Quantum field inspired model of decision making: Asymptotic stabilization of the belief state via interaction with surrounding mental environment

Fabio Bagarello [1]
Irina Basieva [2]
Andrei Khrennikov [3]

Abstract

This paper is devoted to justification of the quantum-like model of the process of decision making based on theory of open quantum systems: decision making as decoherence. This process is modeled as interaction of a decision maker, Alice, with a mental (information) environment $\mathcal{R}$ surrounding her. Such an interaction generates "dissipation of uncertainty" from Alice's belief-state $\rho(t)$ into $\mathcal{R}$ and asymptotic stabilization of $\rho(t)$ to a steady belief-state. The latter is treated as the decision state. Mathematically the problem under study is about finding constraints on $\mathcal{R}$ guaranteeing such stabilization.

[1]Dipartimento di Energia, Ingegneria dell'Informazione e Modelli Matematici, Scuola Politecnica, Università di Palermo, I-90128 Palermo, INFN, Sezione di Napoli, ITALY and
Department of Mathematics and Applied Mathematics, Cape Town, South Africa
e-mail: fabio.bagarello@unipa.it Home page: www1.unipa.it/fabio.bagarello
[2]Department of Psychology, City University, London, UK
e-mail: Irina.Basieva@gmail.com
[3]International Center for Mathematical Modeling, in Physics and Cognitive Science
Linnaeus University, Växjö, Sweden
National Research University of Information Technologies, Mechanics and Optics (ITMO)
St. Petersburg, Russia
e-mail: Andrei.Khrennikov@lnu.se

We found a partial solution of this problem (in the form of sufficient conditions). We present the corresponding decision making analysis for one class of mental environments, so-called “almost homogeneous environments”, with the illustrative examples: a) behavior of electorate interacting with the mass-media “reservoir”; b) consumers’ persuasion. We also comment on other classes of mental environments.

keywords: decision making; quantum-like model; mental (information) environment; open quantum systems; dissipation of uncertainty; voters’ behavior; consumers’ persuasion

1 Introduction

The recent years were characterized by explosion of interest in applications of the mathematical formalism of quantum theory to studies in cognition, decision making, psychology, economics, finance, and biology, see, e.g., the monographs [1]-[6] and a few representative papers [7]-[33] (the first steps in this direction were done long time ago, see, e.g., [34]).

The approach explored in such mathematical modeling is known as *quantum-like.* In this approach an agent (human, animal, or even cell) is considered as a *black box* processing information in accordance with the laws of quantum information and probability theories. Thus the quantum-like modeling is basically quantum informational modeling (although this characteristic feature of such modeling is typically not emphasized, cf., however, with [35]).[1]

The quantum-like models have to be sharply distinguished from genuinely quantum physical models of cognition which are based on consideration of quantum physical processes in the brain, cf. with R. Penrose [38] and S. Hameroff [39]. Although the quantum physical models have been criticized for mismatching between the temperature and space-times scales of the quantum physical processes and neuronal processing in the brain, see especially Tegmark [40], they cannot be rejected completely and one may expect that the quantum-like model of cognition would be (soon or later) coupled with real physical processes in the brain, see [41]-[47] for some steps in this direction.

The quantum-like approach generated a variety of models of cognition and decision making. In the simplest model [34], [1], the mental state (the belief state) of an agent, Alice, is represented as a quantum state ψ and questions or tasks as quantum observables (Hermitian operators). Probabilities of answers are determined by Born’s rule. For Hermitian operator A

[1] See, e.g. D’ Ariano [36] and Plotnitsky [37] for the information approach to quantum mechanics.

with the purely discrete spectrum, Born's rule can be written as

$$p(A = \alpha_k) = \|P_{\alpha_k}\psi\|^2 = \langle P_{\alpha_k}\psi, \psi\rangle, \tag{1.1}$$

where α_k is an eigenvalue of A and P_{α_k} is the projector on the eigenspace corresponding to this eigenvalue.

This model does not describe the dynamics of the belief state in the process of decision making. This dynamics was invented in the works of Khrennikov [7, 8], and Pothos and Busemeyer [10]. In this dynamical model, an observable A corresponding to a question (task) to Alice is also represented as a Hermitian operator. Then Hamiltonian H generating the unitary dynamics

$$\psi(t) = U(t)\psi_0,\ U(t) = e^{-itH} \tag{1.2}$$

of the initial belief state ψ_0 is introduced, and Alice's decision is represented as measurement of the observable A at some instant of time. The authors of [10] presented cognitive arguments supported by experimental studies to determine the instant t_m of measurement. Here the probability of an answer is also determined by the Born rule, but for the belief state $\psi(t_m)$. Of course, this is an essential issue, since different values ot t_m can give rise to completely different results. In spite of the partial progress in determination of t_m in the article of Pothos and Busemeyer [10], this complex problem cannot be considered as solved completely. Moreover, to solve this problem Pothos and Busemeyer have to go beyond the quantum theory and to appeal to psychological theoretical and experimental studies. It would be attractive to solve this problem entirely in the quantum framework.

We remark that the Schrödinger equation describes the dynamics of an isolated system. In the presence of an environment, the dynamics of the system is non-unitary. Approximately (under some sufficiently natural conditions) it is described by Gorini-Kossakowski-Sudarshan-Lindblad (GKSL) equation (often called simply Lindblad equation), the simplest version of the quantum master equation. One of the main distinguishing features of such dynamics is that it does not preserve the pure state structure: it transforms (immediately) the pure initial state ψ_0 into a mixed state given by the density operator:

$$\rho(t) = U(t)\rho_0,\ U(t) = e^{-itL}, \tag{1.3}$$

where L is the generator of the GKSL-evolution and the $\rho_0 = |\psi_0\rangle\langle\psi_0|$. This dynamics is in general non-unitary. This equation describes the process of *adaptation of a system to the surrounding environment.* This is the complex dynamical process combining the internal state dynamics of a system with adaptation to signals received from the environment. If the dynamics is discrete with respect to time, then it can be represented as *a chain of unitary evolutions and (generalized) quantum Bayesian updates.*[2] In this paper we cannot discuss this interesting issue in more detail, see [16] for detailed consideration of the two dimensional example with application to the evolution theory.

In fact, the GKSL-dynamics does not contradict to the Schrödinger equation structure of the quantum evolution. Let S denote a system and let $\mathcal{R}$ denote the surrounding environment ("reservoir" for S.) Suppose that the initial state of the compound system $S + \mathcal{R}$ is pure and that it is separable. The dynamics in the state space of $S + \mathcal{R}$ is still unitary and it is given by the Hamiltonian of $S + \mathcal{R}$. The main distinguishing feature of this unitary dynamics is that (in the presence of the interaction between S and $\mathcal{R}$) it induces *entanglement* and the state of the compound system is not more separable. This Hamiltonian is very complex, since it includes in general the infinite number degrees of freedom of $\mathcal{R}$. Typically it is impossible to solve the Schrödinger equation for the state of $S+\mathcal{R}$. (Although in some special cases, as those considered in this paper and others discussed in [4] analytic solutions can be found.) Therefore studies are typically restricted to the dynamics of the state $\rho(t)$ of S described (approximately) by the GKSL-equation. But even if one were able to solve the Schrödinger equation for $S + \mathcal{R}$, its solution is a very complex infinite-dimensional state vector. Since we are interested in behavior of S, it is natural to take the trace with respect to all degrees of freedom of $\mathcal{R}$ and obtain the state of S. The simple mathematical theorem implies that in the presence of entanglement this trace-state cannot be pure. One has to consider the dynamics of the density operator. This dynamical process is known as *decoherence:* decreasing of state's purity in the process of interaction with an environment.

Since consideration of an isolated cognitive system is even a higher degree idealization than consideration of an isolated physical system, it is natural to modify the dynamical scheme of decision making based on the unitary Schrödinger dynamics [7, 8, 10] and consider the general dynamics of the belief-state, either by using the approximative GKSL-dynamics or by tracing the state of the compound system. Roughly speaking there is no choice: either one

[2] As was quickly understood in quantum physics, the Lüders projection postulate describes only one very special class of the quantum state updates resulting from measurements. We remark that already von Neumann accepted applicability of this straightforward form of the state update only for observables with non-degenerate spectra. Generally, for an observable with degenerate spectrum, a pure pre-measurement state can be transferred into a mixed state [48]. Later these considerations of von Neumann were elaborated in the form of the *theory of quantum instruments*, see [49] for non-physicist friendly presentation. The most consistent justification of this theory is obtained in the framework of the *theory of open quantum systems*, see [50] for non-physicist friendly presentation.

has to ignore the presence of environment or consider non-unitary dynamics, e.g., (1.3). Of course, such non-unitary dynamical model of decision making is essentially more complicated mathematically. However, it has one very important advantage. Here the aforementioned problem of determination of t_m is solved automatically, although not in the straightforward way.

