Speculation Under Unawareness*

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Abstract

"No trade" theorems establish that, in various trading environments, investors who share a common prior will not engage in speculation, as long as expected utility, Bayesian updating and full awareness are imposed. We relax the last assumption by allowing for asymmetric unawareness and examine under which conditions speculative behavior emerges. We find that if common knowledge is assumed (as in the settings of Aumann [1976] and Milgrom and Stokey [1982]), unawareness cannot generate speculation. This is not true, however, in settings where no common knowledge is assumed, such as speculation in equilibrium (Geanakoplos [1989]) and betting that is always beneficial (Morris [1994]), unless stronger conditions on awareness are imposed.

JEL-Classifications: C70, C72, D80, D82.

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1 Introduction

A well established behavioral implication of the common prior assumption (Harsanyi [1968]) is that it precludes speculation, a result which is robust to the particular details of the trading environment and stems from the no-agreeing-to-disagree result of Aumann [1976]. To focus on just four, Morris [1994], Samet [1998] and Feinberg [2000] show that there cannot be a bet that makes everyone strictly better off and this is common knowledge (we henceforth call this speculative betting) or it is true at all states (always beneficial bet). Geanakoplos [1989] shows that there cannot be trade from a Pareto efficient allocation in a Bayesian Nash equilibrium (speculation in equilibrium)

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and Milgrom and Stokey [1982] show that if an allocation is ex ante Pareto efficient it cannot be common knowledge in the interim stage that there is another allocation that Pareto dominates it (speculative trade).[1]

One reason why the details of these trading environments do not matter is that several idealised assumptions are imposed, such as expected utility, Bayesian updating (implying Dynamic Consistency) and full awareness of the relevant dimensions of the environment. In this paper we examine whether and how speculation occurs when the last assumption is relaxed, so that some investors might be partially aware. We find that the details of the trading environment matter for speculation and, in particular, unawareness is compensated only by the property of common knowledge, which is sufficient to ensure no speculation.

Specifically, we show that common priors imply the absence of speculative betting. This result was also established in Heifetz et al. [2013a], however they also impose an additional property in their framework, Projections Preserve Posteriors, which is not needed.[2] We also show that the converse is true. In particular, we identify a condition, Enlargements Preserve Common Priors (EPCP), which requires that whenever there is a "local" common prior generating beliefs within all public or self evident events at each state space, there is also a common prior across all state spaces. This condition is automatically satisfied in the standard model without unawareness and a unique state space. We then show that no common priors and EPCP are equivalent to speculative betting.

Second, as long as the payoff relevant state space, where allocations depend on, coincides with the "common" state space, which is the most expressive state space that it is common knowledge that everyone is aware of, speculative trade (which also imposes common knowledge) cannot occur. This assumption would be true, for example, in the case where the payoff relevant state space describes the prices of all possible stocks, and this coincides with the common state space.

Unawareness, however, does break the connection between common priors and no speculation in environments where common knowledge is not assumed. Examples 1 and 2 show that no common priors are neither sufficient nor necessary for the existence of an always beneficial bet.[3] However, under Conditional Independence, an always beneficial bet implies no common priors. This property requires that investors do not misunderstand the signal created by the information revealed by their varying awareness.[4] Moreover, under the stronger condition of Projections Preserve Posteriors, no common "local" priors (in the state space where the bet is formulated) imply an always beneficial bet. Finally, we show that speculation does not occur in equilibrium

[1] Although common priors are not required for the speculative trade theorem, a Pareto efficient allocation necessitates some form of common priors, as we explain in footnote 23. Also, the setting is slightly different from that of Milgrom and Stokey [1982], as we do not employ a signal structure but a type or belief structure.

[2] This property requires that posterior beliefs do not change as we project down to a state space describing lower awareness.

[3] Note that, under unawareness, what is always true may not always be common knowledge, because some unaware investors may fail to deduce it, due to their limited perception. As a result, an always beneficial bet does not imply speculative betting, as in the standard setting.

[4] It was first studied by Galanis [2015, 2016a] in the context of analysing the value of information in single-investor and multi-investor environments.

if each investor's information structure either satisfies Conditional Independence or path-independence. This last property specifies that each attained level of awareness specifies a unique path of successively lower levels of awareness that the investor has attained in other states.

When common knowledge is not assumed, an always beneficial bet can occur with a common prior because investors cannot reason properly about the information of others, as their awareness may be too low. This forces them to take information at face value, without being able to completely comprehend why others are selling when they are buying. This "bias" in reasoning is consistent with empirical evidence in psychology which shows that individuals are in general slow to incorporate additional information because of their confidence in their existing assumptions and opinions (Fischhoff et al. [1977]).[5] Related is also the confirmatory bias, which suggests that once investors form strong hypotheses, they tend to ignore new information that contradicts them (Rabin [1998]).

These and other psychological biases have inspired a large literature in finance, which generates speculative behavior with overconfident investors who overestimate the precision of some signals and underestimate the precision of others.[6] For example, Scheinkman and Xiong [2003] explain speculative bubbles and large trading volumes using two groups of investors and two signals that are publicly available. Each group is overconfident about one signal, regarding the other signal (and the fact that the other group is overconfident about it) as noise.

Most of these models assume that investors have a "wrong" perception of the signal structure, which can be formalised by having different priors over it. The main difference of the present paper is that it endogenizes the investors' speculative behavior by explicitly modelling their awareness, without altering the common prior assumption, thus providing an insight into why certain types of speculation occur whereas other do not. Moreover, as we argue in Section 3.5, at least in some settings (e.g. correlated equilibria), static models with different priors can always be reinterpreted as models with common priors and investors with significant information processing errors. This means that a model with different priors, which does not provide foundations for the investors' errors or limited perception, may nevertheless have implications about them which are not clear.

Finally, examining speculation under various trading environments allows us to differentiate, in terms of behavioral implications, between unawareness models, different priors models (such as models with overconfidence) and models with information processing errors represented by non-partitional structures (e.g. Geanakoplos [1989]).

1.1 Related Literature

The literature on no trade theorems stems from Aumann [1976], who shows that a common prior implies that it cannot be common knowledge that the posteriors are different. Investors trade because they have different priors, they have no common

[5] See also Oskamp [1965], Mahajan [1992] and Paese and Kinnaly [1993] in the psychology literature.

[6] See Daniel and Hirshleifer [2015] for a survey. I thank a referee for pointing out the connection to the overconfidence literature.

knowledge or they make information processing errors, for example by being unaware. In the context of the standard model where investors make no mistakes, Morris [1994], Bonanno and Nehring [1996], Samet [1998], Feinberg [2000], Ng [2003] and Heifetz [2006] show that a necessary and sufficient condition for the existence of a common prior is that there is no bet for which all investors expect a positive gain always.[7] Milgrom and Stokey [1982] and Sebenius and Geanakoplos [1983] show that common priors imply that there cannot be speculative trade in a two-period model. Ma [2001] and Halevy [2004] generalize this result for general preferences that satisfy Dynamic Consistency, whereas Galanis [2016c] shows that Dynamic Consistency can be weakened.

Although there are several explanations of why investors trade (e.g. liquidity shocks) the consensus is that they cannot account for the great magnitude of trading that we observe, unless we allow for investors to agree to disagree or speculate.[8] Harrison and Kreps [1978] generate excessive trading in a model without common priors and no short sales. Scheinkman and Xiong [2003] use the same framework to model trading in an environment where each investor is overconfident, overestimating the informational content of some signals. Similarly, Daniel et al. [1998, 2001] show how overconfidence increases trading volume. Eyster et al. [2015] show that disagreement neglect, the inability to properly account for the informational content of prices or others' trades, is key in generating trading volume in large markets with dispersed private information.

Geanakoplos [1989] examines speculation in an environment where investors share a common prior but make information processing errors, modelled in a single state space and non-partitional information structures.[9] Dekel et al. [1998] argue that mistakes due to unawareness cannot be modelled in the one state space framework of Geanakoplos [1989]. Galanis [2013], Heifetz et al. [2013a] and Meier and Schipper [2014b] generalize the no-agreeing-to-disagree result of Aumann [1976] in a model with unawareness, using multiple state spaces.[10]

The paper implicitly assumes that investors are unaware of their own unawareness. That is, they do not entertain the possibility that they might be missing something, which could lead them, for example, to not trade even if their posterior beliefs specify that this is beneficial, or conversely to trade when it is not. This assumption is made in most of the unawareness literature. However, there are some papers (e.g. Halpern and Rêgo [2009], Board and Chung [2011a,b]) which explicitly model agents who are aware of their own unawareness. Board and Chung [2011b] show that there is no trade if, either everyone believes that there is nothing they are unaware of, or everyone believes that there is something that they are unaware of. In a more applied setting and without

[7]Note that in the standard model without unawareness, what is always true is also always common knowledge. Conversely, if something is common knowledge, it is always true within this common knowledge event.

[8]See discussion in Hong and Stein [2007] and Daniel and Hirshleifer [2015], for example.

[9]We compare his model with ours in more detail in Section 3.5.

[10]There is now a large literature on unawareness, beginning with Fagin and Halpern [1988]. For interactive models with unawareness, see Modica and Rustichini [1994, 1999], Halpern [2001], Heifetz et al. [2006], Halpern and Rêgo [2008], Li [2009], Heifetz et al. [2008], Galanis [2011a], Chen et al. [2012] and Galanis [2013]. For games with unawareness, see Feinberg [2012], Rêgo and Halpern [2012], Heifetz et al. [2013b], Meier and Schipper [2014a] and Halpern and Rêgo [2014].

providing a formal model of unawareness, Tirole [2009] examines contracting between a seller and a buyer who are aware that they are unaware. Gabaix and Laibson [2006] show that firms will optimally shroud high-priced add ons from unaware consumers, a fact which is exploited by fully aware consumers. Moreover, informational shrouding occurs even in highly competitive markets or markets with costless advertising.

The paper proceeds as follows. Section 2 presents the model. In Section 3 we examine speculative behavior in four different environments, namely speculative betting, always beneficial bet, speculation in equilibrium and speculative trade. Section 3.5 compares our approach with models that generate speculation through different priors or non-partitional information structures. Section 4 concludes. Proofs are contained in the Appendix.

2 The model

2.1 Preliminaries

Different levels of awareness are represented by disjoint state spaces. Let $\mathcal{S} = \{S_a\}_{a \in A}$ be the finite collection of all these state spaces. We assume that $\mathcal{S}$ is a complete lattice and endow it with a partial order $\preceq$.[11] If $S \preceq S'$, we say that S' is (weakly) more expressive than S or, equivalently, that an investor whose state space is S' is more aware than an investor whose state space is S. By construction, there is a top and a bottom state space, which we call the full state space S^* and the payoff relevant state space S_0, respectively. That is, for all $S \in \mathcal{S}$, $S_0 \preceq S \preceq S^*$.

A state s is an element of some state space S. Let $\Sigma = \bigcup_{S \in \mathcal{S}} S$ be the set of all states. We assume that every state space $S \in \mathcal{S}$ has finitely many states. An *event* E is a subset of some state space $S \in \mathcal{S}$, which we denote by $S(E)$.

If $S \preceq S'$, so that S' is more expressive than S, we require that each state $s' \in S'$ can be mapped to its "restricted" image in the less expressive S. Formally, we require that there is a surjective projection $r_S^{S'} : S' \to S$. Projections are required to commute: if $S \preceq S' \preceq S''$, then $r_S^{S''} = r_S^{S'} \circ r_{S'}^{S''}$.

