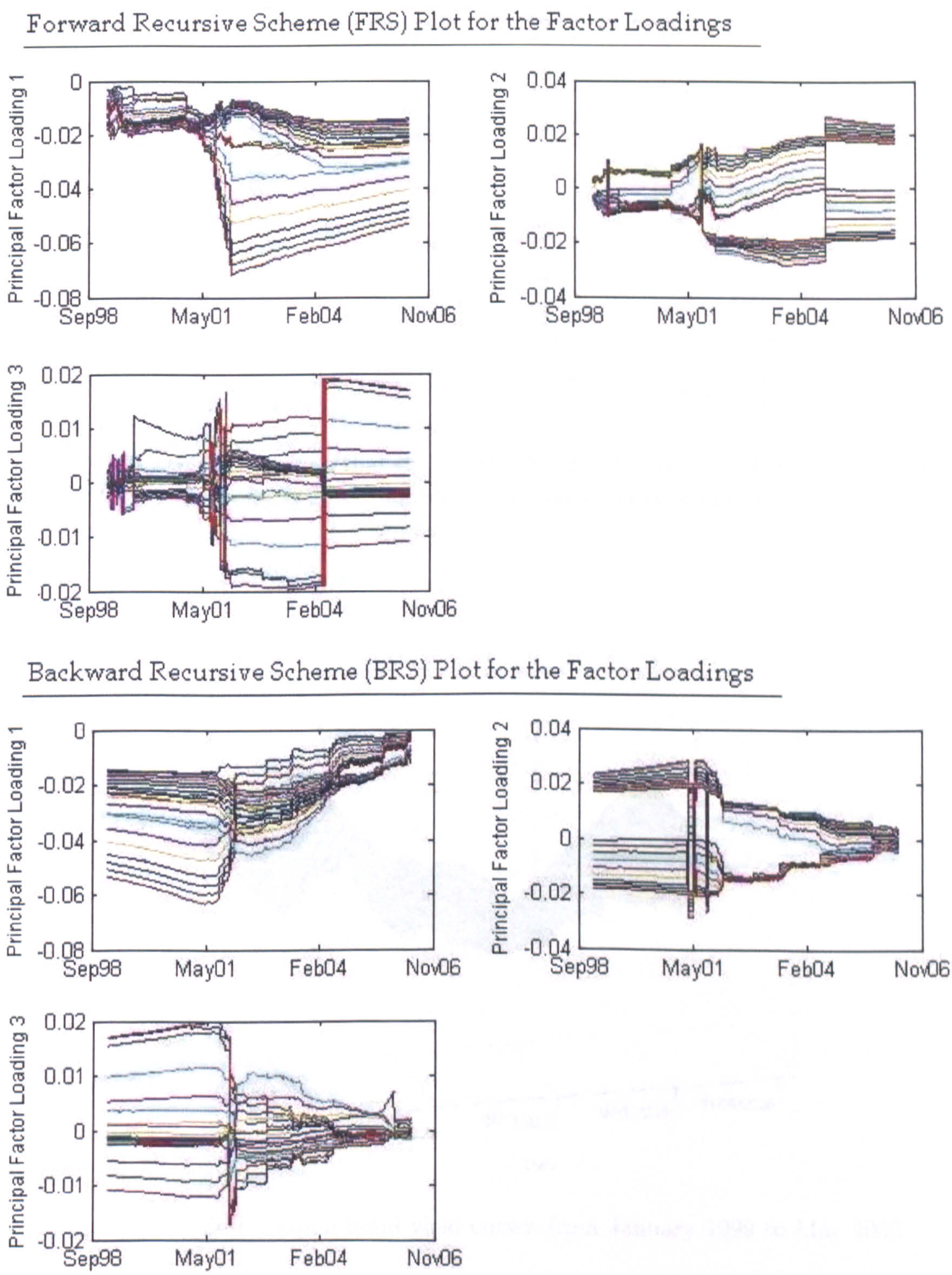


FIGURE 2.7. Forward recursive scheme (FRS) and backward recursive scheme (BRS) of the factor loadings

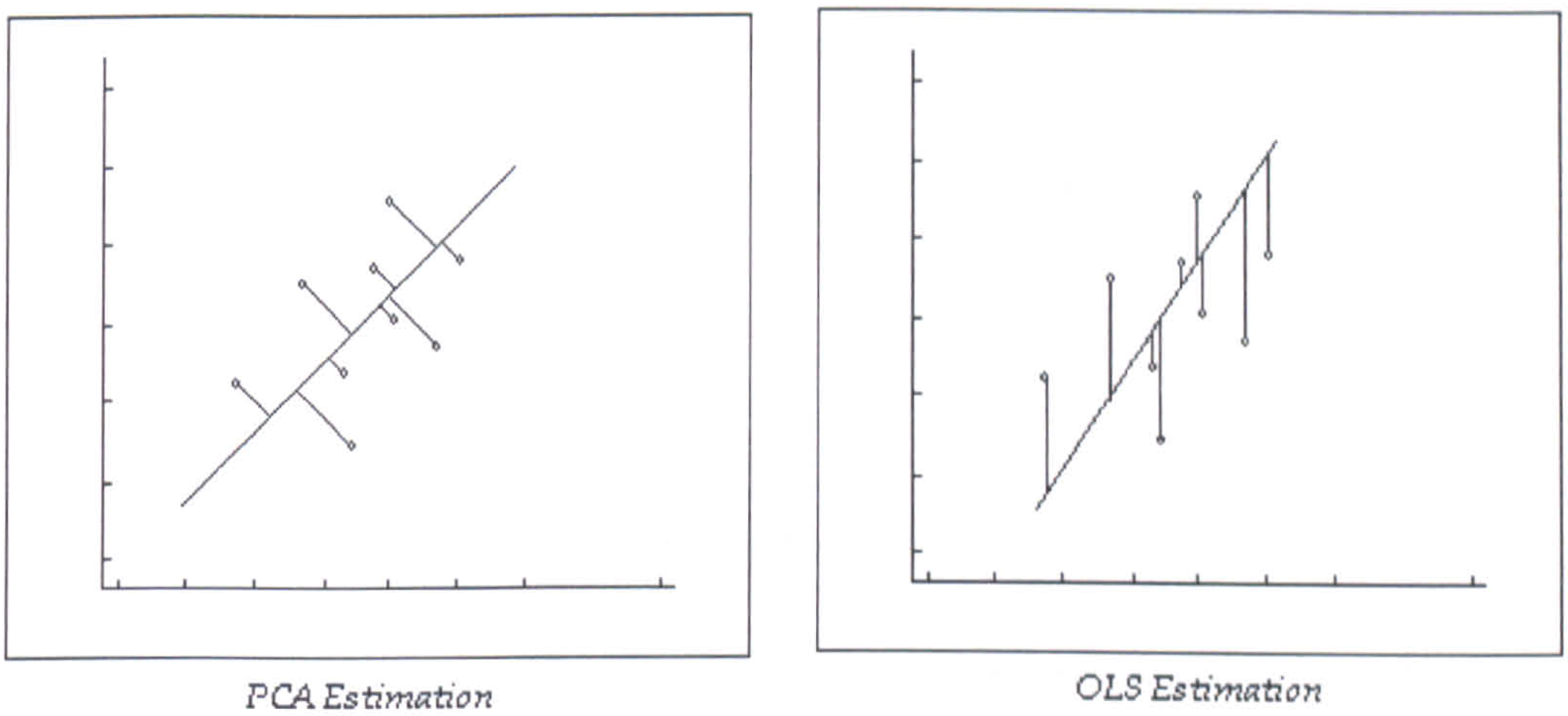


FIGURE 2.8. Illustration diagram that shows the difference between the Principal Component Analysis (PCA) and Ordinary Least Squares (OLS) estimation methods

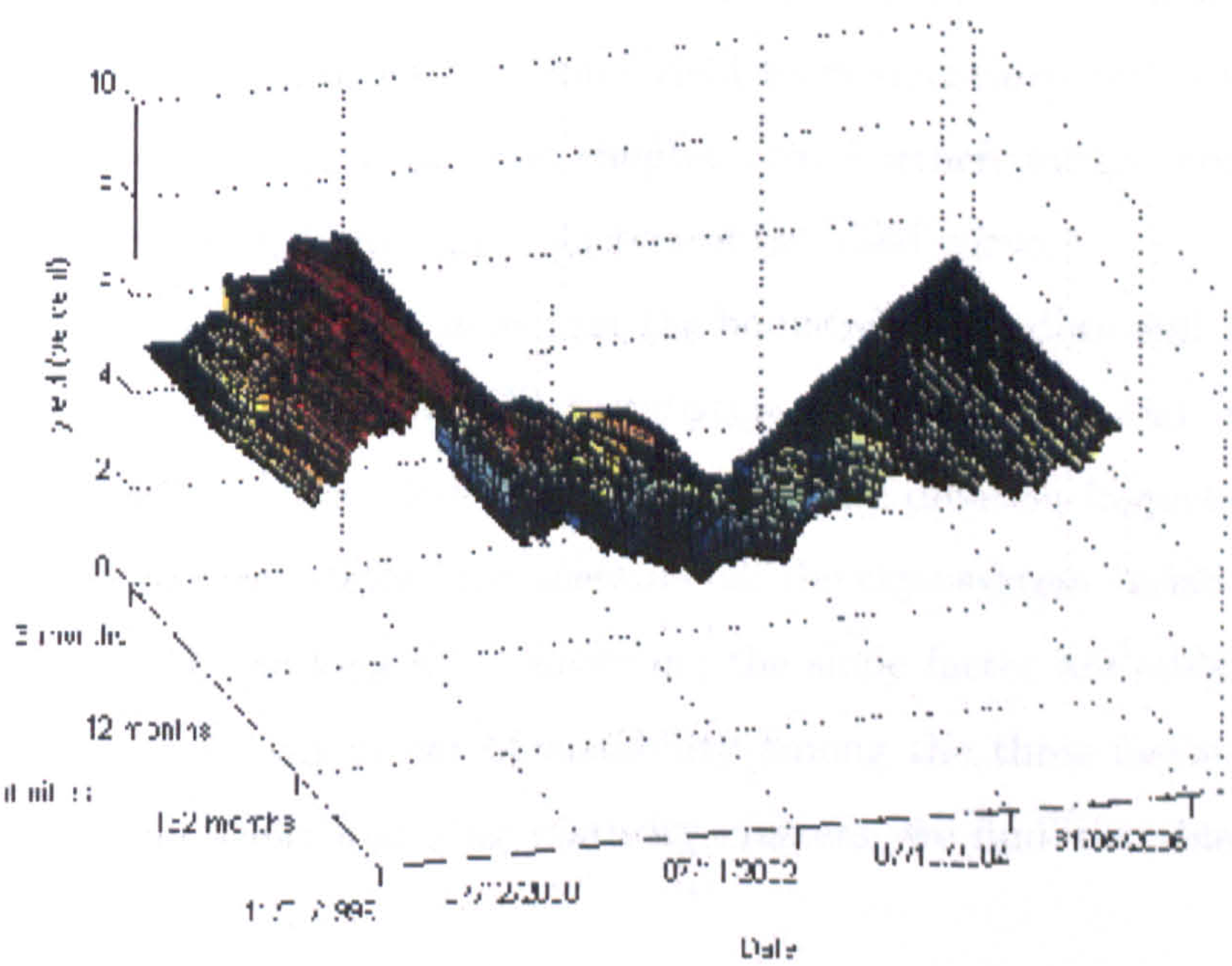


FIGURE 2.9. Zero coupon bond yield curves from January 1999 to May 2006

Chapter 3

Bond Yield Factor Stability Test Validation

ABSTRACT

Factor structure stability in the case of US zero coupon bond yields have been evaluated in chapter two. Several hypotheses were formulated and bootstrapped critical values were used in order draw inferences on stability in the eigenspace variables governing the factors of the zero coupon yield curves. The goal of this chapter is two-fold. First, we evaluate the size and power properties of the tests and provide the validity in the use of bootstrapped critical values for testing the eigensystem. Second, we conduct stability tests on widely used Fama-Bliss and Federal Reserve zero coupon bond yield term structures with daily and monthly frequencies and compare the results from chapter two. Further, we also consider evaluating stability in the short and long maturity clusters of the yield curve.

Monte Carlo simulation results show that the bootstrap procedure well approximates the finite sample null distribution of the test statistics with good size and power properties. Conducting the stability tests on four term structures with different frequencies, we find that overall eigensystem of three factors are unstable, all the eigensystem variables governing the level factor is unstable, the long rates governing the slope factor are stable over time, and there is evidence of common points of instability among the three factors. In testing for instabilities within the short and long maturity clusters, we find unstable eigensystem for both clusters.

3.1 Introduction

It is well known that term structure of yield curves are subject to instabilities such as structural breaks, regime switches, and parameter inconstancies. Authors have therefore modelled and tested for such concerns in literature. The instabilities observed in yields can also seep into the factors governing these yields. There is common consensus that the three factors, namely level, slope, and curvature are sufficient to explain most of the variations in yields. Instabilities present within the factor structures would evidently mean that the dependence structure of these factors have incurred a structural change at some point in time.

Instability risks present within the factor structure of yield curves have commonly been assumed to be nil. Since these latent factors are generally extracted using rotations such as the principal component analysis, authors have seemingly assumed that these rotations are robust to structural changes. Some authors informally test for instabilities using graphical analysis and conclude stability of the factors if the cumulative variations explained by the three factors have remained stable over time. In chapter two, we conduct a formal stability test on the eigenspace variables (eigenvalues, eigenvectors, and factor loadings) associated with the level, slope, and curvature factors. Bootstrapped critical values from the tests were used in order to draw test inferences. We find evidence of structural changes affecting all the eigenspace variables of yield curve factors.

This chapter aims at validating the conclusions drawn in chapter two. Particularly, we show that the bootstrap procedure implemented by the paper well approximates the null distribution of the test statistics and that the bootstrap distribution converges to the asymptotic distribution of the test statistic in probability. A Monte Carlo investigation of the finite sample size and power performance of the tests have also been undertaken. Further, we validate the empirical results drawn in chapter two by considering widely used constant maturity zero coupon bond term structures of Fama-Bliss and Federal Reserve. In this, we consider term structures with daily and monthly frequencies and consider short and long maturity clusters governing the term structure.

To summarize the simulation results, we find good size and power properties for all the bootstrap test statistics. We find that the overall empirical size of the bootstrap tests is very close to the nominal size for almost all the test statistics. In examining restrictions on the curvature factor, the size performance of the tests shows substantial size improvements as the cross sectional dimensions increase. The performance in terms of power is essentially close to one for majority of the test statistics. In the case of test statistic $W_{IV}(\cdot)$ evaluating the curvature factor, we find low power in capturing small structural changes for small cross sectional dimension N panels. However we find power gain as the magnitude of structural change increase.

To summarize the empirical findings for bond yield curves, we find the overall eigensystem unstable for the three factors independent of the term structure analyzed. All the eigenspace variables governing the level factor have been unstable in all term structures considered. The long rates governing the slope factor has remained unstable across all term structures. In investigating common structural change points, we find that within the five term structures considered, three conclude the presence of common change points within level, slope, and curvature.

The chapter is organized as follows. In section 3.2 we summarize the stability testing procedure implemented in chapter two. Section 3.3 presents the bootstrap algorithm used to approximate the null distribution of the test statistics. We provide an outline of the consistency for the bootstrap procedure and show that the bootstrap methodology for pivotal test statistics provides refinements to asymptotic approximations. In section 3.4 we carry out Monte Carlo simulations in order to investigate the performance of the bootstrap test statistics in finite samples. In section 3.5 we investigate stability in four different zero coupon bond yield term structures commonly used in literature and discuss the testing results. Section 3.6 discusses stability analysis in the case of term structures with correlation clusters and conducts stability tests on the short and long maturity clusters separately. Section 3.7 concludes.

3.2 Estimation and testing framework

Consider the optimization problem governing the principal component analysis

$$\begin{aligned} \underset{\beta_i}{max}\ \beta_i'\Sigma\beta_i \\ \text{subject to } \beta_i'\beta_i &= 1 \text{ and } \beta_i'Y' \perp \beta_j'Y' \quad \text{for } i,j = 1,...,N \text{ and } i < j \end{aligned} \tag{3.1}$$

where Σ the covariance matrix of a stationary $N \times T$ panel Y. The estimated matrix $\boldsymbol{\beta} = (\beta_1, \beta_2, ., \beta_i, .., \beta_N)$ is such that each vector $\beta_i'Y'$ $(i = 1, ..., N)$ is the directional vector that captures the maximum variability in Y and are orthogonal to each other. The solution to this optimization problem satisfies the eigen-decomposition system

$$\Sigma\beta_i = \lambda_i\beta_i \tag{3.2}$$

where λ_i are the i^{th} eigenvalue of the matrix Σ and β_i the corresponding eigenvector. Hence the eigenvalues $\Lambda = (\lambda_1, \lambda_2, ., \lambda_i, .., \lambda_N)$ summarize the amount of variability captured by their corresponding eigenvectors. The framework pertains to the classical PCA framework, allowing for the cross-sectional correlation among units. Thus the framework is comparable to the approximate factor models originally introduced by Chamberlain and Rothschild (1983).

We define the i^{th} factor loading of matrix Σ as

$$\gamma_i = \beta_i\lambda_i^{1/2} \tag{3.3}$$

The factor loadings matrix $\boldsymbol{\gamma} = (\gamma_1, \gamma_2, ., \gamma_i, ..., \gamma_N)$ load the factors on to the variables, explaining the correlation between the factors and the variables. The factor loadings matrix $\boldsymbol{\gamma}$ is computed as the unit length eigenvectors matrix multiplied by its singular value, which is the square-root of eigenvalues. Thus it characterizes the unit length eigenvectors in its true size and encompasses in them the information of direction as well as magnitude.

We formulate six hypotheses for testing the stability of the eigensystem around the unknown breakpoint τ that splits the sample into two subsamples: subsample a and subsample b. The parameters Λ, β, γ are estimated for the two subsamples a and b (denoted in superscript of parameters) and we test for the equality of the parameters. The hypotheses pertain to testing the null of stability against the alternative of instability induced by one or more structural breaks.[1]

A Wald type statistic $W(\tau) = (W_I(\tau), W_{II}(i,\tau), W_{III}(i,\tau), W_{IV}(i,\tau), W_V(i,\tau), W_{VI}(i,\tau))$ associated with the six hypotheses are formulated (as outlined in chapter two), which for a given value of τ are chi-squared distributed.

In testing the hypotheses, since the break point τ is unknown, we use the three commonly used weighted measures Sup, Avg,and Exp for the Wald test statistics $W(\tau)$

$$W_{Sup}(\tau) = \max_{t_1<\tau<t_2} W(\tau) \tag{3.4}$$

$$W_{Avg}(\tau) = \frac{1}{t_2 - t_1 + 1}\sum_{\tau=t_1}^{t_2} W(\tau) \tag{3.5}$$

$$W_{Exp}(\tau) = \ln\left[\frac{1}{t_2 - t_1 + 1}\sum_{\tau=t_1}^{t_2} \exp\left(\frac{1}{2}W(\tau)\right)\right] \tag{3.6}$$

where the change point τ is unknown but lies between t_1 and t_2 such that $t_1 = [T\epsilon_1]$, $t_2 = [T\epsilon_2]$, $t_1 \neq t_2$, $\epsilon_1 = [0.15, 0.85], \epsilon_2 = 1 - \epsilon_1$, and t_1 is bounded away from zero and t_2 is bounded away from boundary, T.

When the break point τ is unknown, the weighted test statistics converge to the following

[1]The tests are detailed in chapter two

quantities:

$$W_{Sup}(\tau) \xrightarrow{d} \max_{\epsilon_1<\epsilon<\epsilon_2} Q(\epsilon) \tag{3.7}$$

$$W_{Avg}(\tau) \xrightarrow{d} \int_{\epsilon_1}^{\epsilon_2} Q(\epsilon)d\epsilon \tag{3.8}$$

$$W_{Exp}(\tau) \xrightarrow{d} \ln\left[\int_{\epsilon_1}^{\epsilon_2} \exp\left(\frac{1}{2}Q(\epsilon)\right) d\epsilon\right] \tag{3.9}$$

where if we know the break point fraction ϵ, $Q(\epsilon)$ will be $\chi^2(q)$ with the degrees of freedom q corresponding to the number of restrictions being tested.

3.3 Bootstrap procedure

The bootstrap method has been used to approximate the null distribution of the various instability test statistics proposed in the section above. In order to construct the bootstrap distribution of the test statistics, we undertake the following steps:

1. Randomly draw the vector of maturities from the $T \times N$ term structure data in order to construct the $T \times N$ bootstrapped data, for time series dimension T and number of interest rate maturities N.

2. Construct the covariance matrix for the bootstrapped data and conduct the principal component analysis in order to estimate the eigenspace variables $\hat{\Lambda}, \hat{\beta}, \hat{\gamma}$.

3. Compute the six Wald statistics $W(\tau)$ and calculate the weighted measures; *Sup*, *Avg*, and *Exp* of $W(\tau)$.

4. Repeat steps 1 through 3 for BR number of bootstrap replications.

The procedure generates BR number of bootstrap statistics of *Sup*, *Avg*, and *Exp* of $W(\tau)$; $\{W^*_{Sup}(\tau)_m\}_{m=1}^{BR}, \{W^*_{Avg}(\tau)_m\}_{m=1}^{BR}, \{W^*_{Exp}(\tau)_m\}_{m=1}^{BR}$ from the bootstrapped samples. We calculate the bootstrap p-value for $W^*_{Sup}(\tau), W^*_{Avg}(\tau), W^*_{Exp}(\tau)$ as the proportion of bootstrap replications that yielded a statistic value greater than the test statistic values using the original data. Since we are interested in considering only the upper tail of the bootstrap distribution, the bootstrap p-value is

$$p^* = 1 - F^*(\cdot)$$

where $F^*(\cdot)$ is the distribution function of the bootstrap test statistics $W^*_{Sup}(\tau)$, $W^*_{Avg}(\tau)$, $W^*_{Exp}(\tau)$. We determine the significance level α for the tests and reject the null hypotheses when the bootstrap p-value does not exceed the significance level.

3.3.1 Validity of the bootstrap

The Wald test statistic is asymptotically pivotal and thus the asymptotic distribution of the test statistic does not depend on a particular data generating process under the null. Therefore bootstrap distribution can consistently estimate the asymptotic distribution of the test statistic and provide more reliable inference than asymptotically based inferences by removing the finite sample biases. However, in the case of the framework above, we do not bootstrap the Wald test statistic but the weighted test statistics; *Sup*, *Avg*, and *Exp*. As from Andrews (1993) and Andrews and Ploberger (1994) we know that these test statistics are functions of partial sums processes since the time of the change point is unknown and depends on parameter τ. The functional form of the partial sums process is implicit but unknown in the case of the principal component analysis framework.

Here we use the notion of weak convergence that guarantees accuracy of the bootstrap approximations for large number of iterations. For a known change point τ, the Wald statistics $W(\tau)$ defined above are known to be chi-squared in limit for large time dimensions. Since the test statistics under the null are free of nuisance parameters, for large number of replications, the bootstrapped statistic

$$W^*(\tau) \xrightarrow{W} W(\tau) \tag{3.10}$$

for a given value of τ where $\xrightarrow{W}$ denotes weak convergence in probability.

Consider the weighted statistics *Sup*, *Avg*, and *Exp* for $W^*(\tau)$ that are continuous transformations of the $W^*(\tau)$. According to the continuous mapping theorem,

$$\begin{aligned} W^*_{Sup}(\tau) &= \max_{t_1<\tau<t_2} W^*(\tau) \\ &\xrightarrow{d} \max_{\epsilon_1<\epsilon<\epsilon_2} Q(\epsilon) \qquad (3.11) \\ W^*_{Avg}(\tau) &= \frac{1}{t_2-t_1+1}\sum_{\tau=t_1}^{t_2} W^*(\tau) \\ &\xrightarrow{d} \int_{\epsilon_1}^{\epsilon_2} Q(\epsilon)d\epsilon \qquad (3.12) \\ W^*_{Exp}(\tau) &= \ln\left[\frac{1}{t_2-t_1+1}\sum_{\tau=t_1}^{t_2} \exp\left(\frac{1}{2}W^*(\tau)\right)\right] \\ &\xrightarrow{d} \ln\left[\int_{\epsilon_1}^{\epsilon_2} \exp\left(\frac{Q(\epsilon)}{2}\right)d\epsilon\right] \qquad (3.13) \end{aligned}$$

If we define $F^*(\cdot)$ as the distribution function of the weighted statistics *Sup*, *Avg*, and *Exp* of $W^*(\tau)$ given the bootstrapped data, then

$$F^*(\cdot) \xrightarrow{d} F(\cdot) \qquad (3.14)$$

and by continuous mapping theorem

$$p^* = 1 - F^*(\cdot) \xrightarrow{d} 1 - F(\cdot) = p$$

Thus, the bootstrap distributions of the weighted measures of the test statistics provide a good approximation of their asymptotic distributions for sufficiently large number of replications.

3.3.2 Bootstrap refinements

It is well known that, under certain conditions, bootstrap methodology provides refinements to the asymptotic approximations. The size distortions or the error committed in the rejection probability for an asymptotic test is, in general, $O(T^{-1/2})$ for one-sided tests and $O(T^{-1})$ for two-sided tests. Based on the theory of Edgeworth expansion, one can show that, for most asymptotically pivotal test statistics, these errors can be further refined of $O(T^{-1/2})$ or more under bootstrap sampling (see Hall (1996) and Horowitz (1997)).

Consider the term structure panel data $\{X_{ij} : i = 1, ..., T \text{ and } j = 1, ..., N\}$ from a multivariate distribution with CDF denoted by G. Using the bootstrap methodology we can generate the empirical CDF, denoted by G^*. Under some regulatory conditions, $G^* - G = O\left(T^{-1/2}\right)$ almost surely.

Let the CDF of the stability test statistics $\bar{W}(\tau)$ under the null be given by $F^0(z_\alpha, G) = P\left(\bar{W}(\tau) \leq z_\alpha\right)$ where $\bar{W}(\tau)$ is the weighted test statistic $W_{Sup}(\tau), W_{Avg}(\tau), W_{Exp}(\tau)$ and z_α is the critical value from the null distribution. Suppose we approximate $F^0(z_\alpha, G)$ using the first order asymptotic CDF $F^\infty(z_\alpha, G)$, we can show that $F^\infty(z_\alpha, G)$ converge to $F^0(z_\alpha, G)$ with an error of size $O(T^{-1/2})$. If instead we use the bootstrapped CDF $F^*(z_\alpha, G^*)$ to approximate the empirical CDF of $\bar{W}(\tau)$, using the theory of Edgeworth expansion, we can show that bootstrap provides further refinements, where the empirical size of the test converges to the true one faster than asymptotic tests. We develop this notion below.

Consider higher order approximation (Edgeworth expansion) of $F^0(z_\alpha, G)$ and $F^*(z_\alpha, G^*)$

$$F^0(z_\alpha, G) = F^\infty(z_\alpha, G) + \frac{1}{\sqrt{T}} f_1(z_\alpha, G) + \frac{1}{T} f_2(z_\alpha, G) + O\left(T^{-3/2}\right) \tag{3.15}$$

and

$$F^*(z_\alpha, G^*) = F^\infty(z_\alpha, G^*) + \frac{1}{\sqrt{T}} f_1(z_\alpha, G^*) + \frac{1}{T} f_2(z_\alpha, G^*) + O\left(T^{-3/2}\right) \tag{3.16}$$

where f_1 is an even function of z_α, and f_2 is an odd function of z_α.

To evaluate the accuracy of the bootstrap approximation, we evaluate

$$\begin{aligned} F^*(z_\alpha, G^*) - F^0(z_\alpha, G) &= [F^\infty(z_\alpha, G^*) - F^\infty(z_\alpha, G)] + \frac{1}{\sqrt{T}}[f_1(z_\alpha, G^*) - f_1(z_\alpha, G)] \\ &\quad + \frac{1}{T}[f_2(z_\alpha, G^*) - f_2(z_\alpha, G)] + O\left(T^{-3/2}\right) \\ &= O\left(T^{-1/2}\right) + O\left(T^{-1}\right) + O\left(T^{-3/2}\right) + O\left(T^{-3/2}\right) \\ &= O\left(T^{-1/2}\right) \end{aligned} \tag{3.17}$$

Thus we see that the bootstrap approximations provide the same accuracy (maximum error of size $O\left(T^{-1/2}\right)$), as in the case of asymptotic approximations.

For the case of asymptotically pivotal test statistics, since the asymptotic distributions $F^\infty(\cdot)$ do not depend on G or G^* [that is, $F^\infty(z_\alpha, G^*) = F^\infty(z_\alpha, G)$],

$$\begin{aligned} F^*(z_\alpha, G^*) - F^0(z_\alpha, G) &= \frac{1}{\sqrt{T}}[f_1(z_\alpha, G^*) - f_1(z_\alpha, G)] \\ &\quad + \frac{1}{T}[f_2(z_\alpha, G^*) - f_2(z_\alpha, G)] + O\left(T^{-3/2}\right) \\ &= O\left(T^{-1}\right) + O\left(T^{-3/2}\right) + O\left(T^{-3/2}\right) \\ &= O\left(T^{-1}\right) \end{aligned} \tag{3.18}$$

Thus we show that in general cases, the bootstrap provides approximations as good as their asymptotic counterparts. However, in the case of test statistics that are asymptotically pivotal, we can see that the bootstrap based tests would provide faster convergence compared to their asymptotic approximations.

3.4 Monte Carlo experiments

In this section, we perform a simulation study to examine the finite-sample size and power of the bootstrap test statistics developed above.

3.4.1 Monte Carlo design

Below we outline the simulation study undertaken in order to study the bootstrap size performance and power properties of the stability test statistics developed in chapter two. We estimate the eigenvalues, eigenvectors, and factor loadings via principal component analysis on the covariance matrix of the simulated data. We generate independent multivariate normal random variables with an arbitrarily chosen covariance structure. Figure *3.1* presents the algorithm implemented for evaluating the size and power properties of the test statistics.

[Insert Figure 3.1 here]

In evaluating the size performance, we carry out the following steps:

1. For a given value of T, N, τ, and for a given covariance structure (Σ), generate a panel of *iid* multivariate normal random variables
2. Conduct the PCA, compute the six test statistics $W(\tau)$, and calculate the weighted measures *Sup*, *Avg*, and *Exp* of $W(\tau)$
3. Generate bootstrapped data from the simulated data, compute the six test statistics $W(\tau)$, and estimate *Sup*, *Avg*, and *Exp* of $W(\tau)$
4. Repeat step 3 BR times and estimate the empirical distribution of *Sup*, *Avg*, and *Exp* of $W(\tau)$
5. Compute the $\alpha\%$ critical value of the bootstrapped distribution for each Monte Carlo run

6. Compare the statistic values of the weighted measures from step 2 to the critical value α and determine whether the critical value has exceeded the statistic values (rejection frequency of the tests)

7. Repeat steps 1-6 MCR times

8. Compute the relative rejection frequency across all the Monte Carlo runs and this (empirical size) should equal α, with a Monte Carlo error occurring approximately $\alpha\%$ times

We determine the power of the test statistics as the relative rejection frequencies when the data generating process is generated under the alternative. In order to evaluate power, we carry out the above procedure but perform step two on panel of multivariate normal random variables, imposing for a structural change in the eigensystem. We generate data under the alternative hypothesis of structural change in both the eigenvalues and eigenvectors. For this, we impose a structural break in the covariance structure of the data. Suppose the original data is iid $N(0, \Sigma)$. For a structural change happening at time τ, we define

$$\Sigma = \begin{cases} \Sigma & for\ t = 1, ..., \tau \\ \Sigma^* & for\ t = \tau + 1, ..., T \end{cases}$$

where $\Sigma^* = U\Sigma U$ where U is a $N \times N$ diagonal matrix with constants $u_1, ..., u_N$ generated once from a uniform distribution. In assessing the power of the tests, the matrix U dictates the magnitude of the structural change.

3.4.2 Monte Carlo results

We fix the time series dimension $T = 2000$, and vary the cross sectional dimension $N = 5$, 10, and 20. We trim the sample by discarding 15% of observations at the boundaries. Within each Monte Carlo replications, we conduct bootstrap runs in order to estimate the empirical bootstrap distribution of the test statistics and evaluate the relative size and power of the test statistics. The number of Monte Carlo replications (MCR) is chosen to be 5000 and the number of bootstrap runs (BR) is chosen to be 500. In accessing the power properties under the alternative hypothesis of instability, we consider structural changes occurring at three possible change points $\tau = T/3$, $T/2$, and $2T/3$ into the sample. We consider three break-size matrices U_1 uniformly generated between the intervals 1 and 3, U_2 uniformly generated between the intervals 3 and 5, and U_3 uniformly generated between the intervals 5 and 7. The size and power properties are evaluated at 5% level.

[Insert Table 3.1, 3.2, and 3.3 here]

Tables *3.1*, *3.2*, and *3.3* report the bootstrap size performance of the weighted measures *Sup*, *Avg*, and *Exp* of the six Wald type statistics respectively. Overall, we find that the empirical size of the bootstrap tests is very close to the nominal size of 5% almost for all the test statistics. In the case of level and slope factors, we find that the weighted measures *Sup*, *Avg*, and *Exp* do not show any significant size distortions. The size performance is around 5% and for $N = 5, 10$, and 20 we find the bootstrap test statistics size remained essential very similar. In the case of the curvature factors, we find that the empirical sizes of the weighted measures *Sup*, *Avg*, and *Exp* vary enormously (under-sizing) from the nominal size of 5%. However, increasing the cross sectional dimensions N from 5 to 20, we see significant improvements in distortions. For $N = 20$ we can actually find that the size distortions are minimal with empirical sizes close to 5%. These large size improvements can be mainly spotted in the weighted measures of $W_{III}(3, \tau)$ (testing instabilities in curvature eigenvectors), $W_{IV}(3, ., \tau)$ (testing instabilities in interest rate maturities governing the curvature eigenvectors), $W_V(3, \tau)$ (testing instabilities in curvature factor loadings), $W_{VI}(1, 3, \tau)$ (testing common instabilities in level and curvature factor loadings), and $W_{VI}(2, 3, \tau)$ (testing common instabilities in slope and curvature factor loadings).

