The Analysis of Actuarial Investment Risk

by

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Abstract

The Analysis of Actuarial Investment Risk considers the theoretical foundations of different measures of investment risk in an historical context. It begins by looking at the justifications of variance of investment returns as a measure of risk. Utility functions which are not as restrictive as those which lead to the choice of variance of returns as a measure of risk are then discussed. Actuaries have long considered risk control from a practical viewpoint. Canons of investment have been proposed by various authors. The underlying utility functions and investment risk measures which relate to practical and theoretical contributions in the field of actuarial science applied to institutional investment are analysed. Criticisms of utility theory are then discussed and the application of expected utility maximisation models, as methods of risk control, is proposed. Practical applications of those models are considered.

THE ANALYSIS OF ACTUARIAL INVESTMENT RISK

1. Introduction

1.1 In recent years, in the actuarial literature, there has been some discussion of the meaning of investment risk. Some of this discussion has concentrated on asset risk, for example Clarkson (1989) and some of the discussion has emphasised the importance of the interaction of assets and liabilities. Much of the United Kingdom actuarial literature on investment risk has bypassed the wealth of financial economics literature on the subject. This is probably, in part, due to the unfamiliarity with the work in that area. However, it is also due to the fact that the applications of financial economics, emphasised in the literature, have seemed remote from the problems faced by actuaries.

1.2 In particular, the capital asset pricing model, developed from work by Sharpe (1963), Tobin (1958) and Sharpe (1964) is often seen to be built on unrealistic assumptions and to employ an unrealistic view of risk (variance of short term returns). Also, work by Markowitz (1952, 1987 and 1991) has tended to emphasis variance of short term investment returns as a measure of risk. However, much of the work undertaken by financial economists does allow for a broader definition of risk than variance of investment returns. There is, for example, important work, undertaken by Markowitz, demonstrating that many other measures of risk, approximate to variance of investment returns in most situations [see Levy and Markowitz (1979)]. Financial economics has also been criticised by actuaries for ignoring liabilities although some of the problems have, more recently, been addressed by Sharpe and Tinte(1990).

1.3 These criticisms of some of the applications of financial economics and of some of the simplifying assumptions should not prevent actuaries from using some of the precepts of the discipline for the analysis of investment decisions. This paper therefore takes a step backwards and discusses the body of literature on utility theory, which is the backbone not only of financial economics but also of insurance and investment risk. The use of variance of investment returns as a measure of risk is analysed and criticised by relating it to its original utility theory origins. The implicit assumptions which justify the use of variance as a measure of risk are analysed explicitly.

1.4 The different measures of risk, which have been proposed in an actuarial context, are analysed in a utility theory framework. The applications of utility theory to actuarial investment problems are then discussed. Non-utility theory approaches to risk control are also considered. Indications are given as to how future research can lead to a more analytical approach to investment risk being taken by actuaries. This paper can be seen as taking one aspect of the papers by Smith (1995) and Clarkson (1996) on financial economics, developing that aspect and drawing out its uses to actuaries. It can also be seen as a review paper, tracing the origins of investment risk measurement and reviewing the applications of different approaches to risk to the solution of current actuarial problems.

2. The Basis of Utility Theory

2.1 Utility theory was first used by Bernoulli (1738) to explain the "St. Petersburg paradox". It appeared, from empirical observation, that gamblers were unwilling to stake money on a "fair game", which had a finite probability of a large loss. Bernoulli, in offering an explanation for this paradox, suggested that the determination of the value of wealth was not based on its monetary amount but on the utility that monetary amount of wealth provides. An investment corollary of this observation is that the value of an investment is not based upon its expected pay off but on the expected utility that may be yielded from the various pay offs. Bernoulli suggested that to value a set of possible pay-offs from a risk we should determine the amount of money which, when paid with certainty, has the same value as the risky set of pay offs.

2.2 The utility function is an attempt to put a numerical value on different levels of wealth. The legitimacy of attempts to do this will be discussed in Section 8. There are two principles which are normally used when postulating a utility function. If the utility of a given level of wealth (X) is defined as $U(X)$, then it is normally assumed that:

$$U'(X) \geq 0$$

That is, we assume that we do not prefer less wealth to more. Any individual who did prefer less wealth to more could presumably find ways of disposing of wealth until this position was rectified. It is also normally assumed that:

$$U''(X) \leq 0$$

That is, the value we put on a given increment in wealth does not increase as our level of wealth increases. If the value we put on a given increment in wealth decreases as wealth increases, then we have diminishing marginal utility of wealth.

2.3 The utility functions described in Section 3 have the above properties. They also have the property that the same functional form is assumed throughout relevant regions of income. The assumptions of a continuous functional form and of diminishing marginal utility of wealth are later relaxed.

3. Utility Functions: The Basis of Variance as a Measure of Investment Risk

3.1 Consider the simplest form of a utility function is such that $U''(X) = 0$, to give:

$$U(X) = a + bX \qquad b > 0.$$

An investor taking decisions involving risk will look to maximise the expected value of the function $a + bX$ where X denotes the wealth after the outcome of the investment.

3.2 Let the starting value of wealth = W and the random accumulation of an investment = A. We choose an investment portfolio so that we:

$$Max\ E\left[a + b(W + A)\right]$$

$$\text{i.e. } Max\left[a + bW + bE(A)\right]$$

Given that A is the only random component, the utility maximising investment strategy is the one which maximises the expected investment return. This arises because a linear utility function indicates that the investor is not at all risk averse. Such a trivial form of utility function may have applications over certain ranges of wealth or in special situations.

3.3 Now consider the utility function:

$$U(X) = a + bX + cX^2 \text{ with } c < 0, b > 0$$

Again, if we let the starting value of wealth = W and the random accumulation = A, we choose a portfolio that maximises the expected utility of the accumulation so that we:

$$Max\ E\left[a + b(W + A) + c(W + A)^2\right]$$

$$\text{i.e. } Max\left[a + bW + bE(A) + cW^2 + 2cW\,E(A) + cE(A^2)\right]$$

3.4 Three points are worthy of note here. The elements of the function to be maximised are functions of $E(A)$ and $E(A^2)$: the maximisation of utility therefore depends only on the mean and variance of the probability distribution of the accumulation (and therefore of investment returns). Secondly, W simply becomes a constant and, although the value of W helps determine the portfolio which maximises utility, for a given level of W we do not require the initial wealth explicitly in the function of mean and variance of return which we maximise. Thirdly, for a given level of expected return, a reduction in variance increases expected utility and for a given level of variance an increase in expected return increases expected utility.

