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Appendix A: Long Replenishment Lead Time

We analyze the case when the replenishment lead time is long so that both (instead of one) retailers place their orders “before” the market size M is realized. We show numerically that the structural results continue to hold.

A.1. Setting 2: Selling two substitutable products through one retailer

Observe that Setting 2 (Figure 4-2) corresponds to the base case when $s_{m1} = s_{m2} = 0$. Hence, $D_1 = m \cdot \frac{(p_2 - s_{b_2}) - \delta(p_1 - s_{b_1})}{\delta - 1}$, and $D_2 = m \cdot [1 - \frac{(p_2 - s_{b_2}) - (p_1 - s_{b_1})}{\delta - 1}]$.

Retailer’s pricing problem. Because the retailer’s pricing problem occurs after the orders z_1 and z_2 are placed and the market size m is realized, the ordering costs (i.e., $w_1 \cdot z_1, w_2 \cdot z_2$) are “sunk” and the sales $S_1 = \min\{D_1, z_1\}$ and $S_2 = \min\{D_2, z_2\}$; respectively. Therefore, the retailer’s problem is: $\max_{p_1, p_2} \{(p_1 + s_{r_1}) \cdot D_1 + (p_2 + s_{r_2}) \cdot D_2\}$, s.t. $D_1 \leq z_1, D_2 \leq z_2$. Let $\mathcal{M}_2 = \frac{2z_2 \cdot (\delta - 1)}{\delta - 1 - s_1 + s_2}$ and $\mathcal{M}_3 = \frac{2(z_1 + z_2)}{1 + s_1}$, we can show that:

$$p_1^* = \begin{cases} \frac{1}{2}(1 + s_{b_1} - s_{r_1}) & \text{if } m \leq \mathcal{M}_3 \\ 1 + s_{b_1} - \frac{z_1 + z_2}{m} & \text{if } m \geq \mathcal{M}_3 \end{cases}, \quad p_2^* = \begin{cases} \frac{1}{2}(\delta + s_{b_2} - s_{r_2}) & \text{if } m \leq \mathcal{M}_2 \\ \frac{m(2\delta - 1 - s_1 + 2s_{b_2}) - 2z_2(\delta - 1)}{2m} & \text{if } \mathcal{M}_2 < m < \mathcal{M}_3 \\ \delta + s_{b_2} - \frac{z_1 + \delta \cdot z_2}{m} & \text{if } m \geq \mathcal{M}_3 \end{cases}$$

$$S_1^* = \begin{cases} \frac{m \cdot (\delta \cdot s_1 - s_2)}{2(\delta - 1)} & \text{if } m \leq \mathcal{M}_2 \\ \frac{1}{2}m(1 + s_1) - z_2 & \text{if } \mathcal{M}_2 < m < \mathcal{M}_3 \\ z_1 & \text{if } m > \mathcal{M}_3 \end{cases}, \quad S_2^* = \begin{cases} \frac{m \cdot (\delta - 1 + s_2 - s_1)}{2(\delta - 1)} & \text{if } m \leq \mathcal{M}_2 \\ z_2 & \text{if } m > \mathcal{M}_2 \end{cases}$$

Retailer’s ordering problem. By using (p_1^*, p_2^*) and (S_1^*, S_2^*) , the retailer’s problem is:

$$\max_{z_1, z_2} E_M[\Pi_r(m)] = \int_0^{\mathcal{M}_2} \Pi_{r,1}(m) \cdot f(m) dm + \int_{\mathcal{M}_2}^{\mathcal{M}_1} \Pi_{r,2}(m) \cdot f(m) dm + \int_{\mathcal{M}_1}^{\infty} \Pi_{r,3}(m) \cdot f(m) dm, \text{ where}$$

$$\Pi_r(m) = (p_1^* + s_{r_1}) \cdot S_1^* + (p_2^* + s_{r_2}) \cdot S_2^* - w_1 \cdot z_1 - w_2 \cdot z_2 = \begin{cases} \Pi_{r,1}(m) & \text{if } m \leq \mathcal{M}_2 \\ \Pi_{r,2}(m) & \text{if } \mathcal{M}_2 < m < \mathcal{M}_1 \\ \Pi_{r,3}(m) & \text{if } m \geq \mathcal{M}_1 \end{cases}$$

Donor’s problem. When offering uniform subsidy $s_1 = s_2 = s$, the donor’s problem is: $\max_s E_M[S_1^* + S_2^*]$ s.t. $E_M[s \cdot (S_1^* + S_2^*)] \leq K$, where

$$E_M[S_1^* + S_2^*] = \int_0^{\mathcal{M}_2} \frac{m(\delta \cdot s + \delta - 1)}{2(\delta - 1)} \cdot f(m) dm + \int_{\mathcal{M}_2}^{\mathcal{M}_1} \left(\frac{m \cdot s}{2} + z_2^*\right) \cdot f(m) dm + \int_{\mathcal{M}_1}^{\infty} (z_1^* + z_2^*) \cdot f(m) dm.$$

A.2. Setting 3: Two manufacturers sell two products separately through two retailers

We now consider Setting 3 (Figure 4-3) that corresponds to the base case when $s_{m1} = s_{m2} = 0$ and the wholesale price is exogenous.