[Insert Table 3.4, 3.5, and 3.6 here]

Tables *3.4*, *3.5*, and *3.6* report the bootstrapped power of the weighted measures of the six test statistics. In the case of $W_I(\cdot) - W_{III}(\cdot)$ and $W_{VI}(\cdot)$, we observe that the bootstrapped power of the weighted measures *Sup*, *Avg*, and *Exp* are essentially 100% for any magnitude of structural change and for changes occurring at any point within the sample. For $W_{IV}(1, \cdot, \tau)$ testing the level factor, we find the test performs well in terms of power for $N = 5$ and for structural changes occurring towards the second half of the sample. This means that the tests are more likely to capture changes when the time series is large. For $W_{IV}(2, \cdot, \tau)$ testing the slope factor, we find highest overall power for structural changes occurring towards the last 30% of the sample. We observe that as N increases and the change magnitude U increases, the power of the slope tests increase. In the case of $W_{IV}(3, \cdot, \tau)$, for testing the curvature factor, we observe very low power for lower number of cross-sections ($N = 5$). The power of the tests increase as N increase and for higher magnitude of changes (U). For $N = 10$, we observe that the bootstrapped power is very close to one for all possible change point magnitudes. Further, we find very similar bootstrapped performance in terms of power for all the three weighted measures *Sup*, *Avg*, and *Exp* of the six Wald type statistics.

3.5 Empirical validation of bond yield factor stability[2]

We investigate stability in several zero coupon bond yield term structures commonly used in literature. We obtain the term structures from Datastream, Federal Reserve, and Fama-Bliss. Various approaches have been proposed in the construction of bond yield term structures. In estimating the yield maturities, we need a pricing function and a functional form of the discount rate function. Suppose we consider the pricing function, where the price of a bond is discounted coupon and principal payments occuring at dates $k = 1, \ldots K$. The price of a

[2] I thank our paper discussant Christophe Villa at the French Finance Association meeting (Paris, Mar 17-19, 2008) who suggested to also analyse Federal Reserve and Fama Bliss term structures.

bond is given by

$$P = \sum_{k=1}^{K} c_k \cdot e^{-r(k).k}$$

Various functional forms for the discount rate function have been proposed in literature such as cubic splines (McCullock (1975)), step functions (Ronn (1987)), piecewise linear functions (Fama-Bliss (1987)), and exponential forms (Nelson-Siegel (1987)). Using different estimation methodologies would produce different term structures. Bliss (1996) compares the different forms of estimation and suggest that there is minimum estimation errors in using the Fama-Bliss(1987) methodology of constructing yields. Several econometric techniques have been proposed to estimate these functions. Fama-Bliss(1987) uses the iterative method of forward rate extraction. McCullock (1975) use ordinary least squares to estimate the splines. Datastream uses the bootstrapping methodology in constructing the unobserved yields from neaby maturities.

We test for stability in the eigensystem of the yield curves using the testing procedure outlined above and we present the results in this section.

3.5.1 Federal Reserve constant maturity zero coupon bond yields

The Federal Reserve constant maturity term structure dataset includes the US yields with maturities 3, 6, 12, 24, 36, 60, 84, and 120 months over the years January 4, 1982 to February 4, 2008. We use the dataset with daily frequency (6806 observations) as well as the monthly frequency (314 observations). This is the same dataset with monthly and weekly frequencies are used in the paper by Reisman and Zohar (2004) that extracts the principal factors using PCA and graphically validates (in)stability of the factor loadings. The paper concludes that the factor loadings of level and slope are quite stable, but the curvature factor changes rapidly over time. We conduct a statistical test on the eigenspace variables and report the results below.

[Insert Table 3.7 here]

Table *3.7* provides the stability test results for the term structure with daily frequency. We test the six hypotheses on the eigensystem variables and calculate the weighted measures *Sup*, *Avg*, and *Exp* of tests and present the value of the test statistic with its p-value. Using a significance level of 5%, we gather inference on the stability of the term structure variables. Testing for stability in the vector of eigenvalues of level, slope and curvature, we find rejection of the null hypothesis. This means that the overall eigensystem explaining the variance process of the term structure has remained unstable over time. Investigating stability in the eigensystem variables of the individual factors, we find instability in the eigenvalues, eigenvectors and factor loading matrices of level and curvature. In the case of slope factor, the eigenvector is stable and the factor loading seems to be marginally stable (p-value around 0.05). Testing for stability in the interest rates governing the three individual factors; we find all interest rate maturities of level factor unstable. For the slope factor, the short end maturities up to 12 months have been unstable, but the long end rates were stable. Testing for common instability points in the factors, we find that the factors have common dates of instability.

[Insert Table 3.8 here]

Table *3.8* records the stability results in the case of Fed monthly term structure of yields. We find the results are almost the same with the overall eigensystem unstable, all three eigensystem variables of the level factor unstable, presence of common break points among the three factors, and short rates unstable for the slope factor. Unlike the daily data, we now find all the three eigensystem variables of the slope factor unstable. Reisman and Zohar (2004) found the slope factor graphically stable over time. When using lower frequency, we find the long rates of the level factor have become stable.

3.5.2 Fama-Bliss constant maturity zero coupon bond yields

The Fama-Bliss term structure of unsmoothed US zero coupon bond yields include 21 maturities of 3, 6, 9, 12, 15, 18, 21, 24, 30, 36, 48, 60, 72, 84, 96, 108, 120, 180, 240, 300, and 360 months. The dataset extends over the period January 1970 to December 2003 (408 observations). The dataset is widely used in term structure empirical papers for the purpose of forecasting (see Diebold and Li (2006)). Bliss (1997) and Perignon and Villa (2006) use this dataset in order to study the time-varying movements of factors in yield curves. Bliss (1997) evaluate the stability patterns of level, slope, and curvature factors for the period January 1970 to December 1995 and find that though the factors vary in detail, the cumulative explanatory power of the factors have remained same over the entire period. Perignon and Villa (2006) studied the sources of time variation in the covariance matrices of bond yields. The paper found the variances (eigenvalues) have significant time variations but the factor loadings remain same over time. Conducting a statistical test for stability in eigenvalues, eigenvectors, and factor loadings, we summarize the results below.

[Insert Table 3.9 here]

Table *3.9* presents the stability results for the Fama-Bliss term structure. We find that the overall eigensystem, measured by the eigenvalues of level, slope, and curvature factors, have been unstable. Testing for stability in the individual three factors, we find all three factors have unstable eigenvalues, eigenvectors, and factor loadings with the exception of the slope factor. The slope factor has stable factor loadings, as in the case of using monthly term structure from the Federal Reserve database. We find that the long end maturities governing the slope factor have been stable over time. This result corroborates the results found using other datasets. Further, we find presence of common structural changes among all the three factors of the Fama-Bliss term structure.

3.5.3 Datastream US zero coupon bond yields

[Insert Table 3.10 here]

The term structure of US zero coupon bond yields from Datastream include 21 maturities of 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 24, 36, 48, 60, 72, 84, 96, 108, 120, 132, and 144 months. We use daily frequency of data extending from 11 January 1999 to 31 May 2006 (1927 observations). Table *3.10* records the stability results using the dataset. The results are very similar to results presented in chapter two of the dissertation that uses weekly term structure data in order to carry out the empirical application. We therefore do not discuss the results here.

3.6 Stability analysis for term structure with correlation clusters[3]

In applying PCA to the whole yield curve would mean estimating the principal factors explaining the risk profile of the whole yield curve. In this case we assume one contemporaneous correlation structure among interest rates. However, within a term structure, one may find several correlation clusters linearly or non-linearly correlated in the same way. Conducting PCA decomposition to term structure with multiple correlation clusters might not be able to pick up the true systematic yield shifts of the yield curve.

In the case of discount bond yield curves, the nature of uncertainties influencing the short rates is indeed different than the ones affecting the long rates. One reason for this is the diverse nature and varied preferences of market participants influencing the different ends of the yield curve. In associating macroeconomic variables to unobserved latent factors, various authors have used variables such as inflation changes, business cycles, and monetary policy surprises that define the state of the economy. Ang and Piazzesi (2003) finds that inflation and real activity were significant in explaining movements along yield curve maturities up to one year.

[3] I thank our paper discussant Umberto Cherubini at the Term Structure Modelling International Conference (Verona, Jun 25-26, 2007) who suggested to analyse short term and long term interest rates seperately.

The paper find that the macroeconomic variables explained a great deal of variations in the slope factor but explained less variations of the level factor. Similar conclusions were drawn by Evans and Marshall (2007) in which they find that the macroeconomic variables significantly explained not only the short and medium term yields but also the long term yields. Since expectations about future short rates are influenced by changes in macroeconomic variable, this in turn influences the movements in the long rates. Therefore the factors influencing the short and long rates have a dependence structure. The magnitude of variations along the yield curve maturities may however vary and therefore we have non-parallel shifts such as slope shifts and curvature shifts.

Suppose we consider the short end and long end of the yield curve as two separate correlation clusters. We can then adapt the PCA decomposition for the two clusters in order to extract the systematic yield shifts governing the short end and the long end of the yield curve. In grouping the interest rate maturities into short term and long term maturity clusters, we essentially aim to understand the contemporaneous relationship between the factors governing the short and long end of the yield curve. Consider in this case portfolios whose positions depend on more than one underlying factor and therefore sensitive to correlations between those factors. An example of such portfolio would be bonds with varying coupon rates. Bonds with higher coupons would be more sensitive to movements in the short end of the yield curve than bonds with lower coupons.

In the presence of correlation clusters, one can estimate the factors for the individual clusters and study the correlations between the factors extracted from the two clusters. Below we elaborate this notion. Consider a $(p+q)$ dimensional vector $z_t = (x_t', y_t')'$ for $t = 1, ..., T$ where z_t is partitioned into p - dimensional subvector x_t and q - dimensional subvector y_t explaining different aspects of the yield curve. Let z_t be multivariate normal with zero mean and a covariance structure Σ such that

$$\begin{pmatrix} x_t \\ y_t \end{pmatrix} \sim N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{bmatrix} \Sigma_{xx} & \Sigma_{xy} \\ \Sigma_{yx} & \Sigma_{yy} \end{bmatrix} \right) \tag{3.19}$$

For given two partitions X and Y, using principal component analysis we find orthogonal linear transformations of X and Y independently by minimizing the sum of squared Euclidean distances between principal axes obtained and the datapoints. The minimization problem can equivalently considered as maximization of sum of squared projections onto these principal axes. Therefore, the maximization for the two partitions are as follows:

$$\underset{\beta_x:|\beta_x|=1}{\arg\max}\,(\beta_x'X)\,(\beta_x'X)' \text{ and } \underset{\beta_y:|\beta_y|=1}{\arg\max}\,(\beta_y'Y)\,(\beta_y'Y)' \tag{3.20}$$

The constraint optimization problem reduces to the eigen decomposition problem

$$\hat{\Sigma}_{xx}\beta_x = \lambda_x\beta_x \tag{3.21}$$

$$\hat{\Sigma}_{yy}\beta_y = \lambda_y\beta_y \tag{3.22}$$

where λ is the eigenvalues and β is the corresponding eigenvector. The solution to this optimization yields the first principal component for the two partitions: $U_1 = \beta_{1x}'X$ and $V_1 = \beta_{1y}'Y$ and the magnitude of variation explained by the first principal component for the two partitions is given by their eigenvalues λ_{1x} and λ_{1y}. This procedure can be repeated in order to find the successive principal components estimated with the additional optimization constraint that the new principal component is orthogonal to the previous one. The decomposition in equation *3.21* estimates $\lambda_{1x}, ..., \lambda_{px}$ (the ordered eigenvalues) and $\boldsymbol{\beta}_{1x}, ..., \boldsymbol{\beta}_{px}$ (the corresponding ordered eigenvectors) of $\hat{\Sigma}_{xx}$. Similarly, equation *3.22* estimates $\lambda_{1y}, ..., \lambda_{qy}$ (the ordered eigenvalues) and $\boldsymbol{\beta}_{1y}, ..., \boldsymbol{\beta}_{qy}$ (the corresponding ordered eigenvectors) of $\hat{\Sigma}_{yy}$.

3.6.1 Stability testing results

In this section, we implement the stability testing procedure outlined in chapter two for the case of long and short maturity clusters. We use the term structure of US zero coupon bond yields including ten maturities of 3, 4, 5, 6, 7, 8, 9, 10, 11, and 12 months as the short maturity cluster and eleven maturities of 24, 36, 48, 60, 72, 84, 96, 108, 120, 132 and 144 months as the long maturity cluster. We use daily frequency of data extending from 11 January 1999 to 31 May 2006 (1927 observations) from Datastream.

[Insert Tables 3.11 - 3.12 here]

Tables *3.11* - *3.12* records the stability results for the two clusters. We find that all the three factors (level, slope, and curvature) governing short and long rates have been statistically unstable. In chapter two, the stability results for term structures with short and long rates combined conclude instability in the level factor loadings but stable factor loadings for slope and curvature. The implication of this finding may be interpreted as the presence of "Cobreaking" between clusters, which needs to be statistically tested for in future work. Cobreaking was first introduced by Prof. David Hendry in 1996 as a modelling technique for non-stationary time series, where the non-stationary property induced by location shifts can be annihilated by taking linear combinations of the variables (see Hendry (1996) and Hendry and Massmann (2007)).

3.7 Conclusion

This chapter evaluates the validity of the bootstrap methodology proposed in chapter two, which formulates several hypotheses in order to test for instabilities present within the eigenspace variables (namely, eigenvalues, eigenvectors, and factor loadings) governing the term structure of interest rates. A set of Monte Carlo experiments are conducted in order to study the finite sample performance of the tests proposed. Further, several commonly used zero coupon bond term structures have been empirically tested for robustification of stability results.

From the Monte Carlo simulations, we find that the weighted measures *Sup*, *Avg*, and *Exp* of the six Wald type test statistics show good size and power properties at 5% significance level. We find that the empirical size of the bootstrap tests is very close to the nominal size for all the test statistics. In the case of testing the curvature factors, we find undersizing for small N. However, we see substantial size improvements as N increase. In terms of power performance, we find that the test statistics $W_I(\cdot) - W_{III}(\cdot)$ and $W_{VI}(\cdot)$ show power essentially close to one. The test statistic $W_{IV}(\cdot)$ however show low power in evaluating the curvature factor for small N and small structural change magnitudes. There is however power gain as the magnitude of structural change increase.

In summarizing the US zero coupon yield stability test results for the various datasets, we find qualitatively very similar results. We find the overall eigensystem of three factors are unstable, all the eigensystem variables governing the level factor is unstable, the long rates governing the slope factor are stable over time and there is evidence of common points of instability among the three factors. In testing for instabilities within factors governing the short and long maturity clusters, we find that all the factors for the two clusters have been statistically unstable. The implication of this finding may be interpreted as the presence of "Cobreaking" between maturity clusters of yield curves, which needs to be statistically tested for in future work.

TABLE 3.1. Bootstrapped Size of *Sup* (5% level)

	$N=5$	$N=10$	$N=20$		$N=5$	$N=10$	$N=20$
$W_{I}(\tau)$	0.0468	0.0508	0.0456	$W_{IV}(1,1,\tau)$	0.0524	0.0564	0.0504
				$W_{IV}(1,2,\tau)$	0.0568	0.0568	0.056
$W_{II}(1,\tau)$	0.0498	0.0488	0.0486	$W_{IV}(1,3,\tau)$	0.055	0.0582	0.0554
$W_{II}(2,\tau)$	0.0508	0.0518	0.051	$W_{IV}(1,4,\tau)$	0.0544	0.0562	0.057
$W_{II}(3,\tau)$	0.0498	0.0494	0.0542	$W_{IV}(1,5,\tau)$	0.055	0.0554	0.0564
				$W_{IV}(1,6,\tau)$		0.0574	0.057
$W_{III}(1,\tau)$	0.0572	0.0506	0.0458	$W_{IV}(1,7,\tau)$		0.0564	0.0576
$W_{III}(2,\tau)$	0.0508	0.0536	0.0418	$W_{IV}(1,8,\tau)$		0.0552	0.0562
$W_{III}(3,\tau)$	0.0018	0.046	0.0458	$W_{IV}(1,9,\tau)$		0.0544	0.055
				$W_{IV}(1,10,\tau)$		0.0524	0.0556
$W_{V}(1,\tau)$	0.0538	0.0482	0.0466	$W_{IV}(1,11,\tau)$			0.0518
$W_{V}(2,\tau)$	0.0456	0.0484	0.0422	$W_{IV}(1,12,\tau)$			0.056
$W_{V}(3,\tau)$	0.0032	0.055	0.0484	$W_{IV}(1,13,\tau)$			0.058
				$W_{IV}(1,14,\tau)$			0.0556
$W_{VI}(1,2,\tau)$	0.0488	0.0482	0.0398	$W_{IV}(1,15,\tau)$			0.0544
$W_{VI}(1,3,\tau)$	0.0032	0.0472	0.0452	$W_{IV}(1,16,\tau)$			0.0546
$W_{VI}(2,3,\tau)$	0.0032	0.0544	0.0438	$W_{IV}(1,17,\tau)$			0.054
				$W_{IV}(1,18,\tau)$			0.0554
				$W_{IV}(1,19,\tau)$			0.0536
				$W_{IV}(1,20,\tau)$			0.0544
$W_{IV}(2,1,\tau)$	0.0576	0.0566	0.0472	$W_{IV}(3,1,\tau)$	0.0286	0.0486	0.0532
$W_{IV}(2,2,\tau)$	0.0522	0.0566	0.0538	$W_{IV}(3,2,\tau)$	0.0282	0.045	0.053
$W_{IV}(2,3,\tau)$	0.0504	0.0542	0.0518	$W_{IV}(3,3,\tau)$	0.0046	0.0494	0.0438
$W_{IV}(2,4,\tau)$	0.0552	0.0524	0.0536	$W_{IV}(3,4,\tau)$	0.004	0.0538	0.0418
$W_{IV}(2,5,\tau)$	0.0522	0.0476	0.0532	$W_{IV}(3,5,\tau)$	0.0048	0.0454	0.0474
$W_{IV}(2,6,\tau)$		0.0542	0.0524	$W_{IV}(3,6,\tau)$		0.047	0.0518
$W_{IV}(2,7,\tau)$		0.052	0.0488	$W_{IV}(3,7,\tau)$		0.053	0.052
$W_{IV}(2,8,\tau)$		0.0522	0.0516	$W_{IV}(3,8,\tau)$		0.0508	0.055
$W_{IV}(2,9,\tau)$		0.0538	0.051	$W_{IV}(3,9,\tau)$		0.0502	0.0536
$W_{IV}(2,10,\tau)$		0.0534	0.0472	$W_{IV}(3,10,\tau)$		0.052	0.0482
$W_{IV}(2,11,\tau)$			0.0454	$W_{IV}(3,11,\tau)$			0.0438
$W_{IV}(2,12,\tau)$			0.0512	$W_{IV}(3,12,\tau)$			0.0472
$W_{IV}(2,13,\tau)$			0.0524	$W_{IV}(3,13,\tau)$			0.053
$W_{IV}(2,14,\tau)$			0.0532	$W_{IV}(3,14,\tau)$			0.0518
$W_{IV}(2,15,\tau)$			0.0508	$W_{IV}(3,15,\tau)$			0.0472
$W_{IV}(2,16,\tau)$			0.0512	$W_{IV}(3,16,\tau)$			0.05
$W_{IV}(2,17,\tau)$			0.0488	$W_{IV}(3,17,\tau)$			0.048
$W_{IV}(2,18,\tau)$			0.0504	$W_{IV}(3,18,\tau)$			0.049
$W_{IV}(2,19,\tau)$			0.0496	$W_{IV}(3,19,\tau)$			0.049
$W_{IV}(2,20,\tau)$			0.0486	$W_{IV}(3,20,\tau)$			0.0496

TABLE 3.2. Bootstrapped Size of Avg (5% level)

	$N=5$	$N=10$	$N=20$		$N=5$	$N=10$	$N=20$
$W_I(\tau)$	0.0542	0.0494	0.0502	$W_{IV}(1,1,\tau)$	0.057	0.058	0.0568
				$W_{IV}(1,2,\tau)$	0.0552	0.0576	0.0572
$W_{II}(1,\tau)$	0.0498	0.0496	0.0502	$W_{IV}(1,3,\tau)$	0.0546	0.0566	0.0536
$W_{II}(2,\tau)$	0.0508	0.0512	0.0538	$W_{IV}(1,4,\tau)$	0.0586	0.0554	0.0556
$W_{II}(3,\tau)$	0.0484	0.0462	0.0466	$W_{IV}(1,5,\tau)$	0.0588	0.0556	0.0582
				$W_{IV}(1,6,\tau)$		0.0566	0.056
$W_{III}(1,\tau)$	0.0542	0.0544	0.0536	$W_{IV}(1,7,\tau)$		0.0576	0.0546
$W_{III}(2,\tau)$	0.0476	0.0514	0.0472	$W_{IV}(1,8,\tau)$		0.0574	0.0536
$W_{III}(3,\tau)$	0.005	0.0506	0.0514	$W_{IV}(1,9,\tau)$		0.0566	0.0552
				$W_{IV}(1,10,\tau)$		0.0592	0.0558
$W_V(1,\tau)$	0.0514	0.0556	0.0528	$W_{IV}(1,11,\tau)$			0.0534
$W_V(2,\tau)$	0.049	0.0546	0.0482	$W_{IV}(1,12,\tau)$			0.0548
$W_V(3,\tau)$	0.0096	0.0494	0.053	$W_{IV}(1,13,\tau)$			0.0556
				$W_{IV}(1,14,\tau)$			0.058
$W_{VI}(1,2,\tau)$	0.0506	0.0528	0.0482	$W_{IV}(1,15,\tau)$			0.0602
$W_{VI}(1,3,\tau)$	0.01	0.0556	0.0478	$W_{IV}(1,16,\tau)$			0.0594
$W_{VI}(2,3,\tau)$	0.01	0.057	0.0532	$W_{IV}(1,17,\tau)$			0.0582
				$W_{IV}(1,18,\tau)$			0.0568
				$W_{IV}(1,19,\tau)$			0.056
				$W_{IV}(1,20,\tau)$			0.0574
$W_{IV}(2,1,\tau)$	0.053	0.0548	0.0498	$W_{IV}(3,1,\tau)$	0.0156	0.0514	0.0532
$W_{IV}(2,2,\tau)$	0.0564	0.0526	0.054	$W_{IV}(3,2,\tau)$	0.0148	0.0444	0.0518
$W_{IV}(2,3,\tau)$	0.0548	0.056	0.0512	$W_{IV}(3,3,\tau)$	0.0138	0.0494	0.0416
$W_{IV}(2,4,\tau)$	0.0562	0.0518	0.0516	$W_{IV}(3,4,\tau)$	0.013	0.0534	0.0434
$W_{IV}(2,5,\tau)$	0.0506	0.0518	0.0526	$W_{IV}(3,5,\tau)$	0.0138	0.0474	0.0472
$W_{IV}(2,6,\tau)$		0.052	0.053	$W_{IV}(3,6,\tau)$		0.0496	0.0556
$W_{IV}(2,7,\tau)$		0.0508	0.0522	$W_{IV}(3,7,\tau)$		0.0548	0.055
$W_{IV}(2,8,\tau)$		0.054	0.0524	$W_{IV}(3,8,\tau)$		0.0504	0.0544
$W_{IV}(2,9,\tau)$		0.0496	0.0504	$W_{IV}(3,9,\tau)$		0.049	0.0504
$W_{IV}(2,10,\tau)$		0.0476	0.0482	$W_{IV}(3,10,\tau)$		0.053	0.0472
$W_{IV}(2,11,\tau)$			0.0476	$W_{IV}(3,11,\tau)$			0.0452
$W_{IV}(2,12,\tau)$			0.0538	$W_{IV}(3,12,\tau)$			0.0484
$W_{IV}(2,13,\tau)$			0.0508	$W_{IV}(3,13,\tau)$			0.0532
$W_{IV}(2,14,\tau)$			0.0502	$W_{IV}(3,14,\tau)$			0.0522
$W_{IV}(2,15,\tau)$			0.0476	$W_{IV}(3,15,\tau)$			0.0488
$W_{IV}(2,16,\tau)$			0.051	$W_{IV}(3,16,\tau)$			0.0508
$W_{IV}(2,17,\tau)$			0.048	$W_{IV}(3,17,\tau)$			0.0498
$W_{IV}(2,18,\tau)$			0.0482	$W_{IV}(3,18,\tau)$			0.0488
$W_{IV}(2,19,\tau)$			0.0468	$W_{IV}(3,19,\tau)$			0.0494
$W_{IV}(2,20,\tau)$			0.0484	$W_{IV}(3,20,\tau)$			0.0504

TABLE 3.3. Bootstrapped Size of *Exp* (5% level)

	$N=5$	$N=10$	$N=20$		$N=5$	$N=10$	$N=20$
$W_I(\tau)$	0.0542	0.0494	0.0502	$W_{IV}(1,1,\tau)$	0.0572	0.0582	0.0568
				$W_{IV}(1,2,\tau)$	0.0554	0.0576	0.0572
$W_{II}(1,\tau)$	0.0498	0.0496	0.0502	$W_{IV}(1,3,\tau)$	0.0546	0.0566	0.0536
$W_{II}(2,\tau)$	0.0508	0.0512	0.0538	$W_{IV}(1,4,\tau)$	0.0586	0.0554	0.0556
$W_{II}(3,\tau)$	0.0482	0.0462	0.0466	$W_{IV}(1,5,\tau)$	0.0588	0.0556	0.0582
				$W_{IV}(1,6,\tau)$		0.0566	0.0558
$W_{III}(1,\tau)$	0.0542	0.0544	0.0536	$W_{IV}(1,7,\tau)$		0.0576	0.0546
$W_{III}(2,\tau)$	0.0476	0.0514	0.047	$W_{IV}(1,8,\tau)$		0.0574	0.0538
$W_{III}(3,\tau)$	0.0016	0.0506	0.0514	$W_{IV}(1,9,\tau)$		0.0566	0.0552
				$W_{IV}(1,10,\tau)$		0.0592	0.0558
$W_V(1,\tau)$	0.0514	0.0556	0.0528	$W_{IV}(1,11,\tau)$			0.0534
$W_V(2,\tau)$	0.049	0.0546	0.0478	$W_{IV}(1,12,\tau)$			0.0548
$W_V(3,\tau)$	0.0088	0.0494	0.053	$W_{IV}(1,13,\tau)$			0.0556
				$W_{IV}(1,14,\tau)$			0.058
$W_{VI}(1,2,\tau)$	0.0506	0.0528	0.048	$W_{IV}(1,15,\tau)$			0.0602
$W_{VI}(1,3,\tau)$	0.0086	0.0556	0.048	$W_{IV}(1,16,\tau)$			0.0594
$W_{VI}(2,3,\tau)$	0.0086	0.0568	0.0532	$W_{IV}(1,17,\tau)$			0.058
				$W_{IV}(1,18,\tau)$			0.0568
				$W_{IV}(1,19,\tau)$			0.056
				$W_{IV}(1,20,\tau)$			0.0574
$W_{IV}(2,1,\tau)$	0.053	0.0548	0.0498	$W_{IV}(3,1,\tau)$	0.016	0.0514	0.0532
$W_{IV}(2,2,\tau)$	0.0564	0.0526	0.054	$W_{IV}(3,2,\tau)$	0.0164	0.0444	0.0518
$W_{IV}(2,3,\tau)$	0.0548	0.056	0.0512	$W_{IV}(3,3,\tau)$	0.0138	0.0494	0.0416
$W_{IV}(2,4,\tau)$	0.0562	0.0518	0.0514	$W_{IV}(3,4,\tau)$	0.013	0.0534	0.0434
$W_{IV}(2,5,\tau)$	0.0506	0.0518	0.052	$W_{IV}(3,5,\tau)$	0.0138	0.0474	0.0472
$W_{IV}(2,6,\tau)$		0.052	0.053	$W_{IV}(3,6,\tau)$		0.0496	0.0556
$W_{IV}(2,7,\tau)$		0.0508	0.0522	$W_{IV}(3,7,\tau)$		0.0548	0.055
$W_{IV}(2,8,\tau)$		0.054	0.0522	$W_{IV}(3,8,\tau)$		0.0504	0.0542
$W_{IV}(2,9,\tau)$		0.0496	0.0504	$W_{IV}(3,9,\tau)$		0.049	0.0504
$W_{IV}(2,10,\tau)$		0.0476	0.048	$W_{IV}(3,10,\tau)$		0.053	0.0472
$W_{IV}(2,11,\tau)$			0.0476	$W_{IV}(3,11,\tau)$			0.0452
$W_{IV}(2,12,\tau)$			0.0538	$W_{IV}(3,12,\tau)$			0.0484
$W_{IV}(2,13,\tau)$			0.0508	$W_{IV}(3,13,\tau)$			0.0532
$W_{IV}(2,14,\tau)$			0.0502	$W_{IV}(3,14,\tau)$			0.0522
$W_{IV}(2,15,\tau)$			0.0476	$W_{IV}(3,15,\tau)$			0.0488
$W_{IV}(2,16,\tau)$			0.051	$W_{IV}(3,16,\tau)$			0.0508
$W_{IV}(2,17,\tau)$			0.048	$W_{IV}(3,17,\tau)$			0.0498
$W_{IV}(2,18,\tau)$			0.0482	$W_{IV}(3,18,\tau)$			0.0488
$W_{IV}(2,19,\tau)$			0.0468	$W_{IV}(3,19,\tau)$			0.0494
$W_{IV}(2,20,\tau)$			0.0484	$W_{IV}(3,20,\tau)$			0.0504

TABLE 3.4. Bootstrapped Power of *Sup* (5% level). The table reports the power of the weighted measure *Sup*, for the various test statistics.

