SOLUTIONS OF CERTAIN BOUNDARY INTEGRAL EQUATIONS IN POTENTIAL THEORY

BY

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Certain Fredholm integral equations are studied which arise from boundary value problems of potential theory. It is shown how these may be solved numerically to a good approximation. The results are applied to the calculation of electrostatic capacities and to the computation of velocity potentials.
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INTRODUCTION

This thesis deals with certain three-dimensional boundary value problems of potential theory. Such problems may be formulated in many different ways, both numerically and analytically. Here we use exclusively the method of boundary integral equations. The idea goes back nearly 100 years but was not systematically exploited until after 1955. One reason is that, generally speaking, the equations can only be solved numerically utilising fast digital computers which were not available before about 1955. In some cases the existence of the solution did not seem to be clearly established. Difficulties also arise from the presence of weakly singular kernels.

Part I of the thesis deals with the formulation of certain boundary integral equations arising in electrostatics and in potential fluid motion. As regards the determination of electrostatic capacity of conductors, we introduce simple sources on the boundary which generate a unit potential everywhere in the interior and on the boundary. This leads to a Fredholm integral equation of the first kind for the source density distribution on the boundary. We establish the existence and uniqueness of the solution of this equation, which does not seem to be readily available elsewhere. The electrostatic problem can alternatively be formulated by a normal derivative condition, leading to a Fredholm integral equation of the second kind for the source density distribution on the boundary. We have made a comparison between the two approaches, which seem to be interesting both on analytical and numerical grounds.

In Part I we also treat the velocity potential of potential fluid flows past various rigid obstacles, of shapes which can not be handled analytically. Two distinct formulations are studied. The first, due to A.M.O. Smith in U.S.A., represents the velocity potential as generated by a simple source distribution on the boundary. This source distribution satisfies a Fredholm integral equation of the second kind expressing a normal derivative condition on the boundary. The second formulation, due to M. A. Jaswon, utilises Green's formula on the boundary to determine the velocity potential directly. Here again we have made a comparison between the two approaches, which seems to be interesting both on numerical and analytical grounds.
Part II shows how to discretise the preceding equations. Our main problem here concerns the subdivision of a given surface into small intervals, i.e. sub-areas. Special complications arise when the boundary has sharp edges and corners. We also show how the presence of homogeneous equations affects the discretisation procedures.

In Part III we compute the electrostatic capacity of a cube. Our results lie within all known bounds. We also compute the capacity of circular discs of varying thicknesses. Our results converge to the exact known result for a thin circular disc.

In Part IV we compute the velocity potential for shapes of cylindrical symmetry, as well as for a thick delta wing. In all these cases we first work with a suitable test velocity potential. This is a necessary precaution against any errors which may arise in the discretisation procedures and in our computer programs. In this section we also deal with the thin delta wing problem discussed by Brown and Stewartson. The thin delta may be considered as a limiting case of a thick delta, but such an approach is not practicable numerically for reasons given in Chapter 18. Accordingly we attack the problem by analysing the velocity potential of a thick delta wing into symmetric and antisymmetric components. The symmetric part arise from the thickness effect, and the antisymmetric part accordingly solves the problem of a thin delta wing.

In the final Chapter we experiment with a method of successive approximations. In effect this amounts to obtaining an approximate analytical solution of a Fredholm integral equation of the second kind using a perturbation technique. Although it works very well for smooth boundaries, e.g. a sphere, it appears not to work well with boundaries having sharp edges and corners.
PART I

THE FORMULATION OF BOUNDARY INTEGRAL EQUATIONS IN POTENTIAL THEORY
CHAPTER I

PROPERTIES OF POTENTIALS GENERATED BY SIMPLE AND DOUBLE LAYERS

Simple Layer Potential

Let \( B_1 \) denote a finite domain bounded by a smooth regular surface \( \delta B \). The infinite region exterior to \( B_1 \) is denoted by \( B_e \). Let there be a surface distribution of simple sources on \( \delta B \) of density \( \sigma \), which is a continuous function on \( \delta B \) and satisfies a Hölder condition at every point on \( \delta B \). This distribution generates a Newtonian potential at any point \( p \) which is given by

\[
v(p) = \int_{\delta B} \frac{\sigma(q) dq}{|p - q|} ; \quad q \in \delta B , \quad \text{either } p \in B_1 \text{ or } p \in B_e ,
\]

where \( p \) and \( q \) are vector variables such that \( p \) specifies a field point in \( B_1 \) or in \( B_e \) and \( q \) specifies a source point on \( \delta B \); \( dq \) and \( \sigma(q) \) denote the area differential and source density, respectively, at the point \( q \) on \( \delta B \); \( |p - q| \) denotes the distance from \( p \) to \( q \) (Fig. 1). The simple layer integral (1) remains continuous as \( p \) crosses \( \delta B \), and therefore on \( \delta B \)

\[
v(p) = \int_{\delta B} \frac{\sigma(q) dq}{|p - q|} ; \quad p, q \in \delta B .
\]

Although \( V \) remains continuous at \( \delta B \), its normal derivative is discontinuous. The interior normal derivative of \( V \) at a point \( p \) of \( \delta B \) (Fig. 2) is given by (Kellogg)

\[
V^1_\text{i}(p) = -2\pi \sigma(p) + \int_{\delta B} \frac{\sigma(q) dq}{i|p - q|} ; \quad p, q \in \delta B
\]

where \( \frac{1}{|p - q|} \) stands for the interior normal derivative of \( |p - q|^{-1} \) at \( p \) keeping \( q \) fixed. The exterior normal derivative of \( V \) at \( p \) on \( \delta B \) (Fig. 2) is given by

\[
V^1_\text{e}(p) = -2\pi \sigma(p) + \int_{\delta B} \frac{\sigma(q) dq}{e|p - q|} ; \quad p, q \in \delta B
\]
where \( \frac{1}{|p - q|} \) stands for the exterior normal derivative of \(|p - q|^{-1}\) at \( p \) keeping \( q \) fixed. Bearing in mind (Fig. 2) that

\[
\frac{1}{|p - q|} - \frac{1}{|p - q|} = 0 \quad ; \quad p \in \delta B,
\]

we find

\[
V_i^1(p) + V_e^1(p) = -4\pi \sigma(p) \quad .
\]

Further,

\[
V(p) \rightarrow |p|^{-1} \int_{\delta B} \sigma(q) dq \quad \text{as} \quad |p| \rightarrow \infty \quad .
\]

More precisely

\[
V(p) = |p|^{-1} \int_{\delta B} \sigma(q) dq + O|p|^{-2}
\]

\[
= 0|p|^{-1} \quad \text{as} \quad |p| \rightarrow \infty \quad .
\]

**Double Layer Potential**

Let there be a surface distribution of double layer sources on \( \delta B \) of density \( \mu \) which is a piecewise continuous function at every point of \( \delta B \). The potential \( W \) generated by this distribution is given by

\[
W_i(p) = \int_{\delta B} \frac{\mu(q) dq}{|p - q|} \quad ; \quad q \in \delta B, \ p \in \delta B_i \quad ,
\]

where \( \frac{1}{|q - p|} \) stands for the interior normal derivative of \(|p - q|^{-1}\) at \( q \) keeping \( p \) fixed. Unlike the simple layer integral, the double layer integral is discontinuous at \( \delta B \) whereas its normal derivative is continuous. If we approach a surface point \( p \) from the interior, \( W_i \) jumps by an amount \(-2\pi \mu(p)\). Let \( W \) represent a continuous function \( \phi \), in which case

\[
\phi(p) = W(p) \quad ; \quad \text{either} \ p \in B_i \text{ or } p \in B_e \quad .
\]

when \( p \) approaches \( \delta B \) along the normal at \( p \) to \( \delta B \), either from the interior or from the exterior, if follows that

\[
\phi(p) = W(p) + 2\pi \mu(p) \quad ; \quad p \in \delta B \quad .
\]

The above sign conventions are those adopted by Jaswon (1963). This convention has the advantage of ensuring that the interior and the exterior formulae carry the same signs i.e. (3) and (4) for the normal derivative expression and (8) and (9) for the double layer surface relations.
Fig. 1

Fig. 2
Green's Formula

Since every Newtonian potential is a harmonic function, it follows that the potentials $V$ and $W$ are harmonic. Now the question arises whether an arbitrary harmonic function $\phi$ in $B$ can be represented by potentials such as (1) or (7). According to Kellogg, if $\phi$ is given on $\partial B$ (Dirichlet problem), it may be represented by (7). If $\phi^1$, the interior normal derivative of $\phi$, is given on $\partial B$ (Neumann problem), it may be represented by (1). However a more general representation is provided by Green's formula. Given a harmonic function $\phi$ defined throughout $B$, which assumes values $\phi$ on $\partial B$ and normal derivatives $\phi^1$ on $\partial B$, Green's formula states that

$$
\int_{\partial B} G(p,q) \frac{1}{2} \phi^1(q) dq - \int_{\partial B} G(p,q) \phi^1(q) dq = \phi(p); \quad q \in \partial B, \quad p \in B \quad (10)
$$

where $G^{-1} = \frac{4 \pi}{|p - q|}$ and $G(p,q)\frac{1}{2}$ represents the interior normal derivative of $\phi$ at $q$ keeping $p$ fixed. When $p$ lies on $\partial B$, (10) becomes

$$
\int_{\partial B} G(p,q) \frac{1}{2} \phi^1(q) dq - \int_{\partial B} G(p,q) \phi^1(q) dq = \frac{1}{2} \phi(p); \quad q, p \in \partial B \quad (11)
$$

by virtue of the jump $-\frac{1}{2} \phi$ in the double layer integral

$$
\int_{\partial B} G(p,q) \frac{1}{2} \phi(q) dq.
$$

This is Green's boundary formula for the interior harmonic $\phi$. When $p$ lies in $B$, (10) becomes

$$
\int_{\partial B} G(p,q) \frac{1}{2} \phi^1(q) dq - \int_{\partial B} G(p,q) \phi^1(q) dq = 0; \quad p \in B \cap \partial B, \quad q \in \partial B \quad (12)
$$

by virtue of the further jump in the integral

$$
\int_{\partial B} G(p,q) \frac{1}{2} \phi(q) dq.
$$

Our sign conventions ensure that all exterior equations carry the same signs as their interior counterparts. Hence for the exterior harmonic $\phi$ defined in $B$, such that $\phi \rightarrow \mathcal{O}|p|^{-1}$ as $|p| \rightarrow \infty$, which assumes values $\phi$ on $\partial B$ and normal derivatives $\phi^1$ on $\partial B$, Green's formula take the form
\[ \int_{\delta B} G(p,q)^{1} \phi(q) dq - \int_{\delta B} G(p,q) \phi_e^1(q) dq = \phi(p); \ p, q \in \delta B, p \in \delta B_e \] (13)

where \( G(p,q)^{1} \) represents the exterior normal derivative of \( G \) at \( q \) keeping \( p \) fixed. When \( p \) is a point on \( \delta B \), as before,

\[ \int_{\delta B} G(p,q)^{1} \phi(q) dq - \int_{\delta B} G(p,q) \phi_e^1(q) dq = \frac{1}{2} \phi(p); \ p, q \in \delta B \] (14)

and for a point \( p \) in \( B_1 \), by virtue of a further jump in the double layer integral,

\[ \int_{\delta B} G(p,q)^{1} \phi(q) dq - \int_{\delta B} G(p,q) \phi_e^1(q) dq = 0; \ p \in B_1, q \in \delta B. \] (15)

It is interesting to examine the behavior of \( \phi \) defined by (13), at infinity. Given \( \phi \) and \( \phi_e^1 \) on \( \delta B \), from (13) we have

\[ \phi(p) = 0 |p|^{-2} \int_{\delta B} \phi(q) dq - 0 |p|^{-1} \int_{\delta B} \phi_e^1(q) dq, \] (16)

since \( G = 0 |p|^{-1} \) and \( G^1 = 0 |p|^{-2} \) when \( |p| \rightarrow \infty \).

In contrast with the interior problem, where

\[ \int_{\delta B} \phi^1(q) dq = 0 \]

for the interior harmonic \( \phi \) (Gauss condition)

\[ \int_{\delta B} \phi_e^1(q) dq \neq 0 \] (17)

necessarily. This does not contradict the Gauss condition if we bear in mind the compensating contribution from a large sphere at infinity.
Dirichlet Problem

If the simple source potential \( V \) represents a harmonic function characterised by the boundary values \( \phi \), it must, from (2), satisfy the boundary equation

\[
\phi(p) = \int_{\delta B} G(p,q)\sigma(q)\,dq ; \quad p, q \in \delta B
\]  

which is a Fredholm integral equation of the 1st kind for \( \sigma \) in terms of \( \phi \) on \( \delta B \). If a solution of (18) exists, it generates an interior harmonic function

\[
\phi(p) = \int_{\delta B} G(p,q)\sigma(q)\,dq ; \quad q \in \delta B, \quad p \in \delta B^1
\]

and an exterior harmonic function

\[
\phi(p) = \int_{\delta B} G(p,q)\sigma(q)\,dq ; \quad q \in \delta B, \quad p \in \delta B_e
\]

such that \( \phi(p) = 0 \) as \( |p| \to \infty \). Similarly if the double source potential \( W \) represents a harmonic function \( \phi \) in \( B_1 \), it must, from (9) satisfy the boundary equation

\[
\phi(p) = \frac{1}{2} \mu(p) + \int_{\delta B} G(p,q)\mu(q)\,dq ; \quad p, q \in \delta B
\]  

which is a Fredholm integral equation of the 2nd kind for \( \mu \) in terms of \( \phi \) on \( \delta B \).

Interior Neumann Problem

In the case of the interior Neumann problem, where \( \phi^i_1 \) is given on \( \delta B \), it follows from (3) that

\[
\phi^i_1(p) = -\frac{1}{2} \sigma(p) + \int_{\delta B} G^i_1(p,q)\sigma(q)\,dq ; \quad p, q \in \delta B .
\]  

This is a Fredholm integral equation of the 2nd kind for the unknown boundary function \( \sigma \).
Equation (19) is fully discussed by Kellogg but is not utilised in this thesis. Leaving aside (18) to be discussed later, the necessary and sufficient condition for the existence of a solution of (20), by Kellogg, is

\[ \int_{\delta B} \mu(p) \phi_1(p) dp = 0 \]  

(21)

where \( \mu \) is a solution of the transpose (or adjoint) homogeneous equation

\[ -\frac{1}{2} \mu(p) + \int_{\delta B} G(p,q) \phi_1(q) dq = 0 ; \quad p, q \in \delta B . \]  

(22)

This admits the non-trivial solution \( \mu = 1 \) by virtue of the Gauss flux theorem for the field point on \( \delta B \) viz.

\[ \int_{\delta B} G(p,q) \phi_1(q) dq = - \int_{\delta B} G(p,q) \phi_1(q) dq = \frac{1}{2} ; \quad p, q \in \delta B . \]  

(23)

Setting \( \mu = 1 \) in (21) we have the expected Gauss condition for the interior harmonic \( \phi_1 \) viz.

\[ \int_{\delta B} \phi_1(p) dp = 0 . \]  

(24)

The general solution of (20) is then given by

\[ \sigma = \sigma_0 + k \lambda \]

where \( \sigma_0 \) is a particular solution of (20); \( k \) is an arbitrary constant and \( \lambda \) is a solution of the corresponding homogeneous equation

\[ -\frac{1}{2} \lambda(p) + \int_{\delta B} G_1(p,q) \lambda(q) dq = 0 ; \quad p, q \in \delta B . \]  

(25)

An alternative proof of (24) is as follows. Integrating both sides of (20) with respect to \( p \) and bearing in mind the Theorem (23):

\[ \int_{\delta B} \phi_1(p) dp = - \int_{\delta B} \frac{1}{2} \sigma(p) dp + \int_{\delta B} \int_{\delta B} G_1(p,q) \sigma(q) dq dp \]

\[ = - \int_{\delta B} \frac{1}{2} \sigma(p) dp + \int_{\delta B} \frac{1}{2} \sigma(q) dq = 0 \]  

(26)
Exterior Neumann Problem

In the case of the exterior Neumann problem, where \( \phi_1 \) is given on \( \delta B \), it follows from (4) that

\[
\phi_1(p) = -\frac{i}{2} \sigma(p) + \int_{\delta B} \nabla'(p, q) \sigma(q) dq ; \quad p, q \in \delta B .
\]  

(27)

This has a solution, by Kellogg, if

\[
\int_{\delta B} \phi_1(p) \mu(p) dp = 0
\]  

(28)

where \( \mu \) is a solution of the corresponding transpose homogeneous equation

\[
-\frac{i}{2} \mu(p) + \int_{\delta B} \nabla(p, q) \mu(q) dq = 0 ; \quad p, q \in \delta B .
\]  

(29)

The equation (29) in three dimensions has no non-trivial solution, since its transpose

\[
-\frac{i}{2} \lambda(p) + \int_{\delta B} \nabla'(p, q) \lambda(q) dq = 0 ; \quad p, q \in \delta B .
\]  

(30)

as shown in the next chapter, has no non-trivial solution. As a result, by Kellogg, (27) has always a unique solution.

Integrating both the sides of (27) with respect to \( p \), we find

\[
\int_{\delta B} \phi_1(p) dp = -\frac{i}{2} \int_{\delta B} \sigma(p) dp + \int_{\delta B} \int_{\delta B} \nabla'(p, q) \sigma(q) dq dp
\]

\[
= -\frac{i}{2} \int_{\delta B} \sigma(p) dp -\frac{i}{2} \int_{\delta B} \sigma(q) dq = - \int_{\delta B} \sigma(q) dq
\]  

(31)

which, in contrast with (26), does not equal zero necessarily. This is completely in accordance with (17) in Chapter 1.

Green's Boundary Formula

In Green's formula (10), the interior harmonic function \( \phi \) is expressed in terms of the values of \( \phi \) and \( \phi_1 \) on \( \delta B \). These over-prescribe the boundary data and, therefore, the formula cannot be used directly to solve the boundary value problems. This is because \( \phi \) alone on \( \delta B \), or \( \phi_1 \) alone on \( \delta B \), or any admissible local relation between \( \phi \) and \( \phi_1 \) on \( \delta B \), suffices to determine \( \phi \) throughout \( B \). One way out of this difficulty is to take the field point \( p \) on \( \delta B \) itself, which is Green's boundary formula (11). This may be viewed as a constraint between \( \phi \) and
\( \phi \) on \( \delta B \) that defines one in terms of the other. Given \( \phi \) on \( \delta B \) (Dirichlet problem), (11) becomes a Fredholm integral equation on the 1st kind for \( \phi \) viz.

\[
\int_{\delta B} G(p, q) \phi_i(q) dq = -\frac{1}{2} \phi(p) + \int_{\delta B} G(p, q) \phi_i(q) dq ; p, q \in \delta B . \tag{32}
\]

Conversely, given \( \phi \) on \( \delta B \) (Neumann problem), (11) becomes a Fredholm integral equation of the 2nd kind for \( \phi \) viz.

\[
-\frac{1}{2} \phi(p) + \int_{\delta B} G(p, q) \phi_i(q) dq = \int_{\delta B} G(p, q) \phi_i(q) dq ; p, q \in \delta B . \tag{33}
\]

Leaving equation (32) to be discussed in the next chapter, we come to equation (33) which, by Kellogg, has a solution if

\[
\int_{\delta B} \lambda(p) dp \int_{\delta B} G(p, q) \phi_i(q) dq = 0 ; p, q \in \delta B . \tag{34}
\]

Here \( \lambda \) is a solution of (25), which is the transpose of homogeneous part of (33) viz.

\[
-\frac{1}{2} \phi(q) + \int G(p, q) \phi_i(q) dq = 0 ; p, q \in \delta B . \tag{35}
\]

This equation (35), by virtue of (23), exhibits a non-trivial solution \( \phi = 1 \). Hence (25), which is the transpose of (35), has a non-trivial solution \( \lambda \) . Interchanging the order of integration in the left hand side of (34), we have

\[
\int_{\delta B} \lambda(p) dp \int_{\delta B} G(p, q) \phi_i(q) dq = \int_{\delta B} \phi_i(q) dq \int_{\delta B} G(q, p) \lambda(p) dp = 0 ,
\]

since \( G(p, q) = G(q, p) \) and \( \int_{\delta B} G(q, p) \lambda(p) dp = \phi = 1 \).

The above condition is in agreement with the condition derived in (24).
The general solution of (33) is given by

$$\phi = \phi_0 + k \eta$$

where $\phi_0$ is a particular solution of (33); $k$ is an arbitrary constant and $\eta = 1$ is a solution of the corresponding homogeneous equation (35).

Given $\phi_0$ on $\delta B$ (the exterior Neumann problem), (14) becomes a Fredholm integral equation of the 2nd kind for $\phi$ viz.

$$-\frac{i}{2} \phi(p) + \int_{\delta B} G(p,q) \phi(q) dq = \int_{\delta B} G(p,q) \phi_0(q) dq ; p, q \in \delta B . \tag{36}$$

This has a solution, by Kellogg, if

$$\int_{\delta B} \lambda(p) dp \int_{\delta B} G(p,q) \phi_0(q) dq = 0 ; p, q \in \delta B$$

where $\lambda$ is a solution of (30) which is a transpose of the homogeneous part of (36). It is discussed earlier that in three dimensions, equation (30) has no non-trivial solution. Hence in three dimensions, by Kellogg, (36) has a unique solution.

Confining our discussion to three dimensions, we find the exterior Neumann problem, in contrast with the interior Neumann problem, has always a solution and that it is unique.
The Electrostatic Equation

If we put \( \phi = 1 \) in (18), we obtain the electrostatic equation

\[
\int_{\delta B} G(p,q)\lambda(q)\,dq = 1 ; \, p, q \in \delta B .
\] (37)

Since \( \phi = 1 \) on \( \delta B \), it follows that \( \phi = 1 \) everywhere in \( B_1 \). Hence taking the interior normal derivative of (37), we have

\[
-\frac{1}{2}\lambda(p) + \int_{\delta B} G_1(p,q)\lambda(q)\,dq = 0 ; \, p, q \in \delta B .
\] (38)

This equation exhibits a non-trivial solution \( \lambda \), since its transpose

\[
-\frac{1}{2}\mu(p) + \int_{\delta B} G(p,q)\mu(q)\,dq = 0 ; \, p, q \in \delta B ,
\]

has, by virtue of (23), a non-trivial solution \( \mu = 1 \). The solution of (38) generates an interior simple source potential

\[
X(p) = \int_{\delta B} G(p,q)\lambda(q)\,dq ; \, q \in \delta B, \, p \in B_1
\] (39)

characterised by

\[
X_1(p) = 0 ; \, p \in \delta B .
\] (40)

It follows that \( X = \text{Constant on } \delta B \). Hence the solution \( \lambda \) either satisfies (37) or possibly satisfies

\[
\int_{\delta B} G(p,q)\lambda(q)\,dq = 0 ; \, p, q \in \delta B .
\] (41)

Let us assume the non-trivial solution \( \lambda \) of (38) satisfies (41). Hence it generates an exterior simple source potential
\[ \chi(p) = \int_{\delta B} G(p,q) \chi(q) dq ; \quad q \in \delta B, p \in B_e \]  
characterised by

\[ \chi(p) \rightarrow 0 |p|^{-1} \int_{\delta B} \lambda(q) dq \quad \text{as} \quad |p| \rightarrow \infty . \]  

(43)

The combination of (43) with (41), i.e. \( \chi = 0 \) on \( \delta B \), implies by a classical existence theorem that \( \chi = 0 \) everywhere in \( B_e \).

Hence

\[ \chi_1^p(p) = 0 ; \quad p \in \delta B . \]  

(44)

Bearing in mind \( \chi_1^p(p) = 0 ; \quad p \in \delta B \), by (6), it follows that

\[ -\lambda(p) = \chi_1^p(p) + \chi_e(p) = 0 . \]

This shows that the equation (41) has no non-trivial solution. Hence \( \lambda \) satisfies (37), and the solution of (37) is unique. In two-dimensional potential theory

\[ \chi(p) = 0(\log |p|) \int_{\delta B} \lambda(q) dq \quad \text{as} \quad |p| \rightarrow \infty . \]

and hence we cannot conclude that \( \chi = 0 \) everywhere in \( B_e \) even though \( \chi = 0 \) on \( \delta B \). In two-dimensions, therefore, equation (41) may exhibit a non-trivial solution (\( \Gamma \) contour case, Jaswon 1963).

**Generalisation of Electrostatic Equation**

To show that the more general equation (18) has a unique solution, let us consider its equivalent normal derivative equation (20).

It has already been shown that the general solution of (20) is

\[ \sigma = \sigma_0 + k\lambda \]  

(45)
where $\sigma_0$ is a particular solution of (20); $k$ is an arbitrary constant and $\lambda$ is a solution of the corresponding homogeneous equation (25). This solution generates a simple source potential that differs from $\phi(p)$ of (18) only by a constant, which may be eliminated by choosing a suitable value of $k$. Hence $\sigma_0 + k\lambda$ provides a unique solution of (18). This discussion covers the equation (32) though it remains to be proved that $\phi_1(p)$ of (32) satisfies (24). Operating on both sides of (32) by $\int \cdots \cdots \lambda(p)dp$ and interchanging the order of integration (Fubini's theorem), we have

$$\int_{\delta B} \lambda(p)dp \int_{\delta B} G(p,q) \phi_1(q)dq = \int_{\delta B} \lambda(p)dp \int_{\delta B} G(p,q) \phi(q)dq - \frac{1}{2} \int_{\delta B} \lambda(p)\phi(p)dp$$

i.e.

$$\int_{\delta B} \phi_1(q)dq \int_{\delta B} G(q,p)\lambda(p)dp = \int_{\delta B} \phi(q)dq \int_{\delta B} G(p,q) \lambda(p)dp - \frac{1}{2} \int_{\delta B} \lambda(p)\phi(p)dp$$

i.e.

$$\int_{\delta B} \phi_1(q)dq = \frac{1}{2} \int_{\delta B} \phi(q) \lambda(q)dq - \frac{1}{2} \int_{\delta B} \lambda(p)\phi(p)dp$$

i.e.

$$\int_{\delta B} \phi_1(q)dq = 0$$

Relation between formulations

We have two formulations (18) and (32) of the Dirichlet problem, both of which are Fredholm integral equations of the 1st kind. Neither of these coincide with the classical formulation (19), which is a Fredholm integral equation of the 2nd kind. To establish a connection between (18), (19) and (32) let us introduce Green's identity (15) for an exterior function $\psi$ in $B_e$, characterised by the behaviour $0|p|^{-1}$ as $|p| \to \infty$.
\[ \int_{\delta B} G(p,q) \psi(q) dq - \int_{\delta B} G(p,q) \psi_e(q) dq = 0 \ ; \ p \in B, \ q \in \delta B. \quad (46) \]

Superimposing (10) and (46), and bearing in mind the relation (5), we find

\[ \int_{\delta B} G(p,q) \left[ \phi(q) - \psi(q) \right] dq - \int_{\delta B} G(p,q) \left[ \phi_e(q) + \psi_e(q) \right] dq = \phi(p) . \quad (47) \]

There are two distinct possibilities for \( \psi \):

(i) \( \psi = \phi \) on \( \delta B \), whence

\[ \phi(p) = - \int_{\delta B} G(p,q) \left[ \psi_e(q) + \phi_e(q) \right] dq ; \ q \in \delta B, \ p \in B. \quad (48) \]

Putting \( \psi_e(q) + \phi_e(q) = - \sigma(q) \), (48) identifies with (18).

(ii) \( \psi_e(q) = - \phi_e(q) \) on \( \delta B \), whence

\[ \phi(p) = \int_{\delta B} \left[ \phi(q) - \psi(q) \right] G(p,q) \psi_e(q) dq ; \ p \in B, \ q \in \delta B. \quad (49) \]

Putting \( \phi(q) - \psi(q) = \mu(q) \), (49) identifies with (19) when \( p \) is taken on \( \delta B \).
CHAPTER 4
SOME APPLICATIONS OF POTENTIAL THEORY

Electrostatic Capacity

It has been shown that the 'electrostatic' equation

\[ \int_{\delta B} G(p,q)\lambda(q)\,dq = 1 ; \quad p, q \in \delta B \]  

exhibits a unique solution \( \lambda \). To prove that \( \lambda \) has the same sign everywhere on \( \delta B \), we note that \( \delta B \) is an equipotential of the exterior harmonic function \( \chi \) of (42). Hence \( \chi_{e}^{1} \) has the same sign everywhere on \( \delta B \). Now

\[ \chi = O\left|\frac{1}{k}\right|^{-1} \quad \text{as} \quad |\frac{1}{k}| \to \infty , \]

Therefore, \( \chi_{e}^{1}(p) < 0 \)

everywhere on \( \delta B \). Bearing in mind \( \chi_{i}^{1}(p) = 0 \), we see that, by (6),

\[ \chi(p) = -(\chi_{i}^{1}(p) + \chi_{e}^{1}(p)) > 0 \]

on \( \delta B \). The quantity

\[ |\chi| = \int_{\delta B} \chi(p)\,dp > 0 \]

is defined to be the electrostatic capacity of \( \delta B \).

The electrostatic density \( \lambda \) which generates the potential \( \phi = 1 \) on \( \delta B \) can be obtained by solving the equation (50). The capacity \( |\chi| \) then may be computed using this \( \lambda \) in (52).

Potential Fluid Motion

An inviscid incompressible fluid is flowing from infinity with uniform velocity \( \vec{U} \). In the finite region it passes round a fixed obstacle \( B \) which disturbs the flow. If \( \psi \) is the velocity potential of the free flow, and if \( \phi \) is the perturbation of this potential by the presence of \( B \), then the total velocity potential is

\[ \phi = \phi + \psi \]
where \(-\nabla \psi = \nabla = \) a constant, 

\[
\phi = 0 \quad \text{or} \quad -\nabla \psi = \nabla = \text{constant} \quad \text{as} \quad \|p\| \to \infty ,
\]

The normal velocity component is zero at the boundary \(\delta B\), and so

\[
\psi_e(p) = \psi_e(p) + \frac{1}{2} \psi_e(p) = 0 ; \quad p \in \delta B .
\]  

(55)

From (55), \(\phi_e(p) = -\psi_e(p) ; \quad p \in \delta B\). Since \(\psi_e\) is known, \(\phi_e\) is therefore known on \(\delta B\). Hence the determination of \(\phi\) becomes an exterior Neumann problem which, as shown earlier, has always a unique solution.

Since \(\phi_e = -\psi_e \) on \(\delta B\), we have

\[
\int_{\delta B} \phi_e(p) dp = -\int_{\delta B} \psi_e(p) dp = 0 , \quad \text{by (54)} .
\]  

(56)

Putting (56) in (16), we find in the case of potential flow, that the perturbation \(\phi\) is of order \(\|p\|^2\) as \(\|p\| \to \infty\).

Given \(\psi e\) on \(\delta B\), the perturbation \(\phi\) can be obtained in two ways:

(i) It can be generated by a simple source distribution of density \(\sigma\) on \(\delta B\) such as (18) i.e.

\[
\phi(p) = \int_{\delta B} G(p,q) \sigma(q) dq ; \quad q \in \delta B
\]  

(57)

where \(\sigma\) is obtained by solving the integral equation (27) viz.

\[
-\frac{1}{2} \sigma(p) + \int_{\delta B} G_e(p,q) \sigma(q) dq = \phi_e(p) ; \quad p,q \in \delta B
\]  

(58)

in which \(\phi_e(p)\) is given by (55). The \(\sigma\)'s in (58) have the property, by (31),

\[
\int_{\delta B} \sigma(q) dq = -\int_{\delta B} \phi_e(p) dp = 0 , \quad \text{by (56)} .
\]  

(59)

(ii) \(\phi\) can directly be obtained by solving the integral equation (36) viz.

\[
-\frac{1}{2} \phi(p) + \int_{\delta B} G(p,q) \phi_e(q) dq = \int_{\delta B} G(p,q) \phi_e(q) dq ; \quad p,q \in \delta B
\]  

(60)

where \(\phi_e\) on \(\delta B\) is given by (55).
PART II

NUMERICAL PROCEDURES
CHAPTER 5

NUMERICAL SOLUTION OF FREDHOLM INTEGRAL EQUATION

First Kind

To solve a boundary integral equation analytically is, generally speaking, out of question. A straightforward numerical approach replaces the equation by a system of simultaneous linear algebraic equations, referring to a set of nodal points spaced over the boundary. The equations are then assembled and solved by writing a digital computer program.