Let $E, E' \subseteq \Sigma$ and suppose that for each $s \in E$, $s' \in E'$, where $s \in S$ and $s' \in S'$, we have $S' \succeq S$. We then write $E' \succeq E$, because all states in E' describe higher awareness than all states in E. We denote the projection of set $E' \succeq S$ to the less expressive S by $E'_S = \bigcup \{r_S^{S'}(s') \in S : s' \in E'\}$. We denote the enlargement of $E' \preceq S''$ to the more expressive S'' by ${E'}^{S''} = \bigcup \{s'' \in S'' : r_{S'}^{S''}(s'') \in E'\}$.

2.2 Type structures

Let S_μ be the state space $S \in \mathcal{S}$ such that $\mu \in \Delta S$. If $\mu \in \Delta S$ and $S' \preceq S$ then $\mu_{|S'}$ is the marginal of μ on S', so that $\mu_{|S'}(E) = \mu(E^S)$, where $E \subseteq S'$. For all events $E, F \subseteq S$ with $\mu(F) > 0$, let $\mu(E|F) = \frac{\mu(E \cap F)}{\mu(F)}$ be the probability of E conditional on F.

[11] A complete lattice is a partially ordered set in which all subsets $\mathcal{G} \subseteq \mathcal{S}$ have both a supremum (or join, denoted $\bigvee \mathcal{G}$) and an infimum (or meet, denoted $\bigwedge \mathcal{G}$).

Definition 1. *For each investor $i \in I$, there is a type mapping $t^i : \Sigma \to \bigcup_{S \in \mathcal{S}} \Delta S$, satisfying the following properties:*

0. *Confinement: If $s \in S$ then $t^i(s) \in \Delta S'$ for some $S' \preceq S$.*
1. *If $S'' \succeq S' \succeq S$, $s \in S''$, and $t^i(s) \in \Delta S$ then $t^i(s_{S'}) = t^i(s)$.*
2. *If $S'' \succeq S' \succeq S$, $s \in S''$, $s' \in S'$ and $t^i(s) = t^i(s') \in \Delta S'$ then $t^i(s_S) = t^i(s'_S)$.*
3. *If $S'' \succeq S' \succeq S$, $s \in S''$, and $t^i(s_{S'}) \in \Delta S$ then $S_{t^i(s)} \succeq S$.*
4. *If $S'' \succeq S' \succeq S$, $s \in S''$, and $t^i(s) \in \Delta S'$ then $S_{t^i(s_S)} = S$.*

Properties 0,1 and 3 are identical to the respective properties in Definition 1 of Heifetz et al. [2013a]. Property 2 specifies that if s and s' specify the same type for i and therefore are indistinguishable by him, their projection to a lower state space will also specify that the projections are indistinguishable. In other words, projections preserve ignorance, which is a property first proposed by Heifetz et al. [2006] and adopted by Galanis [2013]. Similarly, 4 corresponds to the property that projections preserve awareness.[12]

Properties 2 and 4 are implied by Property 2 in Definition 1 of Heifetz et al. [2013a], which we call Projections Preserve Posteriors.[13]

Definition 2 (Projections Preserve Posteriors)**.** *If $S'' \succeq S' \succeq S$, $s \in S''$, $t^i(s) \in \Delta S'$ then $t^i(s_S) = t^i(s)_{|S}$.*

The requirement that $t^i(s_S) = t^i(s)_{|S}$ is twofold. First, since $t^i(s)_{|S}$ is the marginal of $t^i(s)$ on S, it is specified that $t^i(s_S) \in \Delta S$, or equivalently $S_{t^i(s_S)} = S$. This is exactly Property 4 of Definition 1. Second, if both s and its projection to S, s_S, describe that investor i is aware of event $E \subseteq S$, then both s and s_S specify the same posterior beliefs about E. This is stronger than Property 2 of Definition 1, as the latter only requires that the support of $t^i(s)$ (projected on S) cannot be larger than the support of $t^i(s_S)$.

Let $S^i(s)$ denote investor i's state space at $s \in \Sigma$. In particular, $S^i : \Sigma \to \mathcal{S}$ is such that for any $s \in \Sigma$, $S^i(s) = S$ if $t^i(s) \in \Delta S$. If $S^i(s) \succ S^j(s)$ then we say that *investor i is more aware than investor j at s.*

Let

$$P^i(s) = \{s_1 \in S^i(s) : t^i(s) = t^i(s_1)\}$$

be the event in $S^i(s)$ describing that i has the same beliefs as in s. This is the standard possibility correspondence that here is not a primitive but derived by the type mapping. The following Assumption allows us to interpret $P^i(s)$ as the event that i considers possible at s.[14]

Assumption 1. *If $P^i(s) \subseteq E$ for some event E, then $t^i(s)(E) = 1$.*

[12] Proposition 1 in the Appendix shows how properties about types translate to properties of possibility correspondences, which are the primitives in Heifetz et al. [2006] and Galanis [2013].

[13] The name Projections Preserve Posteriors is not used by Heifetz et al. [2013a].

[14] This is similar to Assumption 1 in Heifetz et al. [2013a].

Definition 3. *An interactive unawareness belief structure is a tuple* $\underline{\mathcal{S}} = \langle \mathcal{S}, \{r^{S_\alpha}_{S_\beta}\}_{S_\beta \preceq S_\alpha}, \{t^i\}_{i \in I} \rangle$, *where* $\mathcal{S}$ *and each* $S \in \mathcal{S}$ *are finite.*

We say that $\underline{\mathcal{S}}$ is a positive belief structure if $s' \in P^i(s)$ implies $t^i(s)(s') > 0$, for all $s \in \Sigma$ and $i \in I$. A similar assumption is made by Morris [1994] in his Definition 2.1, where all possible signals are given a positive posterior belief by each investor. For simplicity, in all of our results we require a positive belief structure, which implies that each investor considers the true state (up to his awareness) to be possible.

2.3 Events, belief and common knowledge

For state space $S \in \mathcal{S}$ and event $E \subseteq S(E) \preceq S$, define

$$A^i_S(E) = \{s \in S : S^i(s) \succeq S(E)\}$$

to be the event "investor i is aware of E", expressed in the language or vocabulary of state space S. Define

$$B^i_S(E) = \{s \in A^i_S(E) : t^i(s)(E^S) = 1\}$$

to be the event "investor i believes (with probability 1) event E", expressed in the language or vocabulary of state space S.

The mutual belief operator of order 1 given S and event $E \subseteq S(E) \preceq S$ is

$$B^1_S(E) = \bigcap_{i \in I} B^i_S(E).$$

It expresses the event "up to order 1, everyone believes event E", using the language of state space S. The event "investor i believes that, up to order 1, everyone believes event E", using the language of S is given by operator

$$B^{i,1}_S(E) = \{s \in S : s \in B^i_S(B^1_{S^i(s)}(E))\}.$$

Event $B^1_{S^i(s)}(E)$ specifies that there is mutual belief of order 1 (that is, everyone believes E), expressed in $S^i(s)$, which is the highest or most expressive state space that i is aware of at s. If $s \in B^i_S(B^1_{S^i(s)}(E))$, then i believes that there is mutual belief of order 1 at s.

Inductively, for every $n > 1$ we define

$$B^n_S(E) = B^{n-1}_S(E) \bigcap_{i \in I} B^{i,n-1}_S(E)$$

to be the event "up to order n, everyone believes event E", using the language of state space S. In other words, everyone believes E, everyone believes that everyone believes E, everyone believes that everyone believes that everyone believes E, up to order n, where

$$B^{i,n-1}_S(E) = \{s \in S : s \in B^i_S(B^{n-1}_{S^i(s)}(E))\}$$

is the event "investor i believes that, up to order $n-1$, everyone believes event E", using the language of S.

The common knowledge operator given S and event $E \subseteq S(E) \preceq S$ is

$$CB_S(E) = \bigcap_{n=1}^{\infty} B_S^n(E).$$

Definition 4. *Event $E \subseteq S(E) \preceq S$ is common knowledge at $s \in S$ if $s \in CB_S(E)$.*[15]

Although investors may have different awareness at a state s, there is a well defined notion of a "common" state space, which is the "highest" state space that everyone is aware of and this is common knowledge. Define

$$S^{\wedge}(s) = \bigvee \{S' \in \mathcal{S} : s \in CB_S(S')\}$$

to be the join of all state spaces that are common knowledge at $s \in S$. Galanis [2013] shows that $S^{\wedge}(s)$ is the most expressive state space that it is common knowledge at s that everyone is aware of.[16]

2.4 Common priors

In this section we define the notions of a positive, local and common prior. Let tuple $\pi = \{\pi_S\}_{S \in \mathcal{S}} \in \underset{S \in \mathcal{S}}{\times} \Delta S$ be a generalized prior if it is projective, so that for all $S'' \succeq S'$ the marginal of $\pi_{S''}$ on S' is $\pi_{S'}$. It is a local prior for i given S if π_S is a convex combination of i's beliefs in S, so that it generates i's posterior beliefs for types that some state in S describes they are aware of S. It is a prior given S if it is a local prior for all less expressive $S' \preceq S$. It is positive given S if it assigns positive probability to all events $P^i(s)$, where $s \in S$ specifies that i is aware of S.[17]

Definition 5. *Generalized prior $\pi = \{\pi_S\}_{S \in \mathcal{S}}$ is a local prior for $i \in I$ given $S \in \mathcal{S}$ if, for every event $E \subseteq S$, $E \cap A_S^i(E) \neq \emptyset$ implies*

$$\pi_S(E \cap A_S^i(E)) = \sum_{s \in A_S^i(E)} t^i(s)(E)\pi_S(\{s\}).$$

It is a positive local prior for i given S if, additionally, $s \in A_S^i(S)$ implies $\pi_S(P^i(s)) > 0$. It is a (positive) prior for i given S if it is a local (positive) prior given every $S' \preceq S$.

A common prior π given S generates the posterior beliefs of each investor, for each state s that belongs to S or to a less expressive state $S' \preceq S$.

[15] As explained in the previous section, we require a positive belief structure for all of our results, implying that every investor considers the true state (up to his awareness) to be possible. Because of this property, we define the common knowledge, rather than the common belief, operator.

[16] Proposition 1 in the Appendix shows that the interactive unawareness belief structure implies possibility correspondences with the properties assumed in Galanis [2013], so we can use his results.

[17] Aumann [1976] and Heifetz et al. [2013a] provide a similar restriction.

Definition 6. *Belief structure $\underline{S}$ has a (positive) common (local) prior given S if there exists π that is a (positive, local) prior given S for each $i \in I$.*

An nonempty event $E \subseteq S$ is public (or self evident) if $s \in E$ implies $P^i(s) \subseteq E$ for all $i \in I$. We say that E is a smallest public event if there does not exist public event $E' \neq E$ such that $E' \subseteq E$. The following property, Enlargements Preserve Common Priors (EPCP), specifies that if there exist common local priors that can generate the posterior beliefs within each smallest public event in common state space $S^\wedge(s)$, for any $s \in S$, then there exists a positive common prior given S.