		$N=5$			$N=10$			$N=20$		
		U_1	U_2	U_3	U_1	U_2	U_3	U_1	U_2	U_3
$W_I(\tau)$	$T/3$	1	1	1	1	1	1	1	1	1
	$T/2$	1	1	1	1	1	1	1	1	1
	$2T/3$	1	1	1	1	1	1	1	1	1
$W_{II}(1,\tau)$	$T/3$	1	1	1	1	1	1	1	1	1
	$T/2$	1	1	1	1	1	1	1	1	1
	$2T/3$	1	1	1	1	1	1	1	1	1
$W_{III}(1,\tau)$	$T/3$	1	1	1	1	1	1	1	1	1
	$T/2$	1	1	1	1	1	1	1	1	1
	$2T/3$	1	1	1	1	1	1	1	1	1
$W_V(1,\tau)$	$T/3$	1	1	1	1	1	1	1	1	1
	$T/2$	1	1	1	1	1	1	1	1	1
	$2T/3$	1	1	1	1	1	1	1	1	1
$W_{II}(2,\tau)$	$T/3$	1	1	1	1	1	1	1	1	1
	$T/2$	1	1	1	1	1	1	1	1	1
	$2T/3$	1	1	1	1	1	1	1	1	1
$W_{III}(2,\tau)$	$T/3$	1	1	1	1	1	1	1	1	1
	$T/2$	1	1	1	1	1	1	1	1	1
	$2T/3$	1	1	1	1	1	1	1	1	1
$W_V(2,\tau)$	$T/3$	1	1	1	1	1	1	1	1	1
	$T/2$	1	1	1	1	1	1	1	1	1
	$2T/3$	1	1	1	1	1	1	1	1	1

(TABLE 3.4 CONTINUED)

		N = 5			N = 10			N = 20		
		U_1	U_2	U_3	U_1	U_2	U_3	U_1	U_2	U_3
$W_{II}(3,\tau)$	$T/3$	1	1	1	1	1	1	1	1	1
	$T/2$	1	1	1	1	1	1	1	1	1
	$2T/3$	1	1	1	1	1	1	1	1	1
$W_{III}(3,\tau)$	$T/3$	0.9996	1	1	1	1	1	1	1	1
	$T/2$	1	1	1	1	1	1	1	1	1
	$2T/3$	1	1	1	1	1	1	1	1	1
$W_V(3,\tau)$	$T/3$	1	1	1	1	1	1	1	1	1
	$T/2$	1	1	1	1	1	1	1	1	1
	$2T/3$	1	1	1	1	1	1	1	1	1
$W_{VI}(1,2,\tau)$	$T/3$	1	1	1	1	1	1	1	1	1
	$T/2$	1	1	1	1	1	1	1	1	1
	$2T/3$	1	1	1	1	1	1	1	1	1
$W_{VI}(1,3,\tau)$	$T/3$	1	1	1	1	1	1	1	1	1
	$T/2$	1	1	1	1	1	1	1	1	1
	$2T/3$	1	1	1	1	1	1	1	1	1
$W_{VI}(2,3,\tau)$	$T/3$	1	1	1	1	1	1	1	1	1
	$T/2$	1	1	1	1	1	1	1	1	1
	$2T/3$	1	1	1	1	1	1	1	1	1
$W_{IV}(1,1,\tau)$	$T/3$	1	1	1	1	1	1	0.9696	1	1
$W_{IV}(1,2,\tau)$		0.9916	1	1	0.9822	1	1	1	1	1
$W_{IV}(1,3,\tau)$		0.9994	1	1	0.1968	1	1	0.9996	1	1
$W_{IV}(1,4,\tau)$		0.9992	1	1	0.6746	1	1	0.9972	1	1
$W_{IV}(1,5,\tau)$		0.9486	1	1	0.9986	1	1	1	1	1
$W_{IV}(1,6,\tau)$					0.999	1	1	0.137	1	1
$W_{IV}(1,7,\tau)$					0.8068	0.9964	1	1	1	1
$W_{IV}(1,8,\tau)$					0.9994	1	1	0.809	1	1
$W_{IV}(1,9,\tau)$					1	1	1	1	1	1
$W_{IV}(1,10,\tau)$					0.9312	1	1	1	1	1
$W_{IV}(1,11,\tau)$								0.9996	1	1
$W_{IV}(1,12,\tau)$								0.3888	0.9686	1
$W_{IV}(1,13,\tau)$								0.918	1	1
$W_{IV}(1,14,\tau)$								0.1352	0.996	1
$W_{IV}(1,15,\tau)$								0.999	1	1
$W_{IV}(1,16,\tau)$								0.3936	0.9992	1
$W_{IV}(1,17,\tau)$								0.5442	1	1
$W_{IV}(1,18,\tau)$								0.8548	1	1
$W_{IV}(1,19,\tau)$								0.8324	0.9998	1
$W_{IV}(1,20,\tau)$								0.1544	0.9866	1

(TABLE 3.4 CONTINUED)

		$N=5$			$N=10$			$N=20$		
		U_1	U_2	U_3	U_1	U_2	U_3	U_1	U_2	U_3
$W_{IV}(1,1,\tau)$	$T/2$	1	1	1	1	1	1	1	1	1
$W_{IV}(1,2,\tau)$		1	1	1	1	1	1	1	1	1
$W_{IV}(1,3,\tau)$		1	1	1	0.9944	0.9996	1	1	1	1
$W_{IV}(1,4,\tau)$		1	1	1	1	1	1	1	1	1
$W_{IV}(1,5,\tau)$		1	1	1	1	1	1	1	1	1
$W_{IV}(1,6,\tau)$					1	1	1	0.9956	1	1
$W_{IV}(1,7,\tau)$					0.999	1	1	1	1	1
$W_{IV}(1,8,\tau)$					1	1	1	1	1	1
$W_{IV}(1,9,\tau)$					1	1	1	1	1	1
$W_{IV}(1,10,\tau)$					1	1	1	1	1	1
$W_{IV}(1,11,\tau)$								1	1	1
$W_{IV}(1,12,\tau)$								0.7588	1	1
$W_{IV}(1,13,\tau)$								1	1	1
$W_{IV}(1,14,\tau)$								0.3892	1	1
$W_{IV}(1,15,\tau)$								1	1	1
$W_{IV}(1,16,\tau)$								0.7888	1	1
$W_{IV}(1,17,\tau)$								0.9954	1	1
$W_{IV}(1,18,\tau)$								0.999	1	1
$W_{IV}(1,19,\tau)$								0.999	0.9996	1
$W_{IV}(1,20,\tau)$								0.3008	1	1
$W_{IV}(1,1,\tau)$	$2T/3$	1	1	1	1	1	1	1	1	1
$W_{IV}(1,2,\tau)$		1	1	1	1	1	1	1	1	1
$W_{IV}(1,3,\tau)$		1	1	1	1	1	1	1	1	1
$W_{IV}(1,4,\tau)$		1	1	1	1	1	1	1	1	1
$W_{IV}(1,5,\tau)$		1	1	1	1	1	1	1	1	1
$W_{IV}(1,6,\tau)$					1	1	1	1	1	1
$W_{IV}(1,7,\tau)$					0.999	1	1	1	1	1
$W_{IV}(1,8,\tau)$					1	1	1	1	1	1
$W_{IV}(1,9,\tau)$					1	1	1	1	1	1
$W_{IV}(1,10,\tau)$					1	1	1	1	1	1
$W_{IV}(1,11,\tau)$								1	1	1
$W_{IV}(1,12,\tau)$								0.8526	1	1
$W_{IV}(1,13,\tau)$								1	1	1
$W_{IV}(1,14,\tau)$								0.828	1	1
$W_{IV}(1,15,\tau)$								1	1	1
$W_{IV}(1,16,\tau)$								0.8952	1	1
$W_{IV}(1,17,\tau)$								1	1	1
$W_{IV}(1,18,\tau)$								1	1	1
$W_{IV}(1,19,\tau)$								1	1	1
$W_{IV}(1,20,\tau)$								0.4698	1	1

(TABLE 3.4 CONTINUED)

		N = 5			N = 10			N = 20		
		U_1	U_2	U_3	U_1	U_2	U_3	U_1	U_2	U_3
$W_{IV}(2,1,\tau)$	$T/3$	0.9748	1	1	0.3946	1	1	0.1226	0.9464	0.9998
$W_{IV}(2,2,\tau)$		0.9988	1	1	0.4236	1	1	0.303	0.9988	1
$W_{IV}(2,3,\tau)$		0.9922	1	1	0.9736	1	1	0.1224	0.9946	1
$W_{IV}(2,4,\tau)$		0.9714	1	1	0.7102	1	1	0.7562	1	1
$W_{IV}(2,5,\tau)$		1	1	1	1	1	1	0.4896	0.9998	1
$W_{IV}(2,6,\tau)$					0.464	1	1	0.408	0.991	1
$W_{IV}(2,7,\tau)$					0.9812	1	1	0.136	0.9872	1
$W_{IV}(2,8,\tau)$					0.1586	1	1	0.761	1	1
$W_{IV}(2,9,\tau)$					0.9036	0.993	1	0.9846	1	1
$W_{IV}(2,10,\tau)$					0.439	1	1	1	1	1
$W_{IV}(2,11,\tau)$								1	1	1
$W_{IV}(2,12,\tau)$								1	0.9872	1
$W_{IV}(2,13,\tau)$								1	1	1
$W_{IV}(2,14,\tau)$								0.8084	1	1
$W_{IV}(2,15,\tau)$								1	1	1
$W_{IV}(2,16,\tau)$								1	1	1
$W_{IV}(2,17,\tau)$								0.9046	1	1
$W_{IV}(2,18,\tau)$								1	1	1
$W_{IV}(2,19,\tau)$								1	1	1
$W_{IV}(2,20,\tau)$								0.9968	1	1
$W_{IV}(2,1,\tau)$	$T/2$	1	1	1	0.99	1	1	0.334	0.9972	0.9994
$W_{IV}(2,2,\tau)$		1	1	1	0.9888	1	1	0.6784	0.9998	1
$W_{IV}(2,3,\tau)$		1	1	1	1	1	1	0.3784	1	1
$W_{IV}(2,4,\tau)$		1	1	1	1	1	1	0.9986	1	1
$W_{IV}(2,5,\tau)$		1	1	1	1	1	1	0.9886	1	1
$W_{IV}(2,6,\tau)$					1	1	1	0.907	0.9998	1
$W_{IV}(2,7,\tau)$					1	1	1	0.4328	0.9998	1
$W_{IV}(2,8,\tau)$					0.9976	0.9998	1	0.9986	1	1
$W_{IV}(2,9,\tau)$					0.69	1	1	1	1	1
$W_{IV}(2,10,\tau)$					1	1	1	1	1	1
$W_{IV}(2,11,\tau)$								1	1	1
$W_{IV}(2,12,\tau)$								1	1	1
$W_{IV}(2,13,\tau)$								1	1	1
$W_{IV}(2,14,\tau)$								0.843	1	1
$W_{IV}(2,15,\tau)$								1	1	1
$W_{IV}(2,16,\tau)$								1	1	1
$W_{IV}(2,17,\tau)$								1	1	1
$W_{IV}(2,18,\tau)$								1	1	1
$W_{IV}(2,19,\tau)$								1	1	1
$W_{IV}(2,20,\tau)$								0.997	1	1

(TABLE 3.4 CONTINUED)

		$N=5$			$N=10$			$N=20$		
		U_1	U_2	U_3	U_1	U_2	U_3	U_1	U_2	U_3
$W_{IV}(2,1,\tau)$	$2T/3$	1	1	1	1	1	1	0.5826	0.9994	0.9984
$W_{IV}(2,2,\tau)$		1	1	1	0.9998	1	1	0.923	1	1
$W_{IV}(2,3,\tau)$		1	1	1	1	1	1	0.678	1	1
$W_{IV}(2,4,\tau)$		1	1	1	1	1	1	1	1	1
$W_{IV}(2,5,\tau)$		1	1	1	1	1	1	1	1	1
$W_{IV}(2,6,\tau)$					1	1	1	0.9982	1	0.9998
$W_{IV}(2,7,\tau)$					1	1	1	0.7068	1	0.9998
$W_{IV}(2,8,\tau)$					1	1	1	1	1	0.9998
$W_{IV}(2,9,\tau)$					0.9676	1	1	1	1	0.9998
$W_{IV}(2,10,\tau)$					1	1	1	1	1	1
$W_{IV}(2,11,\tau)$								1	1	1
$W_{IV}(2,12,\tau)$								1	1	1
$W_{IV}(2,13,\tau)$								1	1	1
$W_{IV}(2,14,\tau)$								1	1	1
$W_{IV}(2,15,\tau)$								1	1	1
$W_{IV}(2,16,\tau)$								1	1	1
$W_{IV}(2,17,\tau)$								1	1	1
$W_{IV}(2,18,\tau)$								1	1	1
$W_{IV}(2,19,\tau)$								1	1	1
$W_{IV}(2,20,\tau)$								0.9954	1	1
$W_{IV}(3,1,\tau)$	$T/3$	0.0062	0.0728	0.0076	0.9996	0.9996	0.9996	1	1	1
$W_{IV}(3,2,\tau)$		0.0064	0.0992	0.0058	0.2812	0.9996	0.9994	1	1	1
$W_{IV}(3,3,\tau)$		0.0186	0.9996	0.6626	0.9986	0.9996	0.9996	0.385	1	1
$W_{IV}(3,4,\tau)$		0.0114	0.0178	0.312	0.9994	0.9996	0.9996	0.9998	1	1
$W_{IV}(3,5,\tau)$		0.0264	0.9996	0.6794	0.9992	0.9996	0.9996	0.3272	1	1
$W_{IV}(3,6,\tau)$					0.9996	0.9996	0.9996	0.7286	1	1
$W_{IV}(3,7,\tau)$					0.9996	0.9996	0.9996	1	1	1
$W_{IV}(3,8,\tau)$					0.7646	0.9996	0.9996	0.4116	1	1
$W_{IV}(3,9,\tau)$					0.9996	0.9996	0.9996	0.9992	1	1
$W_{IV}(3,10,\tau)$					0.9996	0.9996	0.9996	0.4306	1	1
$W_{IV}(3,11,\tau)$								0.253	0.9998	1
$W_{IV}(3,12,\tau)$								0.3328	0.9686	1
$W_{IV}(3,13,\tau)$								0.7602	0.9986	1
$W_{IV}(3,14,\tau)$								0.7664	0.9994	0.9998
$W_{IV}(3,15,\tau)$								0.977	1	1
$W_{IV}(3,16,\tau)$								0.41	0.9714	0.9944
$W_{IV}(3,17,\tau)$								0.6634	1	1
$W_{IV}(3,18,\tau)$								0.2172	0.9476	0.9974
$W_{IV}(3,19,\tau)$								0.208	0.9766	0.9998
$W_{IV}(3,20,\tau)$								0.3348	0.989	0.9976

(TABLE 3.4 CONTINUED)

		N = 5			N = 10			N = 20		
		U_1	U_2	U_3	U_1	U_2	U_3	U_1	U_2	U_3
$W_{IV}(3,1,\tau)$	$T/2$	0.0202	0.0278	0.0192	0.9996	0.9996	0.9996	1	1	1
$W_{IV}(3,2,\tau)$		0.0434	0.0116	0.024	0.9996	0.9966	0.9996	1	1	1
$W_{IV}(3,3,\tau)$		0.9992	0.9986	0.0372	0.9996	0.9996	0.9996	0.9914	1	1
$W_{IV}(3,4,\tau)$		0.96	0.017	0.0242	0.9996	0.9996	0.9996	1	1	1
$W_{IV}(3,5,\tau)$		0.9992	0.9986	0.0398	0.9996	0.9996	0.9996	1	1	1
$W_{IV}(3,6,\tau)$					0.9996	0.9996	0.9996	1	1	1
$W_{IV}(3,7,\tau)$					0.9996	0.9996	0.9996	1	1	1
$W_{IV}(3,8,\tau)$					0.8046	0.9996	0.9996	1	1	1
$W_{IV}(3,9,\tau)$					0.9996	0.9996	0.9996	1	1	1
$W_{IV}(3,10,\tau)$					0.9996	0.9996	0.9996	0.9556	1	1
$W_{IV}(3,11,\tau)$								0.767	1	1
$W_{IV}(3,12,\tau)$								0.4876	1	1
$W_{IV}(3,13,\tau)$								0.9666	1	0.9998
$W_{IV}(3,14,\tau)$								0.9706	1	1
$W_{IV}(3,15,\tau)$								0.9996	1	1
$W_{IV}(3,16,\tau)$								0.834	0.989	0.9936
$W_{IV}(3,17,\tau)$								0.994	1	1
$W_{IV}(3,18,\tau)$								0.5656	0.9816	0.9972
$W_{IV}(3,19,\tau)$								0.5478	0.9974	0.9998
$W_{IV}(3,20,\tau)$								0.8438	0.9918	0.997
$W_{IV}(3,1,\tau)$	$2T/3$	0.1882	0.0378	0.3778	0.9996	0.9996	0.9996	1	1	1
$W_{IV}(3,2,\tau)$		0.335	0.008	0.5326	0.9996	0.9912	0.9996	1	1	1
$W_{IV}(3,3,\tau)$		0.9988	0.9922	0.0064	0.9974	0.9988	0.9996	1	1	1
$W_{IV}(3,4,\tau)$		0.996	0.8264	0.0072	0.9996	0.9996	0.9996	1	1	1
$W_{IV}(3,5,\tau)$		0.9988	0.9946	0.0064	0.9996	0.9996	0.9996	1	1	1
$W_{IV}(3,6,\tau)$					0.9996	0.9996	0.9996	1	1	1
$W_{IV}(3,7,\tau)$					0.9996	0.9996	0.9996	1	1	1
$W_{IV}(3,8,\tau)$					0.9996	0.9996	0.9996	1	1	1
$W_{IV}(3,9,\tau)$					0.9996	0.9996	0.9996	1	1	1
$W_{IV}(3,10,\tau)$					0.9996	0.9996	0.9996	0.9986	1	0.9998
$W_{IV}(3,11,\tau)$								0.9482	1	1
$W_{IV}(3,12,\tau)$								0.6044	1	0.9998
$W_{IV}(3,13,\tau)$								0.9926	1	0.999
$W_{IV}(3,14,\tau)$								0.9914	1	0.9984
$W_{IV}(3,15,\tau)$								1	1	1
$W_{IV}(3,16,\tau)$								0.9332	0.995	0.9874
$W_{IV}(3,17,\tau)$								1	1	0.9994
$W_{IV}(3,18,\tau)$								0.7136	0.9938	0.9946
$W_{IV}(3,19,\tau)$								0.7092	0.9996	0.9956
$W_{IV}(3,20,\tau)$								0.9716	0.991	0.9954

TABLE 3.5. Bootstrapped Power of *Avg* (5% level). The table reports the power of the weighted measure *Avg*, for the various test statistics.

		$N=5$			$N=10$			$N=20$		
		U_1	U_2	U_3	U_1	U_2	U_3	U_1	U_2	U_3
$W_I(\tau)$	$T/3$	1	1	1	1	1	1	1	1	1
	$T/2$	1	1	1	1	1	1	1	1	1
	$2T/3$	1	1	1	1	1	1	1	1	1
$W_{II}(1,\tau)$	$T/3$	1	1	1	1	1	1	1	1	1
	$T/2$	1	1	1	1	1	1	1	1	1
	$2T/3$	1	1	1	1	1	1	1	1	1
$W_{III}(1,\tau)$	$T/3$	1	1	1	1	1	1	1	1	1
	$T/2$	1	1	1	1	1	1	1	1	1
	$2T/3$	1	1	1	1	1	1	1	1	1
$W_V(1,\tau)$	$T/3$	1	1	1	1	1	1	1	1	1
	$T/2$	1	1	1	1	1	1	1	1	1
	$2T/3$	1	1	1	1	1	1	1	1	1
$W_{II}(2,\tau)$	$T/3$	1	1	1	1	1	1	1	1	1
	$T/2$	1	1	1	1	1	1	1	1	1
	$2T/3$	1	1	1	1	1	1	1	1	1
$W_{III}(2,\tau)$	$T/3$	1	1	1	1	1	1	1	1	1
	$T/2$	1	1	1	1	1	1	1	1	1
	$2T/3$	1	1	1	1	1	1	1	1	1
$W_V(2,\tau)$	$T/3$	1	1	1	1	1	1	1	1	1
	$T/2$	1	1	1	1	1	1	1	1	1
	$2T/3$	1	1	1	1	1	1	1	1	1

(TABLE 3.5 CONTINUED)

		N = 5			N = 10			N = 20		
		U_1	U_2	U_3	U_1	U_2	U_3	U_1	U_2	U_3
$W_{II}(3,\tau)$	$T/3$	1	1	1	1	1	1	1	1	1
	$T/2$	1	1	1	1	1	1	1	1	1
	$2T/3$	1	1	1	1	1	1	1	1	1
$W_{III}(3,\tau)$	$T/3$	0.9994	1	1	1	1	1	1	1	1
	$T/2$	1	1	1	1	1	1	1	1	1
	$2T/3$	1	1	1	1	1	1	1	1	1
$W_V(3,\tau)$	$T/3$	1	1	1	1	1	1	1	1	1
	$T/2$	1	1	1	1	1	1	1	1	1
	$2T/3$	1	1	1	1	1	1	1	1	1
$W_{VI}(1,2,\tau)$	$T/3$	1	1	1	1	1	1	1	1	1
	$T/2$	1	1	1	1	1	1	1	1	1
	$2T/3$	1	1	1	1	1	1	1	1	1
$W_{VI}(1,3,\tau)$	$T/3$	1	1	1	1	1	1	1	1	1
	$T/2$	1	1	1	1	1	1	1	1	1
	$2T/3$	1	1	1	1	1	1	1	1	1
$W_{VI}(2,3,\tau)$	$T/3$	1	1	1	1	1	1	1	1	1
	$T/2$	1	1	1	1	1	1	1	1	0.9956
	$2T/3$	1	1	1	1	1	1	1	1	0.9894
$W_{IV}(1,1,\tau)$	$T/3$	0.9996	1	1	1	1	1	0.9216	1	1
$W_{IV}(1,2,\tau)$		0.9756	1	1	0.9532	1	0.9998	1	1	1
$W_{IV}(1,3,\tau)$		0.997	1	1	0.112	0.9998	0.9996	0.9992	1	1
$W_{IV}(1,4,\tau)$		0.9966	1	1	0.5536	1	1	0.9896	1	1
$W_{IV}(1,5,\tau)$		0.8898	1	1	0.9962	1	1	1	1	0.9998
$W_{IV}(1,6,\tau)$					0.9956	1	1	0.0598	1	1
$W_{IV}(1,7,\tau)$					0.6702	0.8476	1	1	1	1
$W_{IV}(1,8,\tau)$					0.9954	1	1	0.677	1	1
$W_{IV}(1,9,\tau)$					1	1	1	1	1	1
$W_{IV}(1,10,\tau)$					0.8616	1	1	0.9994	1	1
$W_{IV}(1,11,\tau)$								0.9984	1	1
$W_{IV}(1,12,\tau)$								0.2534	0.6062	1
$W_{IV}(1,13,\tau)$								0.8414	1	0.9988
$W_{IV}(1,14,\tau)$								0.0596	0.876	1
$W_{IV}(1,15,\tau)$								0.996	1	1
$W_{IV}(1,16,\tau)$								0.261	0.9854	1
$W_{IV}(1,17,\tau)$								0.3892	1	1
$W_{IV}(1,18,\tau)$								0.7486	1	0.9996
$W_{IV}(1,19,\tau)$								0.7146	0.999	0.8724
$W_{IV}(1,20,\tau)$								0.0718	0.797	0.9998

(TABLE 3.5 CONTINUED)

		N = 5			N = 10			N = 20		
		U_1	U_2	U_3	U_1	U_2	U_3	U_1	U_2	U_3
$W_{IV}(1,1,\tau)$	$T/2$	1	1	1	1	1	1	1	1	1
$W_{IV}(1,2,\tau)$		1	1	1	1	1	1	1	1	1
$W_{IV}(1,3,\tau)$		1	0.9882	1	0.9924	0.9864	0.9996	1	1	1
$W_{IV}(1,4,\tau)$		1	1	1	1	1	1	1	1	1
$W_{IV}(1,5,\tau)$		1	1	1	1	1	1	1	1	1
$W_{IV}(1,6,\tau)$					1	0.9892	1	0.9808	1	1
$W_{IV}(1,7,\tau)$					0.9968	0.9826	1	1	0.9946	1
$W_{IV}(1,8,\tau)$					1	1	1	1	1	1
$W_{IV}(1,9,\tau)$					1	1	1	1	1	1
$W_{IV}(1,10,\tau)$					1	1	1	1	1	1
$W_{IV}(1,11,\tau)$								1	1	1
$W_{IV}(1,12,\tau)$								0.6602	0.991	1
$W_{IV}(1,13,\tau)$								1	1	0.9964
$W_{IV}(1,14,\tau)$								0.1752	0.9988	1
$W_{IV}(1,15,\tau)$								1	1	1
$W_{IV}(1,16,\tau)$								0.7008	0.9994	1
$W_{IV}(1,17,\tau)$								0.9856	1	1
$W_{IV}(1,18,\tau)$								0.9992	1	0.9886
$W_{IV}(1,19,\tau)$								0.998	0.8846	0.934
$W_{IV}(1,20,\tau)$								0.0938	0.9986	0.992
$W_{IV}(1,1,\tau)$	$2T/3$	1	1	1	1	1	1	1	1	1
$W_{IV}(1,2,\tau)$		1	1	1	1	1	1	1	1	1
$W_{IV}(1,3,\tau)$		1	1	1	1	0.9998	0.9982	1	1	1
$W_{IV}(1,4,\tau)$		1	1	1	1	1	1	1	1	1
$W_{IV}(1,5,\tau)$		1	1	1	1	1	1	1	0.9992	1
$W_{IV}(1,6,\tau)$					1	1	1	1	1	1
$W_{IV}(1,7,\tau)$					0.9968	1	0.999	1	0.9984	0.9998
$W_{IV}(1,8,\tau)$					1	1	1	1	1	1
$W_{IV}(1,9,\tau)$					1	1	1	1	1	1
$W_{IV}(1,10,\tau)$					1	1	1	1	1	1
$W_{IV}(1,11,\tau)$								1	1	1
$W_{IV}(1,12,\tau)$								0.777	1	1
$W_{IV}(1,13,\tau)$								1	0.9996	0.9842
$W_{IV}(1,14,\tau)$								0.6704	1	1
$W_{IV}(1,15,\tau)$								1	1	1
$W_{IV}(1,16,\tau)$								0.842	1	1
$W_{IV}(1,17,\tau)$								1	1	1
$W_{IV}(1,18,\tau)$								1	1	0.9624
$W_{IV}(1,19,\tau)$								1	0.7736	0.9484
$W_{IV}(1,20,\tau)$								0.1492	1	0.9662

(TABLE 3.5 CONTINUED)

		$N=5$			$N=10$			$N=20$		
		U_1	U_2	U_3	U_1	U_2	U_3	U_1	U_2	U_3
$W_{IV}(2,1,\tau)$	$T/3$	0.9356	1	1	0.273	0.9986	0.9994	0.0512	0.5768	0.736
$W_{IV}(2,2,\tau)$		0.9922	1	1	0.2808	1	0.969	0.1746	0.8066	0.9878
$W_{IV}(2,3,\tau)$		0.9724	1	1	0.9484	1	1	0.0646	0.9382	0.996
$W_{IV}(2,4,\tau)$		0.9378	1	1	0.5948	1	1	0.5874	0.999	0.9998
$W_{IV}(2,5,\tau)$		0.9992	1	1	1	1	1	0.3926	0.9996	0.9934
$W_{IV}(2,6,\tau)$					0.3594	1	1	0.2654	0.8938	0.9814
$W_{IV}(2,7,\tau)$					0.9592	1	1	0.0916	0.912	0.993
$W_{IV}(2,8,\tau)$					0.097	1	1	0.6	1	1
$W_{IV}(2,9,\tau)$					0.8132	0.9816	0.9988	0.95	1	1
$W_{IV}(2,10,\tau)$					0.3154	1	1	1	1	1
$W_{IV}(2,11,\tau)$								1	1	1
$W_{IV}(2,12,\tau)$								1	0.8392	1
$W_{IV}(2,13,\tau)$								1	1	1
$W_{IV}(2,14,\tau)$								0.6938	1	1
$W_{IV}(2,15,\tau)$								1	1	1
$W_{IV}(2,16,\tau)$								1	1	1
$W_{IV}(2,17,\tau)$								0.8186	1	1
$W_{IV}(2,18,\tau)$								1	1	1
$W_{IV}(2,19,\tau)$								1	1	1
$W_{IV}(2,20,\tau)$								0.9884	1	1
$W_{IV}(2,1,\tau)$	$T/2$	1	1	1	0.9708	0.994	1	0.1518	0.7052	0.6952
$W_{IV}(2,2,\tau)$		1	1	1	0.9754	0.9948	0.9978	0.4618	0.8918	0.9976
$W_{IV}(2,3,\tau)$		1	1	1	1	1	1	0.2338	0.9818	0.9994
$W_{IV}(2,4,\tau)$		1	1	1	1	0.9996	1	0.9942	0.9934	0.9998
$W_{IV}(2,5,\tau)$		1	1	1	1	1	1	0.9834	0.9894	0.9954
$W_{IV}(2,6,\tau)$					1	1	1	0.8272	0.9332	0.9734
$W_{IV}(2,7,\tau)$					1	1	1	0.3666	0.9362	0.9872
$W_{IV}(2,8,\tau)$					0.997	0.994	1	0.9968	1	1
$W_{IV}(2,9,\tau)$					0.514	0.9982	0.9938	1	1	0.9998
$W_{IV}(2,10,\tau)$					1	1	0.999	1	1	1
$W_{IV}(2,11,\tau)$								1	1	1
$W_{IV}(2,12,\tau)$								1	1	1
$W_{IV}(2,13,\tau)$								1	1	1
$W_{IV}(2,14,\tau)$								0.5286	1	1
$W_{IV}(2,15,\tau)$								1	1	1
$W_{IV}(2,16,\tau)$								1	1	1
$W_{IV}(2,17,\tau)$								1	1	1
$W_{IV}(2,18,\tau)$								1	1	1
$W_{IV}(2,19,\tau)$								1	1	0.9998
$W_{IV}(2,20,\tau)$								0.9904	1	1

(TABLE 3.5 CONTINUED)

		N = 5			N = 10			N = 20		
		U_1	U_2	U_3	U_1	U_2	U_3	U_1	U_2	U_3
$W_{IV}(2,1,\tau)$	$2T/3$	1	1	1	1	0.9996	1	0.4084	0.7948	0.6598
$W_{IV}(2,2,\tau)$		1	0.9998	1	0.9996	0.8624	0.9996	0.7724	0.9384	0.9972
$W_{IV}(2,3,\tau)$		1	1	1	1	1	1	0.542	0.9878	0.999
$W_{IV}(2,4,\tau)$		1	1	1	1	1	1	1	0.992	0.9998
$W_{IV}(2,5,\tau)$		1	1	1	1	1	1	1	0.9716	0.987
$W_{IV}(2,6,\tau)$					1	1	1	0.9934	0.9828	0.9548
$W_{IV}(2,7,\tau)$					1	1	1	0.6504	0.9886	0.9568
$W_{IV}(2,8,\tau)$					1	0.9866	1	1	1	0.9994
$W_{IV}(2,9,\tau)$					0.9226	1	0.9982	1	1	0.9952
$W_{IV}(2,10,\tau)$					1	1	0.9988	1	1	1
$W_{IV}(2,11,\tau)$								1	1	1
$W_{IV}(2,12,\tau)$								1	1	1
$W_{IV}(2,13,\tau)$								1	1	0.9992
$W_{IV}(2,14,\tau)$								1	1	1
$W_{IV}(2,15,\tau)$								1	1	1
$W_{IV}(2,16,\tau)$								1	1	0.9998
$W_{IV}(2,17,\tau)$								1	1	1
$W_{IV}(2,18,\tau)$								1	1	0.9996
$W_{IV}(2,19,\tau)$								1	1	0.9992
$W_{IV}(2,20,\tau)$								0.9784	1	0.9996
$W_{IV}(3,1,\tau)$	$T/3$	0.1112	0.395	0.2308	0.9998	1	1	1	1	1
$W_{IV}(3,2,\tau)$		0.1624	0.4928	0.129	0.3378	0.9998	0.7814	1	1	1
$W_{IV}(3,3,\tau)$		0.0746	0.8232	0.4452	0.9996	0.9994	0.7462	0.2384	1	1
$W_{IV}(3,4,\tau)$		0.0868	0.1244	0.4716	0.9992	1	0.9998	0.9974	1	1
$W_{IV}(3,5,\tau)$		0.09	0.7278	0.5196	0.9984	1	0.9998	0.2488	1	1
$W_{IV}(3,6,\tau)$					1	1	1	0.57	1	1
$W_{IV}(3,7,\tau)$					0.9998	1	1	0.9994	1	1
$W_{IV}(3,8,\tau)$					0.7176	1	1	0.2164	1	1
$W_{IV}(3,9,\tau)$					0.9998	0.9998	0.9982	0.9904	1	1
$W_{IV}(3,10,\tau)$					0.9998	0.9996	0.9998	0.2306	0.9976	1
$W_{IV}(3,11,\tau)$								0.0904	0.8372	0.998
$W_{IV}(3,12,\tau)$								0.2124	0.3062	0.8246
$W_{IV}(3,13,\tau)$								0.5562	0.9026	0.7088
$W_{IV}(3,14,\tau)$								0.6592	0.9578	0.936
$W_{IV}(3,15,\tau)$								0.8978	1	1
$W_{IV}(3,16,\tau)$								0.2768	0.7206	0.4078
$W_{IV}(3,17,\tau)$								0.517	0.998	0.9982
$W_{IV}(3,18,\tau)$								0.0978	0.4948	0.5402
$W_{IV}(3,19,\tau)$								0.0914	0.6642	0.9102
$W_{IV}(3,20,\tau)$								0.1956	0.8766	0.585

(TABLE 3.5 CONTINUED)

		N = 5			N = 10			N = 20		
		U_1	U_2	U_3	U_1	U_2	U_3	U_1	U_2	U_3
$W_{IV}(3,1,\tau)$	$T/2$	0.3182	0.3374	0.2282	1	1	1	1	1	1
$W_{IV}(3,2,\tau)$		0.4466	0.3034	0.2214	0.9996	0.931	0.904	1	1	1
$W_{IV}(3,3,\tau)$		0.866	0.8448	0.0492	0.9998	0.9804	0.9372	0.9532	1	1
$W_{IV}(3,4,\tau)$		0.6514	0.119	0.055	0.9998	0.9998	0.9998	1	1	1
$W_{IV}(3,5,\tau)$		0.8708	0.7976	0.0688	0.9998	0.9998	0.9998	1	1	1
$W_{IV}(3,6,\tau)$					1	1	1	0.998	1	1
$W_{IV}(3,7,\tau)$					1	1	1	1	1	0.9998
$W_{IV}(3,8,\tau)$					0.5572	1	1	0.9972	1	1
$W_{IV}(3,9,\tau)$					0.9998	0.9992	0.9992	1	1	1
$W_{IV}(3,10,\tau)$					0.9998	0.9996	0.9998	0.855	0.9998	1
$W_{IV}(3,11,\tau)$								0.3888	0.9596	0.9996
$W_{IV}(3,12,\tau)$								0.3292	0.547	0.8074
$W_{IV}(3,13,\tau)$								0.8568	0.8874	0.5664
$W_{IV}(3,14,\tau)$								0.9494	0.9762	0.7834
$W_{IV}(3,15,\tau)$								0.9984	1	1
$W_{IV}(3,16,\tau)$								0.7376	0.597	0.2956
$W_{IV}(3,17,\tau)$								0.9794	0.9992	0.9836
$W_{IV}(3,18,\tau)$								0.2882	0.5278	0.4848
$W_{IV}(3,19,\tau)$								0.2708	0.8642	0.7834
$W_{IV}(3,20,\tau)$								0.7112	0.7434	0.5412
$W_{IV}(3,1,\tau)$	$2T/3$	0.4274	0.339	0.3682	1	1	1	1	1	1
$W_{IV}(3,2,\tau)$		0.5632	0.1886	0.5138	0.9998	0.5396	0.955	1	1	1
$W_{IV}(3,3,\tau)$		0.843	0.7786	0.0184	0.9984	0.6082	0.9948	1	1	1
$W_{IV}(3,4,\tau)$		0.6236	0.7258	0.0178	1	0.9998	0.9998	1	1	1
$W_{IV}(3,5,\tau)$		0.8248	0.8398	0.0178	1	0.9998	0.9998	1	1	1
$W_{IV}(3,6,\tau)$					1	1	1	0.9998	1	0.9994
$W_{IV}(3,7,\tau)$					1	1	1	1	1	0.9992
$W_{IV}(3,8,\tau)$					0.9998	1	1	1	1	1
$W_{IV}(3,9,\tau)$					0.9998	0.9996	0.9996	1	1	1
$W_{IV}(3,10,\tau)$					0.9998	0.9998	0.9998	0.9888	0.9998	0.9992
$W_{IV}(3,11,\tau)$								0.6996	0.9906	0.9982
$W_{IV}(3,12,\tau)$								0.352	0.7738	0.6936
$W_{IV}(3,13,\tau)$								0.9276	0.8226	0.3666
$W_{IV}(3,14,\tau)$								0.9804	0.9886	0.5236
$W_{IV}(3,15,\tau)$								1	1	0.9986
$W_{IV}(3,16,\tau)$								0.8824	0.5358	0.2322
$W_{IV}(3,17,\tau)$								0.999	0.9996	0.8898
$W_{IV}(3,18,\tau)$								0.4488	0.5766	0.4544
$W_{IV}(3,19,\tau)$								0.43	0.96	0.5774
$W_{IV}(3,20,\tau)$								0.9328	0.6002	0.5178

TABLE 3.6. Bootstrapped Power of *Exp* (5% level). The table reports the power of the weighted measure *Exp*, for the various test statistics.