For a numerical solution we divide the surface $\partial B$ into $N$ intervals i.e. sub-areas, and then we make the fundamental assumption that

\[ \text{(1) \ THE \ SOURCE \ DENSITY \ REMAINS \ CONSTANT \ OVER \ A \ SUB-AREA.} \]

On the basis of this assumption, for a particular field point $p$, equation (18) becomes

\[
\sum_{j=1}^{N} \sigma_j \int G(k, y) \, dy = \phi(k) ,
\]

where $\sigma_j$ stands for the constant value of $\sigma$ over the $j$th sub-area. To make further progress, we introduce a pivotal point $q_k$ within the $k$th sub-area, which is normally the centroid of the sub-area, and we put

\[ k = q_1, q_2, q_3, \ldots, q_N \]

successively. As a result (61) becomes

\[
\sum_{j=1}^{N} \sigma_j \int G(q_k, y) \, dy = \phi(q_k) ; \quad k = 1, 2, \ldots, N .
\]

This is a discrete system of $N$ linear algebraic equations for the $N$ unknowns.

Equations (62) can be put in the matrix form

\[
[A] [\sigma] = [\phi]
\]

where $[\sigma]$ is a column vector with the elements $\sigma_j$ and $[\phi]$ is a column vector with $N$ elements $\phi(q_k)$; $[A]$ is a $N \times N$ matrix with elements
\( Q_{kj} = \int_{j} G(\mathbf{q}, q) \, dq \)  \hfill (64)

i.e. the integral of \( G(q, q) \) over the \( j \)th sub-area keeping \( q_k \) fixed. In principle this can be computed as it stands, but simple approximations to it suffice for our purposes. Two distinct cases arise:

(a) when \( j \neq k \), the integrand is finite. To approximate it we make the assumption that

(2) _THE KERNEL REMAINS CONSTANT THROUGHOUT THE SUB-AREA, ITS VALUE BEING ASSOCIATED WITH THE PIVOTAL POINT \( q_k \)._

On this basis, we find

\[ Q_{kj} = G(q_k, q) \int_{j} dq_j \quad ; \quad j \neq k \]  \hfill (65)

(b) when \( j = k \), the integrand is singular, but integrable, and it may be evaluated analytically (Appendix I).

Given \( \phi \) on \( \partial \mathcal{B} \), (63) represents a system of \( N \) linear algebraic equations for \( \sigma_k \). These can be solved either by the matrix inversion method or, since \( \alpha_{kk} \neq 0 \), by the Gauss-Seidel iterative method.

**Second Kind**

Following the basic assumptions and procedures adopted with equation (18), we write equations (20) and (27) as

\[ -\frac{\sigma_k}{2} + \sum_{j=1}^{N} \int_{j} G'(q_k, q) \, dq = \phi'(q_k) \quad ; \quad k = 1, 2, \ldots, N \].  \hfill (66)

Given \( \phi'(q_k) \) on \( \partial \mathcal{B} \), this represents a system of \( N \) linear algebraic equations for the unknowns \( \sigma_1, \sigma_2, \ldots, \sigma_N \) of the form

\[ [B][\sigma] = [\phi'] \]  \hfill (67)

The element \( b'_{kj} \) of the \( N \times N \) matrix \([B]\) is given by

\[ b'_{kj} = \int_{j} G'(q_k, q_j) \, dq_j \quad ; \quad j \neq k \]  \hfill (68)
The integrand in (68) is finite and on the basis of the assumption 2 it becomes

\[ \phi_{kj} = G^t(q_k, q_j) dq_j \quad ; \quad j \neq k \]  

The integrand in (69) is apparently indeterminate but integrable, and may be evaluated analytically (Appendix II).

Similarly following the same assumptions and procedures, we write equations (33) and (36) as

\[ -\frac{1}{2} + \sum_{j=1}^{N} \phi_j \int G(q_k, q_j) dq_j = \sum_{j=1}^{N} \phi_j \int G(q_k, q_j) dq_j ; \quad k = 1, 2, \ldots, N \]  

Given \( \phi_j(q_k) \) on \( \partial B \), this equation represents a system of \( N \) linear algebraic equations for the unknowns \( \phi(q_1), \phi(q_2), \ldots, \phi(q_N) \), of the form

\[ [C][\phi] = [D] \]  

The element \( C_{kj} \) of the \( N \times N \) matrix \([C]\) is given by

\[ C_{kj} = \int G(q_k, q_j) dq_j \quad ; \quad j \neq k \]  

\[ = -\frac{1}{2} + \int G(q_k, q_j) dq_j \quad ; \quad j = k \]
The integrand in (73) is finite and, as before, on the basis of assumption 2, it becomes

$$G_{Rj} = G\left(\gamma_R, \gamma_j\right) \int_{\gamma_j} \frac{d\gamma}{\gamma} ; \quad j \neq k \quad (75)$$

When \( j = k \), the integrand is apparently indeterminate but integrable, and may be evaluated analytically (Appendix II). The column vector \([D] \) has \( N \) elements

$$d_k = \sum_{j=1}^{N} \phi_j^i \left[ G\left(\gamma_R, \gamma_j\right) \right] \quad k = 1, 2, \ldots, N \quad (75)$$

The above integral for \( j = 1, 2, \ldots, N \) is evaluated in the same way as (64). From (69) and (74) we find that the diagonal element in any of the matrices \([B] \) and \([C] \) is a fairly large element in a row. This makes the equations amenable to solution by the Gauss-Seidel iterative method.

**Singular Matrix**

In the electrostatic problem \( \phi_j^1 = 0 \) on \( \partial B \), so that (20) becomes the homogeneous equation

$$\frac{-1}{2} \lambda(\gamma) + \int_{\partial B} G_{\text{int}} \left(\gamma, \gamma_j\right) \lambda(\gamma) d\gamma = 0 \quad (76)$$

where \( \text{int.} \) stands for interior normal (replacing \( i \) of Part I).

On discretisation (76) gives (67) with \( \phi_j^1 = 0 \), and \([C] \) is replaced by \([\lambda] \)

i.e.

$$[B][\lambda] = 0$$

It has already been shown in Chapter 2, that the equation (76) has a non-trivial solution. Hence the matrix \([B] \) must be singular. This property must be ensured by our numerical procedure. How can this be done? Since (76) has a non-trivial solution, it follows that

$$\int_{\partial B} \left[ \frac{-1}{2} \lambda(\gamma) + \int_{\partial B} G_{\gamma} \left(\gamma, \gamma_j\right) \lambda(\gamma) d\gamma \right] = 0$$
i.e. \[ \int_{\partial \Omega} - \frac{1}{2} \chi(h) \, d\nu + \int_{\partial \Omega} \int_{\partial \Omega} G^{(h, \nu)} \chi(\overline{y}) \, d\gamma \, d\nu = 0 \ . \]

\[ \int_{\partial \Omega} - \frac{1}{2} \chi(h) d\nu + \int_{\partial \Omega} \chi(\overline{y}) d\gamma \int_{\partial \Omega} G^{(h, \nu)} \, d\nu = 0 \ . \]

Our numerical approach should theoretically ensure that

\[ \int_{\partial \Omega} G^{(h, \nu)} d\nu = \frac{1}{2} \ . \quad (77) \]

This result suggests that we should define \( b^* \) given by (69), so that

\[ b_{kk}^* + \sum_{j=1}^{N} b_{j}^* = 0 \ ; \ k = 1, 2, \ldots N \ , \quad (78) \]

where \( \sum_{j=1}^{N} \) indicates omission of \( j = k \) in the sequence \( j = 1, 2, \ldots N \).

This means that the sum of each column of \( [B] \) is zero, and hence evaluation of \( b_{kk}^* \) by (78) ensures that the matrix \( [B] \) is singular.

The homogeneous part of (33), i.e.

\[ -\frac{1}{2} \phi(h) + \int_{\partial \Omega} \phi(\overline{y}) G^{(h, \nu)} \, d\gamma = 0 \ , \quad (79) \]

on discretisation gives (72) with \( [D] = 0 \), i.e.

\[ [C][\phi] = 0 \ . \]
Since (79) has a non-trivial solution, shown in Chapter 2, the matrix \( [C] \) must be singular. This property must be ensured by our numerical procedure. Adding all the elements in a \( k \)th row of \( [C] \), we obtain, by 23,

\[
\sum_{j=1}^{N} C_{kj} = -\frac{1}{2} + \sum_{j=1}^{N} \int_{\partial S} G(\nu, \nu)_{\nu} d\sigma
\]

\[
= -\frac{1}{2} + \int_{\partial S} G(\nu, \nu)_{\nu} d\sigma = 0; \quad k = 1, 2, \ldots N.
\]

Hence our numerical approach should ensure that

\[
\int_{\partial S} G(\nu, \nu)_{\nu} d\sigma = \frac{1}{2}; \quad k = 1, 2, \ldots N. \tag{80}
\]

This result suggests that we should define \( C_{kk} \), given by (74), so that

\[
C_{kk} + \int_{\partial S} G(\nu, \nu)_{\nu} d\sigma = 0 \tag{81}
\]

where \( \int_{\partial S} \) indicates omission of the \( k \)th interval. Evaluation of \( C_{kk} \) by (81) ensures that the matrix \( [C] \) is singular.

In the case of the exterior Neumann problem, the homogeneous equations (29) and (30), shown in Chapter 2, have no non-trivial solutions. Hence the matrix \( [C] \) and \( [B] \), obtained on discretisation of (29) and (30) respectively, are not generally singular. In such cases we assume

(3) THE SUB-AREAS ARE PIECEWISE FLAT.

On this basis, by Appendix II., we find

\[
\int_{\partial S} G(\nu, \nu)_{\nu} d\sigma = \int_{\partial S} G(\nu, \nu)_{\nu} d\sigma = 0. \tag{82}
\]
CHAPTER 6

PRINCIPLES OF DIVISION OF A SURFACE INTO SUB-AREAS

Introduction

We now consider in detail the problem of dividing a surface into sub-areas. In the case of a flat surface, say the surface of a cube no difficulty arises. We simply divide each side into equal squares. On the other hand, in the case of a sphere, it is not immediately obvious how to proceed. The possible sub-division of a spherical surface is given in Figure 3. This suggests that the optimum sub-division will be a mixture of squares and triangles.

It is also necessary to consider that, generally, the charge density is not constant. It varies over an interval and therefore our fundamental assumption i.e. the charge density is constant over a sub-area, brings in some error. The question now arises how we can minimise this error by a suitable choice of sub-area.

Variation of density

To carry out our numerical analysis, we divide a curve, a surface or a volume into smaller intervals and we assume that the density \( \sigma \) over each interval is a continuous function which spreads uniformly in all directions from the centroid. Following Weirstrass's theorem, we know, any continuous function can be represented as accurately as we please, over a finite range of its arguments, by a polynomial of sufficiently high degree. Hence \( \sigma \) at a point \((x, y, z)\) can be approximated by

\[
\sigma = \sum_{n} F_{n}(x, y, z), \tag{83}
\]

where \( F_{n} \) is a piecewise continuous symmetric polynomial function of degree \( n \).

On rearrangement, \( F \) can be written as

\[
F = \sum_{j} \sum_{k} A_{kj} P_{kj}, \tag{84}
\]

where \( P_{kj} \) is a homogeneous polynomial of degree \( j \) which remains invariant under any permutation of \( x, y \) and \( z \).
Fig. 3

SUB-AREAS ON THE SURFACE OF A SPHERE
To simplify our numerical calculations, we approximate the value of $\sigma$ over an interval by taking only the first term in its Taylor expansion about the centroid of the interval. As a result, the approximate value of $\sigma$ becomes a constant over an interval, which agrees with our fundamental assumption.

At any point $q(x_0 + \delta x, y_0 + \delta y, z_0 + \delta z)$, we may write

$$\sigma(q) = \sum_n \left[ F_n(x_0, y_0, z_0) + \delta x \frac{\partial F_n}{\partial x} + \delta y \frac{\partial F_n}{\partial y} + \delta z \frac{\partial F_n}{\partial z} + \cdots \right] , \quad (85)$$

where $(x_0, y_0, z_0)$ are the co-ordinates of the centroid $q_0$ of the $q$th interval. According to our approximation,

$$\sigma(q) = \sum_n F_n(x_0, y_0, z_0) . \quad (86)$$

Neglecting higher-order quantities, the significant part of the error in (86), as compared with (85), is given by

$$\epsilon = \sum_n \left( \int \left( \frac{\partial F_n}{\partial x} \delta x + \frac{\partial F_n}{\partial y} \delta y + \frac{\partial F_n}{\partial z} \delta z \right) dq \right) , \quad (87)$$

where the integral is taken over the $q$th interval and $dq$ stands for the volume element $dx dy dz$ at $(x, y, z)$. Transferring the origin of the reference frame to $(x_0, y_0, z_0)$, we find, by Euler's theorem on homogeneous functions,

$$\epsilon = \sum_n \left( \int \left( x \frac{\partial F_n}{\partial x} + y \frac{\partial F_n}{\partial y} + z \frac{\partial F_n}{\partial z} \right) dx dy dz \right) ,$$

$$= \sum_n \int \int \int F_n(x, y, z) dx dy dz . \quad (88)$$
Some special Intervals

Taking \( z = 0 \), a typical term of \( \zeta \) is given by

\[
I = \int \left( x^m y^n + x^n y^m \right) \, dx \, dy .
\] (89)

(i) For a circular area with radius 'a',

\[
I = 4 \int \left( x^m y^n + x^n y^m \right) \, dx \, dy
\]

\[
= 4 \int_0^{\pi/2} \int_0^a \left( r^{m+n+1} \cos^m \theta \sin^n \theta + \cos^n \theta \sin^m \theta \right) \, dr \, d\theta
\]

\[
= 8 \frac{a^{m+n+2}}{m+n+2} \int_0^{\pi/2} \cos^m \theta \sin^n \theta \, d\theta .
\] (90)

This can be computed for any choice of \( m \) and \( n \):

(a) \( n = 1, m \geq 0 \) or \( m = 1, n \geq 0 \) gives

\[
I_c = \frac{8 \frac{a^{m+n+2}}{m+n+2}}{a^{m+n+2}} \cdot \frac{1}{(m+n)} ;
\]

(b) \( n > 1, m = 0 \) gives

\[
I_c = \frac{8 a^{m+n+2}}{m+n+2} \left[ \frac{(n-1)(n-3) \cdots 3 \cdot 1}{n(n-2) \cdots 4 \cdot 2} \right] ; \text{ when } n \text{ is an even integer,}
\]

\[
= \frac{8 a^{m+n+2}}{m+n+2} \left[ \frac{(n-1)(n-3) \cdots 4 \cdot 2}{n(n-2) \cdots 5 \cdot 1} \right] ; \text{ when } n \text{ is an odd integer.} 
\]
(c) \( n > 1, \ m > 1 \) gives

\[
I_c = \frac{8a^{m+n+2}}{m+n+2} \left[ \frac{1\cdot3\cdot5\cdot\ldots\cdot(n-1)\cdot1\cdot3\cdot5\cdot\ldots\cdot(m-1)}{2\cdot4\cdot6\cdot\ldots\cdot(m+n)} \cdot \frac{\pi}{2} \right]; \quad \text{when both} \ m \ \text{and} \ n \ \text{are even,}
\]

\[
= \frac{8a^{m+n+2}}{m+n+2} \left[ \frac{2\cdot4\cdot6\cdot\ldots\cdot(m-1)}{(n+1)(n+2)\cdot\ldots\cdot(m+n)} \right];
\]

\[\quad \text{when any one of them, say} \ m, \ \text{is an odd integer.}\]

(ii) For a rectangular area with sides 2a and 2b,

\[
I_R = 4 \int_0^a \int_0^b \left( x^m y^n + x^n y^m \right) dx \ dy,
\]

\[
= 4 \frac{1}{(m+1)(n+1)} \left[ a^{m+1} b^{n+1} + a^{n+1} b^{m+1} \right]. \quad (91)
\]

(iii) For an isosceles triangular area (Fig. 4)

\[
I_T = \int_0^\infty \left( x^m y^n + x^n y^m \right) dy,
\]

Putting \( x = r \cos \theta \), \( y = r \sin \theta \)

and \( H = r \sec \theta \)

\[
= 2 \sum_{j=1}^3 \int_0^\infty \left[ (\cos^m \theta \sin^n \theta + \sin^m \theta \cos^n \theta) \int_{r=0}^{H_j \sec \theta} r^{m+n+1} dr \right] d\theta,
\]

\[
= 2 \sum_{j=1}^3 \frac{H_j^{m+n+2}}{m+n+2} \int_0^\infty \left( \tan^n \theta + \tan^m \theta \right) \sec^2 \theta d\theta.
\]
Fig. 4
Optimum Choice of Interval

The integral $I_R$ in (91) varies with $a$ and $b$. For a given rectangular area, $I_R$ has a minimum when $a = b$. Hence by (91)

$$I_R = I_{sq} = \frac{8}{(m+1)(n+1)} a^{m+n+2}, \text{ when } a = b \quad (93)$$

Similarly, $I_T$ in (92) attains a minimum when the triangular interval is an equilateral triangle. This is evident from Table 1 in which $I_T(\Theta)$ refers to the value of $I_T$ for the isosceles triangle with base angles $\Theta$. The triangular areas considered therein are each of unit area. From Table 1 it is clear that for all values of $m$ and $n$, for a given area

$$I_T(35^\circ) > I_T(30^\circ) > I_T(45^\circ) > I_T(60^\circ) < I_T(75^\circ). \quad (94)$$

Further for a given area

$$I_c < I_{sq} < I_T(60^\circ). \quad (95)$$

This is evident from Table 2 in which the areas considered are each of unit area.

All the relations mentioned above are true for every term of (88) and therefore these are true for (88) itself.
ESTIMATION OF ERROR ON UNIT TRIANGULAR AREAS

<table>
<thead>
<tr>
<th>m</th>
<th>n</th>
<th>m+n</th>
<th>$I_T(15^\circ)$</th>
<th>$I_T(30^\circ)$</th>
<th>$I_T(45^\circ)$</th>
<th>$I_T(60^\circ)$</th>
<th>$I_T(75^\circ)$</th>
</tr>
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<td>0</td>
<td>1</td>
<td>1</td>
<td>0.80765</td>
<td>0.64138</td>
<td>0.56929</td>
<td>0.54574</td>
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<tr>
<td>1</td>
<td>1</td>
<td>2</td>
<td>0.25881</td>
<td>0.21605</td>
<td>0.17901</td>
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<td>0.17328</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>3</td>
<td>0.16290</td>
<td>0.10582</td>
<td>0.07344</td>
<td>0.06302</td>
<td>0.07260</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>4</td>
<td>0.16603</td>
<td>0.06883</td>
<td>0.03601</td>
<td>0.02673</td>
<td>0.03972</td>
</tr>
<tr>
<td>3</td>
<td>2</td>
<td>5</td>
<td>0.04918</td>
<td>0.02713</td>
<td>0.01453</td>
<td>0.01067</td>
<td>0.01361</td>
</tr>
<tr>
<td>4</td>
<td>2</td>
<td>6</td>
<td>0.05655</td>
<td>0.01993</td>
<td>0.00805</td>
<td>0.00499</td>
<td>0.00856</td>
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Table 1

ESTIMATION OF ERROR ON DIFFERENT FIGURES OF UNIT AREA

<table>
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<tr>
<th>m</th>
<th>n</th>
<th>m+n</th>
<th>$I_c$</th>
<th>$I_{sq}$</th>
<th>$I_T(60^\circ)$</th>
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<td>0</td>
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</table>

Table 2
General Rules for Sub-division

In numerical analysis, the entire curve or surface or volume under consideration should be covered by intervals keeping no gap between them. The spherical intervals cannot be fitted together to cover a volume in the above fashion. The next most suitable intervals are the regular polygons, of which the simplest interval is a cube. Similarly, in the case of a surface, the most suitable interval i.e. sub-area is a square or an equilateral triangle.

Bearing these considerations in mind, we lay down the following general procedures:

1. It is recommended to cover a curve, a surface or a volume by the same type of intervals as far as possible.

2. In case of a surface it needs, in general, a mixture of triangular and rectangular sub-areas to fit together. The triangular sub-areas should be as far as near to equilateral form. The rectangular sub-areas should be kept near to the square form.

3. It is found, in general, that $\mathcal{O}$ changes rapidly as we approach a sharp edge or a corner on a surface. In fact it cannot properly be represented by (83) near a sharp edge or a corner. Hence in general, one should not expect to obtain an accurate measure of $\mathcal{O}$ at these points by our numerical methods. To achieve a tolerable approximation to $\mathcal{O}$ near such a point:
   (a) The sub-areas should become smaller in size as we approach such a point.
   (b) The reduction in size of the sub-area should be gradual.
CHAPTER 7
APPROXIMATE INTEGRATION

Introduction

To evaluate analytically an integral of the form

\[ I = \int_{\partial B} F(h, q) \, dq \]  \hspace{1cm} (96)

the first requirement, in general, is that the integrand should have an analytic expression in terms of the co-ordinates of \( q \). Further, the boundary surface \( \partial B \) should also have an analytic expression. For bodies with definite regular geometrical shapes, there are analytic expressions for \( \partial B \), but sometimes it happens that even for these the evaluation of (96) becomes very complicated. For a body with an irregular boundary different parts of it may require different analytic expressions, in which case the evaluation of (96) becomes extremely complicated. Often in practice, only the numerical values of the integrand are available at the pivotal points of \( \partial B \). Accordingly this is not generally possible to evaluate (96) analytically.

In view of the above difficulties, we must think of an operation to approximate (96) over any surface \( \partial B \) over which \( F(p, q) \) is defined. It is desirable that the operation should be simple on the one hand and, on the other hand, it should be capable of approximating (96) within a tolerable error.

When \( F(p, q) \) is a function of a single variable, as happens with the plane curves, the Simpson and the Trapezoidal rules of approximate integration produce results to a sufficient degree of accuracy. Unfortunately, in general, there are no such analogous rules to effect an approximate integration when the integrand is a function of two or more independent variables.

Approximation Methods

When \( F(p, q) \) is a function of two or more variables, we propose two methods to approximate the integral over \( \partial B \):
(i) The AVERAGING method of approximation.

(ii) The CENTROID method of approximation.

In both these methods, we divide $\partial B$ into $N$ intervals i.e. sub-areas, operate on each of the sub-areas separately, and then add them up to approximate the integral over $\partial B$.

(i) AVERAGING METHOD

If the $k$ th interval, i.e. sub-area, is an $m$ sided polygon, the averaging approximation to (96) over this area is defined by

$$I_A = \sum_{k=1}^{N} \frac{\sum_{j=1}^{m+1} F(k, j)}{m+1} \int_{q_k}^{p} \; dq \; ; \; p \notin \partial B,$$

where $q_1, q_2, \ldots, q_m$ define the $m$ corner points of the polygon and $q_0$ defines the pivotal point (centroid) of the polygon.

(ii) CENTROID METHOD

If $I_c$ represents the approximation to (96) by the centroid method of approximation, then $I_c$ is defined by

$$I_c = \sum_{k=1}^{N} F(k, q_0) \frac{1}{m+1} \int_{q_k}^{p} \; dq \; ; \; p \notin \partial B,$$

where, as before, $q_0$ defines the centroid of the $k$ th sub-area.

The centroid method of approximation is nothing but the application of assumption 2 in the evaluation of (96). The averaging method may well be looked upon as an extension of the above principle.

If the integrand has a factor $|p - q|$ then, depending upon the position of $p$ two distinct cases arise:

(i) If $p \neq q$, the integrand is finite and evaluation of (97) as well as of (98) is straightforward.

(ii) If $p = q$, the integrand is singular and, the integral must be evaluated analytically.

A Comparative Study of the Two Methods

To make a comparative study of the merits of the two approximations, we consider the analytic value of the integral (96) for a particular $F$. In this thesis, we deal mainly with integrals of the type

$$I = \int_{\partial B} \frac{dq}{|k - y|}, \quad J = \int_{\partial B} \frac{dq}{|p - y|}.$$
Let us therefore take

\[ I = \int_{\partial B} \frac{d\eta}{|\nu - \eta|} \]  

(99)

as a test case for a comparative study of the two approximations.

Using a cartesian frame of reference, let \( \partial B \) be a rectangular area defined by \( z = 0, x = \pm a, y = \pm b \). Since \( q \in \partial B \), we may write \( q = (x, y, 0) \) and a field point may be represented by \( p = (X,Y,Z) \).

By appendix I,

\[ I = \int_{\partial B} \frac{d\eta}{|\nu - \eta|} \]

\[ = \int_{y=-b}^{b} \int_{x=-a}^{a} \frac{dx \, dy}{\sqrt{(x-x')^2 + (y-y')^2 + Z^2}} \]  

(100)

\[ \log \left\{ \frac{C + \sqrt{R^2 + D^2}}{E} \right\} + Z \left\{ \sin^2 \left( \frac{2}{F + E \sqrt{R^2 + D^2}} \right) \right\} \]

where 

\[ C = x+a, \quad D = C+Z^2, \quad E = x-a, \quad F = E^2 + Z^2 \]

Choosing \( a = b = 1 \) to ease the numerical work and treating \( \partial B \) as a single sub-area and not sub-dividing it any further, we compute \( I_A \), \( I_c \) for (100) for various locations of \( p \) as indicated in Table 3.

From Table 3 we find that, for all locations of the field point,

\[ |I-I_c| < |I-I_A| \]  

(101)
## APPROXIMATIONS TO AN INTEGRAL

<table>
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<th>CO-ORDINATES OF p</th>
<th>DISTANCE FROM CENTRE</th>
<th>I</th>
<th>Ic</th>
<th>Ia</th>
<th>% ERROR IN Ic</th>
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<td></td>
<td></td>
<td></td>
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<td>2.00000</td>
<td>1.46667</td>
<td>26.00</td>
</tr>
<tr>
<td>0, 0, 1.750</td>
<td>1.750</td>
<td>0.55671</td>
<td>0.57143</td>
<td>0.33885</td>
<td>2.64</td>
</tr>
<tr>
<td>0, 0, 2.750</td>
<td>2.750</td>
<td>0.35972</td>
<td>0.36364</td>
<td>0.17195</td>
<td>1.09</td>
</tr>
<tr>
<td>0, 0, 4.500</td>
<td>4.500</td>
<td>0.22132</td>
<td>0.22222</td>
<td>0.08300</td>
<td>0.41</td>
</tr>
</tbody>
</table>

Table 3
This means that the centroid method produces a better approximation than that of the averaging method. When we divide \( \Omega_B \) into \( N \) sub-areas to evaluate \( I_A \) and \( I_C \) by a more general application of (97) and (98), relation (101) remains valid for each of the \( N \) sub-areas. Hence (101) remains valid when these are added over the whole of \( \Omega_B \). It may be mentioned that the centroid method not only yields a better approximation than the averaging method but is also simpler to compute.

**Error in the Centroid Method**

If \( \Omega_B \) forms a single sub-area, and \( p \notin \Omega_B \), we see from Table 3 and Fig. 5 that:

(i) The error in \( I_C \) diminishes asymptotically to zero as \( p \) tends to infinity.

(ii) For a given distance from the centroid of \( \Omega_B \), the error is a maximum when \( p \) lies on the normal to \( \Omega_B \) through its centroid. Further, it is evident that, for all locations of \( p \),

\[
\epsilon_{\text{max}} \leq 1\% \quad \text{when} \quad \left| p - q_0 \right| > 2 D_{\text{max}},
\]

where \( \epsilon_{\text{max}} \) represents the maximum of the errors in \( I_C \) for various positions of \( p \) and \( D_{\text{max}} \) represents the greatest diagonal of the largest interval i.e., sub-area.

Now let us divide \( \Omega_B \) into \( N \) sub-areas and examine the behaviour of the error in \( I_C \) as \( N \) gradually increases. We define \( I_C \) at a point \( p \ (p \notin \Omega_B) \) by

\[
I_{C,p} = \sum_{j=1}^{N} I_{C,j},
\]

where \( I_{C,j} \) represents the value of \( I_C \) over the \( j \)th sub-area. The field point \( p \) lies outside \( \Omega_B \) at a perpendicular distance \( d \) from the boundary point \( p_B \) [Fig. 6 (a)], such that

\[
d = \left| p - p_B \right| = \frac{L_{\text{min}}}{2},
\]

where \( L_{\text{min}} \) is the minimum distance between the two nodal points of the \( N \) sub-areas. Hence, as \( N \) increases, \( p \to p_B \). Taking \( \Omega_B \) to be a unit area
$\alpha B = BC = 1
\alpha C = 1.414$

$ABCD$ is a single square sub-area.

Fig. 5

Estimation of error in the approximation to $\int_{|z|=1} \frac{1}{|z|} \, dz$ over a single sub-area.
and dividing it into $N$ sub-areas, $I_{cp}$ and $I$ are evaluated for different values of $N$. For each choice of $N$, the field point $p$ always satisfies the relation (104). The values of $I_{cp}$ and $I$ for different values of $N$ and for different positions of $p$ are given in Table 4.

This Table shows clearly that, as $N$ increases, though the field point $p$ approaches the boundary of $\partial B$, the percentage error in $I_{cp}$ gradually decreases. The same conclusion holds good when $\partial B$ is a triangular surface [Fig. 6 (b)] of unit area with $N$ triangular sub-areas.

If $\partial B$ is divided into $N$ sub-areas, and $p \in \partial B$, $p$ will either be an interior point of a sub-area or it will be a boundary point of two or more sub-areas. In such a case, as stated earlier, we must evaluate the integral analytically over the sub-area for which $p$ is an interior or a boundary point. Evaluating the integral over the rest of the sub-areas by (103), we find

$$I_{cp} = \sum_j I_{cj} + \sum_k I_k ,$$  \hspace{1cm} (105)

where $I_{cj}$ refers to the sub-area not containing $p$ and $I_k$ refers to the sub-area for which $p$ is an interior or a boundary point.

If $p$ satisfies (104), from Fig. 6(1) and Fig. 6(5) it follows that

$$\epsilon_{IN} \leq \epsilon_{OUT}$$ \hspace{1cm} (106)

for the same sub-division of $\partial B$ and for all values of $N$, where $\epsilon_{IN}$, $\epsilon_{OUT}$ respectively stand for the % errors in $I_{cp}$ when $p \in \partial B$ and $p \notin \partial B$. Accordingly, when dealing with boundary value problems, the above approximations produce a better result when the field point $p$ is on the boundary itself than when it is outside the boundary and obeys relation (104).

Application of the Approximation to Some Test Cases

We know, by the Gauss flux theorem (23) of Chapter 2, that

$$J = \int_{\partial B} \frac{\partial y}{l_k \cdot \frac{y}{l_k} \parallel y_{int.}} = 2\pi \cdot \parallel y \parallel_{\partial B} ,$$ \hspace{1cm} (107)
ERROR IN APPROXIMATION DEPENDING UPON THE POSITION OF THE FIELD POINT

| CO-ORDINATES OF p | TOTAL SUB-AREAS N | d=|p-P_B| | I | I_C | % ERROR IN I_C |
|------------------|------------------|-----|---------|-----|-----|----------------|
| 0.6250, 0.1250, 0 | 16               | 0.1250 | 1.23059 | 1.22175 | 0.718 |
| 0.5625, 0.0625, 0 | 64               | 0.0625 | 1.41726 | 1.41122 | 0.359 |
| 0.5417, 0.0417, 0 | 144              | 0.0417 | 1.50030 | 1.49673 | 0.238 |
| 0.5250, 0.0250, 0 | 400              | 0.0250 | 1.58055 | 1.57832 | 0.141 |
| 0.5179, 0.0179, 0 | 784              | 0.0179 | 1.62084 | 1.61922 | 0.099 |

Table 4
ERROR IN THE APPROXIMATION AS THE NUMBER OF SUB-AREAS INCREASES

Fig. 6
where $\frac{|p - q|}{|p - q|_{\text{int.}}}$ represents the interior derivative of $|p - q|$ at the point $q$ keeping $p$ fixed. On discretisation, (107) can be represented as

$$J_k = \sum_{j=1}^{N} \left( \int_{\frac{d\tilde{q}_j}{|q_j - q_j'|_{\text{int.}}}} \right),$$

(108)

where $q_k = p$. When $j$ successively assume values 1, 2, ... $k$, ..., $N$ there arise two distinct cases:

(i) when $j \neq k$, by assumption 2

$$\int_{\frac{d\tilde{q}_j}{|q_j - q_j'|_{\text{int.}}}} = \hat{n}_{\text{ext.}}(q_j) \cdot \hat{n}_{\text{ext.}}(q_j) \int_{\frac{d\tilde{q}_j}{|q_j - q_j'|_{\text{int.}}}}$$

(109)

where $\hat{n}_{\text{ext.}}(q_j)$ represents the exterior unit normal at the pivotal point $q_j$.