3.5 Utility theory thus provides a justification for mean-variance efficient frontier analysis. It should be noted that the quadratic utility function should not peak in the relevant range of investment returns.

3.6 The utility function suggested by Bernoulli (1738) was logarithmic. In its simplest form, this would require:

$$U(X) = \ln(X)$$

Letting the starting value of wealth = W and the random accumulation = A, we are required to choose a portfolio which maximises the expected utility of the accumulation so that we:

$$Max\ E[\ln(W + A)]$$

if we use Taylor's expansion [see Markowitz (1991)]:

$$E[U(W+A)] = E[\ln(W+A)] = U[E(W+A)] + U'[E(W+A)][E(W+A-\mu)] + U''[E(W+A)]\frac{E(W+A-\mu)^2}{2!} + \ldots$$

where μ is the first moment of $W+A$ i.e. $E(W+A)$

therefore $E[\ln(W+A)] \cong \ln[W+E(A)] - \dfrac{Var(W+A)}{2[W+E(A)]^2} + \ldots$

3.7 Thus, ignoring higher order terms, we can approximate a log utility function by a quadratic utility function which relies on the maximisation of a function of the mean and variance of the accumulation. The approximation will tend to work quite well at most realistic levels of investment returns: this is an issue to which we will return later.

3.8 The particular feature of a log utility function, which may make it attractive in practical use over significant ranges of wealth, is that it assumes that an investor believes that a given proportionate increase in wealth is equally valuable at all initial levels of wealth. It therefore leads people, over such levels of wealth, to take the same investment decisions if they are investing the same proportion of their wealth.

3.9 An alternative is a fractional utility function of the form $U(X) = X^a$

$0 < a < 1$. Specifically Cramer proposed a fractional utility function with $a = \frac{1}{2}$. Thus, if the investor has wealth W and a random accumulation A, he should choose the portfolio so that it:

$$Max\ E\left(\sqrt{W + A}\right)$$

3.10 Another class of utility function which has well defined properties is the class of exponential functions. Defining W and A as before, we can consider two possible exponential functions:

$$U(X) = -e^{-cx} \qquad c > 0 \text{ and}$$

$$U(X) = e^{-cx} \qquad c < 0$$

therefore, we:

$$Max\ E\left[-\exp\left(-c(W + A)\right)\right] \quad c > 0$$

$$\text{or } Max\ E\left[\exp - c(W + A)\right] \qquad c < 0.$$

Where c is known as the 'risk tolerance parameter' that is, it indicates the investor's tolerance to risk. The most important feature of exponential utility functions is that they exhibit "constant absolute risk aversion". The concept of risk aversion will be discussed in Section 4. This property means that all investors will take exactly the same decisions if they invest the same amount of money, regardless of their starting value of wealth W.

3.11 For any given statement about the moments of the distribution of investment returns, which are considered by an investor, we can postulate the class of utility functions which the investor will consider and vice versa. It is useful to consider the rationale of the linear, quadric and log utility functions. An investor who is simply interested in maximising expected return and is not at all risk averse will have a linear utility function. An investor who solely considers expected return and variance of returns (so that variance is the measure of risk) will prefer a quadratic utility function: such an investor would only ever work in a mean variance framework. An investor who is risk averse and judges an equal proportionate gain (or loss) in his income to be of equal value, whatever his initial income, would prefer a logarithmic utility function. Such a utility function may be approximated by one depending only on mean and variance, over a certain range of income.

3.12 A further rationale, which can be put forward to justify mean-variance analysis, arises from the results obtained if we assume that the accumulation of an investment follows a normal distribution. Let the utility function $U(X)$ be any function of X which fulfills the properties stated in Paragraph 2.2. If the investment accumulation is normally distributed with mean μ and variance σ^2, the expected utility from a financial decision which involves the adding of a random quantity A to wealth W is:

$$E[U(W+A)] = \int_{-\infty}^{\infty} \left[\frac{1}{\sqrt{2\pi\sigma}}\right] \exp\left[-\tfrac{1}{2}\left(\frac{a-\mu}{\sigma}\right)^2\right] U(W+A)\,da$$

where the random variable A is the accumulation of the investment.

$$E[U(W+A)] = U[E(W+A)] + \frac{U''[E(W+A)]\sigma_1^2}{2!} + \frac{U'''[E(W+A)]\mu_3}{3!} + \ldots$$

Where μ_3 is the third central moment of $W+A$. In general σ_1^2 is the second central moment of $W+A$, but this is equivalent to σ^2, the second central moment of A. Because A, and therefore $W+A$, follow a normal distribution, all the central moments of $W+A$ depend only on μ and σ^2: the mean and variance of the random accumulation (and hence depend only on the mean and variance of investment returns). Thus, we choose our investment portfolio to maximise a function of the mean and variance of investment returns. The particular function of mean and variance that we maximise depends on the utility function and its differentials.

3.13 If the investment accumulation from an investment decision follows a lognormal distribution, finding a utility maximising portfolio also only depends on the values of the mean and variance of the accumulation as all other parameters can be defined from these. However, combinations of investments which each have a lognormal distribution do not necessarily give rise to portfolios the accumulation from which follows the lognormal distribution.

3.14 Thus, we have two justifications for the use of variance of returns as a measure of investment risk: quadratic utility functions and normal returns. The latter is an empirical matter but, given that, in an actuarial context the variable under consideration is likely to derive from the complex interaction of assets and liabilities and the actuary will be more interested in extreme events, it would be bold to assume that the normality assumption is reasonable, *prima facie*. We should therefore turn to whether a quadratic utility function is a good approximation to the investor's preference set. One way of considering this is by looking at the risk premium and pattern of risk aversion which is implied by a quadratic utility function.

4. Risk Aversion and Risk Premiums

4.1 Risk aversion was defined by Pratt (1964). For a given utility function $U(X)$, it is necessary to require $U'(X) > 0$ for an individual to prefer more money to less. Furthermore, the requirement $U''(X) < 0$ ensures that an individual is risk averse. However, neither $U'(X)$ nor $U''(X)$, indicate how risk aversion is changing. However, we can see how risk aversion changes by looking at the risk premium an investor demands for an actuarially neutral investment.