For a natural class of quantum master equations, in the limit $t \to \infty$ system's state $\rho(t)$ approaches the *steady state* $\rho_{\rm out}$. The diagonal elements of this operator with respect to the "pointer basis" corresponding to the observable A under measurement give the probabilities of the outputs of measurements. The model of decision making as decoherence was represented in a series of papers of Asano et al., see, e.g., [51], [52] [13], [6]. The model is purely informational. Both a "quantum-like system", Alice, and her mental (or information) environment ("bath", reservoir") are represented by quantum states, ψ and ϕ. (It can be assumed that initially these states are pure.) We are not interested in their physical or neurophysiologic realizations. Asano et al. [51] applied the *theory of open quantum systems* and the GKSL-equation to model experimental data collected in decision making experiments. The initial belief state is typically represented as a pure state of complete uncertainty,

$$\psi = (|0\rangle + |1\rangle)/\sqrt{2}, \tag{1.4}$$

where $|i\rangle, i = 0, 1$, are eigenstates of the dichotomous observable A representing a "yes"/"no" question. A proper decision making dynamics should asymptotically ($t \to \infty$) drive the belief state of Alice $\rho(t)$ to the output (mixed) state represented by the density operator $\rho_{\rm out}$. Its diagonal elements in the basis of eigenvectors of the operator A give probabilities of possible decisions.

We remark that the open quantum systems approach to decision making can be considered as a possible realization of the contextual treatment of cognition, cf. Khrennikov [1], [2] and Dzhafarov et. al. [12], [17]. The surrounding mental environment represents a measurement context for decision making.

Of course, Alice cannot wait for $t \to \infty$ to make a decision. And, as we know in physics, a quantum system relaxes to a steady state very quickly. Here we have to point to the difference between a mathematical model and its applications to real phenomena. The notion of limit is a mathematical abstraction. Of course, it is not applicable to real physical or cognitive systems. In practice, the limit procedure encodes the process of approaching of some quantity. The existence of a limit for the density operator $\rho(t)$ guarantees that its fluctuations decrease. If the magnitude of fluctuations becomes smaller than some $\epsilon > 0$, then such $\rho(t_\epsilon)$ can be selected as a good approximation of the steady state (the latter is also a mathematical abstraction, in real physical processes it is never approached exactly). This t_ϵ plays the role t_m from the unitary dynamical model of decision making. Moreover, more complicated considerations based on theory of decoherence demonstrate that this instant of time t_ϵ can be identified with the *relaxation time* T. The latter is determined by the structure of interactions between a system and its environment. Typically in quantum physics the time interval $[0, T]$ is very short and a system relaxes very quickly to its steady state. Since such considerations would make the paper even more complex mathematically, we shall not proceed in this direction. (See, e.g., [53] for derivation of an analytical expression for the relaxation time as a function of the heat-bath and interaction parameters.) The main message of these considerations to a reader is that $\lim_{t\to\infty}$ is just an abstraction (at $t = \infty$ all fluctuations disappear completely). In reality a system relaxes very quickly to its steady state.

The main problem of the quantum(-like) decision theory is to construct an operator representation of Hamiltonians and "Lindbladians" (the latter represents interactions of Alice with her mental reservoir). In contrast to quantum physics, there is no analog of the classical phase formalism for cognition and decision making, i.e., we cannot use the quantization procedure to transform functions on the classical phase space into operators - the procedure of Schrödinger's quantization is not applicable.

In quantum theory one can also use the quantization procedure based on the algebra of operators of creation and annihilation. It is especially useful for second quantization (quantum field theory), see appendix 1 for brief presentation of its basics. This sort of quantization can be successfully applied to quantum-like modeling. Consider again a dichotomous question A, with the values "no"/ "yes". Creation operator $a^\star$ creates "yes" from "no, annihilation operator a transforms "yes" to "no". Then we can compound Hamiltonians and observables with the aid of creation and annihilation operators (similarly e.g. to quantum optics) . This approach to construction of operators for problems of decision making was pioneered by Bagarello [4] and coauthors who applied it, see, e.g., [28], [26], [27], to a variety of problems. Bagarello et al. were interested in the time dynamics of averages; in particular, probabilities were treated statistically. In series of works [28], [26], [27], a system was interpreted as an agent and the reservoir had a purely information interpretation, see especially [54] for a discussion.

The same approach based on creation-annihilation operators can be explored to model decision making at individual level (with the subjective interpretation of probabilities). Here one can explore the scheme which was approved in the works of Asano et al. [6]: consideration of the asymptotic dynamics of the belief state and using the asymptotically output state (for

$t \to \infty$) as the basis of decision making. The first step in this direction was done in paper [51]. The aforementioned decision making scheme (based on asymptotic stabilization of the mental (belief) state of an agent, Alice) generates interest to study the asymptotic dynamics of a system interacting with reservoir. This is a nontrivial mathematical problem and its partial solution (for specially designed information reservoirs) is presented in this paper.

In section 2 we discuss several aspects of the process of decision making for a dichotomous question A through belief-state stabilization is considered. Then, in section 3, the mathematical problem is discussed in details. More explicitly, in section 3.1 we prove stabilization of the belief state in the case of the "almost homogeneous mental environment" of Alice. We start with the case of a question' A having the infinite number of possible outcomes labeled as $n_a = 0, 1, ..., n, ...$. Then in section 3.3 we consider the case of a "dichotomous question" A having two possible outcomes, $n_a = 0, 1$. In section 3.2 we generalize our theory to the case of environments having a more complex structure. Section 4 contains our conclusions, while appendices 1 and 2 are devoted to few introductory remarks on canonical anti-commutation relations (CAR) and to some extra mathematical details.

2 Decision making as decoherence

Consider the case of a dichotomous questions asked to Alice, $A = 0, 1$ ("no", "yes") and mental environment $\mathcal{R}$ combined as well of dichotomous degrees of freedom, in section 3.3 this situation will be modeled by using operators $a, a^\dagger$ (Alice's operators) and $b, b^\dagger$ ($\mathcal{R}$'s operators) satisfying the CAR:

$$\{a, a^\dagger\} = \mathbb{I}, \{b, b^\dagger\} = \mathbb{I}, \{a, b\} = 0, \tag{2.1}$$

where the anti-commutator of two operators x, y is defined as $\{x, y\} = xy + yx$. Moreover, $a^2 = b^2 = 0$. In quantum field theory, see appendix 1, these operators are known as *the operators of creation and annihilation.*[3] The operators related to $\mathcal{R}$ can depend on some parameter k (discrete or continuous) representing degrees of freedom of $\mathcal{R}, b = b(k), b^\dagger = b^\dagger(k)$. Typically environment $\mathcal{R}$ has a huge number of degrees of freedom and it is represented by infinite-dimensional Hilbert state space $\mathcal{K}$. In the case of dichotomous questions asked to Alice, it can be assumed that her state space $\mathcal{H}$ has the dimension two, *the qubit space of quantum information theory.*

In quantum field theory the terminology "creation-annihilation operators" has coupling to real physical systems, they represent the processes of creation-annihilation of quantum particles

[3] In physics these operators represents the processes of creation and annihilation of fermions, e.g., electrons.

(e.g., photons or electrons), see appendix 1. As was emphasized in introduction, our model is of the purely informational nature. Therefore the operators $a, a^\dagger$ and $b, b^\dagger$ have to be treated as the formal representation of the processes of "creation" and "annihilation" of information states.[4]

Consider, e.g., Alice's operators $a, a^\dagger$. Let $\phi_0 = |0\rangle, \phi_1 = |1\rangle$ be the orthonormal basis in Alice's state space $\mathcal{H}$ corresponding to the answers "no"/"yes" to the question A. As a consequence of the CAR, the operators $a, a^\dagger$ act to these basis vectors in the following way: $a^\dagger|0\rangle = |1\rangle, a^\dagger|1\rangle = 0; a|1\rangle = |0\rangle, a|0\rangle = 0$.

The operators a and $a^\dagger$ modify Alice's attitude to selection of alternatives. In this framework these operators are considered as the *reflection operators* [54]. We represent the question A as the *number operator* of Alice, $\hat{n}_a = a^\dagger a$. The eigenvalues of this operator, $n_a = 0, 1$, correspond to the choices operated by Alice at $t = 0$. Therefore it is natural to name $\hat{n}_a$ the *decision operator*.[5]

Consider the simplest situation which can be modeled in this framework: Alice is asked a question A and she is surrounded by a population $\mathcal{R}$ whose members are asked the same question A. Alice interacts with this population $\mathcal{R}$ and she gets to know behavior of its members with respect to this question. Of course, she cannot "scan" $\mathcal{R}$ completely to know the concrete answers of people to A. She just gets to know a sort of average n_b of answers to A. In the simplest case of a homogeneous population this average is a constant. In the general case average n_b can non-trivially depend on the parameter k encoding various population clusters $\mathcal{R}(k): \; n_b = n_b(k)$. Here Alice (through interaction with this population) obtains information about averages for the answers to A corresponding to different clusters $\mathcal{R}(k)$ of $\mathcal{R}$.