Definition 7 (Enlargements Preserve Common Priors (EPCP) given $S \in \mathcal{S}$)**.** *If, for each $s \in S$ and smallest public event $E \subseteq S^\wedge(s)$ with $s_{S^\wedge(s)} \in E$, there exists a common local prior π^E given $S^\wedge(s)$ with supp $\pi^E_{S^\wedge(s)} = E$, then there exists a positive common prior given S.*

2.5 Awareness

In this section we group properties that describe different patterns of awareness.

Definition 8. *Awareness for i is path-independent given $S \in \mathcal{S}$ if, for all $s_1, s_2, s_3 \in S$, if $S^i(s_3) \succeq S^i(s_1)$ and $S^i(s_3) \succeq S^i(s_2)$ then $S^i(s_1) \succeq S^i(s_2)$ or $S^i(s_2) \succeq S^i(s_1)$.*

To interpret this property, consider the following example. Suppose that $S^i(s_3)$ denotes that investor i is aware of the theory of relativity at s_3, whereas $S^i(s_1)$ denotes being aware of basic math at s_1 and $S^i(s_2)$ denotes being aware of basic physics at s_2. Path-independence specifies that if all three states of awareness are possible for i and being aware of the theory of relativity implies being aware of basic math and basic physics, then either being aware of basic math implies being aware of basic physics or vice versa. In other words, it is not possible of being aware of basic math but not basic physics (or vice versa), yet there exists a higher state of awareness (relativity theory) that encompasses both math and physics.

For the next property, let

$$\mathcal{E}^i_S(s) = \{s_1 \in S : S^i(s) = S^i(s_1)\}$$

be the event in S describing that the investor has the same awareness as in $s \in S$. Operator $\mathcal{E}^i_S$ provides a partition of state space S, where each partition cell contains all states in S that describe the same awareness for i. Hence, we can interpret $\mathcal{E}^i_S$ as an awareness signal structure of i, given S, where signal k reveals the level of i's awareness. Note that if an investor is unaware of S, he is also unaware of $\mathcal{E}^i_S$ and therefore cannot comprehend this signal structure. We call $\mathcal{E}^i_S$ the awareness signal of i, given S.

The next property, Conditional Independence, specifies that beliefs, conditional on the information signal P^i, do not change when we also condition on the awareness signal $\mathcal{E}^i_S$.

Definition 9. *Investor i satisfies Conditional Independence given $S \in \mathcal{S}$ and prior π if, for any $s \in S$ with $\pi_S(s) > 0$, for any $E \subseteq S^i(s)$,*

$$t^i(s)(E) = \pi_S(E^S | \mathcal{E}^i_S(s) \cap P^i(s)^S).$$

Galanis [2016a] shows that if $\underline{\mathcal{S}}$ satisfies Projections Preserve Posteriors and π is a prior given S for i, then i satisfies Conditional Independence given S and π.

3 Results

Speculation is examined in four different settings. First, we characterize speculative betting with respect to no common priors and EPCP. Second, under Conditional Independence an always beneficial bet implies no common priors, whereas no common local priors and Projections Preserve Posteriors imply no such trade. Third, under either Conditional Independence or path-independence, there cannot be speculation in equilibrium. Finally, in a dynamic setting, there can be no speculative trade if, either the common and the payoff relevant state spaces coincide, or Projections Preserve Posteriors is assumed.

3.1 Speculative betting

We first characterize speculative betting in terms of no common priors and EPCP. A bet given S generates, at each state $s \in S$, gains or losses for each investor, that add up to zero.

Definition 10. *A bet $b = \{b^i\}_{i \in I}$ given $S \in \mathcal{S}$ is a collection of functions $b^i : S \to \mathbb{R}$, such that $\sum_{i \in I} b^i(s) = 0$ for each $s \in S$.*

Fix a bet b given S'. Investor i is aware of b at $s \in S$ if his state space, $S^i(s)$, is more expressive than S', so that $S^i(s) \succeq S'$. Investor i expects positive gains from bet b at $s \in S$ if he is aware of it and $\sum_{s' \in S^i(s)} t^i(s)(s') b^i(s'_{S'}) > 0$.

Let

$$B^b_S = \{s \in S : S^i(s) \succeq S' \text{ and } \sum_{s' \in S^i(s)} t^i(s)(s') b^i(s'_{S'}) > 0, \forall i \in I\}$$

be the event, expressed in S, specifying that all investors are aware of and expect positive gains from bet b given S'.

Recall that in the standard model without unawareness speculative betting characterizes no positive common priors. The following theorem shows that speculative betting characterizes no positive common priors *and* EPCP.[18]

Theorem 1. *Positive belief structure $\underline{\mathcal{S}}$ has no positive common prior and satisfies EPCP given S if and only if $B^b_{S^\wedge(s)}$ is common knowledge at some $s \in S$ and bet b given $S^\wedge(s)$.*

[18] Property EPCP is automatically satisfied in the standard environment without unawareness and a unique state space. See Galanis [2016b] for Theorem 1 in the case of no unawareness.

Heifetz et al. [2013a] show that speculative betting implies no positive common priors, but they assume Projections Preserve Posteriors.[19] This assumption is not needed here but has behavioral implications. For example, Galanis [2015, 2016a] shows that without Conditional Independence (and therefore without Projections Preserve Posteriors), the value of information can be negative (positive) in single-investor (multi-investor) environments with unawareness, which is the opposite of what is true in an environment without unawareness.

Moreover, Heifetz et al. [2013a] do not provide a characterization of speculative betting. In their Example 1, they show that no common prior *does not* imply the existence of speculative betting. This is not inconsistent with Theorem 1, because their example fails EPCP, hence Theorem 1 does not apply.

3.2 Betting that is always beneficial

What are the implications on beliefs if, instead of speculative betting, we assume that there is a bet that always provides all investors with positive expected gains? We call this an always beneficial bet. In the standard model where everyone is always fully aware, an always beneficial bet implies speculative betting, because what is always true is always common knowledge. Moreover, an always beneficial bet is equivalent to no common priors (e.g. Samet [1998]).

With unawareness, however, what is always true may not be common knowledge. As we illustrate in Example 1 below, the reason is that if an investor is not fully aware, he may fail to realize that something is always true, as he can only reason up to his awareness. Together with Example 2, we establish that in an environment with unawareness no common priors are neither sufficient nor necessary for the existence of a bet that makes all investors better off always.

We say that there is an always beneficial bet given S if there exists bet b given S_0 such that $B^b_S = S$. Note that we define a bet given the lowest state space S_0, rather than state space $S^\wedge(s)$ which is common knowledge at s, as in the previous section. The reason is that, as we do not invoke common knowledge, the "lowest" state space that everyone is aware of may be S_0. The results would be the same if, instead of S_0, we defined the bet on any other state space that all investors are always aware of.

Example 1. *We show that the existence of an always beneficial bet does not imply no common priors. The information structure is depicted in Figure 1. There are two investors and four state spaces, S_0, S_1, S_2 and S_3, where $S_3 \succeq S_1, S_2 \succeq S_0$. Each state space S_i has three states, s_i^1, s_i^2 and s_i^3. The projections are given by the thin arrows, so that for $k = 1,2,3$, s_3^k projects to s_1^k and to s_2^k, whereas both project to s_0^k. The common prior on S_3 is $(1/3, 1/3, 1/3)$.*

Investor 1's information structure specifies that $P^1(s_k^j) = S_k$, where $k, j = 1,2,3$. It is depicted in Figure 1 by the discontinuous lines, so that at each state he is completely uninformed. Investor 2's information structure, depicted by the solid lines,

[19] They refer to speculative betting as speculative trade in their Theorem 1. They also show that the converse is not true. As they only show one direction, they do not need to assume that the belief structure is positive.

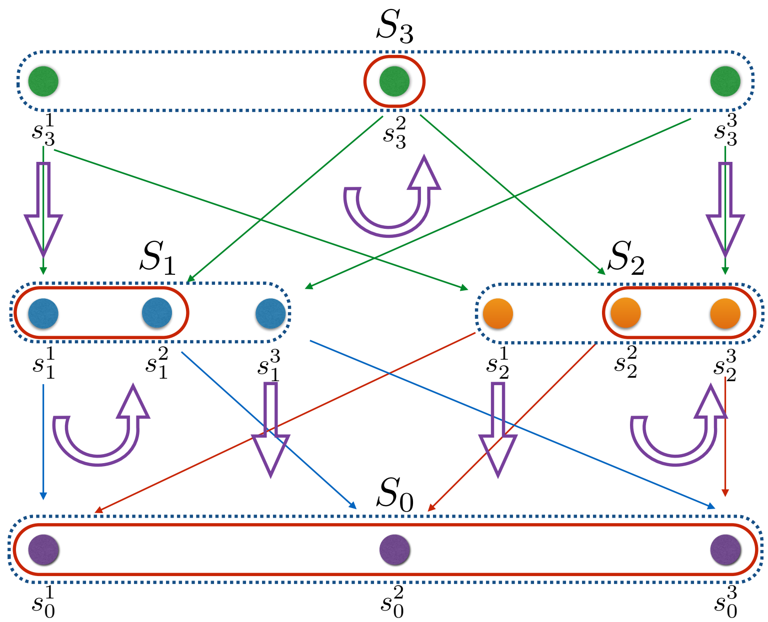

Figure 1: Investors have common priors but bet always

specifies that $P^2(s_3^1) = \{s_1^1, s_1^2\} = P^2(s_1^1) = P^2(s_1^2), P^2(s_3^2) = \{s_3^2\}, P^2(s_3^3) = \{s_2^2, s_2^3\} = P^2(s_2^2) = P^2(s_2^3), P^2(s_1^3) = P^2(s_2^1) = P^2(s_0^j) = S_0$*, where* $j = 1, 2, 3$*. The thick straight arrows specify that at a state* $s \in S$ *the investor 2's awareness is at a lower state space* $S' \prec S$*, whereas the curved arrows show that* $S' = S$*.*

Consider bet $b = \{b^i\}_{1,2}$ *given* S_0 *such that* $b^1(s_0^1) = 1, b^1(s_0^2) = -1.5, b^1(s_0^3) = 1$*, whereas* $b^2 = -b^1$*. It is an always beneficial bet given* S_3 *if for all states in* S_3*, each investor expects a strictly positive payoff from* b*.*

At each s_3^k*,* $k = 1, 2, 3$*, investor 1 is fully aware but completely uninformed. His posterior on* S_0 *is* $(1/3, 1/3, 1/3)$*. His expected payoff from* b^1 *is* $1/6$*. At* s_3^1*, investor 2 is aware of* S_1 *and he considers* $\{s_1^1, s_1^2\}$ *to be possible. His posterior on* S_0 *is* $(1/2, 1/2, 0)$*. His expected payoff from* b^2 *is* $1/4$*. At* s_3^2*, investor 2 is fully aware and knows that* s_3^2 *has occurred. His expected payoff is 1.5. At* s_3^3*, he is aware of* S_2 *and he considers* $\{s_2^2, s_2^3\}$ *to be possible. His posterior on* S_0 *is* $(0, 1/2, 1/2)$ *and his expected payoff is* $1/4$*. Hence, both investors always expect a strictly positive payoff from* b*. Theorem 2 shows that such a bet cannot exist, as long as all investors satisfy Conditional Independence.*

Although there exists a bet that **always** *makes everyone strictly better off, such a fact cannot be common knowledge. The only event that can be common knowledge is the least expressive state space,* S_0*. To see this, note that at* s_3^1*, investor 2 is aware of* S_1 *and thinks that investor 1 is aware of* S_1 *and his information is* S_1*. Because investor 1 is completely uninformed at all states in* S_1*, he does not know whether investor 2 is aware of* S_1*. In fact, he only knows that 2 is aware of* S_0*. Investor 2 also knows that investor 1 is aware of (and knows)* S_0*.*

The following theorem shows that an always beneficial bet implies that there cannot be a common prior that satisfies Conditional Independence for each investor. Moreover, the converse does not hold. As we show in Example 2, even if Conditional Independence and EPCP are satisfied, no common priors do not imply an always beneficial bet. However, if we strengthen Conditional Independence to Projections Preserve Posteriors and no common priors to no common local priors given S_0, then all state spaces describe that there is an always beneficial bet. In other words, if the investors' posteriors in the payoff relevant state space S_0 are not common, then all state spaces describe an always beneficial bet.