4.2 Consider an individual with a given amount of wealth W and random accumulation A. The individual can be said to be risk averse if:

$$E[U(W+A)] < U[W+E(A)]$$

That is, the individual prefers an investment with a known payout of $E(A)$ to one which has a random payout A with the same expected value. The extent of this aversion to risk can be determined by looking at the risk premium the investor requires from a risky investment. Adapting Pratt's notation, the risk premium can be said to be equal to $\pi(W, A)$: for a given utility function, it is a function of the starting wealth (W) and the distribution of the random additions (A). The risk premium for an investment is such that:

$$U[W+E(A)-\pi(W, A)] = E[U(W+A)]$$

That is, $\pi(W, A)$ is the deduction which can be made from the expected value of the investment return which leads the investor to be indifferent between $E(A)-\pi(W, A)$ with certainty and the random variable A under conditions of uncertainty. $E(A)-\pi(W, A)$ can be regarded as the risk free accumulation equivalent to the risky investment. It is they way in which $\pi(W, A)$ varies with W that determines the pattern of changing risk aversion.

4.3 The investor's local aversion to risk is measured first by looking at the risk premium for a small, actuarially neutral risk (so that $E(A) = 0$). Thus we look at $\pi(W, A)$ such that:

$$U[W-\pi(W, A)] = E[U(W+A)]$$

To measure local aversion to risk, we also require the small variance of the investment σ_A^2. Thus, we consider $\pi(W, A)$ with $E(A) = 0$, $\sigma_A^2 \to 0$ and the third central moment of A being of smaller order σ_A^2.

$$U[W - \pi(W, A)] = U(W) - \pi(W, A)U'(W) + [\pi(W, A)]^2 U''(W) - \dots$$

$$E[U(W + A)] = U(W) + E(A)U'(W) + \tfrac{1}{2}E(A^2)U''(W) + \dots$$

$$= U(W) + \tfrac{1}{2}\sigma_A^2 U''(W)$$

Therefore: $\pi(W, A) \simeq -\tfrac{1}{2}\sigma_A^2 \dfrac{U''(W)}{U'(W)}$.

Thus, the risk premium is proportionate to:

$$-\frac{U''(W)}{U'(W)}$$

when considering risks of given variance $-\dfrac{U''(W)}{U'(W)} \equiv r(W)$ is defined as the measure of risk aversion in Pratt (1964). The measure of risk aversion will indicate how the risk premium for similar risks will change with wealth. As $r(W)$ becomes higher, it implies a higher required risk premium and therefore lower risk tolerance.

4.4 It is therefore of interest to consider the risk aversion properties of some of the utility functions we have looked at so far. With a starting value of wealth W and random accumulation A, the linear utility function is defined by:

$$U(W + A) = a + b(W + A)$$

$$U'(W) = b \quad \text{and} \quad U''(W) = 0$$

$$r(W) = -\frac{U''(W)}{U'(W)} = 0 \qquad \text{for all values of } W.$$

Thus, using Pratt's measure of risk aversion, the well known result that the linear utility function has constant (zero) risk aversion is confirmed.

4.5 Considering now the quadratic form:

$$U(W+A) = a + b(W+A) + c(W+A)^2 \qquad \begin{array}{l} c < 0 \\ b > 0 \end{array}$$

we have

$$U'(W) = b + 2cW \quad \text{and}$$

$$U''(W) = 2c$$

$$r(W) = -\frac{2c}{b+2cW}$$

This measure of risk aversion (given $c < 0$) increases until such point that $b = -2cW$. This is the point at which the quadratic utility function peaks and we are not interested in the quadratic utility function after it has peaked, as it implies marginal disutility of income. Thus the quadratic utility function implies increasing risk aversion over all relevant values of the starting value of wealth. Therefore, for initial starting values of wealth W_1 and W_2, where $W_2 > W_1$ the measure of local risk aversion will always be higher at $W_2 + A$ than at $W_1 + A$, for any particular value taken by the random variable A. This is a serious disadvantage of the quadratic function as it would appear intuitive that investor's become less risk averse (or at least no more risk averse) as they become wealthier. This perhaps undermines one of the foundations of the mean-variance framework. Indeed, Pratt (1964) states, "This (feature) severely limits the usefulness of quadratic utility, however nice it would be to have expected utility depend only on the mean and variance of the probability distribution" (Page 132).

4.6 Let us now consider one of the exponential forms.

$$U(W+A) = -\exp[-c(W+A)] \qquad c > 0$$

$$U'(W) = c\exp(-cW)$$

$$U''(W) = -c^2\exp(-cW) \qquad \text{and therefore}$$

$$r(W) = \frac{c^2}{c} = c.$$

Thus, we have constant risk aversion, measured by the investor's risk tolerance parameter c. This feature of constant risk aversion would seem to imply that, whatever the starting value of wealth, W, the investor would take the same investment decision if a given amount of the initial wealth was invested. However, this result is only intuitive because the measure of risk aversion is a differential which only considers the risk premium for an infinitesimal, actuarially neutral investment. Let us consider the relationship between the portions of the utility function allowing the investor to invest in a random risk A which is not constrained to be actuarially neutral. If the investor has initial wealth, W_1, the utility of wealth can be written as:

$$U(W_1 + A) = -\exp[-c(W_1 + A)] \quad c > 0.$$

If the investor has initial wealth W_2, the utility of wealth can be written as:

$$U(W_2 + A) = -\exp[-c(W_2 + A)] \quad c > 0.$$

A is a random variable which can take positive or negative values.

$$\begin{aligned} U(W_2 + A) &= -\exp(-cW_2)\exp(-cA) \\ &= -\exp-(cW_2)\exp(cW_1)\exp[-c(W_1 + A)] \\ &= b\exp[-c(W_1 + A)] \end{aligned}$$

where $b = -\exp-(cW_2)\exp(cW_1)$.