In principle, $\mathcal{R}$ needs not be combined of physical agents. As was emphasized in introduction, our model is of the purely informational nature. The environment $\mathcal{R}$ can, for example, represent mass-media's image of the question under consideration: the image created in TV-debates and shows, web-blogs, newspapers. For example, A can be a referendum question: *"vote for Brexit or against?"* (or the recent USA-election question: *"vote for Trump or not?"*). By analysing (generally unconsciously) mass-media's output Alice estimates the average n_b (or in general the averages $n_b(k)$) and she makes her decision.

In a more general situation the members of $\mathcal{R}$ express their attitude with respect to a variety of dichotomous questions and A, the question asked to Alice, can be among them, but need not.

[4]In principle, we can simply call these operators ladder operators as is done in formal mathematical theory.

[5]In quantum field theory the number operator has the meaning of the number of physical particles, see appendix 1. In our model it represents states indexed by numbers.

In the latter case Alice makes her decision based on behavior of the surrounding environment $\mathcal{R}$ (physical or informational) without even understanding that her answer is elaborated through influence of $\mathcal{R}$.

Now we consider the dynamics of decision making. In section 3 we shall use the Heisenberg picture, i.e., the dynamics of operators. We are interested in the dynamics of the reflection operators, $a^\dagger(t), a(t)$, with the initial conditions $a^\dagger(0) = a^\dagger, a(0) = a$. By knowing those dynamics we can find the dynamics of the decision operator $\hat{n}_a(t) = a^\dagger(t)a(t)$ and, hence, its average

$$n_a(t) :=< \hat{n}_a(t) \otimes \mathbb{I} >=< a^\dagger(t)a(t) \otimes \mathbb{I} > \tag{2.2}$$

with respect to some initial state of the compound system, Alice interacting with her mental environment, see section 3 for details.

In the case of dichotomous questions the average $n_a(t)$ has the straightforward probabilistic meaning. To see this, we rewrite expression (2.2) with more details. Denote the state space of the compound system, Alice and her mental environment, by the symbol $\mathcal{H} \otimes \mathcal{K}$ and the operator of Heisenberg evolution of this system by U_t. For an arbitrary initial state $\langle ., . \rangle_R$, formula (2.2) reads as

$$n_a(t) =< U_t(a^\dagger \otimes \mathbb{I})U_t^\dagger U_t(a \otimes \mathbb{I})U_t^\dagger >_R = \mathrm{Tr U_t(a^\dagger a \otimes \mathbb{I})U_t^\dagger R}.$$

By using the cyclic property of the trace we get:

$$n_a(t) = \mathrm{Tr(a^\dagger a \otimes \mathbb{I})U_t^\dagger R U_t} = \mathrm{TrR(t)(a^\dagger a \otimes \mathbb{I})} =< \mathrm{a^\dagger a} \otimes \mathbb{I} >_{\mathrm{R(t)}},$$

where $R(t) = U_t^\dagger R U_t$ represents the state dynamics (so we transferred the operator dynamics into the the state dynamics, from the Heisenberg picture to the Schödinger picture).[6] Consider now the partial trace of $R(t)$ with respect to the environmental degrees of freedom, $\rho(t) = \mathrm{Tr}_{\mathcal{K}}\mathrm{R(t)}$. It represents the dynamics of Alice's belief-state in the process of her interaction with the environment. Consider the average of the decision operator $\hat{n}_a = a^\dagger a = a^\dagger(0)a(0)$ with respect to this belief-state:

$$< \hat{n}_a >_{\rho(t)} = \mathrm{Tr}\rho(\mathrm{t})\hat{\mathrm{n}}_{\mathrm{a}}.$$

This average has direct coupling with probability. Since the decision operator $\hat{n}_a$ has two eigenvalues, $n_a = 0, 1$, the average coincides with the probability:

$$P_t(A = 1) =< \hat{n}_a >_{\rho(t)} =< \rho(t)\phi_1, \phi_1 > . \tag{2.3}$$

[6] The Schrödinger picture providing the belief-state dynamics is basic for the probabilistic interpretation. However, the equations in the Heisenberg picture are easier for analytic treatment. Therefore in section 3 we shall work in the Heisenberg picture.

We recall that here the question A is represented by the operator $\hat{n}_a$. The eigenvector ϕ_1 corresponds to the answer "yes" to this question. $P_t(A = 1)$ is the subjective probability in favor of this answer which Alice assigns at the moment of time t. We remark that this assignment can be done unconsciously. Alice will consciously report only her final decision, see considerations below.

Now we make a very general remark about the partial trace. For any operator M acting in $\mathcal{H}$, the following equality holds:

$$\mathrm{Tr}\rho\mathrm{M} = \mathrm{Tr}\mathrm{R}(\mathrm{M} \otimes \mathbb{I}), \text{ where } \rho = \mathrm{Tr}_{\mathcal{K}}\mathrm{R}.$$

Therefore the average $n_a(t)$ equals to the average of Alice's decision operator:

$$n_a(t) =< \hat{n}_a >_{\rho(t)} . \tag{2.4}$$

In section 3 we obtain the averages of the decision operator $\hat{n}_a(t)$ for its eigenstates ϕ_{n_a} (for operators satisfying CAR, we have: $n_a = 0, 1$). Thus initially Alice has the definite belief about a possible answer to the question A.

In section 3.1 we consider the case of the "almost homogeneous reservoir" - a mental (information) environment $\mathcal{R}$ which is behaviorally almost homogeneous in the following sense (see section 3.1 for the mathematical formalization). The basic parameter characterizing $\mathcal{R}$ is deduced by $< b^\dagger(k)b(q) >= n_b(k)\delta(q - k)$, the average for the agents of the k-type. In the almost homogeneous $\mathcal{R}$, $n_b(k)$ is constant. In general, however, it can vary with k [7]. It must be stressed that the reservoir considered in section 3.1 is not completely homogeneous. Its members are not all identical. They have different internal dynamical scales represented by the function $\omega(k) = \omega_b k$ which really depends on k, even if this dependence is *relatively mild* ($\omega(k)$ is linear in k, so that $\omega(k_1) \simeq \omega(k_2)$ if $k_1 \simeq k_2$), see again section 3.1.

For such environment $\mathcal{R}$, the average $n_a(t)$ of the decision operator $\hat{n}_a$ was found, see formula (3.7) section 3.1 (with the corresponding generalization to the CAR-case in section 3.3). This formula in combination with equalities (2.3), (2.4) gives the probabilistic dynamics:

$$P_t(A = 1) = n_a(t) = n_a\, e^{-\frac{2\lambda^2\pi}{\omega_b}t} + n_b\left(1 - e^{\frac{2\lambda^2\pi}{\omega_b}t}\right) = n_b + (n_a - n_b)\, e^{-\frac{2\lambda^2\pi}{\omega_b}t}, \tag{2.5}$$

where λ is the constant of interaction between Alice and $\mathcal{R}$, see (3.1). As we will see in the

[7] In the simplest model all agents in $\mathcal{R}$ are asked the same question A as Alice and $n_b(k)$ is the average of their answers to A. In a more complex environment $\mathcal{R}$ each cluster $\mathcal{R}(k)$ of agents labeled by the parameter k is asked its own question $A(k)$ and $n_b(k)$ is the average of their answers to $A(k)$.

next section, this is the result of a reasonable choice of the Hamiltonian operator, where the interactions between Alice and her reservoir are included[8].

Consider, e.g., the case $n_a = 1$, e.g., initially Alice was for Brexit. Suppose that her surrounding environment is almost homogeneous[9] and $n_b = 1/3$, i.e., its members express sufficiently strong anti-Brexit attitude. Then Alice following the dynamics expressed by (2.5) would lose her interest to vote for Brexit and finally she will will accept the surrounding attitude to vote against Brexit. It is interesting to notice that the rapidity of the decision process is directly related to the strength of the interaction between Alice and her environment, and inversely connected to ω_b. This is in agreement with the fact, see [4], that ω_b describes a sort of inertia of the system.

The "decision probability" is given by the expression:

$$P(A = 1) = \lim_{t\to\infty} P_t(A = 1) = n_b. \tag{2.6}$$

We remark that dependence on the initial belief-state of Alice disappeared. It does not matter whether she was in the belief-state ϕ_0 (she was firmly determined to reply "no") or in the belief-state ϕ_1 (she was firmly determined to reply "yes"). Finally, she set the *subjective probability* $P(A = 1) = n_b$ to reply "yes". Moreover, as we will discuss later, even if initially Alice was in superposition belief-state (i.e., she started with a belief-state of uncertainty), it would lead to setting of the same probability in favor to reply "yes". This behavior of Alice is natural: she interacts with a homogeneous reservoir, an ensemble of agents, where all agents have the same probability to favor of the "yes" answer; Alice simply follows them.

In spite of its simplicity, the dynamics of a decision maker interacting with an almost homogeneous information environment can have non-trivial applications. It describes well not only the modern mass-media campaigns at the political arena, but it can also be used in economics and finance.

[8]This approach, which is used everywhere in [4], is the standard way in which the dynamics of any system, micro- or macroscopic, is deduced in classical and in quantum mechanics. What we are doing, in fact, is to adapt the same basic idea also to decision making processes.