		$N=5$			$N=10$			$N=20$		
		U_1	U_2	U_3	U_1	U_2	U_3	U_1	U_2	U_3
$W_I(\tau)$	$T/3$	1	1	1	1	1	1	1	1	1
	$T/2$	1	1	1	1	1	1	1	1	1
	$2T/3$	1	1	1	1	1	1	1	1	1
$W_{II}(1,\tau)$	$T/3$	1	1	1	1	1	1	1	1	1
	$T/2$	1	1	1	1	1	1	1	1	1
	$2T/3$	1	1	1	1	1	1	1	1	1
$W_{III}(1,\tau)$	$T/3$	1	1	1	1	1	1	1	1	1
	$T/2$	1	1	1	1	1	1	1	1	1
	$2T/3$	1	1	1	1	1	1	1	1	1
$W_V(1,\tau)$	$T/3$	1	1	1	1	1	1	1	1	1
	$T/2$	1	1	1	1	1	1	1	1	1
	$2T/3$	1	1	1	1	1	1	1	1	1
$W_{II}(2,\tau)$	$T/3$	1	1	1	1	1	1	1	1	1
	$T/2$	1	1	1	1	1	1	1	1	1
	$2T/3$	1	1	1	1	1	1	1	1	1
$W_{III}(2,\tau)$	$T/3$	1	1	1	1	1	1	1	1	1
	$T/2$	1	1	1	1	1	1	1	1	1
	$2T/3$	1	1	1	1	1	1	1	1	1
$W_V(2,\tau)$	$T/3$	1	1	1	1	1	1	1	1	1
	$T/2$	1	1	1	1	1	1	1	1	1
	$2T/3$	1	1	1	1	1	1	1	1	1

(TABLE 3.6 CONTINUED)

		$N=5$			$N=10$			$N=20$		
		U_1	U_2	U_3	U_1	U_2	U_3	U_1	U_2	U_3
$W_{II}(3,\tau)$	$T/3$	1	1	1	1	1	1	1	1	1
	$T/2$	1	1	1	1	1	1	1	1	1
	$2T/3$	1	1	1	1	1	1	1	1	1
$W_{III}(3,\tau)$	$T/3$	0.9994	1	1	1	1	1	1	1	1
	$T/2$	1	1	1	1	1	1	1	1	1
	$2T/3$	1	1	1	1	1	1	1	1	1
$W_V(3,\tau)$	$T/3$	1	1	1	1	1	1	1	1	1
	$T/2$	1	1	1	1	1	1	1	1	1
	$2T/3$	1	1	1	1	1	1	1	1	1
$W_{VI}(1,2,\tau)$	$T/3$	1	1	1	1	1	1	1	1	1
	$T/2$	1	1	1	1	1	1	1	1	1
	$2T/3$	1	1	1	1	1	1	1	1	1
$W_{VI}(1,3,\tau)$	$T/3$	1	1	1	1	1	1	1	1	1
	$T/2$	1	1	1	1	1	1	1	1	1
	$2T/3$	1	1	1	1	1	1	1	1	1
$W_{VI}(2,3,\tau)$	$T/3$	1	1	1	1	1	1	1	1	1
	$T/2$	1	1	1	1	1	1	1	1	1
	$2T/3$	1	1	1	1	1	1	1	1	1
$W_{IV}(1,1,\tau)$	$T/3$	0.9996	1	1	1	1	1	0.9218	1	1
$W_{IV}(1,2,\tau)$		0.9756	1	1	0.9532	1	0.9998	1	1	1
$W_{IV}(1,3,\tau)$		0.997	1	1	0.112	0.9998	0.9996	0.9992	1	1
$W_{IV}(1,4,\tau)$		0.9966	1	1	0.5542	1	1	0.9896	1	1
$W_{IV}(1,5,\tau)$		0.89	1	1	0.9962	1	1	1	1	0.9998
$W_{IV}(1,6,\tau)$					0.9956	1	1	0.0598	1	1
$W_{IV}(1,7,\tau)$					0.6702	0.848	1	1	1	1
$W_{IV}(1,8,\tau)$					0.9954	1	1	0.6774	1	1
$W_{IV}(1,9,\tau)$					1	1	1	1	1	1
$W_{IV}(1,10,\tau)$					0.8616	1	1	0.9994	1	1
$W_{IV}(1,11,\tau)$								0.9984	1	1
$W_{IV}(1,12,\tau)$								0.2534	0.6068	1
$W_{IV}(1,13,\tau)$								0.8414	1	0.9988
$W_{IV}(1,14,\tau)$								0.0598	0.8762	1
$W_{IV}(1,15,\tau)$								0.996	1	1
$W_{IV}(1,16,\tau)$								0.2612	0.9854	1
$W_{IV}(1,17,\tau)$								0.3892	1	1
$W_{IV}(1,18,\tau)$								0.7486	1	0.9996
$W_{IV}(1,19,\tau)$								0.7148	0.999	0.8742
$W_{IV}(1,20,\tau)$								0.0718	0.797	0.9998

(TABLE 3.6 CONTINUED)

		$N=5$			$N=10$			$N=20$		
		U_1	U_2	U_3	U_1	U_2	U_3	U_1	U_2	U_3
$W_{IV}(1,1,\tau)$	$T/2$	1	1	1	1	1	1	1	1	1
$W_{IV}(1,2,\tau)$		1	1	1	1	1	1	1	1	1
$W_{IV}(1,3,\tau)$		1	0.9886	1	0.9924	0.9864	0.9996	1	1	1
$W_{IV}(1,4,\tau)$		1	1	1	1	1	1	1	1	1
$W_{IV}(1,5,\tau)$		1	1	1	1	1	1	1	1	1
$W_{IV}(1,6,\tau)$					1	0.9892	1	0.9808	1	1
$W_{IV}(1,7,\tau)$					0.9968	0.9826	1	1	0.9946	1
$W_{IV}(1,8,\tau)$					1	1	1	1	1	1
$W_{IV}(1,9,\tau)$					1	1	1	1	1	1
$W_{IV}(1,10,\tau)$					1	1	1	1	1	1
$W_{IV}(1,11,\tau)$								1	1	1
$W_{IV}(1,12,\tau)$								0.6602	0.991	1
$W_{IV}(1,13,\tau)$								1	1	0.9966
$W_{IV}(1,14,\tau)$								0.1752	0.9988	1
$W_{IV}(1,15,\tau)$								1	1	1
$W_{IV}(1,16,\tau)$								0.7008	0.9994	1
$W_{IV}(1,17,\tau)$								0.9856	1	1
$W_{IV}(1,18,\tau)$								0.9992	1	0.9888
$W_{IV}(1,19,\tau)$								0.998	0.8846	0.9368
$W_{IV}(1,20,\tau)$								0.0938	0.9986	0.992
$W_{IV}(1,1,\tau)$	$2T/3$	1	1	1	1	1	1	1	1	1
$W_{IV}(1,2,\tau)$		1	1	1	1	1	1	1	1	1
$W_{IV}(1,3,\tau)$		1	1	1	1	0.9998	0.9982	1	1	1
$W_{IV}(1,4,\tau)$		1	1	1	1	1	1	1	1	1
$W_{IV}(1,5,\tau)$		1	1	1	1	1	1	1	0.9992	1
$W_{IV}(1,6,\tau)$					1	1	1	1	1	1
$W_{IV}(1,7,\tau)$					0.9968	1	0.999	1	0.9984	0.9998
$W_{IV}(1,8,\tau)$					1	1	1	1	1	1
$W_{IV}(1,9,\tau)$					1	1	1	1	1	1
$W_{IV}(1,10,\tau)$					1	1	1	1	1	1
$W_{IV}(1,11,\tau)$								1	1	1
$W_{IV}(1,12,\tau)$								0.7772	1	1
$W_{IV}(1,13,\tau)$								1	0.9996	0.9844
$W_{IV}(1,14,\tau)$								0.6706	1	1
$W_{IV}(1,15,\tau)$								1	1	1
$W_{IV}(1,16,\tau)$								0.842	1	1
$W_{IV}(1,17,\tau)$								1	1	1
$W_{IV}(1,18,\tau)$								1	1	0.9636
$W_{IV}(1,19,\tau)$								1	0.7748	0.9502
$W_{IV}(1,20,\tau)$								0.1492	1	0.9672

(TABLE 3.6 CONTINUED)

		N = 5			N = 10			N = 20		
		U_1	U_2	U_3	U_1	U_2	U_3	U_1	U_2	U_3
$W_{IV}(2,1,\tau)$	$T/3$	0.9356	1	1	0.273	0.9986	0.9994	0.0512	0.5768	0.7376
$W_{IV}(2,2,\tau)$		0.9922	1	1	0.2808	1	0.9702	0.1748	0.8068	0.9882
$W_{IV}(2,3,\tau)$		0.9726	1	1	0.9484	1	1	0.0646	0.9384	0.996
$W_{IV}(2,4,\tau)$		0.9378	1	1	0.595	1	1	0.5876	0.999	0.9998
$W_{IV}(2,5,\tau)$		0.9992	1	1	1	1	1	0.3926	0.9996	0.994
$W_{IV}(2,6,\tau)$					0.3594	1	1	0.2654	0.8952	0.9818
$W_{IV}(2,7,\tau)$					0.9592	1	1	0.0916	0.913	0.9934
$W_{IV}(2,8,\tau)$					0.097	1	1	0.6006	1	1
$W_{IV}(2,9,\tau)$					0.8134	0.9816	0.9988	0.95	1	1
$W_{IV}(2,10,\tau)$					0.3156	1	1	1	1	1
$W_{IV}(2,11,\tau)$								1	1	1
$W_{IV}(2,12,\tau)$								1	0.8394	1
$W_{IV}(2,13,\tau)$								1	1	1
$W_{IV}(2,14,\tau)$								0.6938	1	1
$W_{IV}(2,15,\tau)$								1	1	1
$W_{IV}(2,16,\tau)$								1	1	1
$W_{IV}(2,17,\tau)$								0.8186	1	1
$W_{IV}(2,18,\tau)$								1	1	1
$W_{IV}(2,19,\tau)$								1	1	1
$W_{IV}(2,20,\tau)$								0.9884	1	1
$W_{IV}(2,1,\tau)$	$T/2$	1	1	1	0.9708	0.994	1	0.1518	0.7056	0.6962
$W_{IV}(2,2,\tau)$		1	1	1	0.9754	0.9948	0.998	0.4618	0.893	0.9976
$W_{IV}(2,3,\tau)$		1	1	1	1	1	1	0.2338	0.9818	0.9994
$W_{IV}(2,4,\tau)$		1	1	1	1	0.9996	1	0.9942	0.9934	0.9998
$W_{IV}(2,5,\tau)$		1	1	1	1	1	1	0.9834	0.9896	0.9958
$W_{IV}(2,6,\tau)$					1	1	1	0.8272	0.9348	0.975
$W_{IV}(2,7,\tau)$					1	1	1	0.3668	0.9384	0.9878
$W_{IV}(2,8,\tau)$					0.997	0.994	1	0.9968	1	1
$W_{IV}(2,9,\tau)$					0.5142	0.9982	0.9938	1	1	0.9998
$W_{IV}(2,10,\tau)$					1	1	0.999	1	1	1
$W_{IV}(2,11,\tau)$								1	1	1
$W_{IV}(2,12,\tau)$								1	1	1
$W_{IV}(2,13,\tau)$								1	1	1
$W_{IV}(2,14,\tau)$								0.529	1	1
$W_{IV}(2,15,\tau)$								1	1	1
$W_{IV}(2,16,\tau)$								1	1	1
$W_{IV}(2,17,\tau)$								1	1	1
$W_{IV}(2,18,\tau)$								1	1	1
$W_{IV}(2,19,\tau)$								1	1	0.9998
$W_{IV}(2,20,\tau)$								0.9904	1	1

(TABLE 3.6 CONTINUED)

		N = 5			N = 10			N = 20		
		U_1	U_2	U_3	U_1	U_2	U_3	U_1	U_2	U_3
$W_{IV}(2,1,\tau)$	$2T/3$	1	1	1	1	0.9996	1	0.4084	0.7952	0.6608
$W_{IV}(2,2,\tau)$		1	0.9998	1	0.9996	0.8638	0.9996	0.7728	0.9392	0.9972
$W_{IV}(2,3,\tau)$		1	1	1	1	1	1	0.5422	0.988	0.999
$W_{IV}(2,4,\tau)$		1	1	1	1	1	1	1	0.9922	0.9998
$W_{IV}(2,5,\tau)$		1	1	1	1	1	1	1	0.9728	0.9876
$W_{IV}(2,6,\tau)$					1	1	1	0.9934	0.9834	0.9568
$W_{IV}(2,7,\tau)$					1	1	1	0.6506	0.9892	0.958
$W_{IV}(2,8,\tau)$					1	0.9868	1	1	1	0.9994
$W_{IV}(2,9,\tau)$					0.9226	1	0.9982	1	1	0.9952
$W_{IV}(2,10,\tau)$					1	1	0.9988	1	1	1
$W_{IV}(2,11,\tau)$								1	1	1
$W_{IV}(2,12,\tau)$								1	1	1
$W_{IV}(2,13,\tau)$								1	1	0.9992
$W_{IV}(2,14,\tau)$								1	1	1
$W_{IV}(2,15,\tau)$								1	1	1
$W_{IV}(2,16,\tau)$								1	1	0.9998
$W_{IV}(2,17,\tau)$								1	1	1
$W_{IV}(2,18,\tau)$								1	1	0.9996
$W_{IV}(2,19,\tau)$								1	1	0.9994
$W_{IV}(2,20,\tau)$								0.9784	1	0.9996
$W_{IV}(3,1,\tau)$	$T/3$	0.096	0.3718	0.2028	0.9996	0.9998	0.9998	1	1	1
$W_{IV}(3,2,\tau)$		0.144	0.4714	0.112	0.3378	0.9996	0.7832	1	1	1
$W_{IV}(3,3,\tau)$		0.0668	0.9798	0.463	0.9994	0.9992	0.7506	0.2386	1	1
$W_{IV}(3,4,\tau)$		0.0788	0.1132	0.4482	0.999	0.9998	0.9998	0.9974	1	1
$W_{IV}(3,5,\tau)$		0.0832	0.9238	0.5356	0.9984	0.9998	0.9996	0.2488	1	1
$W_{IV}(3,6,\tau)$					0.9998	1	0.9998	0.5702	1	1
$W_{IV}(3,7,\tau)$					0.9996	0.9998	0.9998	0.9994	1	1
$W_{IV}(3,8,\tau)$					0.7176	0.9998	0.9998	0.2164	1	1
$W_{IV}(3,9,\tau)$					0.9996	0.9996	0.9982	0.9904	1	1
$W_{IV}(3,10,\tau)$					0.9996	0.9996	0.9996	0.2308	0.9976	1
$W_{IV}(3,11,\tau)$								0.0906	0.8376	0.9982
$W_{IV}(3,12,\tau)$								0.2126	0.3068	0.8288
$W_{IV}(3,13,\tau)$								0.5564	0.9026	0.7106
$W_{IV}(3,14,\tau)$								0.6592	0.9578	0.9364
$W_{IV}(3,15,\tau)$								0.898	1	1
$W_{IV}(3,16,\tau)$								0.277	0.7206	0.4086
$W_{IV}(3,17,\tau)$								0.517	0.998	0.9982
$W_{IV}(3,18,\tau)$								0.0978	0.4954	0.5416
$W_{IV}(3,19,\tau)$								0.0914	0.6644	0.9108
$W_{IV}(3,20,\tau)$								0.1954	0.8766	0.587

(TABLE 3.6 CONTINUED)

		N = 5			N = 10			N = 20		
		U_1	U_2	U_3	U_1	U_2	U_3	U_1	U_2	U_3
$W_{IV}(3,1,\tau)$	$T/2$	0.294	0.3106	0.2026	0.9998	0.9998	0.9998	1	1	1
$W_{IV}(3,2,\tau)$		0.4244	0.2776	0.1986	0.9994	0.931	0.9058	1	1	1
$W_{IV}(3,3,\tau)$		0.9714	0.9582	0.0458	0.9996	0.9804	0.9402	0.9532	1	1
$W_{IV}(3,4,\tau)$		0.71	0.106	0.0496	0.9998	0.9998	0.9998	1	1	1
$W_{IV}(3,5,\tau)$		0.9774	0.93	0.0624	0.9998	0.9998	0.9996	1	1	1
$W_{IV}(3,6,\tau)$					0.9998	0.9998	0.9998	0.998	1	1
$W_{IV}(3,7,\tau)$					0.9998	0.9998	0.9998	1	1	0.9998
$W_{IV}(3,8,\tau)$					0.5572	0.9998	0.9998	0.9972	1	1
$W_{IV}(3,9,\tau)$					0.9996	0.9992	0.9992	1	1	1
$W_{IV}(3,10,\tau)$					0.9998	0.9996	0.9996	0.855	0.9998	1
$W_{IV}(3,11,\tau)$								0.389	0.96	0.9996
$W_{IV}(3,12,\tau)$								0.3294	0.5482	0.8122
$W_{IV}(3,13,\tau)$								0.8568	0.888	0.57
$W_{IV}(3,14,\tau)$								0.9496	0.9762	0.7856
$W_{IV}(3,15,\tau)$								0.9984	1	1
$W_{IV}(3,16,\tau)$								0.7376	0.5976	0.297
$W_{IV}(3,17,\tau)$								0.9794	0.9992	0.9838
$W_{IV}(3,18,\tau)$								0.2884	0.5284	0.4866
$W_{IV}(3,19,\tau)$								0.2708	0.8644	0.7844
$W_{IV}(3,20,\tau)$								0.7112	0.7438	0.5416
$W_{IV}(3,1,\tau)$	$2T/3$	0.4074	0.3144	0.354	0.9998	0.9998	0.9998	1	1	1
$W_{IV}(3,2,\tau)$		0.5522	0.167	0.5106	0.9998	0.5406	0.9566	1	1	1
$W_{IV}(3,3,\tau)$		0.978	0.8838	0.0176	0.9984	0.609	0.995	1	1	1
$W_{IV}(3,4,\tau)$		0.7102	0.7264	0.0168	0.9998	0.9998	0.9998	1	1	1
$W_{IV}(3,5,\tau)$		0.9708	0.935	0.0174	0.9998	0.9996	0.9996	1	1	1
$W_{IV}(3,6,\tau)$					0.9998	0.9998	0.9998	0.9998	1	0.9994
$W_{IV}(3,7,\tau)$					0.9998	0.9998	0.9998	1	1	0.9992
$W_{IV}(3,8,\tau)$					0.9996	0.9998	0.9998	1	1	1
$W_{IV}(3,9,\tau)$					0.9998	0.9996	0.9996	1	1	1
$W_{IV}(3,10,\tau)$					0.9996	0.9996	0.9996	0.9888	0.9998	0.9992
$W_{IV}(3,11,\tau)$								0.6996	0.9906	0.9982
$W_{IV}(3,12,\tau)$								0.3524	0.7764	0.699
$W_{IV}(3,13,\tau)$								0.9276	0.8236	0.3678
$W_{IV}(3,14,\tau)$								0.9804	0.9888	0.5258
$W_{IV}(3,15,\tau)$								1	1	0.9988
$W_{IV}(3,16,\tau)$								0.8826	0.5366	0.2328
$W_{IV}(3,17,\tau)$								0.999	0.9996	0.8904
$W_{IV}(3,18,\tau)$								0.4488	0.5776	0.456
$W_{IV}(3,19,\tau)$								0.43	0.9602	0.5776
$W_{IV}(3,20,\tau)$								0.9328	0.6008	0.5196

TABLE 3.7. Testing results for Federal Reserve daily bond yields. The table reports the *Sup*, *Avg*, and *Exp* values for the test statistics $W_I(\tau), W_{II}(i,\tau), W_{III}(i,\tau), W_{IV}(i,\tau), W_V(i,\tau)$, and $W_{VI}(i,\tau)$ associated with the six hypotheses formulated in equations 2.23 - 2.28. The p-values are reported in italics.

Testing overall system

	Sup	*Avg*	*Exp*
$W_I(\tau)$	0.21788	0.024133	-1.8342
	0.000	*0.000*	*0.000*

Testing the level factor

	Sup	*Avg*	*Exp*
$W_{II}(1,\tau)$	0.080904	0.009637	-1.8837
	0.000	*0.000*	*0.000*
$W_{III}(1,\tau)$	0.23128	0.026217	-1.8272
	0.000	*0.000*	*0.000*
$W_V(1,\tau)$	0.24585	0.029767	-1.8151
	0.000	*0.000*	*0.000*

Testing the slope factor

	Sup	*Avg*	*Exp*
$W_{II}(2,\tau)$	0.069336	0.006697	-1.8936
	0.000	*0.000*	*0.000*
$W_{III}(2,\tau)$	248.12	26.206	117.61
	0.753	*0.604*	*0.753*
$W_V(2,\tau)$	1.9809	0.21744	-1.1462
	0.363	*0.052*	*0.058*

Testing the curvature factor

	Sup	*Avg*	*Exp*
$W_{II}(3,\tau)$	0.073677	0.007799	-1.8899
	0.000	*0.000*	*0.000*
$W_{III}(3,\tau)$	0.093286	0.007266	-1.8917
	0.000	*0.000*	*0.000*
$W_V(3,\tau)$	0.18308	0.016498	-1.8602
	0.000	*0.000*	*0.000*

Testing the IRs governing the level factor

	Sup	*Avg*	*Exp*
$W_{IV}(1,1,\tau)$	0.074561	0.006958	-1.8927
	0.000	*0.000*	*0.000*
$W_{IV}(1,2,\tau)$	0.15959	0.015836	-1.8625
	0.000	*0.000*	*0.000*
$W_{IV}(1,3,\tau)$	0.096229	0.009853	-1.8829
	0.000	*0.000*	*0.000*
$W_{IV}(1,4,\tau)$	0.054365	0.005447	-1.8979
	0.000	*0.000*	*0.000*
$W_{IV}(1,5,\tau)$	0.1079	0.011748	-1.8765
	0.000	*0.000*	*0.000*
$W_{IV}(1,6,\tau)$	0.12296	0.013145	-1.8717
	0.000	*0.000*	*0.000*
$W_{IV}(1,7,\tau)$	0.074699	0.007382	-1.8913
	0.000	*0.000*	*0.000*
$W_{IV}(1,8,\tau)$	0.046066	0.004125	-1.9024
	0.000	*0.000*	*0.000*

Testing the common factors

	Sup	*Avg*	*Exp*
$W_{VI}(1,2,\tau)$	2.3662	0.26583	-0.97505
	0.000	*0.000*	*0.000*
$W_{VI}(1,3,\tau)$	0.41223	0.046288	-1.7586
	0.000	*0.000*	*0.000*
$W_{VI}(2,3,\tau)$	2.182	0.2569	-1.013
	0.000	*0.000*	*0.000*

Testing the IRs governing the slope factor			
	Sup	*Avg*	*Exp*
$W_{IV}(2,1,\tau)$	3.4434	0.37308	-0.55909
	0.009	*0.005*	*0.005*
$W_{IV}(2,2,\tau)$	1.2778	0.1228	-1.4827
	0.026	*0.064*	*0.067*
$W_{IV}(2,3,\tau)$	0.28623	0.026416	-1.826
	0.002	*0.025*	*0.023*
$W_{IV}(2,4,\tau)$	0.047793	0.00537	-1.8982
	0.698	*0.225*	*0.227*
$W_{IV}(2,5,\tau)$	0.27393	0.031617	-1.8085
	0.646	*0.156*	*0.164*
$W_{IV}(2,6,\tau)$	0.85967	0.098243	-1.5771
	0.604	*0.120*	*0.143*
$W_{IV}(2,7,\tau)$	0.97655	0.11195	-1.529
	0.578	*0.111*	*0.129*
$W_{IV}(2,8,\tau)$	0.8575	0.09659	-1.5828
	0.536	*0.096*	*0.109*

Testing the IRs governing the curvature factor			
	Sup	*Avg*	*Exp*
$W_{IV}(3,1,\tau)$	0.014698	0.001279	-1.9121
	0.014	*0.001*	*0.001*
$W_{IV}(3,2,\tau)$	0.075935	0.004772	-1.9002
	0.000	*0.000*	*0.000*
$W_{IV}(3,3,\tau)$	0.006669	0.000508	-1.9147
	0.130	*0.050*	*0.050*
$W_{IV}(3,4,\tau)$	0.038835	0.002895	-1.9066
	0.000	*0.000*	*0.000*
$W_{IV}(3,5,\tau)$	0.03513	0.003013	-1.9062
	0.000	*0.000*	*0.000*
$W_{IV}(3,6,\tau)$	0.002986	8.39E-05	-1.9162
	0.073	*0.208*	*0.208*
$W_{IV}(3,7,\tau)$	0.014231	0.001109	-1.9127
	0.001	*0.001*	*0.001*
$W_{IV}(3,8,\tau)$	0.0285	0.002491	-1.908
	0.000	*0.000*	*0.000*

TABLE 3.8. Testing results for Federal Reserve monthly bond yields. The table reports the *Sup*, *Avg*, and *Exp* values for the test statistics $W_I(\tau), W_{II}(i,\tau), W_{III}(i,\tau), W_{IV}(i,\tau), W_V(i,\tau)$, and $W_{VI}(i,\tau)$ associated with the six hypotheses formulated in equations 2.23 - 2.28. The p-values are reported in italics.

Testing overall system

	Sup	*Avg*	*Exp*
$W_I(\tau)$	0.28311	0.62693	1.257
	0.006	*0.000*	*0.000*

Testing the level factor

	Sup	*Avg*	*Exp*
$W_{II}(1,\tau)$	0.125	0.30613	1.2065
	0.002	*0.000*	*0.000*
$W_{III}(1,\tau)$	0.28359	0.62246	1.2565
	0.081	*0.008*	*0.008*
$W_V(1,\tau)$	0.34216	0.82941	1.2888
	0.003	*0.000*	*0.000*

Testing the slope factor

	Sup	*Avg*	*Exp*
$W_{II}(2,\tau)$	0.069375	0.15996	1.1836
	0.242	*0.015*	*0.015*
$W_{III}(2,\tau)$	0.51744	0.8568	1.2935
	0.006	*0.002*	*0.002*
$W_V(2,\tau)$	0.29807	0.77128	1.2797
	0.068	*0.001*	*0.001*

Testing the curvature factor

	Sup	*Avg*	*Exp*
$W_{II}(3,\tau)$	0.10018	0.16084	1.1837
	0.044	*0.004*	*0.004*
$W_{III}(3,\tau)$	0.21639	0.48289	1.2344
	0.131	*0.008*	*0.008*
$W_V(3,\tau)$	0.24129	0.56157	1.2467
	0.036	*0.001*	*0.001*

Testing the IRs governing the level factor

	Sup	*Avg*	*Exp*
$W_{IV}(1,1,\tau)$	0.084724	0.14268	1.1809
	0.300	*0.120*	*0.121*
$W_{IV}(1,2,\tau)$	0.09609	0.15852	1.1834
	0.125	*0.040*	*0.040*
$W_{IV}(1,3,\tau)$	0.056443	0.078843	1.1708
	0.054	*0.026*	*0.026*
$W_{IV}(1,4,\tau)$	0.07872	0.13886	1.1803
	0.091	*0.021*	*0.021*
$W_{IV}(1,5,\tau)$	0.18757	0.33367	1.211
	0.006	*0.001*	*0.001*
$W_{IV}(1,6,\tau)$	0.087837	0.15356	1.1826
	0.143	*0.044*	*0.045*
$W_{IV}(1,7,\tau)$	0.044407	0.064474	1.1686
	0.341	*0.192*	*0.192*
$W_{IV}(1,8,\tau)$	0.021252	0.01793	1.1612
	0.737	*0.695*	*0.696*

Testing the common factors

	Sup	*Avg*	*Exp*
$W_{VI}(1,2,\tau)$	0.6136	1.6169	1.4127
	0.008	*0.000*	*0.000*
$W_{VI}(1,3,\tau)$	0.50214	1.3229	1.3664
	0.004	*0.000*	*0.000*
$W_{VI}(2,3,\tau)$	0.49117	1.3041	1.3634
	0.023	*0.000*	*0.000*

Testing the IRs governing the slope factor	*Sup*	*Avg*	*Exp*	Testing the IRs governing the curvature factor	*Sup*	*Avg*	*Exp*
$W_{IV}(2,1,\tau)$	0.059979	0.12319	1.1778	$W_{IV}(3,1,\tau)$	0.013676	0.014301	1.1607
	0.073	*0.008*	*0.008*		*0.700*	*0.514*	*0.515*
$W_{IV}(2,2,\tau)$	0.10178	0.19236	1.1887	$W_{IV}(3,2,\tau)$	0.070769	0.1332	1.1794
	0.064	*0.013*	*0.013*		*0.100*	*0.020*	*0.020*
$W_{IV}(2,3,\tau)$	0.27441	0.5948	1.2523	$W_{IV}(3,3,\tau)$	0.13232	0.10164	1.1745
	0.006	*0.000*	*0.000*		*0.002*	*0.026*	*0.026*
$W_{IV}(2,4,\tau)$	0.073562	0.12782	1.1785	$W_{IV}(3,4,\tau)$	0.049983	0.025846	1.1625
	0.135	*0.036*	*0.036*		*0.133*	*0.375*	*0.374*
$W_{IV}(2,5,\tau)$	0.052661	0.074805	1.1702	$W_{IV}(3,5,\tau)$	0.038806	0.029134	1.163
	0.252	*0.129*	*0.130*		*0.368*	*0.363*	*0.363*
$W_{IV}(2,6,\tau)$	0.021918	0.028842	1.163	$W_{IV}(3,6,\tau)$	0.042201	0.045078	1.1655
	0.742	*0.553*	*0.553*		*0.264*	*0.231*	*0.231*
$W_{IV}(2,7,\tau)$	0.011033	0.004199	1.1591	$W_{IV}(3,7,\tau)$	0.043421	0.070141	1.1694
	0.842	*0.947*	*0.947*		*0.160*	*0.065*	*0.065*
$W_{IV}(2,8,\tau)$	0.008558	0.002216	1.1588	$W_{IV}(3,8,\tau)$	0.015648	0.014422	1.1607
	0.878	*0.991*	*0.991*		*0.528*	*0.430*	*0.430*

TABLE 3.9. Testing results for Fama Bliss monthly bond yields. The table reports the *Sup*, *Avg*, and *Exp* values for the test statistics $W_I(\tau), W_{II}(i,\tau), W_{III}(i,\tau), W_{IV}(i,\tau), W_V(i,\tau)$, and $W_{VI}(i,\tau)$ associated with the six hypotheses formulated in equations 2.23 - 2.28. The p-values are reported in italics.