Retailers' pricing problem. By using the same approach as before, each retailer solves:

$$\begin{aligned} \max_{p_1} \{ (p_1 + s_{r1}) \cdot D_1 \} \quad \text{s.t.} \quad D_1 &= m \cdot \frac{(p_2 - s_{b2}) - \delta(p_1 - s_{b1})}{\delta - 1} \leq z_1, \quad \text{and} \\ \max_{p_2} \{ (p_2 + s_{r2}) \cdot D_2 \} \quad \text{s.t.} \quad D_2 &= m \cdot \left[1 - \frac{(p_2 - s_{b2}) - (p_1 - s_{b1})}{\delta - 1} \right] \leq z_2. \end{aligned}$$

Let $\mathcal{M}'_2 = \frac{z_2 \cdot (4\delta - 1) \cdot (\delta - 1)}{2\delta^2 - \delta(2 + s_1) + (2\delta - 1)s_2}$ and $\mathcal{M}'_3 = \frac{z_1(2\delta - 1) + z_2\delta}{(1 + s_1) \cdot \delta}$, we get:

$$p_1^* = \begin{cases} \frac{\delta - 1 - s_{b1} - s_2 + 2\delta(s_{b1} - s_{r1})}{4\delta - 1} & \text{if } m \leq \mathcal{M}'_2 \\ \frac{m((\delta - 1)(1 + s_{b1}) - \delta s_{r1}) - z_2(\delta - 1)}{m(2\delta - 1)} & \text{if } \mathcal{M}'_2 < m < \mathcal{M}'_3 \\ 1 + s_{b1} - \frac{z_1 + z_2}{m} & \text{if } m \geq \mathcal{M}'_3 \end{cases}, \quad p_2^* = \begin{cases} \frac{2\delta^2 - 2\delta - s_{b2} - \delta s_1 + 2\delta(s_{b2} - s_{r2})}{4\delta - 1} & \text{if } m \leq \mathcal{M}'_2 \\ \frac{m(\delta(2\delta - 2 - s_1) + (2\delta - 1)s_{b2}) - 2z_2(\delta - 1)\delta}{m(2\delta - 1)} & \text{if } \mathcal{M}'_2 < m < \mathcal{M}'_3 \\ \delta + s_{b2} - \frac{z_1 + z_2\delta}{m} & \text{if } m \geq \mathcal{M}'_3 \end{cases}$$

$$S_1^* = \begin{cases} m \cdot \frac{\delta^2(1 + 2s_1) - \delta(1 + s_1 + s_2)}{(4\delta - 1)(\delta - 1)} & \text{if } m \leq \mathcal{M}'_2 \\ \frac{\delta \cdot (m(1 + s_1) - z_2)}{2\delta - 1} & \text{if } \mathcal{M}'_2 < m < \mathcal{M}'_3 \\ z_1 & \text{if } m \geq \mathcal{M}'_3 \end{cases}, \quad S_2^* = \begin{cases} m \cdot \frac{(2\delta - 1) \cdot s_2 + \delta(2\delta - 2 - s_1)}{(4\delta - 1)(\delta - 1)} & \text{if } m \leq \mathcal{M}'_2 \\ z_2 & \text{if } m > \mathcal{M}'_2 \end{cases}$$

Retailers' ordering problem. By using (p_1^*, p_2^*) and (S_1^*, S_2^*) , each retailer's profit $\Pi_r^i(m)$, $i = 1, 2$ is:

$$\Pi_r^1(m) = (p_1^* + s_{r1}) \cdot S_1^* - w_1 \cdot z_1 = \begin{cases} \Pi_{r,1}^1(m) & \text{if } m \leq \mathcal{M}'_2 \\ \Pi_{r,2}^1(m) & \text{if } \mathcal{M}'_2 < m < \mathcal{M}'_3 \\ \Pi_{r,3}^1(m) & \text{if } m \geq \mathcal{M}'_3 \end{cases}$$

$$\Pi_r^2(m) = (p_2^* + s_{r2}) \cdot S_2^* - w_2 \cdot z_2 = \begin{cases} \Pi_{r,1}^2(m) & \text{if } m \leq \mathcal{M}'_2 \\ \Pi_{r,2}^2(m) & \text{if } \mathcal{M}'_2 < m < \mathcal{M}'_3 \\ \Pi_{r,3}^2(m) & \text{if } m \geq \mathcal{M}'_3 \end{cases}$$

