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A realized volatility approach to option pricing with continuous and jump variance components

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Abstract Stochastic and time-varying volatility models typically fail to correctly price out-of-the-money put options at short maturity. We extend Realized Volatility option pricing models by adding a jump component which provides a rapidly moving volatility factor, and improves on the fitting properties under the physical measure. The change of measure is performed adopting a stochastic discount factor with an equity and two variance risk premia, associated to the continuous and jump components. Our choice preserves analytical tractability and offers a new way of separately estimate variance risk premia by coherently combining high-frequency returns and option data in a multi-factor pricing model.

Keywords High-frequency · Realized volatility · HARG · Option pricing · Variance risk premium · Jumps

1 Introduction

Stochastic and time-varying volatility models, such as [24, 35, 36], are able to qualitatively reproduce the smile (i.e. excess kurtosis) and the smirk (i.e. neg-

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1 active skewness) observed in short term equity options. However, they fail to
2 address these features quantitatively. As a result, they severely underprice out-
3 of-the-money put options. To cope with this problem, a variety of models have
4 been developed to include jumps in returns (see [7–9, 11, 13, 38, 39, 42, 43]
5 in continuous-time, and [18, 25, 40] in discrete-time) and jumps in volatil-
6 ity (see, e.g., [10, 28, 29]). [17] employ a modified version of the two-factor
7 component GARCH in [27] for option pricing, while [8] proposes a two-factor
8 jump-diffusion model to fit the implicit distribution of options on Standard
9 and Poor’s 500 (S&P500) futures. Similarly, the family of Realized Volatility
10 (RV) option pricing models recently proposed by [20] (i.e., ARG, HARG and
11 HARGL) has difficulties in generating realistic level and dynamics of the steep-
12 ness of the implied volatility at short maturity, although, the general shape
13 and dynamics of the smile is much closer to the empirical one compared to
14 the standard GARCH option pricing models. Therefore, the HARGL implies
15 some degree of underpricing for deep out-of-the-money (DOTM) put options.
16 This is a common feature of stochastic volatility option pricing models without
17 jumps, since they cannot completely capture the probability mass in the right
18 tail of the volatility density.

19
20 In this paper we extend the class of RV option pricing models by adding a
21 jump component in volatility and its associated risk premium. The inclusion
22 of jumps in the variance dynamics provides a rapidly moving volatility factor,
23 which will improve on the fitting properties under the physical measure, \mathbb{P} , and
24 on the pricing performance under the risk-neutral measure, \mathbb{Q} . Consequently,
25 our change of measure employs an SDF with three different risk premia: one
26 for equity, and two variance risk premia related to the continuous and jump
27 components. The proposed multiple risk premia SDF allows to improve the
28 flexibility of the option pricing model under the risk neutral dynamics while
29 preserving analytical tractability. In addition, it provides a new methodol-
30 ogy of separate estimation of the continuous and jump variance risk premia
31 which coherently combines information from both high-frequency returns and
32 option data. More specifically, we develop a model where the log-returns are
33 determined by RV dynamics following a process belonging to the HAR-RV
34 family. These processes, introduced by [19], successfully describe the impact
35 that past realized variances aggregated on different time scales (daily, weekly
36 and monthly) have on the current level of realized variance. Recently, [20]
37 have studied the application of these discrete-time models to option pricing
38 introducing the HARGL-RV extension which accounts for transition density
39 specified by noncentered gamma distribution and accounts for the leverage ef-
40 fect through a daily binary component. More recently, [41] have widened the
41 HAR-RV class and included a heterogeneous parabolic structure for leverage,
42 defining the LHARG-RV model.

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44 In this work, we extend the LHARG-RV model to account for a possibil-
45 ity of extreme movements in the evolution of volatility. The newly proposed
46 model is labelled as JLHARG-RV. JLHARG-RV assumes that the dynamics
47 of realized variance is given by the sum of two independent random variables
48 which account for the continuous and the discontinuous components of the
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1 volatility. We model the former as an autoregressive gamma process (see [33])
2 whose conditional mean is assumed to be a linear function of the past real-
3 ized variances and leverage terms aggregated over different time scales (daily,
4 weekly, and monthly). The latter is described as a compound Poisson process
5 where the jump size is sampled from a gamma distribution. For this model
6 we first show how to compute analytically the moment generating function
7 (MGF) of the log-returns, under the physical measure. In order to obtain an
8 analytical option pricing formula, we derive the MGF under the risk neutral
9 measure. The change of measure is performed adopting the same approach as
10 in [17, 20, 31], based on a discrete-time exponential affine SDF which allows to
11 incorporate risk premia for the continuous and discontinuous components of
12 the volatility, in addition to the equity risk premium. We stress the importance
13 of having risk premia for both the volatility factors in order to compensate
14 for two new sources of risk, in addition to the traditional premium related to
15 shocks in the log-return. In particular, including a premium for the jump com-
16 ponent represents an important novel contribution of this work which helps
17 to better understand the negative skew effect implied by out-of-the-money
18 (OTM) option prices quoted on the market. Due to the analytical tractability
19 of exponential-affine forms, we are able to derive the risk-neutral MGF and
20 show that the risk-neutral model still belongs to the JLHARG model class. In
21 particular, we prove the existence of a one-to-one mapping among the parame-
22 ters describing the physical and risk-neutral dynamics of the JLHARG model.
23 An additional advantage of JLHARG is related to the model estimation. This
24 is due to the observability of RV, directly built from the high-frequency time
25 series of log-returns. We compute the RV time series from tick-by-tick returns
26 for the S&P500 futures, from January 1, 1990 to December 31, 2007. In order
27 to separate the two dynamics of volatility, we exploit the Threshold Bipower
28 Variation methodology introduced in [21] which allows to detect the jumps in
29 the RV. Having the time-series for the continuous and discontinuous volatility
30 components, we estimate the parameters of the JLHARG processes employing
31 the Maximum Likelihood Estimator (MLE) on both sets of historical data.

34 To the best of our knowledge, among the approaches available in literature,
35 [14] is the closest to ours. However, a closer look reveals several important
36 differences. The first relevant difference is the method employed to identify
37 and separate the continuous and jump components of the integrated variance.
38 [14] compute a proxy of the continuous component of volatility by means of
39 the Bipower Variation from 5-minute returns and the jump contribution corre-
40 sponds to the difference, when positive, between the Realized Variance and the
41 Bipower Variation. The methodology does not consider any statistical test in
42 order to assess the significance of the jump contribution. The literature warns
43 about the bias in the estimation of the continuous component of the integrated
44 variance in finite sample, especially in presence of successive jump events. A
45 second major difference is that the approach by Christoffersen and co-authors
46 may be viewed as an improved and extended version of the Realized GARCH
47 approach of [34], while the LHARG-ARJ extends the class of RV gamma mod-
48 els [20, 41]. The role played by the observable realized measures in the two
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classes is essentially different. In the former, the conditional variance is a latent process with idiosyncratic shocks given by the RV measure – in the same spirit of the Realized GARCH. The latter directly models the dynamics of the RV components. The impact of the two modeling choices is relevant not only on the estimation methodology – which is based on QMLE for the BPJVM and on MLE for the LHARG-ARJ – but also, and more importantly, on the level of persistence of the conditional variance in the two models. The persistence of the BPJVM latent variance is nearly one, then a miss-specification of the current level of the volatility may lead to the miss-fitting of the term structure of ATM implied volatility.

To assess the pricing performance of our model, we benchmark it with [14]. Our analysis is performed on OTM Plain Vanilla options written on S&P500 Index whose valuation is given each Wednesday from January 1, 1996 to December 31, 2004. We calibrate risk premia on the whole implied volatility surfaces and we compute the option prices using the efficient COS method introduced by [30]. Our results clearly illustrate that JLHARG models represent a valid competitor class to state-of-the-art discrete-time models for the valuation of S&P500 Index OTM options.

The rest of the paper is organized as follows. Section 2 defines our model for log-return and RV under both the physical and risk-neutral probability measures. Section 3 describes the estimation of the model and then analyses its statistical features. In Section 4, we discuss option pricing performances comparing them to the benchmark. Section 5 draws relevant conclusions.

