FREE VIBRATION OF FUNCTIONALLY GRADED BEAMS AND FRAMEWORKS USING THE DYNAMIC STIFFNESS METHOD

J.R. Banerjee* and A. Ananthapuvirajah

School of Mathematics, Computer Science and Engineering, City, University of London, Northampton Square, London EC1V 0HB, U.K.

Abstract

The free vibration analysis of functionally graded beams (FGBs) and frameworks containing FGBs is carried out by applying the dynamic stiffness method and deriving the elements of the dynamic stiffness matrix in explicit algebraic form. The usually adopted rule that the material properties of the FGB vary continuously through the thickness according to a power law forms the fundamental basis of the governing differential equations of motion in free vibration. The differential equations are solved in closed analytical form when the free vibratory motion is harmonic. The dynamic stiffness matrix is then formulated by relating the amplitudes of forces to those of the displacements at the two ends of the beam. Next, the explicit algebraic expressions for the dynamic stiffness elements are derived with the help of symbolic computation. Finally the Wittrick-Williams algorithm is applied as solution technique to solve the free vibration problems of FGBs with uniform cross-section, stepped FGBs and frameworks consisting of FGBs. Some numerical results are validated against published results, but in the absence of published results for frameworks containing FGBs, consistency checks on the reliability of results are performed. The paper closes with discussion of results and conclusions.

Keywords: Free vibration, functionally graded beams, dynamic stiffness method, frameworks, Wittrick-Williams algorithm.

*Corresponding author.

Email address: j.r.banerjee@city.ac.uk

1. Introduction

In recent years interest in functionally graded material (FGM) has grown enormously. The progress made in understanding this material has been phenomenal. One great advantage of FGM is that the properties vary gradually in a continuous manner within the material so that there is no abrupt change or mismatch of the properties which can cause delamination or other problems generally associated with fibre-reinforced composites. Thus, FGM can be designed in a way to have the properties of ceramic at one end and those of metal at the other so that the thermal resistance of ceramic and the mechanical behaviour of metal can be exploited to advantage to guarantee structural integrity. Consequent on this, researchers have been continually motivated to use various techniques and methodologies to deal with this exciting material in order to enhance its state-of-the-art. There are now excellent books [1-4] available on the subject. As potential application of FGM, beams which are extensively used in civil, mechanical, aeronautical and other branches of engineering as principal load carrying structural members can be investigated for their free vibration characteristics. Investigators have expended considerable efforts which have led to the insurgence of massive literature on the free vibration behaviour of Functionally Graded Beams (FGBs). A number of theories and methodologies have been proposed to study the free vibration characteristics of FGBs. Foremost amongst these are the applications of direct analytical procedure using the governing differential equations of motion [5-19], finite element [20-22], Rayleigh-Ritz [23], finite volume [24-26], differential quadrature [27], differential transformation [27, 28] and transfer function [29, 30] methods. Recently the dynamic stiffness method (DSM) has also been proposed [31, 32]. The current paper stems from the previously published DSM theories. The entire formulation using DSM in this paper is accomplished in the real domain as opposed to previous formulations which used complex arithmetic when developing the element dynamic stiffness matrices [31, 32]. Another important further development reported in this paper is the derivation of explicit algebraic expressions for the dynamic stiffness elements using symbolic computation [33-35]. The explicit expressions for the dynamic stiffness elements are particularly useful in optimisation studies and also when some, but not all of the stiffnesses are needed. Of particular significance of this investigation is the application of DSM to analyse the free vibration characteristics of stepped FGBs and frameworks containing FGBs. The substantial advantages of the DSM over the conventional finite element method (FEM) are well known [36-38]. The DSM is often called an exact method because in sharp contrast to chosen approximate shape function assumed in the FEM, the DSM uses exact shape function obtained from the analytical solution of the governing differential equation of motion of the structural element in free vibration. The uncompromising accuracy of the DSM in all frequency ranges and its independency on the number of elements used in the analysis makes it particularly appealing in free vibration analysis. Within this pretext, the application of the DSM in the free vibration analysis of FGBs and frameworks containing FGBs is considered to be a welcome development. The solution technique used in the DSM is robust, particularly when the well-established algorithm of Wittrick and Williams [39], known as Wittrick-Williams algorithm in the literature, is applied. The algorithm ensures that no natural frequency of the structure is missed, and it has featured in literally hundreds of papers. It is worth noting that earlier investigations on the free vibration of FGBs were focused on individual FGBs except for a few isolated cases where stepped FGBs with collinear axes were reported [40, 41]. Accordingly, the literature on the free vibration of frameworks containing FGBs is virtually non-existent. One of the essential purposes of this paper is to fill this gap.

2. Theory

In a right-handed Cartesian coordinate system, Fig. 1 shows a uniform rectangular cross-section FGB of length *L*, width *b* and thickness *h*. The beam material has Young's Modulus *E* and mass density ρ which can vary through the thickness direction (Z) of the cross-section according to the following power law distribution [14, 17, 30, 32]:

$$E(z) = (E_t - E_b)\left(\frac{z}{h} + \frac{1}{2}\right)^k + E_b, \qquad \rho(z) = (\rho_t - \rho_b)\left(\frac{z}{h} + \frac{1}{2}\right)^k + \rho_b \tag{1}$$

where E_t and E_b are the Young's moduli, and ρ_t and ρ_b are the densities at the top and bottom surfaces of the beam, respectively.

In Eq. (1), k $(k \geq 0)$ is the power law index parameter which indicates the material property variation through the beam thickness. The parameter *k* has been extensively discussed in the literature [14-19] and hence it is not elaborated here. However, three special cases maybe observed. Clearly $k = 1$ indicates a linear variation of the composition between the top and bottom surfaces of the beam, $k = 0$ represents the case when the beam is made of full material of the top surface whereas infinite k represents the case when the beam is made of full material of the bottom surface.

2.1 Governing differential equations of motion and solution

The classical Bernoulli-Euler theory is considered here so that the effects of shear deformation and rotary inertia that are relevant to the Timoshenko beam theory are assumed to be small and hence disregarded in the analysis. Referring to Fig. 1, the displacements u_1, v_1 and w_1 along the X, Y and Z directions of a point on the cross-section are given by [6, 30, 32]:

$$u_1 = 0, \qquad v_1(y,z,t) = v(y,t) - z\frac{\partial w(y,t)}{\partial y}, \qquad w_1(y,z,t) = w(y,t) \tag{2}$$

where v and w are the corresponding displacements of a point on the neutral axis of the beam. It should be noted that due to the variation of the material properties through the thickness, the neutral axis would no longer be at the central line of the beam cross-section [42, 43].