Hence, each retailer maximizes its own profit by solves:

$$\begin{aligned} \max_{z_1} E_M[\Pi_r^1(m)] &= \int_0^{\mathcal{M}'_2} \Pi_{r,1}^1(m) \cdot f(m) dm + \int_{\mathcal{M}'_2}^{\mathcal{M}'_3} \Pi_{r,2}^1(m) \cdot f(m) dm + \int_{\mathcal{M}'_3}^{\infty} \Pi_{r,3}^1(m) \cdot f(m) dm, \\ \max_{z_2} E_M[\Pi_r^2(m)] &= \int_0^{\mathcal{M}'_2} \Pi_{r,1}^2(m) \cdot f(m) dm + \int_{\mathcal{M}'_2}^{\mathcal{M}'_3} \Pi_{r,2}^2(m) \cdot f(m) dm + \int_{\mathcal{M}'_3}^{\infty} \Pi_{r,3}^2(m) \cdot f(m) dm \end{aligned}$$

Donor's problem. When offering uniform subsidy $s_1 = s_2 = s$, the donor's problem is: $\max_s E_M[S_1^* + S_2^*]$ s.t. $E_M[s \cdot (S_1^* + S_2^*)] \leq K$, where

$$E_M[S_1^* + S_2^*] = \int_0^{\mathcal{M}'_2} \frac{m(s + \delta(3 + 2s))}{4\delta - 1} \cdot f(m)dm + \int_{\mathcal{M}'_2}^{\mathcal{M}'_3} \left(\frac{\delta(m(1+s) - z_2^*)}{2\delta - 1} + z_2^* \right) \cdot f(m)dm + \int_{\mathcal{M}'_3}^{\infty} (z_1^* + z_2^*) \cdot f(m)dm.$$

A.3. Numerical Analysis

We consider the market size $M \sim N(1, 0.04)$, set $w_1 = 0.5, w_2 = 0.8$, set $\delta = 1.2$, and we get Figure 1.

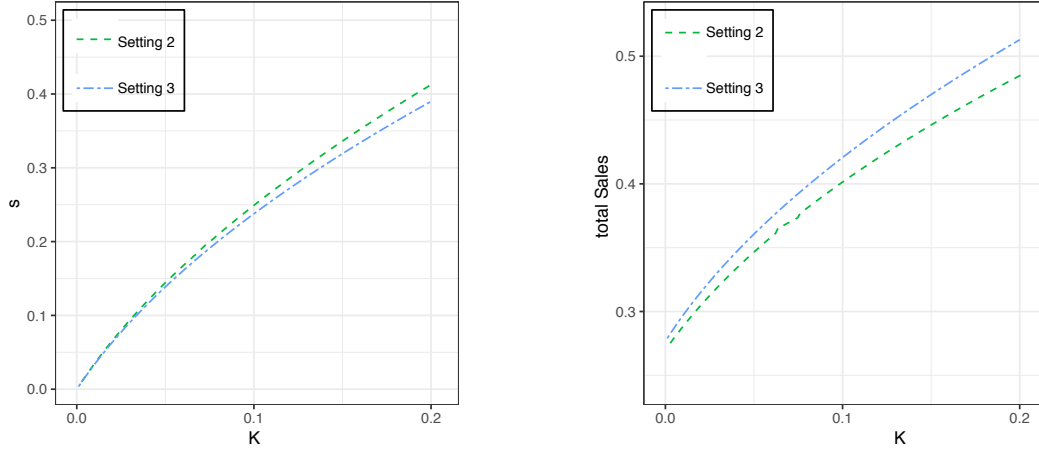


Figure 1 Optimal uniform subsidy (left) and the corresponding total sales (right)

From Figure 1, we find that the optimal per unit subsidy s^* is lower in setting 3, and the total sales ($S_1^* + S_2^*$) is higher in setting 3. Hence, we can conclude that, by using the same budget K , having more retail-channel choice can increase product adoption. Therefore, our structural results obtained in Section 5 continue to hold even when the replenishment lead time is long so that both retailers have to place their orders before the market size is realized.

Appendix B: Proofs

Proof of Proposition 1 By considering the budget constraint, we can obtain that $D \leq \frac{1-w+\sqrt{(1-w)^2+8K}}{4}$. As the objective function is increasing in D , we know that the optimal $D^* = \frac{1-w+\sqrt{(1-w)^2+8K}}{4}$. And we can then calculate the optimal s^* via substitution.

