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# BLOCK ALGEBRAS WITH $HH^1$ A SIMPLE LIE ALGEBRA

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ABSTRACT. The purpose of this note is to add to the evidence that the algebra structure of a  $p$ -block of a finite group is closely related to the Lie algebra structure of its first Hochschild cohomology group. We show that if  $B$  is a block of a finite group algebra  $kG$  over an algebraically closed field  $k$  of prime characteristic  $p$  such that  $HH^1(B)$  is a simple Lie algebra and such that  $B$  has a unique isomorphism class of simple modules, then  $B$  is nilpotent with an elementary abelian defect group  $P$  of order at least 3, and  $HH^1(B)$  is in that case isomorphic to the Witt algebra  $HH^1(kP)$ . In particular, no other simple modular Lie algebras arise as  $HH^1(B)$  of a block  $B$  with a single isomorphism class of simple modules.

## 1. INTRODUCTION

Let  $p$  be a prime and  $k$  an algebraically closed field of characteristic  $p$ . For  $G$  a finite group, a *block of  $kG$*  is an indecomposable direct factor of the group algebra  $kG$ . There is an abundance of stable equivalences in block theory, but it is notoriously difficult to pin down even the most basic numerical invariants - such as the number of isomorphism classes of simple modules - through stable equivalences.

The main motivation for the present note is that on the one hand, the Lie algebra structure of  $HH^1(B)$  of a block  $B$  of a finite group algebra  $kG$  is invariant under stable equivalences of Morita type (cf. [4, Theorem 10.7]), and on the other hand, there is evidence for close structural connections between the algebra structure of  $B$  and the Lie algebra structure of  $HH^1(B)$  (cf. [1]). Understanding those connections might therefore ultimately contribute towards determining block invariants in some cases.

We describe some of the structural connections between  $B$  and  $HH^1(B)$  in two extreme cases for blocks with a single isomorphism class of simple modules.

**Theorem 1.1.** *Let  $G$  be a finite group and let  $B$  be a block algebra of  $kG$  having a unique isomorphism class of simple modules. Then  $HH^1(B)$  is a simple Lie algebra if and only if  $B$  is nilpotent with an elementary abelian defect group  $P$  of order at least 3. In that case, we have a Lie algebra isomorphism  $HH^1(B) \cong HH^1(kP)$ .*

Theorem 1.1 implies in particular that none of the other simple modular Lie algebras occur as  $HH^1(B)$  of some block algebra of a finite group with the property that  $B$  has a single isomorphism class of simple modules. See [7], [8] for details and further references on the classification of simple Lie algebras in positive characteristic. We do not know whether the hypothesis on  $B$  to have a single isomorphism class of simple modules is necessary in Theorem 1.1.

**Theorem 1.2.** *Let  $G$  be a finite group and let  $B$  be a block algebra of  $kG$  having a nontrivial defect group and a unique isomorphism class of simple modules. Then  $\dim_k(HH^1(B)) \geq 2$ .*

The hypothesis that  $B$  has a single isomorphism class of simple modules is necessary in Theorem 1.2. For instance, if  $P$  is cyclic of order  $p \geq 3$  and if  $E$  is the cyclic automorphism group of order  $p-1$  of  $P$ , then  $HH^1(k(P \rtimes E))$  has dimension one. This follows immediately from the centraliser decomposition of Hochschild cohomology; see [3, Theorem 1.4] for a more general result.

## 2. QUOTED RESULTS

We collect in this section results needed for the proof of Theorem 1.1.

**Theorem 2.1** (Okuyama and Tsushima [5]). *Let  $G$  be a finite group and  $B$  a block algebra of  $kG$ . Then  $B$  is a nilpotent block with an abelian defect group if and only if  $J(B) = J(Z(B))B$ .*

