Appendix A: Long Replenishment Lead Time

We analyze the case when the replenishment lead time is long so that both (instead of one) retailers place their orders “before” the market size $M$ is realized. We show numerically that the structural results continue to hold.

A.1. Setting 2: Selling two substitutable products through one retailer

Observe that Setting 2 (Figure 4-2) corresponds to the base case when $s_{m1} = s_{m2} = 0$. Hence, $D_1 = m \cdot \frac{(p_2 - s_{b_2}) - \delta (p_1 - s_{b_1})}{s_{b_1}}$, and $D_2 = m \cdot [1 - \frac{(p_2 - s_{b_2}) - (p_1 - s_{b_1})}{s_{b_1}}].$

Retailer’s pricing problem. Because the retailer’s pricing problem occurs after the orders $z_1$ and $z_2$ are placed and the market size $m$ is realized, the ordering costs (i.e., $w_1 \cdot z_1$, $w_2 \cdot z_2$) are “sunk” and the sales $S_1 = \min \{D_1, z_1\}$ and $S_2 = \min \{D_2, z_2\}$; respectively. Therefore, the retailer’s problem is: $\max_{p_1, p_2} \{(p_1 + s_{r_1}) \cdot D_1 + (p_2 + s_{r_2}) \cdot D_2\}$, s.t. $D_1 \leq z_1$, $D_2 \leq z_2$. Let $M_2 = \frac{2(\delta - 1)}{s_{b_1} + s_{b_2}}$ and $M_3 = \frac{2(1 + \delta)}{1 + s_{b_1}}$, we can show that:

$$p_1^* = \begin{cases} \frac{1}{2}(1 + s_{b_1} - s_{r_1}) & \text{if } m \leq M_3, \\ 1 + s_{b_1} - \frac{s_{b_1} + s_{b_2}}{m} & \text{if } m \geq M_3 \end{cases} \quad p_2^* = \begin{cases} \frac{1}{2}(\delta + s_{b_2} - s_{r_2}) & \text{if } m \leq M_2, \\ \frac{m(2\delta - 1 - s_{b_2}) - 2z_2(\delta - 1)}{2m} & \text{if } M_2 < m < M_3, \\ \delta + s_{b_2} - \frac{s_{b_1} + s_{b_2}}{m} & \text{if } m \geq M_3 \end{cases}$$

$$S_1^* = \begin{cases} \frac{m(\delta + s_{b_1} - s_{b_2})}{2(\delta - 1)} & \text{if } m \leq M_2, \\ \frac{1}{2}m(1 + s_b) - z_2 & \text{if } M_2 < m < M_3, \\ z_1 & \text{if } m \geq M_3 \end{cases} \quad S_2^* = \begin{cases} \frac{m(\delta - 1 + s_{b_2} - s_{b_1})}{2(\delta - 1)} & \text{if } m \leq M_2, \\ z_2 & \text{if } m > M_2 \end{cases}$$

Retailer’s ordering problem. By using $(p_1^*, p_2^*)$ and $(S_1^*, S_2^*)$, the retailer’s problem is:

$$\max_{z_1, z_2} E_M[\Pi_r(m)] = \int_0^{M_2} \Pi_{r,1}(m) \cdot f(m) dm + \int_{M_2}^{M_1} \Pi_{r,2}(m) \cdot f(m) dm + \int_{M_1}^{\infty} \Pi_{r,3}(m) \cdot f(m) dm,$$

where

$$\Pi_r(m) = (p_1^* + s_{r_1}) \cdot S_1^* + (p_2^* + s_{r_2}) \cdot S_2^* - w_1 \cdot z_1 - w_2 \cdot z_2 = \begin{cases} \Pi_{r,1}(m) & \text{if } m \leq M_2, \\ \Pi_{r,2}(m) & \text{if } M_2 < m < M_1, \\ \Pi_{r,3}(m) & \text{if } m \geq M_1 \end{cases}$$

Donor’s problem. When offering uniform subsidy $s_1 = s_2 = s$, the donor’s problem is: $\max_s \quad E_M[S_1^* + S_2^*]$ s.t. $E_M[s \cdot (S_1^* + S_2^*)] \leq K$, where

$$E_M[S_1^* + S_2^*] = \int_0^{M_2} \frac{m(s \cdot z + \delta - 1)}{2(\delta - 1)} \cdot f(m) dm + \int_{M_2}^{M_1} \left( \frac{m \cdot s}{2} + z_2^* \right) \cdot f(m) dm + \int_{M_1}^{\infty} (z_1^* + z_2^*) \cdot f(m) dm.$$

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A.2. Setting 3: Two manufacturers sell two products separately through two retailers

We now consider Setting 3 (Figure 4-3) that corresponds to the base case when \( s_{m1} = s_{m2} = 0 \) and the wholesale price is exogenous.