Proof of Proposition 2 By taking the first order derivative of $f_1(D_1, D_2)$ with respect to D_1, D_2 , we get:

$$\begin{aligned} \frac{\partial f_1}{\partial D_1} &= 4(D_1 + D_2) + (w_1 - 1) = 2s_1 + (1 - w_1) = 2(D_1 + D_2) + s_1 > 0, \\ \frac{\partial f_1}{\partial D_2} &= 4(D_1 + \delta D_2) + (w_2 - \delta) = 2s_2 + (\delta - w_2) = 2(D_1 + \delta D_2) + s_2 > 0, \end{aligned}$$

from which we know that $f_1(D_1, D_2)$ is increasing in both D_1 and D_2 . As the objective function $D_1 + k \cdot D_2$ is also increasing in both D_1 and D_2 , we know that the optimal D_1^* and D_2^* should satisfy the binding budget constraint (i.e., $f_1(D_1^*, D_2^*) = K$). Next, by considering the first order condition of the objective function of the donor's problem given by (10), we obtain $D_2^* = \frac{(\delta - w_2) - (1 - w_1)}{4(\delta - 1)}$. When $\delta - w_2 \geq 1 - w_1$, then D_2^* is feasible, else when $\delta - w_2 < 1 - w_1$, we can find that the objective function is always decreasing in D_2 when $D_2 > 0$, thus we can obtain $D_2^* = 0$. As such, we can get the corresponding D_1^* and optimal subsidy $(s_{b_i}^*, s_{r_i}^*)$ via substitution. Moreover, as $(D_1^*, D_2^*) = (\frac{1}{4}(1 - w_1 + \sqrt{8K + (1 - w_1)^2}), 0)$ is always a feasible solution of donor's problem in setting 2, we know that total demand in setting 2 $D_1^* + D_2^* \geq \frac{1}{4}(1 - w_1 + \sqrt{8K + (1 - w_1)^2})$.

Proof of Proposition 3 By denoting the subsidy cost (i.e., the left hand side of (13)) as $f_2(D_1, D_2)$ and by taking the first order derivative of $f_2(\cdot)$ with respect to D_1, D_2 , we get:

$$\begin{aligned} \frac{\partial f_2}{\partial D_1} &= 2D_1 \cdot \frac{2\delta - 1}{\delta} + 2D_2 + (w_1 - 1) = 2s_1 + (1 - w_1) = \frac{2\delta - 1}{2\delta} \cdot D_1 + D_2 + s_1 > 0, \\ \frac{\partial f_2}{\partial D_2} &= 2(2\delta - 1)D_2 + 2D_1 + (w_2 - \delta) = 2s_2 + (\delta - w_2) = (2\delta - 1)D_2 + D_1 + s_2 > 0, \end{aligned}$$

from which we know that $f_2(D_1, D_2)$ is increasing in both D_1 and D_2 . As the objective function $D_1 + D_2$ is also increasing in both D_1 and D_2 , we know that the optimal D_1^* and D_2^* should satisfy the binding budget constraint (i.e., $f_2(D_1^*, D_2^*) = K$). Also, from (12), we know that D_i only depends on the total subsidy s_i for each product so that we can solve out the unique s_i based on the binding budget constraint, while the optimal $s_{b_i}^*$ and $s_{r_i}^*$ are not uniquely determined.

Proof of Corollary 1 To achieve the same demand (D_1, D_2) , the donor should spend $f_1(D_1, D_2) = 2D_1^2 + 2\delta D_2^2 + 4D_1D_2 + (w_1 - 1)D_1 + (w_2 - \delta)D_2$ in setting 2 and spend $f_2(D_1, D_2) = \frac{2\delta - 1}{\delta}D_1^2 + 2D_1D_2 + (2\delta - 1)D_2^2 + (w_1 - 1)D_1 + (w_2 - \delta)D_2$ in setting 3. By comparing $f_1(D_1, D_2)$ and $f_2(D_1, D_2)$, we obtain:

$$f_1(D_1, D_2) - f_2(D_1, D_2) = (2 - \frac{2\delta - 1}{\delta}) \cdot (D_1^2 + \delta D_2^2) + 2D_1D_2 > 0.$$

Hence we know that to get the same (D_1, D_2) , the donor needs to spend more money in a single retailer case (i.e., setting 2) than two competing retailers case (i.e., setting 3). Recall Proposition 2 and 3, the optimal solutions of the donor's problem all satisfy the binding constraint. Therefore, we know that the optimal solution $(D_{1,1}^*, D_{1,2}^*)$ of setting 2 with a single retailer satisfies $f_1(D_{1,1}^*, D_{1,2}^*) = K$. Meanwhile, we also know that $f_2(D_{1,1}^*, D_{1,2}^*) < K$, which means $(D_{1,1}^*, D_{1,2}^*)$ is not the optimal solution of setting 3 with two competing retailers. As such, we know that the optimal solution of setting 3 yields a greater total demand (i.e., the objective function $D_1 + D_2$) than the optimal solution of setting 2.