Let  $A$  be a finite-dimensional (associative and unital)  $k$ -algebra. A derivation on  $A$  is a  $k$ -linear map  $f : A \rightarrow A$  satisfying  $f(ab) = f(a)b + af(b)$  for all  $a, b \in A$ . The set  $\text{Der}(A)$  of derivations on  $A$  is a Lie subalgebra of  $\text{End}_k(A)$ , with respect to the Lie bracket  $[f, g] = f \circ g - g \circ f$ , for any  $f, g \in \text{End}_k(A)$ . For  $c \in A$ , the map sending  $a \in A$  to the additive commutator  $[c, a] = ca - ac$  is a derivation on  $A$ ; any derivation arising this way is called an *inner derivation on  $A$* . The set  $\text{IDer}(A)$  of inner derivations is a Lie ideal in  $\text{Der}(A)$ , and we have a canonical identification  $HH^1(A) \cong \text{Der}(A)/\text{IDer}(A)$ . See [9, Chapter 9] for more details on Hochschild cohomology. If  $A$  is commutative, then  $HH^1(A) \cong \text{Der}(A)$ . A  $k$ -algebra  $A$  is *symmetric* if  $A$  is isomorphic to its  $k$ -dual  $A^*$  as an  $A$ - $A$ -bimodule; this implies that  $A$  is finite-dimensional.

**Theorem 2.2** ([1, Theorem 3.1]). *Let  $A$  be a symmetric  $k$ -algebra and let  $E$  be a maximal semisimple subalgebra. Let  $f : A \rightarrow A$  be an  $E$ - $E$ -bimodule homomorphism satisfying  $E + J(A)^2 \subseteq \ker(f)$  and  $\text{Im}(f) \subseteq \text{soc}(A)$ . Then  $f$  is a derivation on  $A$  in  $\text{soc}_{Z(A)}(\text{Der}(A))$ , and if  $f \neq 0$ , then  $f$  is an outer derivation of  $A$ . In particular, we have*

$$\sum_S \dim_k(\text{Ext}_A^1(S, S)) \leq \dim_k(\text{soc}_{Z(A)}(HH^1(A)))$$

where in the sum  $S$  runs over a set of representatives of the isomorphism classes of simple  $A$ -modules.

**Corollary 2.3** ([1, Corollary 3.2]). *Let  $A$  be a local symmetric  $k$ -algebra. Let  $f : A \rightarrow A$  be a  $k$ -linear map satisfying  $k \cdot 1 + J(A)^2 \subseteq \ker(f)$  and  $\text{Im}(f) \subseteq \text{soc}(A)$ . Then  $f$  is a derivation on  $A$  in  $\text{soc}_{Z(A)}(\text{Der}(A))$ , and if  $f \neq 0$ , then  $f$  is an outer derivation of  $A$ . In particular, we have*

$$\dim_k(J(A)/J(A)^2) \leq \dim_k(\text{soc}_{Z(A)}(HH^1(A))) .$$

**Theorem 2.4** (Jacobson [2, Theorem 1]). *Let  $P$  be a finite elementary abelian  $p$ -group of order at least 3. Then  $HH^1(kP)$  is a simple Lie algebra.*

The converse to this theorem holds as well.

**Proposition 2.5.** *Let  $P$  be a finite abelian  $p$ -group. If  $HH^1(kP)$  is a simple Lie algebra, then  $P$  is elementary abelian of order at least 3.*

*Proof.* Suppose that  $P$  is not elementary abelian; that is, its Frattini subgroup  $Q = \Phi(P)$  is nontrivial. Since  $P$  is abelian, we have  $HH^1(kP) = \text{Der}(kP)$ . We will show that the set of derivations with image contained in  $I(kQ)kP = \ker(kP \rightarrow kP/Q)$  is a nonzero Lie ideal in  $\text{Der}(kP)$ , where  $I(kQ)$  is the augmentation ideal of  $kQ$ . Indeed, every element in  $Q$  is equal to  $x^p$  for some  $x \in P$ , and hence every element in  $I(kQ)$  is a linear combination of elements of the form

$(x-1)^p$ , where  $x \in P$ . Every derivation on  $kP$  annihilates all elements of this form (using the fact that  $k$  has characteristic  $p$ ), and hence every derivation on  $kP$  preserves  $I(kQ)kP$ . Thus there is a canonical Lie algebra homomorphism  $\text{Der}(kP) \rightarrow \text{Der}(kP/Q)$ . This homomorphism is nonzero; indeed, it is an isomorphism on the components of Hochschild cohomology corresponding to  $H^1(P; k) \cong H^1(P/Q; k)$  under the centraliser decomposition. The kernel of this canonical Lie algebra homomorphism contains all derivations with image in  $\text{soc}(kP)$ , so this kernel is nonzero by Corollary 2.3. Thus  $HH^1(kP)$  is not simple as a Lie algebra, whence the result.  $\square$

**Remark 2.6.** Theorem 1.1 implies that the hypothesis on  $P$  being abelian is not necessary in the statement of 2.5.

### 3. AUXILIARY RESULTS

In order to exploit the hypothesis on  $HH^1$  being simple in the statement of Theorem 1.1, we consider Lie algebra homomorphisms into the  $HH^1$  of subalgebras and quotients.

**Lemma 3.1.** *Let  $A$  be a finite-dimensional  $k$ -algebra and  $f$  a derivation on  $A$ . Then  $f$  sends  $Z(A)$  to  $Z(A)$ , and the map sending  $f$  to the induced derivation on  $Z(A)$  induces a Lie algebra homomorphism  $HH^1(A) \rightarrow HH^1(Z(A))$ .*

*Proof.* Let  $z \in Z(A)$ . For any  $a \in A$  we have  $az = za$ , hence  $f(az) = f(a)z + af(z) = f(z)a + zf(a) = f(za)$ . Comparing the two expressions, using  $zf(a) = f(a)z$ , yields  $af(z) = f(z)a$ , and hence  $f(z) \in Z(A)$ . The result follows.  $\square$

**Lemma 3.2.** *Let  $A$  be a local symmetric  $k$ -algebra such that  $J(Z(A))A \neq J(A)$ . Then the canonical Lie algebra homomorphism  $HH^1(A) \rightarrow HH^1(Z(A))$  is not injective.*

*Proof.* Since  $J(Z(A))A < J(A)$ , it follows from Nakayama's lemma that  $J(Z(A))A + J(A)^2 < J(A)$ . Thus there is a nonzero linear endomorphism  $f$  of  $A$  which vanishes on  $J(Z(A))A + J(A)^2$  and on  $k \cdot 1_A$ , with image contained in  $\text{soc}(A)$ . In particular,  $f$  vanishes on  $Z(A) = k \cdot 1_A + J(Z(A))$ . By 2.3, the map  $f$  is an outer derivation on  $A$ . Thus the class of  $f$  in  $HH^1(A)$  is nonzero, and its image in  $HH^1(Z(A))$  is zero, whence the result.  $\square$

**Lemma 3.3.** *Let  $A$  be a local symmetric  $k$ -algebra and let  $f$  be a derivation on  $A$  such that  $Z(A) \subseteq \ker(f)$ . Then  $f(J(A)) \subseteq J(A)$ .*

*Proof.* Since  $A$  is local and symmetric, we have  $\text{soc}(A) \subseteq Z(A)$ , and  $J(A)$  is the annihilator of  $\text{soc}(A)$ . Let  $x \in J(A)$  and  $y \in \text{soc}(A)$ . Then  $xy = 0$ , hence  $0 = f(xy) = f(x)y + xf(y)$ . Since  $y \in \text{soc}(A) \subseteq Z(A)$ , it follows that  $f(y) = 0$ , hence  $f(x)y = 0$ . This shows that  $f(x)$  annihilates  $\text{soc}(A)$ , and hence that  $f(x) \in J(A)$ .  $\square$

**Lemma 3.4.** *Let  $A$  be a finite-dimensional  $k$ -algebra and  $J$  an ideal in  $A$ .*

- (i) *Let  $f$  be a derivation on  $A$  such that  $f(J) \subseteq J$ . Then  $f(J^n) \subseteq J^n$  for any positive integer  $n$ .*
- (ii) *Let  $f, g$  be derivations on  $A$  and let  $m, n$  be positive integers such that  $f(J) \subseteq J^m$  and  $g(J) \subseteq J^n$ . Then  $[f, g](J) \subseteq J^{m+n-1}$ .*

*Proof.* In order to prove (i), we argue by induction over  $n$ . For  $n = 1$  there is nothing to prove. If  $n > 1$ , then  $f(J^n) \subseteq f(J)J^{n-1} + Jf(J^{n-1})$ . Both terms are in  $J^n$ , the first by the assumptions, and the second by the induction hypothesis  $f(J^{n-1}) \subseteq J^{n-1}$ . Let  $y \in J$ . Then  $[f, g](y) = f(g(y)) - g(f(y))$ . We have  $g(y) \in J^n$ ; that is,  $g(y)$  is a sum of products of  $n$  elements in  $J$ .

Applying  $f$  to any such product shows that the image is in  $J^{m+n-1}$ . A similar argument applied to  $g(f(y))$  implies (ii).  $\square$

**Proposition 3.5.** *Let  $A$  be a finite-dimensional  $k$ -algebra. For any positive integer  $m$  denote by  $\text{Der}_{(m)}(A)$  the  $k$ -subspace of derivations  $f$  on  $A$  satisfying  $f(J(A)) \subseteq J(A)^m$ .*

- (i) *For any two positive integers  $m$  and  $n$  we have  $[\text{Der}_{(m)}(A), \text{Der}_{(n)}(A)] \subseteq \text{Der}_{(m+n-1)}(A)$ .*
- (ii) *The space  $\text{Der}_{(1)}(A)$  is a Lie subalgebra of  $\text{Der}(A)$ .*
- (iii) *For any positive integer  $m$ , the space  $\text{Der}_{(m)}(A)$  is an ideal in  $\text{Der}_{(1)}(A)$ .*
- (iv) *Suppose that  $A$  is local. The space  $\text{Der}_{(2)}(A)$  is a nilpotent Lie subalgebra of  $\text{Der}(A)$ .*

*Proof.* Statement (i) follows from 3.4 (ii). The statements (ii) and (iii) are immediate consequences of (i). Since  $A$  is local and since 1 is annihilated by any derivation on  $A$ , statement (iii) follows from (i) and the fact that  $J(A)$  is nilpotent.  $\square$

#### 4. PROOF OF THEOREMS 1.1 AND 1.2

Let  $G$  be a finite group and  $B$  a block of  $kG$ . Suppose that  $B$  has a single isomorphism class of simple modules. If  $B$  is nilpotent and  $P$  a defect group of  $B$ , then by [6],  $B$  is Morita equivalent to  $kP$ , and hence there is a Lie algebra isomorphism  $HH^1(B) \cong HH^1(kP)$ . Thus if  $B$  is nilpotent with an elementary abelian defect group  $P$  of order at least 3, then  $HH^1(B)$  is a simple Lie algebra by 2.4.

Suppose conversely that  $HH^1(B)$  is a simple Lie algebra. If  $J(B) = J(Z(B))B$ , then  $B$  is nilpotent with an abelian defect group  $P$  by 2.1. As before, we have  $HH^1(B) \cong HH^1(kP)$ , and hence 2.5 implies that  $P$  is elementary abelian of order at least 3.