Theorem 2. *Suppose there is a state space S and a bet b given S_0 such that $B^b_S = S$. Then, positive belief structure $\underline{S}$ has no common prior π given S such that each $i \in I$ satisfies Conditional Independence given S and π. Conversely, if positive belief structure $\underline{S}$ satisfies Projections Preserve Posteriors, then no common local prior given S_0 implies that there is a bet b given S_0 such that $B^b_S = S$, for all $S \in \mathcal{S}$.*

Example 2. *We show that the non existence of an always beneficial bet does not imply common priors, even if Conditional Independence and EPCP are satisfied. The information structure is depicted in Figure 2. There are two investors who have no common local priors given S_0 and therefore no common prior, but there is no bet that ensures positive expected gains at each full state, for both. However, there is speculative betting.*

There are two state spaces, $S = \{s_1, s_2, s_3, s_4\}$ and $S_0 = \{s_5, s_6\}$, such that $S_0 \preceq S$, $s_{1S_0} = s_{2S_0} = s_5$ and $s_{3S_0} = s_{4S_0} = s_6$. Investor 1 is always fully aware and $P^1(s) = S$ for all $s \in S$. For all $s_0 \in S_0$, $P^1(s_0) = S_0$. His information structure is depicted by the discontinuous lines. His prior π^1 on S is $(1/8, 1/2, 2/8, 1/8)$. In fact, this is the only prior that can generate his posteriors. Investor 2's possibility correspondence is as follows: $P^2(s_1) = P^2(s_4) = S_0$, $P^2(s_2) = \{s_2\}$, $P^2(s_3) = \{s_3\}$. For all $s_0 \in S_0$, $P^2(s_0) = S_0$. His information structure is depicted by the solid lines. His prior assigns $1/4$ to each $s \in S$. Since π^1 cannot generate 2's posteriors, the investors have no common priors. Moreover, the investors' priors satisfy Conditional Independence. The thick straight arrows specify that at a state $s \in S$ the investor 2's awareness is at a lower state space $S' \prec S$, whereas the curved arrows show that $S' = S$.

Suppose there is a bet $b = \{b^i\}_{1,2}$ given S_0 such that $\sum\limits_{s' \in S^i(s)} t^i(s)(s')b^i(s'_{S_0}) > 0$, for each $s \in S$, $i = 1, 2$. For investor 2 this means that $b^2(s_5), b^2(s_6) > 0$, because at s_2, s_3 he knows exactly which state in S is true. But since $\sum\limits_{i \in I} b^i(s) = 0$ for each $s \in S_0$, we have $b^1(s_5), b^1(s_6) < 0$, which implies $\sum\limits_{s' \in S^1(s)} t^1(s)(s')b^1(s'_{S_0}) < 0$. Hence, there is no bet b given S_0 such that $B^b_S = S$.

Note that the only public event is S_0 and that there is no common local prior given $S^\wedge(s_1) = S_0$, because 1's unique prior on S_0 is $(5/8, 3/8)$, whereas 2's unique prior is $(1/2, 1/2)$. If we set $b^1(s_5) = 1$, $b^1(s_6) = -1.1$ and $b^2 = -b^1$, then $B^b_{S^\wedge(s_1)}$ is common knowledge at s_1.

It is important to note that, in the preceding examples and throughout the paper, there is the implicit assumption that every investor is unaware that he might be unaware of relevant dimensions of the problem he is facing. This is restrictive, but consistent with most of the literature on unawareness. Relaxing this assumption has

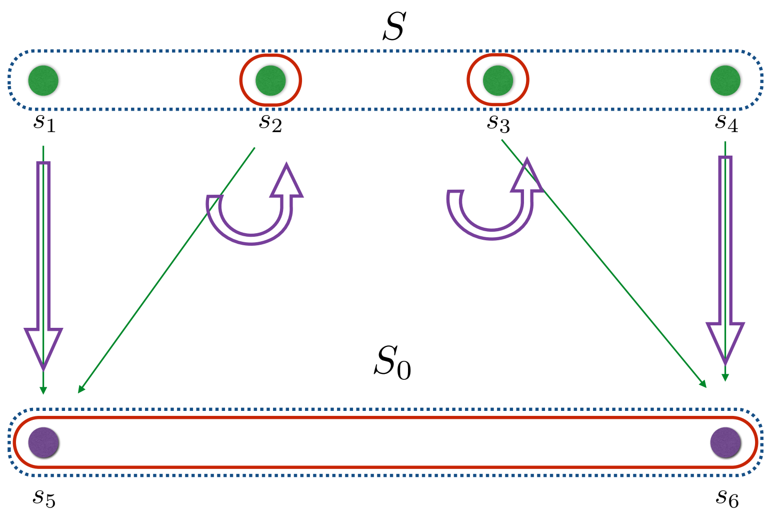

Figure 2: Investors have no common priors but do not bet always

consequences not only on the properties of interactive unawareness, but also on the preferences that an investor might have. For example, the case for having preferences represented by expected utility is less compelling when the investor knows that he is missing some relevant dimensions that he cannot conceptualise. Although there are some papers on interactive unawareness which formally model investors being aware that they are unaware of "something" (e.g. Halpern and Rêgo [2009], Board and Chung [2011a,b]) and there is also a decision theoretic literature on coarse contingencies (e.g. Epstein et al. [2007]), there is still no model that combines both approaches.[20]

3.3 Speculation in equilibrium

We now consider speculation in a game, adapting the framework of Geanakoplos [1989]. With unawareness, investors might speculate in equilibrium because they have an incomplete understanding of other investors' information and therefore of their actions. As a result, they might be playing best response against a fictitious opponent that, in reality, does not exist. However, we show that under Conditional Independence or path-independence, if investors have a correct understanding of their payoffs at each strategy profile, speculation cannot occur.

Definition 11. *A Bayesian game with unawareness is a tuple* $(\underline{S}, C, \pi, (u^i)_{i\in I})$*, where* $\underline{S}$ *is a positive belief structure,* I *is the finite set of investors,* $C = \underset{i\in I}{\times} C^i$ *is the set of all action profiles,* π *is the common prior given* S^* *and* $u^i : C \times \Sigma \to \mathbb{R}$ *denotes* i*'s payoff function.*

[20] See also the discussion in Section 1.1.

We require that investor i's strategy is measurable with respect to his type.

Definition 12. *A strategy for investor $i \in I$ is a function $f^i : \Sigma \to C^i$ such that for all $s, s' \in \Sigma$, $t^i(s) = t^i(s')$ implies $f^i(s) = f^i(s')$.*

A strategy profile is a tuple $f = \{f^i\}_{i \in I}$, where $f(s) = \times_{i \in I} f^i(s)$ and $f^{-i}(s) = \times_{j \in I \setminus i} f^j(s)$ for each $s \in \Sigma$. When choosing a best response at state $s \in S$ and given f^{-i}, investor i who is unaware of S may have a wrong perception about what others will play, because he is only aware of the less expressive state $s_{S'}$, where $S' \prec S$. If he chooses action c^i and thinks he will get payoff $u^i(c^i, f^{-i}(s_{S'}), s_{S'})$ at $s_{S'}$ but receives a different payoff $u^i(c^i, f^{-i}(s), s)$, then he may be surprised, realising that he is unaware of something. Such a surprise cannot be part of an equilibrium, as the investor understands that he may have played suboptimally. We therefore require in equilibrium that $u^i(c^i, f^{-i}(s_{S'}), s_{S'}) = u^i(c^i, f^{-i}(s), s)$. In other words, awareness does not lead to a wrong perception of one's payoffs in equilibrium.

Definition 13 (Projections Preserve Own Payoffs)**.** *A strategy profile $f = \{f^i\}_{i \in I}$ satisfies PPOP if for all $i \in I$, for all $s \in \Sigma$, $u^i(c^i, f^{-i}(s), s) = u^i(c^i, f^{-i}(s_{S_0}), s_{S_0})$, for all $c^i \in C^i$.*

We emphasise that PPOP does not require that an unaware investor can reason correctly about what his opponents are really playing, as $f^{-i}(s) \neq f^{-i}(s_{S_0})$ is allowed. It only requires that if his opponents play $f^{-i}(s)$ at $s \in S$, then his payoff from playing c^i is the same, irrespective of whether his state space is S and reasons that his opponents play $f^{-i}(s)$ at s, or if his state space is $S' \prec S$ and reasons that they play $f^{-i}(s_{S'})$ at $s_{S'}$.[21] Hence, an investor may have a wrong theory of how others are playing, however in equilibrium this does not influence his payoffs. This means that the Bayesian Nash equilibrium which we define below cannot be interpreted as a "self-confirming" equilibrium or as a result of a learning process.[22] A Bayesian Nash equilibrium with unawareness is required to satisfy PPOP.

Definition 14. *Strategy profile $f = \{f^i\}_{i \in I}$ constitutes a Bayesian Nash equilibrium with unawareness if it satisfies PPOP and for all $s \in \Sigma$, all $i \in I$ and all $c^i \in C^i$,*

$$\sum_{s' \in S^i(s)} u^i(f^i(s'), f^{-i}(s'), s') t^i(s)(s') \geq \sum_{s' \in S^i(s)} u^i(c^i, f^{-i}(s'), s') t^i(s)(s').$$

Investor i's ex ante expected utility from strategy profile $f = \{f^j\}_{j \in I}$ is

$$U^i(f) = \sum_{s^* \in S^*} \pi_{S^*}(s^*) u^i(f(s^*), s^*).$$

Note that if investor i's state space in the ex ante stage is $S \prec S^*$, his view of his ex ante expected utility is $U^i_S(f) = \sum_{s \in S} \pi_S(s) u^i(f(s), s)$. Because of PPOP, $U^i_S(f) = U^i(f)$

[21] Note that PPOP is automatically satisfied in Geanakoplos [1989], because nondelusion is assumed.

[22] Similar issues arise and are discussed in Feinberg [2012], Rêgo and Halpern [2012], Grant and Quiggin [2013], Halpern and Rêgo [2014] and Meier and Schipper [2014a]. I thank a referee for pointing this out.

if f is an equilibrium. Hence, we can interpret $U^i_S(f)$ as what the investor thinks ex post his ex ante expected utility had been.