Proof of Proposition 4 It is easy to check that the objective function $D = \frac{1-c}{4} + \frac{s'}{4}$ and the donor's subsidy cost $s' \cdot (\frac{1-c}{4} + \frac{s'}{4})$ are both increasing in s' . Hence we know that the budget constraint is binding at the optimal solution. By solving the binding budget constraint, we obtain $s' = \frac{-(1-c) + \sqrt{(1-c)^2 + 16K}}{2}$ and we then get $D^* = \frac{(1-c) + \sqrt{(1-c)^2 + 16K}}{8}$, $W^* = \frac{[(1-c) + \sqrt{(1-c)^2 + 16K}]^2}{128}$, $\pi_r^* = \frac{[(1-c) + \sqrt{(1-c)^2 + 16K}]^2}{64}$, $\pi_m^* = \frac{[(1-c) + \sqrt{(1-c)^2 + 16K}]^2}{32}$ via substitution.

Proof of Proposition 5 By denoting $f_1(D_1, D_2)$ as the subsidy cost (i.e., the left hand side of (19)) and taking the first order derivative, we obtain:

$$\begin{aligned} \frac{\partial f_1}{\partial D_1} &= [-1 + c_1 + 4(D_1 + D_2)] + 4D_1 + 4D_2 = s'_1 + 4(D_1 + D_2) > 0, \\ \frac{\partial f_2}{\partial D_2} &= [-\delta + c_2 + 4(\delta D_2 + D_1)] + 4(D_1 + \delta D_2) = s'_2 + 4(D_1 + \delta D_2) > 0. \end{aligned}$$

Hence we know that for feasible s'_1, s'_2, D_1, D_2 , the donor's expense $f_1(D_1, D_2)$ is increasing in D_1 and D_2 . As the objective function $D_1 + D_2$ is also increasing in D_1 and D_2 , we know the optimal (D_1^*, D_2^*) satisfies the binding budget constraint (i.e., $[-1 + c_1 + 4(D_1^* + D_2^*)] \cdot D_1^* + [-\delta + c_2 + 4(D_1^* + \delta D_2^*)] \cdot D_2^* = K$). Next, by considering the first order condition of donor's objective function given by (20), we obtain $D_2^* = \frac{\delta - c_2 - (1 - c_1)}{\delta(\delta - 1)}$. When $\delta - c_2 \geq 1 - c_2$, $D_2^* > 0$ so that we can further compute $D_1^* = \frac{c_2 - c_1 \delta}{8(\delta - 1)} + \frac{1}{8} \sqrt{c_1^2 - 2c_2 + 16K + \frac{(c_1 - c_2)^2}{\delta - 1}} + \delta$ via substitution. When $\delta - c_2 < 1 - c_2$, $\frac{\delta - c_2 - (1 - c_1)}{\delta(\delta - 1)} < 0$ so that the objective function is always increasing in D_2 when $D_2 > 0$. Hence we get the optimal $D_2^* = 0$ and $D_1^* = \frac{1}{8}(1 - c_1) + \sqrt{(1 - c_1)^2 + 16K}$. And we can then further compute the optimal subsidy (s_1^*, s_2^*) , and the corresponding π_m^* , π_r^* and W^* via substitution.

Proof of Proposition 6 By denoting $f_2(D_1, D_2)$ as the subsidy cost (i.e., the left hand side of (22)) and taking the first order derivative of $f_2(D_1, D_2)$ with respect to D_1 and D_2 , we get:

$$\begin{aligned} \frac{\partial f_2}{\partial D_1} &= c_1 - 1 + 2D_2 + 2D_1 \cdot (4 + \frac{1}{1 - 2\delta} - \frac{2}{\delta}) = 2s'_1 + 1 - c_1 > 0 \\ \frac{\partial f_2}{\partial D_2} &= c_2 - \delta + 2D_1 + D_2 \cdot (-5 + \frac{1}{1 - 2\delta} + 8\delta) = 2s'_2 + \delta - c_2 > 0 \end{aligned}$$