Suppose that  $J(Z(B))B \neq J(B)$ . Let  $A$  be a basic algebra of  $B$ . Then  $J(Z(A))A \neq J(A)$ . Moreover,  $A$  is local symmetric, since  $B$  has a single isomorphism class of simple modules. Thus  $\text{soc}(A)$  is the unique minimal ideal of  $A$ . We have  $J(A)^2 \neq \{0\}$ . Indeed, if  $J(A)^2 = \{0\}$ , then  $\text{soc}(A)$  contains  $J(A)$ , and hence  $J(A)$  has dimension 1, implying that  $A$  has dimension 2. In that case  $B$  is a block with defect group of order 2. But then  $HH^1(A) \cong HH^1(kC_2)$  is not simple, a contradiction. Thus  $J(A)^2 \neq \{0\}$ , and hence  $\text{soc}(A) \subseteq J(A)^2$ . By 3.2, the canonical Lie algebra homomorphism  $HH^1(A) \rightarrow HH^1(Z(A))$  is not injective. Since  $HH^1(A)$  is a simple Lie algebra, it follows that this homomorphism is zero. In other words, every derivation on  $A$  has  $Z(A)$  in its kernel. It follows from 3.3 that every derivation on  $A$  sends  $J(A)$  to  $J(A)$ . Thus, by 3.4, every derivation on  $A$  sends  $J(A)^2$  to  $J(A)^2$ . This implies that the canonical surjection  $A \rightarrow A/J(A)^2$  induces a Lie algebra homomorphism  $HH^1(A) \rightarrow HH^1(A/J(A)^2)$ . Note that the algebra  $A/J(A)^2$  is commutative, and hence  $HH^1(A/J(A)^2) = \text{Der}(A/J(A)^2)$ . Since  $J(A)^2$  contains  $\text{soc}(A)$ , it follows that the kernel of the canonical map  $HH^1(A) \rightarrow HH^1(A/J(A)^2)$  contains the classes of all derivations with image in  $\text{soc}(A)$ . Since there are outer derivations with this property (cf. 2.3), it follows from the simplicity of  $HH^1(A)$  that the canonical map  $HH^1(A) \rightarrow HH^1(A/J(A)^2) = \text{Der}(A/J(A)^2)$  is zero. Thus every derivation on  $A$  has image in  $J(A)^2$ . But then 3.5 implies that  $\text{Der}(A) = \text{Der}_{(2)}(A)$  is a nilpotent Lie algebra. Thus  $HH^1(A)$  is nilpotent, contradicting the simplicity of  $HH^1(A)$ . The proof of Theorem 1.1 is complete.

*Proof of Theorem 1.2.* Denote by  $A$  a basic algebra of  $B$ . Since  $B$  has a unique isomorphism class of simple modules and a nontrivial defect group, it follows that  $A$  is a local symmetric algebra of dimension at least 2. By 2.3 we have  $\dim_k(HH^1(A)) \geq \dim_k(J(A)/J(A)^2)$ . Thus

$\dim_k(HH^1(A)) \geq 1$ . Moreover, if  $\dim_k(HH^1(A)) = 1$ , then  $\dim_k(J(A)/J(A)^2) = 1$ , and hence  $A$  is a uniserial algebra. In that case  $B$  is a block with a cyclic defect group  $P$  and a unique isomorphism class of simple modules, and hence  $B$  is a nilpotent block. Thus  $A \cong kP$ . We have  $\dim_k(HH^1(kP)) = |P|$ , a contradiction. The result follows.  $\square$

**Remark 4.1.** All finite-dimensional algebras in this paper are split thanks to the assumption that  $k$  is algebraically closed. It is not hard to see that one could replace this by an assumption requiring  $k$  to be a splitting field for the relevant algebras. The statements 3.1 and 3.4 do not require any hypothesis on  $k$ .

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#### REFERENCES

- [1] D. Benson, R. Kessar, and M. Linckelmann, *On blocks of defect two and one simple module, and Lie algebra structure of  $HH^1$* , J. Pure Appl. Algebra **221** (2017), 2953–2973.
- [2] N. Jacobson, *Classes of restricted Lie algebras of characteristic  $p$ , II*, Duke Math. J. **10** (1943), 107–121.
- [3] R. Kessar and M. Linckelmann, *On Blocks with Frobenius Inertial Quotient*, J. Algebra **249** (2002), 127–146.
- [4] S. König, Y. Liu, and G. Zhou, *Transfer maps in Hochschild cohomology and applications to stable and derived invariants and to the Auslander-Reiten conjecture*. Trans. Amer. Math. Soc. **364** (2012), 195–232.
- [5] T. Okuyama and Y. Tsushima, *Local properties of  $p$ -block algebras of finite groups*, Osaka J. Math. **20** (1983), 33–41.
- [6] L. Puig, *Nilpotent blocks and their source algebras*, Invent. Math. **93** (1988), 77–116.
- [7] H. Strade, *Simple Lie Algebras over Fields of Positive Characteristic, I*. de Gruyter Expositions in Mathematics **38** (2004), viii+540pp.
- [8] H. Strade, *Simple Lie Algebras over Fields of Positive Characteristic, II*. de Gruyter Expositions in Mathematics **42** (2009), vi+385pp.
- [9] C. A. Weibel, *An introduction to homological algebra*. Cambridge Studies Adv. Math. **38** (1994), Cambridge University Press.