Using the displacement field given by Eq. (2) and through the application of Hamilton's principle, the governing differential equations of motion in free vibration of the FGB are given by [30, 32]

$$-B_0\ddot{v}+B_1\ddot{w}'+A_0v''-A_1w'''=0, \qquad -B_0\ddot{w}-B_1\ddot{v}'+B_2\ddot{w}''+A_1v'''-A_2w''''=0 \tag{3}$$

where

$$A_i=\int z^i E(z)dA, \qquad B_i=\int z^i \rho(z)dA \qquad (i=0,1,2) \tag{4}$$

The natural boundary conditions from the Hamiltonian formulation [30, 32] give the following expressions for axial force F, shear force S and bending moment M as follows:

$$F=-A_0v'+A_1w'', \qquad S=B_1\ddot{v}-A_1v''-B_2\ddot{w}'+A_2w''', \qquad M=A_1v'-A_2w'' \tag{5}$$

Clearly, due to the use of FGM, the axial (v) and bending motions (w) are coupled as evident from Eqs. (3) and (5).

Assuming harmonic oscillation so that

$$v(y,t)=V(y)e^{i\omega t}, \qquad w(y,t)=W(y)e^{i\omega t} \tag{6}$$

where ω is the angular or circular frequency, $V(y)$ and $W(y)$ are the amplitudes of v and w, respectively.

Introducing the differential operator $\mathrm{D} = \dfrac{\mathrm{d}}{\mathrm{d}\xi}$ and the non-dimensional length ξ as:

$$\xi = \frac{y}{L} \tag{7}$$

The differential equations of motion in Eqs. (3) can now be written as:

$$\left.\begin{aligned}(B_0\omega^2 L^3 + A_0 L\,\mathrm{D}^2)V(\xi) - (B_1\omega^2 L^2\,\mathrm{D} + A_1\,\mathrm{D}^3)W(\xi) = 0\\ (B_1\omega^2 L^3\,\mathrm{D} + A_1 L\,\mathrm{D}^3)V(\xi) + (B_0\omega^2 L^4 - B_2\omega^2 L^2\,\mathrm{D}^2 - A_2\,\mathrm{D}^4)W(\xi) = 0\end{aligned}\right\} \tag{8}$$

By combining the above two differential equations, it is possible to obtain a sixth order ordinary differential equation satisfying both $V(\xi)$ and $W(\xi)$ to give:

$$(\mathrm{D}^6 + a\,\mathrm{D}^4 - b\,\mathrm{D}^2 - c)H = 0 \tag{9}$$

where

$$H = V(\xi) \text{ or } W(\xi) \tag{10}$$

and

$$a = \frac{2A_1B_1 - A_0B_2 - A_2B_0}{A_1^2 - A_0A_2}L^2\omega^2,\ b = \frac{B_0B_2\omega^2 - A_0B_0 - B_1^2\omega^2}{A_1^2 - A_0A_2}L^4\omega^2,\ c = -\frac{B_0^2}{A_1^2 - A_0A_2}L^6\omega^4 \tag{11}$$

By assuming the solution in the form $H = e^{\lambda\xi}$ the characteristic or auxiliary equation of the differential equation Eq. (9) can be expressed as

$$\lambda^6 + a\lambda^4 - b\lambda^2 - c = 0 \tag{12}$$

Equation (12) can be reduced to a cubic equation to give

$$\mu^3 + a\mu^2 - b\mu - c = 0 \tag{13}$$

where

$$\mu = \lambda^2 \tag{14}$$

Equation (13) can now be solved analytically for μ using standard procedure [44]. In earlier investigations [30, 32], it was assumed that the square root of the three roots of the cubic in Eq. (13) could be either real or complex and thus the six roots r_j ($j = 1, 2, \cdots, 6$) of the characteristic equation Eq. (12) give the solutions of the differential equation (Eq. (9)) leading to V and W in the following forms:

$$V(\xi) = \sum_{j=1}^{6} R_j e^{r_j \xi}, \qquad W(\xi) = \sum_{j=1}^{6} Q_j e^{r_j \xi} \tag{15}$$

where R_j and Q_j ($j = 1, 2, \cdots, 6$) are two sets of constants which could be possibly complex.

Clearly a method utilising Eq. (15) as the solution to derive the dynamic stiffness matrix will involve numerical operations using complex arithmetic. This cumbersome and somehow computationally inefficient procedure is circumvented in this paper.

Using an approach similar to the one described in Refs [45, 46], it can be shown that the three roots of the cubic equation Eq. (13) are real and the solution for H in Eq. (9) can be expressed in terms of trigonometric and hyperbolic functions as opposed to complex exponential functions of Eq. (15). This is advantageous when deriving the explicit expressions for the dynamic stiffness elements of the FGB. Explicit expressions are particularly useful when some, but not all of the stiffness elements are needed, e.g. sensitivity analysis in optimisation studies. Thus if the roots [44] of Eq. (13) are α, β and γ, the solution for H is given by

$$H(\xi) = C_1 \cosh \alpha\xi + C_2 \sinh \alpha\xi + C_3 \cos \beta\xi + C_4 \sin \beta\xi + C_5 \cos \gamma\xi + C_6 \sin \gamma\xi \tag{16}$$

where

$$\left.\begin{aligned}
\alpha &= \left[2\left(\frac{q}{3}\right)^{\frac{1}{2}}\cos\frac{\phi}{3}-\frac{a}{3}\right]^{\frac{1}{2}} \\
\beta &= \left[2\left(\frac{q}{3}\right)^{\frac{1}{2}}\cos\frac{\pi-\phi}{3}+\frac{a}{3}\right]^{\frac{1}{2}} \\
\gamma &= \left[2\left(\frac{q}{3}\right)^{\frac{1}{2}}\cos\frac{\pi+\phi}{3}+\frac{a}{3}\right]^{\frac{1}{2}}
\end{aligned}\right\} \tag{17}$$

with

$$q = b+\frac{a^2}{3} \tag{18}$$

and

$$\phi = \cos^{-1}\left[\frac{27c-9ab-2a^3}{2\left(a^2+3b\right)^{\frac{3}{2}}}\right] \tag{19}$$

$H(\xi)$ of Eq. (16) represents the solution for both axial displacement $V(\xi)$ and bending displacement $W(\xi)$, containing different sets of constants as follows

$$V(\xi) = Q_1\cosh\alpha\xi + Q_2\sinh\alpha\xi + Q_3\cos\beta\xi + Q_4\sin\beta\xi + Q_5\cos\gamma\xi + Q_6\sin\gamma\xi \tag{20}$$

$$W(\xi) = R_1\cosh\alpha\xi + R_2\sinh\alpha\xi + R_3\cos\beta\xi + R_4\sin\beta\xi + R_5\cos\gamma\xi + R_6\sin\gamma\xi \tag{21}$$