[9]The latter assumption is not so unusual. Although the modern information (mental) environment represents huge variety of information flows, an individual is typically coupled to the concrete flow, e.g., she has the custom to see only special TV-channels and follow only specially selected Internet information resources. Even her physical human environment is typically homogeneous. A professor of a university is surrounded by rather homogeneous population with liberal views, including Brexit, Trump or Putin. One of the authors of this paper was visiting USA just before the second Bush-vote. The university environment expressed generally the belief that Bush would not be elected once again. For such $\mathcal{R}$, the average n_b could be estimated as approximately equal to one.

For example, in this framework we can model the process of *persuasion of consumers.* A firm wants to persuade consumers to buy some commodity C. So the question A is *"to buy or not to buy?* To approach this aim, this firm creates an information environment $\mathcal{R}$ (almost homogeneous) with the parameter $n_b \approx 1$. [10] Then our model shows that, for consumers, it is possible to approach the same level of confidence to buy C as in the surrounding information environment $\mathcal{R}$. As was already remarked, the information structure of $\mathcal{R}$ need not include directly the question A, i.e., the information campaign need not be straightforwardly oriented to advertising of the commodity C. The $\mathcal{R}$ can be combined of other questions, may be quite different from A. Then the main task of this firm driving the persuasion campaign is to establish interaction of consumers with $\mathcal{R}$ which can be formally represented by the interaction Hamiltonian considered in this paper.

Persuasion of consumers need not be reduced to the concrete commodity C. A group of corporations can perform persuasion of a class of commodities, e.g., a new generation of mobile phones, or wind energy, or electric cars. The strategy is the same: creation of an information environment and coupling of people to it which would generate stabilization of subjective probabilities $P_t(A = 1)$ (in a long run campaign) to the desired value of n_b.

3 The stabilizing procedure

This section is devoted to justification of the model of decision making as decoherence, stabilization of the belief-state $\rho(t)$ of the Alice. In the Heisenberg picture this problem can be transferred to the problem of stabilization of reflection (creation-annihilation) operators $a(t), a^\dagger(t)$ and consequently the decision operator $\hat{n}_a(t) = a^\dagger(t)a(t)$ (the number operator). Of course, such stabilization is possible only for special classes of information (mental) environments and interactions between a decision maker and the surrounding environment. Mathematically the problem is very complicated and we were able to solve it only partially. This section is of the purely mathematical (and quantum physical) nature.

If $\mathcal{S}$ is a closed quantum system with time-independent, self-adjoint, hamiltonian H, it is natural to suspect that only periodic or quasi-periodic effects can take place, during the time evolution of $\mathcal{S}$. This is because the energy of $\mathcal{S}$ is preserved, and this seems to prevent to have any damping effect in $\mathcal{S}$. For instance, if we work in the Schrödinger representation (SR), the time evolution $\Psi(t)$ of the wave-function of the system is simply $\Psi(t) = e^{-iHt}\Psi(0)$ and, since

[10] Typically through a massive campaign in mass-media. However, the information content of $\mathcal{R}$ is not reduced to direct advertising of C, see considerations below.

the operator e^{-iHt} is unitary, we do not expect $\Psi(t)$ to decrease (in some suitable sense) to zero when t diverges. Nevertheless, we will show that a similar decay feature is possible if $\mathcal{S}$ is coupled to a reservoir $\mathcal{R}$, but only if $\mathcal{R}$ is rather *large* as compared to $\mathcal{S}$, or, more explicitly, if $\mathcal{R}$ has an infinite number of degrees of freedom.

To see this in details, we start by considering a first system, $\mathcal{S}$, interacting with a second system, $\tilde{\mathcal{S}}$, and we assume for the time being that both $\mathcal{S}$ and $\tilde{\mathcal{S}}$ are *of the same size*: to be concrete, this means here that $\mathcal{S}$ describes a single particle and, analogously, $\tilde{\mathcal{S}}$ describes a second particle. To model dynamics, we introduce the operators a, $a^\dagger$ and $\hat{n}_a = a^\dagger a$ (for the first particle) and the operators b,$b^\dagger$ and $\hat{n}_b = b^\dagger b$ (for the second particle).[11] These operators obey the following *canonical commutation relations* (CCR):

$$[a, a^\dagger] = [b, b^\dagger] = \mathbb{I},$$

while all the other commutators are assumed to be zero.

A natural choice for the Hamiltonian of compound system $\mathcal{S} \cup \tilde{\mathcal{S}}$ is the following:

$$h = \omega_a \hat{n}_a + \omega_b \hat{n}_b + \mu \left(a^\dagger b + b^\dagger a \right),$$

where ω_a, ω_b and μ must be real quantities for h to be self-adjoint. Recall that loosing self-adjointness of h would produce a non unitary time evolution, and this is out of the scheme usually considered in ordinary quantum mechanics. The hamiltonian h contains a free part plus an interaction which is such that, if the eigenvalue of $\hat{n}_a$ increases of one unit during the time evolution, then the eigenvalue of $\hat{n}_b$ must decrease of one unit, and viceversa. This is because $[h, \hat{n}_a + \hat{n}_b] = 0$, so that $\hat{n}_a + \hat{n}_b$ is an integral of motion.

Consider dynamics of the operators in the Heisenberg representation. The equations of motion for $a(t)$ and $b(t)$, can be easily deduced and turn out to be

$$\dot{a}(t) = i[h, a(t)] = -i\omega_a a(t) - i\mu b(t), \qquad a(0) = a,$$

$$\dot{b}(t) = i[h, b(t)] = -i\omega_b b(t) - i\mu a(t), \qquad b(0) = b,$$

whose solution can be written as $a(t) = a\alpha_a(t) + b\alpha_b(t)$ and $b(t) = a\beta_a(t) + b\beta_b(t)$, where the functions $\alpha_j(t)$ and $\beta_j(t)$, $j = a, b$, are linear combinations of $e^{\lambda_\pm t}$, with $\lambda_\pm = \frac{-i}{2}(\omega_a + \omega_b - \sqrt{(\omega_a - \omega_b)^2 + 4\mu^2})$. Moreover $\alpha_a(0) = \beta_b(0) = 1$ and $\alpha_b(0) = \beta_a(0) = 0$, in order to have $a(0) = a$ and $b(0) = b$. Hence we see that both $a(t)$ and $b(t)$, and $\hat{n}_a(t) = a^\dagger(t)a(t)$ and

[11] The cognitive meaning of these operators in the quantum-like model of decision making will be discussed in section 2.

$\hat{n}_b(t) = b^\dagger(t)b(t)$ as a consequence, are linear combinations of oscillating functions, so that no damping is possible within this simple model.

Suppose now that the system $\tilde{\mathcal{S}}$ is replaced by an (infinitely extended) reservoir $\mathcal{R}$, whose particles are described by an infinite set of CCR-operators $b(k), b^\dagger(k)$ and $\hat{n}(k) = b^\dagger(k)b(k)$, $k \in \mathbb{R}$. Each k labels one of the elements of the reservoir surrounding $\mathcal{S}$. Hence, for $k_1 \neq k_2$, we are considering two different elements which may have different characteristics. This is why, in (3.1), we are introducing two k-depending functions, $\omega(k)$ and $f(k)$. The hamiltonian of $\mathcal{S} \cup \mathcal{R}$ extends h above and is now taken to be

$$H = H_0 + \lambda H_I, \quad H_0 = \omega \hat{n}_a + \int_{\mathbb{R}} \omega(k)\hat{n}(k)\, dk, \quad H_I = \int_{\mathbb{R}} \left(ab^\dagger(k) + a^\dagger b(k)\right) f(k)dk, \tag{3.1}$$

where $[a, a^\dagger] = \mathbb{I}$, $[b(k), b^\dagger(q)] = \mathbb{I}\delta(k-q)$, while all the other commutators are zero. It could be useful to notice that here, rather than the CAR used in section 2, we are assuming CCR. We will see later, in section 3.3, that this does not affect our main conclusions. The reason for these different choices is to show that the output of the model does not really depend on the commutation rules adopted. All the constants appearing in (3.1), as well as the regularizing function $f(k)$, are real, so that $H = H^\dagger$. Notice that an integral of motion exists also for $\mathcal{S}\cup\mathcal{R}$, $\hat{n}_a + \int_{\mathbb{R}} \hat{n}(k)\, dk$, which extends the one for $\mathcal{S}\cup\tilde{\mathcal{S}}$, $\hat{n}_a + \hat{n}_b$. With this choice of H, the Heisenberg equations of motion are

$$\begin{cases} \dot{a}(t) = i[H, a(t)] = -i\omega a(t) - i\lambda \int_{\mathbb{R}} f(k)\, b(k,t)\, dk, \\ \dot{b}(k,t) = i[H, b(k,t)] = -i\omega(k) b(k,t) - i\lambda f(k)\, a(t), \end{cases} \tag{3.2}$$

which are supplemented by the initial conditions $a(0) = a$ and $b(k, 0) = b(k)$. In particular, the last equation can be rewritten in integral form as

$$b(k,t) = b(k)e^{-i\omega(k)t} - i\lambda f(k) \int_0^t a(t_1) e^{-i\omega(k)(t-t_1)}\, dt_1. \tag{3.3}$$

