# Advances in Travel Geometry and Urban Modelling

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# Advances in travel geometry and urban modelling

*Geoffrey Hyman and Les Mayhew*

*Abstract*

Urban travel geometry is a generalization of patterns of movement in cities and regions where route configuration and prevailing traffic speeds constrain or direct movement in distinctive and repeatable patterns. In this paper we use these properties to construct time surfaces on which distance equates to the time of travel in the urban plane. Such surfaces can be two– or three-dimensional and are useful in the study of urban structure, locational analysis, transport planning and traffic management. A particular niche addressed in this paper is non-conformal time surface transformations in which speed or the cost of travel is constrained according to co-ordinate directions. It is argued that such models may be more suited to gridded and orbital-radial cities than previously used conformal transformations. After explaining the rationale behind the approach, a mathematical basis is developed and several calibrated examples are provided based on regions in the UK. The paper concludes with some examples of potential applications, and an annex provides a detailed mathematical framework.

*Key words: Urban geometry, map transformations, time surfaces, isochrones, locational analysis.*

## Introduction

Cities operate in accordance with strong constraints on geographical access and mobility. Taken together with the limited time available for travel, such constraints are a key determinant of where people shop and work and their travel routes, which in turn influence the pattern, degree of congestion and location of major new facilities. For decades geographers have known that the operation of the transport system is an integral factor in cities thriving or stagnating. In this paper we look at these issues from a geometrical standpoint, show how geometry can bring fresh insights into these issues and contribute to an improved understanding of how cities and urban regions may develop in the future.

Travel geometry is a way of representing space that does not depend on a detailed representation of the network. This reduction in detail is essential in order to provide conceptual insights we seek, particularly at a strategic level. The pattern of land use and trip making in an urban area both constrains its transport system and stimulates its development, whilst the changing state of the transport system influences the future patterns of both travel and land use. Different cities exhibit a variety of alternative structures and therefore constraints, which in turn influence the geographical patterns of travel and the modes of transport that are used. One only has to think of the difference between gridded North American cities like New York or Washington and European cities like Paris or Rome.

Variations in traffic concentrations affect speeds and travel times, which in turn are dependent on both location and direction. We wish to represent these situations in the form of maps, based on surfaces that have an appropriate geometry. Our aim in doing so is to show how the physical reality of space and time is altered systematically and how, mathematically, it can be modelled and visualized in a conceptually transparent way (Tobler 1997). But first we need to explore why this approach is worth considering at all.

Travel geometry's analytical origins can be traced back to early location theorists and even before that to early navigators and the study of optics. The geographic renaissance in the 60s and 70s extended the concepts to research topics in economic geography, especially relating to cities. Despite this, the traditional view of a mono-centric city with economic activity located at the centre surrounded by housing still retains a tight grip on conceptual thinking. One of the best examples of this remains Alonso's classic paper published in 1965, in which he develops a micro-economic framework of the urban rental market based on competing land usage and travel costs to the city centre. With urban congestion and environmental issues at the forefront of current concerns it is essential that we update such images of cities so that they are more relevant to today's issues. This paper makes a small start in that direction by reviving ideas that have lain dormant for a couple of decades.

Our starting point is not the conventional map that is found in a road atlas, but a map that represents space in terms of travel time. The early history of cartography was directed towards mapping the Earth's approximately spherical surface onto a plane surface, for use in exploration and territorial expansion. We use that same tradition to project travel times onto a *time surface* (cf. Angel & Hyman, 1972, 1976), in which map distances between urban locations correspond to the travel time between these locations.

The smaller the surface the greater the mobility in the urban plane since journeys are, by virtue of the definition of a time surface, accomplished in a shorter time. In the limit an infinitesimally small surface would, in effect, represent an urban region in which accessibility was instantaneous. Clearly, the nearest any region comes to this would be in the narrow sense of electronic communications, and not as a result of traditional transport means however efficient they may be. Hence, this may be regarded as a qualitatively different case.

Some time surfaces can be represented in a two-dimensional plane co-ordinate system. They can be thought of as flat maps that are scaled to give the correct travel times between locations in an urban area. Locations depicted on the map are shifted from where they were in the urban plane according to the journey times separating them (Warntz, 1967). Another difference between urban and time planes is that, whereas the area of the urban plane is measured in square kilometres or miles, the surface area of a time surface is measured in square hours or minutes. If the time surface is expressed in a three-dimensional co-ordinate system, then a further transformation is needed to create a flat map that represents travel times. Later we shall give an example of a time surface that has a spherical geometry.

The feasibility of creating a flat map that represents travel time in a distortion-free way depends on whether the time surface is developable or not. A surface such as a cone can be slit along a generator and laid out flat but the same cannot be done with a sphere. This is why, of course, it is impossible to create flat maps of the entire Earth's surface that correctly represent distances between locations. Instead, conformal maps of the Earth's surface were originally introduced to show the correct *direction* so as to navigate over long distances…but this is part of discussion that is beyond our immediate concerns.

At this point it is helpful to make a general distinction between projections that start from a flat two-dimensional urban or regional plane and those that start from a spherical representation of the Earth's surface. In either case the projection is to another surface representing travel time. Such projections could portray, for example, a world shrunken and distorted by modern travel. (Tobler, 1999). Such cases are also not considered explicitly in this paper, but several aspects of the theory developed here could be applied to such a situation.

Before proceeding with the detail, we should add two cautionary points about time surface transformations. Firstly not any transformation will be fit for the purpose since, as with any model, there needs to be a fair degree of realism – travel on the transformed surface needs to give a good approximation to the actual travel times. Apart from this there need to be rigorous mathematical rules for the transformations. This requires both general and consistent specifications of the differential equations governing the transformations between the urban area or region and its time surface representation. In order to provide these features, it is also very helpful if these equations admit solutions in closed form and can be readily inverted. In this paper we shall provide several illustrations of this kind.

So far we have used the term 'travel time' loosely as a means of depicting the difficulty in getting from A to B. As is clear from the literature the term travel time can be used in a broad sense of a generalized time and can include many of the factors that influence the choice of travel route, such as fuel costs and road user charges. In other applications, we have used a number of definitions depending on the intended application (for example, see Hyman and Mayhew, 2002) but for this paper we shall restrict our definition to the narrower concept of travel time without any loss of generality.

In previous work on this topic, it was assumed that the local speed of travel is *equal in all directions*, whilst varying between locations to reflect factors such as congestion (Angel & Hyman, 1972, 1976). This assumption is compatible with the use of *conformal* map projections, i.e. transformations to a time surface that preserve local shapes and angles. Hyman and Mayhew (1982), for example, used conformal map transformations derived from time surfaces to model the minimum number of emergency medical facilities required to access any location in London within a given travel time. Wardrop (1969) used conformal transformations to move between the urban plane and a time surface in the complex number plane and recognized that, in reality, local speeds could also vary by direction. Robert Cochrane (private communication) made further original contributions, which have helped to stimulate this paper. Here, we further extend the theoretical basis and applications of map transformations to travel modelling to provide new results that have not appeared previously in the published literature.

## Making the case for non-conformal transformations

A problem with any of these earlier approaches is that when the local cost of travel differs according to *travel direction* conformal transformations do not generally have sufficient flexibility. Take a traditional large city that has developed around a central area, generally over a long period. Typically it will have major radial routes that run into the centre and a limited number of orbital routes that meet the radials orthogonally. For both historical and planning reasons radial and orbital movement would experience routes of different quality and therefore speed, but still be amenable to geometric analysis and provide important insights (Mayhew, 2000). In recent work, we have examined cases where speeds varied in orbital and radial directions, using a generalization of the Karlsruhe metric, named after the German city (Hyman and Mayhew, 2000). Karlsruhe, is just one example of a radial-orbital city of this kind, others include major capital cities such as Moscow, Paris, London and Bejing.

Constraints on the directions on movement also occur in gridded street patterns, which are typical of many North American cities, such as New York (Manhattan) and Washington DC (Krause, 1986). In such cities there is generally a principal axis served by high quality avenues that may also have priority at intersections with lower quality streets. Directional dependency in the ease of movement is also evident in settlements strung out along a coastal corridor, with fast access along the coastal roads and slower local routes inland, where natural barriers such as mountain ranges often further limit the ease of movement. An excellent example of this type is Genoa in Italy.

It is important to distinguish between two quite different aspects of these examples. The first aspect is the difference between the quality and provision of the transport services in different principal travel directions, e.g. fast radial movement versus slow orbital movement, fast avenues versus slow streets and fast coastal movements versus slow inland movement. This requires the urban or regional co-ordinates to be stretched by different amounts in order to represent travel times in the principal directions. It is this aspect that requires the flexibility provided by non-conformal transformation techniques.

The second aspect is simply the effect of constraints imposed by the pattern of land use and natural topography on the permitted directions of movement. The strength of these constraints influences the relationship between the travel times in the principal and general directions of movement. Different specifications of the local travel time metric can be used represent the effects of such constraints, which will vary between different urban areas or regions.

Each of these aspects is represented by mathematical equations, expressed in differential form. The first (set of) differential equations yield a co-ordinate transformation between geographical space and a time surface. The second equation describes the way in which the local travel times (between nearby locations), in a general travel direction, depend on small displacements in the co-ordinates. Given an analytic solution to the transformation equations, together with boundary conditions and the local travel times, the travel times between locations can then be expressed as a function of the geographical co-ordinates, for arbitrary pairs of locations. The resulting (integrated) travel time metric depends on both the differential form for local travel times and the global geometry of the time surface.