(ii) when $j = k$, the integrand is singular. But by assumption 3 and Appendix II, we may approximate this to zero i.e.

$$\int_{\frac{d\tilde{q}_k}{|q_k - q_k'|_{\text{int.}}}} = 0$$

By (109),

$$J_k = \sum_{j=1}^{k-1} \left( \int_{\frac{d\tilde{q}_j}{|q_j - q_j'|_{\text{int.}}}} \right) + \int_{\frac{d\tilde{q}_k}{|q_k - q_k'|_{\text{int.}}}}$$

(110)

where $J_k$ represents the approximated value of $J$ at the point $q_k \in \partial B$ and $\sum_{j=1}^{k-1}$ represents the summation over all the sub-areas except the $k$th sub-area.

Let $\partial B$ be a surface of a unit cube whose 6 sides are given by

$$x = \pm \frac{1}{2}, \quad y = \pm \frac{1}{2} \quad \text{and} \quad z = \pm \frac{1}{2}.$$

Dividing $\partial B$ into $N$ square sub-areas (Fig. 7), the value of $J_k$ is computed by (110). This value, as expected, is most inaccurate when $k$ defines a sub-area nearest to a corner. The value of $J_k$ at the points $q_k$ are computed and exhibited in Table 5 for comparison with the analytic value $2\pi \approx 6.28318.
Fig. 7

SUB-AREAS ON THE SURFACE OF A CUBE
## Testing of the Approximation on the Surface of a Unit Cube

### Table 5

<table>
<thead>
<tr>
<th>CO-ORDINATES OF $q_k$</th>
<th>TOTAL SUB-AREAS $N$</th>
<th>$J_k$</th>
<th>% ERROR</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.000, 0.000, 0.500</td>
<td>216</td>
<td>6.30768</td>
<td>0.389</td>
</tr>
<tr>
<td>0.000, 0.000, 0.500</td>
<td>1944</td>
<td>6.28591</td>
<td>0.043</td>
</tr>
<tr>
<td>0.000, 0.000, 0.500</td>
<td>5400</td>
<td>6.28417</td>
<td>0.017</td>
</tr>
<tr>
<td>0.417, 0.417, 0.500</td>
<td>216</td>
<td>6.19087</td>
<td>1.469</td>
</tr>
<tr>
<td>0.472, 0.472, 0.500</td>
<td>1944</td>
<td>6.18639</td>
<td>1.540</td>
</tr>
<tr>
<td>0.483, 0.483, 0.500</td>
<td>5400</td>
<td>6.18614</td>
<td>1.544</td>
</tr>
</tbody>
</table>
Let \( \mathcal{B} \) be the spherical surface

\[ x^2 + y^2 + z^2 = 1 \]

It is divided into \( N \) sub-areas as in Fig. 3. The sub-areas adjacent to the poles are approximately triangular in form and the rest all are approximately trapezoidal in form. The value of \( J_k \) at the point \( q_k \) is then computed by application of (110) for different values of \( N \) and for different positions of the field point, as exhibited in Table 6.

For a given value of \( N \) the error is a maximum when the field point is nearest to the pole, which is expected because of the size and the form of the sub-areas at that region.
TESTING OF THE APPROXIMATION ON THE SURFACE OF A UNIT SPHERE

<table>
<thead>
<tr>
<th>CO-ORDINATES OF $q_k$</th>
<th>TOTAL SUB-AREAS $N$</th>
<th>$J_k$</th>
<th>% ERROR</th>
</tr>
</thead>
<tbody>
<tr>
<td>$0.9997, 0, 0.0228$</td>
<td>2544</td>
<td>6.16661</td>
<td>1.855</td>
</tr>
<tr>
<td>$0.9998, 0, 0.0175$</td>
<td>9264</td>
<td>6.20372</td>
<td>1.265</td>
</tr>
<tr>
<td>$0.9999, 0, 0.0135$</td>
<td>20184</td>
<td>6.22496</td>
<td>0.927</td>
</tr>
<tr>
<td>$0.9999, 0, 0.0110$</td>
<td>35304</td>
<td>6.23724</td>
<td>0.731</td>
</tr>
<tr>
<td>$0.0906, 0, 0.9959$</td>
<td>2544</td>
<td>6.08119</td>
<td>3.215</td>
</tr>
<tr>
<td>$0.0453, 0, 0.9989$</td>
<td>9264</td>
<td>6.18188</td>
<td>1.612</td>
</tr>
<tr>
<td>$0.0302, 0, 0.9995$</td>
<td>20184</td>
<td>6.21561</td>
<td>1.075</td>
</tr>
<tr>
<td>$0.0227, 0, 0.9997$</td>
<td>35304</td>
<td>6.23325</td>
<td>0.807</td>
</tr>
</tbody>
</table>

Table 6
PART III
CAPACITY OF CONDUCTORS
Recapitulation of Equations

We now regard \( \partial B \) as a closed perfectly conducting surface brought to a unit potential by the introduction of charges. If \( \lambda(\mathbf{y}) \) is the equilibrium charge density at \( \mathbf{y} \), this distribution generates the potential

\[
V(\mathbf{z}) = \int_{\partial B} \frac{\lambda(\mathbf{y}) d\mathbf{y}}{|\mathbf{z} - \mathbf{y}|}
\]

at \( \mathbf{z} \), which exists and is continuous everywhere including \( \partial B \). Hence \( \lambda \) must satisfy the integral equation

\[
\int_{\partial B} \frac{\lambda(\mathbf{y}) d\mathbf{y}}{|\mathbf{z} - \mathbf{y}|} = 1 \quad ; \quad \mathbf{z} \in \partial B,
\]

of which a unique solution has been proved to exist. It has also been proved, in Chapter 4, that \( \lambda \) has the same sign \((>0)\) everywhere on \( \partial B \). This enables us to define the essentially positive quantity

\[
\kappa = \int_{\partial B} \lambda(\mathbf{y}) d\mathbf{y},
\]

which is known as the capacity of \( \partial B \).

On discretisation, (111) gives \( N \) linear algebraic equations for \( \lambda \) viz.

\[
\sum_{j=1}^{N} a_{kj} \lambda_j = 1 \quad ; \quad k = 1, 2 \cdots N,
\]

(112)

where

\[
a_{kj} = \int_{\partial B} \frac{d\mathbf{y}}{|\mathbf{z} - \mathbf{y}|},
\]

(113)
and
\[ \lambda = \sum_{j=1}^{N} \lambda_j dS_j \]  

(114)

where \( dS_j \) represents the area of the \( j \)th sub-area.

Solution of Equations

After evaluation of the \( a_{kj} \) by the procedures discussed in Chapter 5, equations (112) are solved by the Gauss-Seidel iterative method. In this method, after each iteration, we obtain a set of values of \( \lambda_1, \lambda_2, \ldots, \lambda_N \) at the \( N \) pivotal points \( q_1, q_2, \ldots, q_N \) on \( \partial B \). After the \( r \)th iteration, these values are denoted \( \lambda^r_1, \lambda^r_2, \ldots, \lambda^r_N \), and so at the pivotal point \( q_k \) we have, after \( n \) iterations, a sequence of \( n \) values

\[ \left\{ \lambda^r_k \right\} ; \quad r = 1, 2, \ldots, n, \]  

(115)

which are successive approximations to the exact value \( \lambda(q_k) \).

If for a pre-assigned small positive quantity \( \epsilon \) (\( \epsilon = 0.0001 \) say), there exists a number \( M \) such that

\[ \left| \lambda^r_k - \lambda^{r-1}_k \right| \leq \epsilon, \quad \text{for} \quad r = M, \]  

(116)

at every pivotal point \( q_1, q_2, \ldots, q_N \), then at this the approximate solutions are given by

\[ \lambda^M_k ; \quad k = 1, 2, \ldots, N. \]  

(117)

Determination of The Optimum Value of \( N \)

Our preceding analysis has dealt with a fixed number \( N \) of nodal points. From the fundamental assumption that the source density is constant over a sub-area, it appears that the computed source density at a nodal point approaches its analytic value at that point as \( N \to \infty \). But because of the rounding-off errors involved in the computations, after a certain stage, the result becomes distorted as \( N \) increases. Hence the problem arises of finding the optimum value of \( N \). To find this we start with a small value of \( N \) and gradually increase it until a stage comes when either
(i) $\kappa$ ceases to behave monotonically,
or (ii) the density distribution along a line on $\partial B$ changes sign.
At this stage, the optimum value of $N$ is given by the value of $N$ considered
in the previous stage.

**Intrinsic Test of Accuracy**

The solution of (112) yields the numerically generated potential

$$V(k) = \sum_{j=1}^{N} \lambda_j \int_{\frac{dV}{|k-q|}}$$

(118)

at any point $p \in B + \partial B$. This automatically has the value $V = 1$ at
the nodal points on $\partial B$, but will generally deviate from 1 at any other point.
For a particular sub-area,

$$|1 - V(p)| = 0 \quad \text{when } p \text{ is a nodal point on } \partial B,$$

$$> 0 \quad \text{when } p \text{ is not a nodal point on } \partial B.$$  

Since $1 - V(p)$ is a harmonic function in $B$, its modulus $|1 - V(p)|$
attains a maximum for some point $p$ on $\partial B$. We may therefore approximately
determine $|1 - V(p)|_{\text{max}}$ by generating $V(p)$ at a number of representative
non-pivotal points on $\partial B$. 

CHAPTER 9
CAPACITY OF THIN CONDUCTORS

Square Plate

Let the periphery of a thin square conductor ABCD (Fig. 8) of unit area in the plane \( z = 0 \) be given by

\[
x = \pm \frac{1}{2} \quad \text{and} \quad y = \pm \frac{1}{2}.
\]

It is divided into \( N \) equal square sub-areas of area \( ds \) each, where

\[
N = k^2; \quad k = 2m + 1; \quad m = 1, 2, \ldots, n;
\]

and

\[
ds = \frac{N^{-1}}{4}.
\]

Of these \( N \) sub-areas, there is a sub-area with its nodal point at the centroid of the plate, which concides with the origin of the reference frame OXYZ. Further, there are 4 rows of sub-areas with nodal points on the lines \( x = 0, \ y = 0, \ x = y \) and \( x = -y \) respectively. This pattern of sub-division helps us to obtain the density and the potential distribution along these lines directly from (112) and (118) respectively.

According to (112), there are \( N \) linear algebraic equations for \( \lambda \). By symmetry, the number of equations reduces to

\[
N^* = 1 + 2 + \cdots + (k+1)/2
\]

\[
= \frac{k+1}{2} \left( \frac{k+1}{2} + 1 \right)/2
\]

\[
= (k+1)(k+3)/8 .
\]

(120)

In this particular case, the sub-areas are all squares. The elements \( a_{kj} \) of (112) are evaluated as in Chapter 5. The diagonal element \( a_{kk} \), by Appendix I, is

\[
a_{kk} = 4h \log (1 + \sqrt{2}),
\]

where \( h \) denotes the edge length of the square sub-area.

Starting with a small value of \( N \), equations (112) are constructed and solved for \( \lambda \) by the Gauss-Seidel iterative method with \( \epsilon = 0.0001 \), with the help of the I.C.L. 1905 computer at the City University.
FIG. 8

SUB-AREAS ON A SQUARE PLATE
From the \( \lambda \) so computed, we calculate the capacity \( K \) from (114) using (119).

The values of the electrostatic capacity of a thin square conductor of unit area for increasing values of \( N \) are given in Table 7. It is evident from this Table that, when \( N = 361 \), the density distribution at some points becomes negative. This marks the optimum stage in the numerical procedures. At this stage \( K = 0.36188 \) and it is attained for \( N = 289 \) as discussed in Chapter 8.

If 'a' represents the edge length of the thin square plate, then according to Polya and Szego, the capacity lies between the bounds

\[
0.35917a < K < 0.37570a.
\]  

It will be seen that our computed value lies well within the bounds given in (121).

The figures 8(a) and 8(b) show the density distribution along the lines \( x = 0 \) and \( x = y \) respectively. This is a minimum at the centre and it increases gradually as we go towards the rim in any direction. This behaviour compares with the known density behaviour for the circular plate as we move from the centre towards its rim (Chapter 10).

To examine the accuracy attained in generating \( V \) on \( \mathcal{O}B \), \( V \) has been calculated by (118) for \( N = 289 \), taking \( p \) as the corner points of the sub-areas. The \( \lambda \) used in (118) were obtained from (112) for the same value of \( N \) i.e. \( N = 289 \). Table 7(a) shows the generated values of \( V \) at the corner points of the sub-areas along the diagonal of the square.

It is evident from Table 7(a) and from figure 8(c) that \( |V - 1| \) is minimum near the centre of the plate and gradually increases as we move towards the rim. It is maximum, as expected, at a corner of the plate.

**Rectangular Plate**

Let the unit rectangular plate \( ABCD \) (Fig. 9) be in the plane \( z = 0 \). The boundaries of \( ABCD \) are given by \( x = \pm 2a \), \( y = \pm a \). The breadth \( AB \) is divided into \( k \) parts by \((k - 1)\) lines drawn parallel to \( BC \) and the length \( BC \) is divided into \( 2k \) parts by drawing \((2k - 1)\) lines parallel to \( AB \). Hence the rectangular area \( ABCD \) is divided into

\[
N = 2k^2.
\]  

equal square sub-areas.
# ELECTROSTATIC CAPACITY OF A THIN SQUARE CONDUCTOR

<table>
<thead>
<tr>
<th>SUB-AREA</th>
<th>EQUATION</th>
<th>DENSITY AT THE CENTRE OF SUB-AREAS</th>
<th>CAPACITY</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>(a)</td>
<td>(b)</td>
</tr>
<tr>
<td>25</td>
<td>6</td>
<td>0.18792</td>
<td>0.37434</td>
</tr>
<tr>
<td>49</td>
<td>10</td>
<td>0.18515</td>
<td>0.43358</td>
</tr>
<tr>
<td>81</td>
<td>15</td>
<td>0.18369</td>
<td>0.48609</td>
</tr>
<tr>
<td>121</td>
<td>21</td>
<td>0.18269</td>
<td>0.53336</td>
</tr>
<tr>
<td>169</td>
<td>28</td>
<td>0.18198</td>
<td>0.57679</td>
</tr>
<tr>
<td>225</td>
<td>36</td>
<td>0.18144</td>
<td>0.61724</td>
</tr>
<tr>
<td>289</td>
<td>45</td>
<td>0.18093</td>
<td>0.65537</td>
</tr>
<tr>
<td>361</td>
<td>55</td>
<td>0.18029</td>
<td>0.69369</td>
</tr>
</tbody>
</table>

Table 7

(This should be read in conjunction with Fig. 8)

Some values become negative.
Density distribution along a central line of

EQNS = 45

\[ \lambda_0 = 0.18093 \]
Density distribution along a diagonal OA.
POTENTIALS GENERATED AT THE CORNER POINTS OF THE SUB-AREAS LYING ALONG A DIAGONAL OF THE THIN SQUARE PLATE

| CO-ORD. OF THE CORNER POINTS | X    | Y    | V    | | V-1 |
|------------------------------|------|------|------|------|
| 0.02941                      | 0.02941 | 1.00170 | 0.00170 |
| 0.08823                      | 0.08823 | 1.00180 | 0.00180 |
| 0.14706                      | 0.14706 | 1.00180 | 0.00180 |
| 0.20588                      | 0.20588 | 1.00200 | 0.00200 |
| 0.26471                      | 0.26471 | 1.00240 | 0.00240 |
| 0.32353                      | 0.32353 | 1.00310 | 0.00310 |
| 0.38235                      | 0.38235 | 1.00420 | 0.00420 |
| 0.44118                      | 0.44118 | 1.003450 | 0.003450 |
| 0.50000                      | 0.50000 | 0.76845 | 0.23155 |

Table 7(a)
VARIATION OF COMPUTED POTENTIAL ALONG
A DIAGONAL OD.
SUB-AREAS ON A RECTANGULAR PLATE
Proceeding the same way as in the case of a square plate, we form the N equations

\[ \sum_{j=1}^{N} Q_{kj} \lambda_j = 1 \quad ; \quad k = 1, 2, \ldots, N . \]  

(123)

The N equations (123), from symmetry, reduce to

\[ N^k \left( \frac{N}{4} = \frac{k^2}{2} \right) \]  

(124)

The equations are then solved by the Gauss-Seidel iterative method with \( \epsilon = 0.0001 \) and the \( k \) is computed as before by (114). Table 8 exhibits the value of \( k \) as \( N \) increases.

In this case, \( k \) gradually increases from 0.35938 to 0.37431 as \( N \) increases from 32 to 1800. No ill conditioning was noticed in this range of \( N \) but the machine capacity forced us to stop at \( N = 1800 \). For the unit rectangular plate with edge ratio 1:2, we find \( k = 0.37431 \).

2. Isosceles Triangular Plate

Let a thin isosceles triangular conductor ABC (Fig. 10) have its centroid at the origin of a reference frame OXYZ and it lies in the plane \( z = 0 \). Its boundaries are given by

\[ \begin{align*}
  x &= d \\
  y &= x \tan \theta + 2d \tan \theta \\
  y &= -x \tan \theta - 2d \tan \theta,
\end{align*} \]

where the meridian \( AD = 3d \), and \( \theta \) is the angle made by AC with the axis of \( x \).

The plate is divided into

\[ N = k^2 \]  

(125)

equal triangular sub-areas [Fig. 10(a)] by drawing 3 sets of \( (k-1) \)
equidistant parallel lines, parallel to the sides of the triangle, and \( k \) in (125) is given by

\[ k = 1 + (j-1)3 \quad ; \quad j = 2, 3, \ldots, n . \]  

(126)
ELECTROSTATIC CAPACITY OF A THIN RECTANGULAR PLATE
(EDGE RATIO 2:1)

<table>
<thead>
<tr>
<th>SUB-Area N</th>
<th>EQUATION N*</th>
<th>Density at the Points</th>
<th>Capacity</th>
</tr>
</thead>
<tbody>
<tr>
<td>32</td>
<td>8</td>
<td>0.20076</td>
<td>0.62369</td>
</tr>
<tr>
<td>128</td>
<td>32</td>
<td>0.19321</td>
<td>0.99326</td>
</tr>
<tr>
<td>288</td>
<td>72</td>
<td>0.19049</td>
<td>1.31218</td>
</tr>
<tr>
<td>648</td>
<td>162</td>
<td>0.18872</td>
<td>1.73780</td>
</tr>
<tr>
<td>968</td>
<td>242</td>
<td>0.18817</td>
<td>1.99826</td>
</tr>
<tr>
<td>1352</td>
<td>338</td>
<td>0.18780</td>
<td>2.24514</td>
</tr>
<tr>
<td>1800</td>
<td>450</td>
<td>0.18758</td>
<td>2.48107</td>
</tr>
</tbody>
</table>

Table 8
(This should be read in conjunction with Fig. 9)
This pattern of sub-divisions gives us 3 rows of sub-areas whose nodal points lie on the 3 meridians of the triangle along with a sub-area whose nodal point lies at the centroid of the triangle. Further the sub-areas thus formed are all equal in size and in form [Fig. 10(a)].

From symmetry, the number of independent equations reduces to

$$N^* = \frac{1 + (j-1)^3}{2} \left( 2 + (j-1)^3 \right); \quad j = 2, 3 \cdots n. \quad (127)$$

For the equilateral triangular plate

$$N^* = \frac{j}{2} \left( 2 + (j-1)^3 \right); \quad j = 2, 3 \cdots n. \quad (128)$$

The co-efficients $a_{kj}$ of (112) are computed over the sub-areas, as before, by the centroid method when $j \neq k$. When $j = k$, by Appendix I:

$$a_{kk} = \sum_{j=1}^{m} \frac{2A_j}{Q_j} \log \left( \frac{L_j + L_{j+1} + a_j}{L_j + L_{j+1} - a_j} \right), \quad (129)$$

where $A_j$ is the area of the triangle formed by the sides $L_j, L_{j+1}$ and $a_j$ [Fig. 10(b)]. When $j = m$, in (129) $j + 1$ should be replaced by 1 instead of $m + 1$; $m$ denotes the number of sides of the polygon.

**Equilateral Plate of unit area**

For an equilateral plate of unit area $\theta = 30^\circ$, and a side $BC$ is given by

$$\frac{1}{2} BC \cdot AC \sin 60^\circ = 1$$

or

$$\frac{1}{2} BC \cdot BC \sin 60^\circ = 1$$

or

$$BC = \left( \frac{2}{\sin 60^\circ} \right)^{1/2}.$$

The meridian $AD$ is given by

$$AD = 3d = BC \cos 30^\circ.$$
After evaluation of the co-efficients $a_{ij}$ of (112) over the triangular sub-areas, the $N^*$ equations, where $N^*$ is given by (128), are then solved by the Gauss-Seidel iterative method with $\epsilon = 0.0001$. The $\lambda_j$ thus obtained are used in (114) to evaluate the capacity of the plate.

The values of the capacity of the thin equilateral triangular conductor of unit area for increasing values of $N$ are given in Table 9. It is evident from the Table that the capacity of the plate is $K = 0.38308$, and this value is attained when $N = 361$. The density $\lambda$ distributed along a median, for the above value of $N$, is given in the Fig. 10(c).

For $N = 361$, the potential $V$ is calculated at the corner points of the sub-areas along a median of the plate. The reference frame is taken as in Fig. 10. The values of $\left| V - 1 \right|$ thus computed are exhibited in Table 9 (a). It is evident from this Table that the value of $\left| V - 1 \right|$ is the lowest when $p$ is near the centroid of the plate and gradually increases as we move towards the periphery. The maximum value of it, as expected, lies at an apex of the plate.

**Right angled isosceles triangular plate of Unit Area**

A thin isosceles triangular plate of unit area with base angles $45^\circ$ each is divided into $N$ sub-areas by the procedure stated before. In this case $\theta = 45^\circ$ and hence

$$AD = AB \sin 45^\circ$$

and $BC$. $AD = 1$ i.e. $\frac{1}{2} (2 AB \cos 45^\circ) (AB \sin 45^\circ) = 1$.

Hence $AB = \sqrt{2}$ and $BC = 2$.

As in the former case, the $N$ equations (112) are constructed. From symmetry, the $N$ equations reduce to $N^*$ equations where $N^*$ is given by (127). The equations are then solved, as before, by the Gauss-Seidel iterative method and then $K$ is computed by (114). The values of $K$ for a range of values of $N$ are given in Table 10.

In this case $K$ gradually increases from $0.36174$ to $0.40025$ as $N$ increases from 16 to 361. No ill conditioning was noticed in this range of $N$ but the machine capacity forced us to stop at $N = 361$. The value of the capacity attained at this stage is found to be $K = 0.40025$.

Following the same procedure, the capacity of an isosceles triangular plate ($120^\circ, 30^\circ, 30^\circ$) of unit area is computed for increasing value of $N$, and are exhibited in Table 11. The capacity of the plate, from Table 11, is $K = 0.41011$. 
## ELECTRO-STATIC CAPACITY OF A THIN EQUILATERAL PLATE

<table>
<thead>
<tr>
<th>i</th>
<th>SUB-AREA N</th>
<th>EQUATION N*</th>
<th>DENSITY AT CENTROID</th>
<th>CAPACITY</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>16</td>
<td>5</td>
<td>0.18340</td>
<td>0.35361</td>
</tr>
<tr>
<td>3</td>
<td>49</td>
<td>12</td>
<td>0.18610</td>
<td>0.36527</td>
</tr>
<tr>
<td>4</td>
<td>100</td>
<td>22</td>
<td>0.18397</td>
<td>0.37010</td>
</tr>
<tr>
<td>5</td>
<td>109</td>
<td>35</td>
<td>0.18277</td>
<td>0.37273</td>
</tr>
<tr>
<td>6</td>
<td>256</td>
<td>51</td>
<td>0.18200</td>
<td>0.37438</td>
</tr>
<tr>
<td>7</td>
<td>361</td>
<td>70</td>
<td>0.18136</td>
<td>0.38308</td>
</tr>
<tr>
<td>8</td>
<td>484</td>
<td>92</td>
<td>0.18086</td>
<td>0.38139</td>
</tr>
</tbody>
</table>

Table 9
DENSITY DISTRIBUTION ALONG A MEDIAN OC
POTENTIALS GENERATED AT THE CORNER POINTS OF SUB-AREAS LYING ALONG A MEDIAN OF THE THIN EQUILATERAL TRIANGULAR PLATE

| CO-ORD. OF THE CORNER POINTS | X    | Y    | V    | |V - 1| |
|-----------------------------|------|------|------|------|------|
|                             | 0.02309 | 0.03999 | 1.00131 | 0.00131 |
|                             | 0.09236 | 0.15997 | 1.00142 | 0.00142 |
|                             | 0.16162 | 0.27994 | 1.00165 | 0.00165 |
|                             | 0.23089 | 0.39991 | 1.00215 | 0.00215 |
|                             | 0.30016 | 0.51989 | 1.00346 | 0.00346 |
|                             | 0.36942 | 0.63986 | 1.02819 | 0.02819 |
|                             | 0.43869 | 0.75984 | 0.70133 | 0.29867 |

Table 9 (a)
ELECTROSTATIC CAPACITY OF A RIGHT ANGLED ISOSCELES TRIANGULAR PLATE OF UNIT AREA

<table>
<thead>
<tr>
<th>i</th>
<th>SUB-AREA N</th>
<th>EQUATION N*</th>
<th>DENSITY AT CENTROID</th>
<th>CAPACITY</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>16</td>
<td>10</td>
<td>0.1845</td>
<td>0.36174</td>
</tr>
<tr>
<td>3</td>
<td>49</td>
<td>28</td>
<td>0.1890</td>
<td>0.37372</td>
</tr>
<tr>
<td>4</td>
<td>100</td>
<td>55</td>
<td>0.1869</td>
<td>0.37866</td>
</tr>
<tr>
<td>5</td>
<td>169</td>
<td>91</td>
<td>0.1859</td>
<td>0.38135</td>
</tr>
<tr>
<td>6</td>
<td>256</td>
<td>136</td>
<td>0.1850</td>
<td>0.38303</td>
</tr>
<tr>
<td>7</td>
<td>361</td>
<td>190</td>
<td>0.1849</td>
<td>0.40025</td>
</tr>
</tbody>
</table>

Table 10
ELECTROSTATIC CAPACITY OF AN ISOSCELES (120°, 30°, 30°) TRIANGULAR PLATE OF UNIT AREA

<table>
<thead>
<tr>
<th>i</th>
<th>SUB-AREA N</th>
<th>EQUATION N*</th>
<th>DENSITY AT CENTROID</th>
<th>CAPACITY</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>16</td>
<td>10</td>
<td>0.1800</td>
<td>0.38498</td>
</tr>
<tr>
<td>3</td>
<td>49</td>
<td>28</td>
<td>0.1978</td>
<td>0.39829</td>
</tr>
<tr>
<td>4</td>
<td>100</td>
<td>55</td>
<td>0.1932</td>
<td>0.40384</td>
</tr>
<tr>
<td>5</td>
<td>169</td>
<td>91</td>
<td>0.1934</td>
<td>0.40688</td>
</tr>
<tr>
<td>6</td>
<td>256</td>
<td>136</td>
<td>0.1930</td>
<td>0.40879</td>
</tr>
<tr>
<td>7</td>
<td>361</td>
<td>190</td>
<td>0.1930</td>
<td>0.41011</td>
</tr>
</tbody>
</table>

Table 11
General Conclusions

It is interesting to note how $K$ varies for different shaped triangles of unit area. This is exhibited in Table 12. From this, we see that $K$ decreases as the symmetry increases, reaching its minimum for the equilateral plate. Table 12(a) exhibits the capacity of unit plates of different shape. It appears from Table 12(a) that, for regular polygons of unit area, $K$ decreases as the number of sides increases, reaching its minimum for a circular plate (Chapter 10).
### Table 12

<table>
<thead>
<tr>
<th>ANGLES OF THE PLATE IN DEGREES</th>
<th>CAPACITY</th>
</tr>
</thead>
<tbody>
<tr>
<td>60 60 60</td>
<td>0.38139</td>
</tr>
<tr>
<td>90 45 45</td>
<td>0.40025</td>
</tr>
<tr>
<td>120 30 30</td>
<td>0.41011</td>
</tr>
</tbody>
</table>

### Table 12(a)

<table>
<thead>
<tr>
<th>PLATE</th>
<th>CAPACITY</th>
</tr>
</thead>
<tbody>
<tr>
<td>EQUILATERAL TRIANGULAR PLATE</td>
<td>0.38139</td>
</tr>
<tr>
<td>SQUARE</td>
<td>0.36188</td>
</tr>
<tr>
<td>CIRCULAR</td>
<td>0.35917</td>
</tr>
</tbody>
</table>
CHAPTER 10
CAPACITY OF A THIN CIRCULAR DISC

Analytical Solution

Let $V$ be the potential due to an electrified flat circular disc of unit radius. The centre of the disc defines the origin of a system of cylindrical polar co-ordinates, of which the $Z$-axis lies perpendicular to the plane of the disc. In cylindrical co-ordinates, $V$ satisfies

$$\nabla^2 V = \frac{1}{r^2} \frac{\partial^2 V}{\partial r^2} + \frac{1}{r} \frac{\partial V}{\partial r} + \frac{\partial^2 V}{\partial Z^2} = 0,$$

(130)

with boundary conditions, for $z = 0$ (i.e. plane of the disc)

$$V = 1 \quad , \quad 0 \leq r < 1 \quad \text{(131)}$$

$$\frac{\partial V}{\partial Z} = 0 \quad , \quad r > 1 \quad \text{(132)}$$

The 2nd condition (132) comes from the symmetry of $V$ across $z = 0$ and absence of charges outside the disc.

The solution of (130) under the above conditions, according to Tranter, is

$$V = \frac{2}{\pi} \int_0^\infty \frac{e^{-kt}}{r} j_0(r \nu) \sin \nu \, d\nu \quad \text{(133)}$$

For $r < 1$

$$\left( \frac{\partial V}{\partial Z} \right)_{z=0} = \frac{2}{\pi} \int_0^\infty \left( j_0(r \nu) \sin \nu \right) \, d\nu \quad ,$$

which is the imaginary component of
\[
I = \frac{2}{\pi} \int_{0}^{\infty} I_0 (r \mu) e^{-\mu r} d\mu .
\]  
\hspace{1cm} (134)

From Watson,\(^8\)
\[
I = \frac{2}{\pi} \frac{1}{\sqrt{(\nu^2 + r^2)}} = \frac{2}{\pi} \frac{-\nu}{\sqrt{1 - r^2}} ,
\]
\hspace{1cm} (135)

\[
\left( \frac{\partial V}{\partial Z} \right)_{Z=0} = -\frac{2}{\pi} \frac{1}{\sqrt{1 - r^2}} .
\]

By relation (6) of Chapter 1, and from symmetry,
\[
-4\pi \lambda = \left[ \left( \frac{\partial V}{\partial Z} \right)_{\text{int.}} + \left( \frac{\partial V}{\partial Z} \right)_{\text{ext.}} \right]_{Z=0} = 2 \left( \frac{\partial V}{\partial Z} \right)_{Z=0} = -\frac{4}{\pi} \frac{1}{\sqrt{1 - r^2}} .
\]

Therefore
\[
\lambda = \frac{1}{\pi^2} \frac{1}{\sqrt{1 - r^2}} .
\]  
\hspace{1cm} (136)

Where \( \lambda \) is the density at a radial distance \( r \) from the centre. For a disc of radius \( a \), at a distance \( r \) from the centre
\[
\lambda = \frac{1}{\pi a} \frac{1}{\sqrt{1 - (r/a)^2}} .
\]

The capacity \( \kappa \) of the disc of radius \( a \) is
\[
\kappa = \int_{\theta=0}^{2\pi} \int_{r=0}^{a} \lambda (\theta) d\theta \, dr = \int_{\theta=0}^{2\pi} \int_{r=0}^{a} \frac{1}{\pi a} \frac{1}{\sqrt{1 - (r/a)^2}} r \, d\theta \, dr .
\]
\[ V = \frac{2\pi}{\lambda^2} \int_{r=0}^{a} r \frac{dr}{\sqrt{1 - \left(\frac{r}{a}\right)^2}} = a \left(\frac{2}{\lambda}\right) \]

\[ \approx 0.6366203, \quad \text{when} \quad a = 1. \quad (137) \]

**Numerical Approach**

Apart from the analytical approach, already discussed, the integral equation formulae provide a straightforward numerical approach to solve the problem numerically. Let a density distribution \( \lambda \) on \( \partial B \) (thin circular plate of unit radius) generate the potential \( V \) which satisfies the equation (130) i.e.

\[ \nabla^2 V = 0 \]

with boundary conditions (131) and (132) i.e.

\[ V(k) = 1 ; \quad k \in \partial B \]

and on \( Z = 0 \),

\[ V'(k) = 0 ; \quad |k| > \alpha \]

respectively. In the integral equation method the boundary condition (131) is sufficient to solve the problem and hence the condition (132) is redundant in this case. This is essentially because our formulation is a Dirichlet formulation, which confines us to \( \partial B \), whereas Tranter's formulation is a mixed formulation for which we must go outside \( \partial B \).