3.1 An almost homogeneous reservoir

In this section we fix $f(k) = 1$ and $\omega(k) = \omega_b k$, where $\omega_b \in \mathbb{R}_+$. This is a standard choice in quantum optics, [55]. We now insert $b(k,t)$ in (3.3) in the first equation in (3.2), we change the order of integration, and we use the integral expression for the Dirac delta $\int_{\mathbb{R}} e^{-i\omega_b k(t-t_1)}\, dk = \frac{2\pi}{\omega_b}\delta(t-t_1)$, as well as the equality $\int_0^t g(t_1)\delta(t-t_1)\, dt_1 = \frac{1}{2}g(t)$ for any test function $g(t)$. Then, we conclude that

$$\dot{a}(t) = -\left(i\omega + \frac{\pi\lambda^2}{\omega_b}\right) a(t) - i\lambda \int_{\mathbb{R}} b(k)\, e^{-i\omega_b k t}\, dk. \tag{3.4}$$

This equation can be solved, and the solution can be written as

$$a(t) = \left(a - i\lambda \int_{\mathbb{R}} dk\, \eta(k,t) b(k)\right) e^{-\left(i\omega + \frac{\pi\lambda^2}{\omega_b}\right)t}, \tag{3.5}$$

where $\eta(k,t) = \frac{1}{\rho(k)}\left(e^{\rho(k)t} - 1\right)$ and $\rho(k) = i(\omega - \omega_b k) + \frac{\pi\lambda^2}{\omega_b}$. Using complex contour integration it is possible to check, see appendix 2, that $[a(t), a^\dagger(t)] = \mathbb{I}$ for all t: this means that the natural decay of $a(t)$, described in (3.5), is balanced by the reservoir contribution. This feature is crucial since it is a measure of the fact that the time evolution is unitarily implemented in our approach, even if $a(t)$ apparently decays for t increasing.

Let us now consider a state over $\mathcal{S} \cup \mathcal{R}$, $\langle X_{\mathcal{S}} \otimes X_{\mathcal{R}} \rangle = \langle \varphi_{n_a}, X_{\mathcal{S}} \varphi_{n_a} \rangle \, \langle X_{\mathcal{R}} \rangle_{\mathcal{R}}$, in which φ_{n_a} is the eigenstate of the number operator $\hat{n}_a$ and $< >_{\mathcal{R}}$ is a state of the reservoir, which is assumed to satisfy, among other properties (see [55, 4]),

$$\left\langle b^\dagger(k) b(q) \right\rangle_{\mathcal{R}} = n_b(k)\delta(k-q). \tag{3.6}$$

This is a standard choice, see for instance [55], which extends the choice we made for $\mathcal{S}$. Here $X_{\mathcal{S}} \otimes X_{\mathcal{R}}$ is the tensor product of an operator of the system, $X_{\mathcal{S}}$, and an operator of the reservoir, $X_{\mathcal{R}}$. Then, if for simplicity we take the function $n_b(k)$ to be constant in k we get, calling $n_a(t) :=< \hat{n}_a(t) >=< a^\dagger(t) a(t) >$,

$$n_a(t) = n_a\, e^{-\frac{2\lambda^2\pi}{\omega_b}t} + n_b\left(1 - e^{\frac{2\lambda^2\pi}{\omega_b}t}\right), \tag{3.7}$$

which goes to n_b as $t \to \infty$. Hence, if $0 \leq n_b < n_a$, the value of $n_a(t)$ decreases with time. If, on the other hand, $n_b > n_a$, then the value of $n_a(t)$ increases for large t. This is the exponential rule which, as discussed before, cannot be deduced if $\mathcal{R}$ has not an infinite number of degrees of freedom. Notice that, in particular, if the reservoir is originally empty, $n_b = 0$, then $n_a(t) = n_a\, e^{-\frac{2\lambda^2\pi}{\omega_b}t}$ decreases exponentially to zero: the system becomes empty. On the other hand, since $\hat{n}_a + \int_{\mathbb{R}} \hat{n}(k)\, dk$ is a constant of motion, the reservoir starts to be filled up.

Remark 1. The *continuous* reservoir considered here ($k \in \mathbb{R}$) could be replaced by a discrete one, describing again an infinite number of particles, but labeled by a discrete index. In this case, to obtain a Dirac delta distribution, which is the crucial ingredient in the derivation above, we have to replace the integral $\int_{\mathbb{R}} e^{-ik(t-t_1)}\, dk = 2\pi\delta(t-t_1)$ with the *Poisson summation formula*, which we write here as $\sum_{n\in\mathbb{Z}} e^{inxc} = \frac{2\pi}{|c|} \sum_{n\in\mathbb{Z}} \delta\left(x - n\frac{2\pi}{c}\right)$, for all non zero $c \in \mathbb{R}$.

In the above model we see that the bath is essentially characterized by three functions: $f(k)$, $\omega(k)$ and $n_b(k)$. In particular, in what we have done so far, we have taken $f(k) = 1$,

$\omega(k) = \omega_b\, k$ and $n_b(k) = n_b$. This choice can be interpreted as follows: the bath is *almost homogeneous*, meaning with this that the functions $f(k)$ and $n_b(k)$, which in principle could depend on k (remember that each k labels an element of the bath), are constant in k: different *members* of the bath all share the same values of $f(k)$ and of $n_b(k)$. However, these members are not all completely identical, since $\omega(k)$ really depends on k, as we have already observed in section 2.

3.2 Changing reservoir

To keep the situation under control, as much as possible, we now discuss what happens if we fix again $f(k)$ and $\omega(k)$ as before, but we allow for a different dependence of $n_b(k)$ on k. In other words, we do not assume that $n_b(k)$ is constant in k, while we still take $f(k) = 1$ and $\omega(k) = \omega_b\, k$. With this choice, $a(t)$ is given again as in (3.5), and we conclude, in the same way, that $[a(t), a^\dagger(t)] = \mathbb{I}$. The function $n_a(t)$ is the one in (4.5) but, this time, we cannot use (4.4) since $n_b(k)$ is no longer a constant function. The computation of $\int_{\mathbb{R}} n_b(k)|\eta(k,t)|^2 dk$ is quite similar to that of $\int_{\mathbb{R}} |\eta(k,t)|^2 dk$, with the obvious difference due to the presence of $n_b(k)$ inside the integral. What is essential for us is that the integral is real-valued for all real choice of $n_b(k)$. To use again complex integration techniques, it is useful to assume that $n_b(k)$ is analytic in k, and that $|n_b(k)|$ does not diverge when $|k|$ diverges. For concreteness, we take

$$n_b(k) = \frac{n_b}{k^2 + \alpha^2}, \tag{3.8}$$

for some positive n_b and for some $\alpha > 0$. Notice that we could safely take $\alpha < 0$; our choice is needed only to fix the ideas. On the other hand, $n_b > 0$ is forced from the origin of $n_b(k)$, which came from an essentially positive operator. The computation of $\int_{\mathbb{R}} n_b(k)|\eta(k,t)|^2 dk$ is performed again with the same techniques discussed in details in appendix 2, with the only difference that we have two singularities of the integrating functions both in the upper and in the lower complex semi-planes, so the result of the integral is the sum of two residues.

The computation can be performed in details, and the analytic form of $n_a(t)$ could be given for all t. However, in view of our interest for stabilization, what is more relevant for us is the large time limit of $n_a(t)$, which turns out to be

$$n_a(\infty) := \lim_{t,\infty} n_a(t) = \frac{n_b(\alpha\omega_b^2 + \pi\lambda^2)}{\alpha\lambda^2\omega_b^2\left(\frac{\omega^2}{\omega_b^2} + \left(\alpha + \frac{\pi\lambda^2}{\omega_b^2}\right)^2\right)}. \tag{3.9}$$

Notice that this result makes sense only if α and λ are not zero. This suggests that the role of the complex pole in $n_b(k)$, and of the interaction between Alice and her bath, is really essential to get some stabilization. We observe also that the asymptotic value of the decision function $n_a(t)$ strongly depends on the various parameters of the model, which could be adjusted to fit experimental data, if needed.

Notice that the $n_b(k)$ in (3.8) describes a bath which is far from being uniform, or even almost uniform. Here, in particular, $n_b(k) \to 0$ for $|k| \to \infty$. Hence not all the parts of the bath interact with Alice in the same way: serious differences arise, in fact.

Remark 2. In this section we have considered a bath which goes to zero as k^{-2}, see (3.8). In fact, we could think of a function $n_b(k)$ decreasing like k^{-1}, but this is not compatible with the reality of the function $n_b(k)$, at least if we still want to use complex integration techniques to compute $n_a(t)$. The reason is that, to avoid singularities in the real axis, we are forced to choose $n_b(k) = \frac{\mu}{k-i\alpha}$, for some complex μ and for some $\alpha > 0$. This produces $n_a(\infty) = \gamma\omega_b$, at least if μ is chosen as follows: $\mu = \gamma\left(\omega - i\left(\alpha\omega_b + \frac{\pi\lambda^2}{\omega_b}\right)\right)$, for some real γ. So we see that, with this peculiar choice of $n_b(k)$, we still get a real $n_a(\infty)$, but what we cannot exclude, and it is possibly true, is that for finite time $n_a(t)$ could be complex.

