DESCENT OF EQUIVALENCES AND CHARACTER BIJECTIONS

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Abstract. Categorical equivalences between block algebras of finite groups - such as Morita and derived equivalences - are well-known to induce character bijections which commute with the Galois groups of field extensions. This is the motivation for attempting to realise known Morita and derived equivalences over non splitting fields. This article presents various results on the theme of descent to appropriate subfields and subrings. We start with the observation that perfect isometries induced by a virtual Morita equivalence induce isomorphisms of centers in non-split situations, and explain connections with Navarro's generalisation of the Alperin-McKay conjecture. We show that Rouquier's splendid Rickard complex for blocks with cyclic defect groups descends to the non-split case. We also prove a descent theorem for Morita equivalences with endopermutation source.

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1. Introduction

Throughout the paper, p is a prime number. Let $(K, \mathcal{O}, k)$ be a p-modular system; that is, $\mathcal{O}$ is a complete discrete valuation ring with residue field $k = \mathcal{O}/J(\mathcal{O})$ of characteristic p and field of fractions K of characteristic zero. We are interested in capturing equivariance properties of various standard equivalences (such as Morita, Rickard or p-permutation equivalences) in the block theory of finite groups. A common context for these is the notion of virtual Morita equivalence which we recall.

Let A, B, C be $\mathcal{O}$-algebras, finitely generated free as $\mathcal{O}$-modules. Denote by mod-A the category of finitely generated left A-modules and by $\mathcal{R}(A)$ the Grothendieck group of mod-A with respect to split exact sequences. Denoting by $[M]$ the element of $\mathcal{R}(A)$ corresponding to the finitely generated A-module M, $\mathcal{R}(A)$ is a free abelian group with basis the set of all elements $[M]$, where M runs through a set of representatives of the isomorphism classes of finitely generated indecomposable A-modules.

Denote by $\mathcal{R}(A,B)$ the group $\mathcal{R}(A \otimes_{\mathcal{O}} B^{\mathrm{op}})$ and by $\mathcal{P}(A,B)$ the subgroup of $\mathcal{R}(A,B)$ generated by elements $[M]$, where M is an (A,B)-bimodule which is finitely generated projective as left A-module and as right B-module. We let $- \cdot_B - : \mathcal{R}(A,B) \times \mathcal{R}(B,C) \to \mathcal{R}(A,C)$, $M \times N \mapsto M \cdot_B N$ be the group homomorphism induced by tensoring over B, that is such that $[X] \cdot_B [Y] = [X \otimes_B Y]$

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for all finitely generated (A,B)-bimodules X and (B,C)-bimodules Y. If X is an (A,B)-bimodule, then its $\mathcal{O}$-dual $X^\vee = \mathrm{Hom}_{\mathcal{O}}(X,\mathcal{O})$ is a (B,A)-bimodule. The algebra A is called *symmetric* if $A \cong A^\vee$ as (A,A)-bimodules. If A and B are symmetric, and if X is an (A,B)-bimodule which is finitely generated projective as left A-module and as right B-module, then the (B,A)-module $X^\vee$ is again finitely projective as left B-module and as right A-module (this holds more generally if A and B are relatively $\mathcal{O}$-injective). We denote in that case by $M \to M^\vee$ the unique homomorphism $\mathcal{P}(A,B) \to \mathcal{P}(B,A)$ such that $[X]^\vee = [X^\vee]$ for any (A,B)-bimodule X which is finitely generated projective as left A-module and as right B-module. Note that in the above we may replace $\mathcal{O}$ by any complete local ring, and in particular by a field, and we will do so without further comment.

Definition 1.1. Let A and B be $\mathcal{O}$-algebras, finitely generated free as $\mathcal{O}$-modules, and let $M \in \mathcal{P}(A,B)$ and $N \in \mathcal{P}(B,A)$. We say that M and N induce a *virtual Morita equivalence between A and B* if $M \cdot_B N = [A]$ in $\mathcal{R}(A,A)$ and $N \cdot_A M = [B]$ in $\mathcal{R}(B,B)$.

Remark 1.2. Let A and B be symmetric $\mathcal{O}$-algebras. We will use without further comment the following well-known implications between the various levels of equivalences we consider in this paper. If M is an (A,B)-bimodule which is finitely generated projective as a left and right module and which induces a Morita equivalence between A and B, then $[M]$ and $[M^\vee]$ induce a virtual Morita equivalence. More generally, if X is a Rickard complex of (A,B)-bimodules, then $[X] = \sum_{i\in\mathbb{Z}}(-1)^i[X_i]$ and $[X^\vee]$ induce a virtual Morita equivalence between A and B. Following [1], if A, B are blocks of finite group algebras, then a virtual Morita equivalence between A and B given by M and $M^\vee$ is called a *p-permutation equivalence* if M can be written in the form $M = [M_0] - [M_1]$, where M_0, M_1 are p-permutation (A,B)-bimodules which are finitely generated projective as left and right modules. In particular, if X is a splendid Rickard complex of (A,B)-bimodules, then $[X]$ and $[X^\vee]$ induce a p-permutation equivalence.

Let K' be an extension field of K. For an $\mathcal{O}$-algebra A, we denote by $K'A$ the K'-algebra $K' \otimes_{\mathcal{O}} A$, and for any A-module V we denote by $K'V$ the $K'A$-module $K' \otimes_{\mathcal{O}} V$. The functor $K' \otimes_{\mathcal{O}} - :$ mod-$A \to$ mod-$K'A$ induces a group homomorphism $[V] \mapsto [K'V]$ from $\mathcal{R}(A)$ to $\mathcal{R}(K'A)$, for all finitely generated A-modules V. We use analogous notation for bimodules. Let $\mathrm{Aut}(K'/K)$ denote the group of automorphisms of K' which induce the identity on K. For $\sigma \in \mathrm{Aut}(K'/K)$ and a $K'A$-module U we denote by ${}^\sigma U$ the σ-twist of U, that is ${}^\sigma U$ is the $K'A$-module which is equal to U as a KA-module and on which $\lambda \otimes a$ acts as $\sigma^{-1}(\lambda) \otimes a$ for all $\lambda \in K'$ and all $a \in KA$. We use the analogous notation for the induced map on $\mathcal{R}(K'A)$.

If $K'A$ is a semisimple algebra, we denote by $\mathrm{Irr}(K'A)$ the subset of $\mathcal{R}(K'A)$ consisting of the elements $[S]$, where S runs through a set of representatives of isomorphism classes of simple $K'A$-modules. Then $\mathrm{Irr}(K'A)$ is a $\mathbb{Z}$-basis of $\mathcal{R}(K'A)$. For $\chi = [S] \in \mathrm{Irr}(K'A)$ we denote by e_χ the unique primitive idempotent of $Z(K'A)$ such that $e_\chi S \neq 0$.

The following general result on symmetric $\mathcal{O}$-algebras is the starting point of the phenomenon we wish to exhibit.

Theorem 1.3. *Let $(K,\mathcal{O},k)$ be a p-modular system. Let A and B be symmetric $\mathcal{O}$-algebras and let K' be an extension field of K such that $K'A$ and $K'B$ are split* *semisimple. Suppose that* $M \in \mathcal{P}(B,A)$ *is such that* M *and* $M^\vee$ *induce a virtual Morita equivalence between* A *and* B*. Then there exists a bijection* $I : \mathrm{Irr}(K'A) \to \mathrm{Irr}(K'B)$ *and signs* $\epsilon_\chi \in \{\pm 1\}$ *for any* $\chi \in \mathrm{Irr}(K'A)$ *such that*

$$\epsilon_\chi I(\chi) = K'M \cdot_{K'A} \chi$$

for all $\chi \in \mathrm{Irr}(K'A)$*. Moreover, the following holds.*

(a) The algebra isomorphism $Z(K'A) \cong Z(K'B)$ *sending* e_χ *to* $e_{I(\chi)}$ *for all* $\chi \in \mathrm{Irr}(K'A)$ *induces an* $\mathcal{O}$*-algebra isomorphism* $Z(A) \cong Z(B)$*.*

(b) The bijection I *commutes with* $\mathrm{Aut}(K'/K)$*; that is, we have* $I({}^\sigma\chi) = {}^\sigma I(\chi)$ *for all* $\sigma \in \mathrm{Aut}(K'/K)$ *and all* $\chi \in \mathrm{Irr}(K'A)$*.*

Remark 1.4. By a result of Broué [3, 1.2], a virtual Morita equivalence between two blocks of finite group algebras over $\mathcal{O}$ given by a virtual bimodule and its dual induces a perfect isometry. In particular, if A and B in Theorem 1.3 are blocks of finite group algebras, then the bijection I and the signs in the theorem are together a perfect isometry. Not every perfect isometry is induced by a virtual Morita equivalence, but the advantage of virtual Morita equivalences is that they are defined for arbitrary algebras. Broué's abelian defect group conjecture, in the version predicting a Rickard equivalence between a block with an abelian defect group and the Brauer correspondent of that block, in conjunction with Remark 1.2, implies therefore that the induced perfect isometry between a block and its Brauer correspondent is in fact induced by a virtual Morita equivalence.

We are interested in subgroups of $\mathrm{Gal}(\mathbb{Q}_n/\mathbb{Q})$ which lift automorphisms in characteristic p. For a positive integer n, we denote by n_p (respectively $n_{p'}$) the p-part (respectively p'-part) of n.

Definition 1.5. Let $(K, \mathcal{O}, k)$ be a p-modular system such that k is perfect. Let $\bar{K}$ be an algebraic closure of K. Let n be a positive integer, denote by $\mathbb{Q}_n$ the n-th cyclotomic subfield of $\bar{K}$, and let k' be a splitting field of the polynomial $x^{n_{p'}} - 1$ over k. We denote by $\mathcal{H}_n$ the subgroup of $\mathrm{Gal}(\mathbb{Q}_n/\mathbb{Q})$ consisting of those automorphisms α for which there exists a non-negative integer u such that $\alpha(\delta) = \delta^{p^u}$ for all $n_{p'}$-roots of unity δ in $\mathbb{Q}_n$. We denote by $\mathcal{H}_{n,k}$ the subgroup of $\mathcal{H}_n$ consisting of those automorphisms α for which there exists a non-negative integer u and an element $\tau \in \mathrm{Gal}(k'/k)$ such that $\alpha(\delta) = \delta^{p^u}$ for all $n_{p'}$-roots of unity δ in $\mathbb{Q}_n$ and $\tau(\eta) = \eta^{p^u}$ for all $n_{p'}$-roots of unity η in k'.

Note that $\mathcal{H}_n$ is the image under restriction in $\mathbb{Q}_n$ of the Weil subgroup of the absolute Galois group of the p-adic numbers (see Lemma 3.2 and Lemma 3.3). Note also that $\mathcal{H}_{n,k}$ is independent of the choice of a splitting field k' of $x^{n_{p'}} - 1$ over k. With the notation of the above definition, for a finite group G, the set $\mathrm{Irr}(\bar{K}G)$ may be identified with the set of characters $\chi : G \to \bar{K}$ of simple $\bar{K}G$-modules. The group $\mathrm{Aut}(\bar{K})$ acts on $\mathrm{Irr}(\bar{K}G)$ via ${}^\sigma\chi(g) := \chi(\sigma(g))$, $\chi \in \mathrm{Irr}(\bar{K}G)$, $\sigma \in \mathrm{Aut}(\bar{K})$. We say that *a positive integer* n *is large enough for* G if the action of $\mathrm{Aut}(K)$ on $\mathrm{Irr}(\bar{K}G)$ factors through to an action of $\mathrm{Gal}(\mathbb{Q}_n/\mathbb{Q})$ via the surjective homomorphism from $\mathrm{Aut}(\bar{K})$ to $\mathrm{Gal}(\mathbb{Q}_n/\mathbb{Q})$ induced by restriction to $\mathbb{Q}_n$. In particular, if n is a multiple of $|G|$, then n is large enough for G.

By a *block* of $\mathcal{O}G$ for G a finite group we mean a primitive idempotent of the center of the group algebra $\mathcal{O}G$. If b is a block of $\mathcal{O}G$ we denote by $\mathrm{Irr}(\bar{K}Gb)$ the subset of $\mathrm{Irr}(\bar{K}G)$ consisting of the characters of simple $\bar{K}Gb$-modules. There are many open questions and conjectures around bijections between sets of irreducible characters of blocks which commute with the action of the groups $\mathcal{H}_n$ and $\mathcal{H}_{n,k}$, most notably Navarro's refinement of the Alperin-McKay conjecture [24, Conjecture B].

Theorem 1.3 yields the following equivariance result for character bijections. The slogan is: categorical equivalences between blocks over absolutely unramified complete discrete valuation rings give rise to character bijections which commute with the action of $\mathcal{H}_{n,k}$. Recall that $\mathcal{O}$ is said to be *absolutely unramified* if $J(\mathcal{O}) = p\mathcal{O}$.

Theorem 1.6. *Let $(K, \mathcal{O}, k)$ be a p-modular system such that k is perfect and such that $\mathcal{O}$ is absolutely unramified. Let $\bar{K}$ be an algebraic closure of K. Let G and H be finite groups, let b be a block of $\mathcal{O}G$, c a block of $\mathcal{O}H$ and let n be large enough for G and for H. A virtual Morita equivalence between $\mathcal{O}Gb$ and $\mathcal{O}Hc$ given by a virtual bimodule M and its dual $M^\vee$ induces a bijection $I : \mathrm{Irr}(\bar{K}Gb) \to \mathrm{Irr}(\bar{K}Hc)$ satisfying ${}^\sigma I(\chi) = I({}^\sigma\chi)$ for all $\chi \in \mathrm{Irr}(\bar{K}Gb)$ and all $\sigma \in \mathcal{H}_{n,k}$.*

As mentioned before, the bijection I in the above theorem is part of a perfect isometry. Further, Morita, Rickard and p-permutation equivalences all yield virtual Morita equivalences. Thus the conclusion of the Theorem holds on replacing the hypothesis of virtual Morita equivalence by any of these equivalences - in the case of a Morita equivalence the induced bijection between the sets of irreducible Brauer characters also commutes with the action of $\mathcal{H}_{n,k}$ as well as with the decomposition map (see Theorem 3.4).

Recall that a character $\chi \in \mathrm{Irr}(\bar{K}G)$ is said to be *p-rational* if there there exists a root of unity δ in $\bar{K}$ of order prime to p such that $\chi(g) \in \mathbb{Q}[\delta]$ for all $g \in G$. Theorem 1.6 has the following consequence.

Corollary 1.7. *Suppose that $\mathcal{O}$ and $\bar{K}$ are as in Theorem 1.6. Any virtual Morita equivalence between block algebras $\mathcal{O}Gb$ and $\mathcal{O}Hc$ given by a virtual bimodule and its dual induces a bijection between $\mathrm{Irr}(\bar{K}Gb)$ and $\mathrm{Irr}(\bar{K}Hc)$ which preserves p-rationality.*

Recall that for a perfect subfield k' of k, there is a unique absolutely unramified complete discrete valuation ring $W(k')$ contained in $\mathcal{O}$ such that the image of $W(k')$ under the canonical surjection $\mathcal{O} \to k$ is k' (see [32, Chapter 2, Theorems 3, 4 and Prop. 10]). The ring $W(k')$ is called the ring of Witt vectors in $\mathcal{O}$ of k'.

Definition 1.8. Let $(K, \mathcal{O}, k)$ be a p-modular system. Let G be a finite group and b a block of $\mathcal{O}G$. The *minimal complete discrete valuation ring of b in $\mathcal{O}$* denoted $\mathcal{O}_b$ is the ring of Witt vectors in $\mathcal{O}$ of the finite subfield of k generated by the coefficients of the group elements in the image of b under the canonical surjection $\mathcal{O}G \to kG$. If $\mathcal{O}_b = \mathcal{O}$, then we say that $\mathcal{O}$ *is a minimal complete discrete valuation ring of b.*

By idempotent lifting arguments we have $b \in \mathcal{O}_b G$, and if R is any complete discrete valuation ring which is properly contained in $\mathcal{O}_b$ and with $J(R) \subseteq J(\mathcal{O}_b)$, then $b \notin RG$.

The following is a corollary of the special case of Theorem 1.6 in which $\mathcal{O}$ is a minimal complete discrete valuation ring of the blocks involved.

Corollary 1.9. *Suppose that $\mathcal{O}$ and $\bar{K}$ are as in Theorem 1.6. Let G and H be finite groups and let n be large enough for G and for H. Let b be a block of $\mathcal{O}G$ and*

c a block of $\mathcal{O}H$. Suppose that $\mathcal{O}$ is a minimal complete discrete valuation ring for both b and c. A virtual Morita equivalence between $\mathcal{O}Gb$ and $\mathcal{O}Hc$ given by a virtual bimodule and its dual induces a bijection $I : \mathrm{Irr}(\bar{K}Gb) \to \mathrm{Irr}(\bar{K}Hc)$ such that for any $\chi \in \mathrm{Irr}(\bar{K}Gb)$, and any $\sigma \in \mathcal{H}_n$, we have ${}^{\sigma}\chi = \chi$ if and only if ${}^{\sigma}I(\chi) = I(\chi)$.

For a p-subgroup P of G the Brauer homomorphism $\mathrm{Br}_P : (\mathcal{O}G)^P \to kC_G(P)$ is the map which sends an element $\sum_{g\in G} \alpha_g g$ of $(\mathcal{O}G)^P$ to $\sum_{g\in C_G(P)} \bar{\alpha}_g g$, where $\bar{\alpha}$ denotes reduction modulo the maximal ideal $J(\mathcal{O})$ of $\mathcal{O}$. Recall that Br_P is a surjective $\mathcal{O}$-algebra homomorphism and that $\mathrm{Br}_P(Z(\mathcal{O}G)) \subseteq Z(kC_G(P))$. In particular, if b is a central idempotent of $\mathcal{O}G$, then either $\mathrm{Br}_P(b) = 0$ or $\mathrm{Br}_P(b)$ is a central idempotent of $kC_G(P)$. If b is a block of $\mathcal{O}G$, then a defect group of b is defined to be a maximal p-subgroup P of G such that $\mathrm{Br}_P(b) \neq 0$. By Brauer's first main theorem, if b is a block of $\mathcal{O}G$ with defect group P, then there is a unique block c of $\mathcal{O}N_G(P)$ with defect group P such that $\mathrm{Br}_P(b) = \mathrm{Br}_P(c)$ and the map $b \mapsto c$ is a bijection between the set of blocks of $\mathcal{O}G$ with defect group P and the set of blocks of $\mathcal{O}N_G(P)$ with defect group P, and this bijection is known as the Brauer correspondence.

In [24, Conjecture B], Navarro conjectured that if $|G| = n$, b and c are blocks in correspondence as above and K contains $\mathbb{Q}_n$, then for each $\sigma \in \mathcal{H}_n$ the number of height zero characters in $\mathrm{Irr}(\bar{K}Gb)$ fixed by σ equals the number of height zero characters in $\mathrm{Irr}(\bar{K}Hc)$ fixed by σ. Since $\mathcal{O}_b = \mathcal{O}_c$, and since the bijection I of Corollary 1.9 is part of a perfect isometry and hence preserves heights, it follows that a virtual Morita equivalence between $\mathcal{O}_bGb$ and $\mathcal{O}_bHc$ given by a virtual bimodule and its dual implies Navarro's conjecture.

In view of the above discussion, it would be desirable to explore the following question: Given a categorical equivalence, say a Morita equivalence or Rickard equivalence between $\mathcal{O}'Gb$ and $\mathcal{O}'Hc$ for some complete discrete valuation ring $\mathcal{O}'$, for G and H finite groups, b and c blocks of $\mathcal{O}'G$ and $\mathcal{O}'H$ respectively, and a complete discrete valuation ring $\mathcal{O}$ contained in $\mathcal{O}'$ such that b (respectively c) belongs to $\mathcal{O}Gb$ (respectively $\mathcal{O}Hc$), is the equivalence between $\mathcal{O}'Gb$ and $\mathcal{O}'Hc$ an extension of an equivalence between $\mathcal{O}b$ and $\mathcal{O}Hc$? We give a positive answer to this question in the case of blocks with cyclic defect groups.

Let G be a finite group and b a block of $\mathcal{O}G$ with a nontrivial cyclic defect group P. If k is a splitting field for all subgroups of G, then in [31] Rouquier constructed a 2-sided splendid tilting complex of $(\mathcal{O}Gb, \mathcal{O}N_G(P)e)$-bimodules, where e is the Brauer correspondent of b. (The hypotheses in [31] also require the field of fractions K to be large enough, but it is easy to see that Rouquier's construction works with $\mathcal{O}$ absolutely unramified). We show that Rouquier's construction descends to any p-modular system which contains the block coefficients.

Theorem 1.10. *Let $(K', \mathcal{O}', k')$ be a p-modular system such that $\mathcal{O} \subseteq \mathcal{O}'$ and such that $J(\mathcal{O}) \subseteq J(\mathcal{O}')$. Let G be a finite group and b a block of $\mathcal{O}'G$ having a nontrivial cyclic defect group P. Suppose that $b \in \mathcal{O}G$ and that k' is a splitting field for all subgroups of G. Let e be the block of $\mathcal{O}'N_G(P)$ with P as a defect group corresponding to b via the Brauer correspondence. Then $e \in \mathcal{O}N_G(P)$ and the blocks $\mathcal{O}Gb$ and $\mathcal{O}N_G(P)e$ are splendidly Rickard equivalent. More precisely, there is a splendid Rickard complex X of $(\mathcal{O}Gb, \mathcal{O}N_G(P)e)$-bimodules such that $\mathcal{O}' \otimes_{\mathcal{O}} X$ is isomorphic to Rouquier's complex X'.*

Since a Rickard equivalence induces a virtual Morita equivalence, by the above discussion around Navarro's conjecture, we recover the following result of Navarro from Theorem 1.10.

Corollary 1.11 ([24, Theorem 3.4])**.** *Conjecture B of* [24] *holds for blocks with cyclic defect groups.*

General descent arguments from Theorem 6.5 in conjunction with Theorem 1.10 yield a splendid equivalence for cyclic blocks for arbitrary p-modular systems.

Theorem 1.12. *Let $(K, \mathcal{O}, k)$ be a p-modular system. Let G a finite group and b a block of $\mathcal{O}G$ having a nontrivial cyclic defect group P. Let e be the block of $\mathcal{O}N_G(P)$ with P as a defect group corresponding to b via the Brauer correspondence. Then $\mathcal{O}Gb$ and $\mathcal{O}N_G(P)e$ are splendidly Rickard equivalent. In particular, $\mathcal{O}Gb$ and $\mathcal{O}N_G(P)e$ are p-permutation equivalent.*

The above results may be viewed as evidence for a refined version of the Abelian defect group conjecture, namely that for any p-modular system $(K, \mathcal{O}, k)$ and any block b of $\mathcal{O}G$ with abelian defect group P and Brauer correspondent c, there is a splendid Rickard equivalence between $\mathcal{O}Gb$ and $\mathcal{O}N_G(P)c$.

If one is only interested in keeping track of p-rational characters, then by Corollary 1.7 it suffices to descend to any absolutely unramified complete discrete valuation ring. Since p-permutation modules all have forms over absolutely unramified complete discrete valuation rings, any p-permutation equivalence between block algebras of finite groups can be easily seen to be an extension of a p-permutation equivalence between the corresponding blocks over the subring of Witt vectors. We show that such descent is also possible for Morita equivalences induced by bimodules with endopermutation sources.

Theorem 1.13. *Let $(K, \mathcal{O}, k)$ be a p-modular system. Let G and H be finite groups, b a block of $\mathcal{O}G$ and c a block of $\mathcal{O}H$. Denote by $\bar{b}$ the image of b in kG and by $\bar{c}$ the image of c in kH. Assume that k is a splitting field for all subgroups of $G \times H$.*

(a) For any Morita equivalence (resp. stable equivalence of Morita type) between $kG\bar{b}$ and $kH\bar{c}$ given by an indecomposable bimodule $\bar{M}$ with endopermutation source $\bar{V}$ there is a Morita equivalence (resp. stable equivalence of Morita type) between $\mathcal{O}Gb$ and $\mathcal{O}Hc$ given by a bimodule M with endopermutation source V such that $k \otimes_{\mathcal{O}} M \cong \bar{M}$ and $k \otimes_{\mathcal{O}} V \cong \bar{V}$.

(b) For any Morita equivalence (resp. stable equivalence of Morita type) between $\mathcal{O}Gb$ and $\mathcal{O}Hc$ given by an indecomposable bimodule with endopermutation source V there is a Morita equivalence (resp. stable equivalence of Morita type) between $W(k)Gb$ and $W(k)Hc$ given by an indecomposable bimodule with endopermutation source U such that $k \otimes_{W(k)} U \cong k \otimes_{\mathcal{O}} V$.

Remark 1.14. The proof of the above theorem requires a lifting property of fusion stable endopermutation modules from Lemma 8.4 below, which in turn relies on the classification of endopermutation modules. The hypothesis on k being large enough is there to ensure that the fusion systems of the involved blocks are saturated. The well-known Morita equivalences in block theory such as in the context of nilpotent blocks [26], blocks with a normal defect group [12] and blocks of finite p-solvable groups [11], [28] are all given by endopermutation source bimodules hence are defined over the Witt vectors and preserve p-rational characters and p-rational lifts of Brauer characters (cf. Corollary 1.7, Theorem 3.4).

The paper is organised as follows. Section 2 contains the proof of Theorem 1.3 and Section 3 contains the proofs of Theorem 1.6 and its corollaries. Sections 4, 5 and 6 contain general results on descent. Theorems 1.10 and 1.12 are proved in Section 7, and Section 8 contains the proof of Theorem 1.13.

Notation 1.15. We will use the above notation of Galois twists for arbitrary extensions of commutative rings $\mathcal{O} \subseteq \mathcal{O}'$. That is, given an $\mathcal{O}$-algebra A, a module U over the $\mathcal{O}'$-algebra $A' = \mathcal{O}' \otimes_{\mathcal{O}} A$ and a ring automorphism σ of $\mathcal{O}'$ which restricts to the identity map on $\mathcal{O}$, we denote by ${}^{\sigma}U$ the A'-module which is equal to U as a module over the subalgebra $1 \otimes A$ of A', such that $\lambda \otimes a$ acts on U as $\sigma^{-1}(\lambda) \otimes a$ for all $a \in A$ and $\lambda \in \mathcal{O}'$. Note that if $f : U \to V$ is an A'-module homomorphism, then f is also an A'-module homomorphism ${}^{\sigma}U \to {}^{\sigma}V$. The Galois twist induces an $\mathcal{O}$-linear (but not in general $\mathcal{O}'$-linear) self equivalence on mod-A'.

2. On virtual Morita equivalences

This section contains the proof of Theorem 1.3. We start with some background observations. Let $(K, \mathcal{O}, k)$ be a p-modular system. It is well-known that a virtual Morita equivalence between two split semisimple algebras given by a virtual bimodule and its dual is equivalent to fixing a bijection between the isomorphism classes of simple modules of the two algebras together with signs. We sketch the argument for the convenience of the reader.

Lemma 2.1. *Let A and B be split semisimple finite-dimensional K-algebras. Let M be a virtual (A, B)-bimodule in $\mathcal{R}(A, B)$. Then M and $M^\vee$ induce a virtual Morita equivalence between A and B if and only if there is a bijection $I : \mathrm{Irr}(A) \to \mathrm{Irr}(B)$ and signs $\epsilon_S \in \{\pm 1\}$ for all $S \in \mathrm{Irr}(A)$ such that*

$$M = \sum_{S \in \mathrm{Irr}(A)} \epsilon_S S \otimes_K I(S)^\vee$$

in $\mathcal{R}(A, B)$.

Proof. Write $M = \sum_{S,T} a(S, T) S \otimes_K T^\vee$, with integers $a(S, T)$, where S and T run over $\mathrm{Irr}(A)$ and $\mathrm{Irr}(B)$, respectively. Since B is split semisimple, we have $T^\vee \otimes_B T \cong K$ and $T^\vee \otimes_B T' = \{0\}$, where $T, T' \in \mathrm{Irr}(B)$, $T \not\cong T'$. Thus $M \cdot_B M^\vee = \sum_{S,S',T} a(S, T)a(S', T)S' \otimes_K S^\vee$, with S, S' running over $\mathrm{Irr}(A)$ and T running over $\mathrm{Irr}(B)$. We have the analogous formula for $M^\vee \cdot_A M$. Since A is split semisimple, we have $[A] = \sum_S S \otimes_K S^\vee$. Thus M, $M^\vee$ induce a virtual Morita equivalence if and only if $\sum_T a(S, T)^2 = 1$ for all $S \in \mathrm{Irr}(A)$, and $\sum_T a(S, T)a(S', T) = 0$ for any two distinct S, S' in $\mathrm{Irr}(A)$. Since the $a(S, T)$ are integers, the first equation implies that for any S there is a unique $T = I(S)$ such that $a(S, T) \in \{\pm 1\}$ and $a(S, T') = 0$ for $T' \neq T$. The second equation implies that I is a bijection. The result follows with $\epsilon_S = a(S, I(S))$. □

We will use the transfer maps in Hochschild cohomology from [18], specialised in degree 0; we sketch the construction. Let A and B be symmetric $\mathcal{O}$-algebras with fixed symmetrising forms. Let M be an (A, B)-bimodule which is finitely generated projective as left A-module and as right B-module. Then the functors $M \otimes_B -$ and $M^\vee \otimes_A -$ are biadjoint; the choice of the symmetrising forms determines adjunction isomorphisms. Let $y \in Z(B)$. Multiplication by y induces a (B, B)-bimodule endomorphism of B. Tensoring by $M \otimes_B - \otimes_B M^\vee$ yields an A-A-bimodule endomorphism of $M \otimes_B M^\vee$. Composing and precomposing this endomorphism by the adjunction counit $M \otimes_B M^\vee \to A$ and the adjunction unit $A \to M \otimes_B M^\vee$ yields an (A, A)-bimodule endomorphism of A, which in turn yields a unique element $z \in Z(A)$ which induces this endomorphism by multiplication on A. We define the linear map $\mathrm{tr}_M : Z(B) \to Z(A)$ by setting $\mathrm{tr}_M(y) = z$, with y and z as above. The map tr_M is additive in M (cf. [18, 2.11.(i)]), depends only on the isomorphism class of M (cf. [18, 2.12.(iii)]) and is compatible with tensor products of bimodules (cf. [18, 2.11.(ii)]). In general, tr_M depends on the choice of the symmetrising forms (because the adjunction units and counits depend on this choice), but there is one case where it does not:

Lemma 2.2. *Let A be a symmetric $\mathcal{O}$-algebra. Consider A as an (A, A)-bimodule. Then $\mathrm{tr}_A = \mathrm{Id}_{Z(A)}$.*

Proof. Let $s : A \to \mathcal{O}$ be a symmetrising form of A, and let X be an $\mathcal{O}$-basis of A. Denote by X' the dual basis of A with respect to s; for $x \in X$, denote by x' the unique element in X' satisfying $s(xx') = 1$ and $s(yx') = 0$, for all $y \in X \setminus \{x\}$. The well-known explicit description of the adjunction maps (see e. g. [18, Appendix]) implies that the adjunction unit $A \to A \otimes_A A^\vee$ sends 1_A to $1_A \otimes s$, and the adjunction counit $A \otimes_A A^\vee \to A$ sends $1_A \otimes s$ to $\sum_{x \in X} s(x')x$, which is equal to 1_A by [18, Appendix 6.3.3]. One can prove this also without those explicit descriptions, by first observing that the above adjunction maps are isomorphisms, and deduce from this that tr_A is a linear automorphism. Since $\mathrm{tr}_A \circ \mathrm{tr}_A = \mathrm{tr}_{A \otimes_A A} = \mathrm{tr}_A$, this implies that $\mathrm{tr}_A = \mathrm{Id}_{Z(A)}$. □

In order to show that tr_M is well-defined with M replaced by any element in the Grothendieck group $\mathcal{P}(A, B)$, we need the following observation.

Lemma 2.3. *Let A, B be symmetric $\mathcal{O}$-algebras with chosen symmetrising forms. Let M_0, M_1, N_0, N_1 be (A, B)-bimodules which are finitely generated projective as left and as right modules. If $[M_0]-[M_1] = [N_0]-[N_1]$ in $\mathcal{P}(A, B)$, then $\mathrm{tr}_{M_0} - \mathrm{tr}_{M_1} = \mathrm{tr}_{N_0} - \mathrm{tr}_{N_1}$.*

Proof. The equality $[M_0]-[M_1] = [N_0]-[N_1]$ is equivalent to $[M_0 \oplus N_1] = [N_0 \oplus M_1]$. The Krull-Schmidt Theorem implies that this is equivalent to $M_0 \oplus N_1 \cong N_0 \oplus M_1$. The additivity of transfer maps implies that in that case we have $\mathrm{tr}_{M_0} + \mathrm{tr}_{N_1} = \mathrm{tr}_{N_0} + \mathrm{tr}_{M_1}$, whence the result. □

This Lemma implies that if A, B are symmetric $\mathcal{O}$-algebras with chosen symmetrising forms, then for any $M \in \mathcal{P}(A, B)$ we have a well-defined map $\mathrm{tr}_M : Z(A) \to \mathbb{Z}(B)$ given by $\mathrm{tr}_M = \mathrm{tr}_{M_0} - \mathrm{tr}_{M_1}$, where M_0, M_1 are (A, B)-bimodules which are finitely generated as left and right modules such that $M = [M_0] - [M_1]$.

Lemma 2.4. *Let A, B, C be symmetric $\mathcal{O}$-algebras. Let $M \in \mathcal{P}(A, B)$ and $N \in \mathcal{P}(B, C)$. Then $\mathrm{tr}_{M \cdot_B N} = \mathrm{tr}_M \circ \mathrm{tr}_N$. In particular, if M and $M^\vee$ induce a virtual Morita equivalence between A and B, then $\mathrm{tr}_M : Z(A) \to Z(B)$ is a linear isomorphism with inverse $\mathrm{tr}_{M^\vee}$.*

Proof. The first equality follows from the corresponding equality [18, 2.11.(ii)] where M and N are actual bimodules, together with 2.3. The second statement follows from the first and 2.2. □

Remark 2.5. The three lemmas 2.2, 2.3, 2.4 hold verbatim for the transfer maps on the Hochschild cohomology of A in B in any non-negative degree, and with $\mathcal{O}$ replaced by any complete local principal ideal domain.

Proof of Theorem 1.3. We use the notation and hypotheses from Theorem 1.3. Write $M = [M_0] - [M_1]$, where M_0, M_1 are (A, B)-bimodules which are finitely generated projective as left A-modules and as right B-modules. By 2.1 there exist a bijection $I : \mathrm{Irr}(K'A) \to \mathrm{Irr}(K'B)$ and signs $\epsilon_\chi \in \{\pm 1\}$ such that $\epsilon_\chi \chi = K'M \cdot_{K'B} I(\chi)$ in $\mathcal{R}(K'A)$ for all $\chi \in \mathrm{Irr}(K'A)$. By 2.4, the linear map $\mathrm{tr}_M : Z(B) \to Z(A)$ is an isomorphism, with inverse $\mathrm{tr}_{M^\vee}$. Let $v \in Z(B)$ such that $\mathrm{tr}_M(v) = 1_A$ and let $u \in Z(A)$ such that $\mathrm{tr}_{M^\vee}(u) = 1_B$. Define linear maps $\alpha : Z(A) \to Z(B)$ and $\beta : Z(B) \to Z(A)$ by setting

$$\alpha(z) = \mathrm{tr}_{M^\vee}(uz)$$

$$\beta(y) = \mathrm{tr}_M(vy)$$

for all $z \in Z(A)$ and $y \in Z(B)$. By the choice of u and v we have $\alpha(1_A) = 1_B$ and $\beta(1_B) = 1_A$. We extend α and β K'-linearly to maps, still called α, β, between $Z(K'A)$ and $Z(K'B)$. Setting $K'M = K' \otimes_{\mathcal{O}} M$ as before, note that the transfer map $\mathrm{tr}_{K'M} : Z(K'B) \to Z(K'A)$ is the K'-linear extension of tr_M. Note further that $K'M = \sum_{\chi \in \mathrm{Irr}(K'A)} \epsilon_\chi \chi \otimes_{K'} I(\chi)^\vee$ and $\chi \otimes_{K'} I(\chi)^\vee = e_\chi K'M e_{I(\chi)}$. Thus $\mathrm{tr}_{K'M} = \sum_{\chi \in \mathrm{Irr}(K'A)} \epsilon_\chi \mathrm{tr}_{e_\chi K'M e_{I(\chi)}}$. In particular, $\mathrm{tr}_{K'M}$ sends $K'e_\chi$ to $K'e_{I(\chi)}$.

Let $\chi \in \mathrm{Irr}(K'A)$ and $\eta = I(\chi)$. We have $\beta(e_\eta) = \mathrm{tr}_{K'M}(ve_\eta)$. Since $Z(K'B)$ is a direct product of copies of K', it follows that $ve_\eta = \lambda_\chi e_\eta$ for some $\lambda_\chi \in K'$. Thus $\mathrm{tr}_{K'M}(ve_\eta) = \mu_\chi e_\chi$ for some $\mu_\chi \in K'$. Therefore

$$1_A = \mathrm{tr}_M(v) = \sum_{\chi \in \mathrm{Irr}(K'A)} \mathrm{tr}_{K'M}(ve_{I(\chi)}) = \sum_{\chi \in \mathrm{Irr}(K'A)} \mu_\chi e_\chi \ .$$

Since also $1_A = \sum_{\chi \in \mathrm{Irr}(K'A)} e_\chi$, the linear independence of the e_χ implies that all μ_χ are 1, hence that $\alpha(e_\chi) = e_{I(\chi)}$. This shows that α and β are inverse algebra isomorphisms $Z(K'A) \cong Z(K'B)$. By their constructions, α maps $Z(A)$ to $Z(B)$ and β maps $Z(B)$ to $Z(A)$. This proves statement (a). This shows also that the isomorphism $Z(K'A) \cong Z(K'B)$ sending e_χ to $e_{I(\chi)}$ induces an isomorphism $Z(KA) \cong Z(KB)$. In other words, since $Z(K'A) = K' \otimes_K Z(KA)$ and $Z(K'B) = K' \otimes_K Z(KB)$, it follows that the above isomorphism $Z(K'A) \cong Z(K'B)$ is obtained from K'-linearly extending an isomorphism $Z(KA) \cong Z(KB)$, which implies that this isomorphism commutes with the action of $\mathrm{Aut}(K'/K)$, whence statement (b). □

By Lemma 2.1, a virtual Morita equivalence between split semisimple finite-dimensional algebras given by a virtual bimodule and its dual is equivalent to a character bijection with signs. If the compatibility of the character bijection with Galois automorphisms is all one wants to establish, one does not need to descend to valuation rings. For the sake of completeness, we spell this out for block algebras; this is an easy consequence of results of Broué [3].

Proposition 2.6. *Let G and H be finite groups and let K'/K be a finite Galois extension such that K' is a splitting field for both G and H. Let b be a central idempotent of $K'G$ and c a central idempotent of $K'H$ and let $I : \mathrm{Irr}(K'Gb) \to \mathrm{Irr}(K'Hc)$ be a bijection. Suppose that there exist signs $\delta_\chi \in \{\pm 1\}$ for any $\chi \in$*

$\mathrm{Irr}(K'Gb)$, *such that the virtual bicharacter* $\mu := \sum_{\chi\in\mathrm{Irr}(K'Gb)} \delta_\chi(\chi\times I(\chi))$ *of* $G\times H$ *takes values in* K. *Then* $b \in KG$ *and* $c \in KH$. *Moreover, the following hold.*

(a) *For all* $\sigma \in \mathrm{Gal}(K'/K)$ *and all* $\chi \in \mathrm{Irr}(K'Gb)$ *we have* $I({}^\sigma\chi) = {}^\sigma I(\chi)$.
(b) *The* K'*-algebra isomorphism* $Z(K'Gb) \to Z(K'Hc)$ *sending* e_χ *to* $e_{I(\chi)}$ *for all* $\chi \in \mathrm{Irr}(K'Gb)$ *restricts to a* K*-algebra automorphism* $Z(KGb) \to Z(KHc)$.

Proof. The hypothesis that μ takes values in K implies that if $\delta_\chi(\chi \times I(\chi))$ is a summand of μ, then so is $\delta_\chi({}^\sigma\chi \times {}^\sigma I(\chi))$. This shows that b and c are $\mathrm{Gal}(K'/K)$-stable, hence contained in KG and KH, respectively, and it shows that I commutes with the action of $\mathrm{Gal}(K'/K)$ as stated in (a). Again since μ takes values in K, it follows from the explicit formulas of the central isomorphism $Z(K'Gb) \cong Z(K'Hc)$ in the proof of [3, Théorème 1.5] that this isomorphism restricts to an isomorphism $Z(KGb) \cong Z(KHc)$. □

3. Characters and Galois automorphisms

Definition 3.1. An *extension of a* p*-modular system* $(K, \mathcal{O}, k)$ is a p-modular system $(K', \mathcal{O}', k')$ such that $\mathcal{O}$ is a subring of $\mathcal{O}'$, with $J(\mathcal{O}) \subseteq J(\mathcal{O}')$.

In the situation of the above definition, we write $(K, \mathcal{O}, k) \subseteq (K', \mathcal{O}', k')$, and whenever convenient, we identify without further notice K as a subfield of K' and k as a subfield of k' in the obvious way.

In this section we fix a p-modular system $(K, \mathcal{O}, k)$ such that k is perfect. Denote by $\bar{K}$ a fixed algebraic closure of K and $\mathbb{Q}_n$ the n-th cyclotomic extension of $\mathbb{Q}$ in $\bar{K}$. We denote by $\mathcal{G}_n$ the group $\mathrm{Gal}(\mathbb{Q}_n/\mathbb{Q})$. The following lemma combines some basic facts on extensions of complete discrete valuation rings; we include proofs for the convenience of the reader.

Lemma 3.2. *Let* $(K', \mathcal{O}', k')$ *be an extension of the* p*-modular system* $(K, \mathcal{O}, k)$ *such that* K' *is a normal extension of* K*. Then,* $\mathcal{O}'$ *is* $\mathrm{Gal}(K'/K)$*-invariant, and* k'/k *is a Galois extension. Moreover, if* $\mathcal{O}$ *is absolutely unramified, then the homomorphism* $\mathrm{Gal}(K'/K) \to \mathrm{Gal}(k'/k)$ *induced by restriction to* $\mathcal{O}'$ *is surjective.*

Proof. Let π (respectively π') be a uniformiser of $\mathcal{O}$ (respectively $\mathcal{O}'$) and let a be a real number with $0 < a < 1$. Since $\mathcal{O} \subseteq \mathcal{O}'$ and $J(\mathcal{O}) \subseteq J(\mathcal{O}')$, we have $\pi = {\pi'}^e u$, for some positive integer e and some $u \in (\mathcal{O}')^\times$. Let $\nu : K \to \mathbb{R}$ be the absolute value defined by $\nu(x) = a^{ei}$ if $x = \pi^i v$, $v \in \mathcal{O}^\times$ and let $\nu' : K' \to \mathbb{R}$ be the absolute value defined by $\nu'(x) = a^i$ if $x = (\pi')^i v$, $v \in {\mathcal{O}'}^\times$. Then ν', and consequently $\nu' \circ \sigma$ are extensions of ν to K' for any $\sigma \in \mathrm{Gal}(K'/K)$. On the other hand, since K' is an algebraic extension of K and since ν is complete, there is a unique extension of ν to an absolute value on K' (see [25, Chapter 2, Theorem 4.8]). Thus $\nu' = \nu' \circ \sigma$ for all $\sigma \in \mathrm{Gal}(K'/K)$. This proves the first assertion as the valuation ring of ν' is $\mathcal{O}'$. It follows from the first assertion that $\mathcal{O}'$ is integral over $\mathcal{O}$, and consequently that k' is a normal extension of k. Further, since any algebraic extension of a perfect field is perfect, k' is perfect and k'/k is separable, Hence k'/k is Galois as claimed.

Now suppose that $\mathcal{O}$ is absolutely unramified; that is, $\mathcal{O}$ is the ring of Witt vectors $W(k)$. Let $\mathcal{O}_0 = W(k') \subseteq \mathcal{O}'$ be the ring of Witt vectors of k' in $\mathcal{O}'$ and let K_0 be the field of fractions of $\mathcal{O}_0$. Then K_0 is a normal extension of K; to see this it suffices to show that K_0 is $\mathrm{Gal}(K'/K)$-invariant, hence that $\mathcal{O}_0$ is $\mathrm{Gal}(K'/K)$-invariant. This is obvious since $\mathcal{O}_0$ is generated by p and the canonical lift of $(k')^\times$ in $(\mathcal{O}')^\times$, both of which are clearly $\mathrm{Gal}(K'/K)$-invariant.

Applying the first part of the lemma to the extension $(K_0, \mathcal{O}_0, k')$ of $(K, \mathcal{O}, k)$ we obtain, via restriction to $\mathcal{O}_0$, a homomorphism from $\mathrm{Gal}(K_0/K) \to \mathrm{Gal}(k'/k)$. This homomorphism is surjective. Indeed, by [32, Chapter 2, Theorem 4], any automorphism of k' lifts uniquely to an automorphism of $\mathcal{O}_0$, and applying the same theorem again shows that the unique lift of an automorphism of k' which is the identity on k is the identity on $\mathcal{O}$. By the normality of K'/K every element of $\mathrm{Gal}(K_0/K)$ extends to an element of $\mathrm{Gal}(K'/K)$, proving the result. □

Lemma 3.3. *Let $(K', \mathcal{O}', k')$ be an extension of the p-modular system $(K, \mathcal{O}, k)$ such that K' is a normal extension of K contained in $\bar{K}$. Suppose that $\mathcal{O}$ is absolutely unramified.*

(a) *Let $\zeta \in K'$ be a root of unity whose order is a power of p. Then* $\mathrm{Gal}(K[\zeta]/K) \cong \mathrm{Aut}(\langle\zeta\rangle) \cong \mathrm{Gal}(\mathbb{Q}[\zeta]/\mathbb{Q})$.
(b) *Suppose that $\mathbb{Q}_n \subseteq K'$. Then $\mathcal{H}_{n,k}$ is the image of the map* $\mathrm{Gal}(K'/K) \to \mathcal{G}_n$ *induced by restriction to $\mathbb{Q}_n$.*

Proof. (a) Let $m \geq 1$ and let $\Phi_{p^m}(x) \in \mathbb{Z}[x]$ denote the p^m-th cyclotomic polynomial. We have

$$\Phi_{p^m}(x) = \frac{x^{p^m} - 1}{x^{p^{m-1}} - 1} = \Phi_p(x^{p^{m-1}}).$$

Set $f(x) = \Phi_{p^m}(x+1)$. Then,

$$f(x) = \Phi_p((x+1)^{p^{m-1}}) \equiv \Phi_p(x^{p^{m-1}} + 1) \mod p\mathbb{Z}[x].$$

Note that $\Phi_p(x+1) = \frac{(x+1)^p - 1}{x} = \sum_{i=1}^{p} \binom{p}{i} x^{i-1}$, so all but the leading coefficient of this polynomial are divisible by p. Upon replacing x by x^{p^m}, it follows in particular that all intermediate coefficients of

$$\Phi_p(x^{p^{m-1}} + 1) = \frac{(x^{p^{m-1}} + 1)^p - 1}{x^{p^{m-1}}}$$

are divisible by p. Thus all intermediate coefficients of $f(x)$ are divisible by p. Also, $f(x)$ is monic and has constant term p. Since p is prime in $\mathcal{O}$, it follows by Eisenstein's criterion applied to $\mathcal{O}$, that $\Phi_{p^m}(x)$ is irreducible in $\mathcal{O}[x]$, and hence by Gauss's lemma that $\Phi_{p^m}(x)$ is irreducible in $K[x]$. This proves the first assertion.

(b) We first show that the image of $\mathrm{Gal}(K'/K)$ in $\mathcal{G}_n$ is contained in $\mathcal{H}_{n,k}$. By Lemma 3.2, restriction to $\mathcal{O}$ induces a homomorphism $\mathrm{Gal}(K'/K) \to \mathrm{Gal}(k'/k)$. Let $\tau \in \mathrm{Gal}(K'/K)$ and denote by $\bar{\tau}$ the image of $\tau \in \mathrm{Gal}(k'/k)$ under the above map. The restriction of $\bar{\tau}$ to the (finite) splitting field of $x^n - 1$ over $\mathbb{F}_p$ is a power of the Frobenius map $x \to x^p$. Since the canonical surjection $u \to \bar{u}$ from $\mathcal{O}'$ to k' induces an isomorphism between the groups of p'-roots of unity of K' and of k' and since $\bar{\tau}$ is the identity on k, it follows that the restriction of τ to $\mathbb{Q}_n$ is an element of $\mathcal{H}_{n,k}$.

Next we show that $\mathcal{H}_{n,k}$ is contained in the image of $\mathrm{Gal}(K'/K)$ in $\mathcal{G}_n$. Let $\zeta \in \mathbb{Q}_n$ be a primitive n-th root of unity. Write $\zeta = \zeta_p \zeta_{p'}$, where $\zeta_p, \zeta_{p'}$ are powers of ζ, the order of ζ_p is a power of p, and the order of $\zeta_{p'}$ is prime to p. Let $\alpha \in \mathcal{H}_{n,k}$. We will prove that there exists $\beta \in \mathrm{Gal}(K'/K)$ such that the restriction of β to $\mathbb{Q}_n$ equals α. By Lemma 3.2, k' is a normal extension of k. Hence by the definition of $\mathcal{H}_{n,k}$, there exists $\bar{\tau} \in \mathrm{Gal}(k'/k)$ and a non-negative integer u such that $\alpha(\delta) = \delta^{p^u}$ for all $n_{p'}$-roots of unity δ in $\mathbb{Q}_n$ and $\bar{\tau}(\eta) = \eta^{p^u}$ for all $n_{p'}$-roots of unity η in k'. Again by Lemma 3.2, $\bar{\tau}$ lifts to an automorphism $\tau \in \mathrm{Gal}(K'/K)$. By the isomorphism between the groups of p'-roots of unity in K' and in k, we have that

$\tau(\zeta_{p'}) = \alpha(\zeta_{p'})$. By part (a), there exists $\sigma \in \mathrm{Gal}(K_0[\zeta_p]/K_0)$ such that $\sigma(\tau(\zeta_p)) = \alpha(\zeta_p)$. Let σ' be any extension of σ to K' and set $\beta := \sigma'\tau$. Then β has the required properties. □

Theorem 1.6 forms the first part of the statement of the following result. For a finite group G denote by $\mathrm{IBr}(G)$ the set of irreducible Brauer characters of G interpreted as functions from the set of p-regular elements of G to $\bar{K}$. If b is a central idempotent of $\mathcal{O}G$, then we denote by $\mathrm{IBr}(G, b)$ the subset of $\mathrm{IBr}(G)$ consisting of the Brauer characters of simple $k'Gb$-modules for any sufficiently large field k' containing k.

Theorem 3.4. *Let G, H be finite groups and let n be large enough for G and H. Let b, c be blocks of $\mathcal{O}G$ and $\mathcal{O}H$ respectively. Suppose that $\mathcal{O}$ is absolutely unramified. Any virtual Morita equivalence between $\mathcal{O}Gb$ and $\mathcal{O}Hc$ given by a virtual bimodule and its dual induces a bijection $I : \mathrm{Irr}(\bar{K}Gb) \to \mathrm{Irr}(\bar{K}Hc)$ satisfying ${}^{\sigma}I(\chi) = I({}^{\sigma}\chi)$ for all $\sigma \in \mathcal{H}_{n,k}$ and all $\chi \in Irr(\bar{K}Hc)$.*

If the virtual Morita equivalence is induced from a Morita equivalence, then in addition there exists a bijection, $\bar{I} : \mathrm{IBr}(G, b) \to \mathrm{IBr}(H, c)$ such that for all $\chi \in \mathrm{Irr}(G, b)$, $\varphi \in \mathrm{IBr}(G, b)$ and $\sigma \in \mathcal{H}_{n,k}$ the decomposition numbers $d_{I(\chi),\bar{I}(\varphi)}$ and $d_{\chi,\varphi}$ are equal and ${}^{\sigma}\bar{I}(\varphi) = \bar{I}({}^{\sigma}\varphi)$.

Proof. Let $(K', \mathcal{O}', k')$ be an extension of the p-modular system $(K, \mathcal{O}, k)$ such that $K' \subseteq \bar{K}$ and such that the extension K'/K is normal. Suppose that k' is perfect, and that K' contains primitive n-th, $|G|$-th and $|H|$-th roots of unity. We may and will identify $\mathrm{Irr}(\bar{K}G)$ and $\mathrm{Irr}(\bar{K}H)$ with $\mathrm{Irr}(K'G)$ and $\mathrm{Irr}(K'H)$ respectively. By Lemma 3.3, the subgroup $\mathcal{H}_{n,k}$ is the image of the restriction map from $\mathrm{Gal}(K'/K)$ to $\mathcal{G}_n$. It follows from 1.3 that a virtual Morita equivalence between $\mathcal{O}Gb$ and $\mathcal{O}Hc$ given by a virtual bimodule and its dual yields a character bijection I which commutes with $\mathrm{Gal}(K'/K)$, hence with $\mathcal{H}_{n,k}$. By [3, 1.2], the bijection I, together with the signs from 2.1 is a perfect isometry.

Now suppose that X is a $(\mathcal{O}Hc, \mathcal{O}Gb)$-bimodule finitely generated and projective as left $\mathcal{O}Hc$-module and as right $\mathcal{O}Gb$-module such that $X \otimes_{\mathcal{O}Gb} -$ induces a Morita equivalence between $\mathcal{O}Gb$ and $\mathcal{O}Hc$. Then $X' := \mathcal{O}' \otimes_{\mathcal{O}} X$ induces a Morita equivalence between $\mathcal{O}'Gb$ and $\mathcal{O}'Hc$ and $\bar{X} := k \otimes_{\mathcal{O}} X$ and $\bar{X}' := k' \otimes_k \bar{X}$ induce Morita equivalences between kGb and kHc and between $k'Gb$ and $k'Hc$ respectively.

Since k' contains enough roots of unity, we may identify $\mathrm{IBr}(G, b)$ (respectively $\mathrm{IBr}(H, c)$) with the Brauer characters of simple $k'Gb$-modules (respectively $k'Hc$-modules). For a simple $k'Gb$-module (or $k'Hc$-module) S, denote by φ_S the corresponding Brauer character. Let $\bar{I} : \mathrm{IBr}(G, b) \to \mathrm{IBr}(H, c)$ be the bijection induced by $\bar{X}'$, that is such that $\bar{I}(\varphi_S) = \varphi_{\bar{X}' \otimes_k' S}$ for any simple $k'Gb$-module S. Since $\bar{X}' \cong k' \otimes_{\mathcal{O}'} X'$, we have that $d_{I(\chi),\bar{I}(\varphi)} = d_{\chi,\varphi}$ for all $\chi \in \mathrm{Irr}_{K'}(G, b)$, $\varphi \in IBr(b)$.

Let $\sigma \in \mathcal{H}_{n,k}$. By the previous lemma, there exists $\tau \in \mathrm{Gal}(K'/K)$ such that the restriction to $\mathbb{Q}_n$ is σ. Let $\bar{\tau} \in \mathrm{Gal}(k'/k)$ be the image of τ. If S is any simple $k'Gb$-module (respectively $k'Hc$-module), then ${}^{\sigma}(\varphi_S) = \varphi_{{}^{\bar{\tau}}S}$. Thus, it suffices to show that ${}^{\bar{\tau}}(\bar{X}' \otimes_{k'} S) \cong \bar{X}' \otimes_{k'} {}^{\bar{\tau}}S$ for any simple $k'Gb$-module S. Now

$$ {}^{\bar{\tau}}X' = {}^{\bar{\tau}}(k' \otimes_k X) \cong k' \otimes X $$

as $(k'Hc, k'Gb)$-bimodule and it follows that for any simple $k'Gb$-module S that

$$ {}^{\bar{\tau}}(X' \otimes_{k'Hc} S) \cong {}^{\bar{\tau}}X' \otimes_{k'Hc} {}^{\bar{\tau}}S \cong X' \otimes_{k'Hc} {}^{\bar{\tau}}S $$

as $k'Hc$-modules. This proves the result. □

Proof of Corollary 1.7. Let n be a common multiple of $|G|$ and of $|H|$. For any $g \in G$ and $\chi \in \mathrm{Irr}(\bar{K}G)$, $\chi(g) \in \mathbb{Q}_n$. By the basic theory of cyclotomic extensions of $\mathbb{Q}$, χ is p-rational if and only if ${}^{\sigma}\chi = \chi$ for all $\sigma \in \mathcal{G}_n$ such that ${}^{\sigma}(\eta) = \eta$ for all $n_{p'}$-roots of unity $\eta \in \mathbb{Q}_n$ and similarly for the characters of H. On the other hand, if $\sigma \in \mathcal{G}_n$ is such that ${}^{\sigma}(\eta) = \eta$ for all $n_{p'}$-roots of unity $\eta \in \mathbb{Q}_n$, then $\sigma \in \mathcal{H}_{n,k}$. The result is now immediate from Theorem 3.4. □

Proof of Corollary 1.9. The action of $\mathcal{H}_n$ on $\mathrm{Irr}(\bar{K}G)$ induces an action of $\mathcal{H}_n$ on the set of blocks of $\mathcal{O}G$. Since $\mathcal{O}$ is a minimal complete discrete valuation ring for b, k is a finite field and consequently a splitting field of $x^{n_{p'}} - 1$ over k is also finite. Let $|k| = p^d$. Then $\mathcal{H}_{n,k}$ consists of precisely those elements α of $\mathcal{G}_n$ for which there exists a non-negative integer u such that $\sigma(\delta) = \delta^{p^{ud}}$ for all $n_{p'}$-roots of unity in $\mathbb{Q}_n$. It follows that $\mathcal{H}_{n,k}$ is the stabiliser of b in $\mathcal{H}_n$ and similarly for H and c. The result is now immediate from Theorem 3.4. □

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4. Descent for equivalences

Let $\mathcal{O}$, $\mathcal{O}'$ be complete local commutative principal ideal domains such that $\mathcal{O} \subseteq \mathcal{O}'$ and $J(\mathcal{O}) \subseteq \mathcal{O}'$ (so that either $\mathcal{O}$, $\mathcal{O}'$ are complete discrete valuation rings or they are fields, allowing the possibility that $\mathcal{O}$ is a field but $\mathcal{O}'$ is not). Let A be an $\mathcal{O}$-algebra which is finitely generated as an $\mathcal{O}$-module. Then A is in particular noetherian, and hence the category of finitely generated A-modules mod-A is abelian and coincides with the category of finitely presented A-modules. We set $A' = \mathcal{O}' \otimes_{\mathcal{O}} A$. For any A-module U we denote by U' the A'-module $\mathcal{O}' \otimes_{\mathcal{O}} U$, and for any homomorphism of A-modules $f : U \to V$, we denote by f' the homomorphism of A'-modules $\mathrm{Id}_{\mathcal{O}'} \otimes f : U' \to V'$. We extend this notion in the obvious way to complexes; that is, if $X = (X_n)_{n\in\mathbb{Z}}$ is a complex of A-modules with differential $\delta = (\delta_n)_{n\in\mathbb{Z}}$, then we denote by X' the complex of A'-modules $(X'_n)_{n\in\mathbb{Z}}$ with differential $\delta' = (\delta'_n)_{n\in\mathbb{Z}}$.

The following lemmas are adaptations to the situation considered in this paper of well-known results which hold in greater generality; see the references given. We do not require the ring $\mathcal{O}'$ to be finitely generated as an $\mathcal{O}$-module.

Lemma 4.1. *The ring extension $\mathcal{O} \subseteq \mathcal{O}'$ is faithfully flat; that is, the functor $\mathcal{O}' \otimes_{\mathcal{O}} -$ is exact and sends any nonzero $\mathcal{O}$-module to a nonzero $\mathcal{O}'$-module.*

Proof. Since $\mathcal{O}'$ is torsion free as an $\mathcal{O}$-module, it follows from [13, (4.69)] that $\mathcal{O}'$ is flat as an $\mathcal{O}$-module. Since $J(\mathcal{O}) = \mathcal{O} \cap J(\mathcal{O}')$ is the unique prime ideal in $\mathcal{O}$, it follows from [13, (4.74)] or also [13, (4.71)] that $\mathcal{O}'$ is faithfully flat as an $\mathcal{O}$-module. □

Lemma 4.2. *Let A be an $\mathcal{O}$-algebra which is finitely generated as an $\mathcal{O}$-module. Then for any sequence $M \xrightarrow{f} N \xrightarrow{g} U$ of A-modules, the sequence is exact at N if and only if the sequence $M' \xrightarrow{f'} N' \xrightarrow{g'} U'$ of A'-modules is exact at N'.*

Proof. This follows from Lemma 4.1 and [13, (4.70)]. □

Recall that a morphism $\alpha : X \to Y$ in a category $\mathcal{C}$ is *split* if there exists a morphism $\beta : Y \to X$ such that $\alpha\beta\alpha = \alpha$.

Lemma 4.3. *Let A be an $\mathcal{O}$-algebra which is finitely generated as an $\mathcal{O}$-module. A morphism $f : M \to N$ in* mod-A *is split if and only if the morphism $f' : M' \to N'$ in* mod-A' *is split.*

Proof. One checks easily that f is split if and only if the two epimorphisms $M \to \mathrm{Im}(f)$ and $N \to \mathrm{coker}(f)$ are split. By Lemma 4.1, the extension $\mathcal{O} \subseteq \mathcal{O}'$ is faithfully flat, and hence $\mathrm{Im}(f') = \mathrm{Im}(f)'$ and $\mathrm{coker}(f') = \mathrm{coker}(f)'$. Thus it suffices to show that f is a split epimorphism if and only if f' is a split epimorphism. Now f is a split epimorphism if and only of the map $\mathrm{Hom}_A(N, M) \to \mathrm{Hom}_A(N, N)$ induced by composing with f is surjective (since in that case an inverse image of Id_N under this map is a section of f). Again since the extension $\mathcal{O} \subseteq \mathcal{O}'$ is faithfully flat, it follows that f is a split epimorphism if and only if the induced map $\mathcal{O}' \otimes_{\mathcal{O}} \mathrm{Hom}_A(N, M) \to \mathcal{O}' \otimes_{\mathcal{O}} \mathrm{Hom}_A(N, N)$ is surjective. Since M, N are finitely generated, hence finitely presented by the assumptions on A, it follows from [23, Theorem I.11.7] (applied with $\mathcal{O}'$ instead of B) that there is a canonical isomorphism $\mathcal{O}' \otimes_{\mathcal{O}} \mathrm{Hom}_A(N, M) \cong \mathrm{Hom}_{A'}(N', M')$ and a similar isomorphism with N instead of M. Thus the surjectivity of the previous map is equivalent to the surjectivity of the map $\mathrm{Hom}_{A'}(N', M') \to \mathrm{Hom}_{A'}(N', N')$ induced by composing with f'. This is, in turn, equivalent to asserting that f' is a split epimorphism, whence the result. □

Lemma 4.4. *Let A be an $\mathcal{O}$-algebra which is finitely generated as an $\mathcal{O}$-module, let X be a complex of finitely generated A-modules and let M be a finitely generated A-module. Then*

(a) *M is projective if and only if M' is projective as A'-module.*
(b) *X is acyclic if and only if X' is acyclic as complex of A'-modules.*
(c) *X is contractible if and only if X' is contractible as complex of A'-modules.*

Proof. Part (a) and part (b) follow from Lemma 4.2 and Lemma 4.3. By [36, §1.4], a complex of A-modules (respectively A'-modules) is contractible if and only if the complex is acyclic and the differential in each degree is split. Therefore part (c) also follows from Lemma 4.2 and Lemma 4.3. □

Let A and B be symmetric $\mathcal{O}$-algebras. Let M be a finitely generated (A, B)-bimodule which is projective as left A-module and as right B-module. If $M \otimes_B - :$ mod-$B \to$ mod-A is an equivalence, then the symmetry of A and B implies that an inverse of this equivalence is induced by tensoring with the $\mathcal{O}$-dual $M^\vee$ of M; that is, $M \otimes_B M^\vee \cong A$ as (A, A)-bimodules and $M^\vee \otimes_A M \cong B$ as (B, B)-bimodules. Following Broué, we say that M induces a *stable equivalence of Morita type* if there exist a projective (A, A)-bimodule U and a projective (B, B)-bimodule V such that $M \otimes_B M^\vee \cong A \oplus U$ as (A, A)-bimodules and $M^\vee \otimes_A M \cong B \oplus V$ as (B, B)-bimodules. Let X be a bounded complex of finitely generated (A, B)-bimodules which are projective as left A-modules and as right B-modules, and let $X^\vee = \mathrm{Hom}_{\mathcal{O}}(X, \mathcal{O})$ be the dual complex. We say that X induces a *Rickard equivalence* and that X is a *Rickard complex* if there exist a contractible complex of (A, A)-bimodules Y and a contractible complex of (B, B)-bimodules Z such that $X \otimes_B X^\vee \cong A \oplus Y$ as complexes (A, A)-bimodules and $X^\vee \otimes_A X \cong B \oplus Z$ as complexes of (B, B)-bimodules. Let M and N be finitely generated (A, B)-bimodules, projective as left and right modules and let $U = [M] - [N]$. Then $U^\vee = [M^\vee] - [N^\vee]$. Recall that U and $U^\vee$ induce a virtual Morita equivalence between A and B if $U \cdot_B U^\vee = [A]$ in $\mathcal{R}(A, A)$ and $U^\vee \cdot_B U = [B]$ in $\mathcal{R}(B, B)$. We denote by $C^b(A)$ the category of bounded complexes of finitely generated A-modules, by $K^b(A)$ the homotopy category of bounded complexes of finitely generated A-modules and by $D^b(A)$ the bounded derived category of mod-A. For a finitely generated A-module M we denote by $[M]$ the isomorphism class of M as an element of the Grothendieck group of mod-A with respect to split exact sequences. We use the analogous notation for bimodules.

Proposition 4.5. *Let A and B be symmetric $\mathcal{O}$-algebras. Let M, N be finitely generated (A, B)-bimodules which are projective as left A-module and as right B-module and let X be a bounded complex of finitely generated (A, B)-bimodules which are projective as left A-modules and as right B-modules.*

(a) *X' induces a Rickard equivalence between A' and B', if and only if X induces a Rickard equivalence between A and B.*
(b) *M' induces a stable equivalence of Morita type between A' and B' if and only if M induces a stable equivalence of Morita type between A and B.*
(c) *M' induces a Morita equivalence between A' and B' if and only if M induces a Morita equivalence between A and B.*
(d) *$[M'] - [N']$ and $[(M')^\vee] - [(N')^\vee]$ induce a virtual Morita equivalence between A' and B' if and only if $[M] - [N]$ and $[M^\vee] - [N^\vee]$ induce a virtual Morita equivalence between A and B.*

Proof. One direction of the implications is trivial. We verify the reverse implications. We will apply the previous lemmas in this section to the $\mathcal{O}$-algebras $A \otimes_{\mathcal{O}} B^{\mathrm{op}}$, $A \otimes_{\mathcal{O}} A^{\mathrm{op}}$ etc. In what follows, we will freely switch between the terminology of (A, A)-bimodules and $A \otimes_{\mathcal{O}} A^{\mathrm{op}}$-modules.

We prove (a). Suppose that X' induces a Rickard equivalence between A' and B' and let Y be a contractible bounded complex of (A', A')-bimodules such that $X' \otimes_{B'} (X')^\vee = A' \oplus Y$.

The functors $X \otimes_B -$ and $X^\vee \otimes_A -$ define a pair of biadjoint functors between $C^b(B \otimes_{\mathcal{O}} A^{\mathrm{op}})$ and $C^b(A \otimes_{\mathcal{O}} A^{\mathrm{op}})$ (this is well-known; see e. g. [18, Section 6.10]). Denote by $\epsilon_X : X \otimes_B X^\vee \to A$ and $\epsilon_{X^\vee} : X^\vee \otimes_A X \to B$ the counits of these adjunctions. Similarly, denote by $\epsilon_{X'} : X' \otimes_{B'} (X')^\vee \to A'$ and $\epsilon_{(X')^\vee} : (X')^\vee \otimes_{A'} X' \to B$ the counits corresponding to the biadjoint pair $X' \otimes_{B'} -$ and $(X')^\vee \otimes_{A'} -$ between $C^b(B' \otimes_{\mathcal{O}} (A')^{\mathrm{op}})$ and $C^b(A' \otimes_{\mathcal{O}} (A')^{\mathrm{op}})$. Since the terms of X are finitely generated and $\mathcal{O}$-free, we have that $(X')^\vee \cong (X^\vee)'$. Hence we may assume that $\epsilon_{X'} = \epsilon'_X$ and $\epsilon_{(X')^\vee} = \epsilon'_{X^\vee}$. Now the hypothesis implies that $X' \otimes_{B'} -$ and $(X')^\vee \otimes_{A'} -$ define a pair of inverse equivalences between $K^b(A' \otimes_{\mathcal{O}'} (A')^{\mathrm{op}})$ and $K^b(B' \otimes_{\mathcal{O}'} (A')^{\mathrm{op}})$. Thus $\epsilon_{X'} : X' \otimes_{B'} (X')^\vee \to A'$ is an isomorphism in $K^b(A' \otimes_{\mathcal{O}} (A')^{\mathrm{op}})$ (see for instance [22, Chapter 4, §2, Prop. 4]). Since A' is concentrated in a single degree it follows that ϵ_X is split surjective in $C^b(A' \otimes_{\mathcal{O}'} (A')^{\mathrm{op}})$ and $X' \otimes_{B'} (X')^\vee = A' \oplus \mathrm{Ker}(\epsilon_{X'})$ in $C^b(A' \otimes_{\mathcal{O}'} (A')^{\mathrm{op}})$. Since we also have $X' \otimes_{B'} (X')^\vee = A' \oplus Y$ in $C^b(A' \otimes_{\mathcal{O}'} (A')^{\mathrm{op}})$ with Y contractible, by the Krull-Schmidt property of $C^b(A' \otimes_{\mathcal{O}'} (A')^{\mathrm{op}})$ we have that $\mathrm{Ker}(\epsilon_{X'})$ is contractible. By Lemma 4.2 we have that $\mathrm{Ker}(\epsilon_{X'}) = \mathrm{Ker}(\epsilon'_X) = (\mathrm{Ker}(\epsilon_X))'$. Hence by Lemma 4.4 we have that $\mathrm{Ker}(\epsilon_X)$ is contractible as a complex of (A, A)-bimodules. Similarly by Lemma 4.2 we have that ϵ_X is surjective and by Lemma 4.3 that ϵ_X is split (note that since A

is concentrated in a single degree, namely zero, the split surjectivity of ϵ_X as map of complexes is equivalent to the split surjectivity of the degree 0-component of ϵ_X). Thus, we have that $X \otimes_B X^\vee = A \oplus \mathrm{Ker}(\epsilon_X)$ as complexes of (A, A)-bimodules and $\mathrm{Ker}(\epsilon_X)$ is contractible. Arguing similarly for $X^\vee \otimes_A X$ proves (a).

The proof of (b) proceeds along the same lines as that of (a), the contractibility arguments are replaced by the fact that if U is a finitely generated (A, A)-bimodule, then U is projective if U' is a projective (A', A')-bimodule (Lemma 4.4). The proof of (c) is a special case of the proof of (b).

Statement (d) is a consequence of a version, due to Grothendieck, of the Noether-Deuring Theorem for the base rings under consideration. More precisely, if $[M]-[N]$ and its dual induce a virtual Morita equivalence, then

$$[M \otimes_B M^\vee] + [N \otimes_B N^\vee] - [M \otimes_B N^\vee] - [N \otimes_B M^\vee] = [A] \ ,$$

which is equivalent to the existence of an (A, A)-bimodule isomorphism

$$M \otimes_B M^\vee \oplus N \otimes_B N^\vee \cong A \oplus M \otimes_B N^\vee \oplus N \otimes_B M^\vee \ .$$

By [9, Proposition (2.5.8) (i)] such an isomorphism exists if and only if there exists an analogous isomorphism for the corresponding (A', A')-bimodules, whence (d). □

Remark 4.6. While the categorical equivalences in (a), (b), (c) in the theorem above induced by a bimodule or a complex of bimodules have the property that their inverses are automatically induced by the dual of that bimodule or complex, this is not true for virtual Morita equivalences, whence the extra hypothesis in (d). For instance, if A is a split semisimple K-algebra with m isomorphism classes of simple modules, then any matrix $(a_{i,j})$ in $\mathrm{SL}_m(\mathbb{Z})$ with inverse $(b_{i,j})$ yields a virtual self Morita equivalence of A of the form $\sum_{i,j} a_{i,j}[S_i \otimes_k S_j^\vee]$ with inverse $\sum_{i,j} b_{i,j}[S_i \otimes_k S_j^\vee]$, where $\{S_i\}$ is a set of representatives of the isomorphism classes of simple A-modules, and where the indices i, j run from 1 to m.

5. Descent and relative projectivity

Let $(K, \mathcal{O}, k) \subseteq (K', \mathcal{O}', k')$ be an extension of p-modular systems (see Definition 3.1). Let G be a finite group and P a subgroup of G. An $\mathcal{O}G$-module U is called *relatively P-projective*, if U is isomorphic to a direct summand of $\mathrm{Ind}_P^G(V) = \mathcal{O}G \otimes_{\mathcal{O}P} V$ for some $\mathcal{O}P$-module V, where $\mathcal{O}G$ is regarded as an $\mathcal{O}G$-$\mathcal{O}P$-bimodule. Dually, U is *relatively P-injective*, if U is isomorphic to a direct summand of $\mathrm{Hom}_{\mathcal{O}P}(\mathcal{O}G, V)$ for some $\mathcal{O}P$-module V, where $\mathcal{O}G$ is regarded as an $\mathcal{O}P$-$\mathcal{O}G$-bimodule. It is well-known that because $\mathcal{O}G$ is symmetric, the notions of relative projectivity and relative injectivity coincide. Any $\mathcal{O}G$-module is relatively $\mathcal{O}P$-projective, where P is a Sylow p-subgroup of G. Following Green [7], a *vertex of a finitely generated indecomposable $\mathcal{O}G$-module U* is a minimal p-subgroup P of G such that U is relatively P-projective. In that case, U is isomorphic to a direct summand of $\mathrm{Ind}_P^G(V)$ for some finitely generated indecomposable $\mathcal{O}P$-module V, called an *$\mathcal{O}P$-source of V*, and then V is isomorphic to a direct summand of $\mathrm{Res}_P^G(U)$. If P is clear from the context, V is just called a source of U. The vertex-source pairs (P, V) of U are unique up to G-conjugacy. See e. g. [34, §18] for details.

Lemma 5.1 ([6, (III.4.14)]). *Suppose that $\mathcal{O}'$ is finitely generated as an $\mathcal{O}$-module. Let G be a finite group and U a finitely generated $\mathcal{O}$-free indecomposable $\mathcal{O}G$-module. Let P be a vertex of U. Then P is a vertex of every indecomposable direct summand of the $\mathcal{O}'G$-module $\mathcal{O}' \otimes_{\mathcal{O}} U$.*

Lemma 5.2. *Suppose that $\mathcal{O}'$ is finitely generated as an $\mathcal{O}$-module. Let G be a finite group and U a finitely generated $\mathcal{O}$-free indecomposable $\mathcal{O}G$-module. Let P be a vertex of U. Let V be an indecomposable direct summand of the $\mathcal{O}'G$-module $U' = \mathcal{O}' \otimes_{\mathcal{O}} U$ and let Y be an $\mathcal{O}'P$-source of V. Suppose that $Y \cong \mathcal{O}' \otimes_{\mathcal{O}} X$ for some $\mathcal{O}P$-module X. Then X is an $\mathcal{O}P$-source of U, and every indecomposable direct summand of U' has Y as a source. In particular, U has a trivial source if and only if every indecomposable direct summand of U' has a trivial source.*

Proof. This is basically a special case of the Noether-Deuring Theorem; we sketch the argument. Since $\mathcal{O}'$ is finitely generated as an $\mathcal{O}$-module, we have $\mathcal{O}' \cong \mathcal{O}^d$ for some positive integer d. Thus restricting U' to $\mathcal{O}G$ yields an $\mathcal{O}G$-isomorphism $U' \cong U^d$, and hence, as an $\mathcal{O}G$-module, V' is isomorphic to U^c for some positive integer c, by the Krull-Schmidt Theorem. Similarly, we have an $\mathcal{O}P$-isomorphism $Y \cong X^d$. Since Y is isomorphic to a direct summand of $\mathrm{Res}^G_P(V)$, it follows again from the Krull-Schmidt Theorem that X is isomorphic to a direct summand of $\mathrm{Res}^G_P(U)$. By Lemma 5.1, P is a vertex of X and of U, and therefore X is a source of U. Since U is isomorphic to a direct summand of $\mathrm{Ind}^G_P(X)$, it follows that U' is isomorphic to a direct summand of $\mathrm{Ind}^G_P(Y)$. This implies that every indecomposable summand of U' has Y as a source. The last statement follows from the special case where $Y = \mathcal{O}'$. □

We use the following concepts and results from Knörr [10] and Thévenaz [33]. Let G be a finite group, P a p-subgroup of G and U a finitely generated $\mathcal{O}G$-module. A *relative P-projective presentation of U* is a pair (Y, π) consisting of a relatively P-projective $\mathcal{O}G$-module Y and a surjective $\mathcal{O}G$-homomorphism $\pi : Y \to U$ whose restriction to $\mathcal{O}P$ is split surjective. Such a presentation is called a *relatively P-projective cover* if in addition $\ker(\pi)$ has no nonzero relatively P-projective direct summand; by [33, Cor. (1.9)] this is equivalent to π being *essential*; that is, any endomorphism β of Y satisfying $\pi = \pi \circ \beta$ is an automorphism of Y. The results in [10] and [33] imply that U has a relative projective resolution which is unique up to isomorphism and which is additive in U. Moreover, if U is indecomposable and not relatively P-projective, and if (Y, π) is a relatively P-projective cover of U, then $\ker(\pi)$ is indecomposable and not relatively P-projective. These results, together with Lemma 5.1, imply immediately the following.

Lemma 5.3. *Suppose that $\mathcal{O}'$ is finitely generated as an $\mathcal{O}$-module. Let G be a finite group and R a p-subgroup of G. Let U be an $\mathcal{O}$-free $\mathcal{O}G$-module which has no nonzero relatively R-projective direct summand. Let (Y, π) be a relatively R-projective cover of U. Then the $\mathcal{O}'G$-module $U' = \mathcal{O}' \otimes_{\mathcal{O}} U$ has no nonzero relatively R-projective direct summand and $(Y', \pi') = (\mathcal{O}' \otimes_{\mathcal{O}} Y, \mathrm{Id}_{\mathcal{O}'} \otimes \pi)$ is a relatively R-projective cover of U'.*

6. Descent and Galois automorphisms

Let $(K, \mathcal{O}, k) \subseteq (K', \mathcal{O}', k')$ be an extension of p-modular systems (see Definition 3.1). The following Lemma, due to Reiner, makes use of the fact that finite fields have trivial Schur indices.

Lemma 6.1 ([29, Theorem 3], [4, (30.33)]). *Suppose that the field k is finite. Let G be a finite group and U a finitely generated $\mathcal{O}$-free indecomposable $\mathcal{O}G$-module. Then the indecomposable direct summands in a decomposition of $\mathcal{O}' \otimes_{\mathcal{O}} U$ as an $\mathcal{O}'G$-module are pairwise nonisomorphic.*

Denote by Γ the automorphism group of the field extension k'/k. Let A be a finite-dimensional k-algebra, set $A' = k' \otimes_k A$, and let U' be an A'-module. We say that U' is *Γ-stable*, if $U' \cong {}^{\sigma}U'$ as A'-modules, for all $\sigma \in \Gamma$. If $U' \cong k' \otimes_k U$ for some A-module, then U' is Γ-stable. Indeed, the map sending $\lambda \otimes u$ to $\sigma^{-1}(\lambda) \otimes u$ is an isomorphism $k' \otimes_k U \cong {}^{\sigma}(k' \otimes_k U)$, where $\sigma \in \Gamma$, $u \in U$, and $\lambda \in k'$. The following is well-known.

Lemma 6.2. *Suppose that the fields k' and k are finite. Let A be a finite-dimensional k-algebra. Set $A' = k' \otimes_k A$. Suppose that the semisimple quotient $A/J(A)$ is separable. Let Γ be the Galois group of the extension k'/k.*

(a) *Let S be a simple A-module. Then the A'-module $S' = k' \otimes_k S$ is semisimple, isomorphic to direct sum of pairwise nonisomorphic Galois conjugates of a simple A'-module T.*
(b) *Let S' be a semisimple A'-module. There exists a semisimple A-module S satisfying $S' \cong k' \otimes_k S$ if and only if S' is Γ-stable.*
(c) *Let Y' be a finitely generated projective A'-module. There exists a projective A-module Y satisfying $Y' \cong k' \otimes_k Y$ if and only if Y' is Γ-stable.*

For the remainder of this section, assume that k, k' are finite and that $\mathcal{O}$, $\mathcal{O}'$ are absolutely unramified. Set $d = [k' : k]$. Then $\mathcal{O}'$ is free of rank d as an $\mathcal{O}$-module. Let $\sigma : k' \to k'$ be a generator of $\mathrm{Gal}(k'/k)$. Denote by the same letter $\sigma : \mathcal{O}' \to \mathcal{O}'$ the unique ring automorphism of $\mathcal{O}'$ lifting σ.

Let A be an $\mathcal{O}$-algebra which is free of finite rank as an $\mathcal{O}$-module. Set $A' = \mathcal{O}' \otimes_{\mathcal{O}} A$. Let $\tau : \text{mod-}A' \to \text{mod-}A'$ be the functor which sends an A'-module U to the A'-module $\tau(U) := \oplus_{0 \le i \le d-1} {}^{\sigma^i}U$ and a morphism $f : U \to V$ of A'-modules to the morphism $\tau(f) := (f, \dots, f)$. Let $\delta : \text{mod-}A' \to \text{mod-}A$ be the functor which sends an A'-module U to the A-submodule $\delta(U)$ of $\tau(U)$ defined by $\delta(U) = \{(x, \dots, x) : x \in U\}$ and which sends the morphism $f : U \to V$ of A'-modules to the morphism $\delta(f)$ defined to be the restriction of $\tau(f)$ to $\delta(U)$. Finally let $\epsilon : \text{mod-}A \to \text{mod-}A'$ be the extension functor $\mathcal{O}' \otimes_{\mathcal{O}} -$. The functors ϵ, δ and τ are exact functors of $\mathcal{O}$-linear categories where we regard mod-A' as an $\mathcal{O}$-linear category by restriction of scalars.

Proposition 6.3. *With the notation and assumptions above, the functors $\epsilon \circ \delta$ and τ are naturally isomorphic.*

Proof. Let $\alpha \in \mathcal{O}'$ be such that $k' = k[\bar{\alpha}]$ where $\bar{\alpha} = \alpha + J(\mathcal{O}') \in k'$. Then $\{\bar{\alpha}^i : 0 \le i \le d-1\}$ is a k-basis of k'. Since the extension $\mathcal{O} \subseteq \mathcal{O}'$ is unramified $J(\mathcal{O}') = J(\mathcal{O})\mathcal{O}'$. Hence by Nakayama's lemma $\{\alpha^i : 0 \le i \le d-1\}$ is an $\mathcal{O}$-basis of $\mathcal{O}'$. Let U be a finitely generated A'-module and let

$$\eta_U : \epsilon\delta(U) \to \tau(U)$$

be the unique $\mathcal{O}'$-linear extension of the inclusion $\delta(U) \subseteq \tau(U)$. Then $\eta = (\eta_U)$ is a natural transformation from $\epsilon\delta$ to τ. We will show that η is an isomorphism. It suffices to show that this is an $\mathcal{O}$-linear isomorphism; that is, we may assume that $A = \mathcal{O}$ and $A' = \mathcal{O}'$. We show first that η_U is an isomorphism for $U = \mathcal{O}'$. Since

$\{\alpha^i \ : \ 0 \leq i \leq d-1\}$ is an $\mathcal{O}$-basis of $\mathcal{O}'$, it follows that $\{(\alpha^i, \alpha^i, ..., \alpha^i) \ : \ 0 \leq i \leq d-1\}$ is an $\mathcal{O}$-basis of $\delta(\mathcal{O}')$. We claim that this set is an $\mathcal{O}'$-basis of $\tau(\mathcal{O}')$. Since the cardinality of this set is equal to d, which is also the $\mathcal{O}'$-rank of $\tau(\mathcal{O}')$, it suffices to show that the image of this set in $k' \otimes_{\mathcal{O}'} \tau(\mathcal{O}')$ is linearly independent. For notational convenience, assume temporarily that $\mathcal{O}' = k'$. Suppose that

$$\sum_{i=0}^{d-1} \lambda_i(\alpha^i, \alpha^i, .., \alpha^i) = 0$$

for some coefficients $\lambda_i \in k'$. The scalar λ_i acts on the j-th coordinate as multiplication by $\sigma^{-j}(\lambda_i)$, so this is equivalent to the d equations

$$\sum_{i=0}^{d-1} \sigma^{-j}(\lambda_i)\alpha^i = 0$$

for $0 \leq j \leq d-1$. Applying σ^j to the corresponding equation implies that this is equivalent to

$$\sum_{i=0}^{d-1} \lambda_i \sigma^j(\alpha)^i = 0$$

for $0 \leq j \leq d-1$. Note that the $\sigma^j(\alpha)$, with $0 \leq j \leq d-1$, are pairwise different, and hence the Vandermonde matrix $(\sigma^j(\alpha)^i)$ has nonzero determinant. Thus all coefficients λ_i are 0.

Reverting to the ring $\mathcal{O}'$ as before, this shows that η_U is an isomorphism if $U = \mathcal{O}'$. Since η_U is additive in U, it follows that η_U is an isomorphism whenever U is free of finite rank over $\mathcal{O}'$. Let U be an arbitrary finitely generated $\mathcal{O}'$-module and let $Q_1 \to Q_0 \to U \to 0$ be a free presentation of U. By the naturality of η we obtain the following commutative diagram

$$\begin{array}{ccccccc}
\epsilon\delta(Q_1) & \longrightarrow & \epsilon\delta(Q_0) & \longrightarrow & \epsilon\delta(U) & \longrightarrow & 0 \\
\downarrow{\scriptstyle \eta_{Q_1}} & & \downarrow{\scriptstyle \eta_{Q_0}} & & \downarrow{\scriptstyle \eta_U} & & \\
\tau(Q_1) & \longrightarrow & \tau(Q_0) & \longrightarrow & \tau(U) & \longrightarrow & 0
\end{array}$$

By the exactness of δ, τ and ϵ, the horizontal rows are exact. Since Q_1 and Q_0 are $\mathcal{O}'$-free, the vertical maps η_{Q_1} and η_{Q_0} are isomorphisms. It follows that η_U is an isomorphism. □

Let G be a finite group. For $a = \sum_{g \in G} \alpha_g g$ an element of $k'G$ or of $\mathcal{O}'G$, denote by $k[a]$ the smallest subfield of k' containing k and (the images in k' of) all coefficients α_g, $g \in G$.

Lemma 6.4. *Let G be finite group. Let b' be a block of $k'G$ and b a block of kG such that $bb' \neq 0$. Suppose that $k' = k[b']$. Then the extension k'/k is finite. Set $d = [k' : k]$ and let σ be a generator of* $\mathrm{Gal}(k'/k)$. *Then*

$$b = \sum_{0 \leq i \leq d-1} \sigma^i(b')$$

is the block decomposition of b in $k'G$.

Proof. The block idempotent b' of $k'G$ has coefficients contained in a finite subfield of k' (because G has a finite splitting field) and hence $k[b']/k$ is a finite extension. For any i, $0 \leq i \leq d-1$, $\sigma^i(b')$ is a block of $k'G$ satisfying $\sigma^i(b')b = \sigma^i(b'b) \neq 0$. Hence we only need to show that $\sigma^i(b') \neq b'$ for any i, $0 \leq i \leq d-1$. This follows from the fact that $k' = k[b']$ is a finite Galois extension with Galois group $\langle\sigma\rangle$. □

Theorem 6.5. *Suppose that k and k' are finite and that $\mathcal{O}$ and $\mathcal{O}'$ are absolutely unramified. Let G, H be finite groups, b a block of $\mathcal{O}G$ and c a block of $\mathcal{O}H$. Let b' be a block of $\mathcal{O}'G$ satisfying $bb' \neq 0$ and let c' be a block of $\mathcal{O}'H$ satisfying $cc' \neq 0$. Suppose that $k' = k[b'] = k[c']$. The following hold.*

(a) If $\mathcal{O}'Gb'$ and $\mathcal{O}'Hc'$ are Morita equivalent via an $(\mathcal{O}'Gb', \mathcal{O}'Hc')$-bimodule M', then $\mathcal{O}Gb$ and $\mathcal{O}Hc$ are Morita equivalent via an $(\mathcal{O}Gb, \mathcal{O}Hc)$-bimodul M, such that M' is isomorphic to a direct summand of $\mathcal{O}' \otimes_{\mathcal{O}} M$. In particular, if M' has a trivial source, then M has a trivial source.

(b) If there is a Rickard complex X' of $(\mathcal{O}'Gb', \mathcal{O}'Hc')$-bimodules, then there is a Rickard complex X of $(\mathcal{O}Gb, \mathcal{O}Hc)$-bimodules such that X' is isomorphic to a direct summand of $\mathcal{O}' \otimes_{\mathcal{O}} X$. In particular, if b' and c' are splendidly Rickard equivalent, then b and c are splendidly Rickard equivalent.

(c) If there is a virtual Morita equivalence (resp. a p-permutation equivalence) between $\mathcal{O}'Gb'$ and $\mathcal{O}'Hc'$, then there is a virtual Morita equivalence (resp. a p-permutation equivalence) between $\mathcal{O}Gb$ and $\mathcal{O}Hc$.

Proof. Let σ be a generator of $\mathrm{Gal}(k'/k)$. Since $k' = k[b']$ and $k' = k[c']$, it follows from 6.4 that the $\sigma^i(b')$, $0 \leq i \leq d-1$, are pairwise different blocks of $\mathcal{O}'G$ whose sum is b, and the analogous statement holds for $\mathcal{O}'Hc'$ and c. Suppose that $\mathcal{O}'Gb'$ and $\mathcal{O}'Hc'$ are Morita equivalent via an $(\mathcal{O}'Gb', \mathcal{O}'Hc')$-bimodule M'. Then $\mathcal{O}'G\sigma^i(b')$ and $\mathcal{O}'H\sigma^i(c')$ are Morita equivalent via the bimodule ${}^{\sigma^i}M'$. Thus the direct sum $\tau(M') = \oplus_{i=0}^{d-1} {}^{\sigma^i}M'$ induces a Morita equivalence between $\mathcal{O}'Gb$ and $\mathcal{O}'Hc$. By Proposition 6.3, the above direct sum is isomorphic to $\mathcal{O}' \otimes_{\mathcal{O}} M$ for some $(\mathcal{O}Gb, \mathcal{O}Hc)$-bimodule M. By Proposition 4.5, M induces a Morita equivalence. It follows from Lemma 5.2 that if M' has a trivial source, then M has a trivial source. This proves (a). Obvious variations of this argument prove (b) and (c). □

7. On cyclic blocks

We prove in this section the Theorems 1.10 and 1.12. Modules in this section are finitely generated. Let $(K, \mathcal{O}, k) \subseteq (K', \mathcal{O}', k')$ be an extension of p-modular systems as in Definition 3.1.

Let G be a finite group and b a block of $\mathcal{O}'G$ with a nontrivial cyclic defect group P. If k' is a splitting field for all subgroups of G, then Rouquier constructed a 2-sided splendid tilting complex X' of $(\mathcal{O}'Gb, \mathcal{O}'N_G(P)e)$-bimodules, where e is the Brauer correspondent of b. The hypotheses in [31] also require K' to be large enough, but it is easy to see that Rouquier's construction works with $\mathcal{O}'$ absolutely unramified. In order to prove Theorem 1.10, we need to show that Rouquier's complex is defined over the subring $\mathcal{O}$ so long as the block idempotent b is contained in $\mathcal{O}G$. We review Rouquier's construction and other facts on cyclic blocks as we go along. We start with some basic observations regarding automorphisms of Brauer trees.

Remark 7.1. Let G be a finite group and b a block of $\mathcal{O}G$ with a nontrivial cyclic defect group P. Suppose in addition that $\mathcal{O}$ contains a primitive $|G|$-th root of unity. Any ring automorphism σ of $\mathcal{O}Gb$ permutes the sets of isomorphism classes of simple modules, of projective indecomposable modules, and the set of ordinary irreducible characters of $\mathcal{O}Gb$. Thus σ induces an automorphism of the Brauer tree of b. If $|P| = 2$, then $\mathcal{O}Gb$ is Morita equivalent to $\mathcal{O}C_2$, and the Brauer tree has a single edge and no exceptional vertex. Thus there are two automorphisms of this Brauer tree - the identity, and the automorphism exchanging the two vertices, and both are induced by ring automorphisms (the automorphism of $\mathcal{O}C_2$ sending the nontrivial group element t of C_2 to $-t$ in $\mathcal{O}C_2$ exchanges the two vertices of the tree). If $|P| \geq 3$, then the Brauer tree has an exceptional vertex or at least two edges. In both cases, an easy combinatorial argument shows that an automorphism of the Brauer tree is uniquely determined by its effect on the edges of the tree. It follows that the automorphism of the Brauer tree induced by a ring automorphism σ of $\mathcal{O}Gb$ is already determined by the induced ring automorphism $\bar{\sigma}$ of $kG\bar{b}$, where $\bar{b}$ is the image of b in kG. This is the reason for why the following Lemma, which is an immediate consequence of (the proof of) [16, Proposition 4.5, Remark 4.6], is formulated over k rather than $\mathcal{O}$.

Lemma 7.2. *Let G be a finite group and b a block of kG with a nontrivial cyclic defect group P of order at least 3. Suppose that k is a splitting field for the subgroups of G. Let γ be a ring automorphism of kGb. Then γ induces an automorphism of the Brauer tree of b which fixes at least one vertex.*

Proof. The statement is trivial if the Brauer tree has an exceptional vertex (which is necessarily fixed). Suppose that the Brauer tree does not have an exceptional vertex. Then $|P| = p \geq 3$, and the tree has $p-1$ edges; note that $p-1$ is even. An easy argument shows that any tree automorphism fixes an edge or a vertex. In the latter case, we are done, so assume that it fixes an edge, which we label by i. Removing this edge from the Brauer tree yields two disjoint trees. If the two disjoint trees are exchanged by the Brauer tree automorphism, then they have the same number t of edges. But then the number of edges of the Brauer tree itself is $2t+1$, which is odd, a contradiction. Thus the Brauer tree automorphism stabilises the two trees obtained from removing the edge i. But then it stabilises the two vertices connected by i, whence the result. □

Proof of Theorem 1.10. Since any block of a finite group algebra has a finite splitting field, we may assume that k and k' are finite.

Rouquier's splendid Rickard complex is constructed inductively, separating the cases according to whether G has a nontrivial normal p-subgroup or not. The construction of this splendid equivalence is played back to [31, Theorem 10.3]. It suffices therefore to show that the complexes arising in that theorem are defined over $\mathcal{O}$. We start with the case $O_p(G) = \{1\}$. Since p-permutation modules of finite groups lift uniquely, up to isomorphism, from k to $\mathcal{O}$, it is easy to see that we may replace $\mathcal{O}$ and $\mathcal{O}'$ by k and k', respectively. (This simplifies notation, but one could as well write the proof over $\mathcal{O}$ and $\mathcal{O}'$, if desired.)

Let H be the normaliser in G of the unique subgroup Z of order p of P, and let c be the block of $k'H$ corresponding to b via the Brauer correspondence. Since any block idempotent of kH is contained in $kC_G(Z)$, we have $\mathrm{Br}_Z(b) = c$. Since $b \in kG$, it follows that also $c \in kH$.

Set $A = kGb$, $A' = k'Gb$, $B = kHc$ and $B' = k'Hc$. By [31, Theorem 10.3], there is a splendid Rickard complex X' of (A', B')-modules of the form

$$\cdots \longrightarrow 0 \longrightarrow N' \xrightarrow{\pi'} M' \longrightarrow 0 \longrightarrow \cdots$$

for some projective (A', B')-bimodule N' and some bimodule homomorphism π' such that (N', π') is a direct summand of a projective cover of M'. The algebra B' is Morita equivalent to the serial algebra $k'(P \rtimes E)$, where E is the inertial quotient of b. That is, the Brauer tree of B' is a star with $|E|$ edges, and exceptional vertex in the center, if any.

By Proposition 4.5, in order to prove Theorem 1.10, it suffices to prove that there is a complex of A-B-bimodules X satisfying $k' \otimes_k X \cong X'$.

The (A, B)-bimodule $bkGc$ has, up to isomorphism, a unique nonprojective indecomposable bimodule summand M. This bimodule and its dual induce a stable equivalence of Morita type between A and B (this goes back to Green [8]; see [17] for a proof using this terminology). As a $k(G \times H)$-module, the diagonal subgroup ΔP is a vertex of M. The analogous properties hold for $A' = k'Gb$ and $B' = k'Hc$. Lemma 5.1 implies that if M' is the unique (up to isomorphism) nonprojective bimodule summand of $bk'Gc$, then $M' \cong k' \otimes_k M$.

If $|P| = 2$, then M', hence M, induces a Morita equivalence, and so we are done in that case. Suppose now that $|P| \geq 3$.

The bimodule M' is the right term in Rouquier's complex. For the left term, we need to show that $N' \cong k' \otimes_k N$ for some (by 4.4 necessarily projective) (A, B)-bimodule N, and that the map π' is obtained from applying $k' \otimes_k -$ to some map $\pi : N \to M$. To that end, we need to show that N' is Γ-stable, where as before $\Gamma = \mathrm{Gal}(k'/k)$. This will follow from Rouquier's description of N', which we review briefly.

For that purpose, we need some classical facts on blocks with cyclic defect groups which have their origins in work of Brauer, Dade, and Green. We follow the presentation given in [15], [17]. Denote by I a set of representatives of the conjugacy classes of primitive idempotents in A', and by J a set of representatives of the conjugacy classes of primitive idempotents in B'. Set $S_i = A'i/J(A')i$ for all $i \in I$ and $T_j = B'j/J(B')j$ for all $j \in J'$.

Using general properties of stable equivalences of Morita type from [16] and well-known facts on cyclic blocks, it follows that the B'-modules $\mathcal{F}(S_i) = M'^* \otimes_A S_i$ and the A-modules $\mathcal{G}(T_j) = M' \otimes_B T_j$ are indecomposable and uniserial. There are unique bijections $\delta, \gamma : I \to J$ such that $T_{\delta(i)}$ is isomorphic to the unique simple quotient of $\mathcal{F}(S_i)$ and such that $T_{\gamma(i)}$ is isomorphic to the unique simple submodule of $\mathcal{F}(S_i)$. For any $i \in I$ there are unique uniserial submodules U_i and V_i of $A'i$ isomorphic to $\mathcal{G}(T_{\delta(i)})$ and $\mathcal{G}(\Omega(T_{\gamma(i)}))$, respectively. There are unique permutations ρ and σ of I such that the top composition factors of U_i and V_i are isomorphic to $S_{\rho(i)}$ and $S_{\sigma(i)}$, respectively. In particular, $A'\rho(i)$ is a projective cover of $U_i \cong M' \otimes_{B'} T_{\delta(i)}$. Since B' is symmetric, the projective indecomposable right B'-module $\delta(i)B'$ is a projective cover of the simple right B'-module $T^{\vee}_{\delta(i)}$. It follows from the description of projective covers of bimodules in [31, Lemma 10.2.12], that a projective cover of the (A', B')-bimodule M' has the form

$$Z' = \oplus_{i \in I} \; A'\rho(i) \otimes_k \delta(i)B'$$

together with a surjective (A', B')-bimodule homomorphism π' from Z' onto M'.

The permutations ρ and σ determine the Brauer tree as follows. For $i \in I$, denote by i^ρ the $\langle\rho\rangle$-orbit of i in I; use the analogous notation for i^σ. The vertices of the Brauer tree are the $\langle\rho\rangle$-orbits and $\langle\sigma\rangle$-orbits, with exactly one edge labelled i linking i^ρ and i^σ. Denote by v the exceptional vertex with exceptional multiplicity m; if there is no exceptional vertex, we choose for v a Γ-stable vertex (which is possible by 7.2) and set $m = 1$. Note that there is a unique edge $\rho(i)$ which links $i^\rho = \rho(i)^\rho$ and $\rho(i)^\sigma$. Since there is a unique minimal path from v to any other vertex in the Brauer tree, it follows that we have a well-defined notion of distance from v - this is the number of edges of a minimal path from v to any other vertex.

The construction of Rouquier's bimodule complex is based on a partition of I into two subsets. Note that the vertex $i^\rho = \rho(i)^\rho$ is linked to the vertex $\rho(i)^\sigma$ via the edge labelled $\rho(i)$. Thus exactly one of these two vertices is further away from v than the other. We denote by I_0 the set of all $i \in I$ such that the vertex i^ρ of the Brauer tree is further away from the exceptional vertex v than the vertex $\rho(i)^\sigma$. In particular i^ρ is nonexceptional in that case. We set $I_1 = I \setminus I_0$; that is, I_1 consists of all $i \in I$ such that $\rho(i)^\sigma$ is further away from v than i^ρ. In particular, $\rho(i)^\sigma$ is nonexceptional in that case. Then

$$N' = \oplus_{i\in I_1}\ A'\rho(i) \otimes_k \delta(i)B'$$

This is a direct summand of the above projective cover of M', and we denote the restriction of π' again by π'. Since the action of Γ on the Brauer tree fixes v, it follows that the set I_1 is Γ-stable, and hence so is the isomorphism class of N'. It follows from Lemma 6.2 that there is a projective (A, B)-bimodule N such that $N' \cong k' \otimes_k N$. To see that the map π' can also be chosen to be of the form $\mathrm{Id}_{k'} \otimes \pi$ for some bimodule homomorphism $\pi : N \to M$, consider a projective cover $\pi : Z \to M$. Observe that then $k' \otimes_k Z \cong Z'$ yields the projective cover of M' above, and Lemma 6.2 implies that Z has a summand isomorphic to N, so we just need to restrict π to N and the extend scalars to k'.

This shows that A and B are splendidly Rickard equivalent. It remains to show that the complex in [31, Theorem 10.3] is also defined over k in the case where $O_p(G)$ is nontrivial.

Set $R = O_p(G)$ and assume that $R \neq \{1\}$. If $R = P$, there is nothing further to prove; thus we may assume that R is a proper subgroup of P. Let Q be the unique subgroup of P such that $|Q : R| = p$. Changing earlier notation, set $H = N_G(Q)$, and denote by c the block of $k'H$ which is the Brauer correspondent of b. We have $c = \mathrm{Br}_Q(b)$, and hence $c \in kH$. Set $A = kGb$, $A' = k'Gb$, $B = kHc$, and $B' = k'Hc$.

Note that $kG \otimes_{kR} kH \cong \mathrm{Ind}_R^{G\times H}(k)$ as $k(G \times H)$-modules. Thus $A \otimes_{kR} B$, together with the multiplication map $A \otimes_{kR} B \to bkGc$, is a relatively ΔR-projective presentation of $bkGc$, where we regard this bimodule as $k(G \times H)$-module. Thus some bimodule summand of $A \otimes_{kR} B$ yields a relatively ΔR-projective cover of $bkGc$.

Rouquier's splendid Rickard complex of (A', B')-bimodules from [31, Theorem 10.3] is in the present situation a complex X' of the form

$$\cdots \longrightarrow 0 \longrightarrow N' \xrightarrow{\pi'} M' \longrightarrow 0 \longrightarrow \cdots$$

which is a direct summand of the complex

$$\cdots \longrightarrow 0 \longrightarrow A' \otimes_{k'R} B' \xrightarrow{\pi'} bk'Gc \longrightarrow 0 \longrightarrow \cdots$$

where π' is the map induced by multiplication, where M' is the unique (up to isomorphism) indecomposable direct bimodule summand of $bk'Gc$ with vertex ΔP, and where either $N' = \{0\}$ or (N', π') is a relatively ΔR-projective cover of M'. As before, Lemma 5.1 implies that $M' \cong k' \otimes_k M$, where M is the unique indecomposable direct bimodule summand of $bkGc$ with vertex ΔP. If $N' = \{0\}$, then X' is the complex M' concentrated in degree 0, so is trivially of the form $k' \otimes_k X$, where X is the complex M concentrated in degree 0. If $N' \neq \{0\}$, then N' is a relatively ΔR-projective cover of M'. The properties collected in Lemma 5.3 imply that this relative projective cover is isomorphic to one obtained from extending the scalars in a relatively ΔR-projective cover of M, and hence in this case we also get that $X' \cong k' \otimes_k X$ for some complex X. This completes the proof of Theorem 1.10. $\square$

Proof of Theorem 1.12. Denote by $\bar{b}$ the image of b in kG. Write $\bar{b} = \sum_{g \in G} \alpha_g g$ with coefficients $\alpha_g \in k$. By Proposition 4.5 we may assume that $k = \mathbb{F}_p[\bar{b}]$. Since all central idempotents of $\mathcal{O}G$ belong to $W(k)G$, we may assume that $\mathcal{O}$ is absolutely unramified. Let $\tilde{k}$ be a splitting field for G containing k, and let $\bar{b}'$ be a block of $\tilde{k}G$ such that $\bar{b}\bar{b}' \neq 0$. Set $\tilde{\mathcal{O}} = W(\tilde{k})$ and let $\tilde{K}$ be the field of fractions of $\tilde{\mathcal{O}}$. Let $k' = k[\bar{b}'] \subseteq \tilde{k}$, let $\mathcal{O}' = W(k')$ and let K' the field of fractions of $\mathcal{O}'$. Let b' be the block of $\mathcal{O}'G$ lifting $\bar{b}'$. By Lemma 6.4 P is a defect group of b'. Let e' be the block of $\mathcal{O}'N_G(P)$ in Brauer correspondence with b'. Then $\bar{e}'\bar{e} \neq 0$, $k = \mathbb{F}_p[\bar{e}]$ and $k' = k[\bar{e}']$. By Theorem 1.10, applied to the block b' and the extension of p-modular systems $(K', \mathcal{O}', k') \subseteq (\tilde{K}, \tilde{\mathcal{O}}, \tilde{k})$ there is a splendid Rickard complex X' of $(\mathcal{O}'Gb', \mathcal{O}'N_G(P)e')$-bimodules. It follows from Theorem 6.5 that there is a splendid Rickard complex X of $(\mathcal{O}Gb, \mathcal{O}N_G(P)e)$-bimodules, whence the result. $\square$

Remark 7.3. Zimmermann showed in [37] that Rouquier's complex can be extended to Green orders, a concept due to Roggenkamp [30]. This might provide alternative proofs of the Theorems 1.10 and 1.12. In order to apply Zimmermann's result one would need to show that $\mathcal{O}Gb$ and $\mathcal{O}N_G(P)e$ are Green orders whose underlying structure data, as required in [37], coincide.

8. Descent for Morita equivalences with endopermutation source

We briefly recall some notation and general facts about endopermutation modules over p-groups, which we will use without further reference. Let $(K, \mathcal{O}, k)$ be a p-modular system, and let P be a finite p-group. By an endopermutation $\mathcal{O}P$-module we will always mean an endopermutation $\mathcal{O}$-lattice.

By results of Dade [5], the tensor product of two indecomposable endopermutation $\mathcal{O}P$-modules (respectively kP-modules) with vertex P has a unique indecomposable direct summand with vertex P; this induces an abelian group structure on the set of isomorphism classes of indecomposable endopermutation $\mathcal{O}P$-modules (respectively kP-modules) with vertex P. The resulting group is denoted $D_{\mathcal{O}}(P)$ (respectively $D_k(P)$), called the *Dade group of P over $\mathcal{O}$ (respectively k).* Let V be an endopermutation $\mathcal{O}P$-module (respectively kP-module) having an indecomposable direct summand with vertex P. For any subgroup Q of P, the indecomposable direct summands of $\mathrm{Res}^P_Q(V)$ with vertex P are all isomorphic, and we denote by V_Q an indecomposable direct summand of $\mathrm{Res}^P_Q(V)$ with vertex Q. If V is an indecomposable endopermutation $\mathcal{O}P$-module, then $\bar{V} = k \otimes_{\mathcal{O}} V$ is an indecomposable endopermutation kP-module with the same vertices.

Let $\mathcal{F}$ be a saturated fusion system on P. Following the terminology in [21, 3.3] we say that the class $[V]$ of an endopermutation $\mathcal{O}P$-module (respectively kP-module) V in the Dade group $D_{\mathcal{O}}(P)$ is *$\mathcal{F}$-stable* if for every isomorphism $\varphi : Q \to R$ in $\mathcal{F}$ between two subgroups Q, R of P we have $V_Q \cong {}_{\varphi}V_R$. Here ${}_{\varphi}V_R$ is the $\mathcal{O}Q$-module (respectively kQ-module) which is equal to V_R as an $\mathcal{O}$-module (respectively k-module), with $u \in Q$ acting as $\varphi(u)$ on V_R. The $\mathcal{F}$-stable classes of indecomposable endopermutation $\mathcal{O}P$-modules (respectively kP-modules) with vertex P form a subgroup of $D_{\mathcal{O}}(P)$ (respectively $D_k(P)$), denoted $D_{\mathcal{O}}(P, \mathcal{F})$ (respectively $D_k(P, \mathcal{F})$). The $\mathcal{F}$-stability of the class $[V]$ is a slightly weaker condition than the $\mathcal{F}$-stability of the actual module V. More precisely, an $\mathcal{O}P$-module V is $\mathcal{F}$-stable if for every isomorphism $\varphi : Q \to R$ in $\mathcal{F}$ between two subgroups Q, R of P we have $\mathrm{Res}^P_Q(V) \cong {}_{\varphi}\mathrm{Res}^P_R(V)$. If V is an $\mathcal{F}$-stable endopermutation $\mathcal{O}P$-module having an indecomposable direct summand V_P with vertex P, then the class $[V_P]$ in $D_{\mathcal{O}}(P)$ is clearly $\mathcal{F}$-stable. We will need the following result.

Proposition 8.1 ([21, Proposition 3.7])**.** *Let P be a finite p-group, $\mathcal{F}$ a saturated fusion system on P and V and indecomposable endopermutation $\mathcal{O}P$-module with vertex P such that the class of $[V]$ in $D_{\mathcal{O}}(P)$ is $\mathcal{F}$-stable. Then there exists an $\mathcal{F}$-stable endopermutation $\mathcal{O}P$-module V' having a direct summand isomorphic to V. Moreover, we may choose V' to have $\mathcal{O}$-rank prime to p, and the analogous result holds with k instead of $\mathcal{O}$.*

The statement on the rank is not made explicitly in [21, Proposition 3.7], and this Proposition is stated there only over k, but the slightly stronger version above follows immediately from the construction of V' in the proof of that Proposition in [21].

For P, Q finite p-groups, $\mathcal{F}$ a fusion system on P and $\varphi : P \to Q$ a group isomorphism, we set $\Delta\varphi = \{(u, \varphi(u)) \mid u \in P\}$, and we denote by ${}^{\varphi}\mathcal{F}$ the fusion system on Q induced by $\mathcal{F}$ via the isomorphism φ. That is, for R, S subgroups of P, we have $\mathrm{Hom}_{{}^{\varphi}\mathcal{F}}(\varphi(R), \varphi(S)) = \varphi \circ \mathrm{Hom}_{\mathcal{F}}(R, S) \circ \varphi^{-1}$, where we use the same notation φ, φ^{-1} for their restrictions to S, $\varphi(R)$, respectively.

The proof of Theorem 1.13 requires the following Lemma, due to Puig, which summarises some of the essential properties of stable equivalences of Morita type with endopermutation source. We assume in the remainder of this section that k is large enough for all finite groups and their subgroups, so that fusion systems of blocks are saturated.

Lemma 8.2 ([27, 7.6])**.** *Let G, H be finite groups, b, c blocks of $\mathcal{O}G$, $\mathcal{O}H$ with defect groups P, Q, respectively, and let $i \in (\mathcal{O}Gb)^P$ and $j \in (\mathcal{O}Hc)^P$ be source idempotents. Denote by $\mathcal{F}$ the fusion system on P of b determined by i, and denote by $\mathcal{G}$ the fusion system on Q determined by j. Let M be an indecomposable $(\mathcal{O}Gb, \mathcal{O}Hc)$-bimodule inducing a stable equivalence of Morita type with endopermutation source.*

Then there is an isomorphism $\varphi : P \to Q$ and an indecomposable endopermutation $\Delta\varphi$-module V such that M is isomorphic to a direct summand of

$$\mathcal{O}Gi \otimes_{\mathcal{O}P} \mathrm{Ind}^{P\times Q}_{\Delta\varphi}(V) \otimes_{\mathcal{O}Q} j\mathcal{O}H$$

as an $(\mathcal{O}Gb, \mathcal{O}Hc)$-bimodule. For any such φ and V, the following hold.

(a) *$\Delta\varphi$ is a vertex of M and V is a source of M.*

(b) *We have* ${}^{\varphi}\mathcal{F} = \mathcal{G}$, *and when regarded as an* $\mathcal{O}P$-*module via the canonical isomorphism* $P \cong \Delta\varphi$, *the class* $[V]$ *of the endopermutation module* V *is* $\mathcal{F}$-*stable.*

See also [20, 9.11.2] for a proof of the above Lemma.

Lemma 8.3. *Let* G, H *be finite groups,* b, c *blocks of* $\mathcal{O}G$, $\mathcal{O}H$, *respectively, with a common defect group* P, *and let* $i \in (\mathcal{O}Gb)^{\Delta P}$ *and* $j \in (\mathcal{O}Hc)^{\Delta P}$ *be source idempotents. Suppose that* i *and* j *determine the same fusion system* $\mathcal{F}$ *on* P. *Let* V *be an indecomposable endopermutation* $\mathcal{O}P$-*module with vertex* P *such that* $[V]$ *is* $\mathcal{F}$-*stable. Consider* V *as an* $\mathcal{O}\Delta P$-*module via the canonical isomorphism* $\Delta P \cong P$. *Set*

$$X = \mathcal{O}Gi \otimes_{\mathcal{O}P} \mathrm{Ind}_{\Delta P}^{P\times P}(V) \otimes_{\mathcal{O}P} j\mathcal{O}H$$

The canonical algebra homomorphism

$$\mathrm{End}_{\mathcal{O}(G\times H)}(X) \to \mathrm{End}_{k(G\times H)}(k \otimes_{\mathcal{O}} X)$$

is surjective. In particular, for any direct summand $\bar{M}$ *of* $k \otimes_{\mathcal{O}} X$ *there is a direct summand* M *of* X *such that* $k \otimes_{\mathcal{O}} M \cong \bar{M}$.

Proof. Set $A = i\mathcal{O}Gi$ and $B = j\mathcal{O}Hj$, and set

$$U = A \otimes_{\mathcal{O}P} \mathrm{Ind}_{\Delta P}^{P\times P}(V) \otimes_{\mathcal{O}P} B\ .$$

Set $\bar{A} = k \otimes_{\mathcal{O}} A$, $\bar{B} = k \otimes_{\mathcal{O}} B$, and $\bar{U} = k \otimes_{\mathcal{O}} U$. Since multiplication by a source idempotent is a Morita equivalence, it suffices to show that the canonical map

$$\mathrm{End}_{A\otimes_{\mathcal{O}}B^{\mathrm{op}}}(U) \to \mathrm{End}_{\bar{A}\otimes_k \bar{B}^{\mathrm{op}}}(\bar{U})$$

is surjective. By a standard adjunction, we have an isomorphism

$$\mathrm{End}_{A\otimes_{\mathcal{O}}B^{\mathrm{op}}}(U) \cong \mathrm{Hom}_{\mathcal{O}\Delta P}(V, U)\ .$$

Thus we need to show that the right side maps onto the corresponding expression over k. The right side is equal to the space of $\mathcal{O}\Delta P$-fixed points in the module

$$\mathrm{Hom}_{\mathcal{O}}(V, U) \cong V^{\vee} \otimes_{\mathcal{O}} U\ .$$

Thus we need to show the surjectivity of the canonical map

$$(V^{\vee} \otimes_{\mathcal{O}} U)^{\Delta P} \to (\bar{V}^{\vee} \otimes_k \bar{U})^{\Delta P}\ .$$

Fixed points in a permutation module with respect to a finite group action over either $\mathcal{O}$ or k are spanned by the orbit sums of a permutation basis, and hence for the surjectivity of the previous map it suffices to show that

$$V^{\vee} \otimes_{\mathcal{O}} U$$

is a permutation $\mathcal{O}\Delta P$-module. By [19, Proposition 4.1], as an $\mathcal{O}\Delta P$-module, U is an endopermutation $\mathcal{O}\Delta P$-module having V as a direct summand. Thus $U^{\vee} \otimes_{\mathcal{O}} U$ is a permutation $\mathcal{O}\Delta P$-module having $V^{\vee} \otimes U$ is a direct summand. In particular, $V^{\vee} \otimes_{\mathcal{O}} U$ is a permutation $\mathcal{O}\Delta P$-module as required. The last statement follows from lifting idempotents. □

As a consequence of the classification theorem of endopermutation modules over finite p-groups, if U is an endopermutation kP-module having an indecomposable direct summand with vertex P, then there exists an endopermutation $\mathcal{O}P$-module V such that $\bar{V} \cong U$ (see [35, Theorem 14.2]). In particular, the canonical map $D_{\mathcal{O}}(P) \to D_k(P)$ is surjective (cf. [2, Corollary 8.5]). By standard properties of endopermutation modules, the kernel of this map is $\mathrm{Hom}(P, \mathcal{O}^{\times})$. Further, for any saturated fusion system $\mathcal{F}$ on P, if V or its class $[V]$ is $\mathcal{F}$-stable, then $\bar{V}$ or its class $[\bar{V}]$ is $\mathcal{F}$-stable, respectively. In particular, the surjection $D_{\mathcal{O}}(P) \to D_k(P)$ restricts to a map from $D_{\mathcal{O}}(P,\mathcal{F})$ to $D_k(P,\mathcal{F})$. For fusion systems of finite groups, the following result has also been observed by Lassueur and Thévenaz in [14, Lemma 4.1].

Lemma 8.4. *Let P be a finite p-group and $\mathcal{F}$ a saturated fusion system on P. The canonical map $D_{\mathcal{O}}(P,\mathcal{F}) \to D_k(P,\mathcal{F})$ is surjective.*

Proof. Let U be an indecomposable endopermutation kP-module with vertex P such that the class $[U]$ of U is in $D_k(P;\mathcal{F})$. By Proposition 8.1 there is an $\mathcal{F}$-stable endopermutation kP-module U' of dimension prime to p having a direct summand isomorphic to U. By the remarks at the beginning of this section, there is an endopermutation $\mathcal{O}P$-module V' of determinant 1 such that $\bar{V}' \cong U$. Moreover, the determinant 1 condition implies that V' is unique up to isomorphism (see e. g. [34, Lemma (28.1)]). Then, for Q a subgroup of P, the $\mathcal{O}Q$-module $\mathrm{Res}^P_Q(V')$ of V' is also the unique - up to isomorphism - lift of the kQ-module $\mathrm{Res}^P_Q(U')$ with determinant 1, and for $\varphi : Q \to R$ an isomorphism in $\mathcal{F}$, the $\mathcal{O}Q$-module ${}_\varphi\mathrm{Res}^P_R(V')$ is the lift with determinant 1 of the kQ-module ${}_\varphi\mathrm{Res}^P_R(U')$. Thus the $\mathcal{F}$-stability of U' implies that V' is an $\mathcal{F}$-stable $\mathcal{O}P$-module. But then the class of V is $\mathcal{F}$-stable in $D_{\mathcal{O}}(P;\mathcal{F})$. By construction, we have $\bar{V} \cong U$, proving the result. □

Proof of Theorem 1.13. Let $\bar{M}$ be an indecomposable $(kG\bar{b}, kH\bar{c})$-bimodule inducing a Morita equivalence (resp. stable equivalence of Morita type). Suppose that $\bar{M}$ has endopermutation source $\bar{V}$. By Lemma 8.2, we may identify a defect group P of b with a defect group of c, such that $\bar{M}$ is a direct summand of

$$kG\bar{i} \otimes_{kP} \mathrm{Ind}^{P\times P}_{\Delta P}(\bar{V}) \otimes_{kP} \bar{j}kH$$

for some source idempotents $\bar{i}$, $\bar{j}$ of $\bar{b}$, $\bar{c}$. Moreover, still by Lemma 8.2, these two source idempotents determine the same fusion system $\mathcal{F}$ on P, and the class $[\bar{V}]$ in $D_k(P)$ is $\mathcal{F}$-stable, where here $\bar{V}$ is regarded as a kP-module. By Lemma 8.4 there is an endopermutation $\mathcal{O}P$-module V satisfying $\bar{V} \cong k \otimes_{\mathcal{O}} V$ such that $[V]$ is $\mathcal{F}$-stable in $D_{\mathcal{O}}(P)$. It follows from Lemma 8.3 that there is a direct summand M of

$$\mathcal{O}Gi \otimes_{\mathcal{O}P} \mathrm{Ind}^{P\times P}_{\Delta P}(V) \otimes_{\mathcal{O}P} j\mathcal{O}H$$

satisfying $\bar{M} \cong k \otimes_{\mathcal{O}} M$, where i, j are source idempotents lifting $\bar{i}$, $\bar{j}$. By construction, M has vertex ΔP and source V, and by Proposition 4.5, M induces a Morita equivalence (resp. stable equivalence of Morita type). This proves (a). A Morita equivalence (resp. stable equivalence of Morita type) between $\mathcal{O}Gb$ and $\mathcal{O}Hc$ with endopermutation source induces clearly such an equivalence with endopermutation source between $kG\bar{b}$ and $kH\bar{c}$, and by (a), this lifts back to an equivalence between $W(k)Gb$ and $W(k)Hc$ with the properties as stated in (b). □

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