**Retailers’ pricing problem.** By using the same approach as before, each retailer solves:

\[
\begin{align*}
\max_{p_1} \quad & \{(p_1 + s_{r1}) \cdot D_1\} \quad \text{s.t.} \quad D_1 = m \cdot \frac{(p_2 - s_{b2}) - \delta (p_1 - s_{b1})}{\delta - 1} \leq z_1, \quad \text{and} \\
\max_{p_2} \quad & \{(p_2 + s_{r2}) \cdot D_2\} \quad \text{s.t.} \quad D_2 = m \cdot \left[1 - \frac{(p_2 - s_{b2}) - (p_1 - s_{b1})}{\delta - 1}\right] \leq z_2.
\end{align*}
\]

Let \( M'_1 = \frac{z_2 (4\delta - 1) (\delta - 1)}{2 \delta^2 - \delta (2 + s_1) + (2\delta - 1) s_2} \) and \( M'_3 = \frac{z_1 (2\delta - 1) + z_2}{(1 + s_1) \delta} \), we get:

\[
p^*_1 = \begin{cases} \frac{\delta^2 - 2\delta - s_2 + 2\delta^2 (s_1 - s_{r1})}{4\delta - 1} & \text{if } m \leq M'_2 \\
\frac{m (\delta^2 - s_2 - 2\delta^2 (s_1 - s_{r1})) - s_2 (\delta - 1)}{m (1 + s_1) (\delta - 1)} & \text{if } M'_2 < m < M'_3 \\
1 + s_{b1} - \frac{s_1 + s_2}{m} & \text{if } m \geq M'_3
\end{cases}
\]

\[
p^*_2 = \begin{cases} \frac{\delta^2 - 2\delta - s_2 + 2\delta^2 (s_1 - s_{r1})}{4\delta - 1} & \text{if } m \leq M'_2 \\
\frac{m (\delta^2 - s_2 - 2\delta^2 (s_1 - s_{r1})) - s_2 (\delta - 1)}{m (1 + s_1) (\delta - 1)} & \text{if } M'_2 < m < M'_3 \\
\delta + s_{b2} - \frac{s_1 + s_2}{m} & \text{if } m \geq M'_3
\end{cases}
\]

\[
S^*_1 = \begin{cases} \frac{\delta^2 (2\delta - 2) - s_2 + 2\delta^2 (s_1 - s_{r1})}{4\delta - 1} & \text{if } m \leq M'_2 \\
\frac{m (\delta^2 - s_2 - 2\delta^2 (s_1 - s_{r1})) - s_2 (\delta - 1)}{m (1 + s_1) (\delta - 1)} & \text{if } M'_2 < m < M'_3 \\
z_1 & \text{if } m \geq M'_3
\end{cases}
\]

\[
S^*_2 = \begin{cases} \frac{\delta^2 (2\delta - 2) - s_2 + 2\delta^2 (s_1 - s_{r1})}{4\delta - 1} & \text{if } m \leq M'_2 \\
\frac{m (\delta^2 - s_2 - 2\delta^2 (s_1 - s_{r1})) - s_2 (\delta - 1)}{m (1 + s_1) (\delta - 1)} & \text{if } M'_2 < m < M'_3 \\
z_2 & \text{if } m \geq M'_3
\end{cases}
\]

**Retailers’ ordering problem.** By using \((p^*_1, p^*_2)\) and \((S^*_1, S^*_2)\), each retailer’s profit \( \Pi^*_i(m), i = 1, 2 \) is:

\[
\Pi^*_1(m) = (p^*_1 + s_{r1}) \cdot S^*_1 - w_1 \cdot z_1 = \begin{cases} \Pi^*_1,1(m) & \text{if } m \leq M'_2 \\
\Pi^*_1,2(m) & \text{if } M'_2 < m < M'_3 \\
\Pi^*_1,3(m) & \text{if } m \geq M'_3
\end{cases}
\]

\[
\Pi^*_2(m) = (p^*_2 + s_{r2}) \cdot S^*_2 - w_2 \cdot z_2 = \begin{cases} \Pi^*_2,1(m) & \text{if } m \leq M'_2 \\
\Pi^*_2,2(m) & \text{if } M'_2 < m < M'_3 \\
\Pi^*_2,3(m) & \text{if } m \geq M'_3
\end{cases}
\]