Therefore, for feasible s'_1, s'_2, D_1, D_2 , the donor's expense $f_2(D_1, D_2)$ is increasing in D_1 and D_2 . As the objective function $D_1 + D_2$ is also increasing in D_1 and D_2 , we obtain that the optimal (D_1^*, D_2^*) should satisfy the binding budget constraint (i.e., $[c_1 - 1 + D_2^* + D_1^* \cdot (4 + \frac{1}{1 - 2\delta} - \frac{2}{\delta})] \cdot D_1^* + [c_2 - \delta + D_1^* + D_2^* \cdot (-\frac{5}{2} + \frac{1}{2 - 4\delta} + 4\delta)] \cdot D_2^* = K$), which is stated as the first statement of Proposition 6. Next, we know from (21) that D_i only depends on s'_i , which also implies that the total subsidy per unit s'_i for product i is uniquely determined but the optimal subsidy $(s_{b_i}^*, s_{r_i}^*, s_{m_i}^*)$ are not unique. Then we can easily check that π_r^* , π_m^* , and W^* also only depend on s'_i . Finally,

we show the third statement by the following. To achieve the same demand (D_1, D_2) , the donor should spend $f_1(D_1, D_2)$ in the setting 2 and spend $f_2(D_1, D_2)$ in the setting 3. By comparing $f_1(D_1, D_2)$ and $f_2(D_1, D_2)$, we obtain:

$$f_1(D_1, D_2) - f_2(D_1, D_2) = \frac{1}{2} [D_2(12D_1 + 5D_2) + \frac{4D_1^2}{\delta} + \frac{2D_1^2 + D_2^2}{2\delta - 1}] > 0.$$

Hence we know that to get the same (D_1, D_2) , the donor needs to spend more money in setting 2 than setting 3. As the optimal solutions of the donor's problem all satisfy the binding constraint, we know that the optimal solution $(D_{1,1}^*, D_{1,2}^*)$ of setting 2 satisfies $f_1(D_{1,1}^*, D_{1,2}^*) = K$. Meanwhile, we also know that $f_2(D_{1,1}^*, D_{1,2}^*) < K$, which means $(D_{1,1}^*, D_{1,2}^*)$ is not the optimal solution of setting 3. As such, we know that the optimal solution of setting 3 yields a greater total demand (i.e., $D_1 + D_2$) than setting 2.

Proof of Proposition 7 Then by taking the second order derivative of $E_m[\Pi_r(m)]$ with respect to z and using the Leibniz integral rule, we obtain

$$\frac{\partial^2 E_m[\Pi_r(m)]}{\partial z^2} = - \int_{\frac{2z}{1+s}}^{\infty} \frac{2}{m} \cdot f(m) dm < 0$$

Hence we know the expected profit function of the retailer is concave. Hence the optimal z^* satisfies the first order condition (i.e., $\int_{\frac{2z^*}{1+s}}^{\infty} (1+s - \frac{2z^*}{m}) \cdot f(m) dm - w = 0$). We use $g(z, s, w)$ to represent the function $\int_{\frac{2z^*}{1+s}}^{\infty} (1+s - \frac{2z^*}{m}) \cdot f(m) dm - w$, and we have $g(z^*, s, w) = 0$. By taking the first order derivative of $g(z, s, w)$ with respect to z , s and w , we get:

$$\frac{\partial g}{\partial z} = - \int_{\frac{2z}{1+s}}^{\infty} \frac{2}{m} \cdot f(m) dm < 0, \quad \frac{\partial g}{\partial s} = \int_{\frac{2z}{1+s}}^{\infty} f(m) dm > 0, \quad \frac{\partial g}{\partial w} = -1 < 0$$

From the above, we know that $g(z, s, w)$ is increasing in s and decreasing in z and w . Hence to ensure $g(z^*, s, w) = 0$, we can easily know that z^* is increasing in s and decreasing in w .

Proof of Proposition 8 By taking the first order derivative of $E_M[S]$ with respect to s , we get:

$$\begin{aligned} \frac{\partial E_M[S]}{\partial s} &= \frac{1+s}{2} \cdot \frac{2z^*}{1+s} \cdot f\left(\frac{2z^*}{1+s}\right) \cdot \partial\left(\frac{2z^*}{1+s}\right) / \partial s + \int_0^{\frac{2z^*}{1+s}} \frac{m}{2} \cdot f(m) dm \\ &\quad - z^* f\left(\frac{2z^*}{1+s}\right) \cdot \partial\left(\frac{2z^*}{1+s}\right) / \partial s + \int_{\frac{2z^*}{1+s}}^{\infty} \frac{\partial z^*}{\partial s} \cdot f(m) dm \\ &= \int_0^{\frac{2z^*}{1+s}} \frac{m}{2} \cdot f(m) dm + \int_{\frac{2z^*}{1+s}}^{\infty} \frac{\partial z^*}{\partial s} \cdot f(m) dm \end{aligned}$$

From Proposition 7 we know that z^* is increasing in s . Hence we obtain that $\frac{\partial E_M[S]}{\partial s} > 0$, which indicates that the total sale is increasing in the donor's subsidy s . With the objective function $E_M[S]$ and the total subsidy

cost $s \cdot E_M[S]$ both increasing in s , we know that the optimal solution will be achieved at the binding budget constraint. With the binding budget constraint, we know that when the budget K increase, the optimal s^* will increase.