4.7 A utility function is unaffected by any linear transformation in the sense that no linear transformation of a utility function will affect any investment decisions. The utility after any increment in wealth A (for any value of A), if the starting value of wealth is W_2 is a constant multiple of the utility caused by the same increase in wealth A if the starting value of wealth is W_1. The investment decision which leads to the greatest expected utility if the starting value of wealth is W_2 must be the same as that which leads to the greatest expected utility if the starting value of wealth is W_1. Thus, if investors are assumed to be risk averse to the same degree, whatever their wealth (that is they demand the same risk premia from the same size investments) they are likely to have exponential utility functions. They will then take identical investment decisions regardless of their initial level of wealth if they invest the same amount of money. The use of an exponential utility function is intuitively appealing because of constant risk aversion and it has been used in the actuarial literature by Sherris (1992), Booth and Ong (1994) and Ong (1996). However, the risk aversion properties of the log utility function are perhaps even more appealing.

4.8 Consider an investor with a log utility function so that:

$$U(W+A)=\ln(W+A)$$

We have:

$$U'(W)=\frac{1}{W} \quad \text{and}$$

$$U''(W)=-\frac{1}{W^2} \quad \text{therefore:}$$

$$r(W)=\frac{1}{W}$$

As W increases, $r(W)$ falls, thus the required risk premium falls. Interestingly, the required risk premium is inversely proportionate to income.

4.9 The risk premium can also be expressed in proportionate form. Paragraphs 4.1 and 4.2 looked at the absolute risk premium for a fixed investment. Consider an investor, with starting wealth W, who undertakes a risk WA. For a given value of A the risk would give rise to a set of payoffs which would be defined in terms of a given proportion of wealth rather than by a given amount of wealth.

4.10 Adapting the notation of Pratt (1964), $\pi^*(W,A)$ can be defined as the proportionate risk premium. Thus $\pi^*(W,A)$ can be defined such that the investor is indifferent between the risk WA and the certain amount:

$$E(WA)-W\pi^*(W,A)$$

The concept of a proportionate risk premium is perhaps less intuitive than that of an absolute risk premium so it is worthwhile considering a particular example. An individual with initial wealth of W would define the risk in terms of a proportion of that wealth. If we set $W=100$ and $A=0.1$, the risk premium $\pi^*(W,A)$ would be a proportionate deduction from wealth which would be accepted by the investor for taking the expected accumulation of wealth with certainty, rather than the random accumulation. Thus, if the expected value of the return from the random investment is 10%, then $E(WA)$ is $111=(100+0.1\times 100+0.01\times 100)$. The proportionate risk premium could be a number such as 0.001, so that the investor was indifferent between 110.9 with certainty $(111-100\times 0.001)$ and the starting wealth of 100 and risk A which give an expected accumulated wealth of 111. The proportionate risk premium is defined in terms of a proportion of wealth (W) for an investment of a given proportion of wealth (A): for a given value of A, it can also be written in terms of a proportion of A.

5. Utility of What to Whom?

5.1 Most of the financial economics literature has concentrated on the maximisation of the utility of the accumulation of (generally short term) investments. This is clearly inadequate in the actuarial context.

5.2 If we consider the shareholders of a proprietary life company, one of the aims of the company should be to maximise the expected utility of the real value of the distribution to shareholders. This is the approach taken by Sherris (1992) and by Ong (1996). The liabilities of the company will consist of non-profit and with-profit liabilities and shareholders' and policyholders' funds together with future expenses. The investment decisions should be taken so that policyholders' liabilities are met and the dividends paid to shareholders provide them with the highest expected utility, given their utility function.

5.3 Three qualifying points are necessary here. If this policy does not simultaneously provide a close to expected utility maximising situation for with-profit policyholders, it would have to be reconsidered. Secondly the shareholders have limited liability. Thirdly, there can be said to be a number of other interested parties in the life office: non-profit policyholders, with-profit policyholders, management and regulators. This paper, nor other literature in this field, has attempted to reconcile the claims of with-profit policyholders and shareholders, in so far as they conflict: that is a rich research area in its own right.

5.4 The interests of shareholders are satisfied if the expected discounted utility of the real income flow to shareholders is maximised. At first sight, it could be thought that this implies that shareholders are indifferent to how insolvent an insurance company becomes in the event of it becoming insolvent (due to limited liability). This is not a reasonable interpretation. The insurance company is likely to be closed to new business (or lose new business) well before the company has a negative net present value as a going concern. It will therefore be critical not to fall below this particular level of solvency at which it would lose new business or be closed to new business: otherwise expenses would rise and profits fall very quickly.

5.5 The recognition of this position can help us draw inference about the sort of utility function which could be used to accord with the preferences of the professionals who run the company and the management and regulators. The actuaries taking financial decisions are charged with a professional duty to policyholders. In fulfilling this duty they will have a particular aversion to becoming insolvent. Other management will share the aversion to the business ceasing to be a going concern, and so will the regulators. Modelling the sharp drop in profits that occurs when the office is closed to new business and the utility drop of the various parties when the office ceases to be a going concern is difficult. It also leads to philosophical difficulties regarding whose utility we should be maximising. Ong (1996) has suggested some practical solutions.

5.6 The situation pertaining to a non-life office is rather similar to that described above. In many ways the use of utility theory in determining investment strategy is easier as there are no with-profit policyholders to consider and the liabilities will tend to be of shorter duration. The modelling of a non-life company, using utility theory to aid the asset allocation decision has not been pursued by the author: it is, however, a possible fruitful area for further research.

5.7 For a pension fund, a number of variables could be put forward for optimisation. One possibility would be to try to minimize the expected utility of the contributions into the fund. This approach would be very difficult to understand and model. An alternative would be to work within a framework similar to that proposed by Wilkie (1985) and to maximise the expected utility of the ultimate surplus for a given level of initial surplus and contribution rate (the contribution rate could be ignored if we only considered liabilities accrued to date). It is important to express money amounts in real terms.

5.8 As far as personal investors or investors in money purchase pension schemes are concerned, we can look at the expected utility of the accumulation of the real value of the fund. The utility function is likely to have a complex shape with one or more discontinuities. Some experiments in risk measurement for personal pension investors are described in Booth (1995).

5.9 To talk about maximising the expected utility of the various cash flows to interested parties may seem esoteric and impractical. However, every measure of risk proposed by actuaries or financial economists has a utility function behind it, if only implicitly. In the following sections, we will therefore discuss some of the historical actuarial measures of risk and their implicit utility functions; we will then discuss the value of using a utility maximisation approach and alternative measures of investment risk in the context of actuarial management of institutions.