This paper will provide a theoretical framework for each of these aspects. It will illustrate how time surfaces and travel time metrics can be constructed for a substantially wider class of travel assumptions than is available from previous work, providing a geometric foundation for new applications and developments of location theory. The analytical models that we adopt can be calibrated using minimal data and so can be linked to more familiar constructs. One example is the familiar isochrone map. *Isochrones* are the locus of all points that can just be reached in a given time from any fixed location. They are widely used to measure access from (or to) a point, such as a place of employment or shopping centre, medical facility, or a major centre such as an airport or rail terminus.

It is instructive to compare what is proposed here with non-conformal transformations based on azimuthal equidistant projections to a plane surface. Azimuthal maps can be used to turn isochrones into circles, while the underlying geography is stretched or compressed to fit the selected travel time metric (e.g. Tobler, 1963, 2001). However, the problem is that these maps are generally inaccurate at locations other than the fixed point, and so need to be re-calibrated for each travel origin or point of interest. This is a drawback, particularly in applications to location theory, which need to consider alternative facility locations that provide good access to multiple service points (Mayhew 1981, 1986; Hyman & Mayhew 2001).

With our methodology it is possible to construct a unified model of the travel time relationships between several different locations. For example, it permits a single transformation to be used to create isochrones for all of the locations within the study area, and to depict how a range of different locations mutually relates to each other in terms of access. This was one of the principal achievements in Hyman and Mayhew (1983), but based on conformal transformations. In order to develop the same possibilities here we need to construct a parallel framework for dealing not only with non-conformal transformations, but one flexible enough to include Karlsruhe and Manhattan metrics (Anjoumani 1981, Klein 1988, Okabe et al 1992).

Our extended examples of transformations exploit not only the basic geometry of travel but also take into account the physical setting, for example a fast route along a coastal region. These variables are combined to produce outputs in the form of time surfaces, travel time maps, or isochrone maps. The descriptive titles we give to the examples epitomize a particular defining feature of the urban area or region, for example a transport intersection or corridor, or the underlying time surface itself. Although we may use the word ‘city’ in the title of a transformation, note that it may be applied to either an urban area or to a region that behaves like a city in routing terms.

A good example is the *Grid City* transformation, which is applied to a region of England that has trunk roads arranged orthogonally. *Surf City* by contrast is applied to the South Wales coastal region, whilst *Edge City* is applied to the outer fringe of a large conurbation, in this case London. The titles of other transformations are suggested by established urban forms. *Market Place,* for example, is a traditional market centre with radiating roads, whereas *Polaris* is so named because it has a hemispherical time surface where the urban centre is represented at the pole and the urban perimeter by the equator.

In an annex, we provide the mathematical basis for 17 such city types and provide a short description of each, although in the text we describe only five of them in detail. An extended description of all of the examples is available from the authors. The first stage in the development of such a framework is a brief discussion of co-ordinate systems and metrics. We then discuss non-conformal transformations between different sets of co-ordinates and provide a series of practical examples. A final section paints some general conclusions, and implications for urban geography and location theory. The main mathematical apparatus is contained in annexes.

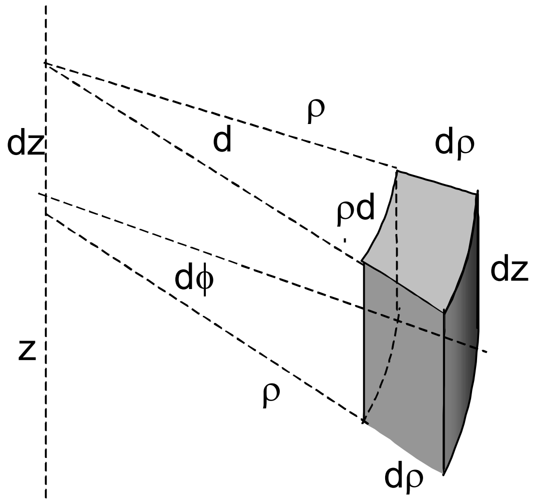
**Co-ordinate systems**

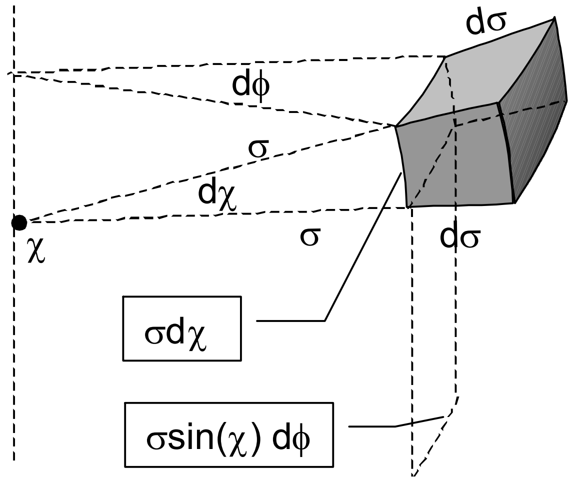
To identify points in both in geographic space and on the time surface we adopt the standard notation shown in Table 1. The notation in the top line of this is always used to represent points in the urban or regional plane. The notation in the bottom line is used to represent locations on the time surface.

| | Cartesian | Polar | Cylindrical | Spherical |
|---|---|---|---|---|
| Urban plane | (x,y) | (r,θ) | | |
| Time surface | (u,v) | (ω,ψ) | (z,ρ,φ) | (σ,χ,φ) |

*Table 1. Notation used to identify points in geographic space and the time surface*

The polar co-ordinates are in the form (radius, angle). Figure 1 shows this how the notation is applied in the cylindrical and spherical cases, which are less frequently encountered in mathematical texts than other cases.




*a) Cylindrical co-ordinates* *b) Spherical co-ordinates*

*Figure 1: a) Cylindrical co-ordinates z: height; ρ: cylindrical radius; φ: azimuth and b) spherical co-ordinates σ : spherical radius; χ colatitude; φ azimuth.*

**Metrics**

A *metric* is a real valued function d of two points that satisfies the laws d(P,P)=0, d(P,Q)>0 if P≠Q, d(P,Q)=d(Q,P) and d(P,R)≤d(P,Q)+d(Q,R). The last condition is known as the *triangle inequality*. Many travel time metrics can be written as differential forms $ds = (|ds_1|^n + |ds_2|^n)^{1/n}$ where $ds_k$ is the element of travel time in direction k and n is a parameter. When n=2 these conform to the familiar Euclidean rules, but other values are permitted.

| Co-ordinate System | $ds_1$ | $ds_2$ | n=2 | n=1 |
|---|---|---|---|---|
| Cartesian Plane (u,v) | $\|du\|$ | $\|dv\|$ | Euclidean Cartesian | Manhattan |
| Polar plane (ω,φ) | $d\omega$ | $\omega\|d\phi\|$ | Euclidean Polar | Karlsruhe |
| Cylindrical (z,ρ,φ) | $(dz^2+d\rho^2)^{1/2}$ | $\rho\|d\phi\|$ | Standard surface | Longitude/ Latitude |
| Spherical (σ,χ,φ) | $(d\sigma^2+\sigma^2 d\chi^2)^{1/2}$ | $\sigma \sin(\chi)\|d\phi\|$ | Standard surface | Longitude/ Latitude |

*Table 2: Local travel time metrics in different co-ordinate systems.*

Table 2 shows how the differential elements of travel time are defined in different co-ordinate systems. The last two columns give names to these metrics, for values of n=1 or n=2. In order to satisfy the triangle inequality, n needs to be at least unity.

The differential (local) form of the travel time metric, in the Cartesian urban plane is: $dt = ((|dx|/V_X)^n + (|dy|/V_Y)^n)^{1/n}$. and, in the polar urban plane: $dt = ((dr/V_R)^n + (r|d\theta|/V_O)^n)^{1/n}$where the $V_X$ and $V_Y$ are the speeds in co-ordinate directions x and y respectively, $V_R$ is the radial speed and $V_O$ is the orbital speed.

The links between geographical and time surface co-ordinates are provided by the differential equations for the transformations, to be discussed next. In order to represent travel times between nearby points, in general directions *the required power of n on the time surface implies an identical value of n in urban plane co-ordinates*. The appropriate value of n should ideally be based on an empirical analysis of travel times, and preliminary checks suggest that a value a little above 1 is realistic in an urban area and values of 2 or more in a rural area.

To compute journey times between locations that are far apart we need to integrate the local travel time metric along a geodesic on the time surface. We therefore require an explicit expression for the *integrated forms* of the above differential travel time metrics. The formula that is obtained will depend on the global geometry of the time surface, which includes the shape of its boundary. Later, we shall give a number of illustrations of the resulting solutions for the travel time formulae. Once (integrated) formulae for the travel times have been obtained in time surface co-ordinates the corresponding expression in the original geographical co-ordinates can be written down by simple substitution of the transformation equations. This will also be done in the illustrations given later.

**Basic transformation techniques**

We now turn to the discussion of the mathematical tools for defining the transformations to the time surface. The basic idea is to project the urban plane onto a time surface so that travel time in the urban or regional plane corresponds to recognised metrics on the projected time surface. We do this by requiring the transformation to give the correct travel times *in the co-ordinate directions of the urban or regional plane*. This will leave open the method for determining travel times in non-co-ordinate directions as a function of the times in the co-ordinate directions. As noted above, these times will depend on the local (differential) form of the travel time metric. The metric could be Euclidean, based a metric power equal to two, Manhattan, based on a metric power equal to unity, or some other value that has been calibrated from data for the study area.