Testing overall system

	Sup	*Avg*	*Exp*
$W_I(\tau)$	0.47223	0.70067	1.0435
	0.140	*0.133*	*0.924*

Testing the level factor

	Sup	*Avg*	*Exp*
$W_{II}(1,\tau)$	0.18327	0.26953	0.95232
	0.000	*0.000*	*0.000*
$W_{III}(1,\tau)$	0.70166	1.0604	1.1158
	0.007	*0.002*	*0.002*
$W_V(1,\tau)$	0.62397	1.0427	1.1123
	0.000	*0.000*	*0.000*

Testing the slope factor

	Sup	*Avg*	*Exp*
$W_{II}(2,\tau)$	0.115	0.15889	0.92948
	0.000	*0.000*	*0.000*
$W_{III}(2,\tau)$	0.53077	0.74771	1.0503
	0.013	*0.004*	*0.005*
$W_V(2,\tau)$	0.41481	0.73693	1.0481
	0.005	*0.000*	*0.000*

Testing the curvature factor

	Sup	*Avg*	*Exp*
$W_{II}(3,\tau)$	0.19333	0.27225	0.95306
	0.000	*0.000*	*0.000*
$W_{III}(3,\tau)$	0.42918	0.64173	1.0282
	0.338	*0.146*	*0.159*
$W_V(3,\tau)$	0.48997	0.82029	1.066
	0.064	*0.003*	*0.004*

Testing the common factors

	Sup	*Avg*	*Exp*
$W_{VI}(1,2,\tau)$	0.92379	1.6091	1.23
	0.003	*0.000*	*0.000*
$W_{VI}(1,3,\tau)$	1.0419	1.7968	1.2706
	0.031	*0.000*	*0.000*
$W_{VI}(2,3,\tau)$	1.164	2.0277	1.3225
	0.021	*0.000*	*0.000*

Testing the IRs governing the level factor

	Sup	*Avg*	*Exp*
$W_{IV}(1,1,\tau)$	0.28486	0.38651	0.97683
	0.000	*0.000*	*0.000*
$W_{IV}(1,2,\tau)$	0.34409	0.45176	0.99064
	0.000	*0.000*	*0.000*
$W_{IV}(1,3,\tau)$	0.43058	0.43963	0.98835
	0.000	*0.000*	*0.000*
$W_{IV}(1,4,\tau)$	0.27042	0.3326	0.96543
	0.000	*0.000*	*0.000*
$W_{IV}(1,5,\tau)$	0.18172	0.20256	0.93849
	0.000	*0.000*	*0.000*
$W_{IV}(1,6,\tau)$	0.11036	0.11305	0.92002
	0.025	*0.016*	*0.015*
$W_{IV}(1,7,\tau)$	0.079312	0.05133	0.90737
	0.012	*0.051*	*0.050*
$W_{IV}(1,8,\tau)$	0.027267	0.010379	0.89899
	0.289	*0.529*	*0.529*
$W_{IV}(1,9,\tau)$	0.015298	0.012722	0.89946
	0.732	*0.537*	*0.537*
$W_{IV}(1,10,\tau)$	0.16416	0.18611	0.93522
	0.000	*0.001*	*0.001*
$W_{IV}(1,11,\tau)$	0.28215	0.37595	0.97472
	0.000	*0.000*	*0.000*
$W_{IV}(1,12,\tau)$	0.39883	0.59148	1.0199
	0.000	*0.000*	*0.000*
$W_{IV}(1,13,\tau)$	0.34209	0.49583	0.9992
	0.000	*0.000*	*0.000*
$W_{IV}(1,14,\tau)$	0.32182	0.4332	0.98599
	0.000	*0.000*	*0.000*
$W_{IV}(1,15,\tau)$	0.30738	0.42135	0.98373
	0.000	*0.000*	*0.000*
$W_{IV}(1,16,\tau)$	0.28336	0.33451	0.96568
	0.000	*0.000*	*0.000*
$W_{IV}(1,17,\tau)$	0.31334	0.28299	0.95532
	0.000	*0.000*	*0.000*
$W_{IV}(1,18,\tau)$	0.17546	0.14729	0.92717
	0.000	*0.001*	*0.001*
$W_{IV}(1,19,\tau)$	0.23686	0.10196	0.9179
	0.000	*0.018*	*0.018*
$W_{IV}(1,20,\tau)$	0.19447	0.080092	0.91334
	0.000	*0.036*	*0.033*
$W_{IV}(1,21,\tau)$	0.14803	0.063431	0.90988
	0.001	*0.041*	*0.040*

Testing the IRs governing the slope factor

	Sup	*Avg*	*Exp*
$W_{IV}(2,1,\tau)$	0.015325	0.019611	0.90087
	0.565	*0.221*	*0.221*
$W_{IV}(2,2,\tau)$	0.012791	0.007502	0.8984
	0.621	*0.590*	*0.590*
$W_{IV}(2,3,\tau)$	0.083969	0.12711	0.92286
	0.014	*0.001*	*0.001*
$W_{IV}(2,4,\tau)$	0.10422	0.11245	0.9199
	0.002	*0.000*	*0.000*
$W_{IV}(2,5,\tau)$	0.16548	0.22846	0.94382
	0.001	*0.000*	*0.000*
$W_{IV}(2,6,\tau)$	0.13564	0.1945	0.93679
	0.001	*0.000*	*0.000*
$W_{IV}(2,7,\tau)$	0.12321	0.19624	0.93706
	0.005	*0.001*	*0.001*
$W_{IV}(2,8,\tau)$	0.18675	0.22926	0.94386
	0.000	*0.000*	*0.000*
$W_{IV}(2,9,\tau)$	0.12094	0.15915	0.92936
	0.004	*0.001*	*0.001*
$W_{IV}(2,10,\tau)$	0.10936	0.16118	0.92984
	0.003	*0.003*	*0.003*
$W_{IV}(2,11,\tau)$	0.08732	0.12848	0.92316
	0.016	*0.004*	*0.004*
$W_{IV}(2,12,\tau)$	0.068445	0.083145	0.91388
	0.027	*0.008*	*0.008*
$W_{IV}(2,13,\tau)$	0.04162	0.039435	0.90492
	0.101	*0.064*	*0.064*
$W_{IV}(2,14,\tau)$	0.058328	0.060814	0.9093
	0.049	*0.036*	*0.036*
$W_{IV}(2,15,\tau)$	0.028223	0.018894	0.90073
	0.170	*0.192*	*0.192*
$W_{IV}(2,16,\tau)$	0.051298	0.037002	0.90443
	0.012	*0.045*	*0.045*
$W_{IV}(2,17,\tau)$	0.008739	0.002974	0.89747
	0.770	*0.894*	*0.894*
$W_{IV}(2,18,\tau)$	0.006821	0.00286	0.89745
	0.889	*0.911*	*0.911*
$W_{IV}(2,19,\tau)$	0.016391	0.008169	0.89853
	0.451	*0.497*	*0.497*
$W_{IV}(2,20,\tau)$	0.011784	0.012301	0.89937
	0.629	*0.324*	*0.324*
$W_{IV}(2,21,\tau)$	0.011639	0.009328	0.89877
	0.605	*0.416*	*0.416*

Testing the IRs governing the curvature factor

	Sup	*Avg*	*Exp*
$W_{IV}(3,1,\tau)$	0.004254	0.001733	0.89722
	0.963	*0.969*	*0.969*
$W_{IV}(3,2,\tau)$	0.01785	0.019643	0.90087
	0.309	*0.108*	*0.110*
$W_{IV}(3,3,\tau)$	0.047549	0.051524	0.90739
	0.064	*0.014*	*0.014*
$W_{IV}(3,4,\tau)$	0.029769	0.038349	0.9047
	0.305	*0.084*	*0.084*
$W_{IV}(3,5,\tau)$	0.062465	0.048166	0.90671
	0.107	*0.092*	*0.092*
$W_{IV}(3,6,\tau)$	0.028814	0.027685	0.90251
	0.393	*0.209*	*0.209*
$W_{IV}(3,7,\tau)$	0.024056	0.02107	0.90117
	0.566	*0.374*	*0.374*
$W_{IV}(3,8,\tau)$	0.04849	0.023871	0.90176
	0.101	*0.203*	*0.202*
$W_{IV}(3,9,\tau)$	0.011174	0.007633	0.89842
	0.716	*0.602*	*0.603*
$W_{IV}(3,10,\tau)$	0.022095	0.021407	0.90123
	0.682	*0.456*	*0.461*
$W_{IV}(3,11,\tau)$	0.008244	0.004866	0.89786
	0.929	*0.873*	*0.873*
$W_{IV}(3,12,\tau)$	0.023356	0.008579	0.89862
	0.397	*0.608*	*0.608*
$W_{IV}(3,13,\tau)$	0.057081	0.016472	0.90024
	0.341	*0.665*	*0.663*
$W_{IV}(3,14,\tau)$	0.030502	0.01019	0.89895
	0.534	*0.743*	*0.742*
$W_{IV}(3,15,\tau)$	0.039536	0.015681	0.90008
	0.325	*0.495*	*0.494*
$W_{IV}(3,16,\tau)$	0.046383	0.030354	0.90307
	0.235	*0.226*	*0.226*
$W_{IV}(3,17,\tau)$	0.030402	0.021695	0.9013
	0.330	*0.258*	*0.258*
$W_{IV}(3,18,\tau)$	0.12516	0.031969	0.90344
	0.027	*0.409*	*0.408*
$W_{IV}(3,19,\tau)$	0.021461	0.020342	0.90102
	0.352	*0.231*	*0.231*
$W_{IV}(3,20,\tau)$	0.029471	0.026966	0.90237
	0.308	*0.210*	*0.211*
$W_{IV}(3,21,\tau)$	0.016953	0.008696	0.89864
	0.441	*0.520*	*0.520*

TABLE 3.10. Testing results for Datastream US daily discount bond yields. The table reports the *Sup*, *Avg*, and *Exp* values for the test statistics $W_I(\tau), W_{II}(i,\tau), W_{III}(i,\tau), W_{IV}(i,\tau), W_V(i,\tau)$, and $W_{VI}(i,\tau)$ associated with the six hypotheses formulated in equations 2.23 - 2.28. The p-values are reported in italics.

Testing overall system

	Sup	*Avg*	*Exp*
$W_I(\tau)$	0.84868	0.25198	-0.40539
	0.000	*0.000*	*0.000*

Testing the level factor

	Sup	*Avg*	*Exp*
$W_{II}(1,\tau)$	0.23488	0.072509	-0.58508
	0.000	*0.000*	*0.000*
$W_{III}(1,\tau)$	3.9197	1.1069	0.56299
	0.006	*0.001*	*0.006*
$W_V(1,\tau)$	1.0012	0.4269	-0.23818
	0.004	*0.000*	*0.000*

Testing the slope factor

	Sup	*Avg*	*Exp*
$W_{II}(2,\tau)$	0.36233	0.10218	-0.55553
	0.000	*0.000*	*0.000*
$W_{III}(2,\tau)$	160.93	5.2745	73.258
	0.500	*0.491*	*0.501*
$W_V(2,\tau)$	1.9027	0.73165	0.069582
	0.296	*0.050*	*0.058*

Testing the curvature factor

	Sup	*Avg*	*Exp*
$W_{II}(3,\tau)$	0.26432	0.077286	-0.57981
	0.000	*0.000*	*0.000*
$W_{III}(3,\tau)$	272	4.5427	128.79
	0.489	*0.524*	*0.490*
$W_V(3,\tau)$	1.7474	0.59921	-0.06578
	0.507	*0.051*	*0.081*

Testing the common factors

	Sup	*Avg*	*Exp*
$W_{VI}(1,2,\tau)$	5.9162	1.9401	1.409
	0.116	*0.001*	*0.081*
$W_{VI}(1,3,\tau)$	2.4814	1.0453	0.37833
	0.001	*0.000*	*0.000*
$W_{VI}(2,3,\tau)$	4.1397	1.4937	0.84192
	0.290	*0.009*	*0.032*

Testing the IRs governing the level factor

	Sup	*Avg*	*Exp*
$W_{IV}(1,1,\tau)$	0.092151	0.00638	-0.64907
	0.006	*0.165*	*0.164*
$W_{IV}(1,2,\tau)$	0.23507	0.054486	-0.60199
	0.006	*0.000*	*0.000*
$W_{IV}(1,3,\tau)$	0.15922	0.037041	-0.61925
	0.009	*0.001*	*0.001*
$W_{IV}(1,4,\tau)$	0.63385	0.13326	-0.52019
	0.006	*0.000*	*0.000*
$W_{IV}(1,5,\tau)$	1.3623	0.27524	-0.35856
	0.006	*0.000*	*0.001*
$W_{IV}(1,6,\tau)$	2.17	0.42767	-0.16188
	0.006	*0.000*	*0.001*
$W_{IV}(1,7,\tau)$	2.7844	0.53535	-0.01017
	0.006	*0.000*	*0.002*
$W_{IV}(1,8,\tau)$	3.0918	0.60334	0.082748
	0.006	*0.000*	*0.002*
$W_{IV}(1,9,\tau)$	3.0438	0.61671	0.086453
	0.006	*0.000*	*0.001*
$W_{IV}(1,10,\tau)$	2.8236	0.59699	0.042164
	0.006	*0.000*	*0.001*
$W_{IV}(1,11,\tau)$	0.59444	0.1581	-0.50021
	0.000	*0.000*	*0.000*
$W_{IV}(1,12,\tau)$	0.69106	0.17597	-0.48064
	0.000	*0.000*	*0.000*
$W_{IV}(1,13,\tau)$	0.99383	0.22111	-0.42984
	0.000	*0.000*	*0.000*
$W_{IV}(1,14,\tau)$	0.97272	0.20116	-0.4492
	0.000	*0.000*	*0.000*
$W_{IV}(1,15,\tau)$	1.0111	0.22775	-0.42259
	0.000	*0.000*	*0.000*
$W_{IV}(1,16,\tau)$	1.0126	0.22804	-0.42248
	0.000	*0.000*	*0.000*
$W_{IV}(1,17,\tau)$	0.9625	0.21737	-0.43405
	0.000	*0.000*	*0.000*
$W_{IV}(1,18,\tau)$	0.91251	0.20748	-0.44469
	0.000	*0.000*	*0.000*
$W_{IV}(1,19,\tau)$	0.84541	0.19331	-0.45993
	0.000	*0.000*	*0.000*
$W_{IV}(1,20,\tau)$	0.84564	0.18812	-0.46463
	0.000	*0.000*	*0.000*
$W_{IV}(1,21,\tau)$	0.82034	0.18007	-0.47289
	0.000	*0.000*	*0.000*

Testing the IRs governing the slope factor

	Sup	*Avg*	*Exp*
$W_{IV}(2,1,\tau)$	0.095025	0.024061	-0.63198
	0.002	*0.000*	*0.000*
$W_{IV}(2,2,\tau)$	0.17489	0.053578	-0.60329
	0.005	*0.000*	*0.000*
$W_{IV}(2,3,\tau)$	0.23434	0.066188	-0.59114
	0.000	*0.000*	*0.000*
$W_{IV}(2,4,\tau)$	0.59242	0.19319	-0.46601
	0.000	*0.000*	*0.000*
$W_{IV}(2,5,\tau)$	0.8257	0.2881	-0.36919
	0.000	*0.000*	*0.000*
$W_{IV}(2,6,\tau)$	1.0392	0.32381	-0.33058
	0.000	*0.000*	*0.000*
$W_{IV}(2,7,\tau)$	1.147	0.30557	-0.34885
	0.000	*0.000*	*0.000*
$W_{IV}(2,8,\tau)$	1.2904	0.323	-0.329
	0.000	*0.000*	*0.000*
$W_{IV}(2,9,\tau)$	1.1901	0.29856	-0.35484
	0.000	*0.000*	*0.000*
$W_{IV}(2,10,\tau)$	1.0135	0.25712	-0.39859
	0.000	*0.000*	*0.000*
$W_{IV}(2,11,\tau)$	0.99718	0.036843	-0.61754
	0.461	*0.523*	*0.541*
$W_{IV}(2,12,\tau)$	1.2779	0.083914	-0.57091
	0.452	*0.420*	*0.455*
$W_{IV}(2,13,\tau)$	4.1216	0.32528	-0.30783
	0.327	*0.308*	*0.403*
$W_{IV}(2,14,\tau)$	4.9877	0.33903	-0.26816
	0.304	*0.324*	*0.421*
$W_{IV}(2,15,\tau)$	5.725	0.8532	0.28911
	0.328	*0.195*	*0.326*
$W_{IV}(2,16,\tau)$	6.399	1.2087	0.76746
	0.323	*0.155*	*0.276*
$W_{IV}(2,17,\tau)$	6.1186	1.1535	0.66721
	0.320	*0.155*	*0.274*
$W_{IV}(2,18,\tau)$	5.6919	0.92114	0.35332
	0.309	*0.173*	*0.288*
$W_{IV}(2,19,\tau)$	5.2038	0.75599	0.14752
	0.289	*0.183*	*0.295*
$W_{IV}(2,20,\tau)$	4.4232	0.69438	0.073502
	0.312	*0.186*	*0.280*
$W_{IV}(2,21,\tau)$	3.7806	0.61647	-0.01507
	0.328	*0.191*	*0.279*

Testing the IRs governing the curvature factor

	Sup	*Avg*	*Exp*
$W_{IV}(3,1,\tau)$	0.51604	0.051075	-0.60558
	0.516	*0.396*	*0.422*
$W_{IV}(3,2,\tau)$	0.82403	0.072583	-0.58319
	0.407	*0.328*	*0.354*
$W_{IV}(3,3,\tau)$	1.2715	0.11658	-0.53863
	0.337	*0.291*	*0.329*
$W_{IV}(3,4,\tau)$	1.4784	0.16218	-0.48281
	0.348	*0.259*	*0.292*
$W_{IV}(3,5,\tau)$	1.1359	0.13989	-0.50373
	0.342	*0.235*	*0.254*
$W_{IV}(3,6,\tau)$	0.48934	0.035675	-0.61938
	0.300	*0.370*	*0.373*
$W_{IV}(3,7,\tau)$	0.15511	0.010385	-0.64525
	0.002	*0.083*	*0.083*
$W_{IV}(3,8,\tau)$	1.0181	0.062826	-0.59261
	0.009	*0.272*	*0.283*
$W_{IV}(3,9,\tau)$	1.4895	0.11718	-0.53549
	0.165	*0.284*	*0.312*
$W_{IV}(3,10,\tau)$	1.2432	0.12581	-0.52895
	0.247	*0.257*	*0.284*
$W_{IV}(3,11,\tau)$	0.5929	0.13048	-0.52454
	0.018	*0.001*	*0.001*
$W_{IV}(3,12,\tau)$	0.34933	0.098085	-0.5585
	0.021	*0.000*	*0.000*
$W_{IV}(3,13,\tau)$	0.55416	0.12692	-0.52678
	0.000	*0.000*	*0.000*
$W_{IV}(3,14,\tau)$	0.14352	0.024502	-0.63153
	0.000	*0.000*	*0.000*
$W_{IV}(3,15,\tau)$	0.049134	0.005316	-0.65015
	0.000	*0.028*	*0.028*
$W_{IV}(3,16,\tau)$	0.097715	0.011626	-0.64405
	0.115	*0.204*	*0.206*
$W_{IV}(3,17,\tau)$	0.29506	0.066789	-0.5897
	0.004	*0.002*	*0.002*
$W_{IV}(3,18,\tau)$	0.52904	0.14248	-0.51194
	0.001	*0.000*	*0.000*
$W_{IV}(3,19,\tau)$	0.76198	0.20256	-0.44785
	0.000	*0.000*	*0.000*
$W_{IV}(3,20,\tau)$	0.70479	0.19966	-0.45123
	0.000	*0.000*	*0.000*
$W_{IV}(3,21,\tau)$	0.63873	0.18	-0.47233
	0.000	*0.000*	*0.000*

TABLE 3.11. Stability results for US zero coupon bond yield short maturity cluster. The table reports the *Sup*, *Avg*, and *Exp* values for the test statistics $W_I(\tau), W_{II}(i,\tau), W_{III}(i,\tau), W_{IV}(i,\tau), W_V(i,\tau)$, and $W_{VI}(i,\tau)$ associated with the six hypotheses formulated in equations 2.23 - 2.28. The p-values are reported in italics.

Testing overall system

	Sup	*Avg*	*Exp*
$W_I(\tau)$	1.1535	0.29814	-0.35658
	0.000	*0.000*	*0.000*

Testing the level factor

	Sup	*Avg*	*Exp*
$W_{II}(1,\tau)$	0.36777	0.10295	-0.55285
	0.000	*0.000*	*0.000*
$W_{III}(1,\tau)$	0.29332	0.082537	-0.57345
	0.008	*0.004*	*0.004*
$W_V(1,\tau)$	0.78625	0.2258	-0.43127
	0.000	*0.000*	*0.000*

Testing the slope factor

	Sup	*Avg*	*Exp*
$W_{II}(2,\tau)$	0.40144	0.11431	-0.54194
	0.000	*0.000*	*0.000*
$W_{III}(2,\tau)$	0.84423	0.14078	-0.51059
	0.000	*0.000*	*0.000*
$W_V(2,\tau)$	1.0648	0.26789	-0.38648
	0.000	*0.000*	*0.000*

Testing the IRs governing the level factor

	Sup	*Avg*	*Exp*
$W_{IV}(1,1,\tau)$	0.16119	0.027609	-0.62634
	0.000	*0.004*	*0.004*
$W_{IV}(1,2,\tau)$	0.14635	0.014668	-0.63897
	0.000	*0.004*	*0.004*
$W_{IV}(1,3,\tau)$	0.16219	0.027949	-0.62613
	0.000	*0.000*	*0.000*
$W_{IV}(1,4,\tau)$	0.20911	0.032481	-0.62173
	0.000	*0.000*	*0.000*
$W_{IV}(1,5,\tau)$	0.13662	0.022785	-0.63119
	0.000	*0.001*	*0.001*
$W_{IV}(1,6,\tau)$	0.06753	0.005748	-0.64765
	0.020	*0.233*	*0.232*
$W_{IV}(1,7,\tau)$	0.12997	0.019584	-0.63415
	0.000	*0.002*	*0.002*
$W_{IV}(1,8,\tau)$	0.21198	0.031176	-0.62291
	0.000	*0.000*	*0.000*
$W_{IV}(1,9,\tau)$	0.22322	0.028553	-0.62549
	0.000	*0.000*	*0.000*
$W_{IV}(1,10,\tau)$	0.22734	0.024714	-0.6292
	0.000	*0.000*	*0.000*

Testing the curvature factor

	Sup	*Avg*	*Exp*
$W_{II}(3,\tau)$	0.38429	0.080879	-0.5744
	0.000	*0.000*	*0.000*
$W_{III}(3,\tau)$	1.2638	0.11471	-0.53653
	0.049	*0.029*	*0.034*
$W_V(3,\tau)$	0.68481	0.15715	-0.49974
	0.029	*0.003*	*0.003*

Testing the common factors

	Sup	*Avg*	*Exp*
$W_{VI}(1,2,\tau)$	1.9795	0.50845	-0.13066
	0.000	*0.000*	*0.000*
$W_{VI}(1,3,\tau)$	1.3586	0.3802	-0.27518
	0.002	*0.001*	*0.001*
$W_{VI}(2,3,\tau)$	1.8104	0.43972	-0.20621
	0.001	*0.000*	*0.000*

Testing the IRs governing the slope factor

	Sup	*Avg*	*Exp*
$W_{IV}(2,1,\tau)$	0.115	0.014545	-0.63906
	0.001	*0.003*	*0.003*
$W_{IV}(2,2,\tau)$	0.030452	0.003056	-0.65026
	0.004	*0.059*	*0.059*
$W_{IV}(2,3,\tau)$	0.15285	0.028869	-0.62524
	0.001	*0.001*	*0.001*
$W_{IV}(2,4,\tau)$	0.15578	0.015671	-0.63801
	0.000	*0.002*	*0.002*
$W_{IV}(2,5,\tau)$	0.18094	0.01812	-0.63564
	0.000	*0.000*	*0.000*
$W_{IV}(2,6,\tau)$	0.19508	0.018849	-0.63493
	0.000	*0.000*	*0.000*
$W_{IV}(2,7,\tau)$	0.15172	0.014427	-0.63923
	0.000	*0.000*	*0.000*
$W_{IV}(2,8,\tau)$	0.1022	0.016751	-0.63699
	0.000	*0.001*	*0.001*
$W_{IV}(2,9,\tau)$	0.065471	0.008475	-0.64501
	0.006	*0.009*	*0.009*
$W_{IV}(2,10,\tau)$	0.031184	0.00475	-0.64863
	0.030	*0.044*	*0.044*

Testing the IRs governing the curvature factor

	Sup	*Avg*	*Exp*
$W_{IV}(3,1,\tau)$	0.04877	0.011848	-0.6418
	0.031	*0.005*	*0.005*
$W_{IV}(3,2,\tau)$	0.13366	0.012844	-0.64072
	0.024	*0.030*	*0.030*
$W_{IV}(3,3,\tau)$	0.098076	0.020779	-0.6331
	0.155	*0.032*	*0.033*
$W_{IV}(3,4,\tau)$	0.17229	0.014599	-0.63906
	0.078	*0.049*	*0.049*
$W_{IV}(3,5,\tau)$	0.41218	0.032071	-0.62194
	0.042	*0.019*	*0.019*
$W_{IV}(3,6,\tau)$	0.5223	0.038496	-0.61536
	0.026	*0.015*	*0.017*
$W_{IV}(3,7,\tau)$	0.11621	0.017317	-0.63643
	0.063	*0.021*	*0.021*
$W_{IV}(3,8,\tau)$	0.045311	0.005598	-0.6478
	0.009	*0.048*	*0.047*
$W_{IV}(3,9,\tau)$	0.33049	0.021065	-0.63266
	0.055	*0.039*	*0.039*
$W_{IV}(3,10,\tau)$	0.39263	0.022499	-0.63117
	0.038	*0.029*	*0.029*

TABLE 3.12. Stability results for US zero coupon bond yield long maturity cluster. The table reports the *Sup*, *Avg*, and *Exp* values for the test statistics $W_I(\tau), W_{II}(i,\tau), W_{III}(i,\tau), W_{IV}(i,\tau), W_V(i,\tau)$, and $W_{VI}(i,\tau)$ associated with the six hypotheses formulated in equations 2.23 - 2.28. The p-values are reported in italics.

Testing overall system

	Sup	*Avg*	*Exp*
$W_I(\tau)$	0.35834	0.10035	-0.55531
	0.000	*0.000*	*0.000*

Testing the level factor

	Sup	*Avg*	*Exp*
$W_{II}(1,\tau)$	0.16774	0.041898	-0.61264
	0.000	*0.000*	*0.000*
$W_{III}(1,\tau)$	0.26598	0.072521	-0.58295
	0.000	*0.000*	*0.000*
$W_V(1,\tau)$	0.41544	0.11307	-0.54289
	0.000	*0.000*	*0.000*

Testing the slope factor

	Sup	*Avg*	*Exp*
$W_{II}(2,\tau)$	0.19124	0.040006	-0.6143
	0.000	*0.000*	*0.000*
$W_{III}(2,\tau)$	1.1325	0.15493	-0.49045
	0.000	*0.000*	*0.000*
$W_V(2,\tau)$	0.62588	0.14121	-0.51243
	0.000	*0.000*	*0.000*

Testing the IRs governing the level factor

	Sup	*Avg*	*Exp*
$W_{IV}(1,1,\tau)$	0.12723	0.026686	-0.62732
	0.000	*0.000*	*0.000*
$W_{IV}(1,2,\tau)$	0.085575	0.02065	-0.63327
	0.000	*0.000*	*0.000*
$W_{IV}(1,3,\tau)$	0.14344	0.035964	-0.6184
	0.000	*0.000*	*0.000*
$W_{IV}(1,4,\tau)$	0.063585	0.01148	-0.64213
	0.000	*0.000*	*0.000*
$W_{IV}(1,5,\tau)$	0.060129	0.006262	-0.64715
	0.000	*0.000*	*0.000*
$W_{IV}(1,6,\tau)$	0.18496	0.027153	-0.62679
	0.000	*0.000*	*0.000*
$W_{IV}(1,7,\tau)$	0.16224	0.035759	-0.61851
	0.000	*0.000*	*0.000*
$W_{IV}(1,8,\tau)$	0.19816	0.038455	-0.61584
	0.000	*0.000*	*0.000*
$W_{IV}(1,9,\tau)$	0.1926	0.036836	-0.61744
	0.000	*0.000*	*0.000*
$W_{IV}(1,10,\tau)$	0.1647	0.033922	-0.62033
	0.000	*0.000*	*0.000*
$W_{IV}(1,11,\tau)$	0.14942	0.032236	-0.62199
	0.000	*0.000*	*0.000*

Testing the curvature factor

	Sup	*Avg*	*Exp*
$W_{II}(3,\tau)$	0.13384	0.018448	-0.63518
	0.000	*0.000*	*0.000*
$W_{III}(3,\tau)$	7.9842	0.63988	0.7395
	0.049	*0.010*	*0.033*
$W_V(3,\tau)$	0.7587	0.15399	-0.50284
	0.097	*0.004*	*0.005*

Testing the common factors

	Sup	*Avg*	*Exp*
$W_{VI}(1,2,\tau)$	0.95086	0.23044	-0.42188
	0.000	*0.000*	*0.000*
$W_{VI}(1,3,\tau)$	1.0502	0.25334	-0.4035
	0.038	*0.002*	*0.003*
$W_{VI}(2,3,\tau)$	1.2629	0.28943	-0.36415
	0.013	*0.001*	*0.002*

Testing the IRs governing the slope factor

	Sup	*Avg*	*Exp*
$W_{IV}(2,1,\tau)$	0.115	0.014545	-0.63906
	0.001	*0.003*	*0.003*
$W_{IV}(2,2,\tau)$	0.030452	0.003056	-0.65026
	0.004	*0.059*	*0.059*
$W_{IV}(2,3,\tau)$	0.15285	0.028869	-0.62524
	0.001	*0.001*	*0.001*
$W_{IV}(2,4,\tau)$	0.15578	0.015671	-0.63801
	0.000	*0.002*	*0.002*
$W_{IV}(2,5,\tau)$	0.18094	0.01812	-0.63564
	0.000	*0.000*	*0.000*
$W_{IV}(2,6,\tau)$	0.19508	0.018849	-0.63493
	0.000	*0.000*	*0.000*
$W_{IV}(2,7,\tau)$	0.15172	0.014427	-0.63923
	0.000	*0.000*	*0.000*
$W_{IV}(2,8,\tau)$	0.1022	0.016751	-0.63699
	0.000	*0.001*	*0.001*
$W_{IV}(2,9,\tau)$	0.065471	0.008475	-0.64501
	0.006	*0.009*	*0.009*
$W_{IV}(2,10,\tau)$	0.031184	0.00475	-0.64863
	0.030	*0.044*	*0.044*

Testing the IRs governing the curvature factor

	Sup	*Avg*	*Exp*
$W_{IV}(3,1,\tau)$	0.04877	0.011848	-0.6418
	0.031	*0.005*	*0.005*
$W_{IV}(3,2,\tau)$	0.13366	0.012844	-0.64072
	0.024	*0.030*	*0.030*
$W_{IV}(3,3,\tau)$	0.098076	0.020779	-0.6331
	0.155	*0.032*	*0.033*
$W_{IV}(3,4,\tau)$	0.17229	0.014599	-0.63906
	0.078	*0.049*	*0.049*
$W_{IV}(3,5,\tau)$	0.41218	0.032071	-0.62194
	0.042	*0.019*	*0.019*
$W_{IV}(3,6,\tau)$	0.5223	0.038496	-0.61536
	0.026	*0.015*	*0.017*
$W_{IV}(3,7,\tau)$	0.11621	0.017317	-0.63643
	0.063	*0.021*	*0.021*
$W_{IV}(3,8,\tau)$	0.045311	0.005598	-0.6478
	0.009	*0.048*	*0.047*
$W_{IV}(3,9,\tau)$	0.33049	0.021065	-0.63266
	0.055	*0.039*	*0.039*
$W_{IV}(3,10,\tau)$	0.39263	0.022499	-0.63117
	0.038	*0.029*	*0.029*

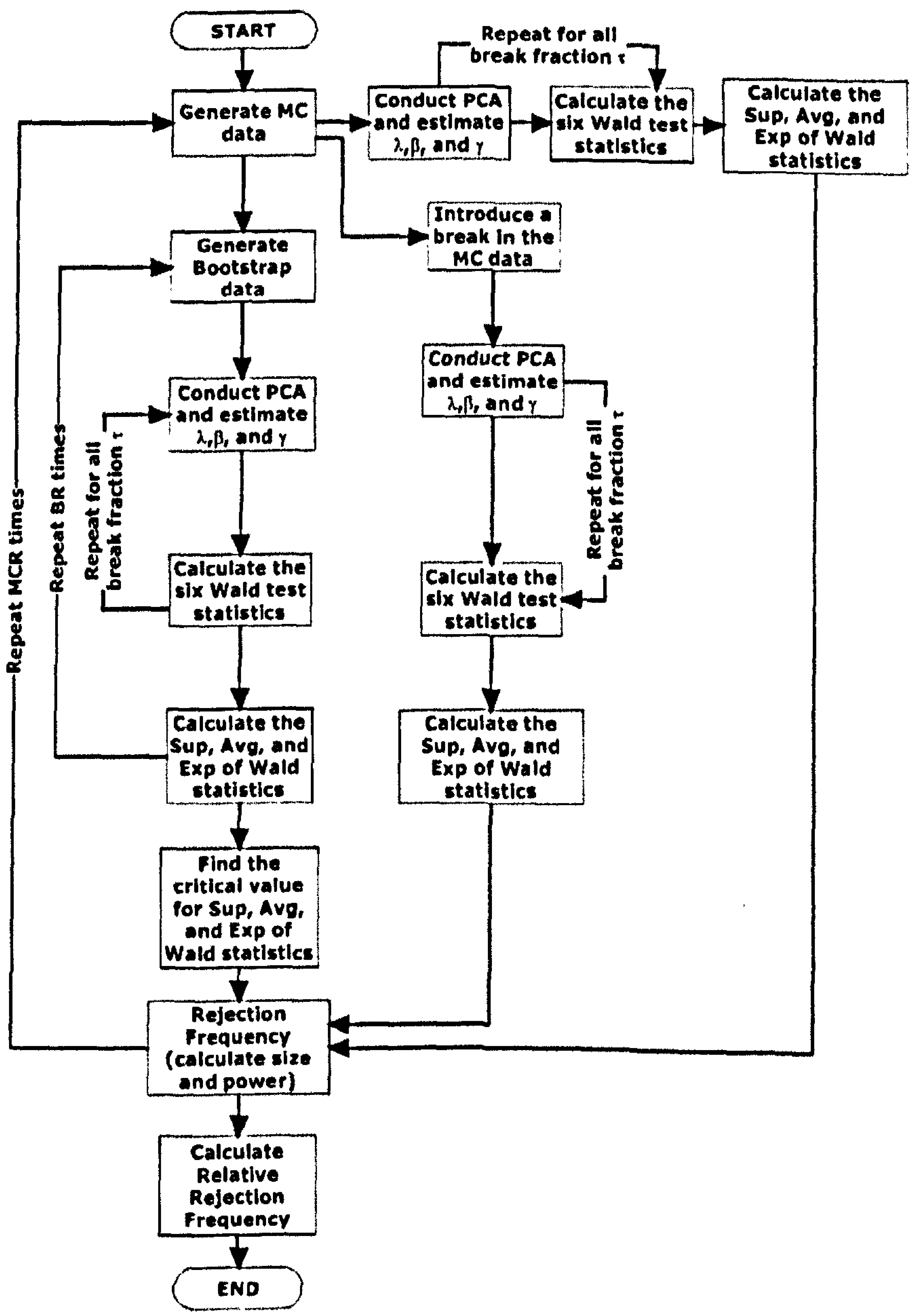


FIGURE 3.1. Flow chart of the simulation study evaluating size and power properties

Chapter 4

ESTIMATION OF FACTORS FOR TERM STRUCTURES WITH DEPENDENCE CLUSTERS

ABSTRACT

In estimating term structure factors, a common dependence structure between maturities is implicitly assumed. In this chapter we study interactions between factors in the presence of multiple dependencies (with short and long maturity clusters) within a term structure. We introduce the block dynamic Nelson-Siegel (1987) model for term structures with maturity clusters for the purpose of forecasting. This new framework generalizes the dynamic representation proposed by Diebold and Li (2006) for constructing yield curve forecasts of the Nelson-Siegel factors and relaxes the assumption of common factor dynamics among clusters within a term structure.

In the case of zero coupon term structures, we identify the factors for the short and long maturity clusters separately. Using dependence graphs such as Chi-plots and recursive Kendall plots, we find that factors governing the short and long clusters show loose dependence and therefore measuring factors over separate maturity clusters would lead to significant information gains. Application of the block dynamic Nelson-Siegel (1987) model on the term structure of daily zero coupon bond yields with short and long maturity clusters show better out-of-sample forecasting performance than the dynamic representation proposed by Diebold and Li (2006).

4.1 Introduction

Modelling of the zero coupon yield curve, particularly interesting to financial economists and central banks, have been towards capturing the dynamics of the entire yield curve. Since expectations about future short rates influence movements in the long rates, changes in the short end and the long end maturity spectrum of the yield curve show a strong dependence structure. However, the magnitudes of changes along the maturity spectrum vary, producing linear and nonlinear shifts in the yield curve.

Since the interest rate maturities portray a strong dependence structure among themselves, latent factor models have been widely used in order to extract the systematic movements in yield curves. There is common consensus that a few systematic yield shifts are sufficient in explaining the fluctuations in yield curves. Generally, the dependence structure among maturities are measured with pairwise correlations or covariances; and the common factors are estimated using statistical tools such as principal component analysis (PCA) and factor analysis. The first three principal components or factors are commonly known as level, slope, and curvature; cumulatively accounting for almost all the variations in the yield curve. Since the correlation structure among yield maturities are, though high, not equal to unity and decreasing with increase in difference in maturities; non-parallel movements such as slope shifts and curvature shifts are significant in measuring the non-parallel risks within the term structure. However, the parallel shift factor (level) explains most of the variations in the term structure.

In describing the evolution of interest rates, multivariate models capturing the time series properties of interest rates have been developed. The aim is to introduce tractable multivariate models that can capture both the inter-temporal properties and also provide a theoretically consistent way of explaining the cross-section of the yields. In estimation and forecasting of the yield curve, a popular function-based factor model is the Nelson and Siegel (1987) model and its extensions such as the Svensson (1994) model. Since the factor models are able to produce typically observed yield curve shapes in the market, these models are widely used among practitioners. Fabozzi et al. (2005), using a three factor Nelson-Siegel model, find statistically and economically meaningful predictions in the shape of the yield curve. The authors implemented systematic trading strategies and find economical gains using the model predictions. For the purpose of constructing term structure forecasts, Diebold and Li (2006) fit a dynamic factor model to the Nelson-Siegel factors and find accurate forecast performance than various standard benchmarks. The authors allow the factors (level, slope, and curvature) to follow a vector autoregressive process capturing the whole yield curve dynamics over time.

In this case, the underlying dynamics of the factors are constrained to be common for all maturities. This is to say that the dynamics underlying the short end and the long end factors are identical. However, empirical evidence suggests that the nature of uncertainties influencing the short rates is different than the ones affecting the long rates. One reason for this is the diverse nature and varied preferences of market participants influencing the different ends of the yield curve. This is further reflected in the fact that volatility of the short rates are higher than the volatility of the long rates. Sarkar and Ariff (2002) investigated the role of maturities in the effect of volatility. The authors found a negative relationship between interest rate volatility and US treasury yields. The effect was seen to be much stronger in the case of long maturities than short maturities. Backus and Zin (1994) and Gong and Remolona (1997) document that the mean reversion for yields near the short end of the yield curve are much faster than for yields near the long end of the curve. Higher mean reversion is implied by the yield curve steepness at the short end and lower mean reversion is implied by flat volatility curve at the long end maturities.

In this chapter, we study interactions between factors governing term structures with maturity clusters. This scenario commonly arise in practice where yield curve segments are proxied by various securities. In the case of zero coupon bond yield curves, one commonly use the yields from treasury bills as a proxy of the short end of the curve and discount bonds or swap rates as proxies to the long end of the curve. Using dependence graphs, we find a loose dependence structure between the principal component factors governing the short and long maturity clusters. Therefore accounting for the two clusters may lead to significant information gains. We study this by developing a block dynamic Nelson-Siegel factor model for term structures with maturity clusters and investigate the forecasting performance of this model. We find significant improvements in forecastability of the term structure over the benchmark model.

The remainder of this chapter is structured as follows. In Section 4.2 and Section 4.3 we estimate the principal component factors governing the maturity clusters in the term structure of zero coupon yields and study the dependence structures between factors using Chi-plots and K-plots. Section 4.4 introduces a new block dynamic specification to the Nelson-Siegel (1987) model for modelling term structures with maturity clusters. In this, we estimate the Nelson-Siegel factors for the various maturity clusters and allow the dynamics of the factors to be common within clusters. This extension would enable us to investigate the information loss in estimating factors of term structures containing multiple dependence structures. Section 4.5 conducts an empirical application of the new block dynamic model and examines the forecasting performance of the model. Section 4.6 concludes.

4.2 PCA for correlation clusters

In estimating common factors governing term structures, presence of multiple data clusters within a single term structure would distort the estimation of true systematic shifts (level, slope, and curvature) in the yield curve. This issue is most eminent in the statistical procedures such as PCA that are implemented in order to extract the true factors. PCA assumes a stable contemporaneous correlation structure among the yield curve maturities. However, within a term structure, one may find several correlation clusters linearly or non-linearly correlated in the same way. Conducting PCA decomposition to term structures with multiple correlation clusters would estimate factors influencing only certain maturities of the yield curve.

We use the US zero coupon yield term structure with constant maturities of 3, 4, ..., 12, 24, ..., 144 months (21 maturities) for our empirical study. Figure *4.1* is the matrix plot of the yield maturities.

[Insert Figure 4.1 here]

We can find that the term structure data obtained show the presence of two distinct maturity clusters, one with maturities 3 to 12 months (we call it short cluster) of treasury bills and the second with maturities 2 years to 12 years (we call it long cluster) of discount bonds. In the presence of correlation clusters, one can estimate the factors for the individual clusters and study the interactions between the factors extracted from the two clusters.

Consider a $(p+q)$ dimensional vector $z_t = (x_t', y_t')'$ for $t = 1, ..., T$ where z_t is partitioned into p - dimensional subvector x_t and q - dimensional subvector y_t explaining different aspects of the yield curve. The covariance matrix of z_t can be partitioned as

$$\Sigma_{zz} = \begin{bmatrix} \Sigma_{xx} & \Sigma_{xy} \\ \Sigma_{yx} & \Sigma_{yy} \end{bmatrix} \tag{4.1}$$

For given two partitions X and Y, using principal component analysis we find orthogonal linear transformations of X and Y independently by minimizing the sum of squared Euclidean distances between principal axes obtained and the datapoints. The minimization problem can equivalently considered as maximization of sum of squared projections onto these principal axes. Therefore, the maximization for the two partitions are as follows:

$$\underset{\beta_x:|\beta_x|=1}{\arg\max} \left(\beta_x' X\right) \left(\beta_x' X\right)' \text{ and } \underset{\beta_y:|\beta_y|=1}{\arg\max} \left(\beta_y' Y\right) \left(\beta_y' Y\right)' \tag{4.2}$$

Let the sample covariance matrix $\hat{\Sigma}_{xx} = \frac{1}{n}(XX')$ and $\hat{\Sigma}_{yy} = \frac{1}{n}(YY')$. Therefore the constraint optimization problem is

$$\arg\max_{\beta_x:|\beta_x|=1} \beta_x' \hat{\Sigma}_{xx} \beta_x \text{ and } \arg\max_{\beta_y:|\beta_y|=1} \beta_y' \hat{\Sigma}_{yy} \beta_y$$

The Lagrangian multipliers for the two optimizations are

$$\beta_x' \hat{\Sigma}_{xx} \beta_x - \lambda_x \left(\beta_x' \beta_x - 1\right)$$
$$\beta_y' \hat{\Sigma}_{yy} \beta_y - \lambda_y \left(\beta_y' \beta_y - 1\right)$$

Differentiating the equations reduces to the eigen decomposition problem

$$\begin{aligned} \hat{\Sigma}_{xx}\beta_x &= \lambda_x \beta_x \\ \hat{\Sigma}_{yy}\beta_y &= \lambda_y \beta_y \end{aligned}$$

where λ is the eigenvalues and β is the corresponding eigenvector. The solution to this optimization yields the first principal component for the two partitions: $U_1 = \beta_{1x}' X$ and $V_1 = \beta_{1y}' Y$ and the magnitude of variation explained by the first principal component for the two partitions is given by their eigenvalues λ_{1x} and λ_{1y}. This procedure can be repeated in order to find the successive principal components estimated with the additional optimization constraint that the new principal component is orthogonal to the previous one.

Let $\Lambda_x = (\lambda_{1x}, ..., \lambda_{px})$ be the ordered eigenvalues of $\hat{\Sigma}_{xx}$ and $\beta_x = \left(\beta_{1x}, ..., \beta_{px}\right)$ be its corresponding eigenvectors. Similarly, let $\Lambda_y = (\lambda_{1y}, ..., \lambda_{qy})$ be the ordered eigenvalues of $\hat{\Sigma}_{yy}$ and $\beta_y = \left(\beta_{1y}, ..., \beta_{qy}\right)$ be its corresponding eigenvectors. The principal components of the partitions X and Y are given by $\mathbf{U} = (U_1, ..., U_p)$ and $\mathbf{V} = (V_1, ..., V_q)$ respectively.

The correlations between the principal components of the two partitions X and Y can be calculated as

$$\hat{\rho}_{ij} \equiv \frac{cov\,(U_i V_j)}{\sqrt{Var\,(U_i)\,Var\,(V_j)}} = \lambda_{ix}^{-1/2}\beta_{ix}'\hat{\Sigma}_{xy}\beta_{jy}\lambda_{jy}^{-1/2} \quad (4.3)$$

where $i = 1, ..., p$ and $j = 1, ..., q$.

A recent paper Yamamoto et al. (2007) provide the limiting distribution for the correlation coefficient $\hat{\rho}_{ij}$ for two sets of variables that are bivariate normal. Suppose $z_1, ..., z_T$ are observations from $N(\mu, \Sigma)$, then

$$\sqrt{T}\left(\hat{\rho}_{ij} - \rho_{ij}\right) \xrightarrow{d} N(0, \zeta_{ij}^2)$$

where the form of ζ_{ij}^2 is shown using the asymptotic expansion of $\hat{\rho}_{ij}$ in the paper.

[Insert Table 4.1 here]

[Insert Figure 4.2 here]

Table *4.1* reports the principal component results for the short and long maturity clusters. We find that the principal three factors cumulative explain 99 percent of the variations in both clusters. Plotting the three factor loadings for the two clusters in Figure *4.2,* we find that they retain the economic interpretations of being level, slope, and curvature. From the table we see that the level factor explains almost all of the variations in the short maturity segment of the yield curve. In the case of the long maturity cluster, we see that the level factor alone explains 95 percent of the variance and cumulatively the three factors are sufficient in capturing most of the variations in the long maturity segment.

4.3 Dependencies between term structure factors

In this section we explore the dependence structure between the factors driving the short and long clusters of the yield curve. We use two graphical diagnostics, namely Chi-plots and K-plots, for assessing the degree of association between the level, slope, and curvature factors governing the two clusters of the yield curve. In this, we aim to graphically study the significance of the dependence structure between factors.

The use of scatter plots are usually employed in understanding the dependence between variables. However, these kinds of graphs are limited in accessing the nature of dependence between variables. Fisher and Switzer (1985, 2001) proposed the use of Chi-plots as an important tool towards revealing complex dependence structures between two variables. These plots are based on the ranks of the observations and able to explain independence, monotonic dependence, asymmetries, tail dependencies, etc. In finance literature, authors have used Chi-plots in order to assess tail properties of securities and choose suitable copulas in modelling the underlying dependencies between variables (see Abberger (2002)). Another measure for studying dependencies is the Kendall Plot (known as K-plots) proposed by Genest and Boies (2003). As in the case of Chi-plots, K-plots are also based on probability integral transforms of observations but easier to interpret that Chi-plots.

Below we briefly describe the two graphical measures of dependence and present the results from implementation of these measures on factors governing the long and short term structure clusters.

Chi-plots

Consider a random sample $(X_1, Y_1), ..., (X_T, Y_T)$ from a bivariate continuous distribution H. Let F and G be the marginal distributions of X and Y respectively. For a given pair (X_i, Y_i) with $1 \leq i \leq T$, we define

$$H_i \equiv H(X_i, Y_i) = \frac{1}{T-1} \sum_{j \neq i} I(X_j \leq X_i, Y_j \leq Y_i) \tag{4.4}$$

$$F_i \equiv F(X_i) = \frac{1}{T-1}\sum_{j\neq i} I(X_j \leq X_i) \tag{4.5}$$

$$G_i \equiv G(Y_i) = \frac{1}{T-1}\sum_{j\neq i} I(Y_j \leq Y_i) \tag{4.6}$$

where $I(A)$ is the indicator function of the event A.

If X and Y are independent, then for a given pair (X_i, Y_i) we have $H_i = F_iG_i$. In evaluating independence between (X_i, Y_i) for $i = 1, ..., T$, Fisher and Switzer (1985) construct a test statistic

$$\chi_i = \frac{1}{\sqrt{T}}(H_i - F_iG_i) \xrightarrow{d} N(0, V_i) \tag{4.7}$$

where

$$V_i = F_i(1-F_i)G_i(1-G_i) \tag{4.8}$$

Therefore for a given pair (X_i, Y_i), the $\chi_i - transform$ defined as

$$\frac{H_i - F_iG_i}{\sqrt{F_i(1-F_i)G_i(1-G_i)}} \tag{4.9}$$

is a measure of departure from bivariate independence. χ_i lie in the interval $[-1, 1]$ and acts as a correlation coefficient between (X_i, Y_i).

Further, Fisher and Switzer (1985) propose the data transform λ_i, a real valued function of marginal frequencies. The authors use

$$\lambda_i = 4 \cdot S_i \cdot \max\left\{\left(F_i - \frac{1}{2}\right)^2, \left(G_i - \frac{1}{2}\right)^2\right\} \tag{4.10}$$

where $S_i = sign\left(F_i - \frac{1}{2}\right) \times \left(G_i - \frac{1}{2}\right)$. The value of λ_i is a measure of distance of the pairs (X_i, Y_i) from the bivariate median of the distribution and $\lambda_i \in [-1, 1]$.

A scatter plot of the T transformed pairs (χ_i, λ_i) defines the Chi-plot and provides a meaningful rank based indication of dependence between X and Y.

The authors recommend avoiding the boundaries of the distribution since asymptotic normality no longer holds then. They propose plotting only pairs (χ_i, λ_i) for which $|\lambda_i| \leq 4\left(\frac{1}{T-1} - \frac{1}{2}\right)^2$ in order to avoid outliers.

K-plots

K-plots or Kendall plots are based on the notion of standard QQ-plots used in order to access the deviations from normality of random variables. Genest and Boies (2003) proposed this graphical tool that assesses the degree of dependence in bivariate random samples. Consider a random sample $(X_1, Y_1), ..., (X_T, Y_T)$ generated from a bivariate continuous distribution H. In this case, the K-plot can be constructed as under:

1. Order H_i such that $H_1 \leq ... \leq H_T$

2. Plot the pairs $(W_{i:T}, H_i)$ for $1 \leq i \leq T$ where $W_{i:T}$ is the expected value of the i^{th} order statistic from a random sample of size T drawn from the distribution K_0, which is the joint distribution under the null of independence between X and Y. From the density of the i^{th} order statistic, we can calculate $W_{i:T}$ as

$$W_{i:T} = \frac{T!}{(i-1)!\,(T-i)!} \int_0^1 w\,\{K_0(w)\}^{i-1}\,\{1 - K_0(w)\}^{T-i}\,dK_0(w) \qquad (4.11)$$

where $K_0(w) = P(UV \leq w) = w - w\log(w)$ where $0 \leq w \leq 1$ and U and V are independent standard uniform random variables.

Implementing the dependence graphs for the factors governing the maturity clusters, we provide the results below. We first estimate the three prinicipal factors $(\beta_1, \beta_2, \beta_3)$ using PCA for the two maturity clusters (denoted by a and b) and graphically analyse the factor dependence structures across the two clusters. Figures *4.4 - 4.7* plot the scatter plots and the dependence graphs for the bond yield factors.

[Insert Figures 4.4 - 4.7 here]

From the Chi-plots in Figures *4.4* - *4.6*, we find that the first factor of the two clusters, β_1^a and β_1^b have a positive dependence structure with the deviations above the horizontal line at $\chi = 0$. The factors β_2 and β_3 governing the two data clusters seen to lie horizontally depicting loose dependence across factors governing the short and long clusters. This means that estimating factors over multiple maturity clusters together would lead to significant information loss. In this case, the estimated factors would not be explaining the whole term structure but be influenced by individual clusters. In Figure *4.7* we plot the non-overlapping recursive K-plot with an estimation window of 100 observations. Points lying on the straight line correspond to the case of independence between the two observations and the data points lying of the smooth curve is associated with perfect positive dependence between observations. We find that in the case of graph (a), the first factor β_1 from the two clusters always lie above the independence line showing significant positive dependence. In the case of graph (b), we find the β_1^a and β_2^b show greater negative dependence. We see that the dependence structure between the factors have changed over the subsamples considered. There seem to be a positive and negative dependence evolving over time.

4.4 Block dynamic factor model

In this section, we propose a new dynamic framework that extends the Dynamic Nelson-Siegel model proposed by Diebold and Li (2006) for the case of modelling term structures with dependence clusters.

4.4.1 Dynamic Nelson-Siegel factor model

In modelling the term structure of interest rates, a class of function-based curve fitting techniques have become most popular in recent years. These techniques specify bond prices as a function of time to maturity and other parameters. A premier to this class of models is the Nelson-Siegel (1987) model that uses exponential polynomial functions for the instantaneous forward rates and proposes a parametric model for the yield curves. The model produces reasonable yield curve shapes observed in the market and thus captures the cross-sectional dependence among rates.

Given $Y(\tau)$ is the yield of a zero coupon bond at a point in time with time to maturity τ, $Y(\tau)$ can be written as

$$Y(\tau) = \frac{1}{\tau}\int_0^{\tau} f(s)\,ds \tag{4.12}$$

where $f(\tau)$ is the instantaneous forward rate with maturity τ. Nelson and Siegel (1987) considered the instantaneous forward rate to be

$$f(\tau) = \beta_1 + \beta_2\left(e^{-\tau/\lambda}\right) + \beta_3\left(\frac{\tau}{\lambda}e^{-\tau/\lambda}\right) \tag{4.13}$$

where $\beta_0, \beta_1, \beta_2$ are the coefficients and λ is a constant decay parameter. Solving for the integral, the Nelson-Siegel (1987) model, as parameterized by Diebold and Li (2006), is given by

$$Y_t(\tau) = \beta_{1t} + \beta_{2t}\left(\frac{1-e^{-\tau/\lambda}}{\tau/\lambda}\right) + \beta_{3t}\left(\frac{1-e^{-\tau/\lambda}}{\tau/\lambda} - e^{-\tau/\lambda}\right) \tag{4.14}$$

Fabozzi et al. (2005) and Diebold and Li (2006) show that the three factors can be economically interpreted as the level, slope, and curvature. β_{1t} is the factor loading associated with the first component which is interpreted as the level factor, β_{2t} is the factor loading for the second component which captures the slope factor mostly influencing the short term factors, and β_{3t} is the factor loading for the third component associated with the medium rates, interpreted as the curvature factor loadings.

The model has been extensively used by central banks for the purpose of modelling and forecasting interest rates. In order to model the time series dynamics of the yield curves, Diebold and Li (2006) dynamized the Nelson-Siegel (1987) model by proposing a VAR(1) process for the factors

$$\beta_t = \mu + A\beta_{t-1} + \xi_t \qquad \xi_t \sim N(0, \Sigma_\xi) \tag{4.15}$$

where β_t is a 3×1 vector of level, slope, and curvature factors. The dynamic model is shown to be well suited for generating forecasts and predominantly beats other benchmark models.

The estimation of the above model can be done with a two-step procedure outlined by Diebold and Li (2006). First, estimate β_{it} coefficients $i = 1, 2, 3$ by cross sectional least squares for each t. The coefficients can be estimated well as long as sufficient number of maturities are available at a given point in time. Second, the time series estimates of β_{it} obtained from the first step is modelled as an autoregressive process and forecasts for β_{it}, and therefore $Y_t(\tau)$, are generated using the specification in equation *4.14*. An alternative estimation approach is a one-step procedure outlined by Diebold, Rudebusch, and Aruoba (2006) in which the equations *4.14* and *4.15* are formulated in a state-space system and estimated iteratively using a Kalman filter. Diebold, Rudebusch, and Aruoba (2006) argue that the two-step approach in the second stage fails to account for the parameter estimation uncertainity associated with the first step and therefore the one-step Kalman filter approach is preferable to the two-step approach. Further, the one-step Kalman filter approach is superior in the fact that various extensions such as allowance of heteroskedasticity, estimation of unbalanced term structures, and allowance for estimation of the decay parameter (λ) can be accommodated in the above framework. Yu and Zivot (2008) compare the forecast performance of the two estimation methods and conclude that there were no considerable forecast improvements in using one approach over the other.

4.4.2 Block dynamic factor representation

In this section, we develop the block dynamic factor representation for estimating the Nelson-Siegel factors of the yield curve. Suppose that the yield term structure Y is identified to contain two sets of data clusters, say a short cluster with maturities up to τ_k and a long cluster with maturities beyond τ_k. Then

$$Y(\tau) = \frac{1}{\tau}\int_0^{\tau} f^a(s)\,ds \cdot I_{(\tau \leq \tau_k)} + \frac{1}{\tau}\int_0^{\tau} f^b(s)\,ds \cdot I_{(\tau > \tau_k)} \tag{4.16}$$

Solving the integral, the Nelson and Siegel (1987) representation for the yield curve maturity clusters can be written as

$$Y(\tau) = \left[\beta_1^a \cdot I_{(\tau \leq \tau_k)} + \beta_1^b \cdot I_{(\tau > \tau_k)}\right] + \left[\beta_2^a \cdot I_{(\tau \leq \tau_k)} + \beta_2^b \cdot I_{(\tau > \tau_k)}\right] \left(\frac{1 - e^{-\tau/\lambda}}{\tau/\lambda}\right) \quad (4.17)$$

$$+ \left[\beta_3^a \cdot I_{(\tau \leq \tau_k)} + \beta_3^b \cdot I_{(\tau > \tau_k)}\right] \left(\frac{1 - e^{-\tau/\lambda}}{\tau/\lambda} - e^{-\tau/\lambda}\right) \quad (4.18)$$

where for $i = 1, 2, 3$; $\beta_i^a \cdot I_{(\tau \leq \tau_k)}$ are the latent factors governing the short cluster and $\beta_i^b \cdot I_{(\tau > \tau_k)}$ are the latent factors governing the long cluster within the yield curve.

We allow for a time-varying dynamic structure of the factors as under:

$$Y_t(\tau) = \left[\beta_{1t}^a \cdot I_{(\tau \leq \tau_k)} + \beta_{1t}^b \cdot I_{(\tau > \tau_k)}\right] + \left[\beta_{2t}^a \cdot I_{(\tau \leq \tau_k)} + \beta_{2t}^b \cdot I_{(\tau > \tau_k)}\right] \left(\frac{1 - e^{-\tau/\lambda}}{\tau/\lambda}\right)$$

$$+ \left[\beta_{3t}^a \cdot I_{(\tau \leq \tau_k)} + \beta_{3t}^b \cdot I_{(\tau > \tau_k)}\right] \left(\frac{1 - e^{-\tau/\lambda}}{\tau/\lambda} - e^{-\tau/\lambda}\right) \quad (4.19)$$

and the demeaned factors $\bar{\beta}_1^{a,b}, \bar{\beta}_2^{a,b}, \bar{\beta}_3^{a,b}$ follow a VAR(1) process as

$$\begin{bmatrix} \bar{\beta}_{1t}^a \\ \bar{\beta}_{1t}^b \\ \bar{\beta}_{2t}^a \\ \bar{\beta}_{2t}^b \\ \bar{\beta}_{3t}^a \\ \bar{\beta}_{3t}^b \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{16} \\ a_{21} & a_{22} & & a_{26} \\ \cdot & & \cdot & \cdot \\ \cdot & & \cdot & \cdot \\ \cdot & & \cdot & \cdot \\ a_{61} & a_{62} & \cdots & a_{66} \end{bmatrix} \begin{bmatrix} \bar{\beta}_{1t-1}^a \\ \bar{\beta}_{1t-1}^b \\ \bar{\beta}_{2t-1}^a \\ \bar{\beta}_{2t-1}^b \\ \bar{\beta}_{3t-1}^a \\ \bar{\beta}_{3t-1}^b \end{bmatrix} + \begin{bmatrix} \xi_{1t}^a \\ \xi_{1t}^b \\ \xi_{2t}^a \\ \xi_{2t}^b \\ \xi_{3t}^a \\ \xi_{3t}^b \end{bmatrix} \quad (4.20)$$

where ξ_{it}^m $(i = 1, 2, 3; m = a, b)$ are the disturbances of the factors, uncorrelated across factors and clusters:

$$E\left(\xi_{it}^m \xi_{jt-s}^n\right) = \begin{cases} 1 & \text{for } i = j, m = n, s = 0 \\ 0 & \text{otherwise} \end{cases} \quad (4.21)$$

In order to estimate the model, one can use the cross sectional regression approach suggested by Diebold and Li (2006). Alternatively, the block dynamic representation above can be formulated in a state space framework and a Kalman filter is adopted in order to estimate the specification. The term structure of yields as functions of the factors act as the measurement equations and the process for the factors act as the state equations.

4.4.3 Model estimation

The block dynamic model can be estimated using a two step approach as suggested by Diebold and Li (2006). We first estimate the measurement equations using cross-sectional regressions and then estimate the factor dynamics in the second step. More precisely, in the first step we regress:

$$\underset{6\times1}{Y_t} = \underset{21\times6}{C} \cdot \underset{6\times1}{\beta_t} + \underset{6\times1}{v_t} \tag{4.22}$$

and estimate $\hat{\beta}_t$. Then we estimate the state equation as a VAR(1) model using least squares

$$\underset{6\times1}{\beta_t} = \underset{6\times6}{A} \cdot \underset{6\times1}{\beta_{t-1}} + \underset{6\times1}{\xi_t} \tag{4.23}$$

Alternatively, one can consider a state space formulation for the block dynamic factor model

$$\beta_t = A \cdot \beta_{t-1} + \xi_t \tag{4.24}$$

$$Y_t = C \cdot \beta_t + v_t \tag{4.25}$$

where the mean zero state vector of factors β_t follow a VAR(1) representation as in equation (*4.20*). The measurement equation (*4.25*) for the vector of yields with N maturities can be written as

$$\begin{pmatrix} y_t(\tau_1) \\ \vdots \\ y_t(\tau_k) \\ y_t(\tau_{k+1}) \\ \vdots \\ y_t(\tau_N) \end{pmatrix} = \begin{bmatrix} 1 & 0 & \frac{1-e^{-\tau_1\lambda_1}}{\tau_1\lambda_1} & 0 & \frac{1-e^{-\tau_1\lambda_1}}{\tau_1\lambda_1} - e^{-\tau_1\lambda_1} & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 0 & \frac{1-e^{-\tau_k\lambda_1}}{\tau_k\lambda_1} & 0 & \frac{1-e^{-\tau_k\lambda_1}}{\tau_k\lambda_1} - e^{-\tau_k\lambda_1} & 0 \\ 0 & 1 & 0 & \frac{1-e^{-\tau_{k+1}\lambda_2}}{\tau_{k+1}\lambda_2} & 0 & \frac{1-e^{-\tau_{k+1}\lambda_2}}{\tau_{k+1}\lambda_2} - e^{-\tau_{k+1}\lambda_2} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 1 & 0 & \frac{1-e^{-\tau_N\lambda_2}}{\tau_N\lambda_2} & 0 & \frac{1-e^{-\tau_N\lambda_2}}{\tau_N\lambda_2} - e^{-\tau_N\lambda_2} \end{bmatrix}$$

$$\begin{pmatrix} \beta_{1t}^a \\ \beta_{1t}^b \\ \beta_{2t}^a \\ \beta_{2t}^b \\ \beta_{3t}^a \\ \beta_{3t}^b \end{pmatrix} + \begin{pmatrix} v_t(\tau_1) \\ \vdots \\ v_t(\tau_k) \\ v_t(\tau_{k+1}) \\ \vdots \\ v_t(\tau_N) \end{pmatrix} \tag{4.26}$$

for a given k. We allow for two different decay parameters λ_1 and λ_2 for the two maturity clusters. The disturbances of the measurement equation and the state equation are assumed to be uncorrelated, zero mean white noise with covariances

$$\begin{aligned} E(\xi\xi') &= Q \\ E(vv') &= R \end{aligned}$$

We assume both Q and R to be diagonal, an assumption we impose for tractability of the model. Diagonality of Q would mean that the errors in estimation of factors within blocks and across blocks are independent and the diagonality of R would mean that the idiosyncrasies of the yield curve maturities are uncorrelated.

The state space formulation above is estimated via Kalman filtering. Assuming normality of the error terms, maximum likelihood estimates of the unknown parameters, say θ, can be obtained by minimizing the negative log likelihood function

$$-l(Y_t,\theta)=\sum_{t=1}^{T}\left[\log|F_{e_t}(\theta)|+e_t'(\theta)F_{e_t}^{-1}(\theta)e_t(\theta)\right] \tag{4.27}$$

where $e_t(\theta)$ is the prediction error vector and $F_{e_t}(\theta)$ is the covariance matrix of the prediction error obtained from the Kalman filter. In estimating the parameter vector θ via Kalman filtering, we do the following:

- The parameters to be estimated are namely A, Q, and R.
- The matrix C is known and given by the Nelson-Siegel framework. In constructing the observation matrix C, we allow the decay parameter λ to vary for the short and long maturity clusters. For the short maturity cluster, the value of λ_1 maximizes the loadings of β_3 at six month maturity.

$$f(\lambda_1)=\frac{1-e^{-6\lambda_1}}{6\lambda_1}-e^{-6\lambda_1}$$

$$\hat{\lambda}_1=\arg\max f(\lambda_1)=0.29888$$

  Similarly, we compute for the long maturity cluster, the value of λ_2 maximizing the loadings of β_3 at sixty month maturity and equal to 0.0299.

- We initialize the unconditional mean vector of the state prediction equation with zero and its initial unconditional covariance matrix as given by Harvey (1989)

$$vec\left(P_{0|0}\right)=(1-A\otimes A)^{-1}vec(Q) \tag{4.28}$$

- We assume that Q and R diagonal. We initialize the covariance matrices Q and R as identity matrices. In order to impose positive definiteness of the estimated Q and R, we maximize the likelihood over the square root of these matrices. Diebold and Li (2006) propose estimating log variances and then convert the log variances by exponentiating them.

- The likelihood is maximized using the Marquardt and BHHH algorithms, with a convergence criteria of 10^{-6} for the value of the norm of difference between parameter estimates obtained at consecutive iterations. The step size at each iteration is determined by the backtracking algorithm with parameters $p = 0.25$ and $q = 0.5$. The details of the numerical optimization are provided in the appendix.

4.5 Modelling and forecasting performance of block dynamic factor model

In this section, we present the goodness of fit results for the Nelson-Siegel block dynamic factor model in terms of modelling and forecasting performance.

4.5.1 Data

The data of US zero coupon bond term structure is collected from Datastream. The sample period consists of daily frequency extending from 11 Jan 1999 to 31 July 2007 (2232 observations) for constant term structure maturities of 3, 4, ..., 12, 24, ..., 144 months (21 maturities). Table *4.2* provides the descriptive statistics for yields at various maturities. We find that the mean value of yields remain around a constant up to 12 month maturity and then increases with maturity. The standard deviation of the yields, in contrast, remains almost flat up to 12 month maturity and then decreases with maturity. The sample autocorrelations are lags 1, 60 and 200 show evidence of high persistence in yields.

The matrix plot of the term structure (Figure *4.1*) show presence of two correlation clusters of treasury bills (3-12 month maturities) and zero coupon bonds (24-144 month maturities).

Therefore we consider extraction of factors from the two clusters separately. In evaluating the goodness of fit of the block dynamic model introduced in Section 4.4, we conduct an in-sample estimation until 08 May 2007 (2172 observations) and out-of-sample forecasting of the last 60 observations.

[Insert Table 4.2 here]

4.5.2 Estimation results

In estimating the block dynamic model, we employ both the two-step cross sectional regressions and the one-step state space approach.

[Insert Table 4.3 here]

Table *4.3* records the estimates of the vector autoregression employed in the two step cross sectional regressions. We find that the estimated factors load significantly on their own lags, with the exception of the level factor governing the short rates that interact with the lags of the slope factor governing short rates and the level factor governing the long rates. In the case of the long rates slope factor and the short rates curvature factor, we find significant interactions with the other factors.

In using the one step approach to estimate the parameters of the block dynamic model, we use a state space framework and employ Kalman filtering to maximize the prediction error decomposition form of the likelihood formulated. The number of parameters estimated are 36 in the transition matrix A, 6 diagonal parameters of the covariance matrix Q, and 21 diagonal parameters of covariance matrix R (totalling 63 parameters).