Hence, each retailer maximizes its own profit by solving:

\[
\max_{z_1} E_M[\Pi^*_1(m)] = \int_{M'_2}^{M'_3} \Pi^*_1,1(m) \cdot f(m) \, dm + \int_{M'_2}^{M'_3} \Pi^*_1,2(m) \cdot f(m) \, dm + \int_{M'_3}^{\infty} \Pi^*_1,3(m) \cdot f(m) \, dm,
\]

\[
\max_{z_2} E_M[\Pi^*_2(m)] = \int_{M'_2}^{M'_3} \Pi^*_2,1(m) \cdot f(m) \, dm + \int_{M'_2}^{M'_3} \Pi^*_2,2(m) \cdot f(m) \, dm + \int_{M'_3}^{\infty} \Pi^*_2,3(m) \cdot f(m) \, dm.
\]
Donor’s problem. When offering uniform subsidy $s_1 = s_2 = s$, the donor’s problem is:

$$\max_{s} \quad E_M[S^*_1 + S^*_2] \quad \text{s.t.} \quad E_M[s \cdot (S^*_1 + S^*_2)] \leq K,$$

where

$$E_M[S^*_1 + S^*_2] = \int_0^{M_2} \frac{m(s + \delta(3 + 2s))}{4\delta - 1} \cdot f(m)dm + \int_{M_2}^{M_3} \left( \frac{\delta(m(1 + s) - z_2^*)}{2\delta - 1} + z_2^* \right) \cdot f(m)dm + \int_{M_3}^{\infty} (z_1^* + z_2^*) \cdot f(m)dm.$$

A.3. Numerical Analysis

We consider the market size $M \sim N(1, 0.04)$, set $w_1 = 0.5, w_2 = 0.8$, set $\delta = 1.2$, and we get Figure 1.

![Figure 1: Optimal uniform subsidy (left) and the corresponding total sales (right)](image)

From Figure 1, we find that the optimal per unit subsidy $s^*$ is lower in setting 3, and the total sales $(S^*_1 + S^*_2)$ is higher in setting 3. Hence, we can conclude that, by using the same budget $K$, having more retail-channel choice can increase product adoption. Therefore, our structural results obtained in Section 5 continue to hold even when the replenishment lead time is long so that both retailers have to place their orders before the market size is realized.

Appendix B: Proofs

Proof of Proposition 1 By considering the budget constraint, we can obtain that $D \leq \frac{1-w+\sqrt{(1-w)^2+8K}}{4}$. As the objective function is increasing in $D$, we know that the optimal $D^* = \frac{1-w+\sqrt{(1-w)^2+8K}}{4}$. And we can then calculate the optimal $s^*$ via substitution.

Proof of Proposition 2 By taking the first order derivative of $f_1(D_1, D_2)$ with respect to $D_1, D_2$, we get:

$$\frac{\partial f_1}{\partial D_1} = 4(D_1 + D_2) + (w_1 - 1) = 2s_1 + (1 - w_1) = 2(D_1 + D_2) + s_1 > 0,$$

$$\frac{\partial f_1}{\partial D_2} = 4(D_1 + \delta D_2) + (w_2 - \delta) = 2s_2 + (\delta - w_2) = 2(D_1 + \delta D_2) + s_2 > 0,$$
from which we know that \( f_1(D_1, D_2) \) is increasing in both \( D_1 \) and \( D_2 \). As the objective function \( D_1 + k \cdot D_2 \) is also increasing in both \( D_1 \) and \( D_2 \), we know that the optimal \( D_1^* \) and \( D_2^* \) should satisfy the binding budget constraint (i.e., \( f_1(D_1^*, D_2^*) = K \)). Next, by considering the first order condition of the objective function of the donor’s problem given by (10), we obtain \( D_2^* = \frac{(\delta - w_2) - (1 - w_1)}{2(\delta - 1)} \). When \( \delta - w_2 \geq 1 - w_1 \), then \( D_2^* \) is feasible, else when \( \delta - w_2 < 1 - w_1 \), we can find that the objective function is always decreasing in \( D_2 \) when \( D_2 > 0 \), thus we can obtain \( D_2^* = 0 \). As such, we can get the corresponding \( D_1^* \) and optimal subsidy \((s_1^*, s_2^*)\) via substitution. Moreover, as \((D_1^*, D_2^*) = \left( \frac{1}{2}(1 - w_1 + \sqrt{8K + (1 - w_1)^2}), 0 \right)\) is always a feasible solution of donor’s problem in setting 2, we know that total demand in setting 2 \( D_1^* + D_2^* \geq \frac{1}{2}(1 - w_1 + \sqrt{8K + (1 - w_1)^2}) \).