Since \( V = 1 \) on \( \partial B \), \( \lambda \) satisfies the equation (111) which, on discretisation, takes the form (112) i.e.

\[ \sum_{j=1}^{N} a_{kj} \lambda_j = 1 ; \quad k = 1, 2, \ldots, N. \]

**Division of a circular domain into sub-areas**

To find a numerical solution of (112), it is necessary to divide \( \partial B \) into sub-areas. To effect the sub-division the circular domain is divided into \( n \) annular rings and each ring in its turn is divided into \( M \) sub-areas except the inner most ring which is divided into \( M_1 \) sub-areas (Fig. 11). For sub-areas of equal area we have in a \( j \)th ring...
Putting \( j = 2, 3, \ldots, n \) in succession and adding them up, we obtain

\[
\mathbf{r}_n^2 = \left( (n-1) \left( \frac{M}{M_1} \right) + 1 \right) \mathbf{r}_1^2.
\]

Since \( r_n = a \) (=1, the radius of the disc),

\[
\mathbf{r}_1^2 = a^2 \left[ \frac{1}{(n-1) \left( \frac{M}{M_1} \right) + 1} \right].
\]

Using this value of \( \mathbf{r}_1 \) in (138) and putting \( j = 2, 3, \ldots, n \) in succession, we obtain \( n \) annular rings on the circular domain. The value of \( M_1 \) usually equals 6 to obtain the sub-areas, nearly of equilateral form, in the inner-most ring and, \( M \) is determined by

\[
M = 6 \left( \frac{2^{k-1}}{2^k} \right); \quad k \leq n.
\]  

\( n \) may have any value but to keep the sub-areas in the outermost ring near to square form, we choose \( n \) such that it approximately satisfies

\[
\frac{2\pi r_n}{M} = \frac{r_n}{n} \approx \frac{6.28318}{6 \left( \frac{2^{k-1}}{2^k} \right)} \approx \frac{1}{n}.
\]

Hence for a particular choice of \( k \), a choice for \( n \), from above, is given by

\[
n = 2^{k-1}
\]  

(140)

This sub-division gives very thin trapezoidal sub-areas in a few of the inner rings, which are not suitable for numerical work. To eliminate the thin sub-areas, the width of the \( m \)th ring is diminished by the adjustment given by

\[
\mathbf{T}_m = \nu \left( T_m^1 \right), \quad m \leq 1
\]
where $T_m^1$ represents the width of the $m$th ring in (138); $m'$ usually equals 4. The width of the rings inner to $m$th ring are then determined by

$$T_j = \nu (T_{j+1}) ; \quad j = \langle m-1, m-2 \rangle \ldots 2 .$$

The validity of (141) depends on the features of the inner rings

$$T_{j}^1 > T_{j+1}^1 \quad \text{and} \quad 0 < \nu < 1 .$$

Now the radii of the above $m - 1$ concentric rings are given by

$$r_j = r_m - \sum_{k=j}^{m} T_k + T_j ; \quad j = 2, 3 \ldots m .$$

(142)

In the circular area of radius $r_2$, given by (142), $k$ annular rings are introduced where $k$ is given by (139). Each of the $k$ rings has $M_j$ equal sub-areas, where

$$M_j = G(j - 1) ; \quad j = 1, 2, 3 \ldots k .$$

(143)

Of these $k$ rings, if $r_0$ be the radius of the first circle, then $r_1$ is given by

$$r_1 = \left( \frac{2\nu r_2}{M_1} \right) q , \quad 0 < q < 1 ,$$

where $r_2$ is given by (142). After determining $r_1$ with a starting value $q = 0.9$, the radii of the remaining $k - 1$ circles are fixed by

$$r_j = r_{j-1} + \left( \frac{2\nu r_{j-1}}{M_j} \right) q ; \quad j = 2, 3 \ldots k ,$$

(144)

where $r_1$ is usually set at 1.5. Now the annular gap, given by

$$D = r_2 - r_k .$$
is divided into \( J \) parts to give \( J \) annular rings such that

\[
\left| T_3 - \left( r_L - r_{R+J-1} \right) \right| \leq \epsilon,
\]

(145)

where \( \epsilon \) is a pre-assigned small positive quantity (usually \(< 0.001\)) and \( J \) is a positive whole number given by the integral part of \( Q \) where,

\[
Q = \frac{D}{\left( \frac{2\pi r_R}{M_R} \right)^{-1}}.
\]

(146)

The radii of these \( J \) concentric circles are given by

\[
P_{k+1} = P_{k+L-1} + SXT + (l-1)U; \ l = 1, 2, \cdots J,
\]

where

\[
SXT = \left( \frac{2\pi P_R}{M} \right) \frac{1}{l}
\]

and

\[
U = \left[ \frac{1}{2} - \left( P_R + (SXT)J \right) \right]^{\frac{1}{2}}.
\]

If \( Q \leq 0 \), \( q \) is gradually made smaller until \( Q > 0 \) and the adjustment is stopped at the stage when (145) is satisfied. At this stage \( P_{k+J} \) is readjusted by setting

\[
P_{k+J} = r_L
\]

(147)

Now the total number of annular rings on the circular face becomes

\[
N^* = (r-2) + R + J,
\]

(148)

and the radii of the concentric circles are given by

\[
R_j = P_j; \ j = 1, 2, \cdots (R+J),
\]

\[
R_{k+J} = r_L; \ l = 3, 4, \cdots m.
\]
The sub-areas in the 1st ring are quadrilaterals with shapes very near to that of an equilateral triangle. From the 2nd up to the \((k-1)\)th ring, the sub-areas are pentagons [Fig. 11(a)] in which slant side BE of a sub-area is \(p_1\) times the side BD. From the \(k\)th up to the \((N-2)\)th ring, the sub-areas are trapezoidal in form. To make sub-areas smaller in size as we approach the rim, the number of sub-areas are doubled in the \(N\)th ring by inserting radial line segments through the middle of each sub-area. The sub-areas in the \((N-1)\)th and in the \(N\)th ring are then made pentagonal in form [Fig. 11(a)].

The total number of sub-areas on the circular plate is

\[
N = \sum_{j=1}^{2} (N-k+1) + 2 \times 6 \times 2^{k-1}.
\]

Formulation and solution of equations

For the \(N\) sub-areas, there are \(N\) algebraic equations in \(N\) unknown \(\lambda_j\) given by (112). The coefficients \(a_{kj}\) of (112), are evaluated over the sub-areas, as before, by the centroid method when \(j \neq k\) and analytically when \(j = k\).

From symmetry, the \(N\) equations reduce to \(N\) equations where \(N\) is given by (148). The equations are then solved, as before by the Gauss-Seidel iterative method with \(\epsilon = 0.0001\).

Table 13 exhibits the value of the capacity of a thin plate of unit radius with increasing value of \(N\). It is evident from the Table that \(N = 0.6351872\). The analytic value of \(N\), by (137), is

\[
\lambda = \frac{2}{\kappa} 
\approx 0.6366203.
\]

Table 14 exhibits the density distribution along a radius compared with that obtained analytically by (136). The numerical \(\lambda\) deviates only slightly from the analytical \(\lambda\) except in the neighbourhood of the rim. This behaviour of \(\lambda\) in the neighbourhood of the rim supports the conclusions drawn in Chapter 6. Fig. 12(a) gives the density profile based upon Table 14.
Fig. 11

SUB-AREAS ON A CIRCULAR DOMAIN

Fig. 11(a)

A SECTION OF FIG. 11.
ELECTROSTATIC CAPACITY OF A THIN CIRCULAR PLATE COMPARED WITH ANALYTICAL VALUE $k = 0.6366$

<table>
<thead>
<tr>
<th>SUB-AREA N</th>
<th>EQUATION N*</th>
<th>NUMERICAL $k$</th>
</tr>
</thead>
<tbody>
<tr>
<td>162</td>
<td>7</td>
<td>0.6239460</td>
</tr>
<tr>
<td>522</td>
<td>12</td>
<td>0.6314764</td>
</tr>
<tr>
<td>2202</td>
<td>25</td>
<td>0.634633</td>
</tr>
<tr>
<td>2682</td>
<td>30</td>
<td>0.6351872</td>
</tr>
<tr>
<td>3162</td>
<td>35</td>
<td>0.6351505</td>
</tr>
</tbody>
</table>

Table 13

DENSITY DISTRIBUTION ON A CIRCULAR PLATE ALONG A RADIAL LINE

<table>
<thead>
<tr>
<th>RADIAL DIST FROM CENTRE $r$</th>
<th>ANALYTICAL $\lambda$</th>
<th>NUMERICAL $\lambda$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.02797</td>
<td>0.10136</td>
<td>0.10306</td>
</tr>
<tr>
<td>0.12294</td>
<td>0.10209</td>
<td>0.10194</td>
</tr>
<tr>
<td>0.19271</td>
<td>0.10326</td>
<td>0.10324</td>
</tr>
<tr>
<td>0.25042</td>
<td>0.10466</td>
<td>0.10456</td>
</tr>
<tr>
<td>0.32629</td>
<td>0.10719</td>
<td>0.10689</td>
</tr>
<tr>
<td>0.41929</td>
<td>0.11161</td>
<td>0.10647</td>
</tr>
<tr>
<td>0.52502</td>
<td>0.11905</td>
<td>0.12033</td>
</tr>
<tr>
<td>0.61273</td>
<td>0.12821</td>
<td>0.12982</td>
</tr>
<tr>
<td>0.72462</td>
<td>0.14702</td>
<td>0.14901</td>
</tr>
<tr>
<td>0.82141</td>
<td>0.17765</td>
<td>0.18016</td>
</tr>
<tr>
<td>0.93499</td>
<td>0.28567</td>
<td>0.27671</td>
</tr>
<tr>
<td>0.98732</td>
<td>0.63816</td>
<td>0.87936</td>
</tr>
</tbody>
</table>

Table 14
Density distribution along a radial line on the circular face of thick discs of unit radius.

Fig. 12

Thickness of the discs:
1) = 0.00
2) = 0.48
3) = 1.00
CHAPTER 11  
CAPACITY OF THICK CIRCULAR DISCS

Introduction

A thick circular disc (Fig. 13) may be viewed as a right circular cylinder with a small ratio H/a, where H defines the height and a defines the radius of the cylinder. Taking the origin of cylindrical polar co-ordinates at the centroid of the cylinder and the Z-axis to coincide with the axis of the cylinder, the plane boundaries at the ends are

\[ Z = \pm \frac{H}{2}, \]

the curved cylindrical boundary is

\[ r = a. \]

If \( V \) be the potential due to an equilibrium charge distribution on \( \partial B \), \( V \) satisfies Laplace's equation

\[ \nabla^2 V = 0 \]

with boundary conditions (131) and (132) i.e.

\[ V(\mathbf{r}) = 1; \quad \mathbf{r} \in \partial B \]

and on \( Z = 0 \),

\[ \nabla V = 0; \quad |\mathbf{r}| > a. \]

Because of the form of \( \partial B \), complications arise in solving the problem analytically. However, the integral equation formulation provides a straightforward numerical approach. In the integral equation method, the boundary condition (131) i.e.

\[ V(\mathbf{r}) = 1; \quad \mathbf{r} \in \partial B, \]

is sufficient to solve the problem and hence, as in the case of a thin plate, the boundary condition (132) is redundant. If the density distribution \( \chi \) generates the potential \( V = 1 \) on \( \partial B \), then \( \chi \) satisfies (111) which, on discretisation, takes the form (112).

Division of the surface into sub-areas

Each of the plane circular faces is divided into sub-areas as the thin plate in the previous case. Hence if \( N_k \) be the number of annular rings and \( N_1 \) be the total number of sub-areas on a plane face, by (148) and (149),
Fig. 13
A THICK CIRCULAR DISC

Fig. 13(a)
SUB-AREAS ON THE CYLINDRICAL SURFACE
If we now insert $N^*_2$ annular rings in the upper half of the cylindrical surface, then $N^*_2$ is given by

$$N^*_2 = \left\lfloor \frac{H}{2} \left( \frac{2\pi a}{4M} \right) \right\rfloor - 1 \tag{150}$$

where $L^*$ is the integral part of

$$\left\lfloor \frac{H}{2} \left( \frac{2\pi a}{4M} \right) \right\rfloor$$

If $h_1$ be the width of the ring nearest to the edge, then

$$h_1 = \left( \frac{2\pi a}{4M} \right) = h \ (\text{say}) \ . \tag{151}$$

Further, if

$$U_j = \left\{ \frac{H}{2} - \left( \frac{2\pi a}{4M} \right) N^*_2 \right\} \div \left( \frac{N^*_2 (N^*_2 - 1)}{2} \right) \ ,$$

the widths of the subsequent rings, as we move towards the plane $z = 0$, are given by

$$h_j = h + (j - 1) U_j \ ; \ j = 1, 2, \ldots, N^*_2 \ ,$$

when the breadth of each sub-area is kept constant at $h_1$ given by (151).

Each of the $N^*_2$ rings contains $4M$ sub-areas, and hence the total number of sub-areas on the upper half of the cylindrical surface is

$$N_2 = 4 \left( M \right) N^*_2 \ . \tag{152}$$
The total number of annular rings on the surface is

\[ 2N^* = 2 \left( N_1^* + N_2^* \right) \]  \hfill (153)

and the total number of sub-areas is

\[ N = 2 \left( N_1 + N_2 \right) , \]

\[ = 2 \left[ \sum_{j=1}^{k} 6 \left( \frac{j-1}{2^{j-1}} \right) + \left( N_1^* - 1 \right) k \left( 2^{k-1} \right) \right] 6 + 2^{k-1} \left( 12 + 24 N_2^* \right) \]  \hfill (154)

For \( H = 0.18 \) and \( k = 3 \), it is found that \( N^*_2 = 2 \), \( n = 4 \), and \( J = 2 \). Hence \( N^*_1 = 7 \) and, by (153) and (154),

\[ N^* = 9 \]  and \( N = 708 \).

The analysis of the sub-areas in each of the annular rings, for the above values of \( H \) and \( k \), is given in Table 15.

Formulation and solution of the equations

For the \( N \) sub-areas, there are \( N \) algebraic equations in \( N \) unknowns given by (112). The co-efficients \( a_{kj} \) of (112), are evaluated over the sub-areas, as before, by the centroid method when \( j \neq k \) and analytically when \( j = k \).

From symmetry, the \( N \) equations reduce to \( N^* \) independent equations where \( N^* \) is given by (153). The equations are then solved, as before, by the Gauss-Seidel iterative method with \( \varepsilon = 0.0001 \).

Table 16 exhibits the value of \( k \) of a thick plate of unit radius and thickness \( H = 0.18 \) with increasing value of \( N \).
### SUB-AREAS ON THE SURFACE OF A THICK DISC (THICKNESS 0.48) OF UNIT RADIUS

<table>
<thead>
<tr>
<th>NODAL POINTS</th>
<th>ARMS OF SUB-AREAS</th>
<th>AREA</th>
</tr>
</thead>
<tbody>
<tr>
<td>RADIAL DIST</td>
<td>BD</td>
<td>GE</td>
</tr>
<tr>
<td>r</td>
<td>z</td>
<td></td>
</tr>
<tr>
<td>ON THE PLANE CIRCULAR SURFACE</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.10066</td>
<td>0.2400</td>
<td>0.00</td>
</tr>
<tr>
<td>0.22636</td>
<td>0.2400</td>
<td>0.08377</td>
</tr>
<tr>
<td>0.34580</td>
<td>0.2400</td>
<td>0.07543</td>
</tr>
<tr>
<td>0.48519</td>
<td>0.2400</td>
<td>0.10505</td>
</tr>
<tr>
<td>0.62816</td>
<td>0.2400</td>
<td>0.14630</td>
</tr>
<tr>
<td>0.77278</td>
<td>0.2400</td>
<td>0.18312</td>
</tr>
<tr>
<td>0.92506</td>
<td>0.2400</td>
<td>0.11103</td>
</tr>
<tr>
<td>ON THE CYLINDRICAL SURFACE</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1.00</td>
<td>0.20728</td>
<td>0.06545</td>
</tr>
<tr>
<td>1.00</td>
<td>0.13455</td>
<td>0.06545</td>
</tr>
<tr>
<td>1.00</td>
<td>0.04728</td>
<td>0.06545</td>
</tr>
</tbody>
</table>

Table 15

[This should be read in conjunction with Fig. 11(a) and Fig. 13(a)]
Following the same procedure, the capacity of circular plates of unit radius with various thickness are evaluated and are given in Table 17.

Fitting of a polynomial through the capacity values

Our numerical approach gives the capacity for some discrete values of the thickness $H$. To approximate the capacity for any value of $H$ in the above range, we attempt to fit a continuous curve through the computed values of capacity utilising the method of least squares.9

It appears from the difference columns (3) and (4) of Table 17 that the smoothest interpolating function may be a log function. Considering the analytic value of $\kappa$ when the thickness is zero, we expect the form of the function to be

$$\kappa = \log \left( C_0 + \sum_{j=1}^{n} C_j H^j \right), \quad (155)$$

where $C_0 = e$, the base of natural logarithms. For the 11 values of $\kappa$ (Table 18), a polynomial of degree 10 will fit exactly through them. Starting with $m = 1$ and gradually increasing $m$ in steps of 1, it is found that, for $m = 5$, the interpolating function (155) fits the computed values to an accuracy of 3 significant figures. Further when $H \to 0$, $\kappa$ in (155) tends to $\frac{2}{\kappa}$ as required.

For $m = 5$, the co-efficients are $C_0 = e = 2.71828$, $C_1 = 2.53801$, $C_2 = -2.78274$, $C_3 = 4.63385$, $C_4 = -3.74689$ and $C_5 = 1.18925$. Fig. 14 shows the relation between the computed values and the fitted values of $\kappa$, based on Table 18, for a disc of unit radius, as thickness varies from 0 to 1.
### ELECTRO-STATIC CAPACITY OF A THICK DISC (THICKNESS 0.18) OF UNIT RADIUS

<table>
<thead>
<tr>
<th>SUB-AREA N</th>
<th>EQUATION N*</th>
<th>CAPACITY</th>
</tr>
</thead>
<tbody>
<tr>
<td>708</td>
<td>9</td>
<td>0.72143804</td>
</tr>
<tr>
<td>1812</td>
<td>14</td>
<td>0.72189708</td>
</tr>
<tr>
<td>8244</td>
<td>30</td>
<td>0.72209634</td>
</tr>
<tr>
<td>10064</td>
<td>35</td>
<td>0.72201394</td>
</tr>
</tbody>
</table>

Table 16

### ELECTRO-STATIC CAPACITY OF THICK CIRCULAR DISCS OF UNIT RADIUS AND THE DIFFERENCE COLUMNS OF CAPACITY

<table>
<thead>
<tr>
<th>THICKNESS H</th>
<th>CAPACITY K</th>
<th>1ST ORDER DIFFERENCE $\delta^1$</th>
<th>2ND ORDER DIFFERENCE $\delta^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.18</td>
<td>0.72209638</td>
<td>0.03615471</td>
<td>-0.0030993</td>
</tr>
<tr>
<td>0.28</td>
<td>0.75825109</td>
<td>0.03314478</td>
<td>-0.00204284</td>
</tr>
<tr>
<td>0.38</td>
<td>0.79139585</td>
<td>0.03110194</td>
<td>-0.00262198</td>
</tr>
<tr>
<td>0.48</td>
<td>0.82249781</td>
<td>0.02847996</td>
<td>-0.0018507</td>
</tr>
<tr>
<td>0.58</td>
<td>0.85197777</td>
<td>0.02829489</td>
<td>-0.00104826</td>
</tr>
<tr>
<td>0.68</td>
<td>0.88027266</td>
<td>0.02724663</td>
<td>-0.00091254</td>
</tr>
<tr>
<td>0.78</td>
<td>0.90751929</td>
<td>0.02633409</td>
<td>-0.00080316</td>
</tr>
<tr>
<td>0.88</td>
<td>0.93385338</td>
<td>0.02553093</td>
<td></td>
</tr>
</tbody>
</table>

Table 17
CAPACITY OF THICK CIRCULAR DISCS FROM A FITTED POLYNOMIAL

<table>
<thead>
<tr>
<th>THICKNESS H</th>
<th>COMPUTED κ</th>
<th>FITTED κ</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.00</td>
<td>0.63519</td>
<td>0.63662</td>
</tr>
<tr>
<td>0.18</td>
<td>0.72210</td>
<td>0.72197</td>
</tr>
<tr>
<td>0.28</td>
<td>0.75825</td>
<td>0.75843</td>
</tr>
<tr>
<td>0.38</td>
<td>0.79140</td>
<td>0.79141</td>
</tr>
<tr>
<td>0.48</td>
<td>0.82250</td>
<td>0.82237</td>
</tr>
<tr>
<td>0.58</td>
<td>0.85198</td>
<td>0.85196</td>
</tr>
<tr>
<td>0.68</td>
<td>0.88027</td>
<td>0.88034</td>
</tr>
<tr>
<td>0.78</td>
<td>0.90752</td>
<td>0.90757</td>
</tr>
<tr>
<td>0.88</td>
<td>0.93385</td>
<td>0.93378</td>
</tr>
<tr>
<td>0.98</td>
<td>0.95938</td>
<td>0.95941</td>
</tr>
<tr>
<td>1.00</td>
<td>0.96440</td>
<td>0.96453</td>
</tr>
</tbody>
</table>

Table 18
CAPACITY OF THICK CIRCULAR DISCS

Fig. 14
CHAPTER 12

ELECTROSTATIC CAPACITY OF A CUBE

Division of surface into sub-areas

We choose a cartesian co-ordinate system so that the six faces of the cube have the equations

\[ x = \pm \frac{a}{2}, \quad y = \pm \frac{a}{2} \quad \text{and} \quad z = \pm \frac{a}{2}. \]

As in the case of a square plate, each face of the cube is divided into \( N_1 = k^2 \) square sub-areas where \( k \) is always an odd integer. The total number of sub-areas on the surface \( \partial \Omega \) of the cube is

\[ N = 6 N_1 = 6 k^2. \quad (156) \]

Dirichlet Formulation

Let an equilibrium charge distribution \( \lambda \) on \( \partial \Omega \) generate a potential \( V = 1 \) on \( \partial \Omega \). Hence \( \lambda \) satisfies the equation (111) i.e.

\[ \int_{\partial \Omega} \frac{\lambda(\psi) \frac{\partial \psi}{\partial \psi}}{|x^2 - y^2|} = 1. \]

on discretisation, as before, equation (111) gives \( N \) linear algebraic equations for the \( N \) unknown \( \lambda_j \) viz.

\[ \sum_{j=1}^{N} \lambda_j \left( \int_{j} \frac{\frac{\partial \psi}{\partial x}}{|y^2 - y^2|} \right) = 1 \quad ; \quad k = 1, 2, 3 \ldots N, \]

which is of the form

\[ \sum_{j=1}^{N} a_{kj} \lambda_j = 1, \quad k = 1, 2, 3 \ldots N. \]

The co-efficients \( a_{kj} \) are evaluated, as before, by the centroid method of approximation when \( k \neq j \). When \( k = j \), the diagonal elements \( a_{kk} \) for a square sub-area of edge length \( h \), is given by (Appendix I)

\[ a_{kk} = 4 h \log (1 + \sqrt{2}). \]
By symmetry, the $N$ equations reduce to $N^*$ independent equations, where, from (120),

$$N^* = \frac{(k+1)(k+3)}{8}.$$  \hspace{1cm} (157)

The equations are then solved by the Gauss-Seidel iterative method with $\varepsilon = 0.0001$ and the capacity $K$ is then computed by (114). The capacity of the unit cube, computed for an increasing $N$ is given in Table 19. The optimum $N$ occurs at $N = 1014$, since the density $\lambda_0$ at the centre of a face has remained constant to the three preceding values of $N$. At this stage, $K = 0.6595$.

The upper and the lower bounds for the capacity, determined by Polya and Szego, are

$$0.6221 \leq K \leq 0.7106.$$  

Our value of $K$ lies well within the bounds given above. The charge density at the centre of any face is approximately

$$\lambda_0 = 0.0687.$$  

**Neumann Formulation**

If the density distribution $\sigma$ of (2) produces a constant potential on $\partial B$, then from (20) of Chapter 2

$$-\frac{1}{2}\sigma(k) + \int_B \left( \delta^1(h) \right) \sigma(v) dv = 0.$$  \hspace{1cm} (158)

On discretisation, (158) gives $N$ linear algebraic equations in $N$ unknown $\delta_j$ which can be represented by (67) with $\phi_0 = 0$ viz.

$$[B][\delta] = 0.$$  \hspace{1cm} (159)

$[B]$ in (159) is a singular matrix, and the co-efficients $b_{kj}$ are evaluated as in Chapter 5. As before, the $N$ equations reduce to $N^*$ independent equations. To solve these equations we delete the $N^*$ th row, and we put $\delta_j = 1$ in the $N^*$ th column. Hence (159) reduces to a system of $N^*-1$ equations in unknown ratios

$$X_j = \frac{\delta_j}{\delta^*_N}; \quad j = 1, 2, \ldots, (N^*-1).$$  \hspace{1cm} (160)
CAPACITY OF A UNIT CUBE BY SOLVING DIRICHLET PROBLEM

<table>
<thead>
<tr>
<th>SUB-AREA N</th>
<th>EQUATION N*</th>
<th>DENSITY AT THE CENTRE OF THE FACE</th>
<th>CAPACITY</th>
</tr>
</thead>
<tbody>
<tr>
<td>150</td>
<td>6</td>
<td>0.0691</td>
<td>0.65384292</td>
</tr>
<tr>
<td>294</td>
<td>10</td>
<td>0.0691</td>
<td>0.65677327</td>
</tr>
<tr>
<td>486</td>
<td>15</td>
<td>0.0687</td>
<td>0.65819403</td>
</tr>
<tr>
<td>726</td>
<td>21</td>
<td>0.0687</td>
<td>0.65899621</td>
</tr>
<tr>
<td>1014</td>
<td>28</td>
<td>0.0687</td>
<td>0.65945535</td>
</tr>
<tr>
<td>1350</td>
<td>36</td>
<td>0.0684</td>
<td>0.65977961</td>
</tr>
</tbody>
</table>

Table 19
Equation (159) now reduces to

$$\sum_{j=1}^{N^*-1} b_{kj} x_j = -\frac{b_k}{N} N^*, \quad k = 1, 2 \cdots (N^*-1), \quad (161)$$

and are solved by the Gauss-Seidel iterative method with $\varepsilon = 0.0001$.

The Neumann formulation (158) does not immediately give the capacity, since it only provides the relative charge density. However we know that if a conductor $\partial B$ is raised to a constant potential $V = c$ by a charge distribution $\sigma$ on $\partial B$, then

$$\int_{\partial B} G(x, y) \sigma(y) dy = C$$

so that

$$k = \frac{1}{C} \int_{\partial B} \sigma(y) dy.$$  \hspace{1cm} (162)

The numerical $\sigma_j$ do not generate a constant $V$ on $\partial B$. We therefore define

$$C = \left( \frac{\sum_{j=1}^{N^*} V_j}{N} \right).$$

Putting this value of $c$ in (162), we obtain $k$. The values of $k$ thus found, for increasing $N$ are exhibited in Table 20. By contrast with the Dirichlet formulation, no ill-conditioning appeared even at $N = 2166$. The capacity of the unit cube obtained by the Neumann formulation is

$$k(\text{Neumann}) = 0.6475,$$

and that obtained by Dirichlet formulation is

$$k(\text{Dirichlet}) = 0.6595.$$  

Each value is well within the bounds given by Polya and Szego. $k(\text{Neumann})$ appears to lie midway between the bounds whereas $k(\text{Dirichlet})$ lies close to the upper bound.

An alternative comparison with the Dirichlet formulation is possible. We scale the Neumann computed $\sigma_j$ by a factor $f$ so that

$$\int_{\partial \Omega} \sigma(y) dy = f \int_{\partial \Omega} \sigma(y) dy = k(\text{Dirichlet}).$$  \hspace{1cm} (163)

From (163),

$$k(\text{Neumann}) = f \sigma.$$  \hspace{1cm} (164)
### CAPACITY OF A UNIT CUBE BY SOLVING NEUMANN PROBLEM

<table>
<thead>
<tr>
<th>SUB-AREA</th>
<th>EQUATION</th>
<th>AVERAGE POTENTIAL</th>
<th>TOTAL RATIO CHARGES</th>
<th>CAPACITY</th>
</tr>
</thead>
<tbody>
<tr>
<td>N</td>
<td>N*</td>
<td>C</td>
<td></td>
<td></td>
</tr>
<tr>
<td>294</td>
<td>10</td>
<td>4.80998</td>
<td>3.07528</td>
<td>0.65677</td>
</tr>
<tr>
<td>486</td>
<td>15</td>
<td>4.07314</td>
<td>2.61327</td>
<td>0.65819</td>
</tr>
<tr>
<td>726</td>
<td>21</td>
<td>3.54252</td>
<td>2.27895</td>
<td>0.65899</td>
</tr>
<tr>
<td>1014</td>
<td>28</td>
<td>3.13809</td>
<td>2.02303</td>
<td>0.65946</td>
</tr>
<tr>
<td>1350</td>
<td>36</td>
<td>2.81929</td>
<td>1.82072</td>
<td>0.65978</td>
</tr>
<tr>
<td>1534</td>
<td>45</td>
<td>2.56049</td>
<td>1.65599</td>
<td>0.66001</td>
</tr>
<tr>
<td>2166</td>
<td>55</td>
<td>2.33703</td>
<td>1.51328</td>
<td>0.64752</td>
</tr>
</tbody>
</table>

Table 20

(-ve density appears)
This allows us to compare the charge densities yielding the same $\kappa$.

Table 21 exhibits the value of $\lambda$ obtained from both formulations for $N = 1014$. The two solutions are in good agreement with one another except at the nodal points near the sharp edge and the corner of the cube.
### COMPARISON OF DENSITY DISTRIBUTION ON THE SURFACE \( z = 0.5 \)

<table>
<thead>
<tr>
<th>FIELD POINT</th>
<th>DENSITY</th>
<th>Dirichlet</th>
<th>Neumann</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x )</td>
<td>( y )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>------------</td>
<td>---------</td>
<td>-----------</td>
<td>---------</td>
</tr>
<tr>
<td>ALONG A DIAGONAL ON A FACE</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.0</td>
<td>0.0</td>
<td>0.069</td>
<td>0.073</td>
</tr>
<tr>
<td>0.077</td>
<td>0.077</td>
<td>0.070</td>
<td>0.074</td>
</tr>
<tr>
<td>0.154</td>
<td>0.154</td>
<td>0.073</td>
<td>0.077</td>
</tr>
<tr>
<td>0.231</td>
<td>0.231</td>
<td>0.080</td>
<td>0.084</td>
</tr>
<tr>
<td>0.308</td>
<td>0.308</td>
<td>0.092</td>
<td>0.098</td>
</tr>
<tr>
<td>0.385</td>
<td>0.385</td>
<td>0.114</td>
<td>0.129</td>
</tr>
<tr>
<td>0.462</td>
<td>0.462</td>
<td>0.272</td>
<td>0.326</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>ALONG THE NODAL POINTS OF THE SUB-AREAS COMPRISING ANY OUTERMOST ROWS ON A FACE</th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>0.462</td>
<td>0.0</td>
<td>0.154</td>
<td>0.150</td>
</tr>
<tr>
<td>0.462</td>
<td>0.077</td>
<td>0.155</td>
<td>0.151</td>
</tr>
<tr>
<td>0.462</td>
<td>0.154</td>
<td>0.158</td>
<td>0.154</td>
</tr>
<tr>
<td>0.462</td>
<td>0.231</td>
<td>0.163</td>
<td>0.160</td>
</tr>
<tr>
<td>0.462</td>
<td>0.308</td>
<td>0.173</td>
<td>0.170</td>
</tr>
<tr>
<td>0.462</td>
<td>0.385</td>
<td>0.189</td>
<td>0.194</td>
</tr>
<tr>
<td>0.462</td>
<td>0.462</td>
<td>0.272</td>
<td>0.326</td>
</tr>
</tbody>
</table>

Table 21
PART IV

POTENTIAL FLOW OF A FLUID
CHAPTER 13
SUMMARY OF FORMULATIONS

Introduction

It has been shown in (57), Chapter 4, that for uniform potential flow perturbed by a fixed obstacle B, the disturbance potential \( \Phi \) can be generated by a simple source distribution of density \( \sigma \) on \( \partial B \), i.e.

\[
\Phi(h) = \int_{\partial B} \sigma(h, \eta) \, d\eta ; \quad h \in B_a + \partial B .
\]  

(165)

The free flow potential \( \Psi \), by (54) of Chapter 4, is

\[
\Psi = -\mathbf{U} \cdot \mathbf{r} + c
\]

(166)

where \( \mathbf{U} \) is the free flow velocity vector and \( c \) is an additive constant which does not affect the flow. The distribution \( \sigma \) in (165) satisfies the normal derivative equation

\[
-2\pi \sigma(h) + \int_{\partial B} \frac{\sigma(\eta) \, d\eta}{|h - \eta|} = \phi_e'(h) ; \quad h \in \partial B ,
\]

(167)

in which \( \phi_e'(h) \) is given by (55), i.e.

\[
\phi_e'(h) = -\phi_e'(h) ; \quad h \in \partial B .
\]

(168)

It has already been shown in Chapter 2 that equation (167) has a unique solution \( \sigma \) which generates \( \Phi \) everywhere (including the surface \( \partial B \)) as the simple source potential

\[
\Phi(h) = \int_{\partial B} \frac{\sigma(\eta) \, d\eta}{|h - \eta|} .
\]

(169)
FLOW PAST A FIXED BOUNDARY

\[ v^2 \phi = 0 \]

\[ \frac{\partial \phi}{\partial n} = -\gamma \]

Fig. 15
Discretising, by Chapter 5, (167) becomes

\[ -2\pi \sigma_k + \sum_{j=1}^{N} \sigma_j \left( \frac{\partial \phi_j}{\partial y_j} \right) = \frac{\partial \phi_j}{\partial y_j} \left( \frac{y_y}{|y_x - y_j|^2} \right) ; \quad k = 1, 2, \ldots, N , \quad (170) \]

and the computed \( \sigma_j \) generate \( \phi \) according to the formula

\[ \phi (\hat{t}) = \sum_{j=1}^{N} \sigma_j \left( \frac{\partial \phi_j}{\partial y_j} \right) . \quad (171) \]

Alternatively, utilising Green's boundary formula (60) of Chapter 4, \( \phi \) satisfies

\[ -2\pi \phi(\hat{t}) + \int_{\partial B} \frac{\phi \phi}{|y_x - y_j|} \phi = \int_{\partial B} \frac{\phi \phi}{|y_x - y_j|} \phi ; \quad y, \hat{y} \in \partial B . \quad (172) \]

It has been shown in (36), Chapter 2, that equation (172) has a unique solution \( \phi \) on \( \partial B \). On discretisation, by Chapter 5, (172) becomes

\[ -2\pi \phi(\hat{t}) + \sum_{j=1}^{N} \frac{\phi(\hat{y})}{|y_x - y_j|^2} \phi_j = \sum_{j=1}^{N} \frac{\phi(\hat{y})}{|y_x - y_j|^2} \phi_j ; \quad k = 1, 2, \ldots, N . \quad (173) \]

The tangential velocity \( v \) at a point \( \hat{p} \) on \( \partial B \) is given by

\[ v(\hat{t}) = \left[ \left( -\frac{\partial \phi}{\partial \lambda_1} \right)^2 + \left( -\frac{\partial \phi}{\partial \lambda_2} \right)^2 \right]^{1/2} , \quad (174) \]

where \( \phi = \phi + \psi \) and \( \lambda_1, \lambda_2 \) are arc lengths along two mutually perpendicular tangential directions at \( \hat{p} \in \partial B \). When \( \phi \) is determined at discrete equidistant points along \( \lambda_1 \), the tangential velocity component along \( \lambda_1 \) at \( \hat{q} \) is given by

\[ v_1(\hat{q}) = \frac{1}{\lambda_1} \left[ \delta_1^4 - \frac{1}{2} \delta_1^2 \delta_4^3 + \frac{3}{60} \delta_4^5 \delta_1^2 \right] , \quad (175) \]
where \( q_j + \frac{1}{2} \) is the mid point between \( q_j \) and \( q_j + 1 \); \( \delta^r_1 \) is the difference of order \( r \) in a central difference table for \( \varphi \), and \( h_1 \) is the distance between any two equally spaced consecutive points along \( A_1 \) on \( \Omega B \) i.e.,

\[
h_1 = \left| q_{j+1} - q_j \right|
\]

**Axial Flow Past A Symmetric Body**

Let \( \Omega B \) be an axially symmetric surface, and suppose the free flow is parallel to its axis of revolution. Let us now divide \( \Omega B \) into \( 2K \) rings such that the plane of each ring is perpendicular to the axis of flow, and for a ring in the upper part of \( \Omega B \) there is a ring of equal width in the lower part of \( \Omega B \). If \( p \) and \( \bar{p} \) represent one such pair of rings in which \( p \) lies in the upper part and \( \bar{p} \) lies in the lower part of \( \Omega B \) (Fig. 16), the serial number of \( \bar{p} \), counting from the top, is given by

\[
\bar{p} = 2K - p + 1
\]

Similarly for a pair \( q, \bar{q} \)

\[
\bar{q} = 2K - q + 1
\]

Since the plane of the rings are perpendicular to the direction of flow, at the nodal points in the \( p \)th ring \( \sigma \) and \( \varphi \) satisfy

\[
(\varphi)_p = (\varphi^1)_p \quad \text{and} \quad (\sigma)_p = (\sigma^1)_p \quad \text{for} \quad j = 1, \ldots, MK(p), \quad (177)
\]

where \( MK(p) \) is the number of sub-areas in the \( p \)th ring.

**(a) Simple Source Formulation**

By virtue of (177), the \( N \) equations (170) for the \( N \) unknown \( \sigma_j \) reduce to \( 2K \) equations viz.

\[
\sum_{\nu=1}^{2K} E_{p\nu} (\sigma)_\nu = (\varphi^1)_p \quad ; \quad p = 1, 2, \ldots, 2K \quad (178)
\]

In (178) \( (\sigma)_\nu \) is the discrete approximation to \( \sigma \) at any nodal point in the \( \nu \)th ring; \( (\varphi^1)_p \) represents the exterior normal derivative of \( \varphi \) at any nodal
Sub-areas on the surface of a sphere

Fig. 16

Sub-areas at the top

Fig. 16(a)

Upper side

Arm

Lower side

Fig. 16(b)
point in the \( p \) th ring, and \( E_{pq} \) stands for

\[
E = \sum_{j=1}^{M_k(p)} \int \frac{d\psi}{\sqrt{1 - \left( \frac{k}{\varphi_0} \right)^2}}, \quad \varphi = \varphi
\]

\[
= -2\pi + \sum_{j=1}^{M_k(p)} \int \frac{d\psi}{\sqrt{1 - \left( \frac{k}{\varphi_0} \right)^2}}, \quad \varphi = \varphi
\]

where \( \varphi \) is any nodal point in the \( p \) th ring.

If we take the \( Z \)-axis as the axis of flow, then for a pivotal point \((X,Y,Z)\) in the \( p \) th ring there is a pivotal point \((X,Y,-Z)\) in the \( \bar{p} \) th ring on \( \partial B \). Hence

\[
E_{pq} = \int \frac{d\psi}{\sqrt{1 - \left( \frac{k}{\varphi_0} \right)^2}}, \quad \text{over the } q \text{ th ring}
\]

\[
= \int \frac{d\psi}{\sqrt{1 - \left( \frac{k}{\varphi_0} \right)^2}}, \quad \text{over the } \bar{q} \text{ th ring}
\]

\[
= E_{\bar{p}\bar{q}} = E_{2k-\varphi+1,2k-\varphi+1}
\]

and \( (\phi')_{pq} = - (\phi')_{\bar{p}\bar{q}} = -(\phi')_{2k-p+1} \).

Further it is interesting to note that

\[
E_{pq} = \int \frac{d\psi}{\sqrt{1 - \left( \frac{k}{\varphi_0} \right)^2}} + \int \frac{d\psi}{\sqrt{1 - \left( \frac{k}{\varphi_0} \right)^2}} = E_{\bar{p}q}
\]

By virtue of (179) and (180), the system of equations (178) becomes
The solution of (182) has the property (Appendix III)

\[
(\sigma)_p = -(\sigma)_{2p} ; \quad p = 1, 2, \ldots, k.
\]  

(183)

Hence (182) reduces to K equations viz.

\[
\sum_{q=1}^{K} \left( E_{k,1} - E_{k,2k-q+1} \right) (\sigma)_q = (\phi')_p ; \quad p = 1, 2, \ldots, k.
\]  

(184)

From (183) we see that for every positive source on \( \partial \sigma \) there is a negative source of equal strength on \( \partial \sigma \) and hence, in accordance with (59), the total source strength on \( \partial \sigma \) is zero.

(b) Green's boundary formula

For a free flow parallel to the axis of revolution of a symmetric surface \( \partial \sigma \), the disturbance potential \( \Phi \) in every ring, shown earlier, satisfies relation (177). As a result the N equations (173), as before, reduce to 2K equations given by
As in the previous case, following the same procedures, it can be shown that

\[ H_{\mu, \nu} (\phi)_\nu = D_\mu \quad ; \quad \mu = 1, 2, \ldots, 2k \quad , \quad (185) \]

where

\[ H_{\mu, \nu} = \sum_{j=1}^{MK(\nu)} \int_j \frac{c_{ij}}{|k-\nu|^2} \quad ; \quad \nu \neq \mu \]

\[ = -2\pi + \sum_{j=1}^{MK(\nu)} \int_j \frac{c_{ij}}{|k-\nu|^2} \quad ; \quad \nu = \mu \]

and

\[ D_\nu = \sum_{\nu=1}^{2k} \int_\nu \frac{\phi_\nu}{|k-\nu|^2} = \sum_{\nu=1}^{2k} \frac{MK(\nu)}{\nu} \int_\nu \frac{\phi_\nu}{|k-\nu|^2} . \]

As in the previous case, following the same procedures, it can be shown that

\[ H_{\mu, \nu} = H_{\mu, \nu} = H_{2k-\nu+1, 2k-\nu+1} \quad (186) \]

and

\[ D_\nu = -D_\nu = -D_{2k-\mu+1} \quad . \]

By virtue of the above results, the 2K equations in (185), as before, reduce to K equations viz.

\[ \sum_{\nu=1}^{k} (H_{\mu, \nu} - H_{\mu, 2k-\nu+1}) (\phi)_\nu = D_\mu \quad ; \quad \nu = 1, 2, \ldots, K . \]

**Test Function**

It has already been shown that the approximation to an integral, over a given surface \( \overline{D} \), approaches the analytic value as the number of sub-areas increases on \( \overline{D} \). Further, by our fundamental assumption, the density distribution over a surface approaches its true value as the sizes of the
sub-areas decrease and their number increases. Now the question arises what should be the minimum number of sub-areas, along with their respective sizes on $\Omega_B$, which will produce a sound value of the unknown on $\Omega_B$. Accordingly we first find a distribution of sub-areas on $\Omega_B$ which will generate a test harmonic function $h$ of the same nature as the required function $\phi$.

The disturbance potential $\phi$ has the property

$$\phi = O|k|^2 \text{ as } |k| \to \infty.$$  

Hence a comparable test function is

$$h = \frac{-v \cdot \hat{U}}{|k|^3},$$  \hspace{1cm} (189)

where $\hat{U}$ defines a unit vector in the direction of the flow and $h$ is a harmonic function with right behaviour at infinity i.e.

$$h = O|k|^{-2} \text{ as } |k| \to \infty.$$  

The test function has been very useful in experimenting with the sub-division of $\Omega_B$ and with our discretisation procedures.
CHAPTER 14
FLOW PAST A SPHERE

Introduction

A rigid sphere of radius 'a' is fixed with its centre at the origin O of spherical polar co-ordinates (Fig. 17). An inviscid incompressible fluid is flowing from infinity with uniform velocity U given by

\[ U = (0, 0, -U) = -\nabla \psi \]  

where \( \psi \) is the free flow potential, and hence

\[ \psi = Uz \]  

taking the constant of integration to be zero. As already noticed in Chapter 4, the disturbance potential \( \phi \) behaves as \( \frac{1}{|p|^{-2}} \) as \( p \to \infty \), satisfies

\[ \nabla^2 \phi (p) = 0 ; \quad p \in B_e \]  

and on \( \partial B \) satisfies the boundary condition

\[ \phi^u (p) = -\psi^u (p) = -U^u z^u.e = -U \cos \Theta . \]  

The solution of (192) subject to boundary condition (193), in spherical polar co-ordinates, is

\[ \phi = \frac{1}{2} \frac{Ua^3z}{r^3} . \]  

The total velocity potential \( \Phi \), by (53), is

\[ \Phi = \phi + \psi = \frac{1}{2} \frac{Ua^3z}{r^3} + Uz . \]  

The fluid velocity on the surface of the sphere, by symmetry, is in the \( \Theta \) increasing direction. This is given by

\[ U_{\theta} = -\frac{1}{r} \frac{\partial \Phi}{\partial \Theta} = \frac{1}{2} Ua \sin \Theta + Ua \sin \Theta = \frac{3}{2} Ua \sin \Theta . \]
FLOW PAST A FIXED SPHERE
Our aim is to compute an accurate approximation to (194) using the formulation of the last Chapter. Since the analytical solution of the problem is known, we have a chance to test the soundness of our numerical and geometrical procedures by obtaining a numerical solution for comparison with (194), taking a = 1. Since
\[ \phi = \frac{1}{2} \int_0^a \frac{U a^3 z}{r^3} = \frac{1}{2} \frac{z}{r^3} \] (taking U = 1, a = 1) (197)

\[ \phi_e^l(p) \] on \( \mathcal{D}B \) is given by
\[ \phi_e^l(p) = \nabla \phi \cdot \hat{n}_e \]
\[ = \frac{1}{2} \left[ \frac{3 z x}{r^5} \right] - \frac{1}{2} \frac{z}{r^3} \left( \frac{3 z^2}{r^5} - \frac{1}{r^3} \right) \cdot (x, y, z) a^{-1} \]
\[ = -\frac{1}{2} \frac{2 z}{r^3} a^{-1} = -\frac{1}{a} \frac{\cos \theta}{r^2} = -\cos \theta \] (\( : a = 1 \) and \( r \to 1 \) on \( \mathcal{D}B \))
\[ = -\phi_e^l(p) \] [ by (163) ] , (198)

where \( \hat{n}_e = (x, y, z) a^{-1} \) at \( p \in \mathcal{D}B \). Introducing (198) in (172), Green's Boundary Formula (Jaswon) defines an equation for \( \phi \) on \( \mathcal{D}B \) with exact solution (197). Alternatively, introducing (198) in (167) the Simple Source Formulation (A.M.O. Smith) defines an equation for \( \sigma \) with exact solution*
\[ \sigma = \frac{3 \cos \theta}{8 \lambda} \] (199)

This \( \sigma \), by (165), generates \( \phi \) in (197).

\* \[ -4 \lambda \sigma = \phi_e^l + \phi_0^l = \frac{1}{2} \left[ \left( \frac{3 z^2}{r^5} + \frac{2 z}{r^3} \right) \right] \] at \( r = a = 1 \)
\[ = \frac{1}{2} \left[ \frac{d}{dr} \left( \frac{\cos \theta}{r^2} \right) \right] - \frac{d}{dr} \left( r \cos \theta \right) \] \( r = 1 \)
\[ = -\frac{3}{2} \cos \theta \]

\[ \therefore \sigma = \frac{3 \cos \theta}{8 \lambda} \] .
Division of the surface into sub-areas

To solve the equations (167) and (172) numerically, the surface is to be divided into sub-areas. The upper half of the spherical surface $x^2 + y^2 + z^2 = a^2$ is divided into $K$ horizontal rings. Each of the 1st $K_N$ ($< K$) rings, starting from the pole, is divided into $M_K_j$ equal sub-areas by $M_K_j$ meridian line segments where $M_K_j$ is given by

$$M_K_j = 6 \left[ 1 + (j-1)^2 \right], \quad j = 1, 2, \cdots, K_N.$$ 

Starting from $(K_N + 1)$ up to the $K$th ring, each ring is divided into $M$ equal sub-areas where

$$M = M_K_j = 6 \left( 1 + 2K_N \right), \quad j = (K_N + 1), \cdots, K.$$ 

The total number of rings on the upper hemispherical surface is given by

$$K = K_N + K_T = K_N + \left( \frac{M+2}{4} \right) = 4K_N + 2.$$  \hspace{1cm} (200)

Any half meridian is divided into $K$ equal parts to give the height $h_k$ of a trapezoidal sub-area (Fig. 16) adjacent to the equatorial line which is divided into $M$ equal parts to give the breadth $b_k$ of the same sub-area. From the above, the ratio $h_k : b_k$ is given by

$$r = \frac{h_k}{b_k} = \frac{\pi a / 2 (2K_N + 1)}{2\pi a / 6 (2K_N + 1)} = \frac{3}{4},$$

where ideally $r = 1$ (see Chapter 6). If $h_k$ is increased keeping $b_k$ fixed and vice versa, the form of the trapezoidal sub-areas near the polar region deviates from the ideal form. This justifies the value of $K$ chosen in (200).

If $d\phi_j$ be the angle between any two consecutive meridian line segments in the $j$th ring, then

$$d\phi_j = \frac{2\pi}{M_K_j}; \quad j = 1, 2, \cdots, K.$$ 

The width of the 1st ring is tentatively taken to be $l_1$ where

$$l_1 = \left( \frac{\pi a}{2} / K_T \right) \delta_1; \quad 1 < \delta_1 < 2.$$ 

This subtends an angle $\theta_1$ at the centre of the sphere where

$$\theta_1 = l_1 a^{-1}.$$ 

The width of the $j$th ring is given by

$$l_j = \delta_j \left( a \sin \theta_{j-1} \right) d\phi_j; \quad j = 2, 3, \cdots, K_N,$$
where $\Theta_j = \ell_j \alpha^{-1}$ and $1 < \ell_j < 2$. Normally $\ell_j$ is kept fixed at 1.5. From the $(KN + 1)$th up to the $K$th ring, the width of a sub-area is given by

$$L_{KN+j} = D + j(Uf) \quad ; \quad j = 1, 2, \ldots, KT,$$

where

$$D = \frac{3}{2}a \left( \sin \Theta_{KN} \right) d\phi_{KN+1},$$

and

$$Uf = \left\{ A \left( \frac{r}{2} - \Theta_{KN} \right) - (KT)D \right\} / kT(kT+1)/2 \right\} .$$

The total number of sub-areas is

$$N = 2 \left\{ 6 \left( \frac{KN}{2} \left( 2 + \frac{KN-1}{2} \right) \right) + M(kT) \right\}$$

$$= 12 \left( 7 \frac{KN}{2} + 7 \frac{KN+2}{2} \right).$$

(201)

The analysis of the sub-areas thus formed, for $KN = 5$, is given in Table 22. It is evident from this Table that:

1. The sides of the triangular sub-areas adjacent to the pole are nearly equal.
2. From the 2nd up to the $KN$th ring the width of any sub-area is nearly 1.5 times its average breadth.
3. The change in the size of the sub-areas, as we move from the top to the equatorial line, follows a continuous pattern.

Computation of Disturbance Potential

Now we proceed to solve the equations (167) and (172) numerically. Dealing first with the Simple Source Formulation (167), we find, on discretisation, that this gives (170), a system of $N$ linear algebraic equations in $N$ unknown $\Theta_j$. From symmetry of $DB$ and for relation (180), the $N$ equations reduce to $K$ equations given by (184). The co-efficients $E_{pq}$ are evaluated as discussed in Chapter 5 and the equations are then solved by the Gauss-Seidel iterative method with $\epsilon = 0.0001$. 

## ANALYSIS OF SUB-AREAS ON A SPHERICAL SURFACE

<table>
<thead>
<tr>
<th>RING</th>
<th>SUB-AREA</th>
<th>AREA</th>
<th>UPPER SIDE AB</th>
<th>LOWER SIDE CD</th>
<th>ARM AD</th>
</tr>
</thead>
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<td>0.00000E 00</td>
<td>0.16382E 00</td>
<td>0.15708E 00</td>
</tr>
<tr>
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</tr>
<tr>
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<td>0.57145E-01</td>
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</tr>
<tr>
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<td>66</td>
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<td>0.46755E-01</td>
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<tr>
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<td>0.53113E-01</td>
<td>0.58993E-01</td>
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<td>0.95099E-01</td>
<td>0.95200E-01</td>
<td>0.45908E-01</td>
</tr>
</tbody>
</table>

Table 22

(This should be read in conjunction with Fig.16, Fig.16(a), Fig.16(b).)
The $\psi_j$ thus obtained satisfy the relation (183). These computed $\psi_j$ when used in (171), generate the required potential $\phi$ given by (197). Table 23 exhibits the computed $\psi (K = 46)$ compared with analytical $\psi$ given by (199). Table 24 exhibits the $\phi$ in (197), generated by the above $\psi_j$, for the same value of $K$ along with the analytical $\phi$ at the respective points on $\partial B$. Fig. 18 exhibits the graphs of analytical and numerical $\psi$ based on Table 23.

The total velocity potential $\phi$ is then obtained by (53) viz.

$$\phi = \phi + \psi,$$  \hspace{1cm} (202)

where $\psi = \sum_{j=1}^{\infty} \psi_j = z \quad (\because \quad \sum_{j=1}^{\infty})$. The graphs in Fig. 19 exhibit $\phi$ and $\psi$, based on Table 24, on the upper hemispherical surface of the sphere.

In (172) $\phi_e$ is given by (198). On discretisation, (172) gives $N$ linear algebraic equations in $N$ unknown $\phi_j$, by virtue of the symmetry of $\partial B$ and for (180), the $N$ equations reduce to $K$ equations given by (188). After evaluation of the $H_{pq}$ and the $D_p$, of (188), the equations are solved by the Gauss-Seidel iterative method with $\epsilon = 0.0001$. The $\phi_k$ thus obtained for $K = 46$, are exhibited in Table 24. The total potential $\phi$ is then obtained by (202). The $\phi$ and the $\psi$ thus obtained, for $K = 46$, are exhibited in Fig. 19.

**Equipotentials**

The $\sigma$ which generates the required disturbance potential $\phi$, for $\phi_e$ given by (193), is obtained by solving the equation (167) numerically, as discussed earlier. These $\sigma_j$ then generate the $\phi_k$ by (171) at any point $p \in \partial \phi + \partial B$. The total velocity potential $\phi$ is then obtained by (202).

For $K = 46$, the total potential $\phi$ is then obtained at $M^2$ points outside $\partial B$ along with those at the nodal points, each lying on a separate ring, on the upper hemispherical part of $\partial B$. The equipotentials are then drawn from the $K$ nodal points $q_1, q_2, \ldots q_K$ of $\partial B$ through those points $p_j$ for which

$$|\phi(q_m) - \phi(p_j)| \leq 0.001, \quad q_m \in \partial B, \quad j = 1, 2, \ldots (M^2 + K).$$

The equipotentials, thus found, are given in Fig. 20.
## DISTRIBUTION OF SOURCE DENSITY ON A UNIT SPHERE

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<thead>
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<th>POLAR DISTANCE IN RADIAN</th>
<th>DENSITY ( \sigma )</th>
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</thead>
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<td>ANALYTICAL</td>
</tr>
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<td>0.11926</td>
</tr>
<tr>
<td>0.179</td>
<td>0.11746</td>
</tr>
<tr>
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<td>0.11439</td>
</tr>
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</tr>
<tr>
<td>0.484</td>
<td>0.10564</td>
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</tr>
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<td>0.09321</td>
</tr>
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<td>0.771</td>
<td>0.08565</td>
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<td>0.964</td>
<td>0.06810</td>
</tr>
<tr>
<td>1.061</td>
<td>0.05826</td>
</tr>
<tr>
<td>1.159</td>
<td>0.04781</td>
</tr>
<tr>
<td>1.257</td>
<td>0.03686</td>
</tr>
<tr>
<td>1.356</td>
<td>0.02550</td>
</tr>
<tr>
<td>1.455</td>
<td>0.01383</td>
</tr>
<tr>
<td>1.554</td>
<td>0.00198</td>
</tr>
</tbody>
</table>

Table 23
Source distribution on the surface of a sphere

Fig. 18

Source distribution on the surface of a sphere
Table 24
1) VELOCITY POTENTIAL.
2) DISTURBANCE POTENTIAL.

Analytic

Simple layer potential (Smith)

Green's boundary formula (Janway)

Sub-area = 11.112
EANS = 46

Fig. 19
EQUIPOTENTIALS AROUND A FIXED SPHERE
Fluid velcoity on the surface

The analytical value of the fluid velocity at a point \( q \in \partial B \), by (196), is

\[
V_0(q) = \frac{3}{2} U a \sin \theta \quad (203)
\]

where \( U = 1 \), \( a = 1 \) and \( \theta \) represents the value of \( \theta \) (Fig. 17) at the point \( q \). The numerically computed value of the velocity at \( q \) is found by (175), using the numerical \( \phi \). Since the nodal points on \( \partial B \) are not equally spaced and the higher order \( \delta \) is very small, the velocity component at \( q \in \partial B \), in any direction \( S^1 \), is obtained by taking only the first term in (175) i.e.,

\[
V_0(q) = -\frac{1}{k_1} \delta_1^1 \quad (204)
\]

It has already been pointed out, that for symmetry, the flow on the surface is along the meridians on \( \partial B \). The velocity at a point \( q(= P_{j+\frac{1}{2}}) \) on \( \partial B \) is determined by (204) from the numerical \( \phi \) given by (202), in which \( \phi \) is obtained by Simple Source Formulation. Table 25 exhibits the \( V_0 \) thus obtained, for \( K = 46 \), along with the analytical \( V_0 \) at the respective points on \( \partial B \). Similarly, \( V_0 \) is obtained from \( \phi \) in which \( \phi \) is determined by Green's Boundary Formula under the same external condition and for the same sub-division of \( \partial B \). The \( V_0 \) thus obtained are exhibited in Table 25. Fig. 21 shows the velocity distribution on \( \partial B \) base on Table 25.

General Discussion

It is evident from Table 24 that both the formulations, i.e. Simple Source Distribution (Smith) and Green's Boundary Formula (Jaswon), are capable of yielding a good approximation. In the case of a flow past a sphere, in this thesis, we obtained \( \phi \) on \( \partial B \) by both the methods, in which the maximum error in \( \phi \) at a nodal point on \( \partial B \) is \( < 0.6\% \). The error in \( \phi \), generated by the Simple Source Formulation is less than that in \( \phi \) obtained by Green's Boundary Formula.
### Table 25

<table>
<thead>
<tr>
<th>POLAR DISTANCE IN RADIANS</th>
<th>ANALYTIC VELOCITY</th>
<th>FROM S.I. POTENTIAL (Smith)</th>
<th>ERROR</th>
<th>FROM G.B. FORMULA (Jackson)</th>
<th>ERROR</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.084E-01</td>
<td>0.10257E 00</td>
<td>0.8714E-01</td>
<td>-0.150E 02</td>
<td>0.10231E 00</td>
<td>0.336E 00</td>
</tr>
<tr>
<td>0.117E 00</td>
<td>0.1753E 00</td>
<td>0.1643E 00</td>
<td>-0.627E 01</td>
<td>0.17534E 01</td>
<td>0.334E-02</td>
</tr>
<tr>
<td>0.150E 00</td>
<td>0.2372E 00</td>
<td>0.2323E 00</td>
<td>-0.207E 01</td>
<td>0.23023E 00</td>
<td>0.420E 00</td>
</tr>
<tr>
<td>0.190E 00</td>
<td>0.2954E 00</td>
<td>0.2936E 00</td>
<td>-0.562E 00</td>
<td>0.29913E 00</td>
<td>0.922E 00</td>
</tr>
<tr>
<td>0.230E 00</td>
<td>0.3566E 00</td>
<td>0.3502E 00</td>
<td>-0.101E 00</td>
<td>0.3540E 00</td>
<td>0.976E 00</td>
</tr>
<tr>
<td>0.659E 00</td>
<td>0.9181E 00</td>
<td>0.9197E 00</td>
<td>0.169E 00</td>
<td>0.9182E 00</td>
<td>0.331E-02</td>
</tr>
<tr>
<td>0.691E 00</td>
<td>0.9555E 00</td>
<td>0.9560E 00</td>
<td>0.150E 00</td>
<td>0.9555E 00</td>
<td>0.144E-01</td>
</tr>
<tr>
<td>0.723E 00</td>
<td>0.9919E 00</td>
<td>0.9933E 00</td>
<td>0.113E 00</td>
<td>0.9920E 00</td>
<td>0.256E-01</td>
</tr>
<tr>
<td>0.755E 00</td>
<td>0.1027E 01</td>
<td>0.1028E 01</td>
<td>0.131E 00</td>
<td>0.10278E 01</td>
<td>0.367E-01</td>
</tr>
<tr>
<td>0.787E 00</td>
<td>0.1061E 01</td>
<td>0.1063E 01</td>
<td>0.126E 00</td>
<td>0.1062E 01</td>
<td>0.477E-01</td>
</tr>
<tr>
<td>0.141E 01</td>
<td>0.1479E 01</td>
<td>0.1481E 01</td>
<td>0.155E 00</td>
<td>0.1482E 01</td>
<td>0.187E 00</td>
</tr>
<tr>
<td>0.144E 01</td>
<td>0.1480E 01</td>
<td>0.1491E 01</td>
<td>0.156E 00</td>
<td>0.1480E 01</td>
<td>0.190E 00</td>
</tr>
<tr>
<td>0.147E 01</td>
<td>0.1492E 01</td>
<td>0.1493E 01</td>
<td>0.157E 00</td>
<td>0.1494E 01</td>
<td>0.192E 00</td>
</tr>
<tr>
<td>0.150E 01</td>
<td>0.1496E 01</td>
<td>0.1499E 01</td>
<td>0.158E 00</td>
<td>0.1499E 01</td>
<td>0.194E 00</td>
</tr>
<tr>
<td>0.154E 01</td>
<td>0.1499E 01</td>
<td>0.1501E 01</td>
<td>0.159E 00</td>
<td>0.1502E 01</td>
<td>0.195E 00</td>
</tr>
</tbody>
</table>
TOTAL SUB-AREA = 11.112
TOTAL EQUATIONS = 46

Fig. 21

VELOCITY ON THE SURFACE OF THE SPHERE
The error in $\phi$, obtained by Green's Boundary Formula, is nearly uniform. As a result, the numerical velocity, near the pole on $\Omega B$, obtained from $\phi$ given by Green's Boundary Formula is nearer to the analytic velocity than obtained from $\phi$ given by Simple Source Formulation in that region (Fig. 21).
Introduction

Let the centroid of the cylinder define the origin of a cartesian reference frame OXYZ, the axis of Z coinciding with the axis of the cylinder (Fig.22). The cylinder is of length 2H and radius 'a', and therefore the cylindrical surface has the equation
\[ x^2 + y^2 = a^2 , \quad |z| \leq H. \]
The two hemispherical surfaces have the equations
\[ x^2 + y^2 + (z + H)^2 = a^2 \quad \text{respectively, with } |z| \geq H. \]
The cylinder is supposed to be fixed in an infinite fluid moving with free velocity
\[ \mathbf{U} = (0, 0, -1) = -\nabla \psi , \]
where \( \psi \) is the free flow potential, and by (205)
\[ \psi = z . \]
As before, the disturbance potential \( \phi \to 0 \) \( \frac{1}{|p|} \) as \( |p| \to \infty \) and satisfies
\[ \nabla^2 \phi (\mathbf{h}) = 0 ; \quad \mathbf{h} \in \mathcal{B}_e , \]
with boundary condition
\[ \phi_{e}^1 = -\nabla \psi \cdot \hat{n}_e = -(\mathbf{z})_e^1 . \]

The integral equation formulation provides a straightforward approach to determine \( \phi \) on the boundary. This is achieved by substituting (208) into (167) or (172) and solving the equations numerically.

Discretisation Procedures

The numerical approach demands that the surface \( \partial \mathcal{B} \) should be divided into sub-areas. To effect the sub-division, the hemispherical part of the surface is divided into \( K_1 \) rings, similar to the surface of the sphere in Chapter 14. Hence, by (200), \( K_1 \) is given by
\[ K_1 = \frac{4}{\pi} kN + 2 . \]
CYLINDER WITH HEMISPHERICAL CAPS
By (201), the total number of sub-areas of each of the hemispherical surfaces, is

\[ N_1 = 6 \left( 7K + 2 \right). \]  

(210)

The M meridian lines which divide the K_1 th ring into M sub-areas are extended on the cylindrical surface. The cylindrical surface from \( z = H \) to \( z = 0 \) is divided into \( K_2 \) rings, such that the width of the ring at the top of the cylindrical surface nearly equals to the breadth of the sub-area in that ring, i.e.

\[ \frac{H}{K_2} \sim \frac{2\pi a}{M}, \]

where, by Chapter 14, \( M = 6(1 + 2KN) \). When \( H = a = 1 \), from above, the approximate value of \( K_2 \) is \((1 + 2KN)\). Since the width of the sub-area in the \( K_1 \) th ring is little less than the breadth \( 2\pi a/M \) (Chapter 14), the value of \( K_2 \), in this case, is taken to be

\[ K_2 = 3KN. \]  

(211)

Each sub-area on the cylindrical surface is of breadth \( b \) and width \( d \), where

\[ b = 2\pi a/M \quad \text{and} \quad d = \frac{H}{3KN}. \]

The total number of rings on \( \partial B \) is \( 2K \), where

\[ 2K = 2(K_1 + K_2) = 2 \left( 7K + 2 \right). \]  

(212)

The total number of sub-areas

\[ N = 2 \left( N_1 + K_2 \right) \]

\[ = 2 \left[ 6 \left( 7K^2 + 7K + 2 \right) + 3KN + 6 \left( 1 + 2KN \right) \right] \]

\[ = 12 \left( 13K^2 + 10KN + 2 \right). \]  

(213)

**Test function**

In order to test our geometrical and numerical procedures, we introduce the test function

\[ h = \frac{z}{r^3}. \]  

(214)
which is a harmonic function of similar behaviour to the disturbance potential $\phi$. On $\partial B$,

$$
\begin{align*}
\hat{h}_e &= \nabla \hat{h}, \\
\hat{\eta}_e &= - \left[ \frac{3x^2}{r^5}, \frac{3y^2}{r^5}, \frac{3z^2}{r^5} - \frac{1}{r^5} \right] \hat{\eta}_e
\end{align*}
$$

(215)

Introducing this into the place of $\phi_e^1$ in (167) and applying our procedures, we solve for $\phi$ and generate $h$ at all the nodal points on the surface. The generated values are exhibited in Table 26 for comparison with the analytic values defined by (214) on the boundary. It will be seen from the Table that the error in the numerically computed values, for $K = 23$, is less than $1.5\%$.

We may compute $h$ directly on $\partial B$ by inserting $\hat{h}_e$ from (215) in (172) and applying our procedures. For $K = 23$ the computed values of $h$ at the nodal points are exhibited in Table 26. It will be seen from Table 26 that at no nodal point the error exceeds $1.5\%$. It will be noted further from the Table that the two approaches yield a comparable accuracy.

**Computation of Disturbance Potential**

In the actual problem $\phi_e^1$ on $\partial B$ is given by (208). Inserting this into (167) and applying our procedures, we solve for $\phi_k$. Using these $\phi_k$ in (171) we generate $\phi$ on $\partial B$. For $K = 23$, the disturbance potential thus obtained are exhibited in Table 27. The total velocity potential is then obtained by using this $\phi$, and $\psi$ given by (206), in (53) i.e.

$$
\Phi = \phi + \psi
$$

(216)

Fig. 23 shows the graphs of $\phi$ and $\Phi$ thus obtained for $K = 23$.

Similarly we insert $\phi_e^1$ given by (208) into (172) and compute $\phi$ directly at the nodal points on $\partial B$. The $\phi_k$ thus obtained, for $K = 23$, are exhibited in Table 27. On the basis of this value of $\phi_k$, $\phi$ is calculated by (216). Fig. 23 shows the graphs of $\phi$ and $\Phi$ thus obtained for $K = 23$.

It will be seen that the two approaches yeild very similar results.
DEVELOPMENT OF TEST FUNCTION ON THE SURFACE ALONG THE PLANE $y = 0$

<table>
<thead>
<tr>
<th>FIELD POINT (Cartesian Co-Ord)</th>
<th>ANALYTICAL</th>
<th>S.L. POTENTIAL (Smith)</th>
<th>ERROR</th>
<th>G.B. FORMULA (Jaswon)</th>
<th>ERROR</th>
</tr>
</thead>
<tbody>
<tr>
<td>ON THE SPHERICAL SURFACE</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.151 1.989 0.25072E 00</td>
<td>0.25355E 00</td>
<td>0.113E 01</td>
<td>0.25707E 00</td>
<td>0.253E 01</td>
<td></td>
</tr>
<tr>
<td>0.343 1.939 0.25307E 00</td>
<td>0.25707E 00</td>
<td>0.126E 01</td>
<td>0.25960E 00</td>
<td>0.204E 01</td>
<td></td>
</tr>
<tr>
<td>0.483 1.876 0.25016E 00</td>
<td>0.26150E 00</td>
<td>0.133E 01</td>
<td>0.26265E 00</td>
<td>0.174E 01</td>
<td></td>
</tr>
<tr>
<td>0.602 1.798 0.26619E 00</td>
<td>0.26725E 00</td>
<td>0.137E 01</td>
<td>0.26775E 00</td>
<td>0.156E 01</td>
<td></td>
</tr>
<tr>
<td>0.901 1.192 0.32352E 00</td>
<td>0.32834E 00</td>
<td>0.146E 01</td>
<td>0.32945E 00</td>
<td>0.174E 01</td>
<td></td>
</tr>
<tr>
<td>0.992 1.126 0.33316E 00</td>
<td>0.33841E 00</td>
<td>0.156E 01</td>
<td>0.33973E 00</td>
<td>0.175E 01</td>
<td></td>
</tr>
<tr>
<td>0.993 1.069 0.31192E 00</td>
<td>0.31781E 00</td>
<td>0.172E 01</td>
<td>0.31804E 00</td>
<td>0.179E 01</td>
<td></td>
</tr>
<tr>
<td>1.000 1.021 0.34902E 00</td>
<td>0.35527E 00</td>
<td>0.156E 01</td>
<td>0.35572E 00</td>
<td>0.169E 01</td>
<td></td>
</tr>
<tr>
<td>ON THE CYLINDRICAL SURFACE</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1.000 0.944 0.36291E 00</td>
<td>0.36734E 00</td>
<td>0.122E 01</td>
<td>0.36594E 00</td>
<td>0.832E 00</td>
<td></td>
</tr>
<tr>
<td>1.000 0.833 0.37314E 00</td>
<td>0.38131E 00</td>
<td>0.924E 01</td>
<td>0.38303E 00</td>
<td>0.663E 00</td>
<td></td>
</tr>
<tr>
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<td>0.777E 00</td>
<td>0.38730E 00</td>
<td>0.575E 00</td>
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<tr>
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<td>0.24895E 00</td>
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<tr>
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<td>0.16030E 00</td>
<td>0.212E 00</td>
<td>0.16015E 00</td>
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<td>0.55394E-01</td>
<td>0.171E 00</td>
<td>0.55347E-01</td>
<td>0.857E-01</td>
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</tr>
</tbody>
</table>

Table 26
GENERATION OF $\phi$ ON THE SURFACE ALONG THE PLANE $y = 0$

<table>
<thead>
<tr>
<th>FIELD POINT (CARTESIAN CO-ORD)</th>
<th>$\phi$ S.L.POTENTIAL (Smith)</th>
<th>$\phi$ G.B.FORMULA (Jaswon)</th>
</tr>
</thead>
<tbody>
<tr>
<td>x, z</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.151 1.989</td>
<td>0.54115 E 00</td>
<td>0.54539 E 00</td>
</tr>
<tr>
<td>0.343 1.939</td>
<td>0.51979 E 00</td>
<td>0.52102 E 00</td>
</tr>
<tr>
<td>0.483 1.876</td>
<td>0.49051 E 00</td>
<td>0.49008 E 00</td>
</tr>
<tr>
<td>0.602 1.798</td>
<td>0.45467 E 00</td>
<td>0.45338 E 00</td>
</tr>
<tr>
<td>0.981 1.192</td>
<td>0.18835 E 00</td>
<td>0.18749 E 00</td>
</tr>
<tr>
<td>0.992 1.126</td>
<td>0.16366 E 00</td>
<td>0.16263 E 00</td>
</tr>
<tr>
<td>0.998 1.069</td>
<td>0.14374 E 00</td>
<td>0.14258 E 00</td>
</tr>
<tr>
<td>1.000 1.021</td>
<td>0.12898 E 00</td>
<td>0.12777 E 00</td>
</tr>
</tbody>
</table>

ON THE CYLINDRICAL SURFACE

<table>
<thead>
<tr>
<th>x, z</th>
<th>$\phi$ S.L.POTENTIAL (Smith)</th>
<th>$\phi$ G.B.FORMULA (Jaswon)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.000 0.944</td>
<td>0.11006 E 00</td>
<td>0.10838 E 00</td>
</tr>
<tr>
<td>1.000 0.833</td>
<td>0.89885 E-01</td>
<td>0.88547 E-01</td>
</tr>
<tr>
<td>1.000 0.722</td>
<td>0.73556 E-01</td>
<td>0.72481 E-01</td>
</tr>
<tr>
<td>1.000 0.278</td>
<td>0.25043 E-01</td>
<td>0.24696 E-01</td>
</tr>
<tr>
<td>1.000 0.167</td>
<td>0.14847 E-01</td>
<td>0.14643 E-01</td>
</tr>
<tr>
<td>1.000 0.056</td>
<td>0.49201 E-02</td>
<td>0.48525 E-02</td>
</tr>
</tbody>
</table>

Table 27
4) DISTURBANCE POTENTIAL

2) VELOCITY POTENTIAL

POTENTIALS ON THE SURFACE OF A CYLINDER WITH HEMISPHERICAL CAPS
Tangential Velocity on the Surface

By symmetry, the tangential velocity is directed along the meridian of \( \mathcal{S} \). The velocity at \( \mathcal{P} \in \mathcal{S} \), neglecting the terms of higher order in (175), is

\[
\nu_1'(r) = \nu_1' \left( \frac{r}{r^*} \right) = -\frac{1}{r^*} \delta_1^1 .
\]

(217)

The velocity thus calculated by (217), on the basis of the two formulations, are exhibited in Table 28. Fig. 24 shows the graphs of the velocities based on Table 28.
DISTRIBUTION OF VELOCITY ON THE SURFACE ALONG THE PLANE $y = 0$

<table>
<thead>
<tr>
<th>FIELD POINT</th>
<th>VELOCITY FROM</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x$</td>
<td>$z$</td>
</tr>
<tr>
<td>ON THE SPHERICAL SURFACE</td>
<td></td>
</tr>
<tr>
<td>0.248</td>
<td>1.969</td>
</tr>
<tr>
<td>0.414</td>
<td>1.910</td>
</tr>
<tr>
<td>0.544</td>
<td>1.839</td>
</tr>
<tr>
<td>0.654</td>
<td>1.757</td>
</tr>
<tr>
<td>0.973</td>
<td>1.229</td>
</tr>
<tr>
<td>0.987</td>
<td>1.159</td>
</tr>
<tr>
<td>0.995</td>
<td>1.098</td>
</tr>
<tr>
<td>0.999</td>
<td>1.045</td>
</tr>
<tr>
<td>ON THE CYLINDRICAL SURFACE</td>
<td></td>
</tr>
<tr>
<td>1.000</td>
<td>0.889</td>
</tr>
<tr>
<td>1.000</td>
<td>0.778</td>
</tr>
<tr>
<td>1.000</td>
<td>0.667</td>
</tr>
<tr>
<td>1.000</td>
<td>0.333</td>
</tr>
<tr>
<td>1.000</td>
<td>0.222</td>
</tr>
<tr>
<td>1.000</td>
<td>0.111</td>
</tr>
</tbody>
</table>

Table 28
DISTRIBUTION OF VELOCITY
ON THE SURFACE

Fig. 24
CHAPTER 16
FLOW PAST A CYLINDER WITH CONICAL CAPS

Introduction

Let the centroid of the cylinder define the origin of a cartesian reference frame OXYZ, the axis of the cylinder coinciding with the Z - axis (Fig.25). The cylindrical surface is of height 2H, and has the equation

\[ x^2 + y^2 = a^2, \quad |z| \ll H. \]

If the vertical height of the cone be \( H_1 \), the conical surfaces have the equations

\[ x^2 + y^2 = [t(H + H_1) - z]^2 \tan^2 \alpha; (H + H_1) \gg |z| \gg H, \]

respectively, where \( \alpha \) is the semivertical angle of the cone.

For a potential fluid motion past the cylinder with free flow velocity \( \mathbf{U} = (0, 0, -1) = \hat{\mathbf{u}} \), the disturbance potential \( \phi \) satisfies

\[ \nabla^2 \phi (z) = 0; \quad \hat{z} \in B_\varepsilon, \]

with boundary condition

\[ \phi_0 = -\nabla \psi \cdot \hat{n}_e = -(z)_2 \quad (\therefore \psi = z). \]  

(218)

As before, the integral equation formulation provides a straightforward approach to determine \( \phi \) on the boundary. This is achieved by substituting (218) into (167) or (172) and solving the equations numerically.

Subdivision of Boundary

The numerical method of solution requires that the surface should be divided into smaller sub-areas. For this purpose let us consider a definite boundary by choosing \( a = 1, \alpha = 45^\circ \) and \( H = 1 \). Hence \( H_1 = a \cot \alpha = 1. \)

The cylindrical surface of radius \( a = 1 \) is divided into \( M \) vertical approximate rectangular slices each of length \( 2H \) and breadth \( 2 \pi a/M \), where

\[ M = 6 \left\{ 1 + (KN -1) 2 \right\}; \quad KN = 2, 4, 6, \ldots \ldots 2m. \]

Following Chapter 15, each of the vertical slices from \( z = 0 \) to \( z = H \), is divided into \( 3 \) \( KN \) rectangular sub-areas.
SUB-AREAS ON THE SURFACE OF A CYLINDER WITH CONICAL CAPS

Fig. 25
The conical surface is divided into \( M \) approximate triangular slices (Fig. 25). Each slice in turn is divided into \( n \) sub-areas, all of which are approximately trapezoidal in form except the one adjacent to the apex which is triangular in form. The trapezoidal sub-areas are so constructed that in every sub-area the length of the arm is equal to the average breadth of the sub-area [Fig. 25 (a)].

The total number of horizontal rings on the surface is

\[
2N^* = 2 \left( n + 3KN \right) .
\]  

\textbf{Smoothing Procedures on Boundary}

It has already been stated in Chapter 6 that, in general, we can not expect a good accuracy near a sharp edge or a corner by the numerical methods used in this thesis. In our sub-division, the sub-areas adjacent to the tip become very thin and, hence, the results obtained will be untrustworthy. To overcome this difficulty, the sharp tip is replaced by a spherical cap of a radius of curvature \( R_1 \) and, though it is not essential, the corner at \( C \) (Fig. 26) is replaced by an arc of revolution of radius of curvature \( R_2 \) such that, as the number of sub-areas increases, both \( R_1 \) and \( R_2 \) tend to zero.

The cap at the top is so placed that the pole of the cap lies on the axis of \( z \) and it touches the slant line \( AC \) and \( A_1 \) where \( AA_1 = h_1 \) (Fig. 26). Hence

\[
\rho_1 = A_0 A_1 = h_1 \tan \alpha .
\]

and the angle \( A_0 A_1 A_1 = 90^\circ - \alpha \).

The arc of revolution is so fitted that it touches \( AC \) at the point \( A \) and \( CE \) at \( C_1 \) (Fig. 26), where

\[
A_{n-1} C = h_n - h_{n-1} = C C_1 .
\]

The radius of curvature \( R_2 \), from Fig. 26, is

\[
R_2 = (C C_1) \cot \left( \frac{\theta}{2} \right) .
\]

The distance \( AP_1 \), i.e. the gap between the tip \( A \) and the pole \( P_1 \) of the cap, is given by

\[
AP_1 = A_0 - \rho_1 = \rho_1 \csc \alpha - \rho_1 (\csc \alpha - 1) .
\]

The distance \( CP_2 \), i.e. the perpendicular distance between the arc \( A_1 P_2 C_1 \) and \( C \), is given by
\[ \frac{b_r + b_{r+1}}{2} = h_r - h_{r-1} \]  \hspace{1cm} (1)

Now \( \frac{b_r}{b_{r+1}} = \frac{h_r}{h_{r-1}} \) or \( \frac{b_r - b_{r+1}}{b_r} = \frac{h_r - h_{r-1}}{h_r} \).

Or \( \frac{b_r - b_{r+1}}{b_r} = \frac{b_r + b_{r+1}}{2b_r} \), or \( \frac{b_r + b_{r+1}}{b_r} = 2h_r - h_{r-1} = 2AB - 2BC \).

Or \( \frac{b_{r+1}}{b_r} = \frac{2AB - BC}{2AB + BC} = \frac{h_{r+1}}{h_r} \).  \hspace{1cm} (2)

From (2), \( h_{r+1} = \beta h_r \), where \( \beta = \frac{2AB - BC}{2AB + BC} \) is a constant.

Hence, \( h_k = \beta^{k-1} h_1 \); \( k = 1, 2, \ldots, n \).  \hspace{1cm} (3)
SMOOTHENING OF THE CONICAL TIPS AND THE ANGULAR EDGES OF THE SURFACE
\[ CP_2 = \varphi_2 \left( \sec(\alpha/2) - 1 \right) \]  

(223)

Table 29 and Fig.27 exhibit the relation between the total number of sub-areas \( N \), radii of curvatures \( \varphi_1 \), \( \varphi_2 \) and the gaps \( AP_1 \) and \( CP_2 \) as \( N \) increases.

The cap at the top, which is a part of a sphere of radius \( \varphi_1 \), can be divided into sub-areas as was done in Chapter 14. To simplify, the angle \( \Theta \) which the arc \( A_1P_1 \) subtend at \( O_1 \), is divided into \( KN \) parts such that

\[ \Theta_j = \Theta_{j-1} + d\Theta \cdot T(KN-j+1) \quad ; \quad j=2,3,\ldots,KN \]

where,

\[ d\Theta = \Theta / (KN + KN/2) \quad , \quad T = \left( d\Theta / (KN) / 2 \right) / \left( KN(2KN+1)/2 \right) \]

and

\[ \Theta_1 = d\Theta \cdot T(KN) \]

The top cap is thus divided into \( KN \) rings of which the \( j \)th ring is of width

\[ d\xi_j = \varphi_1 \cdot d\theta_j \quad , \quad j=1,2,\ldots,KN \]

As before, the \( j \)th ring is divided into \( IK_j \) sub-areas given by

\[ IK_j = 6 \left\{ 1 + (j-1)2 \right\} \quad ; \quad j=1,2,\ldots,KN \]

The curved surface, formed by the arc of revolution \( A_{n-1}P_2C_1 \) of radius of curvature \( \varphi_2 \) is divided into \( KN \) rings each is of width

\[ d\alpha_j = \varphi_2 \left( \alpha / KN \right) \]

and each ring in its turn is divided into \( 2M \) sub-areas.

Because of the rounding off, the number of sub-areas in each of the \( M \) triangular slices reduces by 2. Hence the number of rings on the slant curved conical surface reduces to \( (n-2) \).

The curved cylindrical surface, of height \( (H - CC_1) \) above the plane \( z = 0 \), is divided into \( K_1 \) rings each of width

\[ d\lambda_2 = (H - CC_1) / K_1 \]

where, following Chapter 15, \( K_1 = 3(KN-1) \).

Now the total number of rings on the upper half of the cylinder is

\[ K = KN + (n - 2) + KN + 3(KN - 1) = 5(KN - 1) + n \]  

(224)

The total number of sub-areas on the surface is

\[ N = 2 \left\{ \frac{6KN}{2} \left\{ 2*1 + (KN-1)2 \right\} + (\alpha-2) \cdot 6 \left\{ 1 + (KN-1)2 \right\} \right\} \]
ROUNDING OFF OF THE NODE AND THE EDGE OF THE CYLINDER WITH CONICAL CAPS

<table>
<thead>
<tr>
<th>TOTAL SUBAREA N</th>
<th>NO. OF RINGS 2K</th>
<th>$\rho_1$</th>
<th>$\Delta P_1$</th>
<th>$\rho_2$</th>
<th>$\Delta P_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>6372</td>
<td>90</td>
<td>0.39340</td>
<td>0.16295</td>
<td>0.22235</td>
<td>0.01840</td>
</tr>
<tr>
<td>14268</td>
<td>150</td>
<td>0.20617</td>
<td>0.08540</td>
<td>0.16448</td>
<td>0.01355</td>
</tr>
<tr>
<td>24140</td>
<td>210</td>
<td>0.14183</td>
<td>0.05875</td>
<td>0.13052</td>
<td>0.01075</td>
</tr>
</tbody>
</table>

Table 29

(This should be read in conjunction with fig.26 and fig.27.)
Successive stages of approximations of the cone at the top and the corner at the edge.

\[ R = \text{radius of curvature} \]

**Fig. 27**
\[ +2^* 6KN \left( 1 + (KN-1)^2 \right) + 3(2KN-1)^6 \left( 1 + (KN-1)^2 \right) \]
\[ = 12 \left[ KN^2 + 5(2KN-1)(KN-1) + n \right] \tag{225} \]

**Test Function**

The test function in this case, by (189) of Chapter 13, is

\[ h = \frac{z}{r^3} \]

On the surface \( \partial B \), by (215),

\[ h_e = -\left[ \frac{3xz}{r^5}, \frac{3yz}{r^5}, \frac{3z^2}{r^5} - \frac{1}{r^3} \right] \cdot \hat{n}_a \tag{226} \]

Introducing this into the place of \( \phi^k_e \) in (167) and applying our procedures, we solve for \( \phi \) and generate \( h \) at all nodal points on the surface. The generated values are exhibited in Table 30 for comparison with the analytic values defined by (214) i.e. \( h = \frac{z}{r^3} \) on the boundary. It will be seen that the error in the numerically computed values, for \( K = 45 \), is less than 1.0\%.

As before, we compute \( h \) directly on \( \partial B \) by inserting \( h_e \) from (226) in (172) and applying our procedures. For \( K = 45 \), the computed values of \( h \) at the nodal points are exhibited in Table 30. It will be seen from the Table that the maximum error in the computed \( h \) occurs at a nodal point either at the rounded off tip or corner of \( \partial B \) and the maximum error does not exceed 2.5\%. Further, in this case also, the Simple Source Distribution formulation generates \( h \) at the nodal points of \( \partial B \) which are nearer to the analytic values of \( h \) than those generated by Green's Boundary Formula. It will be noted from the Table that the two approaches yield a comparable accuracy.

**Computation of Disturbance Potential**

In the actual problem, \( \phi^1_e \) on \( \partial B \) is given by (218). Inserting this into (167) and applying our procedures, we solve for \( \phi \). Using these \( \phi_j \) in (169) we generate the \( \phi^k_e \) on \( \partial B \). For \( K = 105 \), the \( \phi^k \) thus obtained are exhibited in Table 31. The total velocity potential is then given by

\[ \Phi = \phi + \psi \]

where \( \psi \) is given by (218). Fig. 28 exhibits the \( \phi \) and \( \Phi \) thus obtained for \( K = 105 \).

Similarly we insert \( \phi^k_e \) given by (218) into (172) and compute \( \phi \) directly. The \( \phi \) thus obtained for \( K = 105 \) are exhibited in Table 31. Fig 28 exhibits the computed \( \phi \) and \( \Phi \) thus obtained for \( K = 105 \).
DEVELOPMENT OF TEST FUNCTION ALONG A MERIDIAN

\[ y = 0 \text{ on the surface} \]

FIELD POINT (Cartesian Co-Ord) | ANALYTICAL \[ h \] | G.B. FORMULA (Jaswon) | \%ERROR | S.L. POTENTIAL (Smith) | \%ERROR
--- | --- | --- | --- | --- | ---

**ON THE ROUNDED SPHERICAL NOSE**

<table>
<thead>
<tr>
<th>( x )</th>
<th>( z )</th>
<th>( h )</th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>0.037</td>
<td>1.835</td>
<td>0.29670</td>
<td>0.30291</td>
<td>2.090</td>
<td>0.29863</td>
</tr>
<tr>
<td>0.095</td>
<td>1.825</td>
<td>0.29989</td>
<td>0.30544</td>
<td>2.190</td>
<td>0.30103</td>
</tr>
<tr>
<td>0.232</td>
<td>1.761</td>
<td>0.31415</td>
<td>0.31972</td>
<td>1.770</td>
<td>0.31641</td>
</tr>
<tr>
<td>0.264</td>
<td>1.735</td>
<td>0.32095</td>
<td>0.32664</td>
<td>1.680</td>
<td>0.32323</td>
</tr>
</tbody>
</table>

**ON THE CONICAL SURFACE**

<table>
<thead>
<tr>
<th>( x )</th>
<th>( z )</th>
<th>( h )</th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>0.288</td>
<td>1.712</td>
<td>0.32720</td>
<td>0.32768</td>
<td>0.146</td>
<td>0.32936</td>
</tr>
<tr>
<td>0.304</td>
<td>1.692</td>
<td>0.33265</td>
<td>0.33344</td>
<td>0.237</td>
<td>0.33484</td>
</tr>
<tr>
<td>0.494</td>
<td>1.506</td>
<td>0.37817</td>
<td>0.37939</td>
<td>0.324</td>
<td>0.38230</td>
</tr>
<tr>
<td>0.528</td>
<td>1.472</td>
<td>0.38499</td>
<td>0.39606</td>
<td>0.304</td>
<td>0.38698</td>
</tr>
<tr>
<td>0.946</td>
<td>1.154</td>
<td>0.39391</td>
<td>0.39410</td>
<td>0.047</td>
<td>0.39585</td>
</tr>
<tr>
<td>0.905</td>
<td>1.095</td>
<td>0.38199</td>
<td>0.38212</td>
<td>0.034</td>
<td>0.38460</td>
</tr>
</tbody>
</table>

**ON THE ROUNDED OFF CORNER**

<table>
<thead>
<tr>
<th>( x )</th>
<th>( z )</th>
<th>( h )</th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>0.945</td>
<td>1.054</td>
<td>0.37151</td>
<td>0.37919</td>
<td>2.070</td>
<td>0.37404</td>
</tr>
<tr>
<td>0.963</td>
<td>1.031</td>
<td>0.36732</td>
<td>0.37472</td>
<td>2.010</td>
<td>0.37019</td>
</tr>
<tr>
<td>0.996</td>
<td>0.951</td>
<td>0.36426</td>
<td>0.37164</td>
<td>2.030</td>
<td>0.36731</td>
</tr>
<tr>
<td>1.000</td>
<td>0.922</td>
<td>0.36661</td>
<td>0.37570</td>
<td>2.480</td>
<td>0.36983</td>
</tr>
</tbody>
</table>

**ON THE CYLINDRICAL SURFACE**

<table>
<thead>
<tr>
<th>( x )</th>
<th>( z )</th>
<th>( h )</th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>1.000</td>
<td>0.878</td>
<td>0.37262</td>
<td>0.37424</td>
<td>0.435</td>
<td>0.37529</td>
</tr>
<tr>
<td>1.000</td>
<td>0.817</td>
<td>0.37942</td>
<td>0.38160</td>
<td>0.576</td>
<td>0.38191</td>
</tr>
<tr>
<td>1.000</td>
<td>0.757</td>
<td>0.38371</td>
<td>0.38586</td>
<td>0.560</td>
<td>0.38603</td>
</tr>
<tr>
<td>1.000</td>
<td>0.151</td>
<td>0.14626</td>
<td>0.14672</td>
<td>0.311</td>
<td>0.14672</td>
</tr>
<tr>
<td>1.000</td>
<td>0.091</td>
<td>0.08968</td>
<td>0.09995</td>
<td>0.300</td>
<td>0.08995</td>
</tr>
<tr>
<td>1.000</td>
<td>0.030</td>
<td>0.03022</td>
<td>0.03031</td>
<td>0.295</td>
<td>0.03031</td>
</tr>
</tbody>
</table>

Table 30
## Table 31: Generation of Disturbance Potential Along Meridian $y = 0$ on the Surface

<table>
<thead>
<tr>
<th>Field Point (Cartesian Co-Ord)</th>
<th>$\phi$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>S.L. Potential</td>
</tr>
<tr>
<td><strong>On the Rounded Spherical Nose</strong></td>
<td></td>
</tr>
<tr>
<td>$x=0.008$, $z=1.941$</td>
<td>0.37332</td>
</tr>
<tr>
<td>$x=0.021$, $z=1.940$</td>
<td>0.37380</td>
</tr>
<tr>
<td>$x=0.091$, $z=1.908$</td>
<td>0.38023</td>
</tr>
<tr>
<td>$x=0.097$, $z=1.902$</td>
<td>0.38185</td>
</tr>
<tr>
<td><strong>On the Conical Surface</strong></td>
<td></td>
</tr>
<tr>
<td>$x=0.102$, $z=1.898$</td>
<td>0.38336</td>
</tr>
<tr>
<td>$x=0.106$, $z=1.894$</td>
<td>0.38476</td>
</tr>
<tr>
<td>$x=0.215$, $z=1.785$</td>
<td>0.41595</td>
</tr>
<tr>
<td>$x=0.468$, $z=1.532$</td>
<td>0.44412</td>
</tr>
<tr>
<td>$x=0.907$, $z=1.093$</td>
<td>0.32913</td>
</tr>
<tr>
<td>$x=0.944$, $z=1.056$</td>
<td>0.30000</td>
</tr>
<tr>
<td><strong>On the Rounded-Off Corner</strong></td>
<td></td>
</tr>
<tr>
<td>$x=0.965$, $z=1.035$</td>
<td>0.27493</td>
</tr>
<tr>
<td>$x=0.972$, $z=1.027$</td>
<td>0.26577</td>
</tr>
<tr>
<td>$x=0.999$, $z=0.961$</td>
<td>0.19524</td>
</tr>
<tr>
<td>$x=1.00$, $z=0.951$</td>
<td>0.18687</td>
</tr>
<tr>
<td><strong>On the Cylindrical Surface</strong></td>
<td></td>
</tr>
<tr>
<td>$x=1.00$, $z=0.928$</td>
<td>0.17262</td>
</tr>
<tr>
<td>$x=1.00$, $z=0.893$</td>
<td>0.15608</td>
</tr>
<tr>
<td>$x=1.00$, $z=0.543$</td>
<td>0.07127</td>
</tr>
<tr>
<td>$x=1.00$, $z=0.053$</td>
<td>0.00619</td>
</tr>
<tr>
<td>$x=1.00$, $z=0.018$</td>
<td>0.00206</td>
</tr>
</tbody>
</table>

Table 31
INDEPENDENT EQUATIONS = 105
TOTAL SUB AREA = 24,140

Green's boundary formula (Jasson)
Simple layer potential (Smith)

1) DISTURBANCE POTENTIAL
2) VELOCITY POTENTIAL
It will be seen that the two approaches yield similar result.

Tangential Velocity on the Surface

By symmetry, the tangential velocity is directed along the meridians of \( \partial B \). The velocity at \( p \in \partial B \), by (175), neglecting the higher order terms, is given by

\[
\nu_1 (\nabla) = \nu_1 (\nabla_1) = -\frac{1}{\mu_1} \delta_1 .
\]

The velocities thus calculated, on the basis of the two formulations, are exhibited in Table 32 and graphed in Fig. 29.

We know from theory of potential flow that the velocity becomes infinite in the neighbourhood of a sharp edge or a corner. Fig. 30 shows the numerically computed velocity in the neighbourhood of a corner \( C \), smoothed out by an arc of a circle of contact of radius \( c_2 \) (Fig. 26). It is interesting to note that the velocity at the corner rises indefinitely as the radius of curvature of the circle of contact decreases.
VELOCITY ON THE SURFACE ALONG A MERIDIAN $y = 0$

<table>
<thead>
<tr>
<th>FIELD POINT (CARTESIAN CO-ORD)</th>
<th>VELOCITY ON THE SURFACE</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>S.L.POTENTIAL</td>
</tr>
<tr>
<td><strong>ON THE ROUNDED SPHERICAL NOSE</strong></td>
<td></td>
</tr>
<tr>
<td>0.0148</td>
<td>1.9405</td>
</tr>
<tr>
<td>0.0278</td>
<td>1.9385</td>
</tr>
<tr>
<td>0.0876</td>
<td>1.9110</td>
</tr>
<tr>
<td>0.0943</td>
<td>1.9053</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th><strong>ON THE CONICAL SURFACE</strong></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1043</td>
<td>1.8957</td>
<td>0.46295</td>
</tr>
<tr>
<td>0.1085</td>
<td>1.8915</td>
<td>0.46450</td>
</tr>
<tr>
<td>0.2104</td>
<td>1.7896</td>
<td>0.54209</td>
</tr>
<tr>
<td>0.4413</td>
<td>1.5587</td>
<td>0.69142</td>
</tr>
<tr>
<td>0.8901</td>
<td>1.1099</td>
<td>1.15854</td>
</tr>
<tr>
<td>0.9255</td>
<td>1.0745</td>
<td>1.27812</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th><strong>ON THE ROUNDED-OFF CORNER</strong></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>0.9687</td>
<td>1.0307</td>
<td>1.65374</td>
</tr>
<tr>
<td>0.9751</td>
<td>1.0226</td>
<td>1.79843</td>
</tr>
<tr>
<td>0.9984</td>
<td>0.9663</td>
<td>1.88886</td>
</tr>
<tr>
<td>0.9996</td>
<td>0.9562</td>
<td>1.81258</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th><strong>ON THE CYLINDRICAL SURFACE</strong></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>1.00</td>
<td>0.9109</td>
<td>1.47207</td>
</tr>
<tr>
<td>1.00</td>
<td>0.8759</td>
<td>1.37907</td>
</tr>
<tr>
<td>1.00</td>
<td>0.5956</td>
<td>1.17657</td>
</tr>
<tr>
<td>1.00</td>
<td>0.0701</td>
<td>1.11817</td>
</tr>
<tr>
<td>1.00</td>
<td>0.0350</td>
<td>1.11773</td>
</tr>
</tbody>
</table>

Table 32
VELOEITY DISTRIBUTION ON THE SURFACE
FLUID VELOCITY AT SHARP EDGE ON SUCCESSIVE APPROXIMATIONS

Fig. 30
CHAPTER 17

POTENTIAL FLOW PAST A THICK DELTA WING

Introduction

A thick isosceles triangular plate of semi-apex angle $60^\circ$, perturbs an otherwise uniform free flow directed approximately parallel to the plane of the plate. The centroid $O$ of the plate defines the origin of a cartesian reference frame $OXYZ$ [Fig. 31(a)], where $OZ$ is perpendicular to the plane of the plate. Relative to the co-ordinates axes, the triangular faces define the planes $z = \pm H$ [Fig. 31(b)].

The free flow is approximately parallel to the $XOY$ plane in the negative $Y$-direction. If velocity vector $U$ makes an angle $-\theta$ with $OY$ [Fig. 31(c)], it follows that

$$\nabla \psi = U(O, -\cos\theta, \sin\theta).$$

Therefore, taking $U = 1$,

$$\psi = (0, \cos\theta, -\sin\theta) \cdot (x, y, z).$$

For a small angle $(= \theta)$ of attack, the flow remains potential. The disturbance potential $\phi \rightarrow 0$ as $|z| \rightarrow \infty$, and satisfies Laplace's equation

$$\nabla^2 \phi = 0; \quad \hat{z} \in B_e,$$

with boundary condition (168), i.e.

$$\phi_e^l = -\psi_e^l = (0, -\cos\theta, \sin\theta) \cdot \hat{n}_e.$$

The integral equation formulation provides a straightforward approach to determine $\phi$ on the boundary. This is achieved by substituting (229) into equations (167) or (172) and solving them numerically.

Subdivision of Boundary

To solve the integral equation numerically, we divide $OB$ into sub-areas. In this case, we shall not be forced, as in the previous case, to deal with thin sub-areas, and hence the rounding-off of the corner is not necessary. Of course, rounding-off of the sharp edge and corner improves the solution at the
Plane section through the centroid $O$, parallel to the triangular surface of the plate, exhibiting orientation of axes and apex $A_0$. The $Z$-axis passes through $O$ and is perpendicular to this plane.
FLOW PAST A THICK DELTA WING

Fig. 3.1(c)
expense of greater complications.

From symmetry, the sub-areas on the plane \( z = -H \) are made exactly similar to those on the plane \( z = H \). Further, for a sub-area to the right of the plane \( x = 0 \), there is a corresponding sub-area on the left.

From the Fig. 32, the number of sub-areas in the zone
\[
(A \bar{A} \bar{F} F + F \bar{F} \bar{C} C \bar{E} E) \text{ is}
\]
\[
N_1 = (2NT - 1) 7 + 10; \ NT = 3, 6 \ldots 3 + (n-1)3 \quad (230)
\]
The number of sub-areas in the region \((D \bar{D} \bar{E} E + \bar{D} \bar{E} F \bar{A} A)\) is
\[
N_2 = \frac{NT-3}{2} \left(2 \times 4 + (\frac{NT-3}{2} - 1)\right) \times 2 + 6 \quad (231)
\]
The half thickness \( H \) of the plate is determined by
\[
H = AA_0 = \left(\frac{AC}{4NT}\right) KN \quad ; \quad KN = 2, 3, \ldots m \quad (232)
\]
The number of sub-areas in the region \( DD \bar{C} C \bar{C} \bar{O} O \) \( \text{Fig. 32 (a)} \) is
\[
N_3 = 2NT \times KN \; \quad (233)
\]
and that on the region \( F C C F \bar{O} O \) \( \text{Fig. 32(b)} \) is
\[
N_4 = (4NT - 2) \times KN \; \quad (234)
\]
If \( N_5 \) is the number of sub-areas on \( AFF_0 \bar{A} \bar{O} \) \( \text{Fig. 32(b)} \), then
\[
N_5 = 2 \times 2 \times 2 \times KN = 8KN \quad .
\]
Hence, the total number of sub-areas \( N \) on O8 is
\[
N = 4 \sum_{j=1}^{5} N_j = 4 \times N^* \quad (236)
\]
\[
= 4 \left[14 NT + (NT-3)(NT+4) + 6KN(NT+1) + 9\right] .
\]

There are 11 different sub-areas on O8 \( \text{Fig. 32, Fig. 32(a), Fig. 32 (b)} \). An analysis of these for \( NT = 6, \ KN = 2 \) are given in Table 33.
DIVISION OF THE TRIANGULAR PLANE FACE INTO SUB-AREAS
ANALYSIS OF SUB-AREA ON THE SURFACE FOR KN=2 and NT=6

<table>
<thead>
<tr>
<th>REGION</th>
<th>FORM</th>
<th>BASE</th>
<th>HEIGHT</th>
<th>AREA</th>
</tr>
</thead>
<tbody>
<tr>
<td>AÀFF</td>
<td>TRIANGULAR (7)</td>
<td>0.02083</td>
<td>0.01203</td>
<td>0.00013</td>
</tr>
<tr>
<td></td>
<td>RECTANGULAR (6)</td>
<td>0.02083</td>
<td>0.01203</td>
<td>0.00025</td>
</tr>
<tr>
<td></td>
<td>&quot; (5)</td>
<td>0.02083</td>
<td>0.02405</td>
<td>0.00050</td>
</tr>
<tr>
<td></td>
<td>&quot; (4)</td>
<td>0.04167</td>
<td>0.02405</td>
<td>0.00100</td>
</tr>
<tr>
<td></td>
<td>&quot; (3)</td>
<td>0.04167</td>
<td>0.04810</td>
<td>0.00201</td>
</tr>
<tr>
<td>FFÈEC</td>
<td>RECTANGULAR (4)</td>
<td>0.04167</td>
<td>0.02405</td>
<td>0.00100</td>
</tr>
<tr>
<td></td>
<td>&quot; (3)</td>
<td>0.04167</td>
<td>0.04810</td>
<td>0.00201</td>
</tr>
<tr>
<td></td>
<td>TRIANGULAR (8)</td>
<td>0.04167</td>
<td>0.02405</td>
<td>0.00050</td>
</tr>
<tr>
<td>ÌFÈED</td>
<td>RECTANGULAR (1)</td>
<td>0.08333</td>
<td>0.09623</td>
<td>0.00802</td>
</tr>
<tr>
<td></td>
<td>&quot; (2)</td>
<td>0.08333</td>
<td>0.04811</td>
<td>0.00401</td>
</tr>
<tr>
<td>DOC₀D₀</td>
<td>RECTANGULAR (9)</td>
<td>0.08333</td>
<td>0.04811</td>
<td>0.00401</td>
</tr>
<tr>
<td>CC₀F₀F</td>
<td>RECTANGULAR (10)</td>
<td>0.04812</td>
<td>0.04812</td>
<td>0.00232</td>
</tr>
<tr>
<td></td>
<td>&quot; (11)</td>
<td>0.02406</td>
<td>0.02406</td>
<td>0.00058</td>
</tr>
</tbody>
</table>

Table 33
(This should be read in conjunction with Fig.32)
We now introduce the test function

\[ h = \frac{y}{r^3} \]  

(237)

which behaves in a comparable way to the disturbance potential \( \phi \) on \( \partial B \),

\[ \hat{h} = -\left[ \frac{3y}{r^5} , \frac{3y}{r^5} - \frac{1}{r^3} , \frac{3y^2}{r^5} \right] \hat{\eta}_e \]  

(238)

Introducing this in place of \( \phi_{e} \) in (167) and applying our procedures, we solve for \( \sigma \) and generate \( h \) at the nodal points on the surface. The values of \( h \) generated for \( KN = 2, NT = 6 \) i.e. \( N^* = 207 \), are exhibited in Table 34 for comparison with the analytic values defined by (237) on \( \partial B \).

Alternatively we find \( h \) directly on \( \partial B \) by inserting (238) into (172) and applying our procedures. A few of the values of \( h \) at the nodal points in the neighbourhood of the apex of the delta, thus determined, for \( N^* = 207 \), are exhibited in Table 34.

It is evident from Table 34 that, for the same sub-division of \( \partial B \) the Simple Layer potential method (Smith) generates an \( h \) which is nearer to analytic value than that obtained by Green's Boundary Formula (Jaswon).

The percentage error in the computed value of \( h \) obtained by Green's Boundary Formula increases, as expected, when it is generated at a nodal point adjacent to the apex or the leading edge of the delta. This error falls rapidly as we move away from the edge.

Computation of Disturbance Potential

In the actual case, \( \phi_{e} \) is given by (229). Substituting this in (167) and applying our procedures we solve for \( \sigma_j \). Using these \( \sigma_j \) in (170) we generate \( \phi \) on \( \partial B \). For \( NT = 12, KN = 2 \) and \( \theta = 1^\circ \), the disturbance potential \( \phi \) is generated at the nodal points on \( \partial B \) and some of these values are exhibited in Table 35. The total velocity potential \( \Phi \) is then obtained by (53) viz.

\[ \Phi = \phi + \psi \]

where \( \psi \) is given by (228).

Similarly, we insert \( \phi_{e} \), given by (229) into (172) and compute \( \phi \) directly on \( \partial B \). \( \phi \) thus computed for \( NT = 12, KN = 2 \) and \( \theta = 1^\circ \) are exhibited in Table 35.
DEVELOPMENT OF TEST FUNCTION h ON THE PLANE z = 0.096

<table>
<thead>
<tr>
<th>FIELD POINT (ΚΟΡΚΕΑ ΚΟ-ΟΤ)</th>
<th>ANALYTICAL h</th>
<th>S.L. POTENTIAL (Smith)</th>
<th>M.P.H.O.R</th>
<th>G.P. FORMULA (Jaswon)</th>
<th>M.P.H.O.R</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.007</td>
<td>0.177</td>
<td>6.4009</td>
<td>6.4597</td>
<td>0.918</td>
<td>8.2628</td>
</tr>
<tr>
<td>0.025</td>
<td>0.365</td>
<td>6.7365</td>
<td>6.8064</td>
<td>0.947</td>
<td>8.5138</td>
</tr>
<tr>
<td>0.049</td>
<td>0.365</td>
<td>7.0264</td>
<td>7.0958</td>
<td>0.988</td>
<td>8.9163</td>
</tr>
<tr>
<td>0.069</td>
<td>0.341</td>
<td>7.2520</td>
<td>7.3252</td>
<td>1.009</td>
<td>9.1888</td>
</tr>
<tr>
<td>0.090</td>
<td>0.367</td>
<td>6.7160</td>
<td>6.7816</td>
<td>0.964</td>
<td>7.2132</td>
</tr>
<tr>
<td>0.031</td>
<td>0.349</td>
<td>7.2810</td>
<td>7.3552</td>
<td>1.020</td>
<td>7.5911</td>
</tr>
<tr>
<td>0.052</td>
<td>0.343</td>
<td>7.3572</td>
<td>7.4124</td>
<td>1.028</td>
<td>7.9761</td>
</tr>
<tr>
<td>0.073</td>
<td>0.325</td>
<td>7.8091</td>
<td>7.9090</td>
<td>1.084</td>
<td>8.4657</td>
</tr>
<tr>
<td>0.100</td>
<td>0.349</td>
<td>7.3534</td>
<td>7.4299</td>
<td>1.027</td>
<td>7.4734</td>
</tr>
<tr>
<td>0.061</td>
<td>0.325</td>
<td>8.2516</td>
<td>8.3402</td>
<td>1.074</td>
<td>8.4150</td>
</tr>
<tr>
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<td>0.325</td>
<td>8.0693</td>
<td>8.1566</td>
<td>1.042</td>
<td>8.3040</td>
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<tr>
<td>0.021</td>
<td>0.301</td>
<td>9.4926</td>
<td>9.5875</td>
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<td>9.5222</td>
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<tr>
<td>0.100</td>
<td>0.325</td>
<td>8.3453</td>
<td>8.4335</td>
<td>1.056</td>
<td>8.4806</td>
</tr>
<tr>
<td>0.062</td>
<td>0.301</td>
<td>9.0133</td>
<td>9.1162</td>
<td>1.075</td>
<td>9.1889</td>
</tr>
<tr>
<td>0.046</td>
<td>0.217</td>
<td>15.5520</td>
<td>15.5900</td>
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<td>15.5510</td>
</tr>
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<td>0.042</td>
<td>0.144</td>
<td>25.4190</td>
<td>24.5260</td>
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<td>24.4960</td>
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<td>0.042</td>
<td>0.048</td>
<td>31.3320</td>
<td>32.7570</td>
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<tr>
<td>0.042</td>
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</tr>
<tr>
<td>0.042</td>
<td>-0.120</td>
<td>-29.6030</td>
<td>-29.2020</td>
<td>1.356</td>
<td>-29.3520</td>
</tr>
</tbody>
</table>

Table 34
GENERATION OF $\phi$ ON THE PLANE SURFACE $z=0.0484$

<table>
<thead>
<tr>
<th>FIELD POINT</th>
<th>DISTURBANCE POTENTIAL</th>
<th>S.L.POTENTIAL (Smith)</th>
<th>G.B.FORMULA (Jaswon)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x$</td>
<td>$y$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.0052</td>
<td>0.3759</td>
<td>0.03367</td>
<td>0.03720</td>
</tr>
<tr>
<td>0.0156</td>
<td>0.3669</td>
<td>0.03185</td>
<td>0.03422</td>
</tr>
<tr>
<td>0.0260</td>
<td>0.3669</td>
<td>0.03287</td>
<td>0.03714</td>
</tr>
<tr>
<td>0.0365</td>
<td>0.3548</td>
<td>0.03139</td>
<td>0.03592</td>
</tr>
<tr>
<td>0.0052</td>
<td>0.3669</td>
<td>0.03057</td>
<td>0.03180</td>
</tr>
<tr>
<td>0.0156</td>
<td>0.3548</td>
<td>0.02833</td>
<td>0.02978</td>
</tr>
<tr>
<td>0.0260</td>
<td>0.3548</td>
<td>0.02943</td>
<td>0.03138</td>
</tr>
<tr>
<td>0.0104</td>
<td>0.3428</td>
<td>0.02574</td>
<td>0.02684</td>
</tr>
<tr>
<td>0.0052</td>
<td>0.3548</td>
<td>0.02774</td>
<td>0.02903</td>
</tr>
<tr>
<td>0.0312</td>
<td>0.3428</td>
<td>0.02719</td>
<td>0.02842</td>
</tr>
<tr>
<td>0.0208</td>
<td>0.3007</td>
<td>0.01948</td>
<td>0.02032</td>
</tr>
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<td>0.0208</td>
<td>0.2646</td>
<td>0.01502</td>
<td>0.01581</td>
</tr>
<tr>
<td>0.0208</td>
<td>0.2165</td>
<td>0.01014</td>
<td>0.01076</td>
</tr>
<tr>
<td>0.0208</td>
<td>0.1684</td>
<td>0.00583</td>
<td>0.00629</td>
</tr>
<tr>
<td>0.0208</td>
<td>0.1203</td>
<td>0.00180</td>
<td>0.00202</td>
</tr>
<tr>
<td>0.0208</td>
<td>0.0722</td>
<td>-0.00212</td>
<td>-0.00197</td>
</tr>
<tr>
<td>0.0208</td>
<td>0.0241</td>
<td>-0.00610</td>
<td>-0.00612</td>
</tr>
<tr>
<td>0.0208</td>
<td>-0.0241</td>
<td>-0.01035</td>
<td>-0.01056</td>
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<tr>
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<td>-0.1203</td>
<td>-0.02130</td>
<td>-0.02204</td>
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<tr>
<td>0.0208</td>
<td>-0.1564</td>
<td>-0.02752</td>
<td>-0.02868</td>
</tr>
</tbody>
</table>

Table 35
Tangential velocity Component on the Surface

The downward tangential velocity component of the fluid on the surface is calculated, as in Chapter 14, by taking only the 1st term in (175). For \( KN = 2 \), \( NT = 12 \) i.e. \( 2N = 954 \) and \( \Theta = 1^\circ \), the velocity thus obtained along a line \( x = \) constant, is exhibited in Table 36. Fig 33 shows the velocity component along \( x = \) constant, on the planes \( z = \pm H \), based on Table 36. Table 37 exhibits the downward velocity component distributed along the lines \( y = \) constant, adjacent to the upper and lower edges \( CD \) and \( CD \) on the planes \( z = \pm H \). Fig. 34 shows the graphs of the above velocities based on Table 37.

Effect of Thickness Variation

To keep the error due to the approximations made in the evaluation of the integrals below \( 1\% \), the distance between the two triangular planes i.e. the thickness \( 2H \), must satisfy (102) of Chapter 7, i.e.

\[
2H \gg 2L ,
\]

where \( L \) is the diagonal of the biggest sub-area on \( \partial B \).

From Fig. 32,

\[
L = 2 \left( \frac{AC}{4NT} \right)
\]

and, by (232),

\[
2H = 2 \left( \frac{AC}{4NT} \right) \cdot \frac{KN}{L} = L \cdot KN.
\]

Hence, by (239), the minimum value of \( KN \) is 2. For a particular \( AC \), keeping \( KN \) fixed if \( NT \) is increased the thickness decreases satisfying (239) at every stage. Alternately, keeping \( NT \) fixed if \( KN \) is increased thickness increases keeping the subdivisions on the triangular planes unchanged.

To find the effect of thickness on \( \phi \), following our procedures, \( \phi \) is calculated for \( \Theta = 0 \) taking \( KN = 2 \), \( NT = 9 \) and again for \( \Theta = 0 \) taking \( KN = 3 \), \( NT = 9 \). The \( \phi_k \) thus computed on \( \partial B \), along a line \( x = \) constant, are exhibited in Fig. 35.

Discussion

It is evident from Table 35 that the two values of \( \phi \) obtained by the two methods at the nodal points on \( \partial B \) approximately agree with one another except, as expected, at those points adjacent to the sharp edge and to the corner.
VELOCITY ON THE SURFACES \( z = \pm 0.0481 \) ALONG THE INTERSECTION OF THE PLANE \( x = 0.0208 \)

<table>
<thead>
<tr>
<th>FIELD POINT ( y )</th>
<th>DOWNWARD VELOCITY ON SURFACE ON UPPER PLANE ( z=0.0481 )</th>
<th>ON LOWER PLANE ( z=-0.0481 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.2827</td>
<td>1.12321</td>
<td>1.09748</td>
</tr>
<tr>
<td>0.2406</td>
<td>1.10146</td>
<td>1.08326</td>
</tr>
<tr>
<td>0.1925</td>
<td>1.08943</td>
<td>1.07649</td>
</tr>
<tr>
<td>0.1443</td>
<td>1.08343</td>
<td>1.07501</td>
</tr>
<tr>
<td>0.0962</td>
<td>1.08136</td>
<td>1.07733</td>
</tr>
<tr>
<td>0.0481</td>
<td>1.08267</td>
<td>1.08314</td>
</tr>
<tr>
<td>0.000</td>
<td>1.08806</td>
<td>1.09345</td>
</tr>
<tr>
<td>-0.0481</td>
<td>1.10019</td>
<td>1.11138</td>
</tr>
<tr>
<td>-0.0962</td>
<td>1.12709</td>
<td>1.14558</td>
</tr>
<tr>
<td>-0.1383</td>
<td>1.17239</td>
<td>1.20533</td>
</tr>
<tr>
<td>-0.1684</td>
<td>1.34726</td>
<td>1.39694</td>
</tr>
</tbody>
</table>

Table 36
Downward velocity component on the upper and lower surfaces along a line x = constant (0.0208) passing through E C D 1.
DOWNWASH ON THE LINE $y = -0.1684$ ON THE SURFACES $z = \pm 0.0481$

<table>
<thead>
<tr>
<th>FIELD POINT $x$</th>
<th>DOWN WASH VELOCITY ON UPPER PLANE $z=0.0481$</th>
<th>DOWN WASH VELOCITY ON LOWER PLANE $z=-0.0481$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0208</td>
<td>1.34726</td>
<td>1.39694</td>
</tr>
<tr>
<td>0.0625</td>
<td>1.34759</td>
<td>1.39713</td>
</tr>
<tr>
<td>0.1042</td>
<td>1.34824</td>
<td>1.39750</td>
</tr>
<tr>
<td>0.1458</td>
<td>1.34921</td>
<td>1.39806</td>
</tr>
<tr>
<td>0.1875</td>
<td>1.35050</td>
<td>1.39880</td>
</tr>
<tr>
<td>0.2292</td>
<td>1.35211</td>
<td>1.39972</td>
</tr>
<tr>
<td>0.2708</td>
<td>1.35406</td>
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</tr>
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<td>0.3125</td>
<td>1.35634</td>
<td>1.40216</td>
</tr>
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<td>0.3542</td>
<td>1.35898</td>
<td>1.40370</td>
</tr>
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<td>1.36200</td>
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<td>1.36547</td>
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<td>0.4792</td>
<td>1.36944</td>
<td>1.41004</td>
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<td>0.5208</td>
<td>1.37402</td>
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<td>1.37937</td>
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<td>0.6042</td>
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<tr>
<td>0.7292</td>
<td>1.42276</td>
<td>1.45227</td>
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</table>

Table 37
DOWNWARD VELOCITY COMPONENT ON THE UPPER AND ON THE LOWER SURFACES OVER THE LINES $y = \text{constant} (-0.17)$.

($v_y$ stands for velocity component in $y$-direction)
Fig. 35

EFFECT OF THICKNESS ON THE DISTURBANCE POTENTIAL

( T stands for thickness)
It is evident from Fig. 33 and Fig. 34 that, near the trailing edge, the downward velocity on the upper surface is less than that at the corresponding point on the lower surface. This clearly indicates that a vortex sheet will be formed behind the delta wing in the case of a real fluid.

Fig. 35 demonstrates that the disturbance, due to the thickness alone, gradually dies out as the thickness diminishes to zero.
CHAPTER 18
FLOW PAST A THIN DELTA WING

Introduction

For symmetric flow past a thick plate the disturbance potential $\phi$ is, in general, the superposition of two functions $\phi_s$ and $\phi_a$, i.e.

$$\phi = \phi_s + \phi_a$$  \hspace{1cm} (240)

where $\phi_s$ is symmetric and $\phi_a$ is antisymmetric about the plane of the plate. The former arises from the thickness of the plate and the latter arises because of the inclination of the plate to the direction of flow. Thus if $2H$ (thickness) $\neq 0$ and $\theta$ (angle of attack) $= 0$, $\phi = \phi_s$. If $2H = 0$ and $\theta \neq 0$, $\phi = \phi_a$.

The formulation (169) and (171) are only valid when the volume enclosed by $\partial B$ differs from zero. Hence they do not apply to a thin delta wing. We may compute $\phi$ as a limit when the volume enclosed by $\partial B$ tends to zero. To proceed directly with this plan, keeping the numerical error within a tolerable range, $2H$ should be determined by (239) of Chapter 17. As a result, to attain a reasonably small value of $H$, the number of independent equations becomes very large and this in turn demands a huge matrix for storage in the computer. For example, when $NT = 15$, for $KN = 2$, $2H = 0.07698$ and the corresponding number of equations, by (236), becomes 1278. Hence because of the storage capacity alone, leaving aside the attainment of a sufficient degree of accuracy in solving such a huge number of equations, we can not proceed beyond a certain limit.

A way out of the difficulty is to separate the symmetric and antisymmetric components of $\phi$ for any thickness $t = 2H$. If $\phi_a^+$, $\phi_a^-$ represent $\phi_a$, and if $\phi_s^+$, $\phi_s^-$ represent $\phi_s$ respectively on the upper and on the lower surfaces of $\partial B$, we have

$$\phi_+ = \phi_a^+ + \phi_s^+$$  \hspace{1cm} (241)

$$\phi_- = \phi_a^- + \phi_s^-$$  \hspace{1cm} (242)
where \( \phi_+ \), \( \phi_- \) represent the values of \( \phi \) on the upper and on the lower surfaces respectively. Since

\[
\phi_a^+ = -\phi_a^- \quad \text{and} \quad \phi_s^+ = \phi_s^- ,
\]

we find

\[
\phi_s^+ = \phi_s^- = \frac{\phi_+ + \phi_-}{2} , \tag{243}
\]

\[
\phi_a^+ = -\phi_a^- = \frac{\phi_+ - \phi_-}{2} . \tag{244}
\]

The relations (243), (244) yield \( \phi_s \) and \( \phi_a \) of (240). We expect this \( \phi_a \) to approximate the value of \( \phi \) for a thin plate inclined at a small angle of attack \( \theta \not= 0 \).

An useful check on \( \phi_s \) is to compute \( \phi \) for a thick plate parallel to the stream (i.e. \( \theta = 0 \)). This computation can be compared with the \( \phi_s \) previously determined on the basis of (243). It will be seen from Table 38 that the two results compare very closely. The same applies to the computation of \( \phi_a \).

**Polynomial Interpolation**

After computation of \( \phi \) on \( OB \) by the methods stated in Chapter 17, \( \phi_a \) at the discrete points on \( OB \) can be found by (244). If we wish to know the value of \( \phi_a \) at any point in the neighbourhood of any discrete point, we shall have to fit a polynomial through the function values at these points, which should represent the function to a certain degree of accuracy. It has been pointed out, in Chapter 6, that this can be done as accurately as we please, by fitting a polynomial of sufficiently high degree to the data.

Let \( \phi(q_1), \phi(q_2), \phi(q_3), \ldots, \phi(q_L) \) represent the values of \( \phi \) at the discrete points \( q_1, q_2, \ldots, q_L \) on \( OB \). A suitable polynomial \( p(x,y,z) \) of degree \( m \) is given by

\[
P = \sum_{j=0}^{m} C_j p_j ; \quad m \leq L \ , \tag{245}
\]
VERIFICATION OF THE SEPARATION OF THE SYMMETRIC AND ANTISYMMETRIC PART OF THE DISTURBANCE POTENTIAL

DISTURBANCE POTENTIAL ON THE PLANE SURFACE $z=0.0962$ IN THE NEIGHBOURHOOD OF THE TIP

<table>
<thead>
<tr>
<th>$x$</th>
<th>$y$</th>
<th>$\phi^+$</th>
<th>$\phi^-$</th>
<th>$\phi_s = \frac{\phi^+ + \phi^-}{2}$</th>
<th>$\phi_s$</th>
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<td>0.0069</td>
<td>0.3769</td>
<td>0.05767</td>
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<td>0.3669</td>
<td>0.05360</td>
<td>0.05707</td>
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<td>0.04924</td>
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<tr>
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<td>0.04215</td>
<td>0.04721</td>
<td>0.04468</td>
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<td>0.04492</td>
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Table 38
where $P_j$ is a homogeneous polynomial of degree $j$ and $C_j$ is the coefficient of $P_j$. If we wish to approximate $\phi$ by $P$ to a certain degree of accuracy, defined by a pre-assigned small quantity $\epsilon$ ($>0$), starting from $m = 2$ we increase $m$ by a step of 1 and at every stage the $C_j$ are determined by the least squares method until a stage comes when

$$|\phi(y_k) - P(y_k)| \leq \epsilon; \quad k = 1, 2, \ldots, L.$$ (246)

In this case, $\phi$ is a harmonic function symmetric with respect to $x$ and antisymmetric with respect to $z$. Hence the $P_j$ are to be so chosen that they must satisfy

$$\nabla^2 P_j = 0,$$

$$P_j(x,y,z) = P_j(-x,y,z)$$

and $\frac{\partial P_j}{\partial z}(x,y,z) = -\frac{\partial}{\partial z} P_j(x,y,-z).$ (247)

Under the above conditions the polynomials $P_j$ may be chosen as

$$P_0 = 1$$

$$P_1 = y + z$$

$$P_2 = x^2 + y^2 + yz$$

$$P_3 = y^3 + z^3 - 3x^2y + x^2z - 4y^2z$$

$$P_4 = x^4 + y^4 - 6x^2y^2 + 6x^2yz - yz^3 - zy^3$$

$$P_5 = x^5 + y^5 + x^4z + y^4z + 5x^4y - 10x^2y^3 - 5x^2z^3 + 5y^2z^3 + 9x^2y^2z$$

$$P_6 = x^6 - y^6 - 15x^4y^2 - x^4yz + 15x^2y^4 + x^2z^4 + x^2yz^3 - y^5z + 3y^3z^3 - 3y^3z^3 - yz^5$$

$$P_7 = y^7 + 2z^7 - 7yx^6 + 4x^6z + 35x^4y^3 - 25x^4y^3 - 21x^2y^5 + 21x^2z^5$$

$$+ 4y^6z - 25y^4z^3 + 21y^2z^5 + 15x^4y^2z + 15x^2y^4z - 60y^2z^3$$

and so on.