Suppose that in the interim stage, when the investors have formulated their awareness and beliefs, they realise that they had been at an ex ante Pareto efficient allocation and have now an action to unilaterally stick to it. This means that each investor can stick to his allocation and can guarantee for himself an ex ante payoff of $\bar{u}^i$, irrespective of what everyone else is doing, by picking an action z^i. Let $z = \underset{i \in I}{\times} z^i$ be the resulting action profile and $[z^i]$ the strategy that plays z^i always. Moreover, there is no strategy profile that ex ante can make everyone weakly better off and at least one strictly better off. That is, if $U^i(f) \geq \bar{u}^i$ for all $i \in I$, then $f(s^*) = z$ for all $s^* \in S^*$. Will they be willing to speculate in equilibrium or stick to the Pareto optimal allocation? As the following theorem shows, they will not speculate, as long as either P^i is path-independent or i satisfies Conditional Independence, for each $i \in I$.

Theorem 3. *Consider Bayesian game with unawareness $(\underline{\mathcal{S}}, C, \pi, (u^i)_{i \in I})$ such that either P^i is path-independent or i satisfies Conditional Independence given π and S^*, for each $i \in I$. Moreover, suppose there exists an action profile $z = \underset{i \in I}{\times} z^i$ such that for all $f = \{f^i\}_{i \in I}$, $U^i([z^i], f^{-i}) = \bar{u}^i$ and if $U^i(f) \geq \bar{u}^i$ for all $i \in I$, then $f(s^*) = z$ for all $s^* \in S^*$. Then, there is a unique Bayesian Nash equilibrium with unawareness, and it has the property that $f(s^*) = z$ for all $s^* \in S^*$.*

3.4 Speculative trade

Milgrom and Stokey [1982] showed that if an allocation is ex ante Pareto efficient it cannot be common knowledge in the interim stage that there is a mutually beneficial trade. In this section we examine under which conditions this no speculative trade result holds in an environment with unawareness. Contrary to previous sections, the current setting is dynamic, not static. More importantly, there is no common prior assumption.[23]

There are two periods, the ex ante period 1 in which there is symmetric information and awareness and the interim period 2 where there is differential awareness and information.[24] In period 1, types are described by positive belief structure $\underline{\mathcal{S}}_1$, where all investors are only aware of a payoff relevant state space S^1_0 and have no information. That is, $\mathcal{S}_1 = \{S^1_0\}$ and the belief structure $\underline{\mathcal{S}}_1$ specifies a unique type $k^i(\cdot)$, so that $k^i(s) = k^i(s') \in \Delta S^1_0$ for all $s, s' \in S^1_0$. Moreover, $\operatorname{supp} k^i(\cdot) = S^1_0$. We set

[23]Note that an initial allocation is Pareto efficient if and only if there is a common subjective belief, as shown by Rigotti et al. [2008] in the context of convex preferences. In their terminology, a subjective belief is different from a prior, as it is the normal (normalized to be a probability) of the supporting hyperplane of the indifference curve at the initial allocation. If the allocation is not a full insurance one, a subjective belief is not equal to the agent's prior in the expected utility model, as it depends also on the derivatives of the Bernoulli utilities. The result of Rigotti et al. [2008] is a generalization of the results of Billot et al. [2000, 2002], who characterize ex ante trading from a full insurance allocation, in terms of disjoint sets of priors, using the Maxmin and Choquet expected utility models, which contain expected utility as a special case.

[24]There is also an unmodelled period 3 where all payoff relevant information is revealed and trades are executed.

$k^i(\cdot) = \pi_1^i \in \Delta S_0^1$ and note that, by construction, π_1^i is a positive local prior for i given S_0^1.

There is a single consumption good, which we interpret as money, and investors trade state contingent claims. Each investor $i \in I$ is equipped with a strictly increasing utility function $u^i : \mathbb{R}_+ \to \mathbb{R}_+$ over monetary outcomes. His endowment in period 1 is a state contingent claim $f^i : S_0^1 \to \mathbb{R}_+$, where S_0^1 is the payoff relevant, or bottom, state space. The aggregate endowment of the economy in period 1 is given by $e : S_0^1 \to \mathbb{R}_{++}$. An allocation is a tuple $f = (f^1, \ldots, f^m)$, where $f^i : S_0^1 \to \mathbb{R}_+$. It is feasible if $\sum_{i=1}^m f^i \leq e$.

In period 2, types are described by positive belief structure $\underline{\mathcal{S}}_2$ and investors receive differential information and awareness. Collection $\mathcal{S}_2$ contains multiple state spaces and belief structure $\underline{\mathcal{S}}_2$ allows for multiple types, where S_0^2 is the bottom, or payoff relevant state space. We assume that the payoff relevant state spaces in periods 1 and 2 are identical, $S_0^1 = S_0^2$, so that there is no ambiguity on how state contingent claims are defined across periods. We also assume that, for each $i \in I$, there exists a positive prior π^i for i given $S \in \mathcal{S}$ such that its marginal on S_0^2 is equal to π_1^i, i's local prior given S_0^1 in period 1.

An economy is a tuple $\mathcal{C} = \langle \underline{\mathcal{S}}_1, \underline{\mathcal{S}}_2, \{u^i\}_{i \in I}, \Theta, e \rangle$ where $\underline{\mathcal{S}}_1$, $\underline{\mathcal{S}}_2$ are positive belief structures. Ex ante Pareto efficiency is defined in the standard way.

Definition 15. *Feasible allocation f of economy $\mathcal{C}$ is ex ante Pareto efficient if there does not exist another feasible allocation g such that* $\sum\limits_{s \in S_0^1} \pi_1^i(s) u^i(g^i(s_{S_0^1})) \geq \sum\limits_{s \in S_0^1} \pi_1^i(s) u^i(f^i(s_{S_0^1}))$ *for all $i \in I$, with strict inequality for some j.*

We say that in the interim period 2 there is speculative trade in economy $\mathcal{C}$ at $s \in S \in \mathcal{S}_2$, if an allocation is ex ante Pareto efficient but it is common knowledge at s that there exists a Pareto improvement. Let

$$H_{S,j}^{g,f} = \left\{ s \in S \in \mathcal{S}_2 : \sum_{s' \in S^i(s)} t^i(s)(s') u^i(g^i(s'_{S_0^2}) \geq \sum_{s' \in S^i(s)} t^i(s)(s') u^i(f^i(s'_{S_0^2}) \ \forall i \in I, > \text{for } j \right\}$$

be the event in S which specifies that each i weakly prefers g^i over f^i and j strictly prefers g^j over f^j.

Definition 16. *There is speculative trade in economy $\mathcal{C}$ at $s \in S \in \mathcal{S}_2$ if there is an ex ante Pareto efficient allocation f, an investor $j \in I$ and feasible allocation g such that event $H_{S^\wedge(s),j}^{g,f}$ is common knowledge at s.*

Note that $H_{S^\wedge(s),j}^{g,f}$ is a subset of state space $S^\wedge(s)$, which is defined in Section 2.3 to be the common state space at s, the most expressive state space that it is common knowledge at s that everyone is aware of.

Theorem 4. *There is no speculative trade in economy $\mathcal{C} = \langle \underline{\mathcal{S}}_1, \underline{\mathcal{S}}_2, \{u^i\}_{i \in I}, \Theta, e \rangle$ at any $s \in S \in \mathcal{S}_2$ if either $S^\wedge(s) = S_0^2$ or $\underline{\mathcal{S}}_2$ satisfies Projections Preserve Posteriors.*

The theorem specifies that there can be no speculative trade at s if either of the following two conditions is met. First, the common state space at s is equal to the payoff relevant state space S_0^2. In other words, even though the investors receive differential awareness and information in period 2, it cannot be common knowledge that everyone has more awareness than in period 1. Second, Projections Preserve Posteriors is assumed.[25]

To provide some intuition on why either of these conditions is needed, consider the following sketch of the proof of the theorem. Suppose that allocation f is Pareto optimal in period 1 but in period 2 it is common knowledge at state s that allocation g Pareto dominates f, so that everyone weakly prefers g over f and at least one investor strictly prefers it. A property of common knowledge, which is also true with unawareness, implies that there is a public event $E \subseteq S^{\wedge}(s)$, where g Pareto dominates f at all $s' \in E$ and, for all i, P^i provides a partition of E. The law of iterated expectations implies that g Pareto dominates f within E also if we replace posterior with prior beliefs.

The last step requires defining a new allocation h that is identical to g within E and identical to f outside of E. But then, allocation h Pareto dominates f in period 1, which is a contradiction. However, this last step may be impossible in an environment with unawareness if event $E \subseteq S^{\wedge}(s)$ is not the enlargement of some event of payoff relevant state space S_0^2, where allocations are defined. If that is the case, we cannot find an allocation h that separates, in terms of payoffs, E from its complement.[26] If $S^{\wedge}(s) = S_0^2$, the payoff relevant state space coincides with the state space where common knowledge is expressed, so separation is possible. Alternatively, Projections Preserve Posteriors implies that the posterior about an event in S_0^2, specified by $s' \in S^{\wedge}(s)$, is identical to the posterior specified by the projection of s' to S_0^2, so again separation is feasible.

The following speculative trade example shows how separation fails if $S^{\wedge}(s) \neq S_0^2$, Conditional Independence (and therefore Projections Preserve Posteriors) is not assumed and the public event E is not an enlargement of an event in S_0^2.

Example 3. *In period 1, $\mathcal{S}_1$ contains state space $S_0^1 = \{s_1, s_2\}$. In period 2, $\mathcal{S}_2$ contains state spaces S_0^2 and $S = \{s_3, s_4, s_5\}$, where S_0^2 is identical to S_0^1 and we denote it in Figure 3 as S_0. Moreover, $S \succeq S_0^2$ and s_3 projects to s_1, s_4 and s_5 project to s_2, where the projections are denoted by the thin arrows. There are two investors, i and j, with identical utility over monetary outcomes $u(x) = log(x)$. Information and awareness are symmetric. In particular, $P(s_1) = P(s_2) = P(s_5) = S_0$, $P(s_3) = P(s_4) = \{s_3, s_4\}$.*

The aggregate endowment is 1 in s_1 and 1 in s_2. Investor i's prior over S is $(1/2, 1/8, 3/8)$ and j's prior is $(1/2, 1/4, 1/4)$. Note that the marginals of both priors on S_0 are $(1/2, 1/2)$. In period 1, where both investors are only aware of S_0, they have the same beliefs $(1/2, 1/2)$. Because they are risk averse, the full insurance allocation f, giving each investor 0.5 at s_1 and s_2, is Pareto optimal. In period 2 and at state s_3, both investors increase their awareness and consider s_3 and s_4 to be possible. Investor i's posterior beliefs are $(4/5, 1/5)$ j's are $(2/3, 1/3)$. Because beliefs are different, the full

[25] This means that whereas Heifetz et al. [2013a] do not allow for speculative trade (as they assume Projections Preserve Posteriors), the current paper does.

[26] In settings without unawareness but with non expected utility preferences, some separation property is assumed, usually in the form of Dynamic Consistency, as in Ma [2001] and Halevy [2004], or a suitable weakening, as in Galanis [2016c].