2 The model

2.1 Real-World dynamics

We consider a risky asset with the following log-return dynamics

$$y_t = r + \left(\lambda - \frac{1}{2} \right) \text{RV}_t + \sqrt{\text{RV}_t} \epsilon_t, \quad (1)$$

where r is the risk-free rate, λ is the market price of risk¹, ϵ_t are *i.i.d.* standard normal innovations, and RV_t is realized variance at day t . The aggregate daily dynamics (1) is formally equivalent to that employed in [15, 20, 41]. As a major difference, in this paper we distinguish two separate components of realized variance: a continuous component RV_t^c and a jump component RV_t^j (details on the measurement of RV components are provided in Section 3).

Our approach is motivated by the empirical analyses of [1], who find that the distributions of daily equity returns standardized by the corresponding

¹ As shown in Appendix B, the more general specification

$$y_t = r + \lambda_c \text{RV}_t^c + \lambda_j \text{RV}_t^j + \sqrt{\text{RV}_t^c + \text{RV}_t^j} \epsilon_t \quad (2)$$

admits consistency with the no-arbitrage principle if and only if $\lambda_c = \lambda_j = \lambda$.

RV is approximately Gaussian and [2] who investigate the deviation from normality ascribed to a jump component in the price process. The latter results indicate that the discontinuous component has a minor impact on the distributional properties, since the jump-adjusted standardized series are not systematically closer to the Gaussian than the $y_t/\sqrt{RV_t}$ standardized returns.² This is especially true for time series generated from futures contracts on the S&P500 Index, which are recognized in [2] to suffer from minimal microstructure distortion and low liquidity effects. As can be seen from the density plots

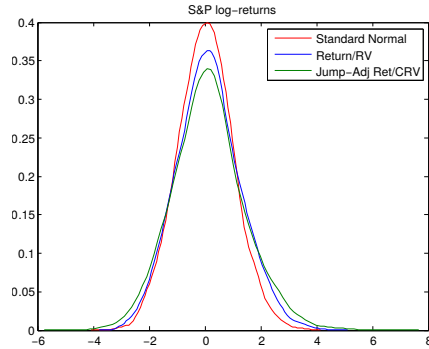


Fig. 1 Standardized log-return distribution. Comparison of the S&P500 futures log-return distribution under different scaling measures: Standard normal distribution (red line), jump-adjusted standardized log-return by RV^c (green line) and standardized log-return by total RV (blue line).

of Figure 1, we observe the same feature for the S&P500 futures in our sampling period. The two-sample Kolmogorov-Smirnov test between the RV standardized and jump-adjusted series indicates that the two distributions cannot be distinguished. If any, by judging on the value of the kurtosis of 3.64 for the jump-adjusted distribution and 3.06 for the RV standardized, we conclude that the latter is closer to a normal distribution than the former one.

Given the information at time t , \mathcal{F}_t , a new realization of the RV components is obtained by sampling at time $t + 1$ from two conditionally independent distributions. The continuous part of RV depends on past realizations of RV^c and of a leverage component ℓ_t , which corresponds to a quadratic function of the total realized variance

$$\ell_t = \left(\epsilon_t - \gamma \sqrt{RV_t^c + RV_t^j} \right)^2.$$

² “Perhaps surprisingly, the results indicate that neither of the jump-adjusted standardized series are systematically closer to Gaussian than the non-adjusted realized volatility standardized returns. [...] One reason is that jumps largely self-standardize: a large jump tends to inflate the (absolute) value of both the return (numerator) and the realized volatility (denominator) of standardized returns, so the impact is muted” [2].

Then, introducing the notation $\mathbf{RV}^c_t = (\text{RV}_{t-21}^c, \dots, \text{RV}_t^c)$ and $\mathbf{L}_t = (\ell_{t-21}, \dots, \ell_t)$, the continuous component of RV is drawn from a noncentred gamma distribution

$$\text{RV}_{t+1}^c | \mathcal{F}_t \sim \bar{\gamma}(\delta, \Theta(\mathbf{RV}^c_t, \mathbf{L}_t), \theta), \quad (3)$$

where δ is the shape parameter, and θ is the scale. The non-centrality is given by

$$\Theta(\mathbf{RV}^c_t, \mathbf{L}_t) = d + \beta_d \text{RV}_t^{c(d)} + \beta_w \text{RV}_t^{c(w)} + \beta_m \text{RV}_t^{c(m)} + \alpha_d \ell_t^{(d)} + \alpha_w \ell_t^{(w)} + \alpha_m \ell_t^{(m)}, \quad (4)$$

where $d \in \mathbb{R}$, $\beta_i \in \mathbb{R}^+$, $\alpha_i \in \mathbb{R}^+$ are constant, and the quantities

$$\begin{aligned} \text{RV}_t^{c(d)} &= \text{RV}_t^c, & \ell_t^{(d)} &= \ell_t, \\ \text{RV}_t^{c(w)} &= \frac{1}{4} \sum_{i=1}^4 \text{RV}_{t-i}^c, & \ell_t^{(w)} &= \frac{1}{4} \sum_{i=1}^4 \ell_{t-i}, \\ \text{RV}_t^{c(m)} &= \frac{1}{17} \sum_{i=5}^{21} \text{RV}_{t-i}^c, & \ell_t^{(m)} &= \frac{1}{17} \sum_{i=5}^{21} \ell_{t-i} \end{aligned}$$

represent the heterogeneous components corresponding to the short-term or daily (d), medium-term or weekly (w) and long-term or monthly (m) realized variance and leverage terms, respectively on the left and right columns above.

The jump component of the realized variance is modelled as a compound Poisson process with intensity $\tilde{\Theta}$ and sizes sampled from a gamma distribution with shape $\tilde{\delta}$ and scale $\tilde{\theta}$

$$\text{RV}_{t+1}^j | \mathcal{F}_t \sim \sum_{i=1}^{n_{t+1}} Y_i \quad \text{with } n_{t+1} \sim \mathcal{P}(\tilde{\Theta}) \text{ and } Y_i \text{ i.i.d. } \sim \gamma(\tilde{\delta}, \tilde{\theta}). \quad (5)$$

Equations (1)-(5) completely characterise the log-return dynamics as an Autoregressive Gamma model in Realized Volatility with Heterogeneous Leverage and Jumps, and we acronym it JLHARG-RV model. The crucial advantage of the JLHARG model is that it satisfies the affine property. The importance of affine processes in finance - due to their analytical tractability - has been acknowledged in many studies (see [23, 26, 41] among others). We prove the following

Proposition 1 *Under \mathbb{P} , the MGF of the log-return $y_{t,T} = \sum_{k=t+1}^T y_k$ for JLHARG model has the following form*

$$\phi^{\mathbb{P}}(t, T, z) = \mathbb{E}^{\mathbb{P}} [e^{zy_{t,T}} | \mathcal{F}_t] = \exp \left(a_t + \sum_{i=1}^p b_{t,i} \text{RV}_{t+1-i}^c + \sum_{i=1}^q c_{t,i} \ell_{t+1-i} \right),$$

where a_t , $b_{t,i}$ and $c_{t,i}$ are given by recursive relations.

Proof: See Appendix A.

2.2 Risk-neutralization

To preserve analytical tractability of the model under the martingale measure we employ an SDF within the family of exponential affine factors, whose high flexibility allows to incorporate multiple factor-dependent risk premia. This approach has been extensively used in literature.³ We propose an SDF of the following form

$$M_{s,s+1} = \frac{e^{-\nu_c RV_{s+1}^c - \nu_j RV_{s+1}^j - \nu_y y_{s+1}}}{\mathbb{E}^{\mathbb{P}} \left[e^{-\nu_c RV_{s+1}^c - \nu_j RV_{s+1}^j - \nu_y y_{s+1}} \mid \mathcal{F}_s \right]}, \quad (6)$$

which represents the Esscher transform from the physical log-return density to the risk neutral one (see [12, 32]). The main advantage of the SDF (6) is to clearly identify the sources of risk and explicitly compensate them with separated risk premia. Specifically, this form allows to have both the continuous (ν_c) and discontinuous (ν_j) variance risk premia, in addition to the standard equity premium (ν_y). The equity premium has to satisfy the following no-arbitrage condition

Proposition 2 *The JLHARG model defined by equations (1) – (5) with SDF given by (6) satisfies the no-arbitrage condition if and only if*

$$\nu_y = \lambda + \frac{1}{2}.$$

Proof: Appendix B.

Moreover, we are able to provide a one-to-one mapping of the parameters under probability measure \mathbb{P} to those under the \mathbb{Q} measure, ensuring that the risk-neutral log-return dynamics is still governed by a JLHARG process.

Proposition 3 *Under risk-neutral measure \mathbb{Q} the realized variance follows a JLHARG process with parameters*

$$\begin{aligned} \beta_d^* &= \frac{\beta_d}{1 - \theta y^{c*}}, & \beta_w^* &= \frac{\beta_w}{1 - \theta y^{c*}}, & \beta_m^* &= \frac{\beta_m}{1 - \theta y^{c*}}, \\ \alpha_d^* &= \frac{\alpha_d}{1 - \theta y^{c*}}, & \alpha_w^* &= \frac{\alpha_w}{1 - \theta y^{c*}}, & \alpha_m^* &= \frac{\alpha_m}{1 - \theta y^{c*}}, \\ \theta^* &= \frac{\theta}{1 - \theta y^{c*}}, & \delta^* &= \delta, & \gamma^* &= \gamma + \lambda + \frac{1}{2}, & d^* &= \frac{d}{1 - \theta y^{c*}}, \\ \tilde{\theta}^* &= \frac{\tilde{\theta}}{(1 - \tilde{\theta} y^{j*})^{\tilde{\delta}}}, & \tilde{\delta}^* &= \tilde{\delta}, & \tilde{\theta}^* &= \frac{\tilde{\theta}}{1 - \tilde{\theta} y^{j*}}, \end{aligned} \quad (7)$$

where $y^{c*} = -\lambda^2/2 - \nu_c + \frac{1}{8}$ and $y^{j*} = -\lambda^2/2 - \nu_j + \frac{1}{8}$.

³ For example, in [3, 16, 20, 31, 41].

1 Proof: Appendix C.

2 Knowing the dynamics of the process under \mathbb{Q} , the moment generating
 3 function under the risk-neutral measure is a straightforward consequence of
 4 Proposition 1.
 5

6 **Corollary 1** *Under \mathbb{Q} the MGF of the JLHARG model is formally the same*
 7 *as in Proposition 1 with equity risk premium $\lambda^* = -0.5$, and $d^*, \delta^*, \theta^*, \tilde{\Theta}^*, \tilde{\delta}^*, \tilde{\theta}^*$*
 8 *$\gamma^*, \alpha_l^*, \beta_l^*$ for $l = d, w, m$ as in (7).*
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10 We point out that the risk premia in the vector (ν_c, ν_j) are the only pa-
 11 rameters that need to be calibrated on option data. Then, all the parameters
 12 governing the dynamics of the process under \mathbb{Q} can be explicitly computed
 13 from the values estimated under \mathbb{P} through the relations given by (7).
 14

15 The JLHARG-RV family nests a variety of RV models as special cases.
 16 The first instance is the JHARG model which preserves the heterogeneous
 17 autoregressive structure for RV but lacks the leverage term. This model can
 18 be seen as a natural extension of the HARG model, by [20], accounting for
 19 a discontinuous component. The second model is the JLHARG model with
 20 Parabolic Leverage (P-JLHARG) that we obtain setting $d = 0$ in (4). The
 21 third one is a JLHARG with zero-mean leverage term (ZM-JLHARG) inspired
 22 by the Component GARCH model of [17]. In that case, heterogeneous leverage
 23 components are given by the following relations
 24

$$\begin{aligned} \bar{\ell}_t^{(d)} &= \epsilon_t^2 - 1 - 2\epsilon_t\gamma\sqrt{\text{RV}_t^c + \text{RV}_t^j}, \\ \bar{\ell}_t^{(w)} &= \frac{1}{4} \sum_{i=1}^4 \left(\epsilon_{t-i}^2 - 1 - 2\epsilon_{t-i}\gamma\sqrt{\text{RV}_{t-i}^c + \text{RV}_{t-i}^j} \right), \\ \bar{\ell}_t^{(m)} &= \frac{1}{17} \sum_{i=5}^{21} \left(\epsilon_{t-i}^2 - 1 - 2\epsilon_{t-i}\gamma\sqrt{\text{RV}_{t-i}^c + \text{RV}_{t-i}^j} \right). \end{aligned}$$

25 The linear $\Theta(\text{RV}_t^c, \mathbf{L}_t)$ reads
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$$\beta_d \text{RV}_t^{c(d)} + \beta_w \text{RV}_t^{c(w)} + \beta_m \text{RV}_t^{c(m)} + \alpha_d \bar{\ell}_t^{(d)} + \alpha_w \bar{\ell}_t^{(w)} + \alpha_m \bar{\ell}_t^{(m)},$$

27 which can be reduced to the form (4) setting $d = -(\alpha_d + \alpha_w + \alpha_m)$, $\beta_l =$
 28 $\beta_l - \alpha_l \gamma^2$ for $l = d, w, m$. The larger flexibility of the leverage term $\bar{\ell}_t$ allows
 29 the model to better describe the skewness and kurtosis of the empirical data.
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33 3 Model estimation and statistical properties

34 The estimation of the parameter under \mathbb{P} is greatly simplified by the direct
 35 observability of RV which avoids the need of latent volatility filtering. In this
 36 paper, the RV time series is obtained from tick-by-tick data for the S&P500
 37 futures, from January 1, 1990 to December 31, 2007. Our estimation procedure
 38 for the continuous and jump component is the following:
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- 1 i) we estimate the total quadratic variation of the log-prices using the Two-
 2 Scale estimator introduced by [45];
 3 ii) we identify the discontinuous component using the Threshold Bipower vari-
 4 ation method by [21] which detects the spikes in RV time series and sepa-
 5 rates it from the continuous component.
 6

7 The RV, so far defined, is built from open-to-close data, thus neglecting the
 8 overnight contribution. We adjust our RV estimator by rescaling the time se-
 9 ries so to match the unconditional mean of the squared daily returns (close-to-
 10 close). We stress that the adopted jump detection method, according to point
 11 (ii) of our procedure, represents a formal statistical test based on asymptotic
 12 theory. This is important to statistically identify days with jumps and subse-
 13 quently associate the most extreme intra-day price movements to jump events
 14 (see for instance [2, 4, 5, 21, 37]).
 15

16 Having the time series for the RV components and log-returns, we can
 17 estimate the parameters of the JLHARG-RV processes via MLE. According
 18 to the model specified in equation (3) and (5), the log-likelihood functions for
 19 the continuous and jump RV components, respectively $l_{t,T}^c$ and $l_{t,T}^j$, are given
 20 by the following series-expansions
 21

$$22 \quad l_{t,T}^c(\lambda, \delta, \theta, d, \beta_d, \beta_w, \beta_m, \alpha_d, \alpha_w, \alpha_m, \gamma) = - \sum_{t=1}^T \left(\frac{\text{RV}_t^c}{\theta} + \Theta(\mathbf{RV}_{t-1}^c, \mathbf{L}_{t-1}(\lambda)) \right)$$

$$23 \quad + \sum_{t=1}^T \log \left(\sum_{k=1}^{\infty} \frac{(\text{RV}_t^c)^{\delta+k-1}}{\theta^{\delta+k} \Gamma(\delta+k)} \frac{\Theta(\mathbf{RV}_{t-1}^c, \mathbf{L}_{t-1}(\lambda))^k}{k!} \right),$$

$$24 \quad l_{t,T}^j(\tilde{\delta}, \tilde{\theta}, \tilde{\Theta}) = - \sum_{t=1}^T \left(\frac{\text{RV}_t^j}{\tilde{\theta}} + \tilde{\Theta} \right) + \sum_{t=1}^T \log \left(\sum_{k=1}^{\infty} \frac{(\text{RV}_t^j)^{k\tilde{\delta}-1}}{\theta^{k\tilde{\delta}} \Gamma(k\tilde{\delta})} \frac{\tilde{\Theta}^k}{k!} \right).$$

25 Both log-likelihoods have a term involving an infinite series. To overcome this
 26 issue we operate a truncation of the infinite sum to the 90th order as suggested
 27 in [20]. The log-likelihood function of returns reads
 28

$$29 \quad l_{t,T}^r(\lambda) = - \sum_{t=1}^T \left(\frac{\left(y_t - r - \left(\frac{1}{2} - \lambda \right) (\text{RV}_t^c + \text{RV}_t^j) \right)^2}{2(\text{RV}_t^c + \text{RV}_t^j)} + \frac{1}{2} \log(2\pi(\text{RV}_t^c + \text{RV}_t^j)) \right).$$

30 The estimation of the parameters is performed maximizing the whole log-
 31 likelihood function $l_{t,T}(\boldsymbol{\theta}) = l_{t,T}^c + l_{t,T}^j + l_{t,T}^r$, with $\boldsymbol{\theta} = (\lambda, \delta, \theta, d, \beta_d, \beta_w, \beta_m, \alpha_d, \alpha_w, \alpha_m, \gamma, \tilde{\delta}, \tilde{\theta}, \tilde{\Theta})$.
 32 In order to reduce the dimension of the space of parameters, we fix δ and $\tilde{\delta}$
 33 by variance targeting, i.e. matching the sample mean of the realized variance.
 34 Fed Funds rate are employed as proxy for the risk-free rate r .
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Parameter	JHARG	P-JLHARG	ZM-JLHARG	Parameter	BPJVM
λ	2.74 (1.50)	2.38 (1.59)	2.69 (1.55)	λ_z	1 (4)
θ	9.75e-06 (9e-08)	9.1e-06 (1e-07)	9.5e-06 (1e-07)	λ_y	4e-05 (8e-05)
δ	1.36	1.25	1.83	γ	1.44e+04 (2e+02)
β_d	4.67e+04 (8e+02)	3.5e+04 (2e+03)	3.9e+04 (1e+03)	ω_z	2.5e-08
β_w	2.9e+04 (1e+03)	3.2e+04 (2e+03)	2.9e+04 (2e+03)	ω_y	0.04
β_m	1.19e+04 (9e+02)	1.4e+04 (3e+03)	1.8e+04 (2e+03)	σ	1.86e-07 (2e-09)
α_d	-	0.28 (0.03)	0.44 (0.04)	θ	1e-05 (5e-05)
α_w	-	0.07 (0.03)	0.42 (0.06)	δ	1.28e-03 (1e-05)
α_m	-	0.00 (0.06)	0.52 (0.10)	ρ	3.3e-01 (2e-02)
γ	-	173 (16)	120 (12)	b_z	6.5e-01 (3e-02)
$\tilde{\theta}$	-	4.7e-05 (3e-06)	-	b_y	9.5e-01 (1e-02)
$\tilde{\delta}$	-	1.15	-	a_z	3.5e-01 (3e-02)
$\tilde{\Theta}$	-	0.299 (0.009)	-	a_y	2.2e+04 (5e+03)
ν_c	756	-1440	-2466		-807
ν_j	-12396	-10239	-7609		-64550
Log-likelihood	10575	10248	10220	Persistence _z	0.999
Persistence	0.85	0.82	0.81	Persistence _y	0.986

Table 1 Maximum likelihood estimates, standard errors, and log-likelihood values. The historical data for the JHARG, P-JLHARG, ZM-JLHARG, and BPJVM models are given by the daily RV computed on tick-by-tick data for the S&P500 futures. For all models, the estimation period ranges from 1990 to 2007.

In Table 1, first four columns, the parameter values estimated under \mathbb{P} are reported. We present estimates for three different models JHARG, P-JLHARG and ZM-JLHARG together with the standard deviations and log-likelihood values. Our results confirm that the impact of the past RV^c components on the current level of RV decreases with the increase of the aggregation horizon. The same evidence has been documented by [19, 22, 41]. Skewness and kurtosis term structures of the underlying distribution play an important role in reproducing the shape of the implied volatility surface and option pricing. Adding a heterogeneous leverage considerably improves the skewness and the excess kurtosis of the log-return probability distribution. In this paper, we not only preserve the heterogeneity of the leverage, but we also add a discontinuous component which captures extreme price movements. With this choice, our JLHARG class of models is able to reproduce a stronger leverage effect. In Figure 2 we show the skewness and the excess kurtosis from a simulation of the P-JLHARG model with parameters from Table 1 at different aggregation time – from one 1 day to 250 days – under both \mathbb{P} and \mathbb{Q} measures. The model is able to reproduce significant negative values of skewness and positive excess kurtosis under the physical measure. When moving to the \mathbb{Q} measure, the effect is strengthened by the presence of the variance risk premia ν_c and ν_j .

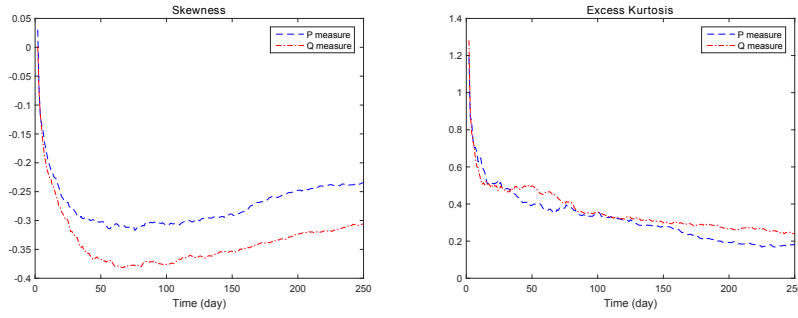


Fig. 2 Skewness and excess kurtosis of the JLHARG process under physical and risk-neutral measures.

4 Option valuation

Our data set consists of Plain Vanilla OTM options on S&P500 Index for each Wednesday from January 1, 1996 to December 31, 2004. We first apply a standard filter removing options with maturity less than 10 days or more than 365 days, implied volatility larger than 70% and prices less than 0.05\$ (see [6], [20] and [41]). Using K/S_t as definition of moneyness, we filter out DOTM options with moneyness larger than 1.3 for call options and less than 0.7 for put options. This choice yields a total number of 46066 observations. For our purposes, put options are identified as DOTM if their moneyness is between $0.7 \leq m \leq 0.9$ and OTM if $0.9 < m \leq 0.98$. On the other hand, call options are said to be DOTM if $1.1 < m \leq 1.3$ and OTM if $1.02 < m \leq 1.1$. Options are called at-the-money (ATM) if $0.98 < m \leq 1.02$. As far as the time to maturity τ is concerned, we identify options as short maturity ($\tau \leq 50$ days), short-medium maturity ($50 < \tau \leq 90$ days), long-medium maturity ($90 < \tau \leq 160$ days), and long maturity ($\tau > 160$ days).

4.1 Model calibration and pricing method

In order to derive the risk-neutral dynamics, the values of risk premium parameters (ν_c, ν_j, ν_y) need to be identified. According to Proposition 2, ν_y is fixed by the no-arbitrage condition, while ν_c and ν_j remain as free parameters to be calibrated on option prices.

Calibration procedure is based on the unconditional minimization of the distance between the market implied and the model implied volatility surface. For this reason, we divide our dataset in different intervals of moneyness and maturity obtaining a 5×4 moneyness-maturity grid. Then, for each subset, we compute the unconditional mean of the market implied volatilities.

In this way, as shown in Table 2, we obtain a 20-point discrete representation of the implied volatility surface. Finally, we compute the same discrete grid

Moneyness	Maturity			
	$\tau \leq 50$	$50 < \tau \leq 90$	$90 < \tau \leq 160$	$160 < \tau$
	Implied Volatility			
$0.7 \leq m \leq 0.9$	0.3564	0.3056	0.2866	0.2662
$0.9 < m \leq 0.98$	0.2353	0.2269	0.2232	0.2230
$0.98 < m \leq 1.02$	0.1958	0.2023	0.2059	0.2108
$1.02 < m \leq 1.1$	0.1767	0.1790	0.1849	0.1923
$1.1 < m \leq 1.3$	0.2317	0.1946	0.1836	0.1842

Table 2 Mean market implied volatilities of S&P500 Index options on each Wednesday from January 1, 1996 to December 31, 2004 (46066 observations) sorted by moneyness and maturity. Moneyness is defined as $m = K/S_t$, where K and S_t are the strike and the underlying price, respectively. Maturity is measured in calendar days.

for the model implied volatility and we identify the optimal values of (ν_c, ν_j) which minimize the distance between the two grids, i.e.

$$\arg \min_{(\nu_c, \nu_j)} \{f_{\text{obj}}(\nu_c, \nu_j)\}.$$

The objective function $f_{\text{obj}}(\nu_c, \nu_j)$ is defined as

$$f_{\text{obj}}(\nu_c, \nu_j) = \sqrt{\sum_{i=1}^5 \sum_{j=1}^4 \left(\text{IV}_{ij}^{\text{mod}}(\nu_c, \nu_j) - \text{IV}_{ij}^{\text{mkt}} \right)^2},$$

and represents the quadratic distance between the model implied volatility surface and the market one, whose elements are $\text{IV}_{ij}^{\text{mod}}(\nu_c, \nu_j)$ and $\text{IV}_{ij}^{\text{mkt}}$, respectively. In order to compute the option prices – and associated implied volatilities – we employ a numerical scheme introduced by [30], termed the COS method. This method, based on Fourier-cosine expansions, efficiently evaluates the price of Plain Vanilla options from the characteristic function of log-returns.

At the bottom of Table 1 we report the calibrated variance risk premia for JHARG models. It is worth recalling that the presence of a positive or a negative value of the risk premium reduces or amplifies the unconditional mean of realized variance, respectively. Moreover, negative premia have the genuine effect to induce more skew in the distribution of returns. The risk premium, ν_c , associated with the continuous component varies from the positive value of the JHARG model to a large negative value for the ZM-JLHARG model. The risk premia, ν_j , associated with the jump component, are all negative and increasing (decreasing in absolute terms) when a better specified form of the leverage is adopted. The most negative jump premium corresponds to the JHARG model and decreases for the P-JLHARG with heterogeneous parabolic leverage. It reaches the highest value (smallest in absolute terms) for the ZM-JLHARG where the heterogeneous leverage is centred. The compensation taking place between ν_c and ν_j is due to the fact that large negative innovations

1 in the price rise the future variance through the leverage term. Then, a better
 2 specification of the leverage component reduces the relative weight of the jump
 3 premium in favour of the premium of the continuous component.
 4

5 6 4.2 Pricing results 7

8 We can summarize the option pricing procedure in four steps: (i) estimation
 9 of the parameters under the physical measure \mathbb{P} ; (ii) unconditional calibration
 10 of the parameter vector (ν_c, ν_j) ; (iii) mapping of parameter values from \mathbb{P} to
 11 \mathbb{Q} using expressions (7); (iv) numerical computation of option prices through
 12 COS method using the MGF recursive formulas in (17).
 13

14 As benchmark approach to assess the pricing performance of JHARG mod-
 15 els, we use the BiPower Jump Variation Model (BPJVM) introduced by [14].⁴
 16 The BPJVM model is a state-of-the-art approach incorporating a GARCH
 17 structure for the latent volatility and jump intensity where bipower and jump
 18 variations play a prominent role as idiosyncratic components. To ensure a fair
 19 comparison with the ZM-JLHARG, risk-neutralization is achieved by means
 20 of a four-dimensional Esscher transform. Two risk premia compensate for the
 21 realized variance components - as in (6) - and two auxiliary premia, μ_c and
 22 μ_j , compensate for the continuous and jump return components. The former
 23 two parameters have to be calibrated on option data, while the latter are
 24 fixed by no-arbitrage. All computational details are available from authors
 25 upon request. The last two columns of Table 1 report the parameter values
 26 for the BPJVM. Following [14], estimation is performed via quasi-maximum
 27 likelihood. At variance with them, we solely use the physical information, and
 28 only afterwards we calibrate the free parameters ν_c and ν_j on market option
 29 prices. This choice is motivated by consistency with the approach proposed
 30 in this paper. Here, we first estimate the physical parameters from historical
 31 information and then we separately assess the impact of risk compensation on
 32 the risk neutral dynamics.
 33

34 As customary in literature ([20, 41, 44]), we employ the Root Mean Square
 35 Error (RMSE) on the percentage IV as performance measure, i.e.

$$36 \quad RMSE_{IV} = \sqrt{\sum_{i=1}^N \frac{(IV_i^{mod} - IV_i^{mkt})^2}{N}},$$

37 where N is the number of options, and IV^{mod} and IV^{mkt} are the model and
 38 the market implied volatility, respectively.
 39

40 Preceding comparison with benchmark models, we perform an internal horse-
 41 race to select the best candidate among JHARG models. In Table 3 we report
 42 the global comparison of the option pricing performances between models be-
 43 longing to the JHARG class. We build ratios between the RMSE of each couple
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 48 ⁴ For practical implementation, we refer to the updated version available on SSRN includ-
 49 ing some corrections to the published version. Link: <http://ssrn.com/abstract=2494379>.
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Implied Volatility RMSE		
Model	Moneyness	
	$0.9 < m < 1.1$	$0.7 < m < 1.3$
JHARG	4.89	6.65
P-JLHARG/JHARG	0.91	0.93
ZM-JLHARG/JHARG	0.83	0.85
ZM-JLHARG/P-JLHARG	0.91	0.92
ZM-JLHARG/BPJVM	0.81	0.83

Table 3 Global option pricing performance for the JLHARG class of models and comparison with the BPJVM model on S&P500 options from January 1, 1996 to December 31, 2004. The RV measure is estimated from 1990 to 2007. Parameter estimates are taken from Table 1.

of models. The table shows that – in terms of RMSE – the performance improves for models accounting for the leverage effect, as expected. Specifically, fixing as benchmark the JHARG model with no leverage, performances in the range of moneyness $0.9 < m < 1.1$ improve by nearly 9% for P-JLHARG and by 17% for ZM-JLHARG. In the range of moneyness $0.7 < m < 1.3$ the improvements are by 7% and by 15% for P-JLHARG and ZM-JLHARG, respectively. These results confirm the well established fact that the inclusion of a leverage component is essential for option pricing. Moreover, ZM-JLHARG always outperforms P-JLHARG independently on the range of moneyness. In accordance with [41] this finding reaffirms that the zero-mean leverage shows more flexibility with respect to the parabolic leverage. The final row of Table 3 reveals the superior performance of the ZM-JLHARG model when benchmarked with the BPJVM. Gains in performance vary from 19%, in the range of moneyness close to ATM, to 17%, when more extreme moneyness are included. The result for the central region of the volatility surface confirms that the heterogeneous structure is a parsimonious and effective way to provide a satisfactory description of the ATM implied volatility dynamics.

In Table 4, the focus is on a more detailed comparison between the ZM-JLHARG model and the competitor BPJVM. Dividing the entire dataset of options according to the grid used for model calibration (see Section 4.1), we observe that for short maturities $\tau \leq 50$ the two models price with almost the same accuracy in the at-the-money region $0.98 < m < 1.02$. BPJVM increases the pricing performance for OTM call options but for DOTM calls the ZM-JLHARG performs better by a factor of 0.71. In the region of short-maturity puts the ZM-JLHARG model consistently over-performs the competitor BPJVM. This result is confirmed with slightly different percentages for options with medium-short maturity $50 < \tau \leq 90$ noting a worsening of BPJVM performance in the ATM region with respect to ZM-JLHARG. As concerns the medium-long maturity region $90 < \tau \leq 160$ BPJVM maintains a higher performance in pricing OTM calls and shows a better valuation of DOTM puts than the ZM-JLHARG. Finally, valuation of long maturity options exhibits the consistent over-performance of the ZM-JLHARG model for

Moneyness	Maturity			
	$\tau \leq 50$	$50 < \tau \leq 90$	$90 < \tau \leq 160$	$160 < \tau$
Panel A				
ZM-JLHARG Implied Volatility RMSE				
$0.7 \leq m \leq 0.9$	12.02	7.53	6.06	4.93
$0.9 < m \leq 0.98$	4.02	3.55	3.72	4.09
$0.98 < m \leq 1.02$	3.43	3.71	4.01	4.52
$1.02 < m \leq 1.1$	4.13	4.59	4.79	4.96
$1.1 < m \leq 1.3$	4.70	3.93	4.58	5.10
Panel B				
ZM-JLHARG/BPJVM Implied Volatility RMSE				
$0.7 \leq m \leq 0.9$	0.89	1.05	1.19	0.71
$0.9 < m \leq 0.98$	0.72	0.81	0.85	0.59
$0.98 < m \leq 1.02$	1.02	0.91	0.89	0.67
$1.02 < m \leq 1.1$	1.03	1.02	0.95	0.63
$1.1 < m \leq 1.3$	0.71	0.61	0.58	0.45

Table 4 Panel A: Percentage $RMSE_{IV}$ of the ZM-JLHARG model sorted by moneyness and maturity. Panel B: $RMSE_{IV}$ ratios computed using BPJVM as benchmark model.

both puts and calls covering all ranges of moneyness under study. The smaller error of ZM-JLHARG for long maturity options suggests that this model has more flexibility than BPJVM to reproduce a realistic term structure of implied volatilities. A possible reason for the rigidity of BPJVM could be the extremely high persistence of both volatility and jump intensity processes, as reported in Table 1. High persistence is a crucial feature to reproduce the long-memory property of the volatility process, nevertheless an extreme level could have the side effect of systematical miss-valuing options – either over-pricing or under-pricing depending on the prevailing high or low level of volatility, respectively.

5 Conclusions

In this paper, we present a class of heterogeneous autoregressive models accounting for a discontinuous component in Realized Volatility. We demonstrate how to analytically characterize the moment generating function of the log-return process under physical and risk-neutral measure. Risk-neutralization is done with a flexible exponential affine pricing kernel which identifies different risks and separately compensates for them introducing three components of risk premium: equity, continuous and jump variance. Then, we show the improvements of the novel class of models in reproducing different features of the implied volatility surface compared with the state-of-the-art of discrete-time pricing models encompassing both continuous and jump dynamics of underlying assets.

As a future perspective, we are aware that equation (5) implicitly assumes that the jump intensity is constant. A stream of literature – consider for instance the empirical conclusions drawn in [14] – advocates an extension considering a time-varying jump intensity. This development is left for future research. Nonetheless, the current contribution shows the ability of JLHARG model to over-perform the benchmark BPJVM model. Moreover, the degree of jump persistence under P measure strongly depends on the methodology employed for the jump identification. We believe that testing for the presence of jumps is an important step for a correct identification procedure. This could partially prevent for over-identification and contamination of the jump component with persistence due to the continuous component.

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A MGF under \mathbb{P} measure

The relations which follow are derived for the log-return dynamics specified in Eq. (2). For the ease of computation, the expression (4) is rewritten as

$$\Theta(\mathbf{RV}^c_t, \mathbf{L}_t) = d + \sum_{i=1}^{22} \beta_i \mathbf{RV}_{t+1-i}^c + \sum_{i=1}^{22} \alpha_i \left(\epsilon_{t+1-i} - \gamma \sqrt{\mathbf{RV}_{t+1-i}^c + \mathbf{RV}_{t+1-i}^j} \right)^2,$$

with

$$\beta_i = \begin{cases} \beta_d & \text{for } i = 1 \\ \beta_w/4 & \text{for } 2 \leq i \leq 5 \\ \beta_w/17 & \text{for } 6 \leq i \leq 22 \end{cases} \quad \alpha_i = \begin{cases} \alpha_d & \text{for } i = 1 \\ \alpha_w/4 & \text{for } 2 \leq i \leq 5 \\ \alpha_w/17 & \text{for } 6 \leq i \leq 22 \end{cases}. \quad (8)$$

We start showing that JLHARG processes satisfy the affine relation

$$\mathbb{E} \left[e^{z y_{s+1} + \mathbf{b} \cdot \mathbf{RV}_{s+1} + c \ell_{s+1}} | \mathcal{F}_s \right] = e^{\mathcal{A}(z, \mathbf{b}, c) + \sum_{i=1}^p \mathcal{B}_i(z, \mathbf{b}, c) \cdot \mathbf{RV}_{s+1-i} + \sum_{j=1}^q \mathcal{C}_j(z, \mathbf{b}, c) \ell_{s+1-j}}, \quad (9)$$

for some functions $\mathcal{A} : \mathbb{R} \times \mathbb{R}^2 \times \mathbb{R} \rightarrow \mathbb{R}$, $\mathcal{B}_i : \mathbb{R} \times \mathbb{R}^2 \times \mathbb{R} \rightarrow \mathbb{R}^2$, $\mathcal{C}_j : \mathbb{R} \times \mathbb{R}^2 \times \mathbb{R} \rightarrow \mathbb{R}$, where $\mathbf{RV}_t = (\mathbf{RV}_t^c, \mathbf{RV}_t^j)$, $\mathbf{b} \in \mathbb{R}^2$, $c \in \mathbb{R}$, and \cdot is the scalar product in \mathbb{R}^2 . To derive the explicit form of the functions \mathcal{A} , \mathcal{B}_i , \mathcal{C}_j which allows to characterise the MGF we show that

$$\begin{aligned} & \mathbb{E}^{\mathbb{P}} \left[e^{z y_t + \mathbf{b} \cdot \mathbf{RV}_t + c \ell_t} | \mathcal{F}_{t-1} \right] \\ &= \mathbb{E}^{\mathbb{P}} \left[e^{z(r + \lambda_c \mathbf{RV}_t^c + \lambda_j \mathbf{RV}_t^j + \sqrt{\mathbf{RV}_t^c + \mathbf{RV}_t^j} \epsilon_t) + \mathbf{b} \cdot \mathbf{RV}_t + c \ell_t} | \mathcal{F}_{t-1} \right] \\ &= \mathbb{E}^{\mathbb{P}} \left[e^{z(r + \lambda_c \mathbf{RV}_t^c + \lambda_j \mathbf{RV}_t^j) + \mathbf{b} \cdot \mathbf{RV}_t} \mathbb{E}^{\mathbb{P}} \left[e^{z \sqrt{\mathbf{RV}_t^c + \mathbf{RV}_t^j} \epsilon_t + c(\epsilon_t - \gamma \sqrt{\mathbf{RV}_t^c + \mathbf{RV}_t^j})} | \mathbf{RV}_t \right] | \mathcal{F}_{t-1} \right] \\ &= \mathbb{E}^{\mathbb{P}} \left[e^{z(r + \lambda_c \mathbf{RV}_t^c + \lambda_j \mathbf{RV}_t^j) + b_1 \mathbf{RV}_t^c + b_2 \mathbf{RV}_t^j - \frac{1}{2} \ln(1-2c) + \left(\frac{\frac{z^2}{2} + \gamma^2 c - 2c\gamma z}{1-2c} \right) (\mathbf{RV}_t^c + \mathbf{RV}_t^j)} | \mathcal{F}_{t-1} \right] \\ &= \mathbb{E}^{\mathbb{P}} \left[e^{zr - \frac{1}{2} \ln(1-2c) + \left(z\lambda_c + b_1 + \frac{\frac{z^2}{2} + \gamma^2 c - 2c\gamma z}{1-2c} \right) \mathbf{RV}_t^c + \left(z\lambda_j + b_2 + \frac{\frac{z^2}{2} + \gamma^2 c - 2c\gamma z}{1-2c} \right) \mathbf{RV}_t^j} | \mathcal{F}_{t-1} \right] \\ &= e^{zr - \frac{1}{2} \ln(1-2c)} \mathbb{E}^{\mathbb{P}} \left[e^{\left(z\lambda_c + b_1 + \frac{\frac{z^2}{2} + \gamma^2 c - 2c\gamma z}{1-2c} \right) \mathbf{RV}_t^c} | \mathcal{F}_{t-1} \right] \mathbb{E}^{\mathbb{P}} \left[e^{\left(z\lambda_j + b_2 + \frac{\frac{z^2}{2} + \gamma^2 c - 2c\gamma z}{1-2c} \right) \mathbf{RV}_t^j} | \mathcal{F}_{t-1} \right]. \end{aligned} \quad (10)$$

In the third line we have used the result that if $Z \sim \mathcal{N}(0, 1)$ then

$$\mathbb{E} \left[\exp \left(x(Z + y)^2 \right) \right] = \exp \left(-\frac{1}{2} \ln(1-2x) + \frac{xy^2}{1-2x} \right).$$

For a noncentred gamma random variable, from [33] we know that

$$\mathbb{E}^{\mathbb{P}} \left[e^{x_1 \mathbf{RV}_t^c} | \mathcal{F}_{t-1} \right] = \exp \left(-\delta \mathcal{W}(x_1, \theta) + \mathcal{V}(x_1, \theta) \left(d + \sum_{i=1}^p \beta_i \mathbf{RV}_{s-i}^c + \sum_{j=1}^q \alpha_j \ell_{s-j} \right) \right),$$

where

$$\mathcal{V}(x_1, \theta) = \frac{\theta x_1}{1 - \theta x_1}, \quad \mathcal{W}(x_1, \theta) = \ln(1 - x_1 \theta),$$

and

$$x_1(z, b_1, c) = z\lambda_c + b_1 + \frac{\frac{1}{2}z^2 + \gamma^2c - 2c\gamma z}{1 - 2c}. \quad (11)$$

For the computation of the last expectation in the final line of (10), we use the property that if Z_t is a compound Poisson process with rate ω and *i.i.d.* jump sizes D_i , then

$$\mathbb{E} \left[e^{xZ_t} | \mathcal{F}_{t-1} \right] = \exp(\omega(M_D(x) - 1)), \quad (12)$$

where $M_D(x)$ is the MGF of the jump size random variable D . Since sizes are distributed according to a gamma distribution, we have

$$M_D(x) = \frac{1}{(1 - x\tilde{\theta})^{\tilde{\delta}}}. \quad (13)$$

From expressions (12) and (13) we obtain

$$\mathbb{E}^{\mathbb{P}} \left[e^{x_2 \text{RV}_t^j} | \mathcal{F}_{t-1} \right] = \exp \left(\tilde{\Theta} \mathcal{J} \left(x_2, \tilde{\theta}, \tilde{\delta} \right) \right),$$

where

$$\mathcal{J}(x_2, \tilde{\theta}, \tilde{\delta}) = \frac{1 - (1 - \tilde{\theta}x_2)^{\tilde{\delta}}}{(1 - \tilde{\theta}x)^{\tilde{\delta}}} \quad \text{and} \quad x_2(z, b_2, c) = z\lambda_j + b_2 + \frac{\frac{1}{2}z^2 + \gamma^2c - 2c\gamma z}{1 - 2c}.$$

Gathering all the previous results, we finally conclude

$$\begin{aligned} & \mathbb{E}^{\mathbb{P}} \left[e^{zy_t + \mathbf{b} \cdot \mathbf{RV}_t + c\ell_t} | \mathcal{F}_{t-1} \right] = \\ & \exp \left[zr - \frac{1}{2} \ln(1 - 2c) + \mathcal{V}(x_1, \theta) \left(d + \sum_{i=1}^p \beta_i \text{RV}_{t-i}^c + \sum_{j=1}^q \alpha_j \ell_{t-j} \right) \right. \\ & \left. - \delta \mathcal{W}(x_1, \theta) + \tilde{\Theta} \mathcal{J}(x_2, \tilde{\theta}, \tilde{\delta}) \right]. \end{aligned}$$

The direct comparison of the last expression with (9) allows to derive the following explicit expressions

$$\mathcal{A}(z, \mathbf{b}, c) = zr - \frac{1}{2} \ln(1 - 2c) - \delta \mathcal{W}(x_1, \theta) + d\mathcal{V}(x_1, \theta) + \tilde{\Theta} \mathcal{J}(x_2, \tilde{\theta}, \tilde{\delta}), \quad (14)$$

$$\mathcal{B}_i(z, b_1, c) = \mathcal{V}(x_1, \theta) \beta_i, \quad (15)$$

$$\mathcal{C}_j(z, b_1, c) = \mathcal{V}(x_1, \theta) \alpha_j. \quad (16)$$

As shown in [41], once we have above expressions we obtain

$$\phi^{\mathbb{P}}(t, T, z) = \mathbb{E}^{\mathbb{P}} \left[e^{zy_{t,T}} | \mathcal{F}_t \right] = \exp \left(\mathbf{a}_t + \sum_{i=1}^p \mathbf{b}_{t,i} \text{RV}_{t+1-i}^c + \sum_{i=1}^q \mathbf{c}_{t,i} \ell_{t+1-i} \right)$$

where

$$\begin{aligned} \mathbf{a}_s &= \mathbf{a}_{s+1} + zr - \frac{1}{2} \log(1 - 2c_{s+1,1}) + d\mathcal{V}(x_{s+1}^c, \theta) - \delta \mathcal{W}(x_{s+1}^c, \theta) + \tilde{\Theta} \mathcal{J}(x_{s+1}^j, \tilde{\theta}) \\ \mathbf{b}_{s,i} &= \begin{cases} \mathbf{b}_{s+1,i} + \mathcal{V}(x_{s+1}^c, \theta) \beta_i & \text{for } 1 \leq i \leq p-1 \\ \mathcal{V}(x_{s+1}^c, \theta) \beta_i & \text{for } i = p \end{cases} \\ \mathbf{c}_{s,i} &= \begin{cases} \mathbf{c}_{s+1,i} + \mathcal{V}(x_{s+1}^c, \theta) \alpha_i & \text{for } 1 \leq i \leq q-1 \\ \mathcal{V}(x_{s+1}^c, \theta) \alpha_i & \text{for } i = q \end{cases} \end{aligned} \quad (17)$$

with

$$x_{s+1}^c = z\lambda_c + b_{s+1,1} + \frac{\frac{1}{2}z^2 + \gamma^2 c_{s+1,1} - 2c_{s+1,1}\gamma z}{1 - 2c_{s+1,1}}, \quad (18)$$

$$x_{s+1}^j = z\lambda_j + \frac{\frac{1}{2}z^2 + \gamma^2 c_{s+1,1} - 2c_{s+1,1}\gamma z}{1 - 2c_{s+1,1}}. \quad (19)$$

The functions \mathcal{V} , \mathcal{W} and \mathcal{J} are defined as above. The terminal conditions read $a_T = b_{T,i} = c_{T,j} = 0$ for $i = 1, 2, \dots, p$ and $j = 1, 2, \dots, q$.

B No-arbitrage condition

The no-arbitrage conditions are

$$\begin{aligned} \mathbb{E}^{\mathbb{P}} [M_{s,s+1} | \mathcal{F}_s] &= 1 \text{ for } s \in \mathbb{N}, \\ \mathbb{E}^{\mathbb{P}} [M_{s,s+1} e^{y_{s+1}} | \mathcal{F}_s] &= e^r \text{ for } s \in \mathbb{N}. \end{aligned} \quad (20)$$

The first relation is satisfied by definition of $M_{s,s+1}$. From a general result in [41], condition (20) is satisfied if, and only if

$$\begin{aligned} \mathcal{A}(1 - \nu_y, -\nu, 0) &= r + \mathcal{A}(-\nu_y, -\nu, 0), \\ \mathcal{B}_i(1 - \nu_y, -\nu, 0) &= \mathcal{B}_i(-\nu_y, -\nu, 0), \\ \mathcal{C}_j(1 - \nu_y, -\nu, 0) &= \mathcal{C}_j(-\nu_y, -\nu, 0), \end{aligned}$$

with $\nu = (\nu_c, \nu_j)$. To conclude, it is sufficient to show under which conditions the following two relations hold true

$$\begin{aligned} x_1(1 - \nu_y, -\nu_c, 0) &= x_1(-\nu_y, -\nu_c, 0), \\ x_2(1 - \nu_y, -\nu_j, 0) &= x_2(-\nu_y, -\nu_j, 0). \end{aligned}$$

Simple computations show that the latter equations are satisfied if and only if

$$\nu_y = \lambda_c + \frac{1}{2} = \lambda_j + \frac{1}{2}.$$

Remarkably, the only specification for the log-return dynamics in Eq. (2) which ensures consistency with no-arbitrage is the dynamics where the equity premia λ_c and λ_j are equal and coincide to λ . Then, we obtain

$$\nu_y = \lambda + \frac{1}{2}.$$

It is important to notice that the no-arbitrage condition for the equity premium does not constrain the value of the variance risk premia ν_c and ν_j .

C Risk-neutral dynamics

JLHARG models imply a risk-neutral MGF for log-returns whose exponential affine terms can be re-parametrized in order to obtain an expression formally equivalent to the physical MGF. Firstly we observe that the risk-neutral MGF can be expressed with a recursive set of expressions, involving a combination of the functions \mathcal{A} , \mathcal{B}_i , \mathcal{C}_j . Then, recalling the results given in [41], the MGF for JLHARG model under measure \mathbb{Q} has the following form

$$\phi_{\nu_c \nu_j \nu_y}^{\mathbb{Q}}(t, T, z) = \mathbb{E}^{\mathbb{Q}} [e^{zy_{t,T}} | \mathcal{F}_t] = \exp \left(a_t^* + \sum_{i=1}^p b_{t,i}^* \text{RV}_{t+1-i}^c + \sum_{i=1}^q c_{t,i}^* \ell_{t+1-i} \right),$$

where

$$\begin{aligned}
a_s^* &= a_{s+1}^* + zr - \frac{1}{2} \log(1 - 2c_{s+1,1}^*) + d\mathcal{V}(x_{s+1}^{c*}, \theta) - d\mathcal{V}(y_{s+1}^{c*}, \theta) \\
&\quad - \delta\mathcal{W}(x_{s+1}^{c*}, \theta) + \delta\mathcal{W}(y_{s+1}^{c*}, \theta) + \tilde{\Theta}\mathcal{J}(x_{s+1}^{j*}, \tilde{\theta}) - \tilde{\Theta}\mathcal{J}(y_{s+1}^{j*}, \tilde{\theta}) \\
b_{s,i}^* &= \begin{cases} b_{s+1,i}^* + (\mathcal{V}(x_{s+1}^{c*}, \theta) - \mathcal{V}(y_{s+1}^{c*}, \theta)) \beta_i & \text{for } 1 \leq i \leq p-1 \\ (\mathcal{V}(x_{s+1}^{c*}, \theta) - \mathcal{V}(y_{s+1}^{c*}, \theta)) \beta_i & \text{for } i = p \end{cases} \\
c_{s,j}^* &= \begin{cases} c_{s+1,j}^* + (\mathcal{V}(x_{s+1}^{c*}, \theta) - \mathcal{V}(y_{s+1}^{c*}, \theta)) \alpha_j & \text{for } 1 \leq j \leq q-1 \\ (\mathcal{V}(x_{s+1}^{c*}, \theta) - \mathcal{V}(y_{s+1}^{c*}, \theta)) \alpha_j & \text{for } j = q \end{cases}
\end{aligned} \tag{21}$$

where

$$\begin{aligned}
x_{s+1}^{c*} &= (z - \nu_y)\lambda + b_{s+1,1}^* - \nu_c + \frac{\frac{1}{2}(z - \nu_y)^2 + \gamma^2 c_{s+1,1}^* - 2c_{s+1,1}^* \gamma(z - \nu_y)}{1 - 2c_{s+1,1}^*} \\
x_{s+1}^{j*} &= (z - \nu_y)\lambda - \nu_j + \frac{\frac{1}{2}(z - \nu_y)^2 + \gamma^2 c_{s+1,1}^* - 2c_{s+1,1}^* \gamma(z - \nu_y)}{1 - 2c_{s+1,1}^*} \\
y_{s+1}^{l*} &= -\nu_y \lambda - \nu_l + \frac{1}{2} \nu_y^2,
\end{aligned}$$

with $l = c, j$ and the terminal conditions are $a_T^* = b_{T,i}^* = c_{T,j}^* = 0$ for $i = 1, 2, \dots, p$ and $j = 1, 2, \dots, q$.

The first passage consists in comparing expression (21) with (17). We have to find a set of new parameters for which the recursive expressions for a_t^*, b_t^*, c_t^* under \mathbb{Q} correspond to the expressions under \mathbb{P} . We start defining

$$\begin{aligned}
x_{s+1,i}^{c**} &= z\lambda^* + b_{s+1,1}^* + \frac{\frac{1}{2}z^2 + (\gamma^*)^2 c_{s+1,1}^* - 2c_{s+1,1}^* \gamma^* z}{1 - 2c_{s+1,1}^*}, \\
x_{s+1,i}^{j**} &= z\lambda^* + \frac{\frac{1}{2}z^2 + (\gamma^*)^2 c_{s+1,1}^* - 2c_{s+1,1}^* \gamma^* z}{1 - 2c_{s+1,1}^*}.
\end{aligned}$$

Then, the following relations have to hold

$$\delta(\mathcal{W}(x_{s+1}^{c*}, \theta) - \mathcal{W}(y_{s+1}^{c*}, \theta)) = \delta^* \mathcal{W}(x_{s+1}^{c**}, \theta^*) \tag{22}$$

$$\beta_i(\mathcal{V}(x_{s+1}^{c*}, \theta) - \mathcal{V}(y_{s+1}^{c*}, \theta)) = \beta_i^* \mathcal{V}(x_{s+1}^{c**}, \theta^*) \tag{23}$$

$$\alpha_j(\mathcal{V}(x_{s+1}^{c*}, \theta) - \mathcal{V}(y_{s+1}^{c*}, \theta)) = \alpha_j^* \mathcal{V}(x_{s+1}^{c**}, \theta^*) \tag{24}$$

$$\tilde{\Theta}(\mathcal{J}(x_{s+1}^{j*}, \tilde{\theta}) - \mathcal{J}(y_{s+1}^{j*}, \tilde{\theta})) = \tilde{\Theta}^* \mathcal{J}(x_{s+1}^{j**}, \tilde{\theta}^*) \tag{25}$$

with $y^{c*} = -\lambda^2/2 - \nu_c + \frac{1}{8}$ and $y^{j*} = -\lambda^2/2 - \nu_j + \frac{1}{8}$.

Equation (22) can be explicitly written as

$$\delta \log \left[1 - \frac{\theta}{1 - \theta y^{c*}} (x_{s+1}^{c*} - y^{c*}) \right] = \delta^* \log (1 - \theta^* x_{s+1}^{c**}),$$

which implies the following three sufficient conditions

$$\begin{aligned}
\delta^* &= \delta \\
\theta^* &= \frac{\theta}{1 - \theta y^{c*}} \\
x_{s+1}^{c**} &= x_{s+1}^{c*} - y^{c*}.
\end{aligned} \tag{26}$$

It can be easily verified that the last condition (26) is satisfied by substituting

$$\lambda_c^* = -\frac{1}{2},$$

$$\gamma^* = \gamma + \lambda + \frac{1}{2}.$$

The equation (23) can be equivalently expressed in the form

$$\frac{\beta_i}{1 - \theta y^{c*}} \frac{\theta}{1 - \theta y^{c*}} \frac{x_{s+1}^{c*} - y^{c*}}{1 - \theta / (1 - \theta y^{c*}) (x_{s+1}^{c*} - y^{c*})} = \beta_i^* \frac{\theta^* x_{s+1}^{c**}}{1 - \theta^* x_{s+1}^{c**}}$$

which gives another sufficient condition for the mapping

$$\beta_i^* = \frac{\beta_i}{1 - \theta y^{c*}}.$$

An analogous consideration about the third condition (24) allows to obtain the condition on α_i^* ,

$$\alpha_i^* = \frac{\alpha_i}{1 - \theta y^{c*}}.$$

Relation (8) gives us the expressions for β_d^* , β_w^* , β_m^* , α_d^* , α_w^* and α_m^* . Finally, equation (25) provides the last sufficient condition

$$\frac{\tilde{\Theta}}{(1 - \tilde{\theta} y^{j*})^{\tilde{\delta}}} \frac{1 - \left((1 - \tilde{\theta} x_{s+1}^{j*}) / (1 - \tilde{\theta} y^{j*}) \right)^{\tilde{\delta}}}{\left((1 - \tilde{\theta} x_{s+1}^{j*}) / (1 - \tilde{\theta} y^{j*}) \right)^{\tilde{\delta}}} = \tilde{\Theta}^* \frac{1 - (1 - \tilde{\theta}^* x_{s+1}^{j**})^{\tilde{\delta}^*}}{(1 - \tilde{\theta}^* x_{s+1}^{j**})^{\tilde{\delta}^*}},$$

which is satisfied if

$$\tilde{\delta}^* = \tilde{\delta},$$

$$\tilde{\Theta}^* = \frac{\tilde{\Theta}}{(1 - \tilde{\theta} y^{j*})^{\tilde{\delta}}},$$

$$\tilde{\theta}^* = \frac{\tilde{\theta}}{1 - \tilde{\theta} y^{j*}},$$

$$x_{s+1}^{j**} = x_{s+1}^{j*} - y^{j*}. \quad (27)$$

As it can be seen the last condition (27) is redundant when compared to the condition (26).