6 Actuarial Utility Functions

6.1 In this section we look at approaches to risk, taken by actuaries, where a specific utility function or risk measure has not been stated explicitly. We use the principles which are used by actuaries in determining preferences for different investments to find implicit utility functions and risk measures. We can derive an implied utility function from any set of investment canons which allow us to order all possible investments. Bailey (1862) proposes a number of canons of investment. The first two, which relate risk and return, are:

1. That the first consideration should invariably be the security of capital.

2. That the highest practicable rate of interest be obtained but this principle should always be subordinate to the previous one: the security of capital.

6.2 It is clear that by "security of capital" Bailey was referring to a requirement to ensure that the capital value of investments varied as little as possible. Pegler (1948) indicated that Bailey's canons had been generally accepted from 1862 to the time of Pegler's paper in 1948. Merely concentrating of "the security of capital" of the investments is, however, inadequate. This emphasis may have arisen from the relatively slow response of liability valuations to changes in asset market values in the middle of the 19th century. If asset market values fell, there may not have been a response in terms of changing the valuation rate of interest of the liabilities and the insurance company may have appeared insolvent. Bailey proposed increased investment in fixed interest mortgages which provided the security of income but, not being marketable, did not suffer from fluctuations in market value. Pegler proposed different canons and suggested that the first two of Bailey's canons were inappropriate. However, Swaminathen, in a written contribution to Pegler (1948) suggested a modernisation of Bailey's canons. A slightly different modernisation, in fact, gives rise to a pair of canons which are remarkably durable.

Thus, updated, Bailey canons might say:

> 1. The first consideration should invariably be the security of the capital of the insurer after all liabilities have been met (including reasonable expectations of with-profit policyholders).
>
> 2. Subject to the above, the aim of life office investment should be to obtain the maximum expected yield.

(The second proposed modernisation is taken directly from Swaminathen).

These revised canons have some similarities with the decision making paradigm for life companies proposed in Section 5. The protection of the insurer's capital is paramount (this reflects the utility of policyholders whom actuaries protect and of the managers and regulators). Having satisfied this, we look for the highest expected return. In utility theory terms, this could be represented as follows:

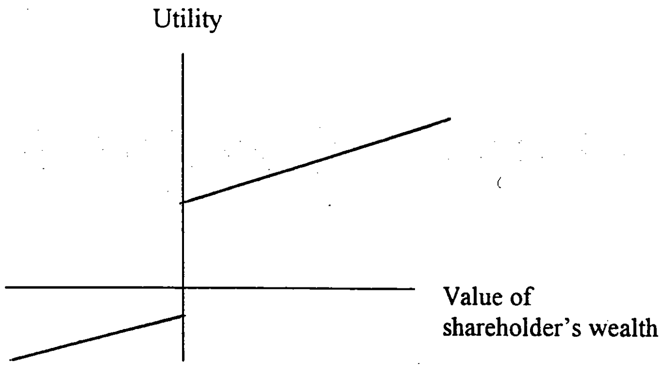


Figure 6.1

Above the discontinuity the function is of the form:

$$U(X) = a+bX \qquad b > 0,\ X > 0.$$

6.3 Any utility function such as this, with a discontinuity, has the difficulty that it implies that the insurance company, once insolvent, will be particularly keen to become solvent again even if this involves risking the remaining policyholders' funds. This is clearly not what Bailey had in mind.

6.4 The best way to deal with this difficulty is to incorporate the ideas of Friedman and Savage (1948). In an attempt to explain why people both insure and gamble (when a continuous concave utility function would imply that rational people should not), Friedman and Savage put forward two hypotheses. Firstly, the utility function was possibly partially convex. Secondly, that people may put a very high utility on being able to break into a higher income bracket (considerably above their current income) but, once there, people face concave utility functions.

6.5 There is an analogy here with actuarial decisions: the actuary puts a very high disutility on becoming insolvent. Having become insolvent, there is not such a high utility on returning to being solvent that the actuary is willing to turn to the roulette wheel. Thus the modernised Bailey canons can be interpreted as implying that the utility function is as shown in Figure 6.1, as long as the company is taking the decision whilst it is solvent.

6.6 Pegler (1948) criticised the emphasis on security of capital in the Bailey canons. He suggested that the life office should obtain the maximum expected return whilst spreading assets across the widest possible range (presumably whilst not reducing the expected return significantly). This principle is implicitly imposing a linear utility function across all ranges of investment return. In the author's view it does not allow sufficiently, either for the security of capital or for diminishing marginal utility of wealth.

6.7 Clarke (1954) suggests that the investor should, "maximise the expected yield with the minimum of error, having regard to the nature and incidence of the liabilities". This approach, in a sense, defines the "efficient" portfolio which an investor may choose without giving any indication as to how the investor may trade risk and return. Clarke believed that this approach could not be avoided. It should be noted that Clarke brings in liabilities in a way similar to Bailey's modernised canons. Also, because "minimum" of error is defined in terms of the risk of not meeting the expected yield (rather than in terms of standard deviation of return) the downside risk is also brought in.

6.8 The above descriptions of investment risk provide indications of how an actuarial utility function may look. I am grateful to Lipman (1990) who made this link although some of Lipman's interpretations are different. For a given utility function and probability distribution of investment returns, we can determine the parameters of the distribution of investment returns which are relevant when taking investment decisions. The other proposed methods of looking at investment risk (see Sections 3 and 7) take us in the other direction. They begin by looking at the parameters of the distribution of investment returns in which we are interested: we can then define the utility function.

6.9 Hemstead (1968), suggested that a reasonable measure of risk would involve looking at the sum of probabilities associated with financial loss of various levels. In particular, he regarded risk as the probability of loss of capital or, at a more acute level, the probability of loss of all capital. This is moving towards the Clarkson (1989) approach in which a downside risk measure was explicitly defined. This will be discussed in Section 7.

6.10 Markowitz (1952) began the flood of literature on financial economics. His principles were well grounded in utility theory. His work included a consideration of semi-variance as a risk measure but the mean variance framework was preferred partly because of the ease of calculation of the variance of returns from a portfolio, given the variance of returns from its constituent parts. Much of Markowitz's work (eg Levy and Markowitz 1979) demonstrated that the mean variance framework is a good approximation in practice, regardless of the distribution of investment returns or the utility function. It may not work well, however, if the utility function of the investor has a discontinuity. Nevertheless the remaining parts of this section consider actuarial research which has analysed investment decisions in a mean-variance framework.

6.11 Wise (1984) made a substantial contribution by considering an investment optimality problem in an asset-liability context. The decision variables Wise considered where the expected value and variance of ultimate surplus, after allowing for liability payments. Problems which had analytical solutions or which could be illustrated by example were discussed. Using simulation it is possible to expand the range of problems that can be analysed in this framework. Covariances, not only between investment returns, but between the asset and liability values can be used to determine optimal investment solutions.

6.12 Inflation linked assets and liabilities were considered alongside fixed money assets and liabilities. The only extension here is the inclusion of a further stochastic variable. This logic could, perhaps, have been extended by Wise to consider, as the decision variable, the mean and variance of the real value of ultimate surplus.

6.13 Wilkie (1985) noted that Wise's decision criterion did not lead to efficient portfolios, in the mean-variance sense, being selected. Wilkie (1985) will be discussed further below. Here, we demonstrate the mean-variance inefficiency of Wise's decision criteria by deriving the characteristics of the underlying utility function.

6.14 In Wise (1984), E_1 was defined as the expected ultimate surplus (at the end of the time horizon) and E_2 as the second moment of the ultimate surplus. There were two suggested decision criteria. The unbiased matched portfolio was defined as that which minimised $[E_2]$ subject to E_1 being zero. The best matched portfolio was defined as that which minimised $[E_2]$.

6.15 The unbiased match condition implies that the portfolio which gave the lowest variance of surplus should be chosen from those portfolios which gave an expected surplus of zero. It is not necessarily the most efficient portfolio as it is conceivable that another portfolio could have a higher expected ultimate surplus but the same variance of surplus: particularly if the initial level of surplus was high. This particular case arises from the limiting nature of the unbiased match of condition and will not be considered further.

6.16 The best match condition involves minimising the second moment of the ultimate surplus i.e. minimising $E_1^2 + V$, where V is the variance of ultimate surplus. If the ultimate surplus is assumed to be normally distributed, there are a number of different utility functions which could give rise to this particular decision criterion. However, normality is not given as a justification for the decision criterion and therefore we will assume that it was to apply generally. We therefore need to find the general utility function which leads to a decision criterion based on minimising the second moment of surplus.

6.17 Again, regarding final wealth X, as the sum of initial wealth W and the random accumulation A, consider a quadratic utility function:

$$U(W+A) = [a+b(W+A) + c(W+A)^2] \qquad (6.1)$$

so that we are required to:

$$\max E[U(W+A)] \text{ or } \max E[a+b(W+A) + c(W+A)^2].$$

This is equivalent to:

$$\max [E(b+2cW)A + cA^2].$$

If we use the quadratic utility function, with $c = -1$ and $b = 2W$ so that we:

$$\max -E(A^2) \text{ or } \min E(A^2).$$

This produces Wise's decision criterion.

6.18 There are a number of features worthy of note. Firstly, the parameters of the utility function are dependent upon initial wealth W. It was shown in Section 3.3 that, if there were a quadratic utility function, the decision criterion (based on the first and second moments of the random accumulation) would vary depending on the starting value of wealth, for given parameters. A corollary is that, if a quadratic utility function is used with the same decision criterion for all investors with different initial wealth, the underlying parameters must be different for the different investors.

6.19 This is not a major difficulty with the utility functions implicit in Wise's decision criteria. It is quite conceivable that investors with different initial wealth will have utility functions of the same functional form but with different parameters. There will be further, possibly offsetting, complications due to investors with different initial wealth who invest different amounts of their starting wealth. The issue is also complicated by the fact that the pension plan sponsors, if we are looking at the surplus of pension schemes, will be companies and not individuals.

6.20 The greater difficulty is with the properties of the utility function. Setting $a = 0$ (this is just a scaling parameter which does not affect investment decisions),

$b = 2W$ and $c = -1$, the utility function, for a given starting value of wealth W is:

$$U(W+A) = 2W^2 - W^2 - A^2. \qquad (6.2)$$

The marginal utility $\dfrac{dU(W+A)}{dA} = -2A$

6.21 This utility function would appear to have negative marginal utility of wealth. This is confirmed if we look at the decision criterion equivalent to Min $E(A^2)$ this would be:

$$\min (E_1^2 + V).$$

A decrease in expected return, for a given variance appears to be beneficial.

6.22 Differentiating again:

$$\frac{d^2U(W+A)}{dA^2} = -2$$

This implies increasing marginal disutility. Although this also runs counter to the usual characteristics of utility functions, it is from this property that the intuitively reasonable aversion to variance arises. As long as the fund sponsor has a given level of E_1, this decision criterion leads to fund sponsors trying to reduce the variance of surplus: this may, in many ways, be a realistic situation. Thus, rather than applying the Wise criterion directly (which leads to inefficient mean variance portfolios being chosen using an illogical utility function), it would probably be better, to apply the Wise criteria by chosing the mean level of surplus initially and then choosing the minimum variance portfolios for a given mean surplus. This would lead to efficient choices. This leads us on to the contribution of Wilkie (1985).

6.23 Wilkie put the asset allocation problem in the presence of liabilities into a more traditional mean-variance framework. The framework can best be described briefly, as follows. If we let S be the ultimate surplus; R_1, R_2 and R_L be the amount at the time the last liability is paid of the accumulation of assets 1 and 2 and of the ultimate liability respectively. E_1, E_2 and E_L are the expected values of those variables; x_1 and x_2 are the number of units nominal of assets 1 and 2 which are purchased; V_1, V_2, V_L, C_{12}, C_{1L} and C_{2L} are the variances and covariances relating to the assets and liabilities. E is the expected ultimate surplus; V is the variance of the ultimate surplus and P is the price of the portfolio. P_1 and P_2 are the prices per unit nominal of securities 1 and 2 respectively. Extension to more than two securities is not difficult in principal.

6.24 In the Wilkie formulation, x_1 and x_2 are chosen in order to find an optimal combination of P, E and V. Thus, the following equations are important:

$$S = x_1R_1 + x_2R_2 - R_L \qquad (6.3)$$

$$E = x_1E_1 + x_2E_2 - E_L \qquad (6.4)$$

$$V = x_1^2V_1 + 2x_1x_2C_{12} + x_2^2V_2 - 2x_1C_{1L} - 2x_2C_{2L} + V_L \qquad (6.5)$$

$$P = x_1P_1 + x_2P_2 \qquad (6.6)$$

Thus x_1 and x_2 are chosen to determine the optimal levels of E, V and P. There is much of the original Markowitz framework in this formulation. The two important advances, as far as investment actuaries are concerned are the inclusion of the price of the portfolio and the inclusion of a stochastic liability which can be correlated with the assets. This second facet can, for example, lead to results which are counter intuitive in the standard portfolio selection models but which are well understood as being intuitive to actuaries. For example, an asset type with a high variability of returns, such as equities, but with a high covariance with a liability (for example equity returns with increases in pension liabilities) can reduce the variability of surplus for a given expected surplus. An increase in the proportion invested in that asset type can lead to a more efficient portfolio.

6.25 It is, perhaps, the price of the portfolio which is the most difficult part of the Wilkie formulation to understand. This is, in fact, an implicit way of bringing in inter-temporal considerations. The prices P_1 and P_2 are the prices of the assets. The expected accumulations per unit invested in asset types 1 and 2 (E_1 and E_2) allow for re-investment or sale before maturity on a different yield basis. Thus, the price of the security and the expected accumulation define the expected return.

6.26 Let us assume that the investor is acting in the face of a fixed budget constraint, so that P is no longer a decision variable. Then equation (6.6) must hold for a fixed value of P. The expected surplus equation (6.4) can then be written as the expected surplus per unit invested:

$$\frac{E}{P} = \frac{x_1 E_1 + x_2 E_2 - E_L}{x_1 P_1 + x_2 P_2} \qquad (6.7)$$

Replacing E_1 with E_1/P_1 and E_2 with E_2/P_2 and multiplying x_1 and x_2 by P_1 and P_2 respectively, we have:

$$\frac{E}{P} = \left(\frac{x_1 P_1}{x_1 P_1 + x_2 P_2}\right)\frac{E_1}{P_1} + \left(\frac{x_2 P_2}{x_1 P_1 + x_2 P_2}\right)\frac{E_2}{P_2} - \frac{E_L}{x_1 P_1 + x_2 P_2}$$

Replacing the coefficients on E_1/P_1 and E_2/P_2 by α_1 and α_2 respectively, where α_1 and α_2 represent the proportion of the total budget invested in each asset type, we obtain:

$$\frac{E}{P} = \alpha_1 \frac{E_1}{P_1} + \alpha_2 \frac{E_2}{P_2} - \frac{E_L}{P} \qquad (6.9)$$

$$\therefore \alpha_1 \frac{E_1}{P_1} + \alpha_2 \frac{E_2}{P_2} = \frac{E + E_L}{P} \qquad (6.10)$$

Thus it can be seen that, taking the expected surplus equation (6.4), we can find an expression for the expected total fund per unit invested in terms of α_1 and α_2 (the proportions invested in each asset type). The particular division of the total fund between E and E_L is not relevant in this case as maximising expected surplus is equivalent to maximising the expected return or total fund for a given price or amount invested.

6.27 Thus for a fixed budget constraint P, the investor has to choose the proportions invested in the asset classes (α_1 and α_2) to find the optimal mix of return on the fund and variance of ultimate surplus (which is equivalent to variance of return on the fund). The "efficient" combinations of proportions invested in each asset type for a given budget constinant, are those which give the highest expected return on the fund for a given variance of ultimate surplus. In economic terms the "optimal" portfolio for the investor is that where the "marginal rate of substitution (of expected surplus for variance of surplus) in investment" is equal to the "marginal rate of transformation (of expected surplus for variance of surplus) in the investment market".

6.28 It is useful to specify the problem in terms of both expected ultimate surplus and variance of ultimate surplus (as Wilkie did) and in terms of expected return and variance of return (as we have reformulated it above), as different parties to the investment decision may see the problem in different ways. Fixing the price of the portfolio, using a budget constraint, illustrates the similarity between the Wilkie approach and traditional portfolio selection approach. The mean-variance emphasis implicitly assumes either a normal distribution of investment returns or a quadratic utility function. In reality, the asymmetric position of trustees, in the face of surpluses and deficits in a pension fund, may make the mean variance approach inappropriate. In particular the difficulties caused by a pension scheme becoming insolvent, and the pressure on companies to use surpluses to increase benefits, together with the Inland Revenue's surplus retention rules are likely to lead to a company funding a pension scheme having a complex utility function with at least one discontinuity. Nevertheless, the inclusion of liabilities is an important step forward.

6.29 The other achievement of Wilkie (1985) was to include the "price of the portfolio". So far, we have hidden this by fixing the budget constraint. The implications of incorporating the price are best uncovered by re-writing equation (6.6) as:

$$C = x_1P_1 + x_2P_2 - L$$

where C is the excess value of the assets over and above the minimum recommended by the actuary to meet future liabilities. L is therefore the actuarially determined liability, in a one period, one contribution model which will be assumed to be independent of the actual asset portfolio held. The price of the portfolio is thus seen to be related to the initial surplus.

6.30 This reformulation has the advantage of treating C and E in a consistent and comparable way. The investor has a choice to increase the excess contribution (and therefore surplus) over and above that which is necessary (i.e. increasing C) and then to expect more surplus at the end (i.e. increasing E, as can be seen from equation 6.4). If we include the initial contribution as a decision variable, it is sensible to concentrate on the cash flows, rather than the investment returns. The investor can now change the values of x_1 and x_2, x_1 and x_2 can be determined in order to choose the optimal level of initial excess contribution, expected ultimate surplus and variance of ultimate surplus. It should be pointed out that L is not a marketable liability, otherwise it could just be treated as a negative asset in the portfolio selection model.

6.31 Three qualifications of the framework are necessary: firstly, using a mean variance framework may be inappropriate; secondly further guidance is needed as to how the fund may deal with variability of surplus and trade it off against expected surplus; thirdly, the preference for C against E is inherently a time preference function: if this is recognised, it is possible to determine the investor's optimal contribution and also extend the model into a multi-period model without increasing the number of decision variables. This brings us to the work of Sherris (1992).

6.32 We can consider the contribution of Sherris in three stages. Firstly, he tackles the problem of optimising investment strategy over a single period where the initial level of surplus is fixed; secondly he considers the optimal level of initial surplus so that initial surplus, expected ultimate surplus and some measure of variability of ultimate surplus are decision variables; thirdly, he extends this into a genuine multi-period model by allowing for the possibility of changing investment policy and varying the level of contributions and/or surplus to be distributed at different times. This framework is a generalisation of the Wilkie approach but the generalisation is limited in its power if a mathematically tractable utility function is to be used.

6.33 If the level of initial surplus (which in the Wilkie formulation corresponds to fixing the price of the portfolio) is fixed, the optimal outcome is that which maximises the expected utility of ultimate surplus. Sherris used an exponential utility function, implying constant absolute risk aversion. Maximising an exponential utility function is equivalent to maximising the moment generating function of the distribution of ultimate surplus: see Booth and Ong (1994) for a demonstration of how this is done. Although Sherris' approach is not necessarily a "mean-variance" framework, his particular application turns out to be one, as the two parameter normal distribution of investment returns is used.

6.34 Sherris then takes account of the size of the initial contribution which must be "traded" against the expected ultimate surplus and the variability of ultimate surplus. Thus the optimal investment allocation is that which maximises the expected discounted utility of all cash flows, with later cash flows being discounted to allow for the investor's time preference. Thus the function to be maximised is:

$$E\left[U(-K)+U(S)/(1+b)\right]$$

where K is the initial cash outflow (or the cash outflow over and above some minimum funding level) and S is the ultimate surplus. b is a discounting factor which relates the utility of money received in one time period to that received in another. b is not necessarily a discount rate of the type used for discounting monetary amounts but it may be observable from a consideration of the market pricing of investments. The decision variables are the proportions to be invested in different asset classes.

6.35 The model was then extended to be a genuine multiperiod model. Here we maximise:

$$[U(-K_0) + \sum_{t=0}^{t=T} E\{U(S_t)/(1 + b)^t\}]$$

where S_t is the surplus withdrawn from the fund (or additional contribution if negative) at time t and K_0 is the initial contribution. The fund is wound up at time T. The optimal investment policy is that which maximises the expected discounted utility of surplus.

6.36 The Sherris approach effectively generalised the Wilkie formulation by suggesting a way in which expected ultimate surplus, variability of ultimate surplus and the initial cash flow could be traded against each other. However, the desire to obtain analytical results led to a restrictive utility function (the exponential function) being used which could be regarded as dealing inadequately with investors' risk preferences. It is also undesirable to impose restrictions on the distribution of the decision variable.

6.37 A different approach was taken by Krinsky (1985). He used data to estimate the utility functions of Canadian insurers. The "profit" of the insurer is defined as the return on the equity of the business and it is assumed that the insurer's utility function is defined in terms of the expected value and standard deviation of the insurer's end net worth. The problem for the insurer is to determine the proportion invested in each asset class to maximise the expected utility. Krinsky finds expressions for the impact of a change in the return on one asset on the demand for the other asset, and of the impact of a change in the riskiness of one asset on the demand for the other asset. This is relatively straightforward, for the two asset case and when the asset allocation problem is defined in terms of maximising a function of the mean and variance of the surplus (indeed, Wilkie (1985) finds similar expressions in the three dimensional model).

6.38 Krinsky then worked backwards through the optimisation problem. The assumption was made that investment returns were normally distributed (this assumption was tested but at a weak level). The expected values and variance of returns from different asset classes were estimated from recent levels of investment returns and current index levels. Non-linear maximum likelihood methods were then used to estimate the utility function which would have led the insurer's asset distribution to be a utility maximising one: the data were used to derive maximum likelihood, utility maximising parameter estimates for different functional forms of the utility function; the functional forms were then tested against each other. The two findings of greatest interest were that: six out of eight insurers appeared to have square root or quadratic utility functions; secondly mutuals appeared to have riskier asset distributions. The two major problems of this work were the normality assumption and the fact that it assumes that the insurers are optimising their investment strategy in an informed manner.

6.39 Thus, we see in this analysis of actuarial contributions to the analysis of investment risk a number of features:

(i) In the early contributions, a recognition that expected return, variability of return and avoidance of insolvency were important.

(ii) A number of important contributions which drafted the consideration of liabilities into a mean-variance framework.

(iii) In the work of Sherris, a move towards consideration of expected utility, which allows us to view risk more generally and more realistically. However, in the particular examples used by Sherris restrictive assumptions were needed to obtain analytical solutions.

6.40 Most of the remaining actuarial contributions to investment risk concentrate on downside risk measures (which recognise the strengths of the early contributions and the limitations of the mean variance framework). We consider these in Section 7.

7. Downside Risk Measures: Relationships with Utility Functions

7.1 In this section, we look at the explicit downside risk measures that have been proposed. Clarkson (1989) proposed a measure of investment risk which was a generalisation of semi-variance. His general axioms regarding risk could be paraphrased as:

1. Risk is a function both of the probabilities of return being below a certain level and the severity of the financial consequences of these values of return.

2. A measure of risk (R) was proposed where:

   $$R = \int_{-\infty}^{L} W(L-r)P(r)dr$$

   where $P(r)$ is the density function of return r; $W(L-r)$ is a function of $(L-r)$ which increases with $(L-r)$ and L is the value of return above which no adverse consequences arise.

3. For a given expected return an investor prefers the investment with the lowest risk.

4. Each investor has a threshold level of risk and will not make an investment which involves risk higher than this level: the investor makes the investment with highest return given that the risk does not exceed the threshold.

7.2 A number of points can be made about this approach. Firstly, when assessing risk, the measure does not use the information above the level of return above which no adverse consequences arise. The utility of income function is therefore linear beyond the level of wealth implied by investment return L. If L is a particularly high level of investment return, the linear portion may be irrelevant in practice. The utility function implied by the risk measure below the threshold is determined by the function $W(L-r)$.

7.3 Considering point 4 in section 7.1, the risk measure is ignored if the function is below the threshold level of risk. Furthermore, the investment is rejected, if the risk measure is above the threshold, even if the expected return is extremely high.

7.4 Clarkson proposes that $W(L-r) = (L-r)^a$; if $a = 2$ then the risk measure is equivalent to semi-variance. An investment decision making criterion based on a linear function of semi-variance and expected return is equivalent to a decision based on variance and expected return when distributions of investment returns are symmetrical. In any event, it is equivