THE IMPACT OF LONGEVITY AND INVESTMENT RISK ON A PORTFOLIO OF LIFE INSURANCE LIABILITIES

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Abstract. In this paper we assess the joint impact of biometric and financial risk on the market valuation of life insurance liabilities. We consider a stylized, contingent claim based model of a life insurance company issuing participating contracts and subject to default risk, as pioneered by Briys and de Varenne (1994, 1997), and build on their model by explicitly introducing biometric risk and its components, namely diversifiable and systematic risk. The contracts considered include pure endowments, deferred whole life annuities and guaranteed annuity options. Our results stress the predominance of systematic over diversifiable risk in determining fair participation rates. We investigate the interaction of contract design, market regimes and mortality assumptions, and show that, particularly for lifelong benefits, the choice of the participation rate must be very conservative if longevity improvements are foreseeable.

1. Introduction

In the last decades, increasing volatility in investment returns coupled with low interest rates regimes and increased expectation of life

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across all developed countries have impacted on life insurance and pension markets, resulting in potential distress for some annuity providers.

The adoption of fair value based accounting standards for insurers, e.g. the full implementation of the Solvency II framework in the European Union in 2016, has enhanced the transparency of their balance sheets by tying assets and liabilities values to the actual (or hypothetical) price they could be exchanged for in a liquid market. On the other hand, the application of these accounting standards has stressed the exposure of life insurers' balance sheets to a variety of financial and biometric factors, with a consequent effect on capital requirements. This is particularly relevant for providers of long-term investment guarantees or lifelong benefits. Traditional life insurance products offering fixed life contingencies have been replaced long ago by more competitive contract structures, with-profits in the UK and participating policies in Europe and the US, where insurers share part of their returns with policyholders. Usually, the policyholder is promised to receive a minimum return even when market performance is poor. This minimum rate of return is set at issuance on a very conservative basis, so that the implicit value of such a guarantee is small. However, given the long-term nature of the contract, guarantees that are initially far out of the money may become highly valuable due to adverse movements in market rates of return and unexpected rise in the length of life. The increasing cost of these guarantees could become unsustainable and eventually compromise the financial stability of their provider. A notable example is given by Equitable Life, the world's oldest life insurer, see **?** and Briys and de Varenne (1994), Grosen and Jørgensen (2002) for other examples of insolvencies in the life insurance industry. Therefore, an accurate contract design and careful assessment of all the risks involved, along with the interaction between them, are crucial.

The aim of this paper is to assess the joint impact of biometric and financial risk on the market valuation of life insurance liabilities. We explicitly incorporate longevity risk on a portfolio level in the stylized, contingent claim model of a life insurance company issuing participating contracts and subject to default risk pioneered by Briys and de Varenne (1994, 1997) and extended, e.g., by Grosen and Jørgensen (2002), Bernard *et al.* (2005), Chen and Suchanecki (2007), Ballotta *et al.* (2006a,b), Ballotta (2005). This stream of literature focuses on financial risks only, as it is implicitly assumed that diversifiable biometric risk can be completely eliminated by pooling a large portfolio and systematic biometric risk, that is longevity risk, is absent. Beyond pure endowments, the participating contracts we consider are deferred whole life annuities and guaranteed annuity options. Longevity risk has been emphasized as a main factor affecting life insurance portfolios only in relatively recent years. Stochastic mortality models have been developed to explicitly allow for the uncertainty surrounding future survival rates, see Barrieu *et al.* (2012) for an overview. The pioneering model of Lee and Carter (1992) has been successfully applied to forecast mortality of different populations and has been extended and improved in several ways, see e.g. Cairns *et al.* (2008) and references therein. In our stylized framework we follow a slightly different approach and introduce a stochastic force of mortality obtained by randomly rescaling a deterministic intensity. This relatively simple formulation allows to clearly separate process risk, represented by the randomness in the times of death of policyholders, from the systematic risk captured by the random rescaling factor.

We conduct a thorough analysis of contract components and fair participation rates, exploring in detail the interplay of guarantees, market regimes, mortality assumptions and portfolio sizes. The main results of the paper can be summarized as follows: first, idiosyncratic biometric risk vanishes even in small portfolios. In other words, when homogeneous contracts are pooled together, diversification becomes fully effective with relatively small portfolio sizes. Further, longevity risk has a very substantial impact on the market value of the participating life insurance liabilities. We show that the relative size of this impact on the fair participation coefficients is particularly relevant when systematic biometric risk is paired with a low interest rate environment, and is preserved when the solvency capital or the pricing rule is adjusted to reflect the portfolio size. This effect has been pointed out by a number of studies: e.g. **?**, and **?** for pension annuities; Ballotta and Haberman (2006), and **?** for pension plans and annuities including guaranteed annuity conversion options; **?** and **?** in the context of traditional/fixed life insurance and annuities products, and **?** for variable annuities with lifetime withdrawal guarantees. Finally, our detailed analysis provides some useful guidance on the possible actions a life insurer could take in order to mitigate the effect of longevity risk.

The remainder of this article is structured as follows. Section 2 sets up the contract structure, the modelling of insurance and financial risk and the extension to a large portfolio. Section 3 focuses on the market valuation of the outstanding liabilities, unbundling them into different components. Section 4 shows how ruin-probability-based capital requirements can be set under our framework. Section 5 is devoted to the numerical analysis and addresses the issue of fair pricing. Section 6 provides some concluding remarks and a short outlook on possible extensions. Proofs and technical results are collected in the Appendices.

2. Model setup

At time $t = 0$ the life insurance company's capital structure can be synthesized through the following simplified balance sheet:

Assets	Liabilities
W_0	$E_0 = (1-\alpha)W_0$ $L_0 = \alpha W_0$
W_0	W_0

Hence, the initial assets W_0 of the firm are financed by two groups of stakeholders. A share α of the assets (with $0 < \alpha < 1$) is contributed by N_0 policyholders that are homogeneous, with the same age x at inception, and are entitled to the same benefits. Therefore, each policyholder pays a single premium $L_0/N_0 = \alpha W_0/N_0$, where L_0 is the overall premium contribution. The remaining fraction $1-\alpha$ is provided by equityholders, whose total contribution is $(1-\alpha)W_0 = E_0$. Clearly, initial assets W_0 and premium income L_0 are related to the portfolio size N_0.

Since we explicitly allow for insurance risk, the outstanding liability at any given time will depend, among other factors, on the demographic evolution of the population of policyholders. However, if all insurance risk can be diversified, for instance when the portfolio is large and there is no systematic risk, then the pool of homogeneous contracts could be treated as a purely financial contract with initial contribution L_0. This point will be picked up again later.

2.1. Contract Structure.

Through their initial investment in the company, policyholders alive at the maturity T of the contract have a claim on the firm's assets. Moreover, we assume that the insurance company issues no further debt, raises no capital and pays no dividends to equityholders before the contract's maturity. Since profits distribution is a common feature of many life insurance contracts, we consider the following version of a participating policy. As we will see, this apparently simple specification encompasses different types of guarantees.

Denote by L and $W = W_0 e^R$ the total liability, respectively the assets value, at time T, where R is the assets log-return over the period $[0, T]$. At maturity, the total outstanding liability the insurance company has to meet depends on the number of alive policyholders N:

$$L = \begin{cases} \Psi & \text{if } N > 0 \\ 0 & \text{if } N = 0 \end{cases} = \Psi 1_{\{N>0\}},$$

where $1_{\mathcal{E}}$ is the indicator of the event $\mathcal{E}$. Then, in the very unlikely case in which no policyholder survives the maturity T, i.e. $N = 0$,[1] the

[1] The probability that a portfolio be completely extinct at maturity is negligible for usual ages and maturities and reasonable portfolio sizes. For instance, with a

company has no liability outstanding. Otherwise, if $N > 0$, the liability depends on the assets value W and the global payoff G guaranteed to surviving policyholders, and is defined as in Briys and de Varenne (1994, 1997) by

$$\Psi = \begin{cases} W & \text{if } W < G \\ G & \text{if } G \leq W \leq \frac{G}{\alpha} \\ G + \delta\left(\alpha W - G\right) & \text{if } \frac{G}{\alpha} < W \end{cases},$$

or, more compactly, by

$$\Psi = G + \delta\alpha \left[W - \frac{G}{\alpha}\right]^{+} - [G - W]^{+}, \tag{2.1}$$

where $\delta \in [0, 1]$ denotes the participation coefficient. Note that the global payoff G guaranteed to surviving policyholders is stochastic since it is proportional to N, that is

$$G = NB,$$

where the individual guaranteed benefit B may depend on other financial or demographic factors and therefore may be random as well. By suitably specifying G (i.e. B) we will obtain different types of provisions payable in case of survival.

In (2.1) three components can be identified: the stochastic guarantee G, the payoff of a call option and that of a shorted put option. Both options are written on the assets of the firm and have a stochastic exercise price depending on G. The call option corresponds to a terminal bonus payment and is usually referred to as the *bonus option*. The participation coefficient δ is the share of the surpluses the policyholders are entitled to as bonus. The shorted put option results from the fact that equityholders have limited liability and is usually known as the *default option*. Unlike the existing literature, this payoff not only depends on the value of financial assets but also on the evolution of the cohort of policyholders under scrutiny and possibly on the realization of demographic risk factors that drive future survival probabilities. The assets W, if insufficient, i.e. $W < G$, will be shared among surviving policyholders. If $G \leq W$, each surviving policyholder will be entitled to the guaranteed amount B and to an additional lump sum bonus if further $\alpha W > G$.

Note that the equityholders' payoff at maturity is residually given by

$$W - L = W 1_{\{N=0\}} + [W - G]^{+} 1_{\{N>0\}} - \delta\alpha \left[W - \frac{G}{\alpha}\right]^{+} 1_{\{N>0\}}. \tag{2.2}$$

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survival probability of 95% (which may be common for a 40-years old policyholder and a 20 years horizon), the probability of extinction is less than 10^{-6} for a group of 5 individuals. When the survival probability is only 50%, the extinction probability is less than 10^{-6} for a group of 20 individuals.

In the present paper we discuss the following alternative specifications for the individual guaranteed benefit B:

(a) $B = b$;
(b) $B = \rho\, a_T$;
(c) $B = b + [b\,\rho^{\mathrm{g}}\, a_T - b]^+ = b \max\{1, \rho^{\mathrm{g}}\, a_T\}$.

Case (a) characterizes pure endowments, where the guarantee is fixed and the individual benefit B is therefore deterministic. We could alternatively consider a stochastic benefit depending on the assets values or some other market related variable.

In case (b) the contracts sold are deferred whole life annuities guaranteeing each survivor the continuous payment at rate ρ per year, starting at time T. The quantity a_T is the market value at time T of an immediate whole life annuity making continuous payments at unitary rate to a life then aged $x + T$. If the market rate a_T were deterministic, from the valuation point of view the contract would be equivalent to that described in case (a). However, the interesting case is when a_T is stochastic as it depends on market conditions prevailing at time T, see Appendix 7.2 where an expression for a_T is worked out. Note that B is then the amount the insurer would need at time T to purchase, on the open market, an immediate annuity matching the future payments guaranteed to each policyholder.