[Insert Table 4.4 here]

Table 4.4 provides the in-sample estimation results of the Nelson-Siegel block dynamic model. The factors appear to be highly persistent to its own lags, apart from the level factor of the short maturity cluster. We find that the level factor from the long maturity cluster depends mainly on its past lag with coefficient close to one. This shows that the level factor of the curve behaves close to a random walk. In the case of the level factor of the short maturity curve, we find that it loads heavily on past lag of the level factor from the short maturity curve. This seems to indicate that the level factor of the two clusters share a common random walk component in them.

Factors estimated from the short cluster seem to have significant cross-factor dynamics unlike the factors from the long cluster that appear not to be influenced by cross-factor dynamics. The estimated covariance matrix of the state equation (Q) is all significant with very small standard errors.

4.5.3 Forecasting performance of models

We compare the out-of-sample performance of the Nelson-Siegel block dynamic model (NS-block) with the Nelson-Siegel dynamic model proposed by Diebold and Li (2006).

In constructing forecasts for the yields, we use the predictions of the state vector into the predictions of the measurement equations. For n period ahead forecasts, we use the forward substitution

$$Y_{T+n|T} = C\beta_{T+n|T} \tag{4.29}$$

$$\beta_{T+n|T} = A^n \beta_T \tag{4.30}$$

We consider five different forecast horizons of $n = 1, 5, 10$, and 30. In order to evaluate the out-of-sample forecastability of the model, we use the Mean Square Error (MSE), Mean Absolute Error (MAE) and the averages (AVG) of the two measures across the whole yield curve maturities. The latter combines the forecast errors from all the maturities for evaluating the overall performance of the models.

[Insert Tables 4.5 - 4.8 here]

Tables *4.5 - 4.8* reports the out-of-sample performance for the various forecast horizons considered. The MSE and MAE measures provide similar forecast evaluation inferences. We find that there is considerable average gain in forecasts in using the NS-block model. The AVG estimator for MSE and MAE show that on average there is a combined forecast improvement over horizons $n = 1, 5, 10$ and 30. For five-period ahead forecasts, we find forecast gains for 3 months to 2 year maturities and very long maturities of 9 to 12 years. The longer maturities of 2 years and above show higher degree of predictability in the case of NS-block for ten-period and thirty-period ahead forecasts.

4.5.4 Testing difference in performance

We use the Diebold and Mariano (1995) statistic in order to formally test whether differences in the out-of-sample performance, outlined above, are statistically significant. In particular, we test the significance of difference in forecasting performance measures MSE and MAE of the Nelson-Siegel block dynamic model (NS-block) against the Nelson-Siegel dynamic model proposed by Diebold and Li (2006). The Diebold and Mariano test aims at testing the null of equal predictive accuracy of the of the two models against the alternative of different forecastability across models. If d_t is the loss differential defined as

$$d_t = g\left(\varepsilon^A_{t+h|t}\right) - g\left(\varepsilon^B_{t+h|t}\right)$$

where $g(\cdot)$ is the loss function measuring the accuracy of the forecasts (MSE and MAE), and $\varepsilon^i_{t+h|t}$ refers to the forecasting error of model i when performing a h period ahead forecasts assumed to be computed for $t = t_0, ..., T$ for a total of k forecasts. The null of equal predictability is

$$H_0 : E[d_t] = 0$$

The test uses autocorrelation corrected sample mean $\bar{d}$ in order to construct the test statistic

$$S = \frac{\bar{d}}{V\left(\bar{d}\right)^{1/2}} \tag{4.31}$$

where $\bar{d} = \frac{1}{k}\sum_{t=t_0}^{T} d_t$ and $V\left(\bar{d}\right) = \frac{1}{k}\left[\gamma_0 + 2\sum_{j=1}^{h-1}\gamma_j\right]$ for $\gamma_j = cov\left(d_t, d_{t-j}\right)$. Diebold and Mariano (1995) show that under the null of equal predictive accuracy, S is asymptotically standard normal.

[Insert Tables 4.9 - 4.10 here]

Tables *4.9 - 4.10* present the Diebold and Mariano test statistic values and the corresponding p-values for the MSE and MAE for 1, 5, 10, and 30 period ahead forecast horizons. In most of the cases, we find the p-values close to zero suggesting significant forecast differences in performance of the two models in favor of the Nelson-Siegel block dynamic model.

4.5.5 Forecast Benchmarking

In order to understand the improvement in forecast performance using the block dynamic factor representation, we compare the forecasts of the two models (NS and NS-block) with a naive model forecasts where the forecasts at $Y_{T+n|T} = Y_T$ for all T where n is the forecast horizon and equal to 1, 5, 10, and 30. Here current yield curve used as a predictor of the future yield curve. We use the relative measures for MSE and MAE calculated as

$$RelMSE = MSE/MSE\_b \tag{4.32}$$

$$RelMAE = MAE/MAE\_b \tag{4.33}$$

where MSE_b and MAE_b are the errors from the benchmark model.

[Insert Tables 4.11 - 4.14 here]

Tables 4.11 - 4.14 reports the out-of-sample performance for the various forecast horizons relative to the benchmark model. The RelMSE and RelMAE measures the improvement possible from the proposed forecast method relative to the benchmark forecast method. We find that the Nelson-Siegel block dynamic model performs relatively better than the Nelson-Siegel model. The forecast accuracy is greater for one-period ahead forecasts but the accuracy of forecasting the long rates diminishes as we increase the forecast horizon.

4.6 Conclusion

This chapter introduces a block dynamic representation for the Nelson-Siegel factors estimated from various maturity clusters within the term structure. The short and long interest rate maturity clusters can be clearly identified using Chi-plots and K-plots. The dependence structure across short and long maturity clusters have been estimated separately, where the factor loadings pertaining to the clusters are allowed to have their own dynamics. The new block dynamic Nelson-Siegel model has been estimated recursively (and via Kalman filter) where the dynamics of the factors are assumed to be a VAR(1) process. The VAR(1) coefficients show that the factors load significantly on the lags of other factors across clusters. This shows that the factors do interact across clusters. We find that the factors governing the clusters are loosely dependent and measuring the factors for the two clusters increase the forecastability of the Nelson-Siegel (1987) model for bond yields.

In the case of US zero coupon bond yield term structures with short maturity cluster including ten maturities of 3, 4, 5, 6, 7, 8, 9, 10, 11, 12 months and eleven maturities of 24, 36, 48, 60, 72, 84, 96, 108, 120, 132 and 144 months as the long maturity cluster, the chapter estimates factors of each cluster in a Nelson-Siegel framework and allow for a block dynamic structure for the evolution of the factors. The out-of-sample forecast evaluation using the

Diebold and Mariano (1995) tests show considerable gains in forecasting performance of the model over the dynamic model of Diebold and Li (2006).

4.7 Appendix

4.7.1 Estimation using Kalman filter

Consider the Kalman filter measurement equation, describing the relation between yields and unobserved factors

$$Y_t = C \cdot \beta_t + v_t \qquad v_t \sim N(0, R_t) \tag{4.34}$$

and the state transition equation, describing the dynamics of the factors

$$\beta_t = A \cdot \beta_{t-1} + \xi_t \qquad \xi_t \sim N(0, Q_t) \tag{4.35}$$

$$E(v_t \xi_s') = 0 \quad \text{for all } t \text{ and } s \ \ (t, s = 1, ...T)$$

The initial state distribution is assumed given by

$$\beta_0 \sim N(\beta_0, P_0)$$

The system matrices A, R, and Q are unknown parameters and we estimate the parameters using maximum likelihood estimation based on the prediction error decomposition obtained via filtering.

For the iteration at time t, the state prediction equation and its variance equation is given by

$$\beta_{t|t-1} = A_t \beta_{t-1} \tag{4.36}$$

$$P_{t|t-1} = A_t P_{t-1} A_t' + Q_t \tag{4.37}$$

The measurement prediction equation and its variance equation is given by

$$\hat{y}_{t|t-1} = C\beta_{t|t-1} \tag{4.38}$$

$$F_t = CP_{t|t-1}C' + R_t \tag{4.39}$$

From the above, the prediction error is $e_t = y_t - \hat{y}_{t|t-1}$ and the Kalman gain is given by $K_t = P_{t|t-1}C'F_t^{-1}$.

The updating equation for the state estimate and its estimated variance equation is given by

$$\beta_t = \beta_{t|t-1} + K_t e_t \tag{4.40}$$

$$P_t = P_{t|t-1} - K_t C P_{t|t-1} \tag{4.41}$$

Let θ denote the vector of unknown parameters. When the observations are normally distributed, the log likelihood function can be obtained as

$$l(\theta) = -\frac{T}{2}\log 2\pi - \frac{1}{2}\sum_{t=1}^{T}\log|F_t| - \frac{1}{2}\sum_{t=1}^{T} e_t' F_t^{-1} e_t \tag{4.42}$$

with e_t being the prediction error and F_t being its variance obtained from the Kalman filter.

At each time t, equations *4.36 - 4.41* provides a filtered estimate of the state vector β_t incorporating past and present information in the measurements Y_t. The filtered estimates are smoothed recursively backwards from time T to time 1 in order to estimate the state variables given the entire sample Y using the smoothing equations

$$\beta_{t|T} = \beta_{t|t} + P_{t|t}^{*}\left(\beta_{t+1|T} - A\beta_{t|t}\right) \tag{4.43}$$

$$P_{t|T} = P_{t|t} + P_{t|t}^{*}\left(P_{t+1|T} - P_{t+1|t}\right) P_{t|t}^{*\prime} \tag{4.44}$$

where $P_{t|t}^{*} = P_{t|t}A'P_{t+1|t}^{-1}$ and $t = T-1, T-2, ..., 1$.

4.7.2 Computational issues

Using numerical optimization techniques, outlined below, we maximize the likelihood in order to find the unknown parameters.

In order to estimate $\hat{\theta}$ given some starting value θ_0, we formulate the maximization as minimization of the negative log likelihood function and find the solution using the Marquardt and Berndt-Hall-Hall-Hausman (BHHH) algorithm. Consider the iterations

$$\theta_{k+1} = \theta_k - \lambda_k H_k G_k \tag{4.45}$$

where G_k is the gradient of the log likelihood function evaluated at θ_k $\left(\frac{\partial l}{\partial \theta}(\theta_k)\right)$, H_k is the direction and λ_k is the step length at iteration k. The matrix H_k is estimated using the BHHH algorithm where

$$H_k = \left[\frac{1}{T}\sum_{i=1}^{T}\frac{\partial l}{\partial \theta}(x_i,\theta_k)\frac{\partial l}{\partial \theta}(x_i,\theta_k)'\right]^{-1} \tag{4.46}$$

as discussed in Berndt et al. (1974). We modify the matrix H_k approximation by incorporating a ridge factor b

$$[H_k + bI] \tag{4.47}$$

where I is the identity matrix and b is a positive constant, as recommended by Levenberg and Marquardt method. Figure *4.3* that presents the algorithm for the Marquardt method. The correction matrix would enable us to reduce the number of iterations by pushing the estimates in the direction of the steepest descent.

The step length λ at every iteration k is chosen using the backtracking line search algorithm where we find λ by minimizing the objective function l along the ray $\{\theta + \lambda\Delta\theta\}$. The algorithm initializes with $\lambda = 1$ and then reduces the value of λ by a fraction q until the stopping condition

$$l(\theta + \lambda\Delta\theta) \leq l(\theta) + p\lambda G'\Delta\theta \tag{4.48}$$

is satisfied. The line search algorithm depends on the parameterization of p and q with $p \in (0, 0.5)$, $q \in (0, 1)$.

TABLE 4.1. Principal component analysis results of short and long maturity cluster factors

	Short Maturity Curve Factors			Long Maturity Curve Factors		
	Factor 1	Factor 2	Factor 3	Factor 1	Factor 2	Factor 3
Eigenvalues	39.33335	0.099677	0.002592	10.54249	0.449597	0.007042
Variance Proportion	0.997345	0.002527	0.000066	0.958408	0.040872	0.00064
Cum. Proportion	0.997345	0.999872	0.999938	0.958408	0.999281	0.999921
Eigenvectors	0.319487	-0.51702	0.663782	-0.28547	0.555138	0.556906
	0.318352	-0.40849	0.017558	-0.29659	0.400101	0.216212
	0.318102	-0.27535	-0.30274	-0.30226	0.282841	-0.31761
	0.31926	-0.14007	-0.33694	-0.30605	0.16008	-0.37169
	0.318212	-0.0262	-0.28227	-0.30768	0.050525	-0.34057
	0.317162	0.087528	-0.22379	-0.30775	-0.04842	-0.24997
	0.316117	0.201224	-0.15509	-0.30671	-0.13432	-0.13899
	0.31394	0.285486	0.026811	-0.3049	-0.2103	-0.01659
	0.311814	0.368708	0.210768	-0.30247	-0.28028	0.117838
	0.309673	0.451782	0.393644	-0.29981	-0.34005	0.238233
				-0.29619	-0.40591	0.367938

TABLE 4.2. Descriptive statistics of daily yields at various maturities. $\hat{\rho}(x)$ is the sample autocorrelations at lag x. The sample period extends from Jan 11, 1999 to Jul 07, 2007.

Maturity (months)	Mean	Std. Dev.	Maximum	Minimum	$\hat{\rho}(1)$	$\hat{\rho}(60)$	$\hat{\rho}(200)$
3	3.839296	2.011128	7.1476	0.9666	0.99963	0.9524	0.69096
4	3.860258	2.001347	7.2198	0.9561	0.99961	0.95344	0.69209
5	3.893181	1.997638	7.3311	0.9353	0.99958	0.95441	0.69713
6	3.919562	2.003349	7.4412	0.9349	0.99954	0.95412	0.69742
7	3.94864	1.996261	7.49	0.9411	0.99944	0.95423	0.70005
8	3.976855	1.989825	7.5553	0.9375	0.99927	0.95415	0.70224
9	4.005398	1.98407	7.6244	0.9339	0.99907	0.9538	0.70392
10	4.035635	1.971424	7.6565	0.937	0.99888	0.95334	0.70559
11	4.065145	1.959516	7.6886	0.9399	0.99866	0.95282	0.70699
12	4.095075	1.947958	7.7227	0.9429	0.9984	0.95201	0.70804
24	4.44159	1.700652	7.8741	1.2639	0.99928	0.94153	0.70693
36	4.720375	1.505919	7.9565	1.6399	0.99898	0.92972	0.70192
48	4.910276	1.32032	7.9052	2.0187	0.99877	0.91549	0.69305
60	5.088792	1.211116	7.9526	2.3815	0.99847	0.9042	0.68362
72	5.234764	1.127338	7.9637	2.697	0.99829	0.89501	0.67469
84	5.358339	1.063532	8.0011	2.9716	0.99813	0.88789	0.66775
96	5.461591	1.011491	8.0124	3.2009	0.99796	0.88119	0.66039
108	5.551528	0.970776	8.0245	3.4055	0.99782	0.87584	0.65434
120	5.634235	0.938075	8.0667	3.5913	0.99767	0.87154	0.65051
132	5.704309	0.910581	8.0639	3.7505	0.99756	0.86905	0.6476
144	5.777376	0.884148	8.0713	3.9148	0.9974	0.86625	0.64424

TABLE 4.3. Vector autoregression estimates from two step regression (with standard errors in ( ) and t-statistics in [ ])

	$\beta^a_{1,t-1}$	$\beta^a_{2,t-1}$	$\beta^a_{3,t-1}$	$\beta^b_{1,t-1}$	$\beta^b_{2,t-1}$	$\beta^b_{3,t-1}$
$\beta^a_{1,t}$	-0.01028	1.012364	-0.118905	0.804234	-0.090574	0.166072
	(-0.03256)	(-0.03246)	(-0.00835)	(-0.02647)	(-0.00727)	(-0.00618)
	[-0.31576]	[31.1839]	[-14.2463]	[30.3817]	[-12.4559]	[26.8764]
$\beta^a_{2,t}$	-0.00033	1.000932	0.002852	0.001057	-0.000111	0.002038
	(-0.01172)	(-0.01169)	(-0.003)	(-0.00953)	(-0.00262)	(-0.00222)
	[-0.02804]	[85.6369]	[0.94912]	[0.11086]	[-0.04259]	[0.91588]
$\beta^a_{3,t}$	0.644825	-0.6464	1.045664	-0.517171	0.078587	-0.103557
	(-0.02585)	(-0.02578)	(-0.00663)	(-0.02102)	(-0.00577)	(-0.00491)
	[24.9409]	[-25.0748]	[157.773]	[-24.6040]	[13.6102]	[-21.1054]
$\beta^b_{1,t}$	0.03684	-0.03792	-0.003877	0.969039	0.005611	-0.007273
	(-0.01415)	(-0.01411)	(-0.00363)	(-0.0115)	(-0.00316)	(-0.00269)
	[2.60368]	[-2.68773]	[-1.06904]	[84.2390]	[1.77578]	[-2.70837]
$\beta^b_{2,t}$	2.154818	-2.16169	0.278709	-1.707826	1.154208	-0.366463
	(-0.07521)	(-0.07499)	(-0.01928)	(-0.06115)	(-0.0168)	(-0.01427)
	[28.6499]	[-28.8251]	[14.4556]	[-27.9291]	[68.7131]	[-25.6736]
$\beta^b_{3,t}$	-0.05249	0.049556	-0.003713	0.040931	-0.014264	0.990219
	(-0.03446)	(-0.03436)	(-0.00883)	(-0.02802)	(-0.0077)	(-0.00654)
	[-1.52333]	[1.44231]	[-0.42033]	[1.46099]	[-1.85342]	[151.417]

TABLE 4.4. Parameter estimates of the state vector (with standard errors in parentheses)

	$\beta^a_{1,t-1}$	$\beta^a_{2,t-1}$	$\beta^a_{3,t-1}$	$\beta^b_{1,t-1}$	$\beta^b_{2,t-1}$	$\beta^b_{3,t-1}$	Q_{ii}
$\beta^a_{1,t}$	-0.017639	-0.13611	-0.12519	0.96884	0.66629	0.18162	0.083993
	(0.06)	(0.02)	(0.02)	(0.06)	(0.05)	(0.01)	(0.00)
$\beta^a_{2,t}$	0.70186	0.80631	0.11571	-0.73198	-0.58664	-0.13296	0.014297
	(0.11)	(0.03)	(0.03)	(0.11)	(0.08)	(0.03)	(0.00)
$\beta^a_{3,t}$	2.0783	0.25409	1.0789	-2.1212	-1.6436	-0.53099	0.14573
	(0.12)	(0.05)	(0.04)	(0.12)	(0.10)	(0.03)	(0.00)
$\beta^b_{1,t}$	-0.070614	-0.0081006	-0.0080661	1.0808	0.085199	0.0195	0.023604
	(0.02)	(0.01)	(0.01)	(0.02)	(0.02)	(0.01)	(0.00)
$\beta^b_{2,t}$	0.023972	0.031228	0.10905	-0.024109	0.89761	0.016883	0.37013
	(0.03)	(0.01)	(0.01)	(0.03)	(0.02)	(0.01)	(0.00)
$\beta^b_{3,t}$	-0.02241	0.07531	0.028457	0.034876	0.010916	1.0009	0.039738
	(0.06)	(0.02)	(0.01)	(0.06)	(0.05)	(0.01)	(0.00)

TABLE 4.5. Out of sample goodness of fit results from one-period ahead forecasts using the Nelson-Seigel block dynamic model (NS-block) introduced in the text and the Nelson-Seigel dynamic model (NS) introduced by Diebold et al (2006)

Maturity	MSE		MAE	
(in months)	NS	NS-block	NS	NS-block
3	6.60E-05	0.00054451	0.0081245	0.023335
4	0.0009199	0.0014425	0.03033	0.037981
5	0.0016591	0.0015179	0.040732	0.03896
6	0.0019552	0.001735	0.044217	0.041654
7	0.0037712	0.0039629	0.06141	0.062952
8	0.0033094	0.0043149	0.057527	0.065688
9	0.0023985	0.0041973	0.048975	0.064787
10	0.0027095	0.0057237	0.052053	0.075655
11	0.0016124	0.0050661	0.040154	0.071176
12	0.0013006	0.0054573	0.036064	0.073873
24	0.0081156	0.00050909	0.090087	0.022563
36	0.012185	0.0024357	0.11039	0.049353
48	0.0086878	0.0022499	0.093208	0.047433
60	0.00372	0.0016469	0.060992	0.040582
72	0.00048939	0.0011786	0.022122	0.034331
84	0.00022484	0.0011431	0.014995	0.033809
96	0.0029125	0.0010755	0.053967	0.032795
108	0.0090988	0.00083486	0.095388	0.028894
120	0.01814	0.00065286	0.13469	0.025551
132	0.029749	0.00046329	0.17248	0.021524
144	0.044576	0.00020053	0.21113	0.014161
AVG	7.483096	0.002207259	0.070430262	0.04319319

TABLE 4.6. Out of sample goodness of fit results from five-period ahead forecasts using the Nelson-Seigel block dynamic model (NS-block) introduced in the text and the Nelson-Seigel dynamic model (NS) introduced by Diebold et al (2006)

Maturity	MSE		MAE	
(in months)	NS	NS-block	NS	NS-block
3	0.0127	0.0042571	0.050399	0.029179
4	0.015793	0.0079638	0.056202	0.039909
5	0.014356	0.0082906	0.053583	0.04072
6	0.013162	0.0083246	0.051308	0.040804
7	0.013993	0.0096755	0.052902	0.04399
8	0.01199	0.0084466	0.048969	0.041101
9	0.010303	0.0072878	0.045395	0.038178
10	0.010117	0.0072718	0.044982	0.038136
11	0.0081669	0.0056918	0.040415	0.03374
12	0.0073642	0.0050341	0.038378	0.03173
24	0.00013882	8.89E-05	0.0052691	0.0042163
36	0.00010463	0.0013681	0.0045744	0.016541
48	2.40E-05	0.0019832	0.002189	0.019916
60	7.09E-05	0.0023464	0.0037647	0.021663
72	0.00061751	0.0026355	0.011113	0.022959
84	0.0016266	0.0031109	0.018037	0.024944
96	0.0032896	0.0033988	0.02565	0.026072
108	0.0053676	0.0037204	0.032765	0.027278
120	0.0081977	0.0037308	0.040491	0.027316
132	0.011409	0.0036969	0.047769	0.027192
144	0.015219	0.0034653	0.05517	0.026326
AVG	0.007810018	0.00484709	0.034729771	0.029614776

TABLE 4.7. Out of sample goodness of fit results from ten-period ahead forecasts using the Nelson-Seigel block dynamic model (NS-block) introduced in the text and the Nelson-Seigel dynamic model (NS) introduced by Diebold et al (2006)

Maturity	MSE		MAE	
(in months)	NS	NS-block	NS	NS-block
3	0.022499	0.012255	0.047433	0.035008
4	0.024432	0.017525	0.049428	0.041863
5	0.022235	0.018407	0.047154	0.042903
6	0.020497	0.018731	0.045274	0.04328
7	0.020634	0.020216	0.045425	0.044963
8	0.018355	0.018851	0.042842	0.043418
9	0.016486	0.01754	0.040603	0.04188
10	0.015962	0.017392	0.039953	0.041704
11	0.013875	0.01546	0.03725	0.039319
12	0.012795	0.014481	0.03577	0.038054
24	0.0031909	0.0018229	0.017863	0.013501
36	0.0016469	4.28E-05	0.012833	0.0020687
48	0.0018943	0.00020414	0.013763	0.0045182
60	0.0027275	0.0008906	0.016515	0.0094372
72	0.0039191	0.001829	0.019797	0.013524
84	0.0054989	0.0027991	0.02345	0.016731
96	0.0073072	0.0037714	0.027032	0.01942
108	0.0093166	0.0046663	0.030523	0.021602
120	0.011524	0.0054177	0.033947	0.023276
132	0.013858	0.0060327	0.037226	0.024562
144	0.016463	0.0063981	0.040575	0.025294
AVG	0.01262459	0.009749178	0.033555048	0.02792029

TABLE 4.8. Out of sample goodness of fit results from thirty-period ahead forecasts using the Nelson-Seigel block dynamic model (NS-block) introduced in the text and the Nelson-Seigel dynamic model (NS) introduced by Diebold et al (2006)

Maturity	MSE		MAE	
(in months)	NS	NS-block	NS	NS-block
3	0.030648	0.029997	0.031963	0.031621
4	0.02903	0.030878	0.031107	0.032082
5	0.027839	0.030479	0.030462	0.031874
6	0.026873	0.029287	0.029929	0.031245
7	0.025523	0.027149	0.029168	0.030083
8	0.024451	0.025065	0.028549	0.028905
9	0.023544	0.023093	0.028014	0.027745
10	0.022316	0.020862	0.027274	0.026371
11	0.021301	0.018951	0.026647	0.025133
12	0.020641	0.017496	0.02623	0.02415
24	0.013818	0.0014873	0.021461	0.007041
36	0.012649	8.44E-07	0.020534	0.00016774
48	0.013034	0.00098297	0.020844	0.0057241
60	0.013968	0.003136	0.021577	0.010224
72	0.015458	0.0056131	0.0227	0.013679
84	0.017031	0.0081872	0.023827	0.01652
96	0.018629	0.01067	0.024919	0.018859
108	0.020297	0.0129	0.026011	0.020736
120	0.021867	0.01495	0.026998	0.022323
132	0.023142	0.016975	0.027774	0.023788
144	0.024539	0.018627	0.0286	0.024918
AVG	0.021266571	0.016513639	0.026408952	0.021580421

TABLE 4.9. Diebold and Mariano (1995) test results evaluating significant out of sample forecast performance for MSE (with p values in italics)

Maturities (in months)	1 period	5 period	10 period	30 period
3	-1.0621	-29.172	-10.258	-7.2947
	0.288	*0.000*	*0.000*	*0.000*
4	-1.8447	-15.467	-6.2079	-3.1287
	0.065	*0.000*	*0.000*	*0.002*
5	-2.3856	-8.9057	-3.6783	-1.4878
	0.017	*0.000*	*0.000*	***0.137***
6	-2.8651	-5.564	-2.107	-2.0315
	0.004	*0.000*	*0.035*	*0.042*
7	-2.0269	-3.7015	-1.1463	-3.6556
	0.043	*0.000*	***0.252***	*0.000*
8	2.3576	-2.5609	-0.52679	-5.3214
	0.018	*0.010*	***0.598***	*0.000*
9	6.4482	-1.6411	-0.074617	-6.4602
	0.000	***0.101***	***0.941***	*0.000*
10	9.5245	-1.0345	0.25637	-6.9814
	0.000	***0.301***	***0.798***	*0.000*
11	11.194	-0.52658	0.51502	-7.2233
	0.000	***0.598***	***0.607***	*0.000*
12	12.177	-0.16732	0.69896	-7.1245
	0.000	***0.867***	***0.485***	*0.000*
24	-2.6461	2.8103	-0.46626	-1.0516
	0.008	*0.005*	***0.641***	***0.293***
36	-9.9705	4.0051	1.9415	1.0595
	0.000	*0.000*	*0.052*	***0.289***
48	-10.836	5.1546	3.0007	2.0908
	0.000	*0.000*	*0.003*	*0.037*
60	-5.6482	5.5742	3.4981	2.7306
	0.000	*0.000*	*0.000*	*0.006*
72	7.5467	5.418	3.7235	3.1578
	0.000	*0.000*	*0.000*	*0.002*
84	6.5546	5.0125	3.8222	3.4672
	0.000	*0.000*	*0.000*	*0.001*
96	2.3762	4.4102	3.7814	3.6888
	0.017	*0.000*	*0.000*	*0.000*
108	-2.0385	3.7743	3.6986	3.8518
	0.042	*0.000*	*0.000*	*0.000*
120	-6.5911	3.0006	3.5111	3.9534
	0.000	*0.003*	*0.000*	*0.000*
132	-10.362	2.2283	3.2828	4.0059
	0.000	*0.026*	*0.001*	*0.000*
144	-14.087	1.3236	2.9527	4.0053
	0.000	***0.186***	*0.003*	*0.000*
AVG	-10.305	1.2593	2.3243	2.9275
	0.000	***0.208***	*0.020*	*0.003*

TABLE 4.10. Diebold and Mariano (1995) test results evaluating significant out of sample forecast performance for MAE (with p values in italics)

Maturities (in months)	1 period	5 period	10 period	30 period
3	-1.4751	-19.322	-9.2082	-9.6879
	0.140	*0.000*	*0.000*	*0.000*
4	-6.3061	-13.528	-5.8887	-3.5663
	0.000	*0.000*	*0.000*	*0.000*
5	-10.171	-9.6375	-3.7956	-1.6013
	0.000	*0.000*	*0.000*	*0.109*
6	-10.426	-7.0344	-2.4631	-2.1323
	0.000	*0.000*	*0.014*	*0.033*
7	-5.8083	-5.2854	-1.6032	-4.174
	0.000	*0.000*	*0.109*	*0.000*
8	0.098736	-4.2148	-1.038	-7.1133
	0.921	*0.000*	*0.299*	*0.000*
9	5.0573	-3.2398	-0.65709	-10.56
	0.000	*0.001*	*0.511*	*0.000*
10	9.3414	-2.6225	-0.39134	-14.268
	0.000	*0.009*	*0.696*	*0.000*
11	12.981	-2.1281	-0.19671	-18.083
	0.000	*0.033*	*0.844*	*0.000*
12	15.067	-1.761	-0.033755	-21.912
	0.000	*0.078*	*0.973*	*0.000*
24	-6.0949	2.5783	-0.59288	-0.92809
	0.000	*0.010*	*0.553*	*0.353*
36	-13.242	4.0234	1.7719	1.0303
	0.000	*0.000*	*0.076*	*0.303*
48	-13.102	5.2831	2.8063	2.1672
	0.000	*0.000*	*0.005*	*0.030*
60	-6.6872	5.8428	3.5514	2.9573
	0.000	*0.000*	*0.000*	*0.003*
72	8.4386	5.8176	3.8803	3.4594
	0.000	*0.000*	*0.000*	*0.001*
84	6.4154	5.3523	4.0246	3.7838
	0.000	*0.000*	*0.000*	*0.000*
96	2.3108	4.6166	3.9539	3.9966
	0.021	*0.000*	*0.000*	*0.000*
108	-2.0514	3.8764	3.8258	4.1464
	0.040	*0.000*	*0.000*	*0.000*
120	-6.3744	3.0333	3.5978	4.2297
	0.000	*0.002*	*0.000*	*0.000*
132	-10.16	2.2191	3.3403	4.2633
	0.000	*0.026*	*0.001*	*0.000*
144	-14.52	1.2915	2.9872	4.2443
	0.000	*0.197*	*0.003*	*0.000*
AVG	-9.7257	1.4284	2.5555	2.8906
	0.000	*0.153*	*0.011*	*0.004*

TABLE 4.11. Relative forecast performance from one-period ahead forecasts using the Nelson-Seigel block dynamic model (NS-block) introduced in the text and the Nelson-Seigel dynamic model (NS) introduced by Diebold et al (2006)

Maturity	RelMSE		RelMAE	
(in months)	NS	NS-block	NS	NS-block
3	2.58901	0.263443	3.930766	11.28985
4	26.91261	42.20181	9.152358	11.46112
5	21.6144	19.77488	8.457817	8.089869
6	13.6689	12.12947	5.752853	5.419393
7	19.3385	20.32152	6.502266	6.665537
8	1.78703	2.329985	3.625347	4.139652
9	6.949354	12.16115	3.719809	4.920781
10	6.781209	14.32501	3.562102	5.177239
11	3.355113	10.54164	2.432396	4.311606
12	2.208562	9.267096	1.955748	4.006128
24	5.727311	0.359273	3.056179	0.765444
36	7.486023	1.496406	3.471602	1.552079
48	5.187675	1.343465	2.864765	1.457862
60	2.225812	0.985401	1.865484	1.24123
72	0.301888	0.727037	0.690924	1.072241
84	0.143594	0.730042	0.47415	1.069059
96	1.916875	0.707845	1.752062	1.064704
108	6.063845	0.556388	3.100032	0.939032
120	12.41615	0.446858	4.456245	0.84536
132	20.69064	0.322221	5.773969	0.720541
144	31.29897	0.140802	7.143389	0.479124
AVG	7.48E+00	2.200877	3.306585	2.027849

TABLE 4.12. Relative forecast performance from five-period ahead forecasts using the Nelson-Seigel block dynamic model (NS-block) introduced in the text and the Nelson-Seigel dynamic model (NS) introduced by Diebold et al (2006)