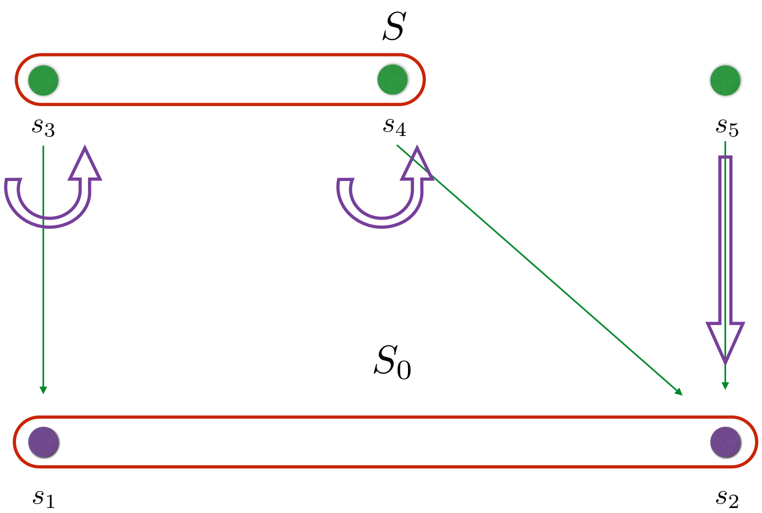

Figure 3: Speculative trade

insurance allocation is no longer Pareto optimal. For example, consider the allocation g *which specifies* $\{0.56, 1/3\}$ *for* i *and* $\{0.44, 2/3\}$ *for* j*. Then, it is common knowledge at* s_3 *that* g *Pareto dominates* f *and there is speculative trade.*

3.5 Speculation with different priors or information processing errors

In this section we briefly examine the behavioral implications in the four trade settings, of models with different priors or information processing errors with non-partitional structures.

Several of the overconfidence models which generate trade assume some form of different priors. For example, Scheinkman and Xiong [2003] model two groups of investors and two signals that are publicly available. Each group is overconfident about one signal, regarding the other signal (and the fact that the other group is overconfident about it) as noise. This can be modelled as having different priors over the signal structure. In the first three settings that we examined, common priors are necessary for a no trade result, hence models similar to Scheinkman and Xiong [2003] with different priors do not preclude speculation. However, Scheinkman and Xiong [2003] preclude the speculative trade of Section 3.4, which means that as long as investors correctly apply Bayes' rule to their subjective prior, there cannot be common knowledge in the interim that there is a Pareto improvement of an ex ante Pareto optimal allocation.

The single state space, non-partitional model of Geanakoplos [1989] is closely related to the present and other multiple state spaces models, in the following sense. If S^*

is the top or full state space, we can define a possibility correspondence $P^*(s^*) = P(s^*)^{S^*}$ for all $s^* \in S^*$. That is, we enlarge to S^* the event $P(s^*)$ that the investor considers possible at s^*, thus generating a non-partitional information structure P^* on S^*. However, the two models differ in that the definition of common knowledge in the present model is not the same as the definition of common knowledge in Geanakoplos [1989], because the former incorporates information about the investors' awareness, which the latter ignores.

The following example examines the behavioral implication of Geanakoplos [1989] in three of the four trade settings.

Example 4. *Let $S = \{s_1, s_2, s_3\}$ be the unique state space and $(1/4, 1/2, 1/4)$ be the common prior. There are two investors, i and j. Investor i has the trivial partition $P^i(s) = S$ for all $s \in S$. Investor j has a non-partitional information structure, so that $P^j(s_1) = P^j(s_2) = \{s_1, s_2\}$ and $P^j(s_3) = \{s_2, s_3\}$. Investor i's posterior beliefs are always $(1/4, 1/2, 1/4)$, whereas j's are $(1/3, 2/3, 0)$ at s_1 and s_2 and $(0, 2/3, 1/3)$ at s_3.*

Consider the trade $b^i(s_1) = b^i(s_3) = 1/4$, $b^i(s_2) = -3/16$, $b^j = -b^1$. At each state both investors expect positive gains, so that $B^b_S = S$, which implies that B^b_S is common knowledge at all states. Because both investors expect positive gains at all states, this is also an always beneficial bet.

Consider now a two-period model, like the one of Section 3.4, where in period 1 both investors have no information and consider all states in S to be possible. Suppose that both investors are risk averse with $u(x) = log x$. The economy has a full insurance allocation $f = \{1, 1\}$, giving 1 to each investor at all states. Then, the common prior $(1/4, 1/2, 1/4)$ and risk aversion imply that this allocation is ex ante Pareto optimal, yielding a utility of 0 for both. In period 2, i's information is given by P^i and j's is given by P^j, with the same posterior beliefs as described above. Then, the allocation $g = \{b^i + 1, b^j + 1\}$ Pareto dominates f at all states, hence it is always common knowledge that f is not Pareto optimal.

This example shows that the model of Geanakoplos [1989] allows for speculative betting (Section 3.1) and for betting that is always beneficial (Section 3.2), with common priors. In the present model, speculative betting cannot occur with common priors. Moreover, an always beneficial bet is not consistent with a common prior that satisfies Conditional Independence. There is no natural analogue of Conditional Independence in a single state space model, because the information about the investor's awareness is suppressed.

The example also shows that speculative trade can occur, even with common priors. In Section 3.4 we show that speculative trade cannot occur if the payoff relevant and the common state space coincide, a condition which is trivially satisfied in the single state space model of Geanakoplos [1989].

Speculation in equilibrium (Section 3.3) is allowed in both models, unless some related conditions are imposed. Geanakoplos [1989] imposes three conditions for no speculation in equilibrium, namely non-delusion, Knowing that You Know (KTYK) and nested. As shown in Galanis [2015], the first two are satisfied by operator P^*, whereas nested is implied by path-independence.

It is interesting to note that models with different priors are related to models with non-partitional information structures. As Brandenburger et al. [1992] show, any correlated equilibrium with different priors can be generated as a correlated equilibrium with common priors but non-partitional information structures, and vice versa. The authors point out that equilibria with different priors can be generated with common priors and *significant* informational processing errors. This means that a model with different priors, which does not provide foundations for the investors' errors or limited perception, may nevertheless have implications about them which are not clear.

4 Concluding remarks

This paper examines speculation in four different trading environments, relaxing the assumption that all investors are always fully aware of all relevant dimensions. We find that what matters for no speculation is whether some form of common knowledge is assumed, which compensates for the lack of awareness.

We also compare our approach with models that generate speculation through different priors (e.g. models with overconfident investors) or information processing errors (e.g. Geanakoplos [1989]). We argue that the main advantage of the current approach is that it endogenizes speculative behavior by explicitly modelling their awareness, without altering the common prior assumption, thus providing an insight into why certain types of speculation occur whereas other do not. On the other hand, the advantage of the overconfidence literature is that it generates several well known empirical findings about speculation, such as large trading volume or value-growth effects (e.g. Daniel et al. [2001]). For future research, it would be interesting to examine whether insights from the unawareness literature could be incorporated in those models so as to provide sharper explanations of these findings.

A Appendix

For an event $E \subseteq S$, denote by $E^{\uparrow} = \bigcup_{S' \in g(S)} E^{S'}$ the enlargements of E to all state spaces which are at least as expressive as S. The following Proposition collects properties that P^i satisfies. Note that properties 0-4 are assumed in Galanis [2013].

Proposition 1. *Consider an interactive unawareness belief structure* $\underline{S} = \langle \mathcal{S}, \{r^{S_\alpha}_{S_\beta}\}_{S_\beta \preceq S_\alpha}, \{t^i\}_{i \in I} \rangle$. *The possibility correspondence P^i satisfies the following properties.*

(0) Confinedness: If $s \in S$ then $P^i(s) \subseteq S'$ for some $S' \preceq S$.

(1) Generalized Reflexivity: $s \in (P^i(s))^{\uparrow}$ for every $s \in \Sigma$.

(2) Stationarity: $s' \in P^i(s)$ implies $P^i(s') = P^i(s)$.

(3) Projections Preserve Ignorance: If $s \in S'$ and $S \preceq S'$ then $(P^i(s))^{\uparrow} \subseteq (P^i(s_S))^{\uparrow}$.

(4) Projections Preserve Awareness: If $s \in S'$, $s \in P^i(s)$ and $S \preceq S'$ then $s_S \in P^i(s_S)$.

(5) For all $s \in \Sigma$, supp $t^i(s) \subseteq P^i(s)$. If $\underline{S}$ is a positive belief structure, then supp $t^i(s) = P^i(s)$.

(6) If π is a positive local prior given S for i then supp $t^i(s) =$ supp $\pi_S \cap P^i(s)$ for all $s \in A^i_S(S)$.

Proof. (0) If $s \in S$ then Property 0 of Definition 1 implies $t^i(s) \in \Delta S'$ for some $S' \preceq S$. By the definition of P^i, we have $P^i(s) \subseteq S'$.

(1) Suppose $s \in S''$. From Property 0 of Definition 1, $t^i(s) \in \Delta S$ for some $S \preceq S''$. Property 1 of Definition 1 implies $t^i(s_S) = t^i(s)$. The definition of P^i implies $s_S \in P^i(s)$, hence $s \in (P^i(s))^{\uparrow}$.

(2) Suppose $s' \in P^i(s)$. From the definition of P^i we have $t^i(s') = t^i(s)$, hence $P^i(s') = P^i(s)$.

(3) Suppose $s \in S'$ and $S \preceq S'$. We need to show that $(P^i(s))_{S^i(s_S)} \subseteq P^i(s_S)$. From Generalized Reflexivity and Stationarity we have $s_{S^i(s)} \in P^i(s_S)$ and $P^i(s_S) = P^i(s_{S^i(s)})$. Suppose $s' \in (P^i(s))_{S^i(s_S)}$. Then, there exists $s'' \in P^i(s)$ such that $s''_{S^i(s_S)} = s'$. From the definition of P^i, $t^i(s) = t^i(s'')$. Property 2 of Definition 1 implies $t^i(s_{S^i(s_S)}) = t^i(s''_{S^i(s_S)})$. The definition of P^i and $s''_{S^i(s_S)} = s'$ imply $s' \in P^i(s_{S^i(s)}) = P^i(s_S)$.

(4) Suppose $s \in S'$, $s \in P^i(s)$ and $S \preceq S'$. Then, $t^i(s) \in \Delta S'$. Property 4 of Definition 1 implies $S_{t^i(s_S)} = S$. Together with Generalized Reflexivity we have $s_S \in P^i(s_S)$.

(5) Suppose $s' \in$ supp $t^i(s)$ but $s' \notin P^i(s)$. Then, $t^i(s)(P^i(s)) < 1$, contradicting Assumption 1. Suppose $\underline{S}$ is a positive belief structure. If $s' \in P^i(s)$ then $t^i(s)(s') > 0$ and $s' \in$ supp $t^i(s)$.

(6) Suppose π is a positive local prior for i given S, so that supp $\pi_S \cap P^i(s) \neq \emptyset$. Take $s \in A^i_S(S)$. Let $s' \in$ supp $\pi_S \cap P^i(s)$, so that $\pi_S(s') > 0$. By definition, $s' \in P^i(s)$ implies $S^i(s') = S^i(s) = S$. From Stationarity we have that $t^i(s'')(s') = 0$ for all $s'' \notin P^i(s)$. If $t^i(s'')(s') = 0$ for some $s'' \in P^i(s)$ then $t^i(s'')(s') = 0$ for all $s'' \in P^i(s)$. From the definition of a local prior, this implies that $\pi_S(s') = 0$, a contradiction. Hence, $t^i(s'')(s') > 0$ for some (and therefore all) $s'' \in P^i(s)$. Conversely, suppose $s' \in$ supp $t^i(s)$, so that $t^i(s)(s') > 0$. We have shown that this implies $s' \in P^i(s)$, so that $t^i(s'')(s') > 0$ for all $s'' \in P^i(s)$. Because π is a positive local prior given S for i we have $\pi_S(P^i(s)) > 0$, so setting $E = s'$ we have $\pi_S(s') > 0$ and $s' \in$ supp $\pi_S \cap P^i(s)$. ∎

Let $E, E' \subseteq \Sigma$ be two sets of states and define $P^i(E') = \bigcup_{s' \in E'} P^i(s')$ to be the set of states that i considers possible if the truth lies in E'. Set $P^i(E')$ is not necessarily an event, as it may contain states of different state spaces. For example, $P^k(P^j(P^i(s)))$ contains all states that i considers possible that j considers possible that k considers possible, when s occurs. To simplify the notation, we write $P^k P^j P^i(s)$ instead of $P^k(P^j(P^i(s)))$. Galanis [2013] uses this notation to provide a definition of common knowledge (referred to as common knowledge) in his Proposition 2. Below we show that this is equivalent to the definition of common knowledge provided in the present paper. We then use the results of Galanis [2013] to show that whenever an event is common knowledge it contains a public event which is partitioned by P^i, for all $i \in I$.