By taking the first order derivative of the subsidy cost $s \cdot E_M[S]$ with respect to z^* , we get $\frac{\partial(s \cdot E_M[S])}{\partial z^*} = s \cdot (\int_{\frac{z^*}{1+s}}^{\infty} f(m)dm) > 0$, from which we know the cost is increasing in z^* . As we have shown in Proposition 7 that z^* is decreasing in the wholesale price w , we obtain that the cost is decreasing in w . To ensure budget constraint is binding, we get that when w increases, the optimal s^* will increase.

Proof of Proposition 9 By taking the second order derivative of $E_M[\Pi_r(m)]$, we get:

$$\frac{\partial E_M^2[\Pi_r(m)]}{\partial z_1^2} = \frac{\partial M_1}{\partial z_1} \cdot 0 + \int_{M_1}^{\infty} \left(\frac{-2(\delta-1)}{m\delta} \right) \cdot f(m)dm < 0,$$

from which we know the retailer's expected profit by selling product 1 is a concave function of z_1 . By considering the first order condition, we obtain that the optimal ordering decision for product 1 (i.e., z_1^*) satisfies

$$\int_{\frac{2z_1^*(\delta-1)}{\delta s_1 + w_2 - s_2}}^{\infty} \left[\frac{-2(\delta-1)z_1^*}{m\delta} + \frac{\delta s_1 - s_2 + w_2}{\delta} \right] \cdot f(m)dm - w_1 = 0.$$

We use $g(z_1, s_1, s_2, w_1, w_2)$ to represent $\int_{\frac{2z_1(\delta-1)}{\delta s_1 + w_2 - s_2}}^{\infty} \left[\frac{-2(\delta-1)z_1}{m\delta} + \frac{\delta s_1 - s_2 + w_2}{\delta} \right] \cdot f(m)dm - w_1$, and we have shown that $g(z_1^*, s_1, s_2, w_1, w_2) = 0$. By taking the first order derivative of $g(z_1, s_1, s_2, w_1, w_2)$ with respect to z_1, s_1, s_2, w_1, w_2 , we get:

$$\begin{aligned} \frac{\partial g}{\partial z_1} &= \int_{M_1}^{\infty} \left(\frac{-2(\delta-1)}{m\delta} \right) \cdot f(m)dm < 0, & \frac{\partial g}{\partial s_1} &= \int_{M_1}^{\infty} f(m)dm > 0, \\ \frac{\partial g}{\partial s_2} &= \int_{M_1}^{\infty} -\frac{1}{\delta} f(m)dm < 0, & \frac{\partial g}{\partial w_1} &= -1 < 0, & \frac{\partial g}{\partial w_2} &= \int_{M_1}^{\infty} \frac{1}{\delta} f(m)dm > 0 \end{aligned}$$

To ensure $g(z_1^*, s_1, s_2, w_1, w_2) = 0$, we can easily obtain that z_1^* is increasing s_1 and w_2 , while is decreasing in s_2 and w_1 .

Proof of Proposition 10 We use $SS_1(m)$ and $SS_2(m)$ to represent the total sales (i.e., $S_1 + S_2$) under cases when $m \leq M_1$ and $m \geq M_1$, respectively; and we have $SS_1(M_1) = SS_2(M_1)$. By taking the first order derivative of $E_M[S_1 + S_2]$ with respect to s , we obtain:

$$\begin{aligned} \frac{\partial E_M[S_1 + S_2]}{\partial s} &= \frac{\partial M_1}{\partial s} \cdot SS_1(M_1) \cdot f(M_1) + \int_0^{M_1} \frac{m}{2} \cdot f(m)dm \\ &\quad - \frac{\partial M_1}{\partial s} \cdot SS_2(M_1) \cdot f(M_1) + \int_{M_1}^{\infty} \left(\frac{\partial z_1^*}{\partial s} + \frac{m}{2\delta} \right) \cdot f(m)dm \\ &= \int_0^{M_1} \frac{m}{2} \cdot f(m)dm + \int_{M_1}^{\infty} \left(\frac{\partial z_1^*}{\partial s} + \frac{m}{2\delta} \right) \cdot f(m)dm \end{aligned}$$