Case (c) describes pure endowments with attached a guaranteed annuity option. These are contracts which provide policyholders with the right to convert, at maturity, a survival benefit into an annuity at a fixed conversion rate ρ^{g}. Conditional on survival, the option is exercised if the benefit b (specified as in case (a)) is less than the market value $b\,\rho^{\mathrm{g}}\, a_T$ of the guaranteed annuity. Indeed, in case the option is exercised, the policyholder will receive an immediate whole life annuity making continuous payments at rate $b\,\rho^{\mathrm{g}}$ per year. Alternatively, the individual benefit can be decomposed into a deferred whole life annuity, as in case (b), making continuous payments at rate $b\,\rho^{\mathrm{g}}$ per year and, in addition, the option to *surrender* the contract at time T. To see this, the individual benefit can be rewritten as follows:

$$B = b\,\rho^{\mathrm{g}}\, a_T + [b - b\,\rho^{\mathrm{g}}\, a_T]^+ .$$

If the surrender option is exercised, the policyholder receives a cash amount equal to b (surrender value).

Although in cases (b) and (c) payments can occur after T, solvency and profit distribution are only assessed at the maturity date by comparing the market values of assets and liabilities, as in Briys and de Varenne (1994, 1997).

It is convenient, especially when analysing (infinitely) large portfolios, to consider quantities at individual, rather than global, level. As policyholders are homogeneous in terms of benefits, the individual liability at maturity T attributed to policyholder i is defined by

$$\ell^i = \frac{L}{N} 1_{\{\tau^i > T\}} = \frac{\Psi}{N} 1_{\{\tau^i > T\}} = \psi 1_{\{\tau^i > T\}}, \quad i = 1, \ldots, N_0, \tag{2.3}$$

where τ^i denotes her residual lifetime.[2] In particular, the liability attributed to each policyholder surviving at time T is then equal to

$$\psi = B + \delta\alpha \left[w - \frac{B}{\alpha} \right]^+ - [B - w]^+ ,$$

with $w = \frac{W}{N}$, on the set $\{N > 0\}$. The interpretation of the three liability components remains unchanged upon considering as underlying of the options the individual share w of the total assets pertaining to each surviving policyholder and, in the exercise price, the individual benefit B instead of the global payoff G. Of course, adding up the individual liabilities recovers the total liability: $L = \sum_{i=1}^{N_0} \ell^i$.

2.2. **Modelling Insurance Risk.** We start this section by observing that the insurance risk affecting our portfolio of homogeneous policyholders arises from the possibility of deviations between actual and expected mortality (survival) rates. As it happens in the case of investment portfolios, this risk can be split into two components. The first component is given by the *unsystematic* risk, that can be diversified away through pooling. In other words, this risk component tends to disappear for large enough portfolios. The second component is instead given by a *systematic* part that hits all policies in the same direction. In our case, this second component can be identified in the so called *longevity risk*, that is the risk of an overall unanticipated decline in mortality rates, see Pitacco *et al.* (2009), Barrieu *et al.* (2012). When it is present, even with a large portfolio there is a residual part of risk that cannot be eliminated.

To model insurance risk, we consider the portfolio of N_0 homogeneous policyholders (each aged x at time 0) introduced in the previous section. The insurer chooses, for pricing purposes, a risk neutral probability Q among the infinitely many equivalent martingale measures existing in incomplete arbitrage-free markets. The probability Q then accounts for both diversifiable and systematic risk inherent to this portfolio, and, in particular, can depend on its size N_0. Recall that τ^i is the residual lifetime of the i-th policyholder in the portfolio. The number of individuals alive in the group at time T is then given by

$$N = \sum_{i=1}^{N_0} 1_{\{\tau^i > T\}}. \tag{2.4}$$

[2]Note that the indicator of the event $\{N > 0\}$ can be omitted in presence of the indicator of the event $\{\tau^i > T\}$.

Assumption 1. *Conditionally on a positive random variable* Δ*, the residual lifetimes* $\tau^i, i = 1, \ldots, N_0$*, are independent, and*

$$Q(\tau^1 > t_1, \ldots, \tau^{N_0} > t_{N_0} | \Delta) = \prod_{i=1}^{N_0} Q(\tau^i > t_i | \Delta) = \prod_{i=1}^{N_0} \mathrm{e}^{-\Delta \int_0^{t_i} m(v) \mathrm{d}v}$$

for any $t_i \geq 0$*,* $i = 1, \ldots, N_0$*, where* m *is a deterministic force of mortality.*[3]

In other words, conditionally on Δ, the residual lifetimes τ^i, $i = 1, \ldots, N_0$, are the first jump times of independent inhomogeneous Poisson processes with common stochastic intensity $\mu_t = \Delta m(t)$, $t > 0$. This framework goes under the name of Cox (or doubly-stochastic) model, see Biffis *et al.* (2010), Brémaud (1981). The positive random variable Δ is a common factor affecting all lifetimes at once and can therefore be interpreted as systematic risk. Its effect is to rescale by a random percentage the deterministic force of mortality m relative to a life aged x at time 0.

Assumption 2. *The random variable* Δ *is part of the information available at the maturity date* T*.*[4]

While the random rescaling amount is unknown at the valuation date (time 0), it is revealed to market participants at time T. In other words, information on demographic risk accumulated by observing mortality experience in this and similar portfolios and/or at national population level allows insurers to resolve the uncertainty related to the systematic risk relative to this specific cohort of individuals. This static and relatively simple parametrization could be extended to a dynamic stochastic mortality model which is updated as new information becomes available.

We remark that a similar multiplicative framework for the force of mortality is sometimes used, although in a different context, in frailty models in order to describe the heterogeneity among individuals in a life insurance portfolio, see for instance Haberman and Olivieri (2008). Our problem, instead, involves completely homogeneous individuals whose lifetime is subject to two layers of uncertainty: a common one due to the randomness of the force of mortality and a specific one resulting from the policyholder's own Poisson process.

From Assumption 1, the t-years survival probability for each individual is

$${}_tp_x = Q(\tau^i > t) = E\left[\mathrm{e}^{-\Delta \int_0^t m(v) \mathrm{d}v}\right]$$

[3]The function m is nonnegative, continuous, and satisfies $\int_0^{+\infty} m(u) \mathrm{d}u = +\infty$.

[4]Formally, the random variable Δ is measurable with respect to the σ-algebra containing the information available to market participants at time T.

for $t \geq 0$ and $i = 1, \ldots, N_0$. In the following we define, for $y \geq x$ and $u \geq 0$,

$$_up^*_y = \mathrm{e}^{-\int_{y-x}^{u+y-x} m(v)\mathrm{d}v}$$

so that in particular, for $0 \leq t \leq s$, we have $_{s-t}p^*_{x+t} = \mathrm{e}^{-\int_t^s m(v)\mathrm{d}v}$. The latter quantity can be thought as the conditional survival probability of a fictitious lifetime τ^* of an individual aged x at 0 having deterministic force of mortality m. More precisely, it is the probability that such individual is still alive at time s conditional on survival at t. When $E[\Delta] < 1$ we have $E[\mu_t] < m(t)$ and, by Jensen inequality, $_tp_x > {}_tp^*_x$. Further, each lifetime τ^i is greater than τ^* in the hazard rate order, see Denuit *et al.* (2006). This and other properties are proved in Appendix 7.1.

To shorten notation, in the following we let $\pi = {}_Tp^*_x$, so that, conditional on Δ, $N \sim \text{Binomial}(N_0, \pi^\Delta)$, while the actual T-years survival probability is $Q(\tau^i > T) = E[\pi^\Delta]$.

The following figures exemplify the versatility of the model in characterizing, despite its simplicity, longevity risk. Figure 1 displays the survival probability $_tp_{40}$, as a function of t, for different choices of the moments of the distribution of Δ. The exact details on the law of Δ and the deterministic force of mortality m employed are provided in Section 5. Note that an increase in $var[\Delta]$ has the same effect (at least in the case $E[\Delta] < 1$) as a decrease in $E[\Delta]$, although survival probabilities are affected mostly at old ages.

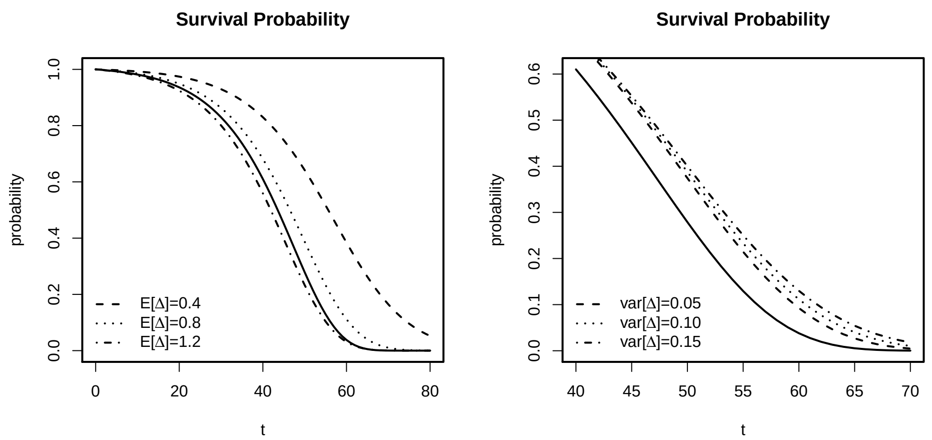

FIGURE 1. Survival probability $_tp^*_{40}$ (continuous line) and $_tp_{40}$ (other lines). The baseline case includes $E[\Delta] = 0.8$ and $var[\Delta] = 0.1$.

Figure 2 displays the percentage change in the expected residual lifetime, $E[\tau^i]$, of a 40-year policyholder with respect to the expectation of the fictitious lifetime τ^* (equal to 41.73 under the same assumptions previously used to construct Figure 1). Note that when

$E[\Delta] = 0.8, 0.4$, the expected lifetime increases by approximately 3 and 13 years respectively. Conversely, when $E[\Delta] = 1.2$, the expected residual lifetime decreases by approximately 1 year. Moreover, the effect of $var[\Delta]$ on the expected residual lifetime is almost linear.

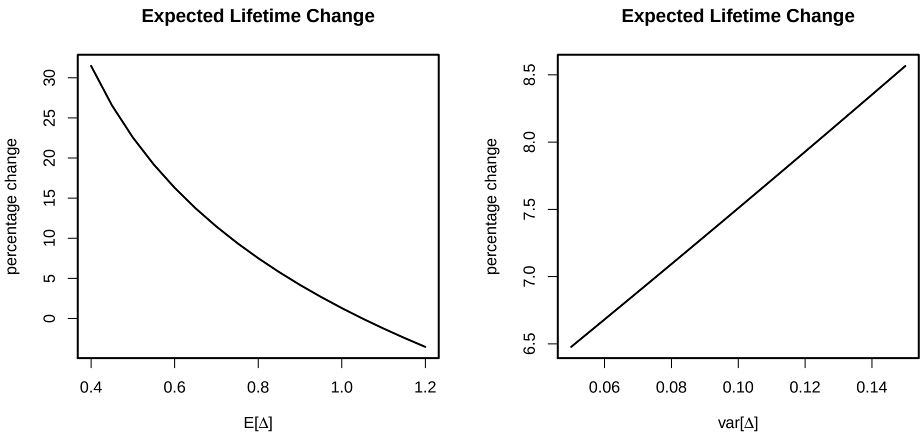

FIGURE 2. Percentage change in $E[\tau^i]$ with respect to $E[\tau^*] = 41.73$ as $E[\Delta]$ and $var[\Delta]$ varies. The baseline case includes $E[\Delta] = 0.8$ and $var[\Delta] = 0.1$.

Finally, in Figure 3 we display the density of $\pi^\Delta = \exp(-\Delta \int_0^{25} m(v)\mathrm{d}v)$, that can be interpreted as the 25 years *stochastic* survival probability for an individual aged 40 at time 0. Although the dependence of this probability on the stochastic reduction factor Δ is not linear, a change in $E[\Delta]$ seems to correspond to a shift in the distribution of π^Δ, except when $E[\Delta]$ is small, in which case the distribution is compressed towards its upper bound.

2.3. **Modelling Large Portfolios Risk.** To represent a portfolio with a large number of homogeneous policyholders, we consider the insurance risk model introduced in the previous section as the portfolio size diverges. More precisely, we assume there are now infinitely many policyholders and, consistently with the previous notation, we denote by τ^i the residual lifetime of the i^{th} policyholder. The risk neutral measure Q now contains an adjustment for systematic risk only, as the portfolio size is large and unsystematic mortality risk has been diversified away.

For each finite sub-portfolio of size N_0, we assume the same contract and capital structure introduced in Section 2.1. In particular, initial assets W_0 and premium income L_0 depend on N_0. It follows that all quantities derived from assets and premiums such as individual and global liabilities, leverage ratio and assets value at time T depend on the sub-portfolio size as well.

The definition of N is still given by (2.4) and now provides the number of survivors at time T within the sub-portfolio of policyholders with

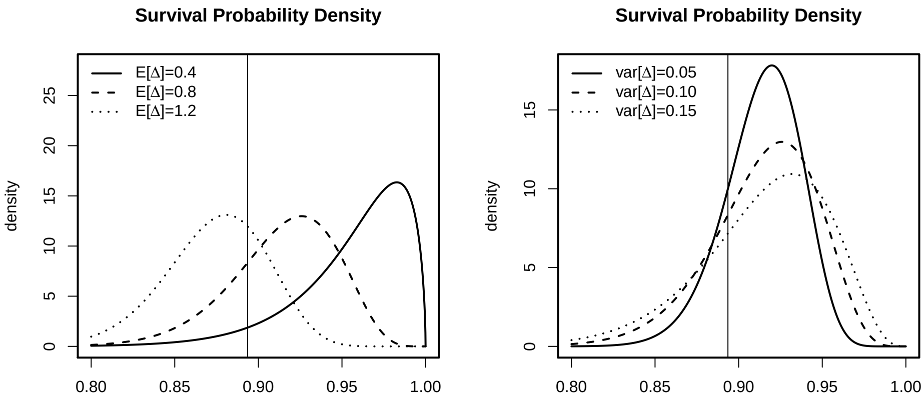

FIGURE 3. Density of $\pi^{\Delta} = \exp(-\Delta \int_0^{25} m(v)\mathrm{d}v)$, the 25 years *stochastic* survival probability at age 40. The vertical line represents the *deterministic* survival probability ${}_{25}p^*_{40}$. The baseline case includes $E[\Delta] = 0.8$ and $var[\Delta] = 0.1$.

index $i = 1, \ldots, N_0$. For each N_0 we keep Assumptions 1-2 under Q, so that the infinitely many random times τ^i, $i = 1, 2, \ldots$, are independent conditionally on Δ. It follows that

$$\frac{N}{N_0} \to \pi^{\Delta} \quad \text{as} \quad N_0 \to +\infty$$

almost surely under Q, see Schervish (1995).[5]

Note that the pricing measure Q, deterministic force of mortality m and random rescaling factor Δ could differ from those introduced in Section 2.2, relative to a finite portfolio. However, in this section and whenever there is no risk of misunderstanding, we stick to this notation. Instead, in Section 5, we will stress the dependence of these quantities and of the corresponding symbols on the portfolio size.

With an infinite portfolio, it only makes sense to consider quantities at individual level. The individual liability for the large portfolio can then be defined by taking the limit in (2.3) as $N_0 \to +\infty$. In order to do so, an assumption on how capital requirements and premium ratings behave as the portfolio size grows is needed. Let then $w_0 = \frac{W_0}{N_0}$ and $\ell_0 = \frac{L_0}{N_0}$ be the individual assets per contract, respectively individual single premium.

[5]This result also holds under any probability measure equivalent to Q, in particular under the physical measure.

Assumption 3. *As $N_0 \to +\infty$,*

$$w_0 \to w_0(\infty) \text{ positive and finite} \tag{2.5}$$

$$\ell_0 \to \ell_0(\infty) \le w_0(\infty). \tag{2.6}$$

It is natural to expect that, in a finite portfolio, the initial assets per contract w_0 decrease with N_0, since they must cover not only the expected individual liability but also its fluctuations. Then, Assumption 3 means that, once the portfolio is large enough for pooling to be fully effective, the assets per contract and the individual single premium stabilize around asymptotic values representing the individual assets and individual premium required in a large portfolio.

The inequality $\ell_0(\infty) \le w_0(\infty)$ will be strict whenever, as in our case, there is systematic risk. The extra capital $w_0(\infty)-\ell_0(\infty)$ provides then a buffer to cover the impact of such risk.

Property (2.5) is satisfied if the initial assets W_0 are set according to a capital requirement criterion guaranteeing a given ruin probability, see Section 4. Property (2.6) holds for many premium calculation principles where the safety loading decreases with the portfolio size. In particular, it is automatically satisfied if one sets premiums using a portfolio based ruin criterion with a higher ruin probability than that used to compute the assets, see Section 4. Of course, the case $\ell_0(\infty) < w_0(\infty)$ might occur in a large portfolio without systematic insurance risk because of systematic financial risk. It also follows from (2.5) and (2.6) that $\alpha = \frac{L_0}{W_0} \to \alpha(\infty) = \frac{\ell_0(\infty)}{w_0(\infty)} \le 1$. The fraction $\alpha(\infty)$ represents the leverage ratio for an insurer supporting a large portfolio.

Under Assumption 3, the individual liability in the large portfolio case for the generic policyholder i is given, on the set $\{\tau^i > T\}$, by

$$\begin{aligned}\ell^i(\infty) = &\lim_{N_0\to+\infty} \ell^i \\ =&B + \delta\alpha(\infty)\left[\frac{w_0(\infty)\mathrm{e}^R}{\pi^\Delta} - \frac{B}{\alpha(\infty)}\right]^+ \\ &- \left[B - \frac{w_0(\infty)\mathrm{e}^R}{\pi^\Delta}\right]^+, \quad i = 1, 2, \dots .\end{aligned} \tag{2.7}$$

We conclude this section by observing that, if no systematic insurance risk affects our portfolio so that Δ is certain, then, by the law of large numbers,

$$\frac{N}{N_0} \to {}_Tp_x \quad \text{as} \quad N_0 \to +\infty$$

almost surely. As all the insurance risk has been diversified away, there is no reason to allow for it when adjusting the physical measure in order to obtain the risk-neutral measure. Therefore, in the absence of both systematic and diversifiable insurance risk, these measures would coincide on events involving insurance risk only, while they may differ on financial related events. Furthermore, if the individual benefit is deterministic, as in case (a), our model could be framed within the original one by Briys and de Varenne (1994, 1997).

2.4. **Modelling Financial Risk.** Since we are primarily concerned with demographic and asset risk, we disregard stochasticity in interest rates and assume that the market short rate is a constant, denoted by r. Then, the financial uncertainty in our model is only due to assets randomness. Beyond the natural requirement that financial and demographic related variables are independent, we do not make any specific assumption on the distribution of the assets value of the firm W under the pricing measure Q.

Assumption 4. *The assets value W is independent of Δ and the residual lifetimes τ^i, $i = 1, \dots, N_0$.*

In the large portfolio case, Assumption 4 holds for each sub-portfolio size N_0.

3. Valuation

3.1. **Finite portfolio case.** Since all policyholders are homogeneous both in terms of benefits and survival probabilities, we consider now the individual liability of the generic policyholder and denote by V_0^ℓ its (initial) market value, given by:

$$\begin{aligned} V_0^\ell =& E\left[\mathrm{e}^{-rT}\ell^i\right] \\ =& E[\mathrm{e}^{-rT}B1_{\{\tau^i>T\}}] \\ &+ \delta\alpha E\left[\mathrm{e}^{-rT}\left[w-\frac{B}{\alpha}\right]^+ 1_{\{\tau^i>T\}}\right] \\ &- E\left[\mathrm{e}^{-rT}\left[B-w\right]^+ 1_{\{\tau^i>T\}}\right] \\ =& V_0^{\mathrm{g}} + \delta\alpha V_0^{\mathrm{b}} - V_0^{\mathrm{d}}, \qquad i = 1, \dots, N_0. \end{aligned} \tag{3.1}$$

The three components, V_0^{g}, V_0^{b} and V_0^{d}, correspond to the values of the guaranteed amount, bonus option and default option, respectively. We derive the above expectations in Appendix 7.3. The value of the total liability is

$$V_0^L = E\left[\mathrm{e}^{-rT}L\right] = E\left[\mathrm{e}^{-rT}\sum_{i=1}^{N_0}\ell^i\right] = N_0V_0^\ell,$$

hence $V_0^\ell = \frac{V_0^L}{N_0}$.

A manipulation of the formulae in (3.1), see Appendix 7.3, shows that the values of the liability components can be expressed in an alternative, yet meaningful, way:

$$\begin{aligned} V_0^{\mathrm{g}} &= E\left[\mathrm{e}^{-\int_0^T \widehat{r}(v)\,\mathrm{d}v} B\right], \\ V_0^{\mathrm{b}} &= E\left[\mathrm{e}^{-\int_0^T \widehat{r}(v)\,\mathrm{d}v} \left[\frac{W}{N^{(i)}} - \frac{B}{\alpha}\right]^+\right], \\ V_0^{\mathrm{d}} &= E\left[\mathrm{e}^{-\int_0^T \widehat{r}(v)\,\mathrm{d}v} \left[B - \frac{W}{N^{(i)}}\right]^+\right], \end{aligned} \tag{3.2}$$

where $\widehat{r}(v) = r + \Delta m(v)$ can be interpreted as the *mortality adjusted* discount rate while $N^{(i)} = 1 + \sum_{h \neq i} 1_{\{\tau^h > T\}}$ is the number of survivors at time T on the set $\{\tau^i > T\}$. Here $\frac{W}{N^{(i)}}$ represents the fraction of assets pertaining to the i-th policyholder, assumed to be alive at time T. The value of each liability component is obtained as an expectation, under the risk-neutral measure Q, of its *adjusted* final payoff discounted at the rate $\widehat{r}$, see also Biffis (2005), Biffis *et al.* (2010).

A contract is fair for the policyholders if the initial market value of the outstanding liabilities equates their initial investment. Alternatively, the contract is fair whenever the equity issuing price is equal to its market value. Fair contracts are then those for which

$$V_0^L = \alpha W_0 \quad \text{or, equivalently,} \quad V_0^\ell = \alpha w_0,$$

i.e., using (3.1), $V_0^{\mathrm{g}} + \delta\alpha V_0^{\mathrm{b}} - V_0^{\mathrm{d}} = \alpha w_0$. Fairness can therefore be defined at global or individual level.

It is particularly relevant to analyse the trade-off between contract parameters that implicitly define a fair policy. These parameters include the participation coefficient and, depending on the type of contract, the survival benefit, the annuity rate and the guaranteed annuity rate. Note that we can explicitly display the participation coefficient δ associated with a fair contract as

$$\delta = \frac{\alpha w_0 - V_0^{\mathrm{g}} + V_0^{\mathrm{d}}}{\alpha V_0^{\mathrm{b}}}, \tag{3.3}$$

while other fair parameters have to be searched for numerically. The fair participation coefficient δ attains its maximum value $Q(N > 0)^{-1}$ when the individual guaranteed benefit B is 0, see Equation (7.2). Then, in principle, the participation coefficient given by (3.3) could exceed 100%, in order to compensate for the low benefit and for the fact that, in the unlikely event that no policyholder survives maturity, the whole assets are passed to the equityholders, see (2.2). On the other hand, if the individual benefit is too high, the default option may be insufficient to compensate the high value of the guarantee and the right hand side of (3.3) could return a negative coefficient. However, we only consider fair contracts for which the participation coefficient δ lies within the interval $[0,1]$.

3.2. **Large Portfolio Case.** Recalling from (2.7) the expression of the individual liability for the generic policyholder in an infinite portfolio, its value is given by

$$\begin{aligned}V_0^\ell(\infty) =& E[\mathrm{e}^{-rT}\ell^i(\infty)]\\ =& E[\mathrm{e}^{-rT}B1_{\{\tau^i>T\}}]\\ &+\delta\alpha(\infty)E\left[\mathrm{e}^{-rT}\left[\frac{w_0(\infty)\mathrm{e}^R}{\pi^\Delta}-\frac{B}{\alpha(\infty)}\right]^+1_{\{\tau^i>T\}}\right]\\ &-E\left[\mathrm{e}^{-rT}\left[B-\frac{w_0(\infty)\mathrm{e}^R}{\pi^\Delta}\right]^+1_{\{\tau^i>T\}}\right]\\ =& V_0^{\mathrm{g}}(\infty)+\delta\alpha(\infty)V_0^{\mathrm{b}}(\infty)-V_0^{\mathrm{d}}(\infty).\end{aligned}$$

The three liability components are computed in Appendix 7.4. Again, it is possible to express them in an alternative way:

$$\begin{aligned}V_0^{\mathrm{g}}(\infty) &= E\left[\mathrm{e}^{-\int_0^T\widehat{r}(v)\,\mathrm{d}v}B\right],\\ V_0^{\mathrm{b}}(\infty) &= E\left[\mathrm{e}^{-\int_0^T\widehat{r}(v)\,\mathrm{d}v}\left[\frac{w_0(\infty)\mathrm{e}^R}{\pi^\Delta}-\frac{B}{\alpha(\infty)}\right]^+\right],\\ V_0^{\mathrm{d}}(\infty) &= E\left[\mathrm{e}^{-\int_0^T\widehat{r}(v)\,\mathrm{d}v}\left[B-\frac{w_0(\infty)\mathrm{e}^R}{\pi^\Delta}\right]^+\right].\end{aligned}$$

We remark that fairness of a contract in an infinitely large portfolio can only be defined at individual level. Fair contracts are then those for which

$$V_0^\ell(\infty)=\alpha(\infty)w_0(\infty).$$

The fair participation coefficient has a similar expression to that in (3.3), namely

$$\delta(\infty)=\frac{\alpha(\infty)w_0(\infty)-V_0^{\mathrm{g}}(\infty)+V_0^{\mathrm{d}}(\infty)}{\alpha(\infty)V_0^{\mathrm{b}}(\infty)}. \tag{3.4}$$

Once again, $\delta(\infty)$ reaches its maximum when $B=0$. However, this maximum is now equal to 1 as the extinction probability is 0. Then, if the guaranteed benefit is 0, policyholders and equityholders have proportional claims on the firm's assets according to their initial contribution.

4. Ruin Probability Capital Requirements

In this section we show how capital requirements and premiums can be calculated, under the physical measure, using a criterion based on the probability of ruin, and discuss their behaviour as the portfolio size diverges. To this end, denote by $\widetilde{Q}$ the physical probability measure and recall that $\widetilde{Q}$ and the pricing measure Q are equivalent. We suppose that Assumptions 1, 2, 4 now hold under $\widetilde{Q}$, with the deterministic force of mortality m replaced by $\widetilde{m}$. We have therefore assumed that the stochastic force of mortality multiplicative structure is preserved under the change of measure, so that $\widetilde{\mu}_t = \Delta\widetilde{m}(t)$ for $t > 0$. Although, in principle, we may have allowed both the deterministic force of mortality and the rescaling factor under Q to be different than those under $\widetilde{Q}$, here, for simplicity, we have maintained the same rescaling factor Δ and modified the deterministic force of mortality only. For a general discussion of change of measure and intensities in Cox processes, see Brémaud (1981) and, in the context of stochastic mortality, see Biffis *et al.* (2010).

Assume that the initial assets W_0 are set according to the following ruin probability criterion (see e.g. Pitacco *et al.* (2009)):

$$\widetilde{Q}(W < G) = \epsilon, \tag{4.1}$$

where ϵ is the ruin probability. We recall that $G = NB$ is the global benefit and $W = W_0 \mathrm{e}^R$, with R the assets log-return over the period $[0, T]$. Hence initial assets are set by forcing the default event — the guaranteed global payoff cannot be covered by the final assets — to have a given confidence level.

Denote now by $\widetilde{F}_R$ and $\widetilde{E}$ the cumulative distribution function of R, respectively the expectation operator, under $\widetilde{Q}$, and assume that R has a continuous distribution with support the real line. The ruin probability Equation (4.1) can be rewritten, upon conditioning on Δ and N, as

$$\widetilde{E}\left[\widetilde{F}_R\left(\log\frac{NB}{W_0}\right)\right] = \epsilon. \tag{4.2}$$

It is immediate to check that for each $0 < \epsilon < 1$ there exists a unique positive solution of (4.1), denoted W_0^ϵ. The following proposition establishes some properties of W_0^ϵ as a function of the portfolio size N_0.

Theorem 1. *The solution W_0^ϵ of (4.1) satisfies the properties:*

(1) *W_0^ϵ is an increasing function of N_0, $\lim_{N_0\to+\infty} W_0^\epsilon = +\infty$,*
(2) *$\lim_{N_0\to+\infty} \frac{W_0^\epsilon}{N_0} = w_0(\infty)$, with $0 < w_0(\infty) < \infty$.*

The proof of Theorem 1 is reported in Appendix 7.5.1.

In the infinite portfolio case, the individual asset allocation can be computed by solving with respect to $w_0(\infty)$ the equation obtained by taking the limit in (4.2), namely

$$\widetilde{E}\left[\widetilde{F}_R\left(\log\frac{\widetilde{\pi}^\Delta B}{w_0(\infty)}\right)\right] = \epsilon, \tag{4.3}$$

with $\widetilde{\pi} = \mathrm{e}^{-\int_0^T \widetilde{m}(v)\mathrm{d}v}$.

To fix the initial overall contribution L_0 (or $\ell_0(\infty)$), one can choose a ruin probability $\epsilon' > \epsilon$ and let $L_0 = W_0^{\epsilon'}$. Hence, the capital provided by equityholders allows to lower the ruin probability from the level ϵ' to ϵ.

Expressions for the expectations in (4.2) and (4.3) are provided in Appendix 7.5.1.

5. Numerical analysis

This section carries out a sensitivity analysis of the various contract components' values as well as of the fair participation rates δ and $\delta(\infty)$ computed according to (3.3) and (3.4). The different contract features introduced in Section 2.1 are considered and compared.

5.1. **Sensitivity analysis - Large portfolio case.** We begin our analysis with the large portfolio case. In particular, we are working under the pricing measure $Q \equiv Q^{(\infty)}$. We assume a deterministic Gompertz law of mortality $m \equiv m^{(\infty)}$,

$$m(t) = \lambda c^{x+t}, \quad t \geq 0,$$

from which

$${}_{s-t}p^*_{x+t} = \mathrm{e}^{-\lambda c^x (c^s - c^t)/\log c}, \quad 0 \leq t \leq s.$$

We set

$$x = 40, \quad \lambda = 2.6743 \cdot 10^{-5}, \quad c = 1.098\,.$$

The values of λ and c were obtained by fitting the survival probabilities ${}_tp^*_{40}$ to the corresponding probabilities implied by the projected life table IPS55 in use in the Italian annuity market. The random variable $\Delta \equiv \Delta^{(\infty)}$ is assumed to be Gamma distributed with $var[\Delta] = 0.1$ while $E[\Delta] \in \{0.4, 0.8, 1.2\}$, corresponding to different mortality pricing assumptions relative to various degrees of conservativeness.

We assume that R, the assets log-return over the interval $[0, T]$, is normally distributed with mean $(r - \sigma^2/2)T$ and standard deviation $\sigma\sqrt{T}$, so that σ is the assets volatility. Unless otherwise mentioned, we fix the following parameter values, which we refer to as baseline case, and, for ease of notation, we suppress all dependencies on ∞:

- maturity $T = 25$;
- initial individual assets per contract $w_0 = 100$;
- initial contribution ratio $\alpha = 0.7$;
- riskless rate $r = 0.03$;
- assets volatility $\sigma = 0.15$;
- in cases (a) and (c), individual survival benefit $b = 150$;
- in case (b), instantaneous annuity amount $\rho = 10$;
- in case (c), guaranteed annuity rate $a^{\mathrm{g}} \doteq 1/\rho^{\mathrm{g}} = 15$.

5.1.1. *Deferred annuities.* Tables 1-3 report the sensitivity of case (b) (deferred annuity) with respect to the parameters ρ, r and σ and show how the different contract components are affected by systematic mortality changes. As expected mortality rates shift downward, both the value of the guaranteed amount and the default option increase, while a reversed impact is observed on the value of the bonus option. Indeed, the insurance company is expected to pay a higher guaranteed amount because both survival probabilities and the annuity value a_T are higher. This in turn implies a lower bonus payment. Overall, for the chosen parameters, the effect of a decrease in $E[\Delta]$ on the value of the guaranteed payment $V_0^{\rm g}$ dominates the other components appearing in (3.4), resulting in a lower participation rate. Furthermore, the impact of such a change becomes much more evident when combined with high annuity payments (see Table 1). When conservative pricing assumptions are adopted, too generous annuity rates are only compatible with less appealing participation coefficients, outside the range $80\% - 100\%$ often practised in the past.[6] Fair contracts may not even exist as, no matter how low is the participation rate, the value of the liabilities cannot match the initial policyholders' contribution. The right hand side of Equation (3.4) produces then a negative value, implying that policyholders should actually transfer part of their assets to the equityholders to compensate for the increased risk. When this happens, the value of the fair participation coefficient δ in the tables is not displayed.

	$E[\Delta] = 0.4$				$E[\Delta] = 0.8$				$E[\Delta] = 1.2$			
ρ	$\delta\%$	$V_0^{\rm g}$	$V_0^{\rm b}$	$V_0^{\rm d}$	$\delta\%$	$V_0^{\rm g}$	$V_0^{\rm b}$	$V_0^{\rm d}$	$\delta\%$	$V_0^{\rm g}$	$V_0^{\rm b}$	$V_0^{\rm d}$
5.0	90.28	43	48	3	95.69	33	57	1	97.65	28	63	1
7.5	69.16	65	33	11	85.11	50	42	5	91.33	42	49	3
10.0	32.76	87	23	22	66.14	67	31	11	79.64	56	38	7
12.5	—	108	17	36	37.33	84	24	20	61.63	70	30	13
15.0	—	130	12	52	—	100	18	30	36.41	84	24	20

TABLE 1. Case (b) for a large portfolio, different annuity rates ρ and values of $E[\Delta]$.

The higher the annuity payment ρ, the lower is the fair participation coefficient δ, as the insurance company is forced to compensate for the increasing cost of the deferred annuity $V_0^{\rm g}$, which is proportional to ρ. The extent of this variation prevails on the increased value of the default option $V_0^{\rm d}$ and the decreased bonus option value $V_0^{\rm b}$. For conservative annuity rates, the bonus portion $\alpha\delta V_0^{\rm b}$ overweighs the other components and constitutes the most sizeable part of the total liability αw_0.

When analysing the dependence on the market interest rate (see Table 2) similar patterns arise, although the effect of r on the different

[6]See for instance Briys and de Varenne (1994, 1997)

	$E[\Delta]=0.4$				$E[\Delta]=0.8$				$E[\Delta]=1.2$			
$r\%$	$\delta\%$	V_0^{g}	V_0^{b}	V_0^{d}	$\delta\%$	V_0^{g}	V_0^{b}	V_0^{d}	$\delta\%$	V_0^{g}	V_0^{b}	V_0^{d}
1	—	196	6	108	—	140	10	59	—	113	15	39
2	—	129	13	52	8.17	97	19	28	45.47	79	25	17
3	32.76	87	23	22	66.14	67	31	11	79.64	56	38	7
4	76.54	59	36	8	87.93	47	45	4	92.75	40	51	2
5	92.21	40	50	3	95.96	33	57	1	97.60	28	62	1

Table 2. Case (b) for a large portfolio, different risk free rates r and values of $E[\Delta]$.

contract components is reversed. The value of the deferred annuity V_0^{g} and that of the default option V_0^{d} are depressed by an increase in the risk free rate, while the bonus (call) option value V_0^{b} increases. Once again, the guaranteed benefit outweighs the other components, resulting in more attractive participation coefficients. On the other hand, under low interest rate regimes comparable to those currently observed in many markets, the insurance company should apply rather uncompetitive participation rates, or may even be unable to offer fair contracts.

	$E[\Delta]=0.4$,			$E[\Delta]=0.8$			$E[\Delta]=1.2$		
σ	$\delta\%$	V_0^{b}	V_0^{d}	$\delta\%$	V_0^{b}	V_0^{d}	$\delta\%$	V_0^{b}	V_0^{d}
0.100	—	14	14	53.37	22	5	78.76	30	2
0.125	9.36	18	18	60.23	27	8	78.58	34	5
0.150	32.76	23	22	66.14	31	11	79.64	38	7
0.175	48.01	28	26	71.08	36	15	81.21	42	10
0.200	58.61	33	30	75.19	40	18	82.93	46	12

Table 3. Case (b) for a large portfolio, different volatilities σ and values of $E[\Delta]$. The values of the guaranteed amount are $V_0^{\mathrm{g}}=87$ for $E[\Delta]=0.4$, $V_0^{\mathrm{g}}=67$ for $E[\Delta]=0.8$, $V_0^{\mathrm{g}}=56$ for $E[\Delta]=1.2$.

A change in the assets volatility only affects the optional contract components, both increasing with σ (see Table 3). However, in most instances, the increase in the default option overshadows that in the bonus option, resulting in richer fair participation coefficients. The opposite only occurs when no longevity improvements are expected and σ is extremely low, as default turns out to be very unlikely (under the risk neutral measure). Nonetheless, the participation coefficients remain in line with those commonly offered by insurance companies. When instead conservative mortality assumptions are in place, the insurer may be tempted to seek highly volatile investment opportunities in order to keep the participation rates within reasonable bounds.

5.1.2. *Pure endowments and guaranteed annuity options.* Tables 4-7 report sensitivities in cases (a) and (c) with respect to the parameters b, a^{g}, r and σ. In case (a) we only display the fair participation rate and the value of the guaranteed individual benefit. Recall that the individual benefit in case (c) can be decomposed into two parts: a pure endowment benefit as in case (a), and a guaranteed annuity option, see Section 2.1. The values of these liabilities are called V_0^{g1}, respectively V_0^{g2}.

	$E[\Delta]=0.4$						$E[\Delta]=0.8$						$E[\Delta]=1.2$					
b	$\delta^{(\mathrm{a})}\%$	$\delta^{(\mathrm{c})}\%$	V_0^{g1}	V_0^{g2}	V_0^{b}	V_0^{d}	$\delta^{(\mathrm{a})}\%$	$\delta^{(\mathrm{c})}\%$	V_0^{g1}	V_0^{g2}	V_0^{b}	V_0^{d}	$\delta^{(\mathrm{a})}\%$	$\delta^{(\mathrm{c})}\%$	V_0^{g1}	V_0^{g2}	V_0^{b}	V_0^{d}
100	89.14	77.39	45	13	37	8	90.54	88.55	43	3	45	4	91.77	91.65	41	0	49	3
125	78.88	58.06	56	16	29	14	81.49	77.91	54	3	37	7	83.80	83.57	52	0	40	5
150	64.29	31.28	68	19	23	22	68.59	62.89	65	4	30	12	72.40	72.04	62	0	34	9
175	44.85	—	79	23	18	31	51.39	42.95	76	5	25	18	57.17	56.65	72	0	28	14
200	20.00	—	90	26	15	41	29.40	17.56	86	6	21	24	37.72	36.98	83	0	24	19

TABLE 4. Cases (a) and (c) for a large portfolio, different guaranteed lump sums b and values of $E[\Delta]$. $\delta^{(\mathrm{a})}$ and $\delta^{(\mathrm{c})}$ are the fair participation rates for cases (a) and (c). V_0^{g1} is the value of the guaranteed survival benefit in cases (a) and (c); V_0^{g2}, V_0^{b} and V_0^{d} are respectively the values of the guaranteed annuity, bonus and default options in case (c).

The results reported in Tables 4-7 help understanding the difference between cases (a) and (c). In general, the cost of adding a guaranteed annuity option to a pure endowment contract translates into lower participation coefficients, and the spread $\delta^{(a)} - \delta^{(c)}$ measures the extra 'premium' required to purchase such option. The values of the different liability components and the fair participation coefficients qualitatively share the same comparative statics observed in case (b) with respect to the mortality assumption. In particular, the guaranteed annuity option value V_0^{g2} is negligible unless some substantial longevity improvements are foreseeable. Exceptions hold when exceedingly generous annuity conversion rates are offered or, more notably, under low interest rate regimes. When the guaranteed annuity option is valueless, there is practically no difference between cases (a) and (c) and the gap between the corresponding fair participation coefficients vanishes.

	$E[\Delta] = 0.4$				$E[\Delta] = 0.8$				$E[\Delta] = 1.2$			
a^g	$\delta^{(c)}\%$	V_0^{g2}	V_0^b	V_0^d	$\delta^{(c)}\%$	V_0^{g2}	V_0^b	V_0^d	$\delta^{(c)}\%$	V_0^{g2}	V_0^b	V_0^d
10.0	—	62	12	52	—	36	18	30	36.41	22	24	20
12.5	—	36	18	33	43.72	16	25	18	64.88	6	31	12
15.0	31.28	19	23	22	62.89	4	30	12	72.04	0	34	9
17.5	50.86	9	27	16	67.97	0	32	10	72.39	0	34	9
20.0	59.64	3	29	13	68.56	0	32	10	72.40	0	34	9

Table 5. Cases (a) and (c) for a large portfolio, different guaranteed conversion rates a^g and values of $E[\Delta]$. In case (c), $\delta^{(c)}$, V_0^{g2}, V_0^b and V_0^d are respectively the fair participation rate and the values of the guaranteed annuity option, bonus and default option. The fair participation rate in case (a) and the value of the guaranteed survival benefit in cases (a) and (c) are $\delta^{(a)} = 64.29$ and $V_0^{g1} = 68$ for $E[\Delta] = 0.4$; $\delta^{(a)} = 68.59$ and $V_0^{g1} = 65$ for $E[\Delta] = 0.8$; $\delta^{(a)} = 72.40$ and $V_0^{g1} = 62$ for $E[\Delta] = 1.2$.

We note that the guarantee components V_0^{g1} and V_0^{g2} are proportional to the lump sum b. When expected longevity improvements are important, the participation coefficient spread widens as the lump sum benefit grows, see Table 4. Unlike case (a), fairness is not achievable in case (c) when a huge lump sum benefit is offered, as the default option value cannot compensate for the guaranteed annuity option cost.

A similar, more striking situation arises when too generous conversion conditions are used (low levels of $a^g = 1/\rho^g$, see Table 5). It should be noted how the participation coefficient spread reacts to changes in the conversion rate. As soon as the annuity conversion option becomes valuable, the premium required to purchase such option takes off and may be unsustainable. Further, when conservative mortality assumptions are used, even offering very low conversion rates still incurs a cost.

	$E[\Delta]=0.4$						$E[\Delta]=0.8$						$E[\Delta]=1.2$					
$r\%$	$\delta^{(a)}\%$	$\delta^{(c)}\%$	V_0^{g1}	V_0^{g2}	V_0^{b}	V_0^{d}	$\delta^{(a)}\%$	$\delta^{(c)}\%$	V_0^{g1}	V_0^{g2}	V_0^{b}	V_0^{d}	$\delta^{(a)}\%$	$\delta^{(c)}\%$	V_0^{g1}	V_0^{g2}	V_0^{b}	V_0^{d}
1	—	—	112	85	6	108	—	—	107	34	10	59	—	—	102	12	15	39
2	27.95	—	87	43	13	52	36.43	5.78	83	14	19	28	43.94	38.51	80	3	24	19
3	64.29	31.28	68	19	23	22	68.59	62.89	65	4	30	12	72.40	72.04	62	0	34	9
4	82.69	75.10	53	7	35	9	84.85	84.30	50	1	41	5	86.76	86.76	48	0	43	4
5	91.89	90.79	41	2	48	3	92.96	92.94	39	0	51	2	93.89	93.89	38	0	52	2

TABLE 6. Cases (a) and (c) for a large portfolio, different risk free rates r and values of $E[\Delta]$. $\delta^{(a)}$ and $\delta^{(c)}$ are the fair participation rates for cases (a) and (c). V_0^{g1} is the value of the guaranteed survival benefit in cases (a) and (c); V_0^{g2}, V_0^{b} and V_0^{d} are respectively the values of the guaranteed annuity, bonus and default options in case (c).

The market interest rate affects directly annuity prices and therefore is the most important factor when discussing guaranteed annuity options. Table 6 is particularly interesting as it helps to single out the pricing scenarios under which the effect of interest rates is most relevant. First, when the market interest rate is 1% (or lower), fairness of the contract cannot be achieved for both the pure endowment and the guaranteed annuity option no matter how low is the share of profits which is released to policyholders, regardless of the mortality assumption. When longevity improvements are anticipated and low to moderate interest rate regimes operate, then the 'perfect storm' scenario is created as fairness can only be obtained at a huge cost in terms of lost share of profits passed back to policyholders, and this is much more the case when the conversion option is present. Both guarantee components strongly react to changes in r, and so do the option components. When instead interest rates are higher, the guaranteed annuity option becomes valueless and the different contract components are insensitive to further interest rate rises.

	$E[\Delta] = 0.4$				$E[\Delta] = 0.8$				$E[\Delta] = 1.2$			
σ	$\delta^{(a)}\%$	$\delta^{(c)}\%$	$V_0^{\rm b}$	$V_0^{\rm d}$	$\delta^{(a)}\%$	$\delta^{(c)}\%$	$V_0^{\rm b}$	$V_0^{\rm d}$	$\delta^{(a)}\%$	$\delta^{(c)}\%$	$V_0^{\rm b}$	$V_0^{\rm d}$
0.100	49.43	—	13	14	58.63	46.37	21	6	66.21	65.51	25	4
0.125	57.57	6.80	18	18	63.72	55.55	25	9	69.02	68.53	29	6
0.150	64.29	31.28	23	22	68.59	62.89	30	12	72.40	72.04	34	9
0.175	69.75	47.07	28	26	72.87	68.74	35	16	75.67	75.41	38	12
0.200	74.21	57.98	32	30	76.53	73.46	39	19	78.65	78.45	42	15

TABLE 7. Cases (a) and (c) for a large portfolio, different volatilities σ and values of $E[\Delta]$. $\delta^{(a)}$ and $\delta^{(c)}$ are the fair participation rates for cases (a) and (c). $V_0^{\rm b}$ and $V_0^{\rm d}$ are the values of the bonus and default option in case (c). The values of the guaranteed survival benefit in cases (a) and (c) and of the guaranteed annuity option in case (c) are $V_0^{\rm g1} = 68$ and $V_0^{\rm g2} = 19$ for $E[\Delta] = 0.4$; $V_0^{\rm g1} = 65$ and $V_0^{\rm g2} = 4$ for $E[\Delta] = 0.8$; $V_0^{\rm g1} = 62$ and $V_0^{\rm g2} = 0$ for $E[\Delta] = 1.2$.

Both option components of the liability $V_0^{\rm b}$ and $V_0^{\rm d}$ increase when the assets volatility does, see Table 7. As in (b), the major effect in both cases (a) and (c) comes from the default option. Overall, the fair participation coefficient increases with the assets volatility, at least for the set of parameters considered here. For the pure endowment, the portion of extra profits transferred to policyholders always stays at reasonable levels. When the guaranteed annuity option is added to the contract and no longevity improvements are expected, the participation spread turns out to be very small and insensitive to volatility changes. If instead moderate to conservative mortality assumptions are in force, the corresponding cost in terms of missed participation to profits can be substantial and even prevent the contract to attain fairness.

5.1.3. *Role of the initial contribution ratio.* Figure 4 displays the sensitivity of the fair participation rate δ with respect to the initial contribution ratio α for the three contracts. We recall that case (c) differs from (a) due to the presence of the guaranteed annuity option, while (c) differs from (b) due to the surrender option. Then the spreads $\delta^{(a)} - \delta^{(c)}$ and $\delta^{(b)} - \delta^{(c)}$ give the extra cost, in terms of missed return of profits, required to add the corresponding option to the contract.

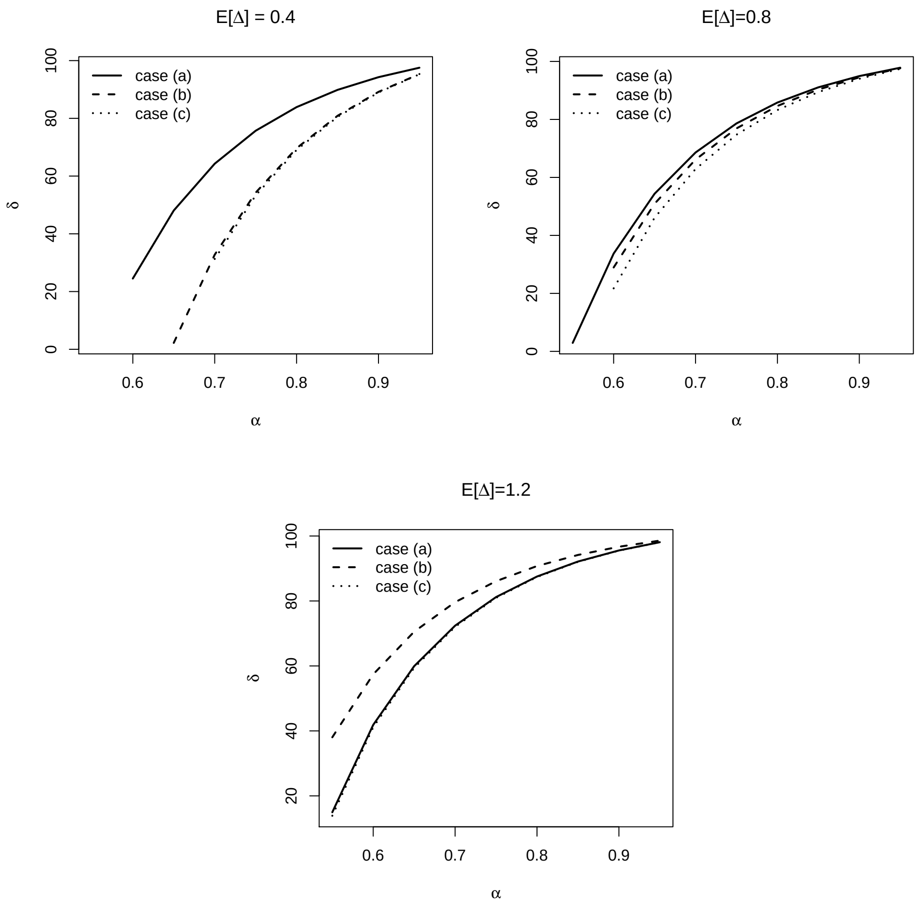

FIGURE 4. Fair participation coefficient δ in cases (a), (b) and (c) for different values of α and $E[\Delta]$.

In all cases (a)-(c), the participation rate increases with the leverage ratio. Indeed, equityholders not only are entitled to a full participation on their quota $1-\alpha$ of the assets, but also to an extra participation, at rate $1-\delta$, of the share of assets α held by policyholders. Therefore, the higher the leverage ratio, the smaller is the extra surplus participation rate yield by them in order to maintain fairness. In the limiting case of $\alpha = 1$ (a mutual company) then $\delta = 1$ as policyholders are entitled to share all profits after benefits have been paid.

Under conservative mortality assumptions, there is practically no difference between cases (b) and (c), as keeping the guaranteed annuity provides a higher value compared to swapping for a lump sum payment. The surrender option is therefore negligible. On the other hand, the possibility of converting a lump sum into an annuity at a guaranteed rate is greatly valuable. This situation is completely reversed when mortality is expected to worsen slightly, and cases (a) and (c) practically coincide, so that the guaranteed annuity option is almost valueless, as previously pointed out. The surrender option instead is sizeable as, at maturity, it may be convenient to give up the annuity and obtain a cash payment. There are no fair contracts for low levels of α, meaning that, as the guaranteed benefit is fixed, there is no way to compensate the low policyholders' contribution by reducing their share of profits. When α increases, the implied cost of the annuity conversion and surrender options decrease as a greater part of the benefits comes from participation to profits.

5.2. **Sensitivity analysis - finite portfolio case.** We move now to the case of finite portfolios. As the qualitative behaviour of the liability components with respect to contractual and market parameters follows the same pattern observed for large portfolios, we limit ourselves to report the fair participation coefficient for different portfolio sizes and longevity assumptions in the baseline case (unless otherwise mentioned).

In Table 8, fair participation coefficients are calculated assuming that the pricing measure Q and the leverage ratio α are independent of N_0, that is $Q \equiv Q^{(N_0)} = Q^{(\infty)}$. We notice that diversifiable risk

	$E[\Delta]=0.4$			$E[\Delta]=0.8$			$E[\Delta]=1.2$		
N_0	$\delta^{(a)}\%$	$\delta^{(b)}\%$	$\delta^{(c)}\%$	$\delta^{(a)}\%$	$\delta^{(b)}\%$	$\delta^{(c)}\%$	$\delta^{(a)}\%$	$\delta^{(b)}\%$	$\delta^{(c)}\%$
1	75.20	44.04	42.68	91.91	89.58	86.92	—	—	—
2	65.49	35.21	33.86	71.18	68.82	65.87	76.81	83.27	76.49
5	64.56	33.56	32.13	69.00	66.58	63.46	72.84	79.77	72.50
10	64.43	33.16	31.70	68.80	66.37	63.18	72.63	79.71	72.28
100	64.30	32.80	31.32	68.61	66.17	62.92	72.42	79.65	72.07
∞	64.29	32.76	31.28	68.59	66.14	62.89	72.40	79.64	72.04

TABLE 8. Cases (a), (b) and (c) for a finite portfolio, different portfolio sizes and values of $E[\Delta]$. $\delta^{(a)}$, $\delta^{(b)}$ and $\delta^{(c)}$ are the fair participation rates for cases (a), (b) and (c).

can be eliminated even by pooling relatively small groups of policyholders, as large portfolios' fair participation rates are achieved very soon. Although the portfolio sizes considered here may appear, at first sight, much lower than actual book dimensions, they are in line with the sizes of completely homogeneous sub-portfolios. The fair participation coefficients decrease with N_0 for all types of benefits, with sizeable change when passing from $N_0 = 1$ to $N_0 = 2$. In fact, if all policyholders die before maturity, the assets are entirely transferred to equityholders, see Equation (2.2). In a small portfolio, the likelihood of such event is not completely negligible and, to achieve fairness, equityholders must agree to release a larger share of profits to policyholders. In particular, in the limiting case of a single policyholder's pool and slight mortality worsening, fairness cannot be achieved as, no matter how high the participation rate is, the initial contribution exceeds the market value of liabilities. The right hand side of (3.3) then produces a value greater than 100%. As the portfolio size grows, the extinction probability decreases and, therefore, fairness can be obtained through lower participation rates.

For the next example, we choose, as before, a pricing measure Q independent of the portfolio size N_0, but we set the initial individual assets w_0 and liabilities ℓ_0 according to the ruin probability criterion described in Section 4. To this end we assume that the deterministic mortality intensity $\widetilde{m}$ driving survival probabilities under the physical measure $\widetilde{Q}$ is such that $m = \gamma\widetilde{m}$, with $\gamma = 0.9$. The risk neutral force of mortality m is then obtained through a proportional reduction of $\widetilde{m}$. Moreover, the systematic risk factor Δ has a Gamma distribution with variance $\widetilde{var}[\Delta] = 0.1$ and expectation $\widetilde{E}[\Delta] = 1$. Therefore, $\widetilde{E}[\widetilde{\mu}_t] = \widetilde{m}(t)$ and $\widetilde{m}$ can be seen as a best estimate force of mortality. The instantaneous assets return is normally distributed with mean 5%, and standard deviation 15%. Finally, to compute the initial assets per contract we fix a ruin probability, over the $T = 25$ years horizon, of 12.5% and, for the initial premium, of 25%. Roughly, if solvency were monitored on a yearly basis, these figures would correspond to an annual ruin probability of 0.53% for the assets and 1.14% for the initial contribution. In Table 9 we display the initial assets w_0, contributions ℓ_0, leverage ratios $\alpha = \ell_0/w_0$ and fair participation coefficients δ for different portfolio sizes N_0. The latter are computed assuming for the rescaling factor Δ, under the pricing measure Q, a Gamma distribution with mean $E[\Delta] = 0.8$ and variance $var[\Delta] = 0.1$ and same assets volatility as under the physical measure $\widetilde{Q}$. As expected, both initial assets and contributions decrease with portfolio size, reflecting the diversification benefit. The corresponding leverage ratio appears to be remarkably stable, even for small portfolios. Adjusting capital and premiums to the size of the pool implies smoother fair participation rates as compared to those in Table 8. Again, in the limiting case of a single policyholder's pool, fair contracts cannot be achieved even under moderate longevity improvement assumptions.

Finally, as a last example, we fix assets and liabilities as in the baseline case, but adjust the pricing measure $Q^{(N_0)}$ to reflect the portfolio

	Case (a)				Case (b)				Case (c)			
N_0	w_0	ℓ_0	$\alpha\%$	$\delta\%$	w_0	ℓ_0	$\alpha\%$	$\delta\%$	w_0	ℓ_0	$\alpha\%$	$\delta\%$
1	127	88	68.74	—	123	84	68.53	—	130	89	68.70	—
2	123	84	68.72	82.54	118	81	68.45	78.17	125	86	68.66	80.25
5	120	83	69.38	80.81	116	80	69.05	76.27	123	85	69.30	78.43
10	120	83	69.66	80.91	115	80	69.31	76.33	122	85	69.58	78.52
100	119	83	69.92	81.04	115	80	69.55	76.44	122	85	69.83	78.65
∞	119	83	69.95	81.05	115	80	69.58	76.45	122	85	69.86	78.66

TABLE 9. Cases (a), (b) and (c) for different portfolio sizes, individual assets w_0 and liabilities ℓ_0 computed using a ruin probability criterion, leverage ratio $\alpha = \ell_0/w_0$ and fair participation rate δ.

size. To keep things simple, we assume that under $Q^{(N_0)}$ the stochastic force of mortality is $\mu_t^{(N_0)} = m(t)\Delta^{(N_0)}$, so that we keep the same deterministic intensity as in the large portfolio case and adjust the systematic rescaling factor $\Delta^{(N_0)}$. More precisely, under $Q^{(N_0)}$, we take for $\Delta^{(N_0)}$ a Gamma distribution with the same variance as in the baseline case and expectation tied to the portfolio size according to the following specification:

$$E^{(N_0)}[\Delta^{(N_0)}] = E^{(\infty)}[\Delta^{(\infty)}]\phi(N_0),$$

where $\phi(N_0) = \frac{N_0}{N_0+1}$. This formulation allows for an adjustment of the systematic risk factor that vanishes as the portfolio size increases. The correction with respect to $Q^{(\infty)}$ is stronger for small portfolios, where diversifiable insurance risk weighs more. In the limiting case of a single policyholder, the effect of the adjustment is to halve the (expected) stochastic mortality, resulting in an extremely prudential liabilities assessment. Then, as N_0 (and $E^{(N_0)}[\Delta^{(N_0)}]$) increases, there are, in all cases (a)-(c), two opposite effects on the fair participation rate. On one hand, as the portfolio size grows the participation rate is pushed down, as it happens in Table 8 when we move downward along a given column. On the other hand, the increase in $E^{(N_0)}[\Delta^{(N_0)}]$ has a positive effect on δ, as it happens in Table 8 when we move rightward along a given row. In particular, the single policyholder

N_0	$\phi(N_0)\%$	$\delta^{(a)}\%$	$\delta^{(b)}\%$	$\delta^{(c)}\%$
1	50	75.20	44.04	42.68
2	67	67.40	50.89	49.32
5	83	67.59	59.30	57.08
10	91	68.05	62.64	60.00
100	99	68.53	65.79	62.60
∞	100	68.59	66.14	62.89

TABLE 10. Fair participation rates δ for cases (a), (b) and (c) with different portfolio sizes and size-adjusted risk neutral measures, $E^{(\infty)}[\Delta^{(\infty)}] = 0.8$.

portfolio case in Table 10 corresponds to the leftmost columns, top row, of Table 8. Conversely, the large portfolio case corresponds to the central columns, bottom row, of Table 8. Looking at cases (b) and (c) in Table 10, as the portfolio size grows the adjustment to the survival rates prevails over the decrease in the extinction probability, resulting in higher fair participation coefficients. The opposite pattern occurs in case (a) for $N_0 \leq 2$, while, for larger pools, the gain in probability extinction exhausts its effects and the fair participation rate remains stable. This different behaviour of cases (b) and (c) with respect to case (a) is due to the fact that individual benefits of the annuity-type highly depend on the mortality assumption, unlike pure endowment-type benefits. This is apparent from Table 8 when comparing the results in cases $E[\Delta] = 0.4$ and $E[\Delta] = 0.8$. The percentage increase in the fair participation coefficients in (b) and (c) is substantial and remarkably stable for any portfolio size. In case (a) the corresponding increase is moderate and comparatively low for $N_0 > 2$.

6. Concluding remarks

This paper aims at shedding some light on the interplay between two key risk factors affecting most life insurance products, namely biometric and investment risk. We enhance the pioneering model by Briys and de Varenne (1994, 1997), featuring a stylized participating life insurance company by explicitly tying benefits to the survivorship of a cohort of policyholders. In particular, we allow for the two main components of biometric risk, that is systematic (longevity) risk and diversifiable (process) risk. The former stems from the uncertainty surrounding future survival rates affecting all policyholders at once, the latter is due to the specific mortality risk associated with each policyholder and can be eliminated after pooling together portfolios of homogeneous contracts.

A first result of our analysis is that systematic risk overshadows process risk even for small portfolios. This fact is not surprising since longevity risk has been recognized as one of the most challenging factors affecting the life insurance business. During the last few decades, demographers and actuaries have made a great effort in trying to develop sound stochastic mortality models that capture trend and variability of survival rates over time. Our base mortality model could then be enhanced by employing a more realistic, dynamic approach. However, we feel that the qualitative nature of our findings will be preserved.

One of the main consequences of the credit crunch crisis has been the transition to a long-lasting phase of extremely low interest rate regimes in many developed countries. This has put some severe strain on life insurers' balance sheets by sensibly inflating the market value of liabilities, even though interest rates are expected to rise again in the near future. We have decided, mostly to preserve the simplicity of the model, to consider constant non-random interest rates. Nonetheless, our results are quite worrying as they show that, under low interest rate levels, yet not even close to those currently experienced, the cost of guarantees offered may be hardly sustainable. A further dimension could then be added by allowing for fluctuations in interest rates through one of the many stochastic term structure models available.

Finally, to keep the transparency of our model to a reasonable level, we have focused on a static, one period approach involving a closed cohort of policyholders and a terminal bonus rate which is decided at the onset. Clearly, a deeper analysis would result from considering the life insurance company as a going concern, including features such as writing new business, setting reversionary bonuses, checking dynamically solvency and updating pricing rules and capital requirements. However, all these aspects could be introduced at the cost of missing some clarity in the results and are left for future research.

7. Appendix

7.1. Properties of τ_i and N.

7.1.1. *Law of τ_i.* The survival probability of a policyholder is given by

$$ {}_tp_x = Q(\tau^i > t) = E\left[\mathrm{e}^{-\Delta \int_0^t m(v)\mathrm{d}v}\right] = \mathcal{L}_\Delta\left(\log {}_tp_x^*\right), $$

where $\mathcal{L}_\Delta$ is the moment-generating function of Δ, i.e. $\mathcal{L}_\Delta(y) = E[\mathrm{e}^{\Delta y}]$.