Proposition 2. *Given a positive belief structure $\underline{S}$, the following are equivalent:*

- *Event E is common knowledge at $s \in S$,*
- *For any $n \in \mathbb{N}$ and any sequence of investors $i_1, \ldots, i_n$, $P^{i_n} \ldots P^{i_1}(s) \succeq S(E)$ and $(P^{i_n} \ldots P^{i_1}(s))_{S(E)} \subseteq E$.*

Suppose $B^b_{S^\wedge(s)}$ is common knowledge at $s \in S$. Then,

- *there is a nonempty public event $E' \subseteq S^\wedge(s)$ such that $s_{S^\wedge(s)} \subseteq E' \subseteq B^b_{S^\wedge(s)}$,*
- *E' is partitioned by P^i, for all $i \in I$.*

Proof. Since $\underline{S}$ is positive and from (5) of Proposition 1, supp $t^i(s) = P^i(s)$ implies that $P^i(s) \succeq S(E)$ and $(P^i(s))_{S(E)} \subseteq E$ for all $i \in I$ if and only if $s \in B^1_S(E) = \bigcap_{i \in I} B^i_S(E)$. We proceed with induction on $n \geq 2$.

- For $n = 2$, we show that $s \in B^2_S(E) = B^1_S(E) \bigcap_{i \in I} B^{i,1}_S(E)$ is equivalent $P^{i_2}(P^{i_1}(s)) \succeq S(E)$ and $(P^{i_2}(P^{i_1}(s)))_{S(E)} \subseteq E$ to all sequences of investors i_1, i_2 with up to 2 elements.
  By definition, $s \in B^2_S(E) \subseteq B^{i_1,1}_S(E)$ implies $s \in B^{i_1}_S(B^1_{S^{i_1}(s)}(E))$, or that $P^{i_1}(s) \subseteq B^1_{S^{i_1}(s)}(E) = \bigcap_{i \in I} B^i_{S^{i_1}(s)}(E) \subseteq B^{i_2}_{S^{i_1}(s)}(E)$. This implies that for all $s' \in P^{i_1}(s)$, $P^{i_2}(s') \succeq S(E)$ and $(P^{i_2}(s'))_{S(E)} \subseteq E$. But this is equivalent to $P^{i_2}(P^{i_1}(s)) \succeq S(E)$ and $(P^{i_2}(P^{i_1}(s)))_{S(E)} \subseteq E$.
  Conversely, suppose that for all sequences of investors i_1, i_2, $P^{i_2}(P^{i_1}(s)) \succeq S(E)$ and $(P^{i_2}(P^{i_1}(s)))_{S(E)} \subseteq E$, which implies that for all $s' \in P^{i_1}(s)$, $P^{i_2}(s') \succeq S(E)$ and $(P^{i_2}(s'))_{S(E)} \subseteq E$ and $s \in B^{i_1}_S(B^{i_2}_{S^{i_1}(s)}(E))$. Because this is true for all sequences of investors with $n = 2$, we have that $s \in B^2_S(E) = B^1_S(E) \bigcap_{i \in I} B^{i,1}_S(E)$.
- Suppose that for $n = k$, $s \in B^k_S(E)$ is equivalent to $P^{i_k} \ldots P^{i_1}(s) \succeq S(E)$ and $(P^{i_k} \ldots P^{i_1}(s))_{S(E)} \subseteq E$ for any sequence of investors $i_1, \ldots, i_k$ with up to k elements.
- We need to show that for $n = k+1$, $s \in B^{k+1}_S(E)$ is equivalent to $P^{i_{k+1}} \ldots P^{i_1}(s) \succeq S(E)$ and $(P^{i_{k+1}} \ldots P^{i_1}(s))_{S(E)} \subseteq E$ for any sequence of investors $i_1, \ldots, i_{k+1}$ with up to $k + 1$ elements.
  By definition, $s \in B^{k+1}_S(E) \subseteq B^{i_1,k}_S(E)$ implies $s \in B^{i_1}_S(B^k_{S^{i_1}(s)}(E))$, or that $P^{i_1}(s) \subseteq B^k_{S^{i_1}(s)}(E)$. Hence, for all $s' \in P^{i_1}(s)$, $s' \in B^k_{S^{i_1}(s)}(E)$. From the induction hypothesis we have that, for all $s' \in P^{i_1}(s)$, $P^{i_{k+1}} \ldots P^{i_2}(s') \succeq S(E)$ and $(P^{i_{k+1}} \ldots P^{i_2}(s'))_{S(E)} \subseteq E$, which is equivalent to $P^{i_{k+1}} \ldots P^{i_1}(s) \succeq S(E)$ and $(P^{i_{k+1}} \ldots P^{i_1}(s))_{S(E)} \subseteq E$.
  Conversely, suppose that for all sequences of investors $i_1, \ldots, i_{k+1}$ with up to $k+1$ elements, $P^{i_{k+1}} \ldots P^{i_1}(s) \succeq S(E)$ and $(P^{i_{k+1}} \ldots P^{i_1}(s))_{S(E)} \subseteq E$, which implies that for all $s' \in P^{i_1}(s)$, $P^{i_{k+1}} \ldots P^{i_2}(s') \succeq S(E)$ and $(P^{i_{k+1}} \ldots P^{i_2}(s'))_{S(E)} \subseteq$

E and, from the induction hypothesis, $s \in B_S^{i_1}(B^k_{S^{i_1}(s)}(E))$. This implies $s \in B_S^{i_1,k}(E)$ and since this is true for all i_1 we have $s \in B_S^{k+1}(E) = B_S^k(E) \bigcap_{i \in I} B_S^{i,k}(E)$, where $s \in B_S^k(E)$ is implied by the induction hypothesis and all sequences of investors with up to k elements.

Galanis [2013] builds a model of interactive unawareness where the possibility correspondence P^i satisfies properties 0-4 of Proposition 1, which are here implied by assumptions on the belief structure. As the definitions of common knowledge are equivalent, we can use the results from that paper. In particular, as S is finite, Theorem 3 in Galanis [2013] states that there is a nonempty public event $E' \subseteq S^\wedge(s)$ such that $s_{S^\wedge(s)} \subseteq E' \subseteq B^b_{S^\wedge(s)}$. The proof of Theorem 4 in Galanis [2013] shows that E' is partitioned by P^i, for each i.

□

In some of the proofs we use the following two notions of "more information", first defined and used in Galanis [2015, 2016a].

Definition 17. *P^2 is more informative than P^1 given $S \in \mathcal{S}$ if $P^2(s)^S \subseteq P^1(s)^S$ for all $s \in S$.*

Definition 18. *P^i is more informative than the awareness signal of P^j given $S \in \mathcal{S}$ if, for all $s \in S$, $P^i(s)^S \subseteq \mathcal{E}^j_S(s)$.*

Proof of Theorem 1. Suppose there exist bet b and state $s \in S$ such that $B^b_{S^\wedge(s)}$ is common knowledge at s. We need to show that there is no positive common prior and EPCP is satisfied given S.

Suppose not, so there is positive common prior π given S. From Proposition 2, there is a nonempty public event $E \subseteq S^\wedge(s)$ such that $s_{S^\wedge(s)} \in E \subseteq B^b_{S^\wedge(s)}$. Moreover, E is partitioned by P^i, for each i. Without loss of generality, take E to be the smallest public event, in the sense that there is no public event $E' \subset E$. Property 0 of Definition 1 and the definition of $B^b_{S^\wedge(s)}$ imply that $S^i(s_1) = S^\wedge(s)$ for each $s_1 \in E$.

Note that $\sum_{s' \in S^i(s_1)} t^i(s_1)(s')b^i(s') > 0$, for all $s_1 \in B^b_{S^\wedge(s)}$ and all $i \in I$. From Proposition 1, point (5), a positive prior implies supp $t^i(s) = P^i(s)$ for all $s \in A^i_S(S)$. We therefore have $\pi_S(s_1) \sum_{s' \in P^i(s_1)} t^i(s_1)(s')b^i(s') > 0$, for all $s_1 \in E \subseteq B^b_{S^\wedge(s)}$ and all $i \in I$. If we add over all $s_1 \in P^i(s)$ and because π is a positive common prior and $t^i(s_1) = t^i(s_2)$ for $s_1, s_2 \in P^i(s)$, we have

$$0 < \sum_{s_1 \in P^i(s)} \sum_{s' \in P^i(s_1)} \pi_S(s_1) t^i(s_1)(s') b^i(s') =$$

$$\sum_{s' \in P^i(s_1)} \sum_{s_1 \in P^i(s)} \pi_S(s_1) t^i(s_1)(s') b^i(s') = \sum_{s' \in P^i(s_1)} \pi_S(s') b^i(s'),$$

where the last equality is due to the definition of a prior. Adding over all $s' \in E$ and since E is partitioned by P^i, we have that $\sum_{s' \in E} \pi_S(s')b^i(s') > 0$. Adding over all $i \in I$ and since $\sum_{i \in I} b^i(s') = 0$ for all $s' \in E$ we have $0 < \sum_{i \in I} \sum_{s' \in E} \pi_S(s') b^i(s') = \sum_{s' \in E} \pi_S(s') \sum_{i \in I} b^i(s') = 0$, a contradiction.

We have shown that there is no positive common prior given S, but the proof also showed that for some $s \in S$, there is no common local prior π given $S^\wedge(s)$ with $\operatorname{supp} \pi_{S^\wedge(s)} = E$ and smallest nonempty public event $E \subseteq S^\wedge(s)$ with $s_{S^\wedge(s)} \in E$. This means that the conditions for EPCP are not satisfied, hence EPCP is trivially true.

Conversely, suppose that $\underline{\mathcal{S}}$ has no positive common prior given S and that it satisfies EPCP given S. We need to show that there exist state $s \in S$ and bet b given $S^\wedge(s)$ such that $B^b_{S^\wedge(s)}$ is common knowledge at s.

There are two cases. First, for all $s \in S$ and smallest nonempty public event $E \subseteq S^\wedge(s)$ with $s_{S^\wedge(s)} \in E$, it has a common local prior π given $S^\wedge(s)$ with $\operatorname{supp} \pi_{S^\wedge(s)} = E$. Applying EPCP given S implies that $\underline{\mathcal{S}}$ has a positive common prior given S, a contradiction.