**Computed Results**

It has already been pointed out that the Simple Layer Formulation yields a tolerably accurate $\phi$, particularly near the edges and the apex. Hence to obtain information about $\phi$ near the apex of the delta, we consider the values of $\phi$ obtained by the Simple Layer potential method only. Further, since values of $\phi$ at the nodal points, adjacent to the tip and the leading edge, are not so reliable, these values are not taken into consideration.

Table 39 exhibits the values of $\phi_0$ near the tip of the delta for $t = 211 = 0.07693$. Leaving the 4 values which are at the nodal points adjacent to the leading edge, the polynomial $P$, given by (245), is fitted...
INTERPOLATION OF POLYNOMIAL THROUGH THE COMPUTED VALUE OF $\phi^+$ NEAR THE TIP

TOTAL SUB-AREAS = 2556
EQUATIONS = 1278

<table>
<thead>
<tr>
<th>$x$</th>
<th>$y$</th>
<th>$z=0.03849$</th>
<th>$z=-0.03849$</th>
<th>$\phi^+$</th>
<th>$\phi^-$</th>
<th>Fitted $\phi_a$</th>
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<td>-0.17427E-02</td>
<td>-0.17427E-02</td>
<td></td>
</tr>
</tbody>
</table>

Table 39
through these values in the least-squares sense, taking \( \ell = 0.0001 \). The condition (246) is satisfied at every point \( q_1, q_2 \ldots q_{10} \) when \( m = 7 \), and at this stage the coefficients are found to be

\[
\begin{align*}
C_0 &= -5.25287, \quad C_1 = 32.82636, \quad C_2 = 92.36869 \\
C_3 &= 218.21335, \quad C_4 = -390.53000, \quad C_5 = 595.55897 \\
C_6 &= 501.92655 \quad \text{and} \quad C_7 = 103.60649.
\end{align*}
\]

The fitted values of \( \phi_a^+ \) thus found are exhibited in Table 39.

**Fluid velocity near the Apex of the Delta Wing**

To find the nature of the flow near the tip of the thin delta, a polynomial \( P \), given by (245) is fitted through the values of \( \phi_a^+ \) in the neighbourhood of the tip. If the tip 0 of the delta OAB defines the origin of a cylindrical polar frame \( O \xi \eta \zeta \) [Fig. 36(a)], the simplest formula for the velocity component on \( OB \) in the \( \eta \) increasing direction is given by

\[
\psi(q_{R+1}) = -\left\{ \phi_a(q_{R+1}) - \phi_a(q_R) \right\}/d ,
\]

(248)

where \( q_R = (r, \eta, z) \), \( q_{R+1} = (r, \eta + d\eta, z) \); \( q_{R+1/2} = (q_{R+1} - q_R)/2 \)

and \( d = rd\eta \). For numerical calculation \( d \) was taken to be 0.001 radians. Table 40 exhibits the tangential velocity component on the plane \( z = H \) for different values of \( r \). It is evident from Table 40 that, on the upper surface, the tangential velocity component is maximum near the leading edge and it gradually falls to zero on the central line OD [fig. 36].

Since \( \phi_a^- = -\phi_a^+ \), the tangential component of the velocity at any point \( (r, \eta, z) \) will be of the same magnitude, but of the opposite sign to that at the point \( (r, \eta, z) \). Hence on the lower surface, near the tip, the fluid is coming away from the central line OD towards the leading edges. Table 40 exhibits the above property of the flow near the apex of the delta. Fig. 36 gives the graphs of the velocity component near the tip based on Table 40.
NATURE OF THE FLOW NEAR THE TIP ON THE SURFACE $Z = \pm 0.0385$

<table>
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<th>FIELD POINT</th>
<th>$\eta$ in degree</th>
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<th>$Z = -0.03849$</th>
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<td>0.13229E-01</td>
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<td>&quot;</td>
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<tr>
<td>&quot;</td>
<td>90</td>
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Table 40
TANGENTIAL COMPONENT OF VELOCITY ALONG THE CIRCULAR ARCS ON THE SURFACES $z = \pm H$
NEAR THE TIP OF THE DELTA.
CHAPTER 19

BEHAVIOUR OF \( \Phi \) NEAR THE TIP OF A DELTA

Introduction

In the case of flow past a thin delta wing with small angle of attack, the flow remains potential. The disturbance potential \( \Phi \) is antisymmetric in character and singular at the tip. According to Brown and Stewartson\(^6\) and to Arscott and Taylor\(^7\) for points sufficiently near the apex, in spherical polar co-ordinates \((r, \xi, \eta)\) with the origin at the tip (Fig. 37),

\[
\Phi = r^\gamma V(\xi, \eta),
\]

where \(V\) is some function of the angular co-ordinates \(\xi\) and \(\eta\). The angular sector lies in the plane \(\xi = 90^\circ\) between the lines \(\eta = 90^\circ - \alpha\) and \(\eta = 90^\circ + \alpha\), where \(\alpha\) is the semi-apex angle of the sector. \(V(\xi, \eta)\) is a constant along any radius vector and hence we may write for points along a radius vector,

\[
\Phi = c r^\gamma.
\]

The exponent \(\gamma\), which determines the order of the singularity, has an infinite set of possible values of which the smallest positive value is of greatest interest for practical purposes. For semi-apex angle \(\alpha = 60^\circ\), \(\gamma = 0.69\).

Computed Values of \(\Phi\)

Following our procedures, we compute for each choice of \(H\), a set of values of \(\phi_a^+\). Omitting the values of \(\phi_a^+\) at the nodal points adjacent to the leading edge and the apex, the polynomial (245) is fitted for \(\xi = 0.0001\) through the \(\Phi_a^+\) in the neighbourhood of the apex. In each case, we have \(L = 10\) and the condition (246) was satisfied for \(m = 7\). The coefficients \(c_j\) of \(P\), thus found for 4 different values of \(2H\) i.e. thickness, are given in Table 41.

For a given \(H\) the line \(\xi, \eta = \text{constant}\) intersects the plane \(Z = H\) at a point \(p (x, y, z)\) [The origin of the cartesian frame is at the centroid
<table>
<thead>
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<th>NT</th>
<th>KN</th>
<th>EQUATIONS</th>
<th>HALF THICKNESS</th>
<th>CO-EFFICIENTS</th>
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<td></td>
<td>-226.61312</td>
</tr>
<tr>
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<td></td>
<td>336.51414</td>
</tr>
<tr>
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<td></td>
<td>33.57290</td>
</tr>
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<td></td>
<td></td>
<td>-206.33711</td>
</tr>
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<td>12</td>
<td>3</td>
<td>1010</td>
<td>0.07217</td>
<td>-2.24340</td>
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<td></td>
<td>11.42473</td>
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<td></td>
<td>6.99570</td>
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<td>147.13780</td>
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<td></td>
<td>-744.44676</td>
</tr>
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<td></td>
<td></td>
<td></td>
<td>1068.63150</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>58.78167</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>-665.38939</td>
</tr>
</tbody>
</table>

Table 41
of the thin plate (Fig. 37). This is not necessarily a nodal point. At this point

\[ r = z \sec \xi, \tag{251} \]

which determines \( r \). Now at \((r, \xi, \eta)\) \( \phi \) is determined by

\[ \phi = P(x, y, z), \tag{252} \]

where \( P \) is the approximate polynomial defined in Chapter 18. For different values of \( z \) the corresponding values of \( r \) and \( \phi \) thus obtained for a given set of \( \xi \) and \( \eta \) are given in Table 42.

**Numerical Determination of \( \phi \)**

Taking log of both sides of (250), we have

\[ \log \phi = \log C + \psi \log r \tag{253} \]

This relation is fitted, in the least square sense, through the set of values of \( \phi \) obtained from (252) for given values of \( r \) along a radial line through the tip. The \( \phi^*_a \) (= \( \phi \) in the case of a thin delta wing) thus fitted along different radial lines are given in Table 43. The \( \psi \) thus found for different radial lines, in the neighbourhood of the tip, are given in Table 44. The perpendicular OD of the triangular sector (Fig. 37), in the case under consideration, is 0.5774. To consider the values of \( \phi \) sufficiently near the tip, the values of \( \gamma \), in this case, is not exceeded beyond 0.065 which is nearly the 10% of OD.

It is evident from Table 44 that the computed \( \gamma \), for the set of computed \( \phi^*_a \) nearest to the tip, closely approximates the theoretically expected value \( \gamma = 0.69 \). The average value of \( \gamma \), from the Table, is 0.71.
Spherical Polar Frame Defined at the Apex

Fig. 37

\[ r = OP \]
COMPUTED VALUES OF $\phi_a^+$ NEAR THE TIP OF THE DELTA

<table>
<thead>
<tr>
<th>r</th>
<th>$\xi$</th>
<th>$\phi_a^+$</th>
<th>$\eta=30^\circ$</th>
<th>$\eta=60^\circ$</th>
<th>$\eta=90^\circ$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.039</td>
<td>5°</td>
<td>-0.721E-03</td>
<td>-0.820E-03</td>
<td>-0.854E-03</td>
<td></td>
</tr>
<tr>
<td>0.048</td>
<td>5°</td>
<td>-0.901E-03</td>
<td>-0.105E-02</td>
<td>-0.107E-02</td>
<td></td>
</tr>
<tr>
<td>0.064</td>
<td>5°</td>
<td>-0.114E-02</td>
<td>-0.121E-02</td>
<td>-0.124E-02</td>
<td></td>
</tr>
<tr>
<td>0.039</td>
<td>10°</td>
<td>-0.821E-03</td>
<td>-0.993E-03</td>
<td>-0.106E-02</td>
<td></td>
</tr>
<tr>
<td>0.049</td>
<td>10°</td>
<td>-0.103E-02</td>
<td>-0.103E-02</td>
<td>-0.120E-02</td>
<td></td>
</tr>
<tr>
<td>0.065</td>
<td>10°</td>
<td>-0.119E-02</td>
<td>-0.134E-02</td>
<td>-0.140E-02</td>
<td></td>
</tr>
</tbody>
</table>

Table 42
THE FITTED VALUE OF $\phi^+_a$ ALONG A RADIUS VECTOR

<table>
<thead>
<tr>
<th>$r$</th>
<th>$\xi$</th>
<th>$\eta$</th>
<th>$\phi^+_a$ READING</th>
<th>$\phi^+_a$ FITTED VALUE</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.038</td>
<td>5°</td>
<td>60°</td>
<td>-0.8196E-03</td>
<td>-0.84215E-03</td>
</tr>
<tr>
<td>0.048</td>
<td>&quot;</td>
<td>&quot;</td>
<td>-0.1045E-02</td>
<td>-0.99609E-03</td>
</tr>
<tr>
<td>0.064</td>
<td>&quot;</td>
<td>&quot;</td>
<td>-0.1211E-02</td>
<td>-0.12368E-02</td>
</tr>
<tr>
<td>0.038</td>
<td>&quot;</td>
<td>90°</td>
<td>-0.8540E-03</td>
<td>-0.87333E-03</td>
</tr>
<tr>
<td>0.048</td>
<td>&quot;</td>
<td>&quot;</td>
<td>-0.1066E-02</td>
<td>-0.10244E-02</td>
</tr>
<tr>
<td>0.064</td>
<td>&quot;</td>
<td>&quot;</td>
<td>-0.1236E-02</td>
<td>-0.12583E-02</td>
</tr>
</tbody>
</table>

Table 43
DETERMINATION OF $\nu$ FOR THE ANGULAR
SECTOR OF SEMIAPEX ANGLE 60°

<table>
<thead>
<tr>
<th>$\xi$</th>
<th>$\eta$</th>
<th>$\nu$</th>
</tr>
</thead>
<tbody>
<tr>
<td>5°</td>
<td>30°</td>
<td>0.88</td>
</tr>
<tr>
<td>6°</td>
<td>60°</td>
<td>0.75</td>
</tr>
<tr>
<td>9°</td>
<td>90°</td>
<td>0.72</td>
</tr>
<tr>
<td>10°</td>
<td>30°</td>
<td>0.72</td>
</tr>
<tr>
<td>6°</td>
<td>60°</td>
<td>0.59</td>
</tr>
<tr>
<td>9°</td>
<td>90°</td>
<td>0.55</td>
</tr>
</tbody>
</table>

Table 44
CHAPTER 20  
SOLUTION BY SUCCESSIVE APPROXIMATION

Introduction

In the case of potential flow past a fixed boundary the disturbance potential $\phi$ is zero at infinity and remains generally small compared with the free flow potential $\psi ( = \psi _f ( \rho _f , \gamma _f ) )$ on $\partial B$. Hence we take $\phi = 0$ as the zeroth approximation to $\phi$ in the right hand side of Green's boundary formula

$$\phi (h) = \frac{1}{2\pi} \int_{\partial B} \frac{\phi (y)}{|h - y|} \, dy - \frac{1}{2\pi} \int_{\partial B} \frac{\phi^1 (y)}{|h - y|} \, dy ; \quad h , y \in \partial B , \quad (254)$$

and so define a better approximation to $\phi$ given by

$$\phi^1 (h) = - \frac{1}{2\pi} \int_{\partial B} \frac{\phi^1 (y) + \phi (h)}{|h - y|} \, dy ; \quad h , y \in \partial B . \quad (255)$$

Insertion of this $\phi^1$ into (254) yields $\phi^2$, given by

$$\phi^2 (h) = - \frac{1}{2\pi} \int_{\partial B} \frac{\phi^2 (y) + \phi^1 (h)}{|h - y|} \, dy + \phi^1 (h) ; \quad h , y \in \partial B . \quad (256)$$

So proceeding, we compute successive approximate $\phi^1, \phi^2, \ldots, \phi^n$ to $\phi$, of which the convergence can be examined.

The integral can be computed numerically as before. On discretisation

$$\phi^1 (q_k) \approx \frac{1}{2\pi} \sum_{j=1}^{N} \phi^1 (q_j) \left( \frac{dy}{|y_k - y_j|} \right) ; \quad k = 1, 2, \ldots, N , \quad (257)$$

$$\phi^2 (q_k) \approx - \frac{1}{2\pi} \sum_{j=1}^{N} \phi^2 (q_j) \left( \frac{dy}{|y_k - y_j|} \right) + \phi^1 (q_k) ; \quad k = 1, 2, \ldots, N . \quad (258)$$

This procedure yields a set of approximation to $\phi$ at the pivotal points $q_1, q_2, \ldots, q_N$. The approximation to $\phi$ at the point $q_m$ after the
The $r$th iteration is written as

$$\phi_r(q_k); \quad r = 1, 2, \ldots, n. \quad (259)$$

For a pre-assigned small positive quantity $\epsilon$, if there exists an $M$ such that

$$\left| \phi^M(q_k) - \phi^{M-1}(q_k) \right| \leq \epsilon \quad ; \quad k = 1, 2, \ldots, N, \quad (260)$$

then $\phi^M$ is said to be the approximate solution of (254).

**Flow past a sphere**

Applying the above approach to the case of flow past a sphere, it appears that the approximation converges to the expected solution (Table 45). Referring to (260), for $KN = 4$, i.e. $N = 2544$ [see (201)] and for $\epsilon = 0.0001$, the approximation converges for $M = 9$ with $B^1 = 0.69493$ and $B^9 = 0.00006$.

The computed value of $\phi$ obtained by the successive approximation method has the worst behaviour at the point $q (0.998, 0.000, 0.070)$ and it is exhibited in Fig. 38.

**Flow past a thick delta wing**

In the case of flow past a thick delta wing, given in Chapter 17, the solution obtained by the successive approximation method does not converge. Fig. 39 exhibits the non-convergence of the computed $\phi$, obtained by the above method, for a plate of thickness $t = 0.0962$ with a number of sub-areas $N = 1908$. Fig. 40 exhibits the behaviour of computed $\phi$ at a point $q (0.005, 0.376, 0.048)$ [see Fig. 31(c), Chapter 17] at which the value of $\phi$, determined by the integral equation method, is 0.0372.

Table 46 exhibits the approximate $\phi$ obtained at some representative points on the surface of the wing of thickness 0.0962 for $\gamma = 38$ [see (259)] compared with the $\phi$ obtained by the integral equation method.
COMPARISON OF SOLUTIONS OBTAINED BY DIFFERENT METHODS IN THE CASE OF FLOW PAST A UNIT SPHERE

DISTURBANCE POTENTIAL ALONG A MERIDIAN

<table>
<thead>
<tr>
<th>FIELD POINT</th>
<th>( \phi )</th>
<th>( \phi )</th>
<th>( \phi )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>ANALYTICAL</td>
<td>INTEGRAL EQN.</td>
<td>SUCCESSIVE APPROX.</td>
</tr>
<tr>
<td>0.091</td>
<td>0.49795</td>
<td>0.50339</td>
<td>0.50340</td>
</tr>
<tr>
<td>0.209</td>
<td>0.48899</td>
<td>0.49421</td>
<td>0.49423</td>
</tr>
<tr>
<td>0.230</td>
<td>0.47703</td>
<td>0.48206</td>
<td>0.48208</td>
</tr>
<tr>
<td>0.382</td>
<td>0.46214</td>
<td>0.46687</td>
<td>0.46688</td>
</tr>
<tr>
<td>0.457</td>
<td>0.44483</td>
<td>0.44905</td>
<td>0.44906</td>
</tr>
<tr>
<td>0.526</td>
<td>0.42533</td>
<td>0.42932</td>
<td>0.42933</td>
</tr>
<tr>
<td>0.590</td>
<td>0.40376</td>
<td>0.40761</td>
<td>0.40762</td>
</tr>
<tr>
<td>0.649</td>
<td>0.38042</td>
<td>0.38414</td>
<td>0.38415</td>
</tr>
<tr>
<td>0.999</td>
<td>0.01140</td>
<td>0.01154</td>
<td>0.01154</td>
</tr>
</tbody>
</table>

Table 45
The computed value of \( \phi \) has the worst behavior.

On successive iterations.

Behavior of the solution at a fixed point:

**Fig. 35**

<table>
<thead>
<tr>
<th>No. of Iterations</th>
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<tbody>
<tr>
<td>25</td>
</tr>
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<td>15</td>
</tr>
<tr>
<td>10</td>
</tr>
<tr>
<td>5</td>
</tr>
</tbody>
</table>

- **\( \phi \) computed**
- **\( \phi \) analytical**

Diagram showing the behavior of \( \phi \) over iterations.
Successive approximation method

Fig. 39

Behaviour of the solution on successive iterations.
behaviour of the solution at a fixed point $p$ on successive iterations: Value of $\phi$ at $p$ is $0.0372$ (integral equation method).
COMPARISON OF SOLUTIONS OBTAINED BY DIFFERENT METHODS IN CASE OF FLOW PAST A THICK DELTA WING

DISTURBANCE POTENTIAL ON THE SURFACE NEAR THE TIP

<table>
<thead>
<tr>
<th>FIELD POINT</th>
<th>$\phi$</th>
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<tbody>
<tr>
<td>$x$</td>
<td>$y$</td>
</tr>
<tr>
<td>0.005</td>
<td>0.376</td>
</tr>
<tr>
<td>0.005</td>
<td>0.367</td>
</tr>
<tr>
<td>0.016</td>
<td>0.367</td>
</tr>
<tr>
<td>0.006</td>
<td>0.355</td>
</tr>
<tr>
<td>0.016</td>
<td>0.355</td>
</tr>
<tr>
<td>0.026</td>
<td>0.364</td>
</tr>
<tr>
<td>0.026</td>
<td>0.355</td>
</tr>
<tr>
<td>0.037</td>
<td>0.355</td>
</tr>
<tr>
<td>0.010</td>
<td>0.343</td>
</tr>
<tr>
<td>0.031</td>
<td>0.343</td>
</tr>
</tbody>
</table>

Table 46
APPENDIX I

Evaluation of

\[ S = \int_{\Omega} \frac{dy}{\sqrt{(x-a)^2 + (y-b)^2}} \]  \tag{1}

when \( \Omega \) is a plane rectangular area.

Let \( \Omega \) define a rectangular area, in the plane \( Z = 0 \) with sides \( 2a \) and \( 2b \), of which the centroid \( 0 \) defines the origin of a reference frame \( OXYZ \) Fig.41(a). Let the co-ordinates of \( p \) be \((X,Y,Z)\) and those of \( q \), since \( q \not\in \Omega \), are \((x,y,0)\).

\[ S = \int_{-b}^{b} \int_{-a}^{a} \frac{dx}{\sqrt{(x-a)^2 + (y-b)^2 + z^2}} \]

\[ = \int_{-b}^{b} \left[ \int_{x-a}^{x+a} \frac{da}{\sqrt{a^2 + (y-b)^2 + z^2}} \right] dy \]

\[ = \int_{-b}^{b} \left[ \log \left( (x+a) + \sqrt{(x+a)^2 + z^2 + (y-b)^2} \right) - \log \left( (x-a) + \sqrt{(x-a)^2 + z^2 + (y-b)^2} \right) \right] dy, \]

\[ = S_1 - S_2 \quad (Say), \text{ Where} \]

\[ S_1 = \int_{-b}^{b} \log \left( (x+a) + \sqrt{(x+a)^2 + z^2 + (y-b)^2} \right) dy \]

\[ = \int_{k=y-a}^{k=y+a} \log \left( C + \sqrt{D + k^2} \right) dk \; \quad \text{C = x+a, \; D = c^2 + z^2, \; k = y - y,} \]

\[ k = y - b \]
Similarly putting \( F = X - \alpha \), \( F^2 = E + Z^2 \)

\[
S_2 = \log \left\{ \left( E + \sqrt{F + E^2} \right) \left( F + \sqrt{F^2 + E^2} \right) \right\} - k - Z \sin^{-1} \frac{F + E \sqrt{F^2 + E^2}}{F \left( E + \sqrt{F^2 + E^2} \right)} \]

Now \( S = S_L - S_2 \),

\[
= \log \left\{ \left( E + \sqrt{F + E^2} \right) \left( F + \sqrt{F^2 + E^2} \right) \right\} + Z \left( \sin^{-1} \frac{F + E \sqrt{F^2 + E^2}}{F \left( E + \sqrt{F^2 + E^2} \right)} - \sin^{-1} \frac{2D + C F^2}{D \left( C + \sqrt{D^2 + F^2} \right)} \right) \]

(2)
When \( DB \) is a square of edge length \( h \) and \( p \) coincides with the centroid of \( DB \), we have
\[
X = Y = Z = 0
\]
and
\[
a = b = \frac{1}{4}h
\]
Hence,
\[
S = 4h \log (1 + \sqrt{2}) \quad (3)
\]

2. Evaluation of \( S \) over a triangular area when \( p \) coincides with a vertex of the triangle.

Let the triangular area \( ABC \), bounded by arms \( r_1, r_2 \) and \( r_3 \), lie in the plane \( Z = 0 \) [Fig. 41 (b)]. The vertex \( C \) defines the origin of a cylindrical polar reference frame with \( CA \) as the initial line. In this frame, the arm \( CB \) is given by
\[
\theta = \theta_1,
\]
and \( p \) coincides with \( C \) and the co-ordinates of \( q \) are \((r, \theta, 0)\).

Now
\[
S = \int_{\theta=0}^{\theta_1} \frac{\frac{1}{r} r \, dr}{|x - y|} = \int_{\theta=0}^{\theta_1} \left[ \int_{r=0}^{f(\theta)} \right] d\theta = \int_{\theta=0}^{\theta_1} f(\theta) d\theta.
\]

Let the equation of \( AB \) in a cartesian frame \( OXYZ \), with the origin at \( C \) and \( Z \)-axis coinciding with the initial line \( CA \), be
\[
y = mx + C,
\]
with conditions
\[
y = 0 \quad \text{when} \quad x = r_2
\]
and
\[
y = r_1 \sin \theta_1 \quad \text{when} \quad x = r_1 \cos \theta_1
\]
Hence, \( C = -mr_2 \) and \( m = r_1 \sin \theta_1 / (r_1 \cos \theta_1 - r_2) \).

In terms of \( r \) and \( \theta \), \( f(\theta) \) stands as
\[
f(\theta) = r = \frac{r_1 r_2 \sin \theta_1}{(r_2 - r_1 \cos \theta_1) \sin \theta + r_1 \sin \theta_1 \cos \theta}
\]
If \( a = r_2 - r_1 \cos \theta_1 \) and \( b = r_1 \sin \theta_1 \), we have
\[
a^2 + b^2 = r_1^2 + r_2^2 - 2r_1r_2 \cos \theta_1 = r_3^2
\]
and
\[
\{\theta\} = \frac{r_1r_2 \sin \theta_1}{a \sin \theta + b \cos \theta}.
\]

Using these in (4)

\[
S = r_1 r_2 \sin \theta_1 \int_{\theta=0}^{\theta_1} \frac{d\theta}{a \sin \theta + b \cos \theta}
\]
\[
= \frac{r_1 r_2 \sin \theta_1}{\sqrt{a^2+b^2}} \left[ \log \left( \frac{b \sin \theta - a \cos \theta + \sqrt{a^2+b^2}}{b \cos \theta + a \sin \theta} \right) \right]_{\theta=0}^{\theta_1}
\]
\[
= \frac{2\Delta}{r_3} \left[ \log \left( \frac{r_1 \sin \theta_1 \sin \theta - (r_2 - r_1 \cos \theta_1) \cos \theta + r_3}{r_1 \sin \theta_1 \cos \theta + (r_2 - r_1 \cos \theta_1) \sin \theta} \right) \right]_{\theta=0}^{\theta_1}
\]
\[
= \frac{2\Delta}{r_3} \log \left( \frac{r_1 - r_2 \cos \theta_1 + r_3}{r_2 \sin \theta_1} \cdot \frac{r_1 \sin \theta_1}{r_3 - r_2 + r_1 \cos \theta_1} \right)
\]
\[
= \frac{2\Delta}{r_3} \log \left( \frac{r_1^2 + r_2^2 - r_1r_2 \cos \theta_1}{r_2r_3 - r_2^2 + r_1r_2 \cos \theta_1} \right)
\]
\[
= \frac{2\Delta}{r_3} \log \left( \frac{r_1^2 + r_2^2 + r_3^2}{r_1 + r_2 + r_3} \right)
\]

where \( \Delta \) is the area of the triangle \( A\!B\!C \).
APPENDIX II

Evaluation of

\[ S^1 = \int_{\partial \Omega} \frac{da_k}{||k - q||} ; \quad k, q \in \partial \Omega \]

Let \( \partial \Omega \) be the part of a sphere of radius \( 'a' \). The centre of the sphere defines the origin of a spherical polar co-ordinates, in which the co-ordinates of \( p \) are \((a, 0, 0)\) and those of \( q \) are \((a, \theta, \eta)\).

Now \( \frac{1}{||k - q||} = \frac{1}{||q_k - q||} \) ,

\[ \left( q_{k} - q \right) \cdot \hat{\eta}_{e}(q_k) = \frac{2 a \sin(\frac{\theta}{2}) \cdot \sin(\frac{\eta}{2})}{8 a^2 \sin^3(\frac{\theta}{2})} \]

\[ = \frac{1}{4a^2 \sin(\frac{\theta}{2})} \]

where \( \hat{\eta}_{e}(q_k) \) denotes the unit vector normal to \( \partial \Omega \) at the point \( q_k \).

\[ S^1 = \int_{0}^{2\pi} \int_{\theta = 0}^{\Theta} \frac{a \sin \theta \, d\eta \, a \, d\theta}{4 a^2 \sin(\frac{\theta}{2})} \]

\[ = 2\pi \int_{0}^{\Theta} \frac{2 a^2 \sin(\frac{\theta}{2}) \cos(\frac{\eta}{2}) \, d\theta}{4 a^2 \sin(\frac{\theta}{2})} \]

\[ = 2\pi \left[ \sin(\frac{\theta}{2}) \right]_{0}^{\Theta} = 2\pi \sin(\frac{\Theta}{2}) \]

where \( 2\Theta \) is the solid angle, subtended by \( \partial \Omega \), at the centre of the sphere [Fig. 41(e)].

If \( \Lambda \Omega \) be the rim of the circular cap \( \partial \Omega \) with its centre at \( q_k \) and if \( q_k \Lambda A = h_k \),
\[ S = \frac{\pi}{a} 2a \sin\left(\frac{\alpha}{2}\right) = \frac{\pi}{a} a^2 \sin\left(\frac{\alpha}{2}\right) \] \hspace{1cm} (5)

For a flat surface \( \partial B \),

\[ (\mathbf{q}_k - \mathbf{q}) \cdot \hat{n} = \mathbf{0} = (\mathbf{q}_k - \mathbf{q}) \cdot \hat{n} \]

\[ \int_{\partial B} d\mathbf{q} \frac{1}{|\mathbf{q} - \mathbf{k}|} = \int_{\partial B} d\mathbf{q} \frac{1}{|\mathbf{q} - \mathbf{q}_k|} = \mathbf{0} \] \hspace{1cm} (6)
APPENDIX III

Nature of the solution of a system of $2k$ linear algebraic equations represented by

$$
\begin{bmatrix}
[A] \cdot [X] = [B] ,
\end{bmatrix}
$$

where $[A]$ is a square matrix of order $2k \times 2k$ with $|A| \neq 0$; the elements of $[A]$ and $[B]$ satisfy

$$
\alpha_{ij} = \alpha_{2k-i+1 \ 2k-j+1}
$$

and $b_i = -b_{2k-i+1}$ respectively.

On the above conditions, the expanded form of (7) is given by

$$
\begin{bmatrix}
\alpha_{11} & \alpha_{12} & \cdots & \alpha_{1k} & \alpha_{1k+1} & \cdots & \alpha_{12k} \\
\alpha_{21} & \alpha_{22} & \cdots & \alpha_{2k} & \alpha_{2k+1} & \cdots & \alpha_{22k} \\
& & \cdots \cdots \cdots \cdots & & & \cdots \cdots \cdots \cdots \\
\alpha_{k1} & \alpha_{k2} & \cdots & \alpha_{kk} & \alpha_{kk+1} & \cdots & \alpha_{k2k} \\
\alpha_{k+1,2k} & \alpha_{k+1,2k-1} & \cdots & \alpha_{k,k+1} & \alpha_{k,k+1} & \cdots & \alpha_{k+1,k} \\
& & \cdots \cdots \cdots \cdots & & & \cdots \cdots \cdots \cdots \\
\alpha_{2k1} & \alpha_{2k2} & \cdots & \alpha_{2kk} & \alpha_{2kk+1} & \cdots & \alpha_{2k1}
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
\vdots \\
x_k \\
x_{k+1} \\
\vdots \\
x_{2k}
\end{bmatrix}
= 
\begin{bmatrix}
x_1 \\
x_2 \\
\vdots \\
x_k \\
x_{k+1} \\
\vdots \\
x_{2k}
\end{bmatrix}
$$

Since the determinant of $|A| \neq 0$, i.e.

$$
D = |A| = |a_{11} \quad a_{22} \quad \cdots \quad a_{kk} \quad \cdots \quad a_{11}| \neq 0
$$

by Cramer's rule,

$$
X_r = \frac{|a_{11} \quad a_{22} \quad \cdots \quad b_r \quad a_{r+1 \ r+1} \quad \cdots \quad a_{kk} \quad \cdots \quad a_{11}|}{|A|}, \quad r < k
$$
Similarly,

\[ X_{2^{k-r+1}} = \frac{|a_{11} a_{22} \cdots a_{rr} \cdots b_{r} a_{r-1 \cdots r-1} \cdots a_{11}|}{|A|}, \]

\[ = -\frac{|a_{11} a_{22} \cdots a_{kk} \cdots b_{r} a_{r-1 \cdots r-1} \cdots a_{11}|}{|A|}, \]

\[ = -\frac{|a_{11} a_{22} \cdots b_{r} a_{r+1 \cdots r+1} a_{rr} \cdots a_{11}|}{|A|}, \]

\[ = -X_{r}, \]

i.e. \( X_{r} = -X_{2^{k-r+1}} \).

The above relation holds good for \( r = 1, \ldots, k \).
REFERENCES