7.1.2. *Ordering between τ_i and τ^*.*

Proposition 1. *If $E[\Delta] \leq 1$ then τ_i is greater than τ^* in the hazard rate order.*

Proof. We need to show that the ratio ${}_tp_x/{}_tp_x^*$ is nondecreasing with t. For $t < s$, we have

$$ \frac{{}_sp_x}{{}_sp_x^*} - \frac{{}_tp_x}{{}_tp_x^*} = \mathcal{M}({}_sp_x^*) - \mathcal{M}({}_tp_x^*) \geq 0, $$

since the function $\mathcal{M}(z) = \mathcal{L}_\Delta(\log z)/z$, $0 < z \leq 1$, is nonincreasing when $E[\Delta] \leq 1$ as can be seen by inspecting its derivative:

$$ z^2\mathcal{M}'(z) = E[z^\Delta(\Delta - 1)] = \mathrm{Cov}\left(z^\Delta, \Delta\right) + E[z^\Delta]\, E[\Delta - 1] \leq 0. $$

□

7.1.3. *Law of N.* The number of survivors N has, conditionally on Δ, a binomial distribution:

$$ N \sim \text{Binomial}\left(N_0, \mathrm{e}^{-\Delta \int_0^T m(v)\mathrm{d}v}\right). $$

Consequently, the unconditional law of N is a mixture of binomial distributions. Denoting by F_Δ the cumulative distribution function of Δ, we have, for $j = 0, 1, \ldots, N_0$,

$$Q(N = j) = E\left[\text{bin}\left(j; N_0, \pi^\Delta\right)\right] = \int_0^\infty \text{bin}\left(j; N_0, \pi^l\right) F_\Delta(\mathrm{d}l),$$

where $\text{bin}(j; M, p) = \binom{M}{j} p^j (1-p)^{M-j}$ is the mass function of a Binomial random variable with parameters $M \geq 1$ and $0 < p < 1$.

7.2. **Market value of the unitary annuity.** Under Assumptions 1 and 2, the market value of the unitary annuity a_T is

$$\begin{aligned} a_T &= E\left[\int_T^\infty \mathrm{e}^{-r(s-T)} 1_{\{\tau^i > s\}} \mathrm{d}s \middle| \tau^i > T,\, \Delta\right] \\ &= \int_T^\infty \mathrm{e}^{-r(s-T)} Q\left(\tau^i > s \middle| \tau^i > T,\, \Delta\right) \mathrm{d}s \\ &= \int_T^\infty \mathrm{e}^{-r(s-T)} \mathrm{e}^{-\Delta \int_T^s m(v)\mathrm{d}v} \mathrm{d}s \\ &= a(\Delta), \end{aligned}$$

where the function a is given by:

$$a(l) = \int_T^\infty \mathrm{e}^{-r(s-T)} \left({}_{s-T}p^*_{x+T}\right)^l \mathrm{d}s.$$

Note that $a(l)$ is the value of a continuous annuity with force of mortality $l\, m$.

7.3. **Valuation Formulae in the Finite Portfolio Case.** We denote by $C(A, r, T, K)$ and $P(A, r, T, K)$ the values at time 0 of a European call, respectively put, option written on the assets of the firm, when time to maturity is T, initial assets value is A, (fixed) interest rate is r and strike is K.

Note that the individual benefit B is a function of Δ, say $B = \beta(\Delta)$, where

$$\beta(l) = \begin{cases} b & \text{in case (a)} \\ \rho\, a(l) & \text{in case (b)} \\ b \max\{1, \rho^{\mathrm{g}} a(l)\} & \text{in case (c)} \end{cases}.$$

7.3.1. *Market value of the guaranteed amount.* Conditioning on Δ, it follows that

$$\begin{aligned} V_0^{\mathrm{g}} &= E[\mathrm{e}^{-rT} B 1_{\{\tau^i > T\}}] \\ &= \mathrm{e}^{-rT} E\left[B \pi^\Delta\right] \\ &= \mathrm{e}^{-rT} \int_0^\infty \beta(l) \pi^l F_\Delta(\mathrm{d}l). \end{aligned} \tag{7.1}$$

7.3.2. *Market value of the bonus option.* Recalling that $N^{(i)} = 1 + \sum_{h \neq i} 1_{\{\tau^h > T\}}$ is independent of τ^i conditionally on Δ and that W is independent of all biometric related factors, we have

$$\begin{aligned} V_0^{\mathrm{b}} &= E\left[\mathrm{e}^{-rT}\left[w - \frac{B}{\alpha}\right]^{+} 1_{\{\tau^i > T\}}\right] \\ &= E\left[\pi^{\Delta} E\left[\mathrm{e}^{-rT}\left[\frac{W}{N^{(i)}} - \frac{B}{\alpha}\right]^{+} \mid \Delta\right]\right]. \end{aligned}$$

By further conditioning on $N^{(i)}$ the inner expectation and exploiting again Assumption 4,

$$\begin{aligned} V_0^{\mathrm{b}} &= E\left[\pi^{\Delta} E\left[C\left(\frac{W_0}{N^{(i)}}, r, T, \frac{B}{\alpha}\right) |\Delta\right]\right] \\ &= \int_0^{\infty} \pi^l \sum_{j=1}^{N_0} C\left(\frac{W_0}{j}, r, T, \frac{\beta(l)}{\alpha}\right) \operatorname{bin}\left(j-1; N_0 - 1, \pi^l\right) F_{\Delta}(\mathrm{d}l). \\ &= \frac{1}{N_0} \int_0^{\infty} \sum_{j=1}^{N_0} C\left(W_0, r, T, \frac{j\beta(l)}{\alpha}\right) \operatorname{bin}\left(j; N_0, \pi^l\right) F_{\Delta}(\mathrm{d}l), \end{aligned} \tag{7.2}$$

where the last equation is obtained after multiplying and dividing by $\frac{j}{N_0}$.

Note that Equation (7.2) immediately highlights the valuation formula for the aggregate bonus option $N_0 V_0^{\mathrm{b}}$.

7.3.3. *Market value of the default option.* Manipulations similar to those in Section 7.3.2 can be used to obtain the following expression for the default option value:

$$\begin{aligned} V_0^{\mathrm{d}} &= E\left[\mathrm{e}^{-rT}\left[B - w\right]^{+} 1_{\{\tau^i > T\}}\right] \\ &= \frac{1}{N_0} \int_0^{\infty} \sum_{j=1}^{N_0} P\left(W_0, r, T, j\beta(l)\right) \operatorname{bin}\left(j; N_0, \pi^l\right) F_{\Delta}(\mathrm{d}l). \end{aligned}$$

7.4. **Valuation Formulae in the Large Portfolio Case.** Recall that now F_{Δ} and E refer to the cumulative distribution function, respectively expectation operator, under the probability $Q = Q^{\infty}$.

7.4.1. *Market value of the guaranteed amount.* This is formally the same expression as in the case of a finite portfolio, Equation (7.1):

$$\begin{aligned} V_0^{\mathrm{g}}(\infty) =& E[\mathrm{e}^{-rT} B 1_{\{\tau^i > T\}}] \\ =& \mathrm{e}^{-rT} \int_0^{\infty} \beta(l) \pi^l F_{\Delta}(\mathrm{d}l). \end{aligned}$$

7.4.2. *Market value of the bonus option.* Conditioning on Δ and exploiting the independence between financial and demographic factors, we obtain

$$\begin{aligned} V_0^{\mathrm{b}}(\infty) &= E\left[\mathrm{e}^{-rT}\left[\frac{w_0(\infty)\mathrm{e}^R}{\pi^\Delta} - \frac{B}{\alpha(\infty)}\right]^+ 1_{\{\tau^i>T\}}\right] \\ &= E\left[C\left(\frac{w_0(\infty)}{\pi^\Delta}, r, T, \frac{B}{\alpha(\infty)}\right)\pi^\Delta\right] \\ &= \int_0^\infty C\left(w_0(\infty), r, T, \frac{\beta(l)\pi^l}{\alpha(\infty)}\right) F_\Delta(\mathrm{d}l). \end{aligned}$$

7.4.3. *Market value of the default option.* Similarly as in Section 7.4.2, we have:

$$\begin{aligned} V_0^{\mathrm{d}}(\infty) &= E\left[\mathrm{e}^{-rT}\left[B - \frac{w_0(\infty)\mathrm{e}^R}{\pi^\Delta}\right]^+ 1_{\{\tau^i>T\}}\right] \\ &= \int_0^\infty P\left(w_0(\infty), r, T, \beta(l)\pi^l\right) F_\Delta(\mathrm{d}l). \end{aligned}$$

7.5. **Results relative to Section 4.**

7.5.1. *Proof of Theorem 1.*

(1) Write $N^{(N_0)}$ to stress the dependence of N on N_0. Note that $N^{(N_0+1)} \geq N^{(N_0)}$ almost surely and $\widetilde{Q}\left(N^{(N_0+1)} > N^{(N_0)}\right) > 0$. It follows that W_0^ϵ increases with N_0. If the limit of W_0^ϵ as $N_0 \to +\infty$ were finite, then, as $N^{(N_0)} \to +\infty$ a.s., we would have
$$\widetilde{E}\left[\widetilde{F}_R\left(\log\frac{NB}{W_0}\right)\right] \to 1,$$
contradicting (4.2).

(2) Recall first that $N^{(N_0)}/N_0 \to \widetilde{\pi}^\Delta > 0$ and note that $B > 0$. If $W_0^\epsilon/N_0 \to w_0(\infty)$ then the expectation in (4.2) converges to
$$\widetilde{E}\left[\widetilde{F}_R\left(\log\frac{\widetilde{\pi}^\Delta B}{w_0(\infty)}\right)\right].$$
As this limit is also equal to $\epsilon \in (0,1)$, it follows that $0 < w_0(\infty) < +\infty$.

Denote explicitly $W_0^\epsilon(N_0)$ the solution of (4.2) with respect to N_0. To prove that the limit of $W_0^\epsilon(N_0)/N_0$ exists, suppose there are two subsequences (N_0') and (N_0'') such that
$$\frac{W_0^\epsilon(N_0')}{N_0'} \to w_0'(\infty), \quad \frac{W_0^\epsilon(N_0'')}{N_0''} \to w_0''(\infty)$$
with $0 < w_0'(\infty) < w_0''(\infty) < \infty$. Taking the limit in the expectation (4.2) under the two subsequences leads to two different

limits while (4.2) states that both limits should coincide with ϵ.

7.5.2. *Calculation of* W_0^ϵ. For a finite portfolio, the expectation in (4.2) can be computed by

$$\widetilde{E}\left[\widetilde{F}_R\left(\log\frac{NB}{W_0}\right)\right] = \int_0^\infty \sum_{j=0}^{N_0} \widetilde{F}_R\left(\log\frac{j\beta(l)}{W_0}\right)\mathrm{bin}(j;N_0,\widetilde{\pi}^l)\widetilde{F}_\Delta(\mathrm{d}l).$$

In the infinite portfolio case, the expectation in (4.3) can be calculated by

$$\widetilde{E}\left[\widetilde{F}_R\left(\log\frac{\widetilde{\pi}^\Delta B}{w_0(\infty)}\right)\right] = \int_0^\infty \widetilde{F}_R\left(\log\frac{\widetilde{\pi}^l\beta(l)}{w_0(\infty)}\right)\widetilde{F}_\Delta(\mathrm{d}l).$$

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