The two different sets of constants $Q_1 - Q_6$ and $R_1 - R_6$ can be related with the help of any one of the two of Eqs. (8) to give

$$\left.\begin{aligned}
&Q_1 = \left(\frac{k_\alpha}{L}\right)R_2, \quad Q_3 = \left(\frac{k_\beta}{L}\right)R_4, \quad Q_5 = \left(\frac{k_\gamma}{L}\right)R_6, \\
&Q_2 = \left(\frac{k_\alpha}{L}\right)R_1, \quad Q_4 = -\left(\frac{k_\beta}{L}\right)R_3, \quad Q_6 = -\left(\frac{k_\gamma}{L}\right)R_5
\end{aligned}\right\} \tag{22}$$

where

$$k_\alpha = \frac{\alpha\left(A_1\alpha^2 + B_1\omega^2L^2\right)}{\left(A_0\alpha^2 + B_0\omega^2L^2\right)}, \quad k_\beta = \frac{\beta\left(A_1\beta^2 - B_1\omega^2L^2\right)}{\left(A_0\beta^2 - B_0\omega^2L^2\right)}, \quad k_\gamma = \frac{\gamma\left(A_1\gamma^2 - B_1\omega^2L^2\right)}{\left(A_0\gamma^2 - B_0\omega^2L^2\right)} \tag{23}$$

With the help of Eqs. (5), (20) and (21), the expressions for bending rotation $\theta(\xi)$, axial force $F(\xi)$, bending moment $M(\xi)$, and shear force $S(\xi)$ can be obtained after some simplification, as

$$\begin{aligned}\theta(\xi) = \frac{W'(\xi)}{L} = \left(\frac{1}{L}\right)\{&R_1\alpha\sinh\alpha\xi + R_2\alpha\cosh\alpha\xi - R_3\beta\sin\beta\xi + R_4\beta\cos\beta\xi \\ &- R_5\gamma\sin\gamma\xi + R_6\gamma\cos\gamma\xi\}\end{aligned} \tag{24}$$

$$\begin{aligned}F(\xi) = -\frac{A_0}{L}\left(V' - \frac{A_1}{A_0L}W''\right) = \left(\frac{A_0}{L}\right)\{&-R_1\frac{e_\alpha}{L}\cosh\alpha\xi - R_2\frac{e_\alpha}{L}\sinh\alpha\xi + R_3\frac{e_\beta}{L}\cos\beta\xi \\ &+ R_4\frac{e_\beta}{L}\sin\beta\xi + R_5\frac{e_\gamma}{L}\cos\gamma\xi + R_6\frac{e_\gamma}{L}\sin\gamma\xi\}\end{aligned} \tag{25}$$

$$\begin{aligned}M(\xi) = -\frac{A_2}{L^2}\left(W'' - \frac{A_1L}{A_2}V'\right) = -\left(\frac{A_2}{L^2}\right)\{&-R_1g_\alpha\cosh\alpha\xi - R_2g_\alpha\sinh\alpha\xi + R_3g_\beta\cos\beta\xi \\ &+ R_4g_\beta\sin\beta\xi + R_5g_\gamma\cos\gamma\xi + R_6g_\gamma\sin\gamma\xi\}\end{aligned} \tag{26}$$

$$\left.\begin{aligned}S(\xi) &= \frac{A_2}{L^3}\left(W''' + \frac{B_2L^2\omega^2}{A_2}W' - \frac{A_1L}{A_2}V'' - \frac{B_1L^3\omega^2}{A_2}V\right) \\ &= \frac{A_2}{L^3}\left\{R_1f_\alpha\sinh\alpha\xi + R_2f_\alpha\cosh\alpha\xi + R_3f_\beta\sin\beta\xi - R_4\cos\beta\xi + R_5\sin\gamma\xi - R_6\cos\gamma\xi\right\}\end{aligned}\right\} \tag{27}$$

where

$$e_\alpha = \alpha k_\alpha - A_1\alpha^2/A_0, \qquad e_\beta = \beta k_\beta - A_1\beta^2/A_0, \qquad e_\gamma = \gamma k_\gamma - A_1\gamma^2/A_0 \tag{28}$$

$$g_\alpha = \alpha^2 - \alpha k_\alpha A_1/A_2, \qquad g_\beta = \beta^2 - \beta k_\beta A_1/A_2, \qquad g_\gamma = \gamma^2 - \gamma k_\gamma A_1/A_2 \tag{29}$$

$$\left.\begin{aligned}f_\alpha &= \alpha^3 - \alpha^2k_\alpha A_1/A_2 + \alpha\omega^2L^2B_2/A_2 - k_\alpha\omega^2L^2B_1/A_2 \\ f_\beta &= \beta^3 - \beta^2k_\beta A_1/A_2 - \beta\omega^2L^2B_2/A_2 + k_\beta\omega^2L^2B_1/A_2 \\ f_\gamma &= \gamma^3 - \gamma^2k_\gamma A_1/A_2 - \gamma\omega^2L^2B_2/A_2 + k_\gamma\omega^2L^2B_1/A_2\end{aligned}\right\} \tag{30}$$

2.2 Dynamic Stiffness Formulation

The dynamic stiffness matrix of the FGB can now be formulated by applying natural boundary conditions for displacements and forces at the ends of the beam. Referring to the sign convention for positive axial force, shear force and bending moment shown in Fig. 2, the boundary conditions for displacements and forces, see Fig. 3, are:

$$\text{At } \xi = 0: \quad V = V_1,\ W = W_1,\ \theta = \theta_1,\ F = F_1,\ S = S_1,\ M = M_1 \tag{31}$$

$$\text{At } \xi = 1: \quad V = V_2,\ W = W_2,\ \theta = \theta_2,\ F = -F_2,\ S = -S_2,\ M = -M_2 \tag{32}$$

The displacement vector $\boldsymbol{\delta}$ and the force vector $\mathbf{P}$ can be expressed as:

$$\boldsymbol{\delta} = [V_1\ W_1\ \theta_1\ V_2\ W_2\ \theta_2]^T, \qquad \mathbf{P} = [\ F_1\ S_1\ M_1\ F_2\ S_2\ M_2\ ]^T \tag{33}$$

where the upper script T denotes a transpose.