Second, for some $s \in S$ and smallest public event $E \subseteq S^\wedge(s)$ with $s_{S^\wedge(s)} \in E$, it does not have a common local prior π given $S^\wedge(s)$ with $\operatorname{supp} \pi_{S^\wedge(s)} = E$. In the standard setting of a unique state space and partitional information structures, the Corollary in Samet [1998] shows that there is a bet such that everyone expects positive gains always if and only if there is no common prior. In our setting, if we treat E as a standard state space and the restriction of each P^i on E as a partition of E, we can apply this Corollary to show that there is a bet b given $S^\wedge(s)$ (that assigns 0 to all investors for states in $S^\wedge(s) \setminus E$) such that $E = B^b_{S^\wedge(s)}$. Using Proposition 2 which shows that the present definition of common knowledge is equivalent to that of Galanis [2013], we can employ Theorem 3 in Galanis [2013] to show that public event $E = B^b_{S^\wedge(s)}$ is common knowledge at s.

$\square$

Lemma 1. *For any state space $S \in \mathcal{S}$, for any possibility correspondence P^i and the resulting awareness signal $\mathcal{E}^i_S$, $\{P^i(s)^S \cap \mathcal{E}^i_S(s)\}_{s \in S}$ is a partition of S. Moreover, if P^2 is more informative than P^1 given $S \in \mathcal{S}$, then partition $\{P^2(s)^S \cap \mathcal{E}^2_S(s)\}_{s \in S}$ is finer than $\{P^1(s)^S \cap \mathcal{E}^1_S(s)\}_{s \in S}$.*

Proof. We use the properties of P^i that were proven in Proposition 1. From Generalized Reflexivity, $s \in P^i(s)^S \cap \mathcal{E}^i_S(s)$. Suppose $s_1 \in P^i(s)^S \cap \mathcal{E}_S(s)$. Then, $S^i(s) = S(s_1)$ and $\{s_1\}_{S^i(s)} \in P^i(s)$. Generalized Reflexivity implies $\{s_1\}_{S^i(s)} \in P^i(s_1)$ and Stationarity implies $P^i(s_1) = P^i(\{s_1\}_{S^i(s)}) = P^i(s)$. Hence, $P^i(s_1)^S \cap \mathcal{E}_S(s_1) = P^i(s)^S \cap \mathcal{E}(s)$.

For the second claim, suppose $s_1 \in P^2(s)^S \cap \mathcal{E}^2_S(s)$. Because P^2 is more informative than P^1 given S, we have $s_1 \in P^1(s)^S$. Stationarity and Projections Preserve Ignorance imply that $S^1(s_1) \succeq S^1(s)$. Suppose $s_1 \notin \mathcal{E}^1_S(s)$. Then, $S^1(s_1) \succ S^1(s)$, which implies that $s \notin P^1(s_1)^S$. Because $\{P^2(s)^S \cap \mathcal{E}^2_S(s)\}_{s \in S}$ is a partition, we have $P^2(s)^S \cap \mathcal{E}^2_S(s) = P^2(s_1)^S \cap \mathcal{E}^2_S(s_1)$, so together with Generalized Reflexivity we have $s \in P^2(s_1)^S \subseteq P^1(s_1)^S$, a contradiction.

$\square$

Proof of Theorem 2. Suppose there is a state space S and a bet b given S_0 such that $B_S^b = S$. To prove by contradiction, suppose $\underline{\mathcal{S}}$ has a common prior π given S such that each $i \in I$ satisfies Conditional Independence given S and π. Fix investor i and a state $s \in S$ such that $\pi_S(s) > 0$. Then, $\pi_{S^i(s)}(P^i(s)) > 0$ and, as we showed in the proof of Theorem 1, $\sum\limits_{s' \in P^i(s)} b^i(s'_{S_0})\pi_{S^i(s)}(s') > 0$. Conditional Independence and Generalized Reflexivity (from Proposition 1) imply that for $s_1, s_2 \in P^i(s)$ such that $\pi_{S^i(s)}(s_1)$, $\pi_{S^i(s)}(s_2) > 0$, we have

$$t^i(s)(s_1) = \pi_{S^i(s)}(s_1 | P^i(s)) = \pi_S(s_1^S | \mathcal{E}_S^i(s) \cap P^i(s)^S),$$

$$t^i(s)(s_2) = \pi_{S^i(s)}(s_2 | P^i(s)) = \pi_S(s_2^S | \mathcal{E}_S^i(s) \cap P^i(s)^S).$$

Rearranging and since $s_1^S \subseteq P^i(s)^S$ we have

$$\frac{\pi_{S^i(s)}(\mathcal{E}_S^i(s) \cap s_1^S)}{\pi_{S^i(s)}(s_1)} = \frac{\pi_{S^i(s)}(\mathcal{E}_S^i(s) \cap s_2^S)}{\pi_{S^i(s)}(s_2)} > 0.$$

Multiplying $\sum\limits_{s_1 \in P^i(s)} b^i(s_{1S_0})\pi_{S^i(s)}(s_1) > 0$ by that number we have

$$\sum_{s_1 \in P^i(s)} b^i(s_{1S_0})\pi_{S^i(s)}(\mathcal{E}_S^i(s) \cap s_1^S) > 0 \implies$$

$$\sum_{s_1 \in P^i(s)} b^i(s_{1S_0}) \sum_{s_2 \in \mathcal{E}_S^i(s) \cap s_1^S} \pi_S(s_2) > 0.$$

Since $\{s_1\}_{S_0} = s_{2S_0}$ for all $s_2 \in s_1^S$ we have

$$\sum_{s_1 \in P^i(s)} \sum_{s_2 \in \mathcal{E}_S^i(s) \cap s_1^S} b^i(s_{2S_0})\pi_S(s_2) > 0 \implies$$

$$\sum_{s_1 \in P^i(s)^S \cap \mathcal{E}_S^i(s)} b^i(s_{1S_0})\pi_S(s_1) > 0.$$

From Lemma 1, $\left\{(P^i(s))^S \cap \mathcal{E}_S^i(s)\right\}_{s \in S}$ generates a partition of S. By adding over all elements of the partition we have that

$$\sum_{s \in S} b^i(s_{S_0})\pi_S(s) > 0.$$

By adding over all investors and since $\sum\limits_{i \in I} b^i(s_0) = 0$ for all $s_0 \in S_0$, we have a contradiction.

Suppose $\underline{\mathcal{S}}$ satisfies Projections Preserve Posteriors and that there is no common local prior given S_0. This means that S_0 has at least two states. Moreover, because S_0 is the bottom state space, each P^i provides a partition of S_0. This means that no common local prior given S_0 is equivalent to no common prior in a standard model with a unique state space S_0. Applying the Corollary of Samet [1998], there is a bet b

given S_0 such that $B^b_{S_0} = S_0$. Take any $S \in \mathcal{S}$. Projections Preserve Posteriors implies that for any $s, s' \in S$ such that $s_{S_0} = s'_{S_0}$, we have $t^i(s)(E) = t^i(s')(E)$ for all events $E \subseteq S_0$. Because b is given S_0, we have $B^b_S = S$.

$\square$

Proof of Theorem 3. The proof is similar to the proof of Theorem 3 in Geanakoplos [1989]. Let $f = \{f^i\}_{i\in I}$ be an equilibrium and look at i's single-investor decision problem that is induced when the strategy of each $j \neq i$ is fixed. Because of PPOP, Assumption 1 in Galanis [2015] is satisfied. Consider a fictitious investor k, who is always fully aware but has no information at all. In particular, P^i is more informative than P^k given any $S \in \mathcal{S}$. Because k's awareness is constant, P^i is more informative than k's awareness signal, given any $S \in \mathcal{S}$. Investor k's available actions are identical to i's . If he were to replace i in the game, his optimal action would be z^i and his ex ante payoff would be $\bar{u}^i$. Because either the awareness of P^i is path-independent or i satisfies Conditional Independence given π and S^*, Theorem 1 in Galanis [2015] implies that i's ex ante payoff is weakly higher than k's, which is $\bar{u}^i$. Since this is true for all investors, by hypothesis we have $f(s^*) = z$ for all $s^* \in S^*$.

$\square$

Proof of Theorem 4. Let π^i_1 be i's positive prior given S^1_0 in period 1 and π^i be i's positive prior given $S \in \mathcal{S}_2$ in period 2. By assumption, $\pi^i_1 = \pi^i_{S^2_0}$. Suppose that f is ex ante Pareto efficient, g is feasible and $H^{g,f}_{S^\wedge(s),j}$ is common knowledge at $s \in S$. Because $\underline{\mathcal{S}}_2$ is a positive belief structure, from Proposition 1 we have that $P^i(s) = \operatorname{supp} t^i(s)$ for all $s \in S \in \mathcal{S}_2$.

Using the same arguments as in the proof of Theorem 1, there is a public event $E \subseteq S^\wedge(s)$, partitioned by each P^i, such that $s_{S^\wedge(s)} \subseteq E \subseteq H^{g,f}_{S^\wedge(s),j}$. By the definition of $S^\wedge(s)$, for all $i \in I$ and $s' \in E$, $S^i(s') = S^\wedge(s)$. By adding the expression in $H^{g,f}_{S^\wedge(s),j}$ for each $s' \in E$ and because P^i partitions E, we have

$$\sum_{s_1 \in E} \pi^i_{S^\wedge(s)}(s_1) u^i(g^i(s_{1S^2_0})) \geq \sum_{s_1 \in E} \pi^i_{S^\wedge(s)}(s_1) u^i(f^i(s_{1S^2_0})),$$

for each $i \in I$, strict inequality for j.

If $\underline{\mathcal{S}}_2$ satisfies Projections Preserve Posteriors then for each $s' \in S^\wedge(s)$ we have $t^i(s')(E') = t^i(s'_{S^2_0})(E')$ for all $E' \in S^2_0$. This implies that we can project $H^{g,f}_{S^\wedge(s),j}$ to S^2_0 without altering the posterior beliefs. We then have that

$$\sum_{s_1 \in E_{S^2_0}} \pi^i_{S^2_0}(s_1) u^i(g^i(s_{1S^2_0})) \geq \sum_{s_1 \in E_{S^2_0}} \pi^i_{S^2_0}(s_1) u^i(f^i(s_{1S^2_0})),$$

for each $i \in I$, strict inequality for j.

In both cases, whether $S^\wedge(s) = S^2_0$ or $\underline{\mathcal{S}}_2$ satisfies Projections Preserve Posteriors, and because all state contingent trades are defined on S^2_0, we can define $h^i : S^2_0 \to \mathbb{R}_+$ such that $h^i(s') = g^i(s')$ if $s' \in E$ and $h^i(s') = f^i(s')$ otherwise. By construction,

$\{h^i\}_{i \in I}$ is feasible and we have

$$\sum_{s_1 \in S_0^2} \pi^i_{S_0^2}(s_1) u^i(h^i(s_{1S_0^2})) \geq \sum_{s_1 \in S_0^2} \pi^i_{S_0^2}(s_1) u^i(f^i(s_{1S_0^2})),$$

for all $i \in I$, strict inequality for j. By assumption $\pi^i_1 = \pi^i_{S_0^2}$, hence

$$\sum_{s_1 \in S_0^1} \pi^i_1(s_1) u^i(h^i(s_{1S_0^1})) \geq \sum_{s_1 \in S_0^1} \pi^i_1(s_1) u^i(f^i(s_{1S_0^1})),$$

for all $i \in I$, strict inequality for j, so that f is not ex ante Pareto efficient given S, a contradiction.

$\square$

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