Remark 3. Both (3.7) and (3.9) suggest that the asymptotic limit of $n_a(t)$ does not really depend on the fact that, at $t = 0$, the system is in a pure state or in a combination of states. This was already anticipated in section 2. Please notice that our approach does not give directly information on the output state, essentially because we are using the Heisenberg representation. This means that the state does not change, while the observables depend, in general, on time.

3.2.1 A remark on $f(k)$ and $\omega(k)$

So far we have fixed $f(k) = 1$ and $\omega(k) = \omega_b\, k$. This, in principle, is not the only natural choice, in particular if we want to stress the difference between different parts of the reservoir. What is technically useful, for us, is that a Dirac's delta function appears when deducing the differential equation for $a(t)$, see appendix 2. In fact, this is possible also under other conditions. For that, let us suppose that the two functions $f(k)$ and $\omega(k)$ are such that:

$$\omega(k) \to \pm\infty \text{ for } k \to \pm\infty, \qquad \text{and} \qquad \frac{d\omega(k)}{dk} = \frac{1}{\beta} f^2(k),$$

for some real $\beta \neq 0$. These assumptions are satisfied if $f(k) = 1$ and $\omega(k) = \omega_b k$, but also in many other situations. Hence

$$\int_{\mathbb{R}} f^2(k) e^{-i\omega(k)(t-t_1)}\, dk = \beta \int_{\mathbb{R}} e^{-i\omega(k)(t-t_1)}\, d\omega(k) = 2\pi\beta\delta(t - t_1),$$

and we can simplify significantly the equation for $a(t)$, which now becomes

$$\dot a(t) = -\left(i\omega + \beta\pi\lambda^2\right) a(t) - i\lambda \int_{\mathbb{R}} f(k) b(k)\, e^{-i\omega(k)t}\, dk. \tag{3.10}$$

This is the equation which replaces (3.4) in this new situation. The solution can be found as before, and we get

$$a(t) = \left(a - i\lambda \int_{\mathbb{R}} dk\, \tilde\eta(k,t) b(k) f(k)\right) e^{-\left(i\omega+\beta\pi\lambda^2\right)t}, \tag{3.11}$$

where $\tilde\eta(k,t) = \frac{1}{\tilde\rho(k)}\left(e^{\tilde\rho(k)t} - 1\right)$ and $\tilde\rho(k) = i(\omega - \omega(k)) + \beta\pi\lambda^2$. So we see that, at a first sight, the situation is not particularly different from those discussed so far, and in appendix 2. However, a serious technical problem now arises: to find $[a(t), a^\dagger(t)]$ and $n_a(t)$, we need to compute integrals and the poles of the integrating functions are strongly dependent on the analytic expression of $\omega(k)$: in particular, if $\omega(k)$ is not linear, the computations become rather hard!

3.3 What if we use CAR?

In the Hamiltonian (3.1) the operators a and $b(k)$ are assumed to satisfy CCR. However, from the point of view of a DM procedure, it might be more interesting to consider the case in which Alice is described by CAR-operators. This means that, first of all $\{a, a^\dagger\} = a\,a^\dagger + a^\dagger a = \mathbb{I}$, with $\{a, a\} = 0$. It is then natural to consider a CAR-bath as well, i.e. to assume that the operator $b(k)$ satisfy now the following rules:

$$\{b(k), b^\dagger(q)\} = \delta(k-q)\,\mathbb{I}, \qquad \{b(k), b(q)\} = 0, \qquad \{a^\sharp, b^\sharp(k)\} = 0,$$

$x^\sharp$ being either x or $x^\dagger$. Assuming the same Hamiltonian (3.1), the differential equations of motion for the annihilation operators $a(t)$ and $b(k,t)$ can be deduced and they turn out to be the same as in (3.2), except that λ must be replaced by $-\lambda$. In other words, we get

$$\begin{cases} \dot a(t) = i[H, a(t)] = -i\omega a(t) + i\lambda \int_{\mathbb{R}} f(k)\, b(k,t)\, dk, \\ \dot b(k,t) = i[H, b(k,t)] = -i\omega(k) b(k,t) + i\lambda f(k)\, a(t). \end{cases} \tag{3.12}$$

As in the CCR-case, we assume that in (3.6) the function $n_b(k)$ is constant, $n_b(k) \equiv n_b \geq 0$. The CAR-setting implies that this constant cannot exceed one, $0 \leq n_b \leq 1$ (this is the average of the CAR-number operator).

As we can see from (3.7), for the almost homogeneous reservoir, that $n_a(t)$ depends on λ^2 and not on λ. The same is true also for the choice (3.8) of the reservoir. This is evident from

(3.9) for large values of t, but can also be checked for any finite t. In other words, even with this different choice of $n_b(k)$, we observe that $n_a(t)$ depends on λ^2 and not on λ. The conclusion, therefore, is clear: the CCR-CAR nature of our model does not affect the main results we have deduced. One choice or the other should be related to the nature of the decision we want to model: in case of a binary question (yes or not), the natural settings is probably the CAR-one. But if we assume that several possible answers (normally infinite!) are possible, then we should use a CCR-version of the model. In this case, the existence of a quadratic integral of motion can limit the number of possible answers to the original questions, giving rise to a sort of finite-dimensional Hilbert space, similarly to what is discussed in details in [24].

4 Concluding discussion

For modeling of the process of decision making, the main output of this paper is the description of the properties of an information reservoir (environment) $\mathcal{R}$ which lead to the asymptotic stabilization. In other words, this is the description of a context surrounding an agent, Alice, which guaranties that she would make some decision, i.e., her mind would not fluctuate for ever between a variety of alternative answers to a question (solutions of a problem) A.

The asymptotic stabilization results obtained in this paper can be used to support mathematically the model of decision making as decoherence - in its quantum field version. Here the theory of open quantum systems is applied straightforwardly in the Hamiltonian framework. The decision making dynamics is given by the Heisenberg equations for operators describing reflections of an agent, Alice, and the operators representing the information reservoir. The most interesting for decision making are constraints imposed on information reservoirs, environments, guaranteeing asymptotic stabilization. Our present study is only the first step in this direction; further studies of sufficient and necessary conditions of asymptotic stabilization are needed.[12] We hope that this paper would attract attention of experts in mathematical physics (especially working on mathematical problems of quantum field theory) to the problem of asymptotic stabilization. Novel applications, to social science, microeconomics, finance, generate new laws for functions $f(k), n_b(k), \omega(k)$ which were not considered in physical applications.

[12]We also remark that applications of quantum field formalism to modeling of decision making generate new dynamical models based on algebras of *qubit creation-annihilation operators* [57]. These operators satisfy the so called qubit commutation relations which combine both CAR and CCR-features. To the best of our knowledge, there are no results about asymptotic stabilization for such dynamical processes and this is an interesting topic for research.

To extend the readership of this paper, the calculations in section 3 were done at the "physical level of rigor". The complete mathematical treatment of the problem should involve consideration of operator-valued distributions (generalized functions). Such treatment has definitely to be done at some stage of further development of the quantum field inspired model of decision making.

In general the great success of quantum field theory of physics rises the expectations that this formalism will contribute essentially to modeling of decision making and cognitive processes.

Appendix 1: Second quantization: CAR

We briefly review some basic facts on the so–called *canonical anti-commutation relations* (CAR), which were originally introduced in connection with what in quantum physics is called *second quantization* for identical particles with half-integer spin. We say that a set of operators $\{a_\ell,\, a_\ell^\dagger, \ell = 1, 2, \ldots, L\}$ satisfy the CAR if the conditions

$$\{a_\ell, a_n^\dagger\} = \delta_{\ell n}\mathbb{I}, \qquad \{a_\ell, a_n\} = \{a_\ell^\dagger, a_n^\dagger\} = 0 \tag{4.1}$$

hold true for all $\ell, n = 1, 2, \ldots, L$. Here, $\mathbb{I}$ is the identity operator and $\{x, y\} := xy + yx$ is the *anticommutator* of x and y. These operators are those which are used to describe L different *modes* of fermions. From these operators we can construct $\hat{n}_\ell = a_\ell^\dagger a_\ell$ and $\hat{N} = \sum_{\ell=1}^L \hat{n}_\ell$, which are both self–adjoint. In particular, $\hat{n}_\ell$ is the *number operator* for the ℓ–th mode, while $\hat{N}$ is the *number operator of* $\mathcal{S}$. compared with bosonic operators, the operators introduced here satisfy a very important feature: if we try to square them (or to rise to higher powers), we simply get zero: for instance, from (4.1), we have $a_\ell^2 = 0$. This is related to the fact that fermions satisfy the Fermi exclusion principle.

The Hilbert space of our system is constructed as follows: we introduce the *vacuum* of the theory, that is a vector $\varphi_{\mathbf{0}}$ which is annihilated by all the operators a_ℓ: $a_\ell \varphi_{\mathbf{0}} = 0$ for all $\ell = 1, 2, \ldots, L$. Then we act on $\varphi_{\mathbf{0}}$ with the operators $a_\ell^\dagger$ (but not with higher powers, since these powers are simply zero!):

$$\varphi_{n_1, n_2, \ldots, n_L} := (a_1^\dagger)^{n_1} (a_2^\dagger)^{n_2} \cdots (a_L^\dagger)^{n_L} \varphi_{\mathbf{0}}, \tag{4.2}$$

$n_\ell = 0, 1$ for all ℓ. These vectors give an orthonormal set and are eigenstates of both $\hat{n}_\ell$ and $\hat{N}$: $\hat{n}_\ell \varphi_{n_1, n_2, \ldots, n_L} = n_\ell \varphi_{n_1, n_2, \ldots, n_L}$ and $\hat{N} \varphi_{n_1, n_2, \ldots, n_L} = N \varphi_{n_1, n_2, \ldots, n_L}$, where $N = \sum_{\ell=1}^L n_\ell$. Moreover, using the CAR, we deduce that

$$\hat{n}_\ell \left(a_\ell \varphi_{n_1, n_2, \ldots, n_L}\right) = (n_\ell - 1)(a_\ell \varphi_{n_1, n_2, \ldots, n_L})$$

and

$$\hat{n}_\ell \left(a_\ell^\dagger \varphi_{n_1,n_2,\ldots,n_L}\right) = (n_\ell + 1)(a_l^\dagger \varphi_{n_1,n_2,\ldots,n_L}),$$

for all ℓ. Then a_ℓ and $a_\ell^\dagger$ are called the *annihilation* and the *creation* operators. In fact, they transform the vector $\varphi_{n_1,n_2,\ldots,n_\ell,\ldots,n_L}$ into a different vector, proportional to $\varphi_{n_1,n_2,\ldots,n_\ell\pm 1,\ldots,n_L}$, annihilating or creating one particle in the ℓ-th mode. However, in some sense, $a_\ell^\dagger$ is **also** an annihilation operator since, acting on a state with $n_\ell = 1$, we destroy that state.

The Hilbert space $\mathcal{H}$ is obtained by taking the linear span of all these vectors. Of course, $\mathcal{H}$ has a finite dimension. In particular, for just one mode of fermions, $dim(\mathcal{H}) = 2$. This also implies that, contrarily to what happens for bosons, the fermionic operators are bounded.

The vector $\varphi_{n_1,n_2,\ldots,n_L}$ in (4.2) defines a *vector (or number) state* over the algebra $\mathfrak{A}$ as

$$\omega_{n_1,n_2,\ldots,n_L}(X) = \langle \varphi_{n_1,n_2,\ldots,n_L}, X\varphi_{n_1,n_2,\ldots,n_L}\rangle, \tag{4.3}$$

where $\langle\, ,\, \rangle$ is the scalar product in $\mathcal{H}$.

Appendix 2: Explicit computations

In this appendix we give some details of our computations just sketched in Section 3.1, useful for those readers which are not familiar with this kind of interacting open systems.

Our starting point is $b(k,t)$ in (3.3), with $f(k) = 1$ and $\omega(k) = \omega_b\, k$. Hence

$$b(k,t) = b(k)e^{-i\omega_b kt} - i\lambda \int_0^t a(t_1)e^{-i\omega_b k(t-t_1)}\, dt_1.$$

When we replace this expression in the equation for $a(t)$, $\dot{a}(t) = -i\omega a(t) - i\lambda \int_{\mathbb{R}} b(k,t)\, dk$, we have to compute, in particular, the double integral

$$\int_{\mathbb{R}} \left(\int_0^t a(t_1)e^{-i\omega_b k(t-t_1)}\, dt_1\right) dk = \int_0^t a(t_1)\left(\int_{\mathbb{R}} e^{-i\omega_b k(t-t_1)}\, dk\right) dt_1 =$$

$$= \frac{2\pi}{\omega_b}\int_0^t a(t_1)\delta(t-t_1)dt_1 = \frac{\pi}{\omega_b}\, a(t),$$

and (3.4) now follows. The easy way to solve (3.4) consists in using the change of variable $a(t) = A(t)\exp\left\{-\left(i\omega + \frac{\pi\lambda^2}{\omega_b}\right)t\right\}$ since, in terms of $A(t)$, (3.4) can be rewritten as

$$\dot{A}(t) = -i\lambda \int_{\mathbb{R}} b(k)\, e^{-i\omega_b kt}\, dk, \qquad \Rightarrow \qquad A(t) = A(0) - i\lambda \int_0^t \left(\int_{\mathbb{R}} b(k)\, e^{-i\omega_b kt_1}\, dk\right) dt_1,$$

which returns, after few simple computation, the solution in (3.5).

To check now that the CCR are preserved during the time evolution, i.e. that $[a(t), a^\dagger(t)] = \mathbb{I}$ for all t, we observe that the operators in the right hand side of (3.5) are the initial ($t = 0$) operators, so that they satisfy the commutation rules $[a, a^\dagger] = \mathbb{I}$, $[b(k), b^\dagger(q)] = \delta(k - q)$, while $[a, b(k)] = [a^\dagger, b(k)] = ... = 0$. Therefore

$$[a(t), a^\dagger(t)] = e^{-\frac{2\pi\lambda^2}{\omega_b}t}\left(1 + \lambda^2 \int_{\mathbb{R}} |\eta(k,t)|^2 dk\right)\mathbb{I}.$$

In our case, after some simple algebra, we deduce that

$$\int_{\mathbb{R}} |\eta(k,t)|^2 dk = \int_{\mathbb{R}} \frac{dk}{\omega_b^2(k-k_+)(k-k_-)}\left[\left(e^{\frac{2\pi\lambda^2}{\omega_b}t} + 1\right) - e^{(-i\omega+\frac{\pi\lambda^2}{\omega_b})t}e^{i\omega_b kt} - e^{(i\omega+\frac{\pi\lambda^2}{\omega_b})t}e^{-i\omega_b kt}\right],$$

where $k_\pm = \frac{\omega}{\omega_b} \pm i\frac{\pi\lambda^2}{\omega_b^2}$. Using complex integration we can compute each contribution here and we conclude that

$$\int_{\mathbb{R}} |\eta(k,t)|^2 dk = \frac{1}{\lambda^2}\left(e^{\frac{2\pi\lambda^2}{\omega_b}t} - 1\right), \tag{4.4}$$

so that our claim easily follows.

To prove now (3.7), we first observe that the mean values of contributions like $a^\dagger b(k)$ and $ab^\dagger(k)$ on our states are zero. Therefore, using (3.5), we get

$$n_a(t) = e^{-\frac{2\pi\lambda^2}{\omega_b}t}\left(\left\langle\varphi_{n_a}, a^\dagger a\varphi_{n_a}\right\rangle \langle \mathbb{I}_{\mathcal{R}}\rangle_{\mathcal{R}} + \lambda^2 \langle\varphi_{n_a}, \mathbb{I}\varphi_{n_a}\rangle \int_{\mathbb{R}}\int_{\mathbb{R}} \overline{\eta(k,t)}\eta(q,t) \left\langle b^\dagger(k)b(q)\right\rangle_{\mathcal{R}} dk\, dq\right) =$$

$$= e^{-\frac{2\pi\lambda^2}{\omega_b}t}\left(n_a + \lambda^2 \int_{\mathbb{R}} n_b(k)|\eta(k,t)|^2 dk\right). \tag{4.5}$$

Then, if $n_b(k) = n_b$, we can use the result in (4.4) and recover the result in (3.7).

Acknowledgments. This paper is based on the discussions between the coauthors during the visit of two of them (IB and KHR) to the university of Palermo in April 2017. IB was supported by Marie Curie Fellowship at City University of London, H2020-MSA-IF-2015, grant N 696331; AKH was supported by the EU-project Quantum Information Access and Retrieval Theory (QUARTZ), Grant No. 721321. FB thanks the Gruppo Nazionale di Fisica Matematica of I.N.d.A.M., and by the University of Palermo, for support. F.B. also thanks the *Distinguished Visitor Program* of the Faculty of Science of the University of Cape Town, 2017.

References

[1] A. Khrennikov, *Information dynamics in cognitive, psychological, social, and anomalous phenomena.* Ser.: Fundamental Theories of Physics, Kluwer, Dordreht, 2004.

[2] A. Khrennikov, Ubiquitous quantum structure: from psychology to finance. Springer, Berlin-Heidelberg-New York, 2010.

[3] Busemeyer J R, Bruza P D. 2012 *Quantum models of cognition and decision.* Cambridge: Cambridge University Press.

[4] Bagarello F. 2012 *Quantum dynamics for classical systems: with applications of the number operator*. New York: J. Wiley.

[5] Haven E, Khrennikov A. 2013 *Quantum social science.* Cambridge: Cambridge University Press.

[6] Asano M, Khrennikov A, Ohya M, Tanaka Y, Yamato I. 2015 *Quantum adaptivity in biology: from genetics to cognition*. Heidelberg-Berlin-New York: Springer.

[7] Khrennikov, A., 2004. On quantum-like probabilistic structure of mental information. Open Systems and Information Dynamics 11 (3), 267-275.

[8] Khrennikov, A., 2006. Quantum-like brain: Interference of minds. BioSystems 84, 225-241.

[9] Busemeyer J R, Wang Z, Townsend J T. 2006 Quantum dynamics of human decision making. *J. Math. Psychol.* **50**,220-241

[10] Pothos E M, Busemeyer J R. 2009 A quantum probability explanation for violation of rational decision theory. *P. Roy. Soc. Lond. B Bio.* **276**, 2171-2178

[11] Busemeyer J R , Pothos E M, Franco R, Trueblood J. 2011 A quantum theoretical explanation for probability judgment errors. *Psychol. Rev.* **118**, 193-218

[12] Dzhafarov, E.N. & Kujala, J.V. 2012 Selectivity in probabilistic causality: Where psychology runs into quantum physics. *Journal of Mathematical Psychology* **56**, 54-63.

[13] Asano M, Ohya M, Tanaka Y, Basieva I, Khrennikov A. 2012 Quantum-like dynamics of decision-making. *Physica A* **391**, 2083-2099

[14] Hawkins R J and Frieden B R. 2012 Asymmetric Information and Quantization in Financial Economics. *Int. J. Math. and Math. Sc.* **32012**, Article ID 470293.

[15] Pothos E M , Busemeyer J R.2013 Can quantum probability provide a new direction for cognitive modeling? *Behav. Brain Sci.* **36**, 255–327

[16] Asano M, Basieva I, Khrennikov A, Ohya M, Tanaka Y, Yamato I 2013. A model of epigenetic evolution based on theory of open quantum systems. *Systems and Synthetic Biology* **7**, 161-173

[17] Dzhafarov E N and Kujala J V 2014 On selective influences, marginal selectivity, and Bell/CHSH inequalities. *Topics in Cognitive Science* **6**, 121-128

[18] Kvam P D, Busemeyer J R, Lambert-Mogiliansky A. 2014 An empirical test of type-indeterminacy in the Prisoners Dilemma. *Lecture Notes in Computer Science* **8369**,213-224

[19] Khrennikov A, Basieva I. 2014 Possibility to agree on disagree from quantum information and decision making. *J. Math. Psychol.* **62–63**, 1–15

[20] Plotnitsky A. 2014 Are quantum-mechanical-like models possible, or necessary, outside quantum physics? *Phys. Scripta* **T163**, 014011

[21] A. Khrennikov, Basieva, I. (2014). Quantum Model for Psychological Measurements : From the Projection Postulate to Interference of Mental Observables Represented As Positive Operator Valued Measures. NeuroQuantology. 12. 324-336

[22] A. Khrennikov, Basieva, I., Dzhafarov, E.N., Busemeyer, J.R. 2014. Quantum Models for Psychological Measurements : An Unsolved Problem. PLoS ONE. 9. Article ID: e110909.

[23] J. R. Busemeyer, Z. Wang, A. Khrennikov, I. Basieva, I. 2014. Applying quantum principles to psychology. Physica Scripta. T163. 014007.

[24] Bagarello F., Oliveri F., *An operator-like description of love affairs*, SIAM Jour. Appl. Math., **70**, No. 8, pp. 32353251, (2010)

[25] Bagarello F., Gargano F., *Modeling interactions between political parties and electors*, Phys. A, dx.doi.org/10.1016/j.physa.2017.04.035

[26] Bagarello F. 2015 An operator view on alliances in politics. *SIAM J. Appl.. Math.* **75(2)**, 564-584

[27] Bagarello F, Haven E. 2016 First results on applying a non-linear effect formalism to alliances between political parties and buy and sell dynamics. *Physica A* **444**, 403-414

[28] Bagarello F. 2015 A quantum-like view to a generalized two players game. *Int. J. Theor. Phys.* **54** (10), 3612-3627

[29] Denolf J and Lambert-Mogiliansky A. 2016. Bohr complementarity in memory retrieval. *J. Math. Psych.* **73**, 28-36

[30] E. Haven and A. Khrennikov, Statistical and subjective interpretations of probability in quantum-like models of cognition and decision making. *J. Math. Psych.* 74, 8291 (2016).

[31] R. J. Hawkins and B. R. Frieden, 2017 Quantization in financial economics: An information-theoretic approach. The Palgrave handbook of quantum models in social science. Applications and Grand Challenges. Haven E and Khrennikov A (eds). Palgrave Macmillan UK, pp. 19-38.

[32] Lambert-Mogiliansky A. 2017. Quantum-like type indeterminacy: A constructive approach to preferences a la Kahneman and Tversky. Haven E and Khrennikov A (eds). Palgrave Maxmilan UK, pp. 229-250.

[33] Danilov V I, Lambert-Mogiliansky A and Vassili Vergopoulos V. 2016 Dynamic consistency of expected utility under non-classical(quantum) uncertainty. https://halshs.archives-ouvertes.fr/halshs-01324046/

[34] A. Khrennikov, Classical and quantum mechanics on information spaces with applications to cognitive, psychological, social and anomalous phenomena. *Found. of Physics*, **29**, N. 7, 1065-1098 (1999).

[35] M. Asano, I. Basieva, A. Khrennikov, M. Ohya, Y. Tanaka, I. Yamato, Quantum information biology: from information interpretation of quantum mechanics to applications in molecular biology and cognitive psychology. *Found. Phys.* **45**, N 10, 1362-1378 (2015).

[36] D' Ariano, G. M.: Physics as information processing. In: Jaeger, G. et al. (eds.), Advances in Quantum Theory, AIP, Melville, NY, **1327**, pp. 7–16 (2011)

[37] Plotnitsky, A.: Niels Bohr and Complementarity: An Introduction. Springer, Berlin and New York (2012)

[38] Penrose, R.: The Emperor's new mind. Oxford Univ. Press, New-York (1989)

[39] Hameroff, S.: Quantum coherence in microtubules. A neural basis for emergent consciousness? J. of Cons. Stud. 1, 91-118 (1994)

[40] Tegmark, M. (2000). Importance of quantum decoherence in brain processes. Phys. Rev. E 61 (4): 41944206

[41] de Barros A J , Suppes P. 2009 Quantum mechanics, interference, and the brain. *J. Math. Psychol.* **53**, 306-313.

[42] A. Khrennikov, Quantum-like model of processing of information in the brain based on classical electromagnetic field. *Biosystems* 105(3), 250-262 (2011).

[43] de Barros, A. J. Quantum-like model of behavioral response computation using neural oscillators. Biosystems 110, 171-182 (2012)

[44] Takahashi T. Toward a physical theory of quantum cognition. Topics in Cognitive Science 6, 104107 (2014).

[45] Melkikh, A.V., 2013. Biological complexity, quantum coherent states and the problem of efficient transmission of information inside a cell. BioSystems. 111, 190-198.

[46] Melkikh, A.V., 2014. Congenital programs of the behavior and nontrivial quantum effects in the neurons work. BioSystems. 119, 10-19.

[47] Busemeyer, J. R., Fakhari, P., Kvam, P. 2017. Neural implementation of operations used in quantum cognition. Progress in Biophysics and Molecular Biology, in press.

[48] von Neumann J., Mathematical Foundations of Quantum Mechanics, Princeton Univ. Press, Princenton, 1955.

[49] Basieva I and Khrennikov A. 2017 Decision-making and cognition modeling from the theory of mental instruments. The Palgrave Handbook of Quantum Models in Social Science. Applications and Grand Challenges. Haven E and Khrennikov A (eds). Palgrave Maxmilan UK, pp. 75-93.

[50] Khrennikov A. 2017 Why quantum? The Palgrave Handbook of Quantum Models in Social Science. Applications and Grand Challenges. Haven E and Khrennikov A (eds). Palgrave Maxmilan UK, pp. 321-334.

[51] Asano M, Ohya M, Tanaka Y, Basieva I, Khrennikov A. 2011 Quantum-like model of brain's functioning: decision making from decoherence. *J. Theor. Biol.* **281**, 56-64

[52] M. Asano, M. Ohya, A. Khrennikov, Quantum-Like Model for Decision Making Process in Two Players Game - A Non-Kolmogorovian Model. *Found. Phys.*, **41**, 538-548 (2011).

[53] Skomski R, Kashyap A and Sellmyer D J 2012 A quantum-mechanical relaxation model. *J. Appl. Phys.*111, 07D507

[54] Bagarello F, Haven E., Khrennikov A., A model of adaptive decision making from representation of information environment by quantum fields. Phil. Trans. A, to be published in 2017.

[55] S.M. Barnett, P.M. Radmore, *Methods in theoretical quantum optics*, Clarendon Press, Oxford, 1997

[56] A. Khrennikov, "Social laser:" action amplification by stimulated emission of social energy. *Phil. Trans. Royal Soc.* **374,** N 2054, 20150094 (2016).

[57] P. Khrennikova, Modeling behavior of decision makers with the aid of algebra of qubit creation-annihilation operators J. Math. Psych. published on line (2016)