The relationship between the displacement $\boldsymbol{\delta}$ and the constant vector $\mathbf{R}$ can be obtained by using Eqs. (20)-(24) and Eqs. (31)-(32) to give,

$$\boldsymbol{\delta} = \mathbf{B}\,\mathbf{R} \tag{34}$$

where

$$\mathbf{B} = \begin{bmatrix} 0 & \frac{k_\alpha}{L} & 0 & \frac{k_\beta}{L} & 0 & \frac{k_\gamma}{L} \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & \frac{\alpha}{L} & 0 & \frac{\beta}{L} & 0 & \frac{\gamma}{L} \\ \frac{k_\alpha S_{h\alpha}}{L} & \frac{k_\alpha C_{h\alpha}}{L} & -\frac{k_\beta S_\beta}{L} & \frac{k_\beta C_\beta}{L} & -\frac{k_\gamma S_\gamma}{L} & \frac{k_\gamma C_\gamma}{L} \\ C_{h\alpha} & S_{h\alpha} & C_\beta & S_\beta & C_\gamma & S_\gamma \\ \frac{\alpha S_{h\alpha}}{L} & \frac{\alpha C_{h\alpha}}{L} & -\frac{\beta S_\beta}{L} & \frac{\beta C_\beta}{L} & -\frac{\gamma S_\gamma}{L} & \frac{\gamma C_\gamma}{L} \end{bmatrix} \tag{35}$$

with

$$C_{h\alpha} = \cosh\alpha, \quad S_{h\alpha} = \sinh\alpha, \quad C_\beta = \cos\beta, \quad S_\beta = \sin\beta, \quad C_\gamma = \cos\gamma, \quad S_\gamma = \sin\lambda \tag{36}$$

Similarly, the relationship between the force vector $\mathbf{P}$ and the constant vector $\mathbf{R}$ can be obtained using Eqs. (25)-(27) and Eqs. (31)-(32) to give

$$\mathbf{P} = \mathbf{A}\,\mathbf{R} \tag{37}$$

where

$$\mathbf{A}=\begin{bmatrix} -\dfrac{W_1 e_\alpha}{L} & 0 & \dfrac{W_1 e_\beta}{L} & 0 & \dfrac{W_1 e_\gamma}{L} & 0 \\ 0 & W_3 f_\alpha & 0 & -W_3 f_\beta & 0 & -W_3 f_\gamma \\ -W_2 g_\alpha & 0 & W_2 g_\beta & 0 & W_2 g_\gamma & 0 \\ \dfrac{W_1 e_\alpha C_{h\alpha}}{L} & \dfrac{W_1 e_\alpha S_{h\alpha}}{L} & -\dfrac{W_1 e_\beta C_\beta}{L} & -\dfrac{W_1 e_\beta S_\beta}{L} & -\dfrac{W_1 e_\gamma C_\gamma}{L} & -\dfrac{W_1 e_\gamma S_\gamma}{L} \\ -W_3 f_\alpha S_{h\alpha} & -W_3 f_\alpha C_{h\alpha} & -W_3 f_\beta S_\beta & W_3 f_\beta C_\beta & -W_3 f_\gamma S_\gamma & W_3 f_\gamma C_\gamma \\ W_2 g_\alpha C_{h\alpha} & W_2 g_\alpha S_{h\alpha} & -W_2 g_\beta C_\beta & -W_2 g_\beta S_\beta & -W_2 g_\gamma C_\gamma & -W_2 g_\gamma S_\gamma \end{bmatrix} \quad (38)$$

By eliminating the constant vector **R** from Eqs. (34) and (37), **P** and $\boldsymbol{\delta}$ can be related to give the dynamic stiffness matrix relationship of the FGB as

$$\mathbf{P}=\mathbf{K}\,\boldsymbol{\delta} \quad (39)$$

where

$$\mathbf{K}=\mathbf{A}\mathbf{B}^{-1} \quad (40)$$

is the required dynamic stiffness matrix with elements k_{ij} (i = 1, 2, 3..6; j=1,2, 3,..6).

With the help of symbolic computation [33-35], the matrix **B** of Eq. (35) was inverted algebraically and the inverted matrix was pre-multiplied by the matrix **A** of Eq. (38) in order to generate the explicit expressions for each of the elements of the dynamic stiffness matrix **K**. The stiffness expressions are simplified very considerably by means of symbolic computations. They are not necessarily in the shortest form, but they are surprisingly concise. The twelve independent terms of the dynamic stiffness matrix **K** are given by

$$
\left.
\begin{aligned}
K_{1,1} &= K_{4,4} = \left(\frac{A_0}{L}\right)\left(\frac{\Phi_1}{\Delta}\right) \\
K_{1,2} &= -K_{4,5} = \left(\frac{A_2}{L^2}\right)\left(\frac{\Phi_2}{\Delta}\right) \\
K_{1,3} &= K_{4,6} = \left(\frac{A_2}{L}\right)\left(\frac{\Phi_3}{\Delta}\right) \\
K_{1,4} &= \left(\frac{A_0}{L}\right)\left(\frac{\Phi_4}{\Delta}\right) \\
K_{1,5} &= -K_{2,4} = \left(\frac{A_2}{L^2}\right)\left(\frac{\Phi_5}{\Delta}\right) \\
K_{1,6} &= K_{3,4} = \left(\frac{A_2}{L}\right)\left(\frac{\Phi_6}{\Delta}\right) \\
K_{2,2} &= K_{5,5} = \left(\frac{A_2}{L^3}\right)\left(\frac{\Phi_7}{\Delta}\right) \\
K_{2,3} &= -K_{5,6} = \left(\frac{A_2}{L^2}\right)\left(\frac{\Phi_8}{\Delta}\right) \\
K_{2,5} &= \left(\frac{A_2}{L^3}\right)\left(\frac{\Phi_9}{\Delta}\right) \\
K_{2,6} &= -K_{3,5} = \left(\frac{A_2}{L^2}\right)\left(\frac{\Phi_{10}}{\Delta}\right) \\
K_{3,3} &= K_{6,6} = \left(\frac{A_2}{L}\right)\left(\frac{\Phi_{11}}{\Delta}\right) \\
K_{3,6} &= \left(\frac{A_2}{L}\right)\left(\frac{\Phi_{12}}{\Delta}\right)
\end{aligned}
\right\} \quad (41)
$$

where

$$\left.\begin{aligned}
&\Phi_1 = \nu_1 S_\beta C_\gamma S_{h\alpha} + \nu_3 C_\beta S_\gamma S_{h\alpha} - \nu_2 S_\beta S_\gamma C_{h\alpha} + \sigma_3 C_\beta C_\gamma C_{h\alpha} + \overline{\nu}_3 C_\beta + \overline{\nu}_1 C_\gamma - \overline{\nu}_2 C_{h\alpha} \\
&\Phi_2 = -\tau_3 S_\beta S_\gamma S_{h\alpha} - \overline{\lambda}_3 S_{h\alpha}\left(1 - C_\beta C_\gamma\right) - \overline{\lambda}_1 S_\beta\left(1 - C_\gamma C_{h\alpha}\right) - \overline{\lambda}_2 S_\gamma\left(1 - C_\beta C_{h\alpha}\right) \\
&\Phi_3 = \rho_1 S_\beta C_\gamma S_{h\alpha} + \rho_3 C_\beta S_\gamma S_{h\alpha} - \rho_2 S_\beta S_\gamma C_{h\alpha} + \sigma_2 C_\beta C_\gamma C_{h\alpha} + \overline{\rho}_3 C_\beta + \overline{\rho}_1 C_\gamma - \overline{\rho}_2 C_{h\alpha} \\
&\Phi_4 = -\nu_1 S_\beta S_{h\alpha} - \nu_3 S_\gamma S_{h\alpha} + \nu_2 S_\beta S_\gamma + \overline{\nu}_3 C_\gamma C_{h\alpha} - \overline{\nu}_2 C_\beta C_\gamma + \overline{\nu}_1 C_\beta C_{h\alpha} + \sigma_3 \\
&\Phi_5 = -\tau_1\left\{\alpha S_{h\alpha}\left(C_\beta - C_\gamma\right) - \beta S_\beta\left(C_\gamma - C_{h\alpha}\right) + \gamma S_\gamma\left(C_\beta - C_{h\alpha}\right)\right\} \\
&\Phi_6 = -\rho_1 S_\beta S_{h\alpha} - \rho_3 S_\gamma S_{h\alpha} + \rho_2 S_\beta S_\gamma + \overline{\rho}_3 C_\gamma C_{h\alpha} - \overline{\rho}_2 C_\beta C_\gamma + \overline{\rho}_1 C_\beta C_{h\alpha} + \sigma_2 \\
&\Phi_7 = -\tau_1\left(\mu_1 S_\beta C_\gamma S_{h\alpha} - \mu_2 S_\beta S_\gamma C_{h\alpha} + \mu_3 C_\beta S_\gamma S_{h\alpha}\right) \\
&\Phi_8 = \tau_2 S_\beta S_\gamma S_{h\alpha} + \lambda_3 S_{h\alpha}\left(1 - C_\beta C_\gamma\right) + \lambda_1 S_\beta\left(1 - C_\gamma C_{h\alpha}\right) + \lambda_2 S_\gamma\left(1 - C_\beta C_{h\alpha}\right) \\
&\Phi_9 = \tau_1\left(\mu_1 S_\beta S_{h\alpha} - \mu_2 S_\beta S_\gamma + \mu_3 S_\gamma S_{h\alpha}\right) \\
&\Phi_{10} = -\tau_1\left\{k_\alpha S_{h\alpha}\left(C_\beta - C_\gamma\right) - k_\beta S_\beta\left(C_\gamma - C_{h\alpha}\right) + k_\gamma S_\gamma\left(C_\beta - C_{h\alpha}\right)\right\} \\
&\Phi_{11} = -\xi_1 S_\beta C_\gamma S_{h\alpha} + \xi_2 S_\beta S_\gamma C_{h\alpha} - \xi_3 C_\beta S_\gamma S_{h\alpha} + \sigma_1 C_\beta C_\gamma C_{h\alpha} - \overline{\xi}_3 C_\beta - \overline{\xi}_1 C_\gamma + \overline{\xi}_2 C_{h\alpha} \\
&\Phi_{12} = \xi_1 S_\beta S_{h\alpha} - \xi_2 S_\beta S_\gamma + \xi_3 S_\gamma S_{h\alpha} - \overline{\xi}_3 C_\gamma C_{h\alpha} + \overline{\xi}_2 C_\beta C_\gamma - \overline{\xi}_1 C_\beta C_{h\alpha} + \sigma_1
\end{aligned}\right\} \quad (42)$$

with

$$\left.\begin{aligned}
&\mu_1 = \alpha k_\beta - \beta k_\alpha,\ \mu_2 = \beta k_\gamma - \gamma k_\beta,\ \mu_3 = \gamma k_\alpha - \alpha k_\gamma \\
&\varepsilon_1 = f_\alpha k_\beta + f_\beta k_\alpha,\ \varepsilon_2 = f_\beta k_\gamma - f_\gamma k_\beta,\ \varepsilon_3 = f_\gamma k_\alpha + f_\alpha k_\gamma \\
&\zeta_1 = f_\alpha \beta + f_\beta \alpha,\ \zeta_2 = f_\beta \gamma - f_\gamma \beta,\ \zeta_3 = f_\gamma \alpha + f_\alpha \gamma \\
&\eta_1 = g_\alpha + g_\beta,\ \eta_2 = g_\beta - g_\gamma,\ \eta_3 = g_\gamma + g_\alpha \\
&\overline{\eta}_1 = e_\alpha + e_\beta,\ \overline{\eta}_2 = e_\beta - e_\gamma,\ \overline{\eta}_3 = e_\gamma + e_\alpha \\
&\lambda_1 = \mu_1\varepsilon_2 - \mu_2\varepsilon_1,\ \lambda_2 = \mu_2\varepsilon_3 + \mu_3\varepsilon_2,\ \lambda_3 = \mu_3\varepsilon_1 - \mu_1\varepsilon_3 \\
&\overline{\lambda}_1 = \mu_1\zeta_2 - \mu_2\zeta_1,\ \overline{\lambda}_2 = \mu_2\zeta_3 + \mu_3\zeta_2,\ \overline{\lambda}_3 = \mu_3\zeta_1 - \mu_1\zeta_3 \\
&\tau_1 = f_\alpha\mu_2 - f_\beta\mu_3 - f_\gamma\mu_1,\ \tau_2 = \mu_1\varepsilon_1 + \mu_2\varepsilon_2 - \mu_3\varepsilon_3,\ \tau_3 = \mu_1\zeta_1 + \mu_2\zeta_2 - \mu_3\zeta_3 \\
&\xi_1 = k_\alpha\mu_3\eta_2 - k_\beta\mu_2\eta_3,\ \xi_2 = k_\beta\mu_1\eta_3 - k_\gamma\mu_3\eta_1,\ \xi_3 = k_\gamma\mu_2\eta_1 + k_\alpha\mu_1\eta_2 \\
&\overline{\xi}_1 = k_\alpha\mu_2\eta_3 + k_\beta\mu_3\eta_2,\ \overline{\xi}_2 = k_\beta\mu_3\eta_1 - k_\gamma\mu_1\eta_3,\ \overline{\xi}_3 = k_\gamma\mu_1\eta_2 - k_\alpha\mu_2\eta_1 \\
&\rho_1 = \alpha\mu_3\eta_2 - \beta\mu_2\eta_3,\ \rho_2 = \beta\mu_1\eta_3 - \gamma\mu_3\eta_1,\ \rho_3 = \gamma\mu_2\eta_1 + \alpha\mu_1\eta_2 \\
&\overline{\rho}_1 = \alpha\mu_2\eta_3 + \beta\mu_3\eta_2,\ \overline{\rho}_2 = \beta\mu_3\eta_1 - \gamma\mu_1\eta_3,\ \overline{\rho}_3 = \gamma\mu_1\eta_2 - \alpha\mu_2\eta_1 \\
&\nu_1 = \alpha\mu_3\overline{\eta}_2 - \beta\mu_2\overline{\eta}_3,\ \nu_2 = \beta\mu_1\overline{\eta}_3 - \gamma\mu_3\overline{\eta}_1,\ \nu_3 = \gamma\mu_2\overline{\eta}_1 + \alpha\mu_1\overline{\eta}_2 \\
&\overline{\nu}_1 = \alpha\mu_2\overline{\eta}_3 + \beta\mu_3\overline{\eta}_2,\ \overline{\nu}_2 = \beta\mu_3\overline{\eta}_1 - \gamma\mu_1\overline{\eta}_3,\ \overline{\nu}_3 = \gamma\mu_1\overline{\eta}_2 - \alpha\mu_2\overline{\eta}_1 \\
&\sigma_1 = -k_\alpha\mu_2\eta_2 - k_\beta\mu_3\eta_3 + k_\gamma\mu_1\eta_1,\ \sigma_2 = \alpha\mu_2\eta_2 + \beta\mu_3\eta_3 - \gamma\mu_1\eta_1,\ \sigma_3 = \alpha\mu_2\overline{\eta}_2 + \beta\mu_3\overline{\eta}_3 - \gamma\mu_1\overline{\eta}_1
\end{aligned}\right\} \quad (43)$$

and

$$\Delta = \left(\mu_1{}^2 - \mu_2{}^2 + \mu_3{}^2\right)S_\beta S_\gamma S_{h\alpha} - 2\mu_1\mu_2 S_\beta\left(1 - C_\gamma C_{h\alpha}\right) - 2\mu_2\mu_3 S_\gamma\left(1 - C_\beta C_{h\alpha}\right) + 2\mu_1\mu_3 S_{h\alpha}\left(1 - C_\beta C_\gamma\right) \quad (44)$$

The above 6×6 frequency dependent dynamic stiffness matrix **K** of Eq. (40) can now be used to compute the natural frequencies and mode shapes of either an individual FGB or an assembly of FGBs for different boundary conditions. A reliable and accurate method of solving the problem is to apply the Wittrick-Williams algorithm [39] which is well suited for the application of DSM. The algorithm uses the Sturm sequence property of the dynamic stiffness matrix to ensure that no natural frequencies of the structure analysed are missed. The Wittrick-Williams algorithm has featured in hundreds of papers in the literature and its details are not repeated here. Basically the algorithm [39] gives the number of natural frequencies of a structure that lie below an arbitrarily chosen trial frequency specified by the user. As successive trial frequencies can be chosen by the user, this simple feature of the algorithm can be exploited to bracket any natural frequency between its upper and lower bounds to any desired accuracy. The results given in the next section were computed by applying the Wittrick-Williams algorithm as the customary solution technique.

3. Results and discussions

The first set of results was obtained for a uniform FGB with different boundary conditions with the letters C, F and S denoting clamped, free and simple-support at each end of the beam. Four classical boundary conditions are investigated, namely, clamped-free (C-F), simply-supported (S-S), clamped-simply support (C-S) and clamped-clamped (C-C). The simple support (S) boundary condition is assumed to be equivalent to a pinned support which prevents both flexural and axial displacements. A wide range of investigations was carried out by varying the length to thickness ratio (L/h) and the power law index k of the FGB which controls the material property distribution through thickness. However, the authors have been highly selective when presenting the results because existing literature already covers a huge amount of data for natural frequencies and mode shapes with the variations of L/h and k, see for example Ref.[30]. For presentational purposes, only the results for L/h =10 and k = 0.5 have been used in this paper, but the theory has been extensively validated against published results.

In order to make the results universal and also to be consistent with the published results, the following non-dimensional natural frequency parameter is defined

$$\lambda_i = \omega_i \frac{L^2}{h}\sqrt{\frac{\rho_b}{E_b}} \tag{45}$$

where ω_i is the i^{th} angular natural frequency in rad/s, ρ_b and E_b are density and Young's modulus of the bottom surface of the FGB.

Table. 1 shows the first five natural frequencies of the FGB with L/h = 10 and k = 0.5 for C-F, S-S, C-S and C-C boundary conditions alongside the results reported in a recently published paper [30]. The close agreement between the results from the current investigation and the published ones is clearly evident. The maximum discrepancy between the two sets of results is less than 4%. The corresponding mode shapes shown in Fig. 4, reveal that for the C-F and C-C boundary conditions, the first, second, fourth and fifth modes are essentially bending modes whereas the third one is axial. By contrast, for the S-S and C-S boundary conditions, the first, second, third and fifth modes are basically bending modes and the fourth one axial.

The next set of results was obtained for a stepped beam (see Fig. 5) made of FGM, for which some comparative results are available in the literature. The DSM theory developed in this paper can easily account for such problems with any step location, thickness variation and boundary conditions. However, for brevity only the results for a cantilever (C-F) stepped FGB with step locations $L_1 = 0.25L$, $L_1 = 0.5L$ and $L_1 = 0.75L$ (see Fig. 5) and the power law index $k = 0.5$ are presented in Table. 2 together with the published results [41]. Note that for consistency, FGM type – II of Ref. [41] is used so that the results are directly comparable. Clearly, the results from the current investigation are in close agreement with those of [41].

The final set of results was obtained for a portal frame consisting of three beam members AB, BC and CD as shown in Fig. 6. The natural frequencies of this frame are available in the literature [47] when all the three beam members of the frame are made of isotropic material and the supports at both points A and D are either clamped (built-in) or pinned (simply-supported). These results are based on exact analytical theory. Using the current theory, results are obtained for both the boundary conditions C-C and S-S at A and D, respectively and making (i) all the three members AB, BC and CD isotropic (which is achieved by substituting the power-law index parameter k to zero and using both the top and bottom surface material properties to be the same and isotropic), (ii) AB and CD isotropic, but BC made of FGM, (iii) BC isotropic and AB and CD made of FGM and (iv) AB, BC and CD are all made of FGM. When computing numerical results, all three members of the portal frame are assumed to have the same rectangular cross-section and length. The width and depth (height) of the cross-section are taken to be 0.04m and 0.02m, respectively and length of each member is set to 1m. When any of the beam members is isotropic, it is considered to be made of steel with Young's modulus 200 GPa and density 7500 kg/m^3 whereas if it is made of FGM, the bottom surface is considered to be steel with the above properties and the top surface ceramic with Young's modulus 380 GPa and density 3960 kg/m^3. The computed natural frequencies are non-dimensionalised with respect to the metallic properties to give

$$\lambda_i = \omega_i \sqrt{\frac{\rho A L^4}{EI}} \tag{46}$$

The results of the investigation when all members of the portal frame (see Fig. 6) are metallic, are given in Table. 3 showing the first three non-dimensional natural frequencies of the frame when the points A and D are clamped (C-C) or simply-supported (S-S), together with the results reported in Ref. [47]. The agreement between the sets of results in Table. 3 is excellent. Table. 4 shows the three non-dimensional natural frequencies when the vertical members AB and CD and the horizontal member BC of the frame are made of either isotropic metal or FGM in turn, as indicated, and the points A and D are clamped (C-C). Similar results are obtained for the case when the points A and D of Fig. 6 are simply-supported (S-S). The results for the S-S case are shown in Table. 5. Clearly the results shown in Tables. 4 and 5 when compared to Table. 3 indicate that significant changes in natural frequencies can occur as a result of using functionally graded beams. This can have profound influence in the design of fire-resistant multi-storey and multi-bay building structures.

For illustrative purposes, representative mode shapes for the portal frame of Fig. 6 are presented in Figs 7 and 8 when the points A and D of the frame are built-in (clamped) and simply-supported, respectively. The power law index parameter k is set to 0.5 when computing the mode shapes. Figs 7(a) and 8(a) represent the mode shapes when all three members of the portal frame are metallic whereas Figs. 7(b) and 8(b) shows the mode shapes when the columns AB and CD are metallic, but the beam BC is made of FGM. By contrast Figs. 7(c) and 8(c) show the mode shapes for the case when the beam BC of the portal frame is metallic, but its columns AB and CD are made of FGM. Finally, Figs. 7(d) and 8(d) show the mode shapes when all three members of the portal frame are made of FGM. Although the basic nature of the mode shapes for the portal frame remains the same depending on the order of the natural frequency on a case to case basis, significant changes in the natural frequencies are found to occur when using FGM as evident from Figs. 7 and 8. As expected the first mode of the portal frame in each case is a sway mode with the frame oscillating between left and right with virtually no elastic displacement of the central beam. The second mode shows elastic deformations of all three members with no nodal point or any point of inflection within any member. By contrast, the third mode reveals somehow a different picture in that a node with zero displacement appears towards the top end of each columns whereas a node for the beam appears near its centre. The mode shapes shown in Figs. 7 and 8 are typical, as expected from the modal analysis of a portal frame and they are in accord with similar mode shapes reported by other investigators [47-49].

4. Conclusions

The dynamic stiffness matrix of a functionally graded beam is developed by deriving explicit expressions for the individual stiffness elements in explicit algebraic form. This is achieved through the application of symbolic computation. The dynamic stiffness theory is applied by using the Wittrick-Williams algorithm as solution technique to compute the natural frequencies and mode shapes of some representative problems of uniform functionally gradient beams, for which the material properties are considered to vary continuously in the thickness direction according to a power law distribution. A stepped beam made of functionally graded material is also investigated for its free vibration characteristics. The results show good agreement with published results. Importantly, the theory has been applied to study the free vibration behaviour of a portal frame with its constituent members made of both isotropic and functionally graded material (FGM). The investigation has shown that significant changes in the free vibration behaviour are possible when using FGM. The developed theory can be applied to analyse high-rise building structures made of FGM which has advantageous mechanical properties of metal and virtuous fire-resistant characteristics of ceramic and it is in this context, the investigation carried out is expected to be most useful.

References

[1] Rasheedat Modupe Mahamood, Esther Titilayo Akinlabi, Functionally Graded Materials, Springer International Publishing AG, 2017.

[2] Snehashish Chakraverty, Karan Kumar Pradhan, Vibration of Functionally Graded Beams and Plates, Academic Press, 2016.

[3] Isaac Elishakoff, Demetris Pentaras, Cristina Gentilini, Mechanics of Functionally Graded Material Structures, World Scientific, Singapore, 2015.

[4] Zheng Zhong, Linzhi Wu, Weiqiu Chen, Mechanics of Functionally Graded Materials and Structures, Nova Science Publishers Inc, 2012.

[5] A.Y. Tang, J.X. Wu, X.F. Li, K.Y. Lee, Exact frequency equations of free vibration of exponentially non-uniform functionally graded Timoshenko beams, Int. J. Mech. Sci. 89 (2014) 1–11.

[6] S.R. Li, Z.Q. Wan, J.H. Zhang, Free vibration of functionally graded beams based on both classical and first-order shear deformation beam theories, Appl. Math. Mech. (English Ed.) 35 (2014) 591–606.

[7] K. Sarkar, R. Ganguli, Closed-form solutions for axially functionally graded Timoshenko beams having uniform cross-section and fixed-fixed boundary condition, Compos. Part B Eng. 58 (2014) 361–370.

[8] T.P. Vo, H.T. Thai, T.K. Nguyen, F. Inam, Static and vibration analysis of functionally graded beams using refined shear deformation theory, Meccanica. 49 (2014) 155–168.

[9] T.K. Nguyen, T.P. Vo, H.T. Thai, Static and free vibration of axially loaded functionally graded beams based on the first-order shear deformation theory, Compos. Part B Eng. 55 (2013) 147–157.

[10] X.F. Li, Y.A. Kang, J.X. Wu, Exact frequency equations of free vibration of exponentially functionally graded beams, Appl. Acoust. 74 (2013) 413–420.

[11] H.T. Thai, T.P. Vo, Bending and free vibration of functionally graded beams using various higher-order shear deformation beam theories, Int. J. Mech. Sci. 62 (2012) 57–66.

[12] G. Giunta, D. Crisafulli, S. Belouettar, E. Carrera, Hierarchical theories for the free vibration analysis of functionally graded beams, Compos. Struct. 94 (2011) 68–74.

[13] J. Murín, M. Aminbaghai, V. Kutiš, Exact solution of the bending vibration problem of FGM beams with variation of material properties, Eng. Struct. 32 (2010) 1631–1640.

[14] M. Şimşek, Fundamental frequency analysis of functionally graded beams by using different higher-order beam theories, Nucl. Eng. Des. 240 (2010) 697–705.

[15] S.A. Sina, H.M. Navazi, H. Haddadpour, An analytical method for free vibration analysis of functionally graded beams, Mater. Des. 30 (2009) 741-747.

[16] X.F. Li, A unified approach for analyzing static and dynamic behaviors of functionally graded Timoshenko and Euler-Bernoulli beams, J. Sound Vib. 318 (2008) 1210-1229.

[17] M. Aydogdu, V. Taskin, Free vibration analysis of functionally graded beams with simply supported edges, Mater. Des. 28 (2007) 1651-1656.

[18] Z. Zhong, T. Yu, Analytical solution of a cantilever functionally graded beam, Compos. Sci. Technol. 67 (2007) 481–488.

[19] B. V. Sankar, An elasticity solution for functionally graded beams, Compos. Sci. Technol. 61 (2001) 689-696.

[20] T.P. Vo, H.T. Thai, T.K. Nguyen, A. Maheri, J. Lee, Finite element model for vibration and buckling of functionally graded sandwich beams based on a refined shear deformation theory, Eng. Struct. 64 (2014) 12–22.

[21] A.E. Alshorbagy, M.A. Eltaher, F.F. Mahmoud, Free vibration characteristics of a functionally graded beam by finite element method, Appl. Math. Model. 35 (2011) 412–425.

[22] A. Chakraborty, S. Gopalakrishnan, J.N. Reddy, A new beam finite element for the analysis of functionally graded materials, Int. J. Mech. Sci. 45 (2003) 519–539.

[23] K.K. Pradhan, S. Chakraverty, Free vibration of Euler and Timoshenko functionally graded beams by Rayleigh-Ritz method, Compos. Part B Eng. 51 (2013) 175–184.

[24] L.L. Jing, P.J. Ming, W.P. Zhang, L.R. Fu, Y.P. Cao, Static and free vibration analysis of functionally graded beams by combination Timoshenko theory and finite volume method, Compos. Struct. 138 (2016) 192-213.

[25] J. Gong, L. Xuan, P. Ming, W. Zhang, Thermoelastic analysis of functionally graded solids using a staggered finite volume method, Compos. Struct. 104 (2013) 134–143.

[26] J.F. Gong, P.J. Ming, L.K. Xuan, W.P. Zhang, Thermoelastic analysis of three-dimensional functionally graded rotating disks based on finite volume method, Proc. Inst. Mech. Eng. Part C J. Mech. Eng. Sci. 228 (2013) 583–598.

[27] S. Rajasekaran, Differential transformation and differential quadrature methods for centrifugally stiffened axially functionally graded tapered beams, Int. J. Mech. Sci. 74 (2013) 15–31.

[28] S. Rajasekaran, Buckling and vibration of axially functionally graded nonuniform beams using differential transformation based dynamic stiffness approach, Meccanica. 48 (2013) 1053–1070.
[29] Y.S. Al Rjoub, A.G. Hamad, Free vibration of functionally Euler-Bernoulli and Timoshenko graded porous beams using the transfer matrix method, KSCE J. Civ. Eng. 21 (2017) 792–806.
[30] J.W. Lee, J.Y. Lee, Free vibration analysis of functionally graded Bernoulli-Euler beams using an exact transfer matrix expression, Int. J. Mech. Sci. 122 (2017) 1-17.
[31] H. Su, J.R. Banerjee, Development of dynamic stiffness method for free vibration of functionally graded Timoshenko beams, Comput. Struct. 147 (2015) 107–116.
[32] H. Su, J.R. Banerjee, C.W. Cheung, Dynamic stiffness formulation and free vibration analysis of functionally graded beams, Compos. Struct. 106 (2013) 854-862.
[33] J. Fitch, Solving algebraic problems with REDUCE, J. Symb. Comput. 1 (1985) 211-227.
[34] A.C. Hearn, REDUCE User's Manual, Version 3.5, RAND Publication, Santa Monica, California, USA, 1993.
[35] J.R. Banerjee, A.J. Sobey, H. Su, J.P. Fitch, Use of computer algebra in Hamiltonian calculations, Adv. Eng. Software, 39 (2008) 521-525.

[36] F.W. Williams, W.H. Wittrick, Exact buckling and frequency calculations surveyed, J. Struct. Eng, ASCE, 109 (1983) 169-187.

[37] F.W. Williams, Review of exact buckling and frequency calculations with optional multi-level substructuring, Comput. Struct., 48 (1993) 547-552.
[38] J.R. Banerjee, Dynamic stiffness formulation for structural elements: A general approach, Comput. Struct. 63 (1997) 101-103.
[39] W.H. Wittrick, F.W. Williams, A general algorithm for computing natural frequencies of elastic structures, Quart. J. Mech. Appl. Math. 24 (1971) 263-284.
[40] D.V. Bambill, C.A. Rossit, D.H. Felix, Free vibrations of stepped axially functionally graded Timoshenko beams, Meccanica. 50 (2015) 1073-1087.
[41] N. Wattanasakulpong, J. Charoensuk, Vibration characteristics of stepped beams made of FGM using differential transformation method, Meccanica. 50 (2015) 1089-1101.
[42] M.A. Eltaher, A.E. Alshorbagy, F.F. Mahmoud, Determination of neutral axis position and its effect on natural frequencies of functionally graded macro/nanobeams, Compos. Struct. 99 (2013) 193-201.

[43] K.S. Al-Basyouni, A. Tounsi, S.R. Mahmoud, Size dependent bending and vibration analysis of functionally graded micro beams based on modified couple stress theory and neutral surface position, Compos. Struct. 125 (2015) 621-630.

[44] L.A. Pipes, L.R. Harvill, Applied mathematics for engineers and physicists, McGraw-Hill, 1970.

[45] J.R. Banerjee, Coupled bending-torsional dynamic stiffness matrix for beam elements, Int. J. Num. Meth. Eng. 28 (1989) 1283-1298.

[46] J.R. Banerjee, Explicit analytical expressions for frequency equation and mode shapes of composite beams, Int. J. Solids Structs. 38 (2001) 2415-2426.

[47] C.P. Filipich, P.A.A. Laura, In-plane vibrations of portal frames with end supports elastically restrained against rotation and translation, J. Sound Vib. 117 (1987) 467–473.

[48] R.O. Grossi, C.M. Albarracin, A variational approach to vibrating frames, Proc. Inst. Mech. Eng., Part K: J. Multibody Dyn. 221 (2007) 247-259.

[49] I. Tatar, Vibration characteristics of portal frames, MSc Thesis, June 2013, Izmir Institute of Technology, Graduate School of Engineering and Sciences, Izmir, Turkey.