The geometry of time surfaces may be split into two major sub-categories depending on whether the time surface is expressed using two plane co-ordinates or is expressed in terms of three co-ordinate directions. In the first case these plane two– dimensional time surfaces provides maps of travel times that can be laid flat on the ground. If the projection uses three-dimensional co-ordinates, then the points on the surface can sometimes, but not always, be represented as a flat map. This can be done for developable time surfaces which include cones and cylinders, as they can be opened up along a generator and laid flat on the ground. With spherical time surfaces, this is not possible without distorting the representation of travel times, i.e. spherical surfaces are non-developable.

Designing specific transformations is a craft as much as a science. The transformations need to be both realistic and mathematically tractable. However, the interpretation of any mathematical solution is far from unique. Pragmatic considerations are needed in order to specify the most appropriate co-ordinate systems, the boundary conditions and the analytic form for the transformation to be applied in a particular application. These depend on the local situation being modelled and the type and purpose of the required map. A considerable element of geographical intuition is also needed. The illustrations given in this paper should assist in making appropriate choices.

Differential equations for the transformations to the time surface have been derived for all of the combinations of co-ordinate systems quoted in Table 1. The details of these are supplied in Annex 1 and are briefly summarized below.

***a) Transformations from the urban plane to a time surface in plane co-ordinates***

For an urban plane in Cartesian co-ordinates (x,y) let ($V_X$,$V_Y$) denote the speeds in the x and y directions. Similarly, for an urban plane in polar co-ordinates (r,θ) let ($V_R$,$V_O$) denote the speeds in the radial and orbital directions. The transformations to the Cartesian time surface (u,v) and the polar time surface (ω,ψ) satisfy the differential equations given in the first table of Annex 1. Note that they are only conformal when directional speeds are equal ($V_X$=$V_Y$ or $V_R$=$V_O$).

***b) Transformations from the polar urban plane to a time surface in cylindrical co-ordinates***

These transformations map the urban plane (r,θ) to the cylindrical co-ordinate system (z,ρ,ϕ) on the time surface. They satisfy the differential equations given in the first entry of the second table of Annex 1. Again the speeds in the r and θ directions are $V_R$ and $V_O$. These transformations are only conformal when $V_R$=$V_O$.

***c) Transformations from the polar urban plane to a time surface in spherical co-ordinates***

These transformations map the urban plane (r,θ) spherical co-ordinate system (σ,χ,ϕ) on the time surface. These transformations satisfy the differential equations given in the second entry of the second table of Annex 1. These are also only conformal when $V_R$=$V_O$.

***d) Transformations from the Cartesian urban plane to a time surface in cylindrical or spherical co-ordinates***

The differential equations for these transformations are given in the third and fourth entries of the second table of Annex 1. These transformations are also only conformal when $V_X$=$V_Y$.

**Restrictions on speed variation**

A key property of these transformations is that each time surface co-ordinate depends on only *one* co-ordinate in the urban plane. This property preserves the structure of the co-ordinate grid, which greatly facilitates the interpretation of the resulting maps. However, this property implies functional restrictions on the form of speed variation in each urban co-ordinate direction. The general specifications of these restrictions are given in the appropriate table entries in Annex 1. The form they take generally depends on both the urban and regional co-ordinate system and the co-ordinate system for the time surface. This provides a range of different possible restrictions and retains a source of flexibility for model designers, as the choice of the co-ordinate system can be tailored to the geographical pattern of speed variation in the study area.

**Illustrations of time surface transformations**

In the following sections we illustrate these techniques with practical examples drawn from a range of typical geographical situations. In each case there are speed differences between travel in the urban and regional co-ordinate directions, requiring the use of non-conformal transformations. The illustrations differ with respect to the restrictions imposed on speed variations, resulting in different time surface geometries. The intended output maps in each case are representations of travel times in which the intermediate stage involves the creation of a time surface. In some cases the time surface are planes, which only require an analysis in two dimensions. In other illustrations the surfaces use all three co-ordinates. We give each city a name like 'Grid City ' or 'Edge City' for reasons we hope will be apparent. We start with the simplest case: a Cartesian to Cartesian transformation.

*Example 1: A Cartesian to Cartesian transformation*

The co-ordinates in the urban (or regional) plane, and on the time surface, are both Cartesian, so that the required transformation is particularly simple. Let $V_x$ and $V_y$ be the speeds in the x and y directions. The restrictions permit speeds in each co-ordinate direction to vary as a function of that co-ordinate alone: $V_X=1/f(x)$ and $V_Y=1/g(y)$.

This class of transformations is illustrated by 'Grid City'. This could represent a city in the conventional sense or a road network linking several cities within a region. In Grid City, we consider the basic case where the speed of travel in each co-ordinate direction is a constant, but the speeds differ in the two co-ordinate directions, so that $V_X=a$ and $V_Y=b$ with b>a. The transformation to the time surface is given by u(x)=x/a and v(y)=y/b. The inverse transformation (back to the urban plane) is: x(u)=au and y(v)=bv.

Consider two geographical locations ($x_0$, $y_0$) and (x,y), corresponding to the points ($u_0$,$v_0$) and (u,v) on the time surface. If we assume that travel is only permitted in the co-ordinate directions, then the metric power would be unity. The integrated travel time metric is:

$$t = \left( | u - u_0 |^n + | v - v_0 |^n \right)^{1/n} = \left( \left| \frac{x}{a} - \frac{x_0}{a} \right|^n + \left| \frac{y}{b} - \frac{y_0}{b} \right|^n \right)^{1/n}$$

where n is the metric power. We take an empirical example based on a region of England separated by a mountain range known as the Pennine Chain. In this region the 'avenues' are fast motorways running north south whereas the 'streets' are the Trans-Pennine roads running east–west. For locations separated by equal crow-fly distances, north-south travel take 25% less time than east-west travel. On the time surface this gives 25% compression along a north-south axis as is shown in Figure 2.

Consider now the construction of isochrones on either surface. In either case they are obtained from equations like the one above by specifying the travel time interval between isochrones. For the urban plane we pick a fixed-point ($x_0$ ,$y_0$), hold t constant while allowing x to vary systematically and solve for y. On the time surface we do the same using u and v. Figure 3 shows isochrones as they would appear for a metric power n=1. On the time surface the isochrones are squares, in the regional plane the inverse transformation stretches the isochrones in the direction of the motorways, so they become diamonds.

However, as this is a predominantly a rural situation, the metric power would be expected to be at least 2 (based on our experience of this particular model). The adoption of a different power would not affect the time surface transformation, but it would change the shape of the isochrones. For n=2 the squares would be replaced by circles and the diamonds by ellipses.

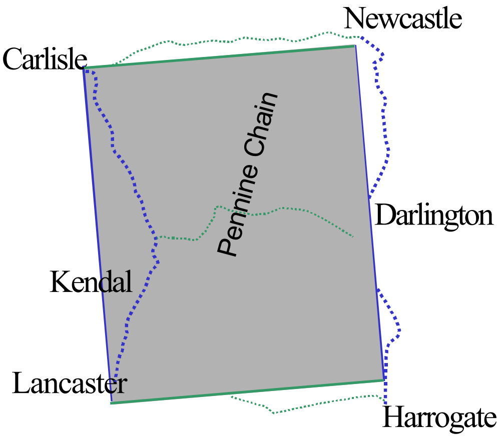



*Figure 2: A map transformation of Northern England showing a) a geographical representation and b) a map of the time surface.*

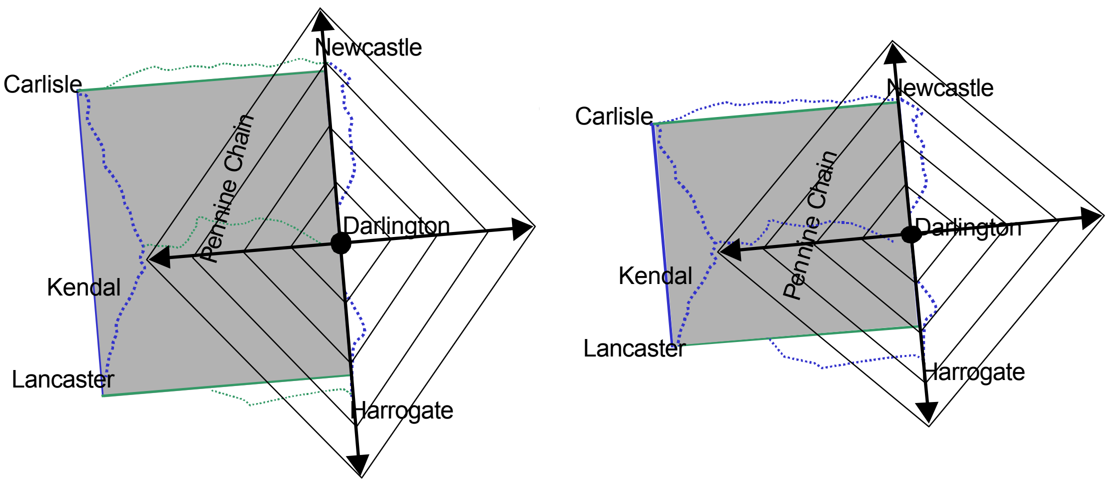



*Figure 3: Isochrone map centered near Darlington, based on this map transformation a) in the regional plane and b) on the time surface. It shows a contour map of locations that take the same time to reach from a location near Darlington. The map assumes a metric power of n=1 and would have a different appearance for other metric powers. The arrows show the directions of the co-ordinate axes.*

<u>*Example 2: A polar to Cartesian transformation*</u>

Our second example is referred to as 'Edge City' because it appears to describe locations that are close to the edge of a major city or conurbation. The urban plane co-ordinates are (r,θ) and the time surface co-ordinates (u,v). The radial speed is restricted to satisfy $V_R$ =1/f(r) and the orbital speed is restricted to satisfy $V_O$ = r/g(θ). These restrictions imply that the orbital component of travel time is proportional to the radius, so that all complete circuits around the city take the same time even though angular speed variations may be involved.