**Proof of Proposition 3**  
By denoting the subsidy cost (i.e., the left hand side of (13)) as \( f_2(D_1, D_2) \) and by taking the first order derivative of \( f_2(\cdot) \) with respect to \( D_1, D_2 \), we get:

\[
\frac{\partial f_2}{\partial D_1} = 2D_1 \cdot \frac{2\delta - 1}{\delta} + 2D_2 + (w_1 - 1) = 2s_1 + (1 - w_1) = \frac{2\delta - 1}{2\delta} \cdot D_1 + D_2 + s_1 > 0,
\]

\[
\frac{\partial f_2}{\partial D_2} = 2(2\delta - 1)D_2 + 2D_1 + (w_2 - \delta) = 2s_2 + (\delta - w_2) = (2\delta - 1)D_2 + D_1 + s_2 > 0,
\]

from which we know that \( f_2(D_1, D_2) \) is increasing in both \( D_1 \) and \( D_2 \). As the objective function \( D_1 + D_2 \) is also increasing in both \( D_1 \) and \( D_2 \), we know that the optimal \( D_1^* \) and \( D_2^* \) should satisfy the binding budget constraint (i.e., \( f_2(D_1^*, D_2^*) = K \)). Also, from (12), we know that \( D_1 \) only depends on the total subsidy \( s_i \) for each product so that we can solve out the unique \( s_i \) based on the binding budget constraint, while the optimal \( s_1^*, s_2^* \) are not uniquely determined.

**Proof of Corollary 1**  
To achieve the same demand \((D_1, D_2)\), the donor should spend \( f_1(D_1, D_2) = 2D_1^2 + 2\delta D_2^2 + 4D_1D_2 + (w_1 - 1)D_1 + (w_2 - \delta)D_2 \) in setting 2 and spend \( f_2(D_1, D_2) = \frac{2\delta - 1}{\delta} D_1^2 + 2D_1D_2 + (2\delta - 1)D_2^2 + (w_1 - 1)D_1 + (w_2 - \delta)D_2 \) in setting 3. By comparing \( f_1(D_1, D_2) \) and \( f_2(D_1, D_2) \), we obtain:

\[
f_1(D_1, D_2) - f_2(D_1, D_2) = (2 - \frac{2\delta - 1}{\delta}) \cdot (D_1^2 + \delta D_2^2) + 2D_1D_2 > 0.
\]

Hence we know that to get the same \((D_1, D_2)\), the donor needs to spend more money in a single retailer case (i.e., setting 2) than two competing retailers case (i.e., setting 3). Recall Proposition 2 and 3, the optimal solutions of the donor’s problem all satisfy the binding constraint. Therefore, we know that the optimal solution \((D_1^*, D_2^*)\) of setting 2 with a single retailer satisfies \( f_1(D_1^*, D_2^*) = K \). Meanwhile, we also know that \( f_2(D_1^*, D_2^*) < K \), which means \((D_1^*, D_2^*)\) is not the optimal solution of setting 3 with two competing retailers. As such, we know that the optimal solution of setting 3 yields a greater total demand (i.e., the objective function \( D_1 + D_2 \)) than the optimal solution of setting 2.
Proof of Proposition 4  
It is easy to check that the objective function \(D = \frac{1+c_i}{s_i} + \frac{c_i}{s_i} \) and the donor’s subsidy cost \(s' \cdot \left(\frac{1+c_i}{s_i} + \frac{c_i}{s_i}\right)\) are both increasing in \(s'\). Hence we know that the budget constraint is binding at the optimal solution. By solving the binding budget constraint, we obtain \(s' = \frac{(1-c_i)\sqrt{16K}}{\sqrt{1-c_i}^2 + 16K}\) and we then get \(D^* = \frac{(1-c_i)\sqrt{16K}}{\sqrt{1-c_i}^2 + 16K}, W^* = \frac{(1-c_i)\sqrt{16K}}{12\sqrt{1-c_i}^2 + 16K}\), \(\pi_i^* = \frac{(1-c_i)\sqrt{16K}}{64\sqrt{1-c_i}^2 + 16K}\), and \(\pi_m^* = \frac{(1-c_i)\sqrt{16K}}{32\sqrt{1-c_i}^2 + 16K}\) via substitution.

Proof of Proposition 5  
By denoting \(f_i(D_1, D_2)\) as the subsidy cost (i.e., the left hand side of (19)) and taking the first order derivative, we obtain:

\[
\frac{\partial f_1}{\partial D_1} = [-1 + c_1 + 4(D_1 + D_2)] + 4D_1 + 4D_2 = s'_1 + 4(D_1 + D_2) > 0,
\]
\[
\frac{\partial f_2}{\partial D_2} = [-\delta + c_2 + 4(\delta D_2 + D_1)] + 4(D_1 + \delta D_2) = s'_2 + 4(D_1 + \delta D_2) > 0.
\]

Hence we know that for feasible \(s'_1, s'_2, D_1, D_2\), the donor’s expense \(f_1(D_1, D_2)\) is increasing in \(D_1\) and \(D_2\). As the objective function \(D_1 + D_2\) is also increasing in \(D_1\) and \(D_2\), we know the optimal \((D_1^*, D_2^*)\) satisfies the binding budget constraint (i.e., \([-1 + c_1 + 4(D_1^* + D_2^*)] \cdot D_1^* + [-\delta + c_2 + 4(D_1^* + \delta D_2^*)] \cdot D_2^* = K\)). Next, by considering the first order condition of donor’s objective function given by (20), we obtain \(D_2^* = \frac{\delta - c_2 - (1-c_i)}{\delta(\delta - 1)}\). When \(\delta - c_2 \geq 1 - c_2, D_2^* > 0\) so that we can further compute \(D_1^* = \frac{c_2 - c_i}{\delta(\delta - 1)} + \frac{1}{\delta} \sqrt{c_1^2 - 2c_1 + 16K + \frac{(1-c_i)^2}{\delta(\delta - 1)} + \delta}\) via substitution. When \(\delta - c_2 < 1 - c_2, \frac{\delta - c_2 - (1-c_i)}{\delta(\delta - 1)} < 0\) so that the objective function is always increasing in \(D_2\) when \(D_2 > 0\). Hence we get the optimal \(D_2^* = 0\) and \(D_1^* = \frac{1}{\delta}(1 - c_1) + \sqrt{(1 - c_i)^2 + 16K}\). And we can then further compute the optimal subsidy \((s'_1, s'_2)\), and the corresponding \(\pi_m^*, \pi_i^*\) and \(W^*\) via substitution.

Proof of Proposition 6  
By denoting \(f_2(D_1, D_2)\) as the subsidy cost (i.e., the left hand side of (22)) and taking the first order derivative of \(f_2(D_1, D_2)\) with respect to \(D_1\) and \(D_2\), we get:

\[
\frac{\partial f_2}{\partial D_1} = c_1 - 1 + 2D_2 + 2D_1 \cdot (4 + \frac{1}{1-2\delta} - \frac{2}{\delta}) = 2s'_2 + 1 - c_1 > 0
\]
\[
\frac{\partial f_2}{\partial D_2} = c_2 - \delta + 2D_1 + D_2 \cdot (-5 + \frac{1}{1-2\delta} + 8\delta) = 2s'_2 + \delta - c_2 > 0
\]

Therefore, for feasible \(s'_1, s'_2, D_1, D_2\), the donor’s expense \(f_2(D_1, D_2)\) is increasing in \(D_1\) and \(D_2\). As the objective function \(D_1 + D_2\) is also increasing in \(D_1\) and \(D_2\), we obtain that the optimal \((D_1^*, D_2^*)\) should satisfy the binding budget constraint (i.e., \([c_1 - 1 + D_2^* + D_1^* \cdot (4 + \frac{1}{1-2\delta} - \frac{2}{\delta})] \cdot D_1^* + [c_2 - \delta + D_1^* + D_2^* \cdot (-5 + \frac{1}{1-2\delta} + 4\delta)] \cdot D_2^* = K\)), which is stated as the first statement of Proposition 6. Next, we know from (21) that \(D_2\) only depends on \(s'_1\), which also implies that the total subsidy per unit \(s'_1\) for product \(i\) is uniquely determined but the optimal subsidy \((s'_1^*, s'_2^*, s'_m^*)\) are not unique. Then we can easily check that \(\pi_m^*, \pi_i^*\), and \(W^*\) also only depend on \(s'_1^*\). Finally,
we show the third statement by the following. To achieve the same demand \((D_1, D_2)\), the donor should spend \(f_1(D_1, D_2)\) in the setting 2 and spend \(f_2(D_1, D_2)\) in the setting 3. By comparing \(f_1(D_1, D_2)\) and \(f_2(D_1, D_2)\), we obtain:

\[
f_1(D_1, D_2) - f_2(D_1, D_2) = \frac{1}{2}[D_2(12D_1 + 5D_2) + \frac{4D_1^2}{\delta} + \frac{2D_1^2 + D_2^2}{\delta - 1}] > 0.
\]

Hence we know that to get the same \((D_1, D_2)\), the donor needs to spend more money in setting 2 than setting 3. As the optimal solutions of the donor’s problem all satisfy the binding constraint, we know that the optimal solution \((D^*_1, D^*_2)\) of setting 2 satisfies \(f_1(D^*_1, D^*_2) = K\). Meanwhile, we also know that \(f_2(D^*_1, D^*_2) < K\), which means \((D^*_1, D^*_2)\) is not the optimal solution of setting 3. As such, we know that the optimal solution of setting 3 yields a greater total demand (i.e., \(D_1 + D_2\)) than setting 2.

**Proof of Proposition 7** Then by taking the second order derivative of \(E_m[\Pi_r(m)]\) with respect to \(z\) and using the Leibniz integral rule, we obtain

\[
\frac{\partial^2 E_m[\Pi_r(m)]}{\partial z^2} = - \int_{\frac{2z^*}{m}}^{\infty} \frac{2}{m} \cdot f(m)dm < 0
\]

Hence we know the expected profit function of the retailer is concave. Hence the optimal \(z^*\) satisfies the first order condition (i.e., \(\int_{\frac{2z^*}{m}}^{\infty} (1 + s - \frac{z^*}{m}) \cdot f(m)dm - w = 0\)). We use \(g(z, s, w)\) to represent the function \(\int_{\frac{2z^*}{m}}^{\infty} (1 + s - \frac{z^*}{m}) \cdot f(m)dm - w\), and we have \(g(z^*, s, w) = 0\). By taking the first order derivative of \(g(z, s, w)\) with respect to \(z, s\) and \(w\), we get:

\[
\frac{\partial g}{\partial z} = - \int_{\frac{2z^*}{m}}^{\infty} \frac{2}{m} \cdot f(m)dm < 0, \quad \frac{\partial g}{\partial s} = \int_{\frac{2z^*}{m}}^{\infty} f(m)dm > 0, \quad \frac{\partial g}{\partial w} = -1 < 0
\]

From the above, we know that \(g(z, s, w)\) is increasing in \(s\) and decreasing in \(z\) and \(w\). Hence to ensure \(g(z^*, s, w) = 0\), we can easily know that \(z^*\) is increasing in \(s\) and decreasing in \(w\).

**Proof of Proposition 8** By taking the first order derivative of \(E_M[S]\) with respect to \(s\), we get:

\[
\frac{\partial E_M[S]}{\partial s} = \frac{1 + s}{2} \cdot \frac{2z^*}{1+s} \cdot f(\frac{2z^*}{1+s}) \cdot \frac{\partial (\frac{2z^*}{1+s})}{\partial s} + \int_{0}^{\frac{2z^*}{m}} \frac{m}{2} \cdot f(m)dm
\]

\[
- z^* f(\frac{2z^*}{1+s}) \cdot \frac{\partial (\frac{2z^*}{1+s})}{\partial s} + \int_{\frac{2z^*}{m}}^{\infty} \frac{\partial z^*}{\partial s} \cdot f(m)dm
\]

\[
= \int_{0}^{\frac{2z^*}{m}} \frac{m}{2} \cdot f(m)dm + \int_{\frac{2z^*}{m}}^{\infty} \frac{\partial z^*}{\partial s} \cdot f(m)dm
\]

From Proposition 7 we know that \(z^*\) is increasing in \(s\). Hence we obtain that \(\frac{\partial E_M[S]}{\partial s} > 0\), which indicates that the total sale is increasing in the donor’s subsidy \(s\). With the objective function \(E_M[S]\) and the total subsidy...
cost \( s \cdot E_M[S] \) both increasing in \( s \), we know that the optimal solution will be achieved at the binding budget constraint. With the binding budget constraint, we know that when the budget \( K \) increase, the optimal \( s^* \) will increase.

By taking the first order derivative of the subsidy cost \( s \cdot E_M[S] \) with respect to \( z^* \), we get \( \frac{\partial E_M[S]}{\partial z^*} = s \cdot (\int_0^\infty f(m)dm) > 0 \), from which we know the cost is increasing in \( z^* \). As we have shown in Proposition 7 that \( z^* \) is decreasing in the wholesale price \( w \), we obtain that the cost is decreasing in \( w \). To ensure budget constraint is binding, we get that when \( w \) increases, the optimal \( s^* \) will increase.

**Proof of Proposition 9** By taking the second order derivative of \( E_M[\Pi_r(m)] \), we get:

\[
\frac{\partial^2 E_M[\Pi_r(m)]}{\partial z_1^2} = \frac{\partial M_1}{\partial z_1} 0 + \int_{M_1}^\infty (-\frac{2(\delta - 1)}{m\delta}) \cdot f(m)dm < 0,
\]

from which we know the retailer’s expected profit by selling product 1 is a concave function of \( z_1 \). By considering the first order condition, we obtain that the optimal ordering decision for product 1 (i.e., \( z_1^* \)) satisfies

\[
\int_{z_1^*}^{\infty} \frac{\int_{M_1}^\infty \left[ -\frac{2(\delta - 1)}{m\delta} \cdot f(m)dm - w_1 \right]}{m^\delta} \cdot f(m)dm - 1 = 0.
\]

We use \( g(z_1, s_1, s_2, w_1, w_2) \) to represent \( \int_{z_1^*}^{\infty} \frac{\int_{M_1}^\infty \left[ -\frac{2(\delta - 1)}{m\delta} \cdot f(m)dm - w_1 \right]}{m^\delta} \cdot f(m)dm - w_1 \), and we have shown that \( g(z_1^*, s_1, s_2, w_1, w_2) = 0 \). By taking the first order derivative of \( g(z_1, s_1, s_2, w_1, w_2) \) with respect to \( z_1, s_1, s_2, w_1, w_2 \), we get:

\[
\frac{\partial g}{\partial z_1} = \int_{M_1}^\infty (-\frac{2(\delta - 1)}{m\delta}) \cdot f(m)dm < 0, \quad \frac{\partial g}{\partial s_1} = \int_{M_1}^\infty f(m)dm > 0,
\]

\[
\frac{\partial g}{\partial s_2} = \int_{M_1}^\infty -\frac{1}{\delta} f(m)dm < 0, \quad \frac{\partial g}{\partial w_1} = -1 < 0, \quad \frac{\partial g}{\partial w_2} = \int_{M_1}^\infty \frac{1}{\delta} f(m)dm > 0.
\]

To ensure \( g(z_1^*, s_1, s_2, w_1, w_2) = 0 \), we can easily obtain that \( z_1^* \) is increasing \( s_1 \) and \( w_2 \), while is decreasing in \( s_2 \) and \( w_1 \).

**Proof of Proposition 10** We use \( SS_1(m) \) and \( SS_2(m) \) to represent the total sales (i.e., \( S_1 + S_2 \)) under cases when \( m \leq M_1 \) and \( m \geq M_1 \), respectively; and we have \( SS_1(M_1) = SS_2(M_1) \). By taking the first order derivative of \( E_M[S_1 + S_2] \) with respect to \( s \), we obtain:

\[
\frac{\partial E_M[S_1 + S_2]}{\partial s} = \frac{\partial M_1}{\partial s} \cdot SS_1(M_1) \cdot f(M_1) + \int_0^{M_1} \frac{m}{2} \cdot f(m)dm
\]

\[
- \frac{\partial M_1}{\partial s} \cdot SS_2(M_1) \cdot f(M_1) + \int_{M_1}^\infty \frac{\partial z_1^*}{\partial s} \cdot m^\delta \cdot f(m)dm
\]

\[
= \int_0^{M_1} \frac{m}{2} \cdot f(m)dm + \int_{M_1}^\infty \frac{\partial z_1^*}{\partial s} \cdot m^\delta \cdot f(m)dm
\]
When \( s_1 = s_2 = s \), we know that the optimal order quantity \( z_1^* \) satisfies 
\[
g(z_1^*, s, w_1, w_2) = \int_{2z_1^*(\omega)}^{\infty} \frac{2(\delta-1)}{m^2} + \frac{\delta s - s + w_2}{4w} \cdot f(m)dm - w_1 = 0.
\]
By taking the first order derivative of \( g(\cdot) \), we find that \( \frac{\partial g}{\partial z_1} < 0 \) and \( \frac{\partial g}{\partial s} > 0 \), from which we can further know \( z_1^* \) is increasing in \( s \) so as to ensure \( g(z_1^*, s, w_1, w_2) = 0 \). As \( z_1^* \) is increasing in \( s \), we can obtain that the total expected sales is increasing in \( s \) (i.e., \( \frac{\partial E_M[S_1+S_2]}{\partial s} > 0 \)). Moreover, it is obvious that the total expense \( E_M[s \cdot (S_1 + S_2)] = s \cdot E_M[S_1 + S_2] \) is also increasing in \( s \). Hence we know that the optimal per unit subsidy \( s^* \) should satisfy the binding budget constraint.

**Proof of Proposition 11** By taking the first order derivative of \( E_M[\Pi_{r_1}(M)] \) with respect to \( z_1 \), we get:
\[
\frac{\partial E_M[\Pi_{r_1}(M)]}{\partial z_1} = \frac{\partial M_2}{\partial z_1} \cdot \Pi_{r_1}(M_2) \cdot f(M_2) + \int_0^{M_2} (-w_1) \cdot f(m)dm
\]
\[
- \frac{\partial M_2}{\partial z_1} \cdot \Pi_{r_1}(M_2) \cdot f(M_2) + \int_{M_2}^{\infty} \left[ - \frac{4(\delta-1)}{m(2\delta-1)} \cdot z_1 + \frac{(\delta - 1 - (s_2 - w_2))}{2\delta - 1} + s_1 - w_1 \right] \cdot f(m)dm
\]
\[
= -w_1 + \int_{M_2}^{\infty} \left[ - \frac{4(\delta-1)}{m(2\delta-1)} \cdot z_1 + \frac{(\delta - 1 - (s_2 - w_2))}{2\delta - 1} + s_1 \right] \cdot f(m)dm
\]
By checking the second order derivative of \( E_M[\Pi_{r_1}(M)] \), we obtain: 
\[
\frac{\partial^2 E_M[\Pi_{r_1}(M)]}{\partial z_1^2} = \frac{\delta - 1}{2\delta - 1} \cdot \left( \frac{1}{2} \cdot f(M_2) - 4 \int_{M_2}^{\infty} \frac{1}{m} \cdot f(m)dm \right) < 0 \text{ when } \frac{1}{2} \cdot f(M_2) < 4 \int_{M_2}^{\infty} \frac{1}{m} \cdot f(m)dm.
\]
Hence we know that \( E_M[\Pi_{r_1}(M)] \) is a concave function of \( z_1 \); and we can obtain Proposition 11 by considering the first order condition.

**Proof of Proposition 12** By taking the first order derivative of \( E_M[S_1 + S_2] \) with respect to \( s \), we get:
\[
\frac{\partial E_M[S_1 + S_2]}{\partial s} = \int_0^{M_2} \frac{1 + 2\delta}{4\delta - 1} \cdot m \cdot f(m)dm + \int_{M_2}^{\infty} \frac{2(\delta-1)}{2\delta - 1} \cdot \frac{\partial z_1^*}{\partial s} + \frac{m}{2\delta - 1} \cdot f(m)dm
\]
From Proposition 11, we know that 
\[
-w_1 + \int_{M_2}^{\infty} \left[ - \frac{4(\delta-1)}{m(2\delta-1)} \cdot z_1^* + \frac{(\delta - 1 - (s_2 - w_2))}{2\delta - 1} + s_1 \right] \cdot f(m)dm = 0.
\]
Hence when \( s_1 = s_2 = s \), we denote \( g(s, z_1) = -w_1 + \int_{M_2}^{\infty} \left[ - \frac{4(\delta-1)}{m(2\delta-1)} \cdot z_1^* + \frac{(\delta - 1 - (s_2 - w_2))}{2\delta - 1} + s_1 \right] \cdot f(m)dm \) and we know \( g(s, z_1^*) = 0 \). It is easy to check that \( \frac{\partial g}{\partial s} < 0 \) and \( \frac{\partial g}{\partial z_1} > 0 \), from which we can obtain that \( z_1^* \) is increasing in \( s \) so as to ensure \( g(s, z_1^*) = 0 \). With \( \frac{\partial z_1^*}{\partial s} > 0 \), we can show \( \frac{\partial E_M[S_1+S_2]}{\partial s} > 0 \). Therefore, we obtain that both the objective function and the subsidy cost shown in the donor’s problem (41) is increasing in \( s \), from which we know that the budget constraint should be binding at the optimal solution.