When $s_1 = s_2 = s$, we know that the optimal order quantity z_1^* satisfies $g(z_1^*, s, w_1, w_2) = \int_{\frac{2z_1^*(\delta-1)}{\delta s + w_2 - s}}^{\infty} \left[\frac{-2(\delta-1)z_1}{m\delta} + \frac{\delta s - s + w_2}{\delta} \right] \cdot f(m) dm - w_1 = 0$. By taking the first order derivative of $g(\cdot)$, we find that $\frac{\partial g}{\partial z_1} < 0$ and $\frac{\partial g}{\partial s} > 0$, from which we can further know z_1^* is increasing in s so as to ensure $g(z_1^*, s, w_1, w_2) = 0$. As z_1^* is increasing in s , we can obtain that the total expected sales is increasing in s (i.e., $\frac{\partial E_M[S_1 + S_2]}{\partial s} > 0$). Moreover, it is obvious that the total expense $E_M[s \cdot (S_1 + S_2)] = s \cdot E_M[S_1 + S_2]$ is also increasing in s . Hence we know that the optimal per unit subsidy s^* should satisfy the binding budget constraint.

Proof of Proposition 11 By taking the first order derivative of $E_M[\Pi_{r_1}(M)]$ with respect to z_1 , we get:

$$\begin{aligned} \frac{\partial E_M[\Pi_{r_1}(m)]}{\partial z_1} &= \frac{\partial M_2}{\partial z_1} \cdot \Pi_{r_1,1}(M_2) \cdot f(M_2) + \int_0^{M_2} (-w_1) \cdot f(m) dm \\ &\quad - \frac{\partial M_2}{\partial z_1} \cdot \Pi_{r_1,2}(M_2) \cdot f(M_2) + \int_{M_2}^{\infty} \left[-\frac{4(\delta-1)}{m(2\delta-1)} \cdot z_1 + \frac{\delta-1-(s_2-w_2)}{2\delta-1} + s_1 - w_1 \right] \cdot f(m) dm \\ &= -w_1 + \int_{M_2}^{\infty} \left[-\frac{4(\delta-1)}{m(2\delta-1)} \cdot z_1 + \frac{\delta-1-(s_2-w_2)}{2\delta-1} + s_1 \right] \cdot f(m) dm \end{aligned}$$

By checking the second order derivative of $E_M[\Pi_{r_1}(m)]$, we obtain: $\frac{\partial^2 E_M[\Pi_{r_1}(m)]}{\partial z_1^2} = \frac{\delta-1}{2\delta-1} \cdot \left[\frac{1}{\delta} \cdot f(M_2) - 4 \int_{M_2}^{\infty} \frac{1}{m} f(m) dm \right] < 0$ when $\frac{1}{\delta} \cdot f(M_2) < 4 \int_{M_2}^{\infty} \frac{1}{m} f(m) dm$. Hence we know that $E_M[\Pi_{r_1}(M)]$ is a concave function of z_1 ; and we can obtain Proposition 11 by considering the first order condition.

Proof of Proposition 12 By taking the first order derivative of $E_M[S_1 + S_2]$ with respect to s , we get:

$$\frac{\partial E_M[S_1 + S_2]}{\partial s} = \int_0^{M_2} \frac{1+2\delta}{4\delta-1} \cdot m \cdot f(m) dm + \int_{M_2}^{\infty} \left[\frac{2(\delta-1)}{2\delta-1} \cdot \frac{\partial z_1^*}{\partial s} + \frac{m}{2\delta-1} \right] \cdot f(m) dm$$

From Proposition 11, we know that $-w_1 + \int_{M_2}^{\infty} \left[-\frac{4(\delta-1)}{m(2\delta-1)} \cdot z_1^* + \frac{\delta-1-(s_2-w_2)}{2\delta-1} + s_1 \right] \cdot f(m) dm = 0$. Hence when $s_1 = s_2 = s$, we denote $g(s, z_1) = -w_1 + \int_{M_2}^{\infty} \left[-\frac{4(\delta-1)}{m(2\delta-1)} \cdot z_1^* + \frac{\delta-1-(s-w_2)}{2\delta-1} + s \right] \cdot f(m) dm$ and we know $g(s, z_1^*) = 0$. It is easy to check that $\frac{\partial g}{\partial z} < 0$ and $\frac{\partial g}{\partial s} > 0$, from which we can obtain that z_1^* is increasing in s so as to ensure $g(s, z_1^*) = 0$. With $\frac{\partial z_1^*}{\partial s} > 0$, we can show $\frac{\partial E_M[S_1 + S_2]}{\partial s} > 0$. Therefore, we obtain that both the objective function and the subsidy cost shown in the donor's problem (41) is increasing in s , from which we know that the budget constraint should be binding at the optimal solution.