The specific form of the radial speed function assumed for Edge City is $V_R = ar^m$ where $m<1$, and the orbital speed function is $V_O = r/\lambda$. These imply average speeds increase at a diminishing rate with distance from the city centre, whilst orbital speeds are proportional to radius and constant with respect to the angle made with the city centre. The equations for the transformation from the urban plane to the time surface are given by:

$$u = \frac{r^{1-m}}{a(1-m)} \quad and \; v = \lambda\theta \quad where - \lambda\pi \le v \le \lambda\pi$$

This maps the urban plane into a vertical strip bounded between $v=-\lambda\pi$ and $v=\lambda\pi$. Under this transformation, movement along the v axis corresponds to orbital movement in the urban plane. It can be visualized by identifying the upper and lower edges $-\lambda\pi$ and $\lambda\pi$, so that the strip becomes a cylinder, and rotating it to align with cylindrical co-ordinates.

The inverse transformation, back to the urban plane, is given by:

$$r = \left(a(1-m)u\right)^{1/(1-m)} \quad and \; \theta = \frac{v}{\lambda}$$

The transformation is depicted in Figure 4. The upper figure represents the urban plane. It has a grid spacing of 10 minutes travel time, in both radial and orbital directions. The radial block size is non-uniform, but the orbital block sizes are all equal in length. The lower figure shows how these blocks are mapped into squares in the travel time plane. Under the transformation, radii map into horizontal lines. The city centre maps into the leftmost vertical line u=0, and circles around the city centre into other vertical lines. The arrow connects a point in the urban plane to its image in the travel time plane. This point is 30 minutes away from the centre and 20 minutes clockwise from due north.

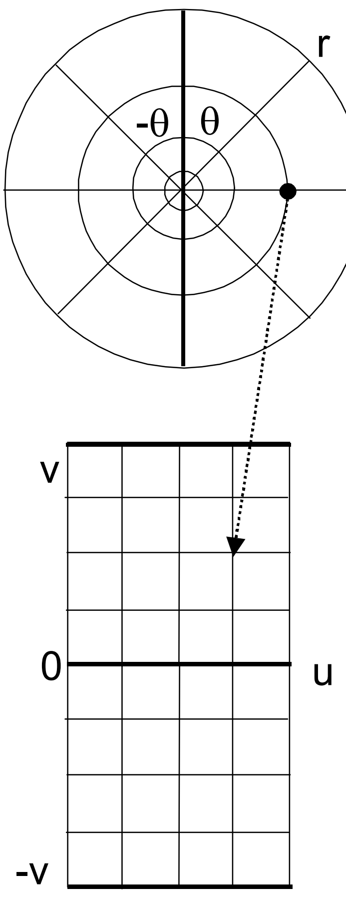



*Figure 4: Mapping from the urban plane in polar co-ordinates to the time surface in Cartesian co-ordinates.*

If the metric power n=1, the travel time metric is of Manhattan type. In time surface co-ordinates the integrated form is:

$$t = Min\{u + u_0, |u - u_0| + Min(|v - v_0|, |v - v_0 + 2\lambda\pi|, |v - v_0 - 2\lambda\pi|)\}$$

The first term corresponds to a 'double radial' movement (via the origin O); the second term corresponds to movements using orbital roads. The terms involving $2\lambda\pi$ represent the effect of identifying the upper and lower edges of the strip, to form a cylinder. Using the transformation, we obtain this metric in urban co-ordinates:

$$t = Min\left(\frac{r^{1-m} + r_0^{1-m}}{a(1-m)}, \frac{|r^{1-m} - r_0^{1-m}|}{a(1-m)} + \lambda Min(|\theta - \theta_0|,. |\theta - \theta_0 + 2\pi|. |\theta - \theta_0| - 2\pi)\right)$$

If the metric power n=2, the travel time metric is a Euclidean type, with integrated form:

$$t = \{(u - u_0)^2 + Min((v - v_0)^2, (v - v_0 + 2\lambda\pi)^2, (v - v_0 - 2\lambda\pi)^2\}^{1/2}$$

$$= \left(\frac{(r^{1-m} - r_0^{1-m})^2}{a^2(1-m)^2} + \lambda^2 Min\left((\theta - \theta_0)^2, (\theta - \theta_0 + 2\lambda\pi)^2, (\theta - \theta_0 - 2\lambda\pi)^2\right)\right)^{1/2}$$

Figure 5 shows this map transformation applied to the case of London. The M25 fast orbital motorway is taken to be a pure circle of radius roughly 25 kilometres as shown on the left. In the transformed travel time plane, shown to the right, the city centre is now a dark strip running from top to bottom of the map. The M25 meanwhile runs parallel to the city centre to the left of the map. The distance between the left edge and the dark strip is the time it takes to travel from the city centre to the M25 along one of the many radial routes (radial speeds are assumed to be constant). The height of the strip represents to time it takes to circumnavigate the M25 (i.e. from Dartford to Dartford).

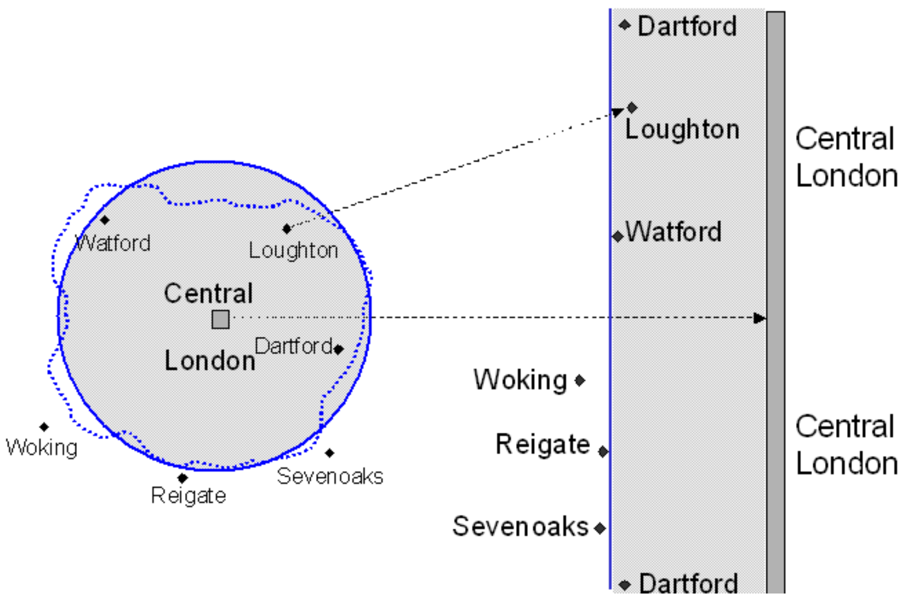



*Figure 5: London urban plane transformed from a disk in the urban plane to a time surface consisting of a plane strip.*

*Example 3: A transformation from a polar urban plane to a time surface in polar or cylindrical co-ordinates*

For these classes of transformation the radial speed is restricted to satisfy $V_R$ =1/f(r) and the orbital speed is restricted to satisfy $V_O$ = 1/h(r)g(θ). The orbital speed variations are clearly very much less restricted than in example 2. The urban plane is defined in polar co-ordinates and the time surface in either plane polar or cylindrical co-ordinates. We are free to do this because these two classes of transformation have compatible restrictions on speed variation. To distinguish the two time surfaces, we shall use the term 'travel time plane' to refer to the representation in the plane polar co-ordinate system.

This illustration is called 'Marketplace', as it appears to describe a city that has developed from a traditional market town and still retains a compact city centre.

The radial speed of travel is assumed to be $V_R$ = a and the orbital speed of travel $V_O$ = b, (a<b). Let λ=a/b. The transformation maps the urban plane (r,θ) into a pie slice in the travel time plane, given by ω = r/a and ψ = λθ, where −λπ<ψ<λπ. The inverse of this transformation is r = aω and θ = ψ/λ.

This mapping can be visualized by identifying the cuts in the pie slice λπ and -λπ, so the travel time plane becomes a conical time surface, as shown in Figure 6. In cylindrical co-ordinates (z,ρ,ϕ) this is gives us the time surface transformation:

$$\varphi = \frac{\psi}{\lambda} = \theta$$

$$\rho = \lambda\omega = \frac{r}{b}$$

$$z = \omega\sqrt{1-\lambda} = \frac{r\sqrt{b-a}}{ab}$$

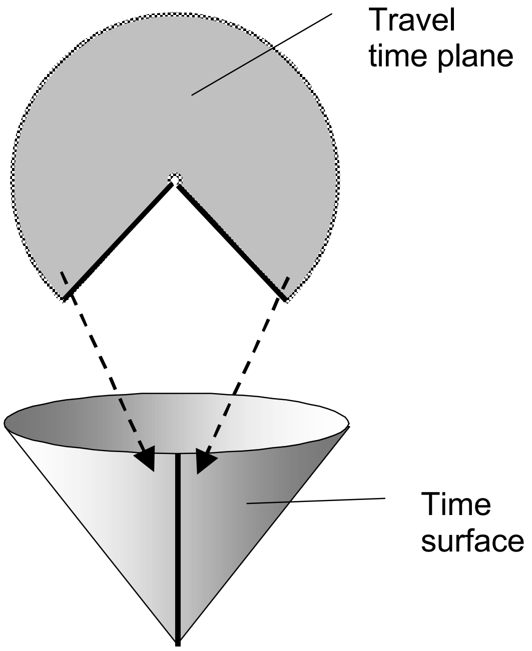



*Figure 6: Illustration showing a transformation from the travel time plane to the time surface.*

When the metric power n =1, the integrated form of the travel time metric is:

$$t = Min\left(\omega + \omega_0, |\,\omega - \omega_0\,| + Min(\omega, \omega_0) Min(|\,\psi - \psi_0\,|, |\,\psi - \psi_0 + 2\lambda\pi\,|, |\,\psi - \psi_0 - 2\lambda\pi\,|)\right)$$

$$= \frac{1}{a} Min\left(r + r_0, |\,r - r_0\,| + (a/b) Min(r, r_0) Min(|\,\theta - \theta_0\,|, |\,\theta - \theta_0 + 2\pi\,|, |\,\theta - \theta_0 + 2\pi\,|)\right)$$

When the metric power n =2, we obtain the travel time metric:

$$t = \left(\omega^2 + \omega_0^2 - 2\omega\omega_0 \cos((1-\lambda) Min(|\,\phi - \phi_0\,|, |\,\phi - \phi_0 + 2\pi\,|, |\,\phi - \phi_0 - 2\pi\,|)\right)^{1/2}$$

$$= \frac{1}{a} Min\left(r^2 + r_0^2 - 2rr_0 \cos((1 - a/b) Min(|\,\theta - \theta_0\,|, |\,\theta - \theta_0 + 2\pi\,|, |\,\theta - \theta_0 + 2\pi\,|)\right)^{1/2}$$

We consider now a traditional European city with radial routes (fanning out from a central point), linked by orbital routes. Travel is possible in either a radial or an orbital direction, or in a combination, so the metric power n=1. In the travel time plane, orbital routes are shorter than radial routes whenever the trip ends are separated by an angle of less than 2 radians (115 degrees). Let $\lambda = V_R/V_O$ be the ratio of radial to orbital speeds. In the urban plane (r,θ), orbital routes take less time than radial routes whenever the trip ends are separated by an angle of less than 2/λ radians (see Figure 7). If λ has a value of less than 2/π=0.637 then radial routes will always take longer than orbital routes. This result can be used to estimate a central area road user charge which diverts through traffic onto the ring road and away from the city centre (Hyman and Mayhew, 2002).

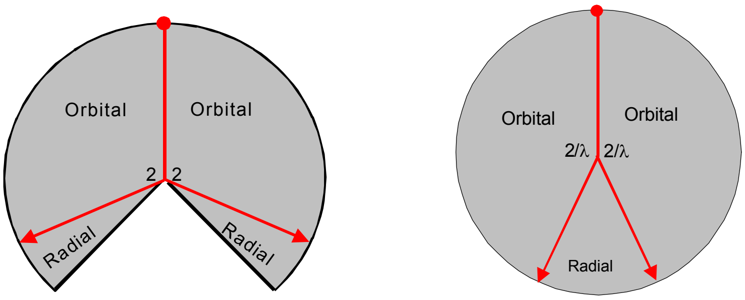
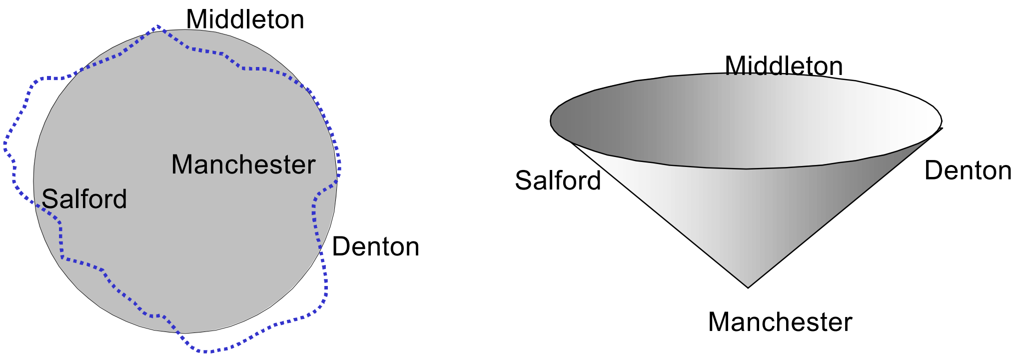



*Figure 7: Two discs showing the travel time plane (left) and the urban plane (right).*

For example, consider the English city of Manchester, where a newly completed orbital motorway has transformed route patterns and accessibility. Between any two locations that are near the ring road, orbital movements will now take less time than radial movements. This cuts out a 25% pie slice in the travel time plane and results in the conical time surface shown in Figure 8.




a) Manchester urban plane

b) Manchester time surface

*Figure 8: a) The urban plane for the Manchester inner ring road, and b) the time surface in cylindrical co-ordinates.*

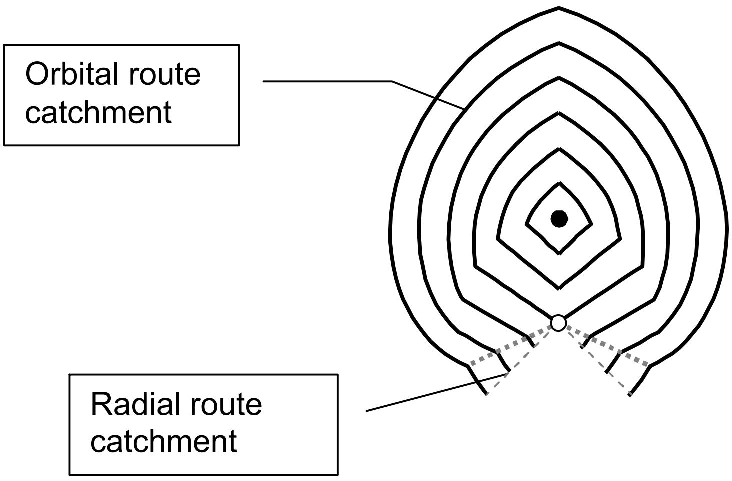



*Figure 9: Isochrones for Middleton, in the Manchester travel time plane*

Figure 9 shows isochrones, at 5 minute intervals, drawn in the travel time plane, for a location near Middleton. The metric power n=1. Also shown are the locations that are quickest to reach via orbital routes, or via radial routes through the city centre.

*Example 4: A transformation from a Cartesian regional plane to a plane time surface in polar co-ordinates*

The application envisioned in this example is an urbanized coastal region with a developed interior consisting of towns and villages. A fast motorway runs parallel to the coast from east to west, whilst the quality of the inland routes parallel to the coastal road are assumed to be of more limited quality with average speeds declining the further one traverses inland. Routes also connect the inland towns to the coastal road via orthogonal routes on which average speeds also decline with distance from the coast. The illustration is based on one given by Cochrane (private communication). Because of its strong coastal focus it is called 'Surf City'.

First, we need to check the restrictions on speed variations for this pair of co-ordinate systems. These are $V_X=1/f(x)$ and $V_Y=1/\omega(x)g(y)$. In Surf City we assume that, on journeys along the coast, speeds are constant in that direction. Speeds in both coastal and inland directions decline at an exponential rate with the distance from the coast. Surf City has the speed functions $V_Y = be^{-mx}$ for movement in the y direction, along the coast, and $V_X = ae^{-mx}$ for movement in the inland direction x. The coast is at x=0 and x cannot be negative.

Let $\lambda$=am/b. The transformation to the time surface is given by:

$$\omega = Ke^{mx} \text{ and } \psi = \lambda y$$

where K is a constant of integration. There are no limits to the angular range as it can describe multiple circuits as the distance along the coast increases. To evaluate K we apply boundary conditions at x=0 (i.e. on the seafront), to give K=1/am. The inverse transformation takes the form**:**

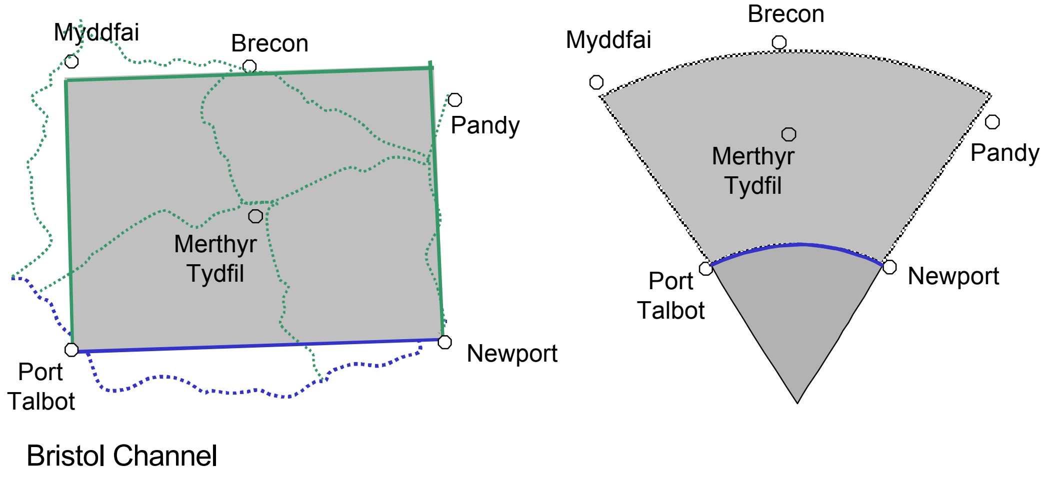
$$x = \frac{\ln(\omega / K)}{m}, \omega \geq K \quad and \;\; y = \psi / \lambda$$

For n=1, the travel time metric, in time surface and regional plane co-ordinates are:

$$t = Min\left(\omega + \omega_0 - 2K + K\,|\,\psi - \psi_0\,|, |\,\omega - \omega_0\,| + Min(\omega, \omega_0)\,|\,\psi - \psi_0\,|\right)$$

$$= \frac{1}{am} Min\left(e^{mx} + e^{mx_0} - 2 + \lambda\,|\,y - y_0\,|, |\,e^{mx} - e^{mx_0}\,| + \lambda Min\left(e^{mx}, e^{mx_0}\right)|\,y - y_0\,|\right)$$

Figure 10 shows an application of this model to a region in South Wales, which has the geographic characteristics we require. Towns inland are connected by lower quality roads whilst the coast is served by a fast motorway, the M4. South of the motorway there is the (inaccessible) Bristol Channel. The pie shaped time surface is shown to the right with the darkly shaded area cut out, representing the Bristol Channel. The upper half of the time surface shows the part in which we are interested. The corridor served by the M4 between Newport and Port Talbot is squeezed together relative to the section joining Pandy and Myddfai in the north. Based on the locations of the towns indicated it would be quicker to use purely local inland routes to travel between Pandy, Brecon and Myddfai rather than to use the M4. However, in travelling from Pandy to Port Talbot, or from Myddfai to Newport, the coastal route the route using the M4 takes less  time than alternative routes using purely local roads.




a) Regional plane (x,y) b) Time surface (ω,ψ)

*Figure 10: The regional plane (a) and time surface (b) for the South Wales coast between Newport and Port Talbot. A fast motorway serves the corridor and runs along the y axis. The positive x axis runs inland from the coast, where $x \geq 0$.*

*Example 5: A transformation from a  polar urban plane to a hemispherical time surface, in cylindrical and spherical co-ordinates*

In our next example we present a case in which orbital speeds are permitted to increase with distance from the city centre. To confirm that this is permissible we can check the speed restrictions for these classes of transformation. They both give us: $V_R=1/f(r)$ and $V_O=1/h(r)g(\theta)$, so the required orbital speed variation can be implemented by means of the function h(r).

We have in mind here a large urban area in which population density declines with radial distance and traffic starts to flow more freely. Travel is also assumed to be restricted to radial and orbital directions, so the metric power will be unity and the travel time metric will be of the type that we have called ‘latitude/longitude’.

On the surface of the Earth, the pole star provides a traditional aid to navigation, but it is only visible from the Northern hemisphere.  This illustration combines elements of Marketplace and Edge City, with the central market corresponding to the North Pole and the edge of the city to the equator. It has therefore been called ‘Polaris’.

The model assumes that radial speeds remain constant, $V_R=a$ , and that orbital speeds vary according to $V_O = abr/\sin(br)$, where r is the distance from the city centre and a and b are parameters. The appropriate transformation to the time surface is defined by:

$$\phi = \theta, \qquad \rho = \frac{\sin(br)}{ab}, \qquad z = K - \frac{\cos(br)}{ab}$$

where K is an arbitrary constant of integration. It follows that the cylindrical radius is related to the height by an equation of the form:

$$(K - z)^2 + \rho^2 = \frac{1}{a^2 b^2}$$

The vertical cross-section of the surface is therefore a circle of radius $\sigma = 1/ab$. Rotating this circle around the azimuth shows that the time surface is a sphere. It is convenient to set the constant K equal to the sphere's radius, giving $(\sigma\text{-}z)^2 + \rho^2 = \sigma^2$. When $r = 0$ we map into the pole: $z = \rho = 0$ and when $r = \pi/2b$ we map onto the equator: $z = \rho = \sigma$. By a suitable choice of origin, $\sigma$ can be set equal to the radial co-ordinate in the spherical co-ordinate system. The specification of the other spherical co-ordinates is completed by noting that $\sin(\chi) =.\rho/\sigma$ and that the spherical azimuth is the same as the cylindrical azimuth. The transformation equation for the cylindrical radius $\rho$ implies that the colatitude is simply $\chi$=br.

Referring to table 2, the element of travel time, along the latitudes of the time surface, is equal to $ds_1=\rho|d\phi|$. The element of travel time, along the longitudes of the time surface, is $ds_2 =\sigma|d\chi|$, since $d\sigma=0$. The mixed co-ordinate system combinations used here are shown in Figure 11.

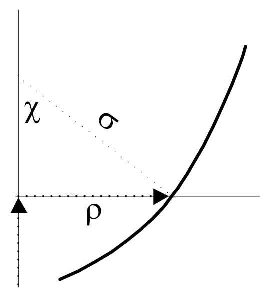



*Figure 11: Co-ordinate combinations used in the Polaris travel time metric.*

The assumptions made above about restrictions on movement to radial and orbital directions imply that movement on the hemisphere is entirely restricted to the curves of latitude and longitude, so the metric power n=1. It follows that the travel time metric, in time surface and urban plane co-ordinates, is given by:

$$t = Min\left(\sigma(\chi_0 + \chi), \sigma|\chi_0 - \chi| + Min(\rho_0, \rho)|\phi_0 - \phi|\right)$$

$$= \frac{1}{a} Min\left(|r_0 + r|, |r_0 - r| + \frac{|\theta_0 - \theta|\, Min(|\sin(br_0)|, |\sin(br)|)}{b}\right)$$

The first argument in the outer minimization corresponds to travel that includes orbital movements in the urban plane (along a circle of latitude on the time surface). The last argument in the outer minimization corresponds to a double radial movement in the urban plane (through the pole of the time surface). The link between the metric on the spherical time surface and the formulae in urban plane co-ordinates uses the three equations: $\chi$=br, $\rho$=sin(br)/ab and $\phi=\theta$, which were obtained earlier.

The time surface could be displayed as a globe similar to one of the Earth's surface, with the latitudes and longitudes marked on it, as found in many schools and offices. The locations of well-known sites or centres within the city could be positioned appropriately, although only one hemisphere would be covered (the other hemisphere could be used to portray another city or be used to display other information). To obtain the travel time between two locations we simply count the least number of latitudes plus longitudes that need to be crossed between the locations. The use of a globe here, and below, is merely to aid visualisation and is not proposed as a means of making journey time calculations

Now take the case for when n=2, the quickest paths would be arcs of great circle routes on the spherical time surface. This would gives us the travel time metric:

$$t = \sigma \cos^{-1}\left(\sin\chi \sin\chi_0 \cos(\varphi - \phi_0) + \cos\chi \cos\chi_0\right)$$

$$= \frac{1}{ab}\cos^{-1}\left(\sin br \sin br_0 \cos(\theta - \theta_0) + \cos br \cos br_0\right)$$

A globe representing the time surface would now need to be able to both rotate on its polar axis and also to pivot on the equatorial axis. In fact, there is a way to do this mechanically so that any two points can be lined up along a calibrated graticule, in the form of half of a great circle, fixed to the frame holding the globe. Travel times between any two points can then be read off by counting regular markings on the graticule, just like using a ruler to measure distance in the plane.

The present application is to a densely built up conurbation, so we assume that n=1 and use the first metric formula given above. Consider two locations on the hemispherical time surface, on opposite sides of its equator. Their co-latitudes are both $\pi/2$ radians and their longitudes differ by $\pi$ radians. The time via either an orbital or a double radial route are both equal to $\pi\sigma$ hours. Now consider more general pairs of locations, with urban co-ordinates $(r_0,\theta_0)$ and $(r,\theta)$. The critical separation in longitude above which polar travel is shorter than travel via a curve of latitude is given by:

$$\phi^* = \frac{2\chi_{\min}}{\sin(\chi_{\min})}$$

where $\chi_{\min} = \mathrm{Min}(\chi_0, \chi)$. In urban plane co-ordinates this is equivalent to a critical angle of:

$$\theta^* = \frac{2br_{\min}}{\sin(br_{\min})}$$

where $r_{\min} = \mathrm{Min}(r_0, r)$ is the innermost radius of the two ends to the trip. For angular separations $|\theta-\theta_0|$ that are below these values, routes using orbital roads take the least time. For angular separations that are above these values the quickest routes are radials via the city centre.

We applied this model to the London region using calibrated parameter values of a=31.8 km/hr, b=0.059 radians/km. This corresponds to an urban radius ($\pi/2b$) of R=26.5 km, which maps into the equator of the time surface. The radius of the time surface (1/ab) is equal to 0.53 hours. The relationship between speeds and distance from the city centre is shown in Figure 12. Whereas radial speeds remain constant, orbital speeds continue to increase until they reach the city perimeter. The urban plane and time London time surface are shown in Figure 13. When n=1 time is given by latitude plus longitude.

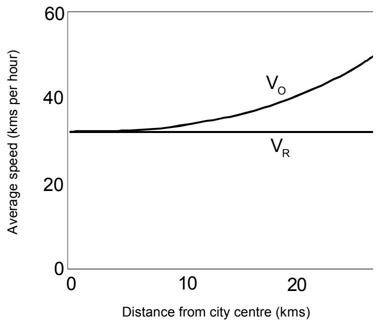
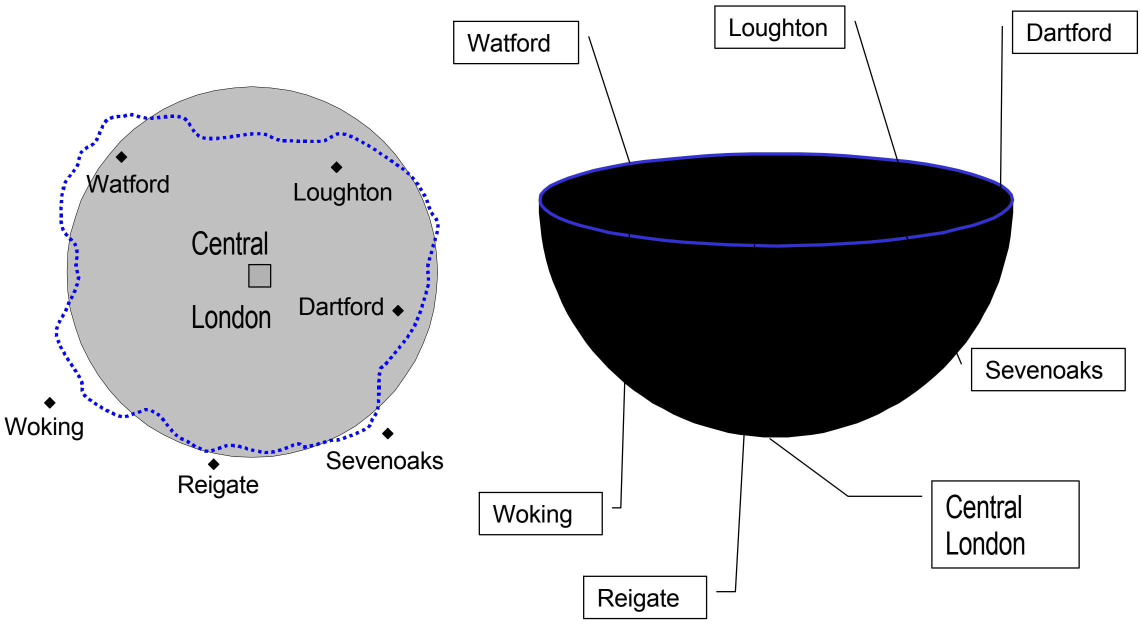



*Figure 12: Orbital and radial speeds in relation to distance from the city centre.*




*a) The London urban plane* *b) The London hemispherical time surface.*

*Figure 13: This illustrates the time surface for London. When n=1 it has the latitude-longitude metric.*

Figure 14 (left-hand map) shows an example of isochrones spaced at 10 minute intervals around a point (grey square) positioned at 10 kms north of central London.

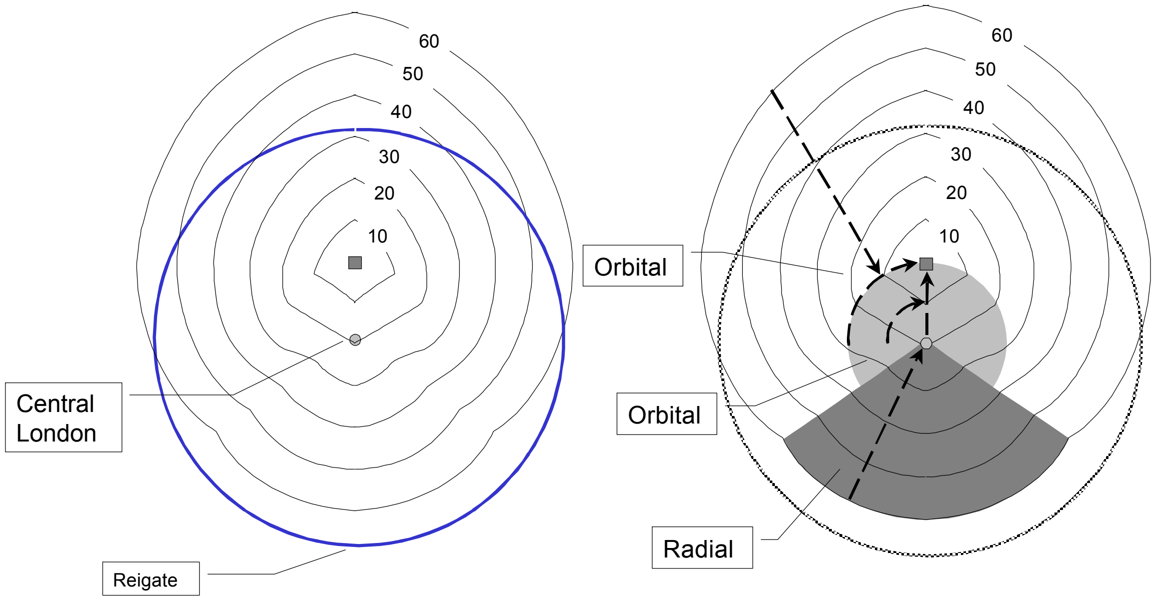



*Figure 14: An isochrone map centred on a fixed point 10 km due north of central London based on the hemispherical travel time surface. The fixed point is shown by a grey square.*

The map on the right shows shaded route catchment areas, which indicate whether the minimum time route is through the city centre or via an appropriate orbital route. The darker shaded locations (a pie slice) show the radial catchment area. A journey starting in this region and terminating at the fixed point would minimize its travel time by using a route via the city centre. The lighter circular grey area are the locations that achieve minimum times to the fixed location by using orbital routes that are less than 10 km from the city centre. The unshaded areas are the locations that are able to minimize their time by using the orbital at a 10 km radius.

Changes to London's transport system, particularly near the central area, could have a significant effect on the size and shape of these catchment areas, impacting on the routing of journeys, and the geographical pattern of trip making and trade. Suppose that it became more costly to travel through Central London, e.g. through a central area road user charge. This would reduce the size of the radial catchment area for locations in Inner London (Hyman and Mayhew, 2002). Some traffic would switch to orbital routes, which could become more congested, further reducing the overall accessibility of locations in Inner London, reducing trips and trade and, potentially, affecting the future location of economic activity.

*Other examples*

In annex 2 we give several further examples of time surface transformations. These are all solutions to the differential equations for the time surface transformations that are given in annex 1, and provide a wide variety of alternative urban and regional models. If required, more details are available from the authors.

## Conclusions

This paper has addressed topics that have lain dormant for a number of years – namely urban geometry, time surfaces and geographical transformations – and has integrated these topics with more recent developments in travel time metrics. The primary purpose has been to inject new thinking into long standing issues in urban modelling using essentially standard geographical theory. The two principal departures were the introduction of non-conformal transformations, which permit a de-coupling of the speeds of travel in orthogonal travel directions, and their application to the construction of a range of alternative travel time metrics.

The purpose in doing so was to develop new ways of modeling cites and regions that are more in accord with actual transport networks than previous geometrical models which assume isotropic variations in speed and Euclidean metrics. In the process we were able to present a mathematical framework that would apply in a variety of geographical contexts, to suit different needs, and to support a series of practical case studies.

The mathematical framework described is not limited to the creation of travel time maps for, say, posters and urban atlases, although this may turn out to be the first practical use made of the research. As important are the theoretical insights that are evident through a relaxation in the limited concepts of Euclidean models of urban space. The visualization capability of the resulting models has major intuitive and presentational advantages over purely numerical, or even purely analytic, representations. The simple basic pictures that the techniques afford has the further advantage that the resultant maps are not cluttered with unnecessary detail, which can be hard to interpret. A sound geometric and analytical framework is essential to this understanding, and this has been provided in a fairly comprehensive manner.

In broader terms, the scope of application of the techniques is as yet uncharted. For example, the authors have recently developed applications of travel geometry and travel time metrics to the siting of urban airports, road congestion charging and route catchment planning. Whilst the published results have made use of travel time metrics they have not employed the concepts of time surfaces for visualization purposes, and have been restricted to a limited class of travel time metrics. It would be well within the capabilities of the framework presented to exploit the insights provided by time surfaces, e.g. in problems relating to facility location, but also to make much less restrictive modelling assumptions, where this is needed. In addition it would also be in the capability of the theory to revisit and re-derive some of the traditional urban models in the geographic literature and investigate the robustness of their implications for policy analysis.

In summary, travel time surfaces and metrics can be constructed for a much wider class of urban travel assumptions than is available from previous work. These constructions provide visualization tools and insights, as well as computational techniques for mapping travel times and the cost of movement, in both an urban and a regional context. The resulting analytical models can be calibrated using minimal data that are generally readily available either from detailed network based models, or from travel time surveys. The approach facilitates simple numerical calculation of solutions, within a unified mathematical approach, and provides a geometric foundation for new developments in urban location theory. Some of these implications have been outlined in the illustrations given in the paper, but further analysis is clearly warranted.

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## Annex 1

This annex gives the differential equations for time surface transformations and the restrictions on speed variations that they require, covering eight major co-ordinate system pairs. The first four describe transformations to a plane time surface.

