Quantum, noncommutative and MOND corrections to the entropic law of gravitation

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Abstract: Quantum and noncommutative corrections to the Newtonian law of inertia are considered in the general setting of Verlinde's entropic force postulate. We demonstrate that the form for the modified Newtonian dynamics (MOND) emerges in a classical setting by seeking appropriate corrections in the entropy. We estimate the correction term by using concrete coherent states in the standard and generalized versions of Heisenberg's uncertainty principle. Using Jackiw's direct and analytic method we compute the explicit wavefunctions for these states producing minimal length as well as minimal products. Subsequently we derive a further selection criterium restricting the free parameters in the model in providing a canonical formulation of the quantum corrected Newtonian law by setting up the Lagrangian and Hamiltonian for the system.

1. Introduction

The weak equivalence principle is a well known concept, see e.g. [1], that identifies the inertial mass m_I occurring in Newton's second law of motion $\vec{F} = m_I \vec{a}$, with the gravitational mass m_G in Newton's inverse square law of gravitation. The latter accounts for the attractive force between a body of mass m_G at the position $\vec{r}$ and n different others specified by their masses m_i occupying positions $\vec{r}_i$, $i = 1, 2, ..., n$, as

$$\vec{F} = -\sum_i \frac{G m_G m_i (\vec{r} - \vec{r}_i)}{|\vec{r} - \vec{r}_i|^3}, \quad i = 1, 2, ..., n, \tag{1.1}$$

where G is the gravitational constant. Equating these two expressions for the force when using $m_I = m_G$ readily yields an expression for the acceleration of a particle in a gravitational field

$$\vec{a} = -G \sum_i \frac{m_i (\vec{r} - \vec{r}_i)}{|\vec{r} - \vec{r}_i|^3}, \quad i = 1, 2, ..., n. \tag{1.2}$$

A consequence of this is the curious feature, known since the time of Galileo, that objects that are dropped from some height, say from the top of a building, will arrive at the same time on the ground as long as their motion is not affected by air resistance or other disturbances, i.e. they fall at the same rate with equal accelerations.

By invoking the holographic principle in the vicinity of a black hole E. Verlinde [2, 3] demonstrated recently that Newton's second law of motion for a particle when confronted with the law of gravitation for celestial bodies, can be viewed as entropic in character. Employing the holographic argument that any information of the black hole, which is imagined as a sphere of Schwarzschild radius R, can emerge only from its surface (because what is inside the black hole remains totally intractable to an outside observer), and using the equipartition rule gave strong thermodynamical evidence to justify such a claim. Note that the boundary of the black hole sphere which is basically an equipotential surface acts as a holographic screen also popularly referred to as the event horizon.

Assuming the change of entropy ΔS near the holographic screen to be linear in the displacement Δx of a test particle, Verlinde suggested for ΔS the relation

$$\Delta S = 2\pi k_B \frac{mc}{\hbar}\Delta x = 2\pi k_B \frac{\Delta x}{\lambda}, \tag{1.3}$$

with $\lambda := \hbar/mc$ denoting the reduced Compton wavelength, k_B the Boltzmann constant, $\hbar$ the reduced Planck constant and c the speed of light. The change in entropy was also assumed to generate an entropic force F acting on the particle to be of the form

$$F\Delta x = T\Delta S, \tag{1.4}$$

where T is the temperature. Taking T to be given by Unruh's temperature [4] for an accelerated observer, namely

$$k_B T = \frac{\hbar}{2\pi}\frac{a}{c}, \tag{1.5}$$

where a stands for the acceleration of the particle, consistency gave Newton's formula for the second law of motion $F = ma$ when combining (1.3)-(1.5). Next, to arrive at the law of gravitation by restricting to the spherical boundary having an area $A = 4\pi R^2$, R being the radius of the sphere, he made use of the holographic principle that the total number of bits making up the maximally storage space is proportional to A. This gave the number N of used bits as $N = Ac^3/G\hbar$, G being identified as Newton's gravitational constant as in (1.1). Using the equipartition rule for the average energy for every bit, $E = \frac{1}{2}Nk_BT = mc^2$, M denoting the mass in the part of the space enclosed by the holographic screen, yielded the well known Newton's law of gravitation: $F = -GMm/R^2$.

It is worthwhile to recall some history behind Verlinde's formulation. First, an early work by Jacobson [5] attempted to derive Einstein's relativistic equations from pure thermodynamical considerations by making the constant of proportionality between the area and entropy universal following a preceding work of Bekenstein [6] who in turn looked at the entropy of any isolated system to be bounded by its area. Second, Padmanabhan [7, 8] arrived at a result of gravitational acceleration by reversing Unruh's temperature-acceleration relation. Both Bekenstein as well as Padmanabhan's discussions were carried out in a fully relativistic frameworks which apparently have no analogue in Verlinde's non-relativistic formulation.

An important observation made by Verlinde was that since the maximally allowed information stored in any continuum volume of space can only be finite, it is not sensible to talk of localizing a particle with an infinite degree of accuracy. Even though in the end he obtained the classical results which were devoid from the appearance of $\hbar$, it is pertinent to bear in mind that while the individual expressions of the change in entropy as well as Unruh's temperature contain an explicit presence of $\hbar$, the latter fortunately cancels out when we look for a force-acceleration relationship. A question then naturally arises as to what happens if we seek higher corrections to the uncertainty principle as is needed to accommodate various modifications of the short distance structure in quantum theories that attempt to incorporate gravity [9]. This in turn calls for an introduction of a so-called minimal length beyond which a localization of space-time events is no longer possible. With a minimal observable length $\Delta x \neq 0$ that is characteristic of a physical quantum state, it is evident that an eigenstate with a zero-uncertainty in position can no longer depict a physical state. Models in string theory [10, 11] as well as in quantum gravity [12] do indeed support the existence of such a minimal length [13]. Alternatively one may also view Δx as a change in the black hole radius [14]. It is also noteworthy that one may derive the Friedmann equations describing the dynamics of the universe with any spatial curvature when considering gravity as an entropic force [15, 16]. The presence of a minimal length with impose corrections on these expressions [17, 18].

Many of the proposed corrections to the entropic force of gravitation are a matter of speculation based on physical plausibility arguments. In particular, constants of proportionality can often only be fixed by reasoning on their dimensionality so that the overall expressions remain to be of a qualitative nature. The main purpose of our paper is to investigate different types corrections more concretely by trying to obtain concrete quantitative values. We achieve this by evaluating these expressions for various types of coherent quantum states. In addition we analyze the Lagrangian ad Hamiltonian formulations for these forces, which imposes further constraints on possible choices of the correction terms.

Our manuscript is organized as follows: In section 2 we discuss how Verlinde's argument can be modified by including quantum or noncommutative corrections in the energy and/or the entropy. We compute the correction terms to the gravitational force for various choices of the free parameters in the standard approach and generalized Heisenberg's uncertainty relations. We use concrete expressions for the uncertainties obtained from different types of coherent states whose wave functions we derive explicitly in section 3 using Jackiw's direct and analytic method. In section 4 we derive some Lagrangians and Hamiltonians for the corrected entropic force, which turn out to be explicitly time-dependent. Demanding that the damping to be small provides a further criterion that allows to exclude certain choices of the free parameters. We state our conclusions in section 5.

2. Quantum corrections to Newton's second law

Let us now see how the above effects might be incorporated into the above reasoning by modifying the equations (1.3)-(1.5) and exploiting Verlinde's observation that a strict localization of the test particle is not possible. As argued by Santos and Vancea [19] the total energy also depends on the momentum p in form of the kinetic energy or possibly in a more general way. This means that the uncertainty in the total energy δE could also acquire a term that depends on the uncertainty in the momentum δp. We assume here the form

$$\delta E = F\delta x + \alpha \frac{p}{m}\delta p = T\delta S, \tag{2.1}$$

to be valid at thermal equilibrium where α is dimensionless, possibly a constant. In other words when $\alpha = 1$ there is correction term associated to the kinetic energy. Note that $T\delta S$ is not a perfect differential. To counterbalance the additional term one also needs to modify the expression for δS. Here we take

$$\delta S = 2\pi k_B \left(\frac{1}{\lambda}\delta x + \frac{\beta}{mc}\delta p \right), \tag{2.2}$$

with a dimensionless parameter β introduced to the equation. We keep the equation for the Unruh temperature (1.5) unchanged. In the limit $\alpha, \beta \to 0$ we recover the equations (1.3)-(1.5). In [19] the options $\alpha = 1$, $\beta = 1$ and $\alpha = 1$, $\beta = p/mc$ were explored. Combining the equations (1.5), (2.1) and (2.2) leads easily to a corrected expression for the force

$$F = ma + \left(\beta\frac{\lambda}{c}a - \alpha\frac{p}{m} \right) \frac{\delta p}{\delta x} = ma + F^{\text{cor}}. \tag{2.3}$$

Our task is now to interpret the additional term F^{cor} and test which choices of α and β are permissible. Keeping in mind that the variations δp and δx are interpreted as the uncertainties in a simultaneous measurement of x and p one can employ the standard Robertson version of Heisenberg's uncertainty relation for a simultaneous measurement of two noncommuting operators A and B

$$(\Delta A)^2 (\Delta B)^2 \geq \frac{1}{4} \left| \langle [A, B] \rangle \right|^2, \tag{2.4}$$

for $A = x$, $B = p$ with $[x, p] = i\hbar$ to make estimations about the ratio $\delta p/\delta x$. In [19, 20] $\delta x\delta p \geq \hbar/2$ was used at saturation point of the lower bound, i.e. it was assumed that the test particle is in a coherent, possibly squeezed, state and δp was traded for $\hbar/(2\delta x)$. This leaves the resulting expression with an unknown factor δx^{-2}. Furthermore it was suggested in [19] that the classical limit is obtained by the simultaneous limit $\hbar \to 0$, $\delta p \to 0$. This gives indeed the classical expression, but the proposed prescription lacks further justification. It addition, demanding $\delta p \to 0$ is ambiguous as one might as well require the simultaneous limit $\hbar \to 0$, $\delta x \to 0$. The question is why is the classical limit not obtainable simply from $\hbar \to 0$? Here we go a step further trying to achieve just that.

Our main assumption is that we take the test particle to be in a specific state so that $\delta p/\delta x$ acquires a concrete value.

2.1 Corrections from canonical coherent states and MOND dynamics

Let us now assume at first the test particle to be in a standard canonical coherent state for which we have the well known expressions, see e.g. [21],

$$\delta x \delta p = \hbar/2, \qquad \text{with } \delta x = \sqrt{\frac{\hbar}{2m\omega}}, \quad \delta p = \sqrt{\frac{m\omega\hbar}{2}}. \tag{2.5}$$

Using these equalities we can evaluate the ratio $\delta p/\delta x$ for these states, such that the quantum corrected force (2.3) becomes

$$F_{\text{coherent}} = ma + \left(\beta\frac{\lambda\!\!\!^{-}}{c}a - \alpha\frac{p}{m}\right)m\omega = ma + F^{\text{cor}}_{\text{coherent}}. \tag{2.6}$$

We have now various options for the choice of α and β. We may take $\alpha \neq 0$, which suggests that the second term in (2.6) becomes a pure quantum correction with $\alpha \sim \hbar$. For instance, $\alpha = \omega\hbar/mc^2$ is an admissible choice corresponding to a quantum correction in the energy (2.1). Having introduced an additional quantum correction it is not a surprise that we obtain also a quantum correction in F.

Taking $\alpha = 0$ the correction term $F^{\text{cor}}_{\text{coherent}}$ becomes a genuine quantum correction and the classical limit is simply reached by taking $\hbar \to 0$. One might take $\beta = 1$ in this case, so that a classical correction in δS has led us to a quantum correction in F.

Finally one may wonder if one can reverse the setting of the previous example and obtain a classical correction to F from an additional quantum term in δS. This is similar to Verlinde's original argument in which also the $\hbar$ from the expression for δS has cancelled the $\hbar$ appearing in the Unruh temperature. An example for such a classical theory of modified Newtonian dynamics (MOND) was proposed in 1983 by Milgrom [22, 23] for situations when the gravitational force shows a marked departure from the conventional Newtonian expectation at low acceleration. The MOND theory, or so it is called, is typically applicable to scales of acceleration (less than the threshold value of $a_0 \approx 1.2 \times 10^{-10} ms^{-2}$) which are rather small compared to what is observed in the solar system and perhaps relevant towards explaining galactic scale phenomena [24]. It has been noted [25] that the Milgrom scheme might be justified as an alternative means to solve for the dark matter problem which is still to find any experimental support, the prime reason being its rather poor coupling with visible matter. A recent paper by Verlinde [3] has sought to explore this issue using the standard thermodynamical arguments as a basis.

A MOND theory has the force form given by a deformed acceleration [26]

$$F_{\text{MOND}} = ma\mu(a_0/a), \qquad \text{with } \mu(a_0/a) = \frac{1}{1 + a_0/a}, \tag{2.7}$$

where a_0 is the aforementioned small acceleration. It is, however, only a phenomenological form but worthwhile to note that in place of the usual expression of the acceleration as is implied by (1.1) namely, $a = MGr^{-2}$, in MOND, a test particle which is at a distance r from a large mass M is subject to the acceleration a given by $a^2/a_0 = MGr^{-2}$, where $a \ll a_0$. Other variants of a modified Newtonian equation have been proposed in the literature [27], but we do not discuss them here.

Choosing now $\alpha = 0$ and $\beta = -c(\omega\lambda)(1 + a/a_0)$ we obtain precisely the form of the MOND force (2.7) with modified acceleration. Remarkably this means with a corrected entropy, just taking the uncertainty of a particles position into account, we may interpret the force in a MOND theory as a classically emerging entropic force.

2.2 Corrections from minimal value and minimal product coherent states

Let us now assume our system to be in a noncommutative space on which the canonical Heisenberg commutation relations are generalized to [9, 28]

$$[x, p] = i\hbar \left(1 + \tau p^2\right), \tag{2.8}$$

with $\tau \in \mathbb{R}^+$ denoting the dimensionful, i.e. $[\tau] = s^2/m^2$, noncommutative constant. Such a generalized commutation relation arises as a particular case of an extended q-deformation given by the commutator [28]

$$[x, p] = i\hbar q^{g(N)} + \frac{i\hbar}{4}(q^2 - 1)\left(\frac{x^2}{\delta^2} + \frac{p^2}{\gamma^2}\right), \quad \delta, \gamma \in \mathbb{R} \tag{2.9}$$

where g is some arbitrary function of the number operator N defined as the product of the creation and annihilation operator for the harmonic oscillator. Taking $g(N) = 0$ and parametrizing the deformation parameter q in the form $q = e^{2\tau\gamma^2}$, we found in the limit $\gamma \to 0$, the τ-corrected form (2.8).

On a noncommutative space one needs to make an important distinction between what we refer to as minimal factor coherent states (mfco) and minimal product coherent states (mpco). The former are the states for which the minimal value is reached for one of the factors in the uncertainty relation, e.g. δx in which case it is referred to as minimal length coherent state (mlco). In contrast,the mpco-states is a state for which the entire product in the uncertainty relation, e.g. $\delta x \delta p$, is minimized. Assuming now the test particle to be in a mlco-state, we have

$$\delta x_{\min} \delta p = \hbar, \qquad \text{with} \;\; \delta x_{\min} = \hbar\sqrt{\tau}, \quad \delta p = \frac{1}{\sqrt{\tau}}. \tag{2.10}$$

These values are easily obtained for (2.8) with (2.4), see the next section for the derivation. We will comment also in more detail on the construction of meaningful explicit states that produce these values. Using (2.10) in (2.3) the noncommutatively corrected force becomes

$$F_{\text{mlco}} = ma + \left(\beta\frac{\lambda}{c}a - \alpha\frac{p}{m}\right)\frac{1}{\hbar\tau} = ma + F_{\text{mlco}}^{\text{cor}}. \tag{2.11}$$

Since $\hbar$ as well as τ are very small, $\hbar\tau \ll 1$, the correction term becomes very large, which does not make sense as we expect only a small modification. However, by demanding that $\alpha \sim (\hbar\tau)^2$ and $\beta \sim \hbar\tau$ this can be achieved. For instance, $\alpha = (\hbar\tau m\omega)^2$ and $\beta = \hbar\tau m\omega$ is an admissible choice from a dimensional point of view. When using this option the modifying terms proportional to δp in (2.1) and (2.2) acquire a new interpretation. They are now noncommutative deformations that give rise to an entropically emergent noncommutative space-time structure.

As is clear from the argument above, the deformed equation (2.8) is only one of many possibilities obtainable from (2.9) or other approaches. In [28] also higher order τ-deformations were explored as for instance

$$[x,p] = i\hbar\left(1+\tau p^2+\frac{1}{2}\tau^2 p^4\right), \tag{2.12}$$

with $\delta x_{\min} = 1.14698\hbar\sqrt{\tau}$, $\delta p = 0.740664/\sqrt{\tau}$ going up to

$$[x,p] = i\hbar e^{\tau p^2}, \tag{2.13}$$

with $\delta x_{\min} = 1.16582\hbar\sqrt{\tau}$, $\delta p = 1/\sqrt{2\tau}$. It was noted in [29] that $1.16582\ldots$ can be expressed as $\sqrt{e/2}$. This means for the higher deformations the corrected force becomes

$$F_{\text{mlco}}(\tau) = ma + \kappa_\tau F_{\text{mlco}}^{\text{cor}}, \tag{2.14}$$

where τ in κ_τ indicates the order in the deformation. So we have $\kappa_1 = 1$, $\kappa_2 = 0.6458$, $\ldots$, $\kappa_\infty = e^{-1/2} = 0.6065$.

One might suspect a different outcome in regard to the previous argument when using mpco-states. Assuming still the emerging space-time structure to be noncommutative with deformed uncertainty relation (2.8) we obtain in this case

$$(\delta x\delta p)_{\min} = \frac{2}{3}\hbar, \qquad \text{with} \;\; \delta x = \frac{2}{\sqrt{3}}\hbar\sqrt{\tau}, \quad \delta p = \frac{1}{\sqrt{3}\sqrt{\tau}}. \tag{2.15}$$

We note that indeed for these values the product $\delta x\delta p$ is smaller and δx is larger in (2.15) when compared to (2.10). As it is less obvious how to derive these values, we will comment in detail on the derivation in the next section. The correction term to the force is in this case half the correction term $F_{\text{mpco}} = ma + F_{\text{mlco}}^{\text{cor}}/2$ which is a further reduction when compared to κ_∞. As the overall dependence on $\hbar$ and τ is unchanged the general discussion and interpretation is the same as for the mlco-states resulting from (2.8).

3. Minimal length and minimal value coherent states

We will now provide the details on how the values for (2.10) and (2.15) are obtained including a derivation of the associated explicit wavefunctions. Since the discussion for the values in (2.15) has not been presented elsewhere before, we provide also a general discussion on the appropriate method to be used.

We distinguish here two fundamental questions regarding the measurement of an observable A in any quantum mechanical system: a) what minimal value can the variance $(\Delta A)^2 := \langle A^2\rangle - \langle A\rangle^2 = \left\langle \hat{A}^2\right\rangle$ with $\hat{A} = A - \langle A\rangle$ or the standard deviation ΔA take and b) what is the associated state $|\psi\rangle_{\min,A}$ in $\langle A\rangle := \langle\psi| A |\psi\rangle$ that realises that minimum? These questions are then naturally extended to simultaneous measurements related to two or more operators. For two operators A and B the questions a) and b) have now three variants, i.e. what are the minimal values and corresponding states for ΔA, ΔB or the product $\Delta A\Delta B$ within the simultaneous measurement of A and B? For three operators A,

B and C this extends to seven variants, i.e. what are the minimal values and corresponding states for ΔA, ΔB, ΔC, $\Delta A \Delta B$, $\Delta A \Delta C$, $\Delta B \Delta C$ or $\Delta A \Delta B \Delta C$ within the simultaneous measurement of the expectation values of all three operators A, B and C? Naturally the states that realise these different possibilities are usually non-identical.

A typical example for the measurement of two operators A and B are the aforementioned position and momentum operators x and p, respectively. To measure the position of a particle in space is an example for three operators corresponding to the coordinate components x, y and z. This measurement is of course trivial in conventional space, but becomes nontrivial and interesting when one considers a noncommutative space in which coordinate component operators do not commute, see e.g. [30, 31] for concrete examples and [32] for a recent review.

The question regarding the minimal values of ΔA and ΔB is not challenging when the commutator on the right hand side of (2.4) is a constant, as one can always achieve $\Delta A \to 0$ or $\Delta B \to 0$ by taking $\Delta B \to \infty$ or $\Delta A \to \infty$, respectively, and still respect the lower bound. This is the standard scenario of Heisenberg's uncertainty relation in which one must give up all the information about A or B to measure B or A, respectively, with absolute precision. However, when the resulting commutator on the right hand side of the inequality involves operators, i.e. when the lower bound becomes a function of A and/or B it is no longer possible to carry out the limits in this trivial manner. Such a scenario arises for theories formulated on certain noncommutative spaces as discussed in section 2.2. In that case one may assume that the minima are reached for *coherent states* that is at equality in (2.4). By defining the function $f(\Delta A, \Delta B) := \Delta A \Delta B - \frac{1}{2} |\langle [A, B] \rangle|$, the critical values are simply obtained by simultaneously solving $f(\Delta A, \Delta B) = 0$ and $\partial_{\Delta B} f(\Delta A, \Delta B) = 0$ for ΔA or $\partial_{\Delta A} f(\Delta A, \Delta B) = 0$ for ΔB from which one can identify the minimum $\Delta A_{\min}$ or $\Delta B_{\min}$, respectively, see e.g. [28]. We note that there is no reason to expect that the product of the individual minimal values $\Delta A_{\min}$ and $\Delta B_{\min}$ is equal to the minimal value of the product $(\Delta A \Delta B)_{\min}$. From this argument we do not obtain any information about the states involved.

3.1 The direct versus the analytic method

Let us now see how to derive the associated minimizing states $|\psi\rangle_{\min,A}$, $|\psi\rangle_{\min,B}$ and $|\psi\rangle_{\min,AB}$ for which these minima are reached. Following Jackiw [33] we recall the difference between the *direct* and *analytic* method that determine the state $|\psi\rangle$ minimizing the uncertainty product. Making the same assumption as above, namely that the minimum is reached for equality, the direct method follows from a comparison of the Schwartz and triangle inequality for $\left| \left\langle \hat{A} \hat{B} \right\rangle \right|^2$. It then consists of solving

$$\left[A - \alpha + \frac{\langle [A, B] \rangle}{2b^2} (B - \beta) \right] |\psi\rangle = 0 \tag{3.1}$$

for $|\psi\rangle$ involving the three free parameters $\alpha := \langle A \rangle$, $\beta := \langle B \rangle$ and $b^2 := \left\langle B^2 \right\rangle - \langle B \rangle^2$. Once the eigenvalue problem in (3.1) is solved, these parameters are just computed in a self-consistent manner via their defining relations. They may be used to convert the solutions into proper square integrable functions and to minimize the desired quantities, that are either the separate minimal values $\Delta A_{\min}$, $\Delta B_{\min}$ or the minimal product $(\Delta A \Delta B)_{\min}$. In the derivation of (3.1) one makes two assumptions: First that the minimal state is reached for the equality sign in (2.4) and second that the commutator $[A, B]$ is a c-number rather than a q-number, that is a constant and not an operator.

The analytic method on the other hand makes no assumptions about the right hand side in the inequality (2.4). In that scheme one treats the left hand side as a functional, minimizing it together with the supplementary assumption that $|\psi\rangle$ is normalizable, i.e. one solves

$$\frac{\delta}{\delta \langle\psi|} \left[\left(\langle\psi| A^2 |\psi\rangle - \langle\psi| A |\psi\rangle^2 \right) \left(\langle\psi| B^2 |\psi\rangle - \langle\psi| B |\psi\rangle^2 \right) - m \left(\langle\psi|\psi\rangle - 1 \right) \right] = 0, \quad (3.2)$$

with Lagrange multiplier m, for $|\psi\rangle$. In a straightforward manner this leads to the eigenvalue problem

$$\left[\frac{(A-\alpha)^2}{a^2} + \frac{(B-\beta)^2}{b^2} \right] |\psi\rangle = 2 |\psi\rangle . \quad (3.3)$$

As no assumption is made about the right hand side in (2.4) an additional parameter $a^2 := \langle A^2 \rangle - \langle A \rangle^2$ enters the scheme when compared to the direct method. By re-expressing the direct method as

$$\left[\frac{(A-\alpha)^2}{a^2} + \frac{(B-\beta)^2}{b^2} \right] |\psi\rangle = 2 \frac{[A,B]}{\langle [A,B] \rangle} |\psi\rangle , \quad (3.4)$$

Jackiw [33] demonstrated that the two schemes coincide if and only if $|\psi\rangle$ is an eigenstate of the commutator $[A, B]$. Thus when this is not the case at least one of the assumptions on which the direct method is based ceases to be valid.

As there are no obvious analogs to the Schwartz and triangle inequality for three operators, it is not evident how to formulate the direct method for that situation. However, it is straightforward to generalize the analytic method to three observables, that is to minimize triple products of variances $(\Delta A)^2 (\Delta B)^2 (\Delta C)^2$. Using the analogue to (3.2), simply with an additional factor $\left(\langle\psi| C^2 |\psi\rangle - \langle\psi| C |\psi\rangle^2 \right)$ on the first term, one easily derives

$$\left[\frac{(A-\alpha)^2}{a^2} + \frac{(B-\beta)^2}{b^2} + \frac{(C-\gamma)^2}{c^2} \right] |\psi\rangle = 3 |\psi\rangle , \quad (3.5)$$

now with two additional free parameters $\gamma = \langle C \rangle$ and $c^2 = \langle C^2 \rangle - \langle C \rangle^2$, see [34] for an example computation and [35] for an experimental verification.

The advantage of the analytic over the direct method is that it is applicable a) irrespective of the nature of $[A, B]$, i.e. resulting to a number or an operator b) even when the minimum is not reached for equality in (3.3) and c) to generalizations of uncertainty relations involving more than two observables.

3.2 Minimal value coherent states from direct method

When one wishes to obtain more information about the wavefunctions $|\psi\rangle_{\min,A}$, $|\psi\rangle_{\min,B}$ and $|\psi\rangle_{\min,AB}$ via the direct or analytic method one needs to specify a concrete representation for the operators involved. For the algebra (2.8) there are various meaningful representations $\Pi_{(i)}$ that we label by i. For instance, with regard to the standard inner product a non-Hermitian and a Hermitian one are

$$x_{(1)} = (1+\tau p_s^2)x_s, \; p_{(1)} = p_s, \quad \text{and} \quad x_{(2)} = x_s, \; p_{(2)} = \frac{1}{\tau}\tan\left(\sqrt{\tau}p_s\right), \tag{3.6}$$

respectively. Here x_s and p_s are standard canonical variables satisfying $[x_s, p_s] = i\hbar$. Naturally for $\tau \to 0$ one recovers the standard Heisenberg commutation relations. Models in terms of these variables and further representations have been studied in more detail in [28, 36]. As demonstrated in [36] the corresponding physical quantities, namely expectation values for adjoint operators, are representation independent and one may simply chose the most suitable one for one's purpose.

Using now the non-Hermitian representation $\Pi_{(1)}$ in the equation for the direct method (3.1) for the observables $A = x$ and $B = p$, we obtain in momentum space simply a first order differential equation

$$\left[i\hbar\left(1+\tau p_s^2\right)\partial_{p_s} + i\hbar\frac{1+\tau b^2+\tau\beta^2}{2b^2}(p_s-\beta) - \alpha\right]\psi_d(p_s) = 0, \tag{3.7}$$

for the minimal state $\psi_d(p_s)$. Setting the constant $\beta = 0$, equation (3.7) is easily solved to

$$\psi_d(p_s) = \left[\frac{\sqrt{\tau}\Gamma\left(\frac{3}{2}+\frac{1}{2\tau b^2}\right)}{\sqrt{\pi}\Gamma\left(\frac{1}{2}+\frac{1}{2\tau b^2}\right)}\right]^{1/2}\left(1+\tau p_s^2\right)^{-\frac{1}{4\tau b^2}-\frac{1}{4}}\exp\left[-\frac{i\alpha\arctan\left(p_s\sqrt{\tau}\right)}{\hbar\sqrt{\tau}}\right]. \tag{3.8}$$

The constant pre-factor is chosen so that these states are normalized with regard to the quasi-Hermitian inner product

$$\left\langle\psi\right|\psi\rangle_\rho := \int_{-\infty}^{\infty}\rho(p_s)\psi^*(p_s)\psi(p_s)dp_s = 1, \tag{3.9}$$

with metric operator $\rho = \left(1+\tau p_s^2\right)^{-1}$. As this is by now well established in the literature we do not justify the choice of ρ here any further, but instead refer the reader to [37, 38, 39, 40, 41, 42, 43] for the general formalism on how to construct meaningful metric operators and how to define consistent inner products for non-Hermitian systems. A well-known argument in [37] states that the metric becomes unique when two observables with specific properties are specified, see also [44][1]. Using the states ψ_d in the expression $\langle . \rangle_\rho = \langle\psi_d|\,.\,|\psi_d\rangle_\rho$, we then easily compute the relevant expectation values

$$\langle x\rangle_\rho = \alpha, \qquad \left\langle x^2\right\rangle_\rho = \alpha^2 + \frac{\hbar^2(1+\tau b^2)^2}{4b^2}, \qquad \langle p\rangle_\rho = 0, \qquad \left\langle p^2\right\rangle_\rho = b^2. \tag{3.10}$$

The values for $\langle x\rangle_\rho$, $\langle p\rangle_\rho$ and $\left\langle p^2\right\rangle_\rho$ are to be expected by definition from the formalism and the explicit computations are just consistency checks. Minimizing $(\Delta x)^2$ as a function of

[1] In [45] this argument was incorrectly attributed to our previous work [28].

b we find $b = 1/\sqrt{\tau}$, such that the minimal length becomes $\Delta x_{\min} = \hbar\sqrt{\tau}$, which coincides with the findings reported in (2.10) and those in [9]. For this value of b we have $\Delta p = 1/\sqrt{\tau}$ so that the product $\Delta x_{\min}\Delta p = \hbar$ does not saturate the lower bound. In the light of the discussion in the previous section this was to be expected and in fact the minimal value for $\Delta x\Delta p = \hbar/2(1+\tau b^2)$, as well for $\Delta p = b$, would be obtained for $b = 0$. However, while the corresponding minimal length state $|\psi\rangle_{\min,p} = \psi_d(b = 1/\sqrt{\tau})$ is well-defined, $\psi_d(p_s)$ is ill-defined for $b = 0$. So the direct method does not allow us to compute the product coherent states $|\psi\rangle_{\min,xp}$. Let us therefore employ the analytical method to determine them.

3.3 Minimal product coherent states from the analytical method

For the representation $\Pi_{(1)}$ the eigenvalue equation for the analytical method (3.3) in momentum space becomes the second order differential equation

$$\left[-\frac{\hbar^2\left(1+\tau p_s^2\right)^2}{a^2}\partial_{p_s}^2 - \frac{2\hbar(i\alpha+\hbar p_s\tau)\left(1+\tau p_s^2\right)}{a^2}\partial_{p_s} + \frac{\alpha^2}{a^2} + \frac{(p_s-\beta)^2}{b^2} - 2\right]\psi_a(p_s) = 0. \tag{3.11}$$

One may verify that $\psi_d(p_s)$ does not satisfy this equation, which is to be expected. Instead, setting $\beta = 0$ this equation is solved in terms of associated Legendre polynomials of $P_\ell^m(x)$ and Legendre functions of the second kind $Q_\ell^m(x)$

$$\psi_a(p_s) = \exp\left[-\frac{i\alpha\arctan\left(p_s\sqrt{\tau}\right)}{\hbar\sqrt{\tau}}\right]\left[c_1 P_\ell^m(ip_s\sqrt{\tau}) + c_2 Q_\ell^m(ip_s\sqrt{\tau})\right], \tag{3.12}$$

with $\ell = \sqrt{4a^2 + \hbar^2\tau^2 b^2}/(2b\tau\hbar) - 1/2$ and $m = a\sqrt{1+2\tau b^2}/(b\tau\hbar)$. Setting ℓ and m to small integer values we find the first meaningful solutions, in the sense of being nonvanishing real numbers, for the parameters a and b for $\ell = 1$ and $m = 2$. For those values we have $P_1^2(ip_s\sqrt{\tau}) = 0$ and $Q_1^2(ip_s\sqrt{\tau}) = 2/(1+\tau p_s^2)$. Suitably normalized with regard to the inner product (3.9) the minimizing solution becomes

$$\psi_a(p_s) = \sqrt{\frac{8}{3\pi}}\frac{\tau^{1/4}}{1+\tau p_s^2}\exp\left[-\frac{i\alpha\arctan\left(p_s\sqrt{\tau}\right)}{\hbar\sqrt{\tau}}\right]. \tag{3.13}$$

Using this solution we then easily compute the expectation values

$$\langle x\rangle_\rho = \alpha, \qquad \left\langle x^2\right\rangle_\rho = \alpha^2 + \frac{4\hbar^2\tau}{3}, \qquad \langle p\rangle_\rho = 0, \qquad \left\langle p^2\right\rangle_\rho = \frac{1}{3\tau}, \tag{3.14}$$

for $\langle\cdot\rangle_\rho = \langle\psi_a|\cdot|\psi_a\rangle_\rho$. For these values the product of uncertainties $(\Delta x\Delta p)_{\min} = 2\hbar/3$ is minimized by construction. Indeed this value is smaller than $\Delta x_{\min}\Delta p$ obtained from the direct method by just minimizing Δx. On the other hand the uncertainty in x computed from these states $\Delta x = 2\hbar\sqrt{\tau/3}$ is corrected by a factor 1.15 and therefore slightly larger than the one computed from the direct method or the minimization of $f(\Delta x, \Delta p)$. Since the contributions from the two observables to the minimal product is not the same, i.e. $\Delta x = 2\hbar\sqrt{\tau/3}$ and $\Delta p = 1/\sqrt{3\tau}$, the states $\psi_a(p_s)$ are squeezed coherent states for all values of α.

4. Lagrangian and Hamiltonian formulation of the entropic force law

Let us now investigate the modified force equation (2.3) further with a special focus on the question of which choices for α and β are meaningful. We observed in the previous section that an important feature of the entropic force law is that it comes in two parts - one corresponding to the inertial term which is the so-called Newtonian or inertial mass m_I times acceleration and the other due to quantum or classical correction F^{cor} which influences it. The entire expression was derived purely on the basis of thermodynamical arguments and the use of the uncertainty principle for particular states. The latter sets a lower limit on the product of the uncertainties of position and canonical momentum but a canonical momentum (defined as the partial derivative of the Lagrangian with respect to the time-derivative of the generalized coordinate) has no place in Newtonian theory which is concerned only with the ordinary momentum given simply by mass times velocity. To keep things in order it would be reasonable to identify the p appearing in (2.3) with $\dot{x}$ whose rate of change is the acceleration $a = \ddot{x}$.

With this understanding we re-express the modified force equation(2.3) as

$$F = P\ddot{x} - Q\dot{x} = -\frac{\partial V}{\partial x}, \tag{4.1}$$

where the quantities P and Q stand for

$$P := m + \beta \frac{\hbar}{mc^2}\frac{\delta p}{\delta x}, \quad Q := -\alpha \frac{\delta p}{\delta x}. \tag{4.2}$$

We also used in (4.1) the definition $F = -\partial V/\partial x$, where V is a potential. It is straightforward to see that when taking

$$\mathcal{L} = \left(\frac{P}{2}\dot{x}^2 - V\right) e^{-\frac{Q}{P}t} \tag{4.3}$$

to be a Lagrangian, the corresponding Euler-Lagrange equation of motion

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) = \frac{\partial \mathcal{L}}{\partial x} \tag{4.4}$$

is equivalent to (2.3) or (4.1) for time-independent P and Q. We notice that $\mathcal{L}$ depends explicitly on the damping factor

$$\frac{Q}{P} = -\frac{\alpha\omega}{m\omega\frac{\delta x}{\delta p} + \beta\frac{\hbar\omega}{mc^2}}, \tag{4.5}$$

which evidently needs to very small or vanishing altogether. We can use this fact to constrain possible choices for α and β even further. For convenience we summarize previously discussed scenarios in table 1[2]