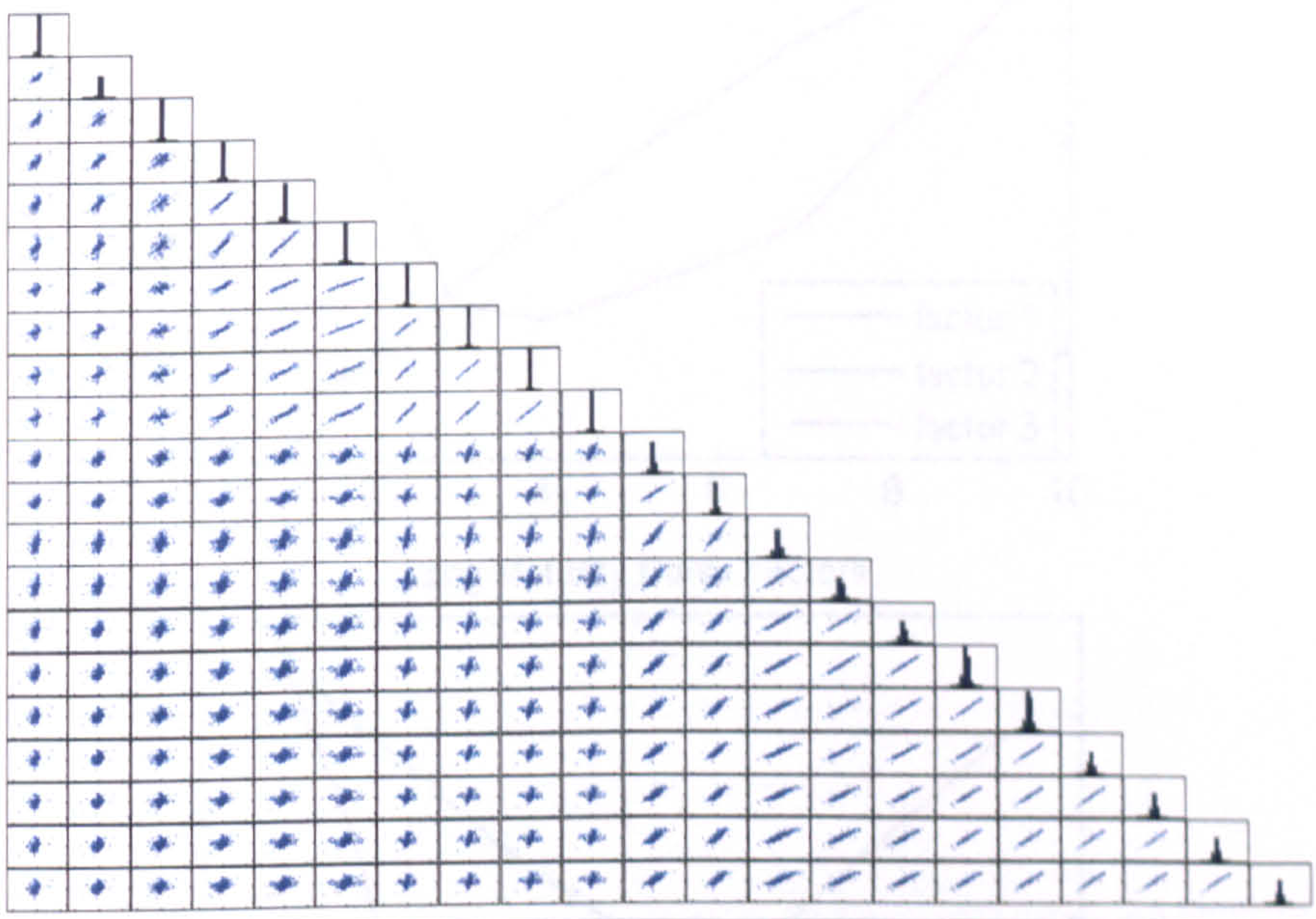
Maturity	RelMSE		RelMAE	
(in months)	NS	NS-block	NS	NS-block
3	215.6526	72.28779	11.13569	6.447115
4	103.3844	52.13276	6.808155	4.834466
5	40.72162	23.51676	4.401791	3.345108
6	20.21005	12.7823	3.104496	2.46893
7	14.21793	9.831027	2.515669	2.091873
8	4.190696	2.952221	1.640448	1.376872
9	5.704873	4.035327	1.511756	1.271413
10	4.539825	3.263092	1.315186	1.115023
11	3.026908	2.109559	1.054259	0.880136
12	2.321993	1.587293	0.907003	0.749888
24	0.019636	0.012573	0.076372	0.061112
36	0.012236	0.159989	0.059688	0.21583
48	0.0026519	0.219517	0.027952	0.254316
60	0.0076677	0.253884	0.047926	0.275776
72	0.066129	0.282234	0.14193	0.293222
84	0.175727	0.336081	0.233928	0.323507
96	0.356759	0.368601	0.336477	0.342013
108	0.591425	0.40993	0.435086	0.362224
120	0.921628	0.419436	0.547457	0.369325
132	1.28613	0.416749	0.64704	0.368321
144	1.714933	0.390483	0.747804	0.356837
AVG	1.459271	0.90566	0.68273	0.582177

TABLE 4.13. Relative forecast performance from ten-period ahead forecasts using the Nelson-Seigel block dynamic model (NS-block) introduced in the text and the Nelson-Seigel dynamic model (NS) introduced by Diebold et al (2006)

Maturity	RelMSE		RelMAE	
(in months)	NS	NS-block	NS	NS-block
3	251.0125	136.7242	7.621963	5.625402
4	91.78062	65.83396	3.90797	3.309851
5	37.6335	31.15448	2.656863	2.417343
6	19.13104	17.48273	1.933134	1.847993
7	12.09638	11.85133	1.478582	1.463544
8	4.679056	4.805496	1.039173	1.053145
9	5.019945	5.340885	0.935014	0.964421
10	3.773166	4.111195	0.792765	0.82751
11	2.616444	2.915331	0.655879	0.692309
12	1.998844	2.262232	0.568066	0.604339
24	0.204597	0.116883	0.175179	0.132402
36	0.0845	0.0021958	0.112927	0.018204
48	0.091147	0.009822	0.117632	0.038617
60	0.129003	0.042123	0.140889	0.080508
72	0.186066	0.086835	0.169815	0.116006
84	0.265917	0.13536	0.204357	0.145804
96	0.357285	0.184403	0.237623	0.17071
108	0.466927	0.233865	0.272697	0.192996
120	0.589553	0.277163	0.30616	0.209921
132	0.723996	0.315172	0.338572	0.223392
144	0.877044	0.34085	0.371736	0.231736
AVG	1.08889	0.840881	0.445311	0.370532

TABLE 4.14. Relative forecast performance from thirty-period ahead forecasts using the Nelson-Seigel block dynamic model (NS-block) introduced in the text and the Nelson-Seigel dynamic model (NS) introduced by Diebold et al (2006)

Maturity	RelMSE		RelMAE	
(in months)	NS	NS-block	NS	NS-block
3	151.956	148.7282	2.632216	2.604052
4	34.46925	36.6635	1.228166	1.266661
5	17.12221	18.74593	0.923147	0.965937
6	9.032334	9.843708	0.64936	0.677913
7	4.979126	5.296332	0.475762	0.490686
8	3.326667	3.410204	0.385287	0.390092
9	2.391856	2.346039	0.323536	0.320429
10	1.751511	1.637391	0.277768	0.268571
11	1.370721	1.219498	0.246321	0.232326
12	1.079607	0.915111	0.219186	0.201805
24	0.24209	0.026057	0.106253	0.03486
36	0.170778	1.14E-05	0.090742	0.000741
48	0.162827	0.01228	0.089521	0.024584
60	0.169045	0.037953	0.091934	0.043562
72	0.186358	0.06767	0.0973	0.058633
84	0.207713	0.099852	0.103375	0.071673
96	0.2298	0.131621	0.109659	0.082992
108	0.256745	0.163178	0.116328	0.092737
120	0.283483	0.193811	0.122523	0.101307
132	0.309149	0.226766	0.127808	0.109466
144	0.337496	0.256186	0.133054	0.115925
AVG	0.485982	0.377368	0.177313	0.144893



[redacted]

FIGURE 4.1. Matrix plot of term structure of zero coupon yields.from January 11, 1999 to July 31, 2007

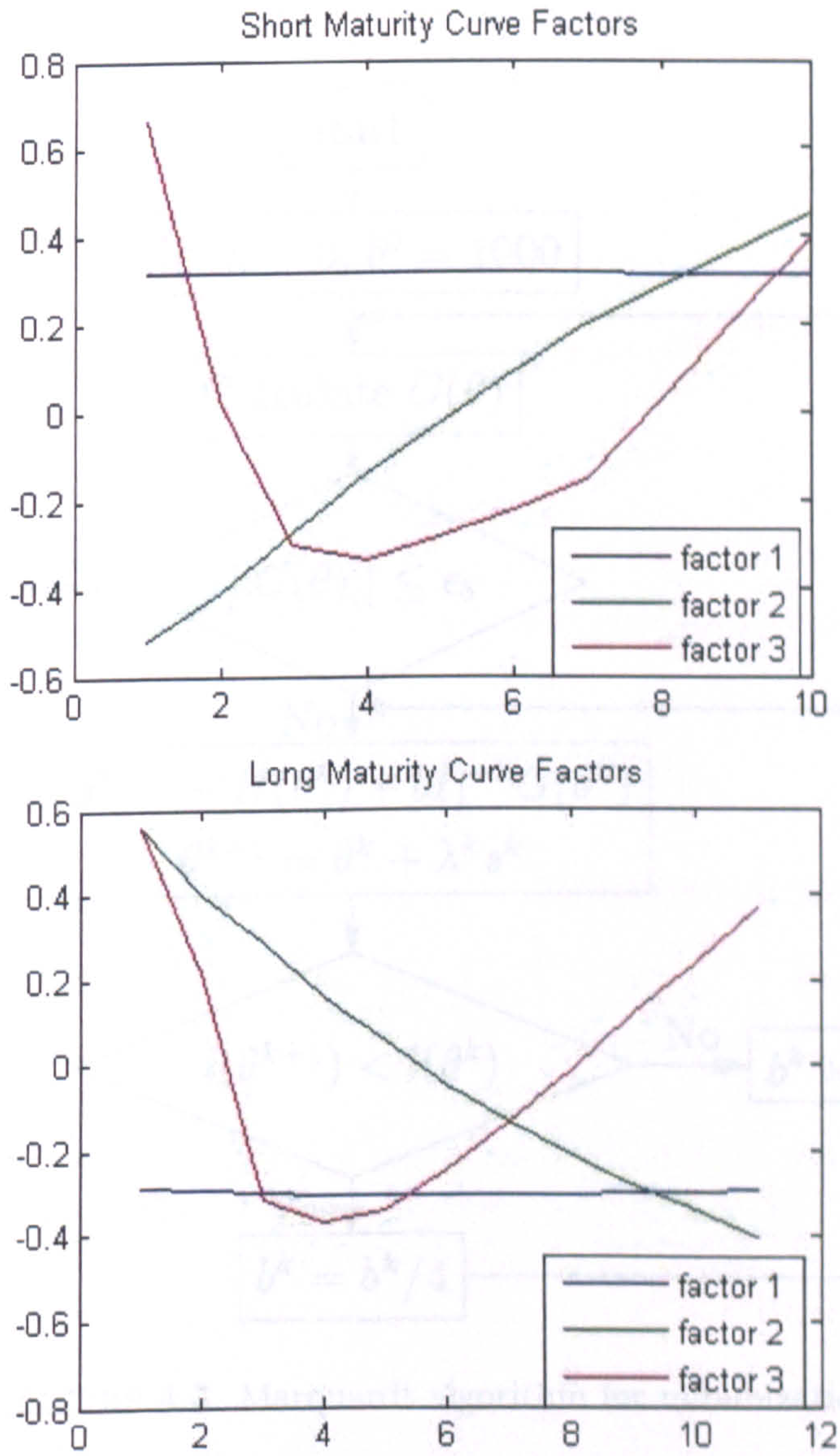


FIGURE 4.2. Long and short maturity curve factors extracted from PCA

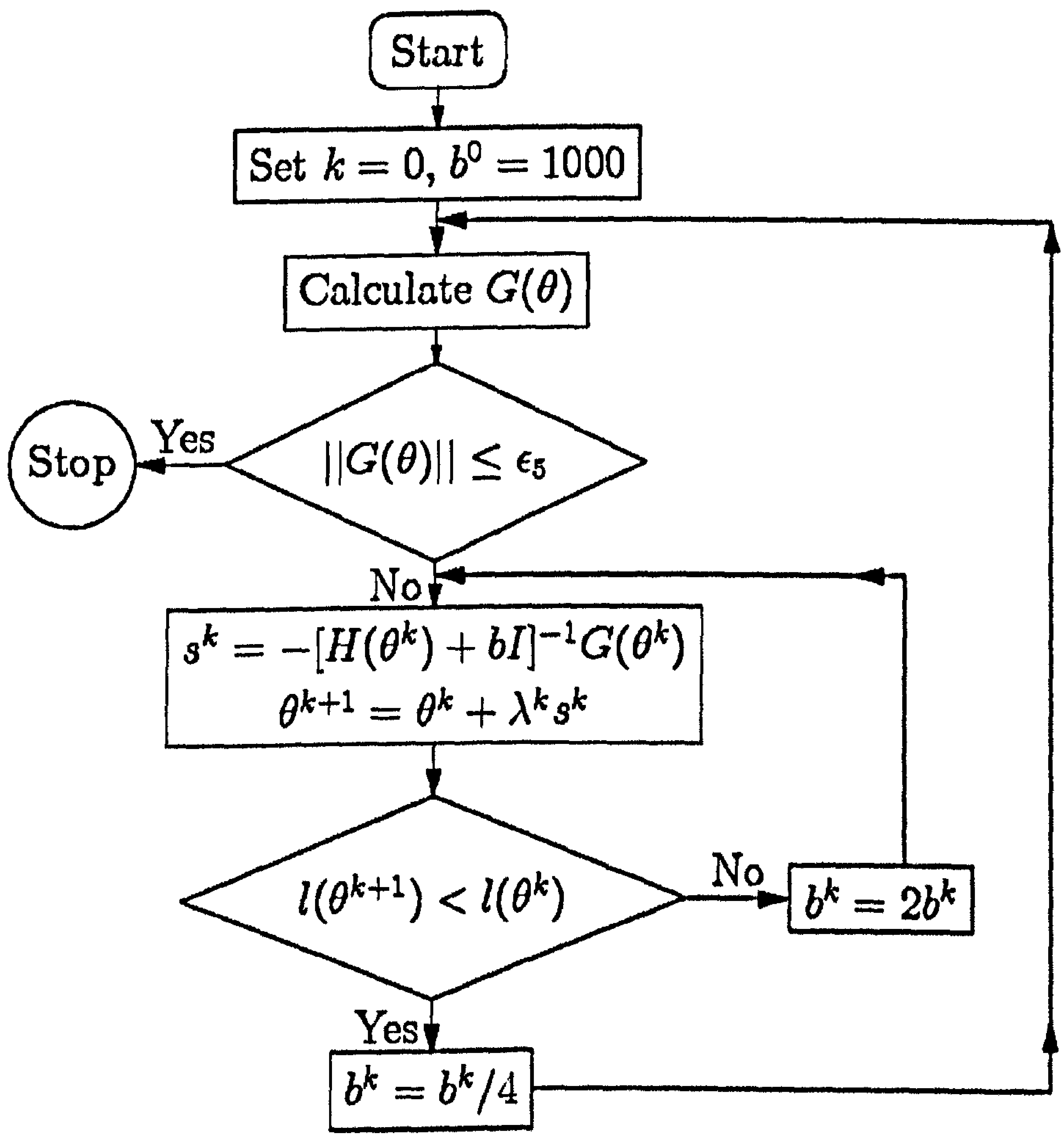


FIGURE 4.3. Marquardt algorithm for minimization

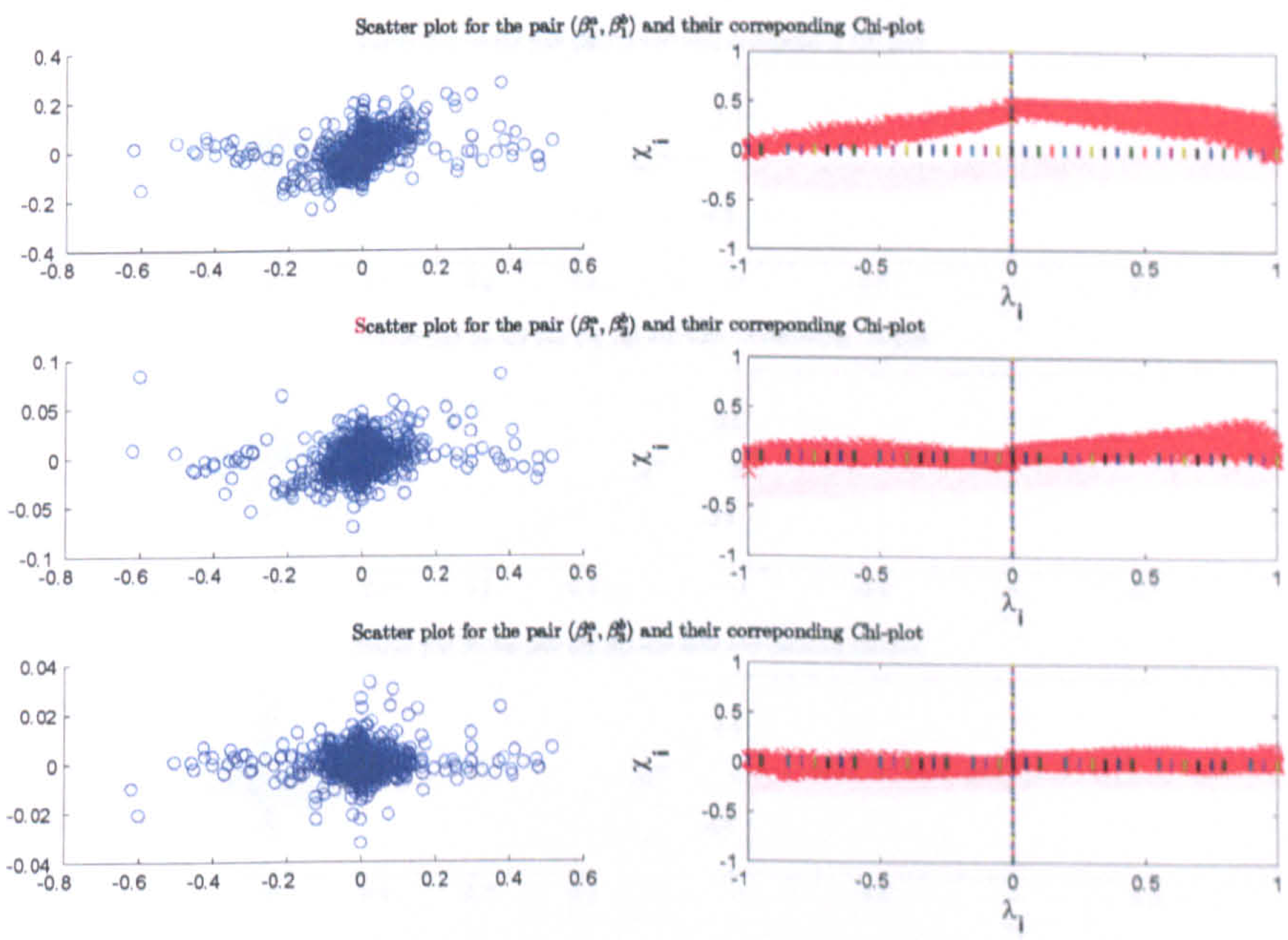


FIGURE 4.4. Scatter plot and Chi-plot of the pairs (β_1^a, β_i^b) for $i = 1, 2, 3$. χ_i lie in the interval $[-1, 1]$ and acts as a correlation coefficient between (X_i, Y_i) and λ_i is a measure of distance of the pairs (X_i, Y_i) from the bivariate median of the distribution.

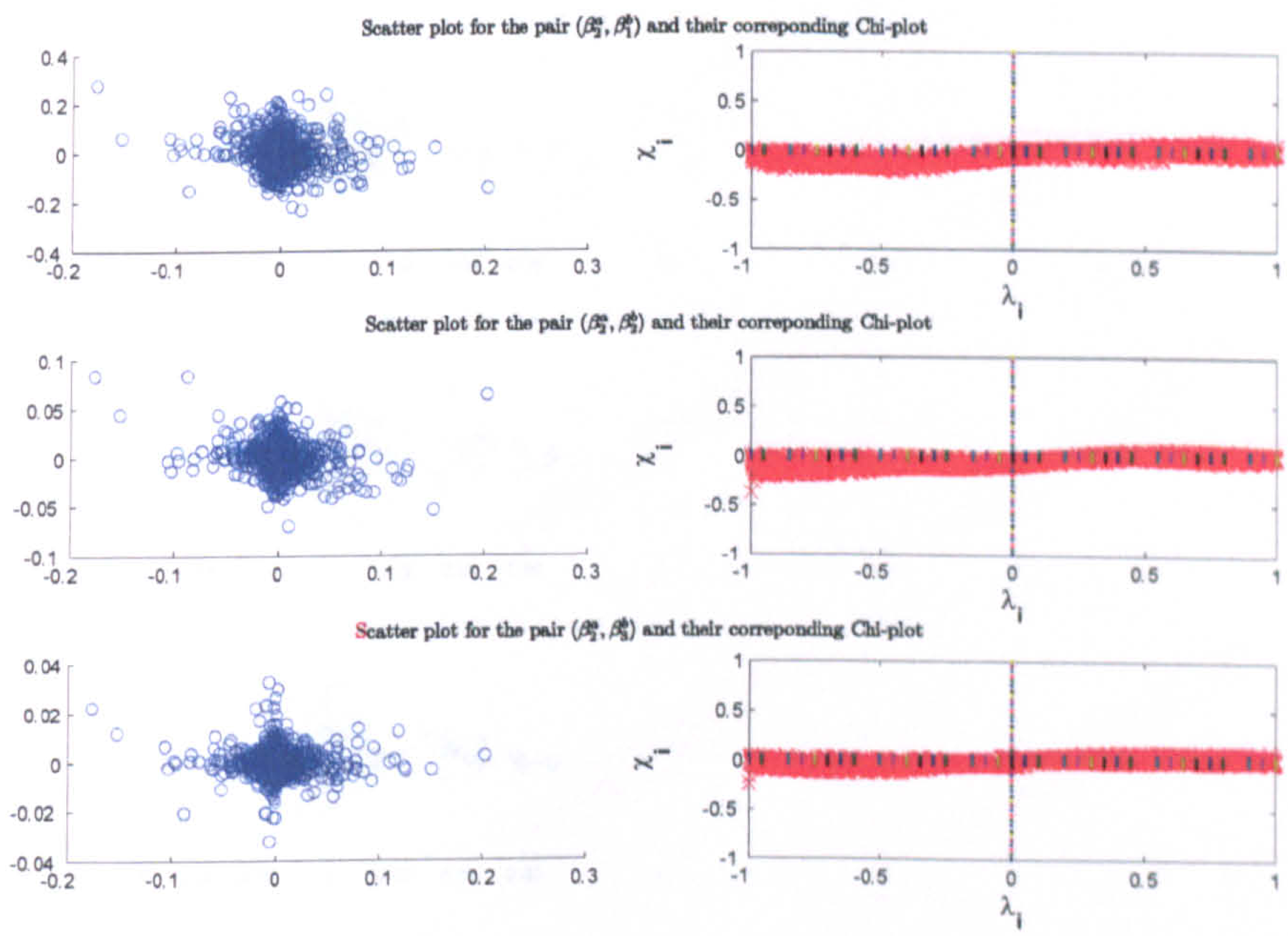


FIGURE 4.5. Scatter plot and Chi-plot of the pairs (β_2^a, β_i^b) for $i = 1, 2, 3$. χ_i lie in the interval $[-1, 1]$ and acts as a correlation coefficient between (X_i, Y_i) and λ_i is a measure of distance of the pairs (X_i, Y_i) from the bivariate median of the distribution.

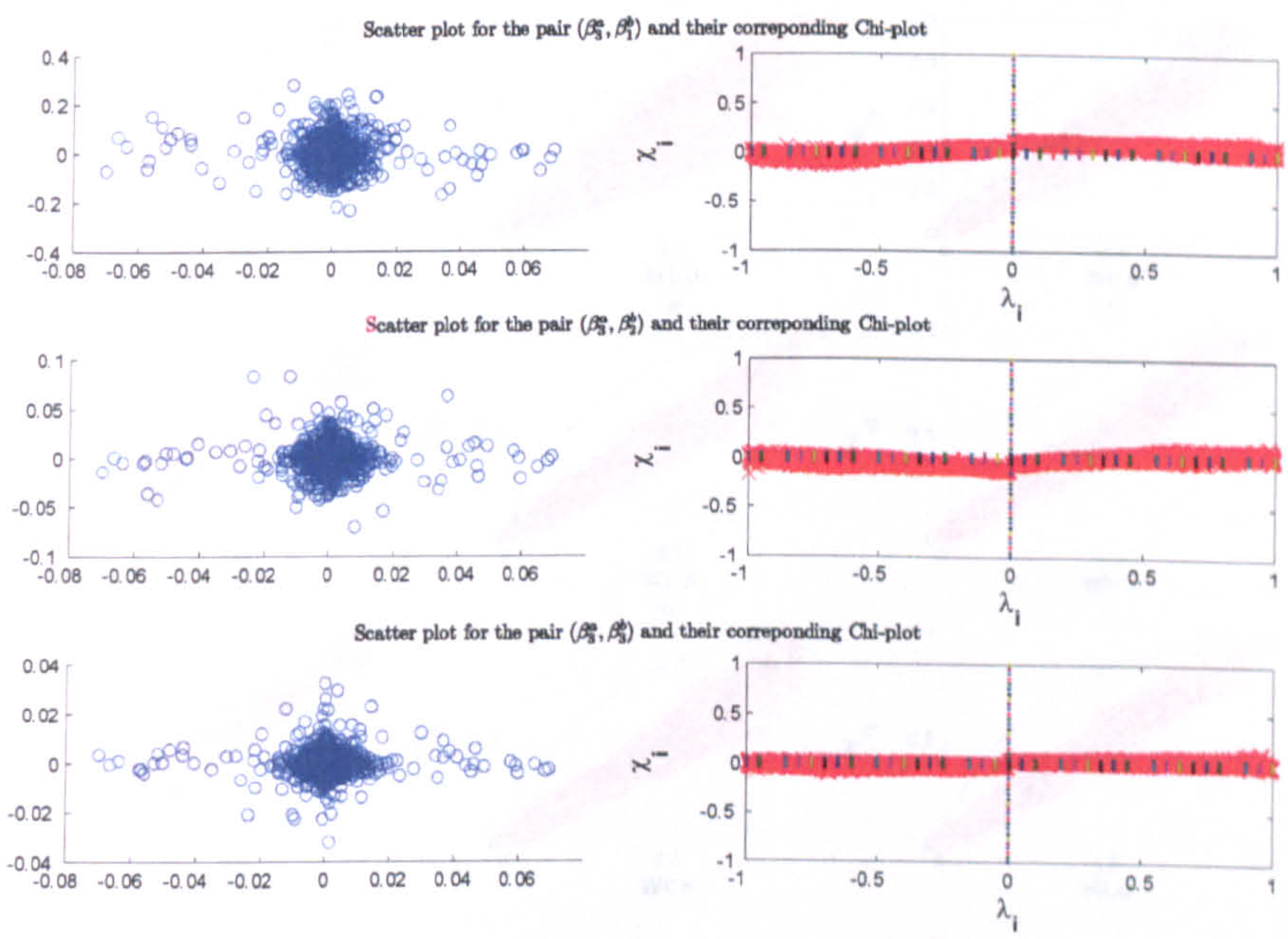


FIGURE 4.6. Scatter plot and Chi-plot of the pairs (β_3^a, β_i^b) for $i = 1, 2, 3$. χ_i lie in the interval $[-1, 1]$ and acts as a correlation coefficient between (X_i, Y_i) and λ_i is a measure of distance of the pairs (X_i, Y_i) from the bivariate median of the distribution.

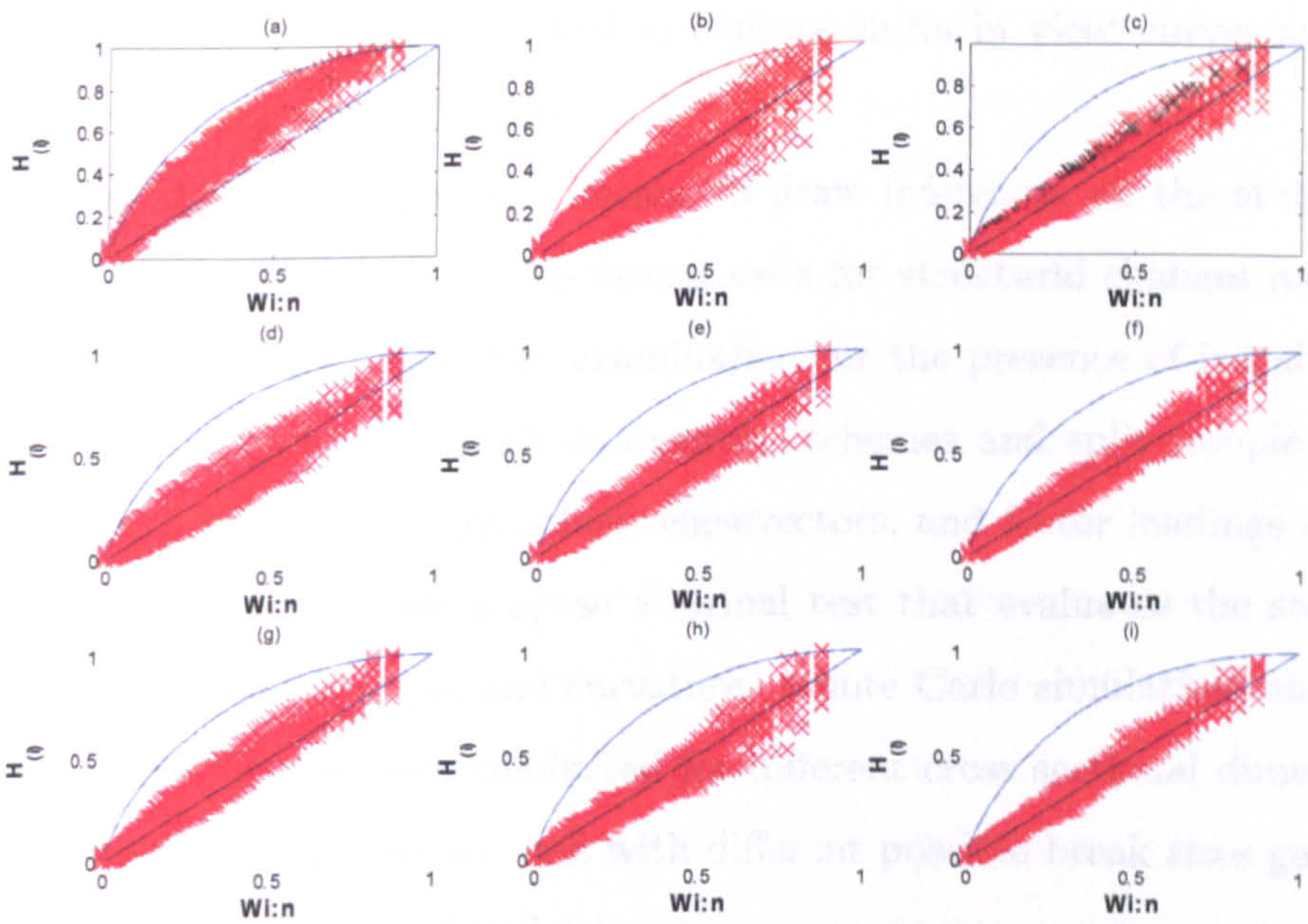


FIGURE 4.7. Recursive K-plot of the pairs (a) (β_1^a, β_1^b) (b) (β_1^a, β_2^b) (c) (β_1^a, β_3^b) (d) (β_2^a, β_1^b) (e) (β_2^a, β_2^b) (f) (β_2^a, β_3^b) (g) (β_3^a, β_1^b) (h) (β_3^a, β_2^b) (i) (β_3^a, β_3^b) . **Points lying on the straight line correspond to the case of independence between the two observations and the data points lying of the smooth curve is associated with perfect positive dependence between observations.**

Conclusions and Implications for Further Research

Panel factor models have been widely applied to modelling yield curves since they capture the cross sectional and time series properties of the term structure. Several parametric and non-parametric factor structures have been proposed in literature. The latent factors governing the term structures have been established to capture shifts in yield curves and commonly referred to as level, slope, and curvature.

Primarily authors employ graphical means to draw inferences on the stability of these factors (level, slope, and curvature). No formal tests for structural changes have been used to study yield curve factors. As a first examination for the presence of instabilities in factors, we use graphical techniques such as recursive schemes and split sample analysis and find significant variations in the eigenvalues, eigenvectors, and factor loadings governing the three factors. In chapter two, we propose a formal test that evaluates the stability in the eigenspace variables of level, slope, and curvature. Monte Carlo simulations show great size and power properties for the tests conducted for different cross sectional dimension panels, with different possible change points, and with different possible break sizes generated from different intervals of the uniform distribution.

We apply the testing procedure to zero coupon bond yield term structures obtained from Datastream, yields (monthly and daily) obtained from Federal Reserve, and the Fama-Bliss monthly yields. We find statistically significant structural changes in the factor structures governing all zero coupon bond yield term structures considered (see chp. 2 and 3). The variance process governing the factors has been found to be unstable in all cases. The findings advocate the use of common principal component analysis which relaxes the assumption of constant eigenvalues governing factors. As further research, we investigate whether the stability results inferred could be translated to the recently raised problem of unstable forecast performance of the three factor Nelson-Siegel model generating level, slope, and curvature factors. In this vein, we plan to investigate the forecast density breaks in the Nelson-Siegel model governing yield curves.

In chapter three, we investigate structural stability of the factors governing the separate correlation clusters, namely short maturity and long maturity clusters, found within the zero coupon term structure. In this case, we find that all the three factors governing short and long rates are statistically unstable. This result contrasts with the case of testing for structural changes in factors for the whole term structure where we find statistically stable slope and curvature factor loadings. The implication to this finding is a possible presence of "Cobreaking" between the short and long maturity clusters, which need to be statistically tested for in future work.

In chapter four, we account for the presence of factors governing short and long maturity clusters, potentially capturing different market information. We model the factors governing the two clusters of rates separately in a dynamic framework proposed by Diebold and Li (2006). In this, we extend and propose a block dynamic Nelson-Siegel model for yield curves with correlation clusters. The proposed new framework seems to forecast well the short end of the curve as compared to the benchmark model, which is the dynamic Nelson-Siegel (1985) model proposed by Diebold and Li (2006). This formulation can allow us to model the instabilities in the short and long rates separately and well suits the empirical objective to pick up interactions between the short and long factors. For further research, we plan to extend the framework and incorporate regime switching to the block dynamic model proposed.

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