### 1) Transformations from the urban plane to a time surface in plane co-ordinates

For an urban plane with Cartesian co-ordinates (x,y) let $(V_X,V_Y)$ denote the speeds in the x and y directions. Similarly, for an urban plane in polar co-ordinates (r, θ) let $(V_R,V_O)$ denote the speeds in the r and θ directions. The time surface is represented by either plane Cartesian co-ordinates (u,v) or by plane polar co-ordinates (ω,ψ). The transformations to the time surface satisfy the following differential equations. *These transformations are only conformal when directional speeds are equal* ($V_X$=$V_Y$ or $V_R$=$V_O$).

| | Transformations from Urban Plane to Time Surface |
|---|---|
| Cartesian (x,y) to Cartesian (u,v) | $V_X = \frac{1}{f(x)}, V_Y = \frac{1}{g(y)}$ <br> $\frac{du}{dx} = f(x), \frac{dv}{dy} = g(y)$ |
| Cartesian (x,y) to Polar(ω,ψ) | $V_X = \frac{1}{f(x)}, V_Y = \frac{1}{\omega(x) g(y)}$ <br> $\frac{dw}{dx} = f(x), \frac{d\psi}{dy} = g(y)$ |
| Polar (r,θ) to Cartesian (u,v) | $V_R = \frac{1}{f(r)}, V_O = \frac{r}{g(\theta)}$ <br> $\frac{du}{dr} = f(r), \frac{dv}{d\theta} = g(\theta)$ |
| Polar (r,θ) to Polar(ω,ψ) | $V_R = \frac{1}{f(r)}, V_O = \frac{r}{\omega(r) g(\theta)}$ <br> $\frac{d\omega}{dr} = f(r), \frac{d\psi}{d\theta} = g(\theta)$ |

**2) Transformations from the urban plane to a time surface in cylindrical and spherical co-ordinates**

The second four describe transformations to time surfaces described in cylindrical co-ordinates (z,ρ,ϕ) and spherical co-ordinates (σ,χ,ϕ). In cylindrical co-ordinates z denotes the height and ρ the cylindrical radius. In spherical co-ordinates σ denotes the spherical radius and χ the colatitude. The cylindrical and spherical systems have identical azimuthal co-ordinates ϕ and are also linked via the identity $\rho=\sigma\sin(\chi)$.

| | Transformations from Urban Plane to Time Surface |
|---|---|
| Polar (r,θ) to Cylindrical (z,ρ,ϕ) | $V_R = \frac{1}{f(r)}, V_O = \frac{1}{h(r)g(\theta)}$<br>$\frac{dz}{dr} = \sqrt{f(r)^2 - \left(\frac{d\rho}{dr}\right)^2}$<br>$\rho = rh(r), \frac{d\phi}{d\theta} = g(\theta)$ |
| Polar (r,θ) to Spherical (σ,χ,ϕ) | $V_R = \frac{1}{f(r)}, V_O = \frac{1}{h(r)g(\theta)}$<br>$\sigma \frac{d\chi}{dr} = \sqrt{f(r)^2 - \left(\frac{d\sigma}{dr}\right)^2}$<br>$\sigma = \frac{rh(r)}{\sin\chi}, \frac{d\phi}{d\theta} = g(\theta)$ |
| Cartesian (x,y) to Cylindrical (z,ρ,ϕ) | $V_X = \frac{1}{f(x)}, V_Y = \frac{1}{h(x)g(y)}$<br>$\frac{dz}{dx} = \sqrt{f(x)^2 - \left(\frac{d\rho}{dx}\right)^2}$<br>$\rho = h(x), \frac{d\phi}{dy} = g(y)$ |
| Cartesian (x,y) to Spherical (σ,χ,ϕ) | $V_X = \frac{1}{f(x)}, V_Y = \frac{1}{h(x)g(y)}$<br>$\sigma \frac{d\chi}{dx} = \sqrt{f(x)^2 - \left(\frac{d\sigma}{dx}\right)^2}$<br>$\sigma = \frac{h(x)}{\sin\chi}, \frac{d\phi}{dy} = g(y)$ |

**Annex 2**

This annex gives a number of specific examples of time surface transformations. These are analytical solutions to the differential equations given in Annex 1. The first table shows the time surfaces that are described in plane co-ordinates. 'Type' refers to the pair of co-ordinates: C for plane Cartesian, and P for plane Polar, with the urban co-ordinates given first. The 'M'columns contain the solved transformation equations. Each example is numbered and a short description of each example follows.

| | **Type** | **$V_1$** | **$V_2$** | **$M_1$** | **$M_2$** | **Surface** |
|---|---|---|---|---|---|---|
| 1 | CC | $V_X=a$ | $V_Y=b$ | $u=x/a$ | $v=y/b$ | Plane |
| 2 | CC | $V_X=a+mx$ | $V_Y=b+ny$ | $u=(1/m)Ln(1+mx/a)$ | $v=(1/n)Ln(1+ny/b)$ | Plane |
| 3 | PP | $V_R=a$ | $V_O=b$ | $\omega=r/a$ | $\psi=a\theta/b$ | Pie Slice |
| 4 | PP | $V_R=a(Ln(br))^2$ $/(Ln(br)-1)$ | $V_O=aLn(br)/\lambda$ | $\omega=r/aLn(br)$ | $\psi=\lambda\theta$ | Pie Slice |
| 5 | PC | $V_R=ar^m$ | $V_O=r/\lambda$ | $u=r^{1-m}/a(1-m)$ | $v=\lambda\theta$ | Plane Strip |
| 6 | PC | $V_R=ar^m$ | $V_O=br$ $Exp(-\lambda\theta)$ | $u=r^{1-m}/a(1-m)$ | $v= (exp(\lambda\theta)-1)/b\lambda$ | Plane Strip |
| 7 | CP | $V_X=a\ exp(-mx)$ | $V_Y=b$ $exp(-mx)$ | $\omega=exp(mx)/am$ | $\psi=amy/b$ | Punctured Plane |

Next we show developable time surfaces, which all use type Y for cylindrical co-ordinates.

| | **Type** | **$V_1$** | **$V_2$** | **$M_1$** | **$M_2$** | **Surface** |
|---|---|---|---|---|---|---|
| 8 | PY | $V_R=a$ | $V_O=b$ | $\rho=r/b$ $z=r(b^2-a^2)^{1/2}/ab$ | $\phi=\theta$ | Cone |
| 9 | PY | $V_R=a(Ln(br))^2$ $/(Ln(br)-1)$ | $V_O=aLn(br)/\lambda$ | $\rho=\lambda r/aLn(br)$ $z=r(1-\lambda^2)^{1/2}/aLn(br)$ | $\phi=\theta$ | Cone |
| 10 | PY | $V_R=ar^m$ | $V_O=r/\lambda$ | $\rho=\lambda$ $z=r^{1-m}/a(1-m)$ | $\phi=\theta$ | Cylinder |
| 11 | CY | $V_X=a$ | $V_Y=b/(1+mx)$ | $\rho=(1+mx)/mb$ $z=(b^2-a^2)x/ab$ | $\phi=my$ | Truncated Cone |

The third table shows non-developable time surfaces. which include the type.S for spherical co-ordinates.

| | **Type** | **$V_1$** | **$V_2$** | **$M_1$** | **$M_2$** | **Surface** |
|---|---|---|---|---|---|---|
| 12 | PY | $V_R=a$ | $V_O=abr$ $/sin(br)$ | $\rho=sin(br)/ab$ $z=(1-cos(br))/ab$ | $\phi=\theta$ | Sphere |
| 13 | PS | $V_R=a$ | $V_O=abr$ $/sin(br)$ | $\sigma=1/ab$ $\chi=br$ | $\phi=\theta$ | Sphere |
| 14 | PY | $V_R=a(R^2-r^2)^{1/2}$ | $V_O=aR$ | $\rho=r/aR$ $z=(1-(1-r^2/R^2)^{1/2})/a$ | $\phi=\theta$ | Sphere |
| 15 | PS | $V_R=a(R^2-r^2)^{1/2}$ | $V_O=aR$ | $\sigma=1/a$ $\chi=sin^{-1}(r/R)$ | $\phi=\theta$ | Sphere |
| 16 | CY | $V_X=a$ | $V_Y=b$ $/sin(mx)$ | $\rho=sin(mx)/am$ $z=(1-cos(mx))/am$ | $\phi=amy/b$ | Sphere |
| 17 | CS | $V_X=a$ | $V_Y=b$ $/sin(mx)$ | $\sigma=1/am$ $\chi=mx$ | $\phi=amy/b$ | Sphere |

| No. | Title | Description |
|---|---|---|
| 1 | Grid City | Fast avenues and slow streets |
| 2 | Crossroads | Near a busy intersection of avenues and streets |
| 3,8 | Marketplace | A traditional market town with a congested city centre |
| 4,9 | Hub City | Long distance travel to or around a central hub |
| 5,10 | Edge City | Near the edge of a big city, with fast orbital travel |
| 6 | Rush hour in Edge City | Edge city with localized congestion on orbital routes caused by intersections or slow moving vehicles |
| 7 | Surf City | Settlements along a coastline, with a fast coast road |
| 11 | Main Street | A thin city, with speeds reducing away from a main axis |
| 12,13 | Polaris | A city region with a central 'pole' and an equator |
| 14,15 | The Lost City | A city 'hidden' by radial speeds that reduce to zero at the city boundary |
| 16,17 | Ridgeway | A slow mountain ridge between two parallel fast routes |

**Note**

When two numbers appear in the first column, the same urban or regional model is represented in two different sets of time surface co-ordinates.