[2] Alternatively there exist also proposals for corrections to the entropic force resulting from rainbow gravity [46], which is not accounted for in our table.

trial states	α	β	$\delta x/\delta p$	$\lvert Q/P \rvert$
canonical coherent states	1	1	$\frac{1}{m\omega}$	$\frac{\omega}{1+\hbar\omega/mc^2} \gg 1$
canonical coherent states	$\frac{\hbar\omega}{mc^2}$	1	$\frac{1}{m\omega}$	$\frac{\omega}{1+mc^2/\hbar\omega} \gg 1$
canonical coherent states	0	1	$\frac{1}{m\omega}$	0
MOND	0	$-c(\omega\lambda)(1+a/a_0)$	$\frac{1}{m\omega}$	0
mlco states	$(\hbar\tau m\omega)^2$	$\hbar\tau m\omega$	$\hbar\tau/\kappa_\tau$	$\frac{\hbar\tau m\omega^2}{1/\kappa_\tau+\hbar\omega/mc^2} \ll 1$
mvco states	$(\hbar\tau m\omega)^2$	$\hbar\tau m\omega$	$2\hbar\tau$	$\frac{\hbar\tau m\omega^2}{2+\hbar\omega/mc^2} \ll 1$

Table 1: Choices for the dimensionless parameters α, β and different types of coherent states leading to admissible ($\ll 1$) and nonphysical ($\gg 1$) damping factors.

The values in table 1 suggest that the framework discussed in [19], based on a quantum correction for the energy is inconsistent. However, two choices survive this simple test. First of all **any** choice with $\alpha = 0$ and arbitrary nonvanishing but dimensionally acceptable β is consistent. In particular that includes the values leading to a classical MOND theory. Furthermore, the deformations of the energy and entropy expressions corresponding to noncommutative deformations also yield consistent damping factors.

Finally let us also comment on a Hamiltonian $\mathcal{H}$ that is readily derived from the Langrangian $\mathcal{L}$ when taking the associated canonical momentum p (not to be confused with $\dot{x}$) to be

$$p = \frac{\partial \mathcal{L}}{\partial \dot{x}} = \dot{x} P e^{-\frac{Q}{P}t}. \tag{4.6}$$

This leads us to

$$\mathcal{H} = \dot{x}\frac{\partial \mathcal{L}}{\partial \dot{x}} - L = \frac{p^2}{2P} e^{\frac{Q}{P}t} + V e^{-\frac{Q}{P}t}, \tag{4.7}$$

so that depending on the sign of Q/P we obtain a damped kinetic term and amplified potential term, or vice versa, unless $Q = 0$.

Evidently the explicit time dependence in the coefficients provides a major hindrance for the quantization of this system. However, one might follow for instance recent work, using a modified Prelle-Singer approach, in which explicit time-independent first integrals have been identified in different parameter regimes for the damped linear harmonic oscillator problem in order to facilitate the quantization procedure [47].

5. Conclusions

We have re-examined Verlinde's argument on the derivation of the entropic law of gravitation by taking into account a possible modification in the form of adding a term linearly dependent on the momentum uncertainty δp in the energy (2.1) and/or the entropy (2.2) uncertainty. These terms involve two free parameters α and β that are only constraint by dimensional arguments. We then computed the general correction term for the entropic force resulting as a consequences of this modification for various choices for the parameters. The uncertainty ratio $\delta p/\delta x$, on which these terms depend, was computed exactly for some concrete coherent states with underlying standard and generalized Heisenberg uncertainty relation. In addition, we derive explicit expressions for the normalizable wave-functions associated to minimal length and minimal value coherent states. Furthermore, we found that the Lagrangian and Hamiltonian associated to the entropic force are explicitly time-dependent in an exponential form. Demanding for obvious physical reasons that this exponential is damped provided us with a further criterion to select possible values for the parameters α and β.

On a standard space with conventional commutation relations for x and p we found consistent correction terms for $\alpha = 0$ accompanied by any choice for β. The latter can be chosen to implement classical or quantum corrections in the entropy. Remarkably a specific version of the latter option allows for the emergence of a classical MOND theory (2.7). Keeping $\alpha = 0$ other choices for β can lead to classical as well as quantum corrections. Any scenario with $\alpha \neq 0$ and $\beta \neq 0$ leads to exponentially growing Langrangians and Hamiltonians and therefore appear to be inconsistent from a physical perspective. This includes the scenario advocated in [19].

However, when considering the equations in a noncommutative setting with deformed canonical commutation relations (2.8), (2.12), (2.13) dimensional consistency together with the requirement that the classical and noncommutative limit can be carried out by $\hbar \to 0$ and $\tau \to 0$, respectively, leads to consist solutions with a very mild damping or amplifying factor in the Langrangian. One may say that this setting leads to an emergent theory of noncommutative space-time.

There are many interesting issues left to investigate, as for instance to clarify the link to a relativistic theory and to extend the analysis to higher dimensions, using (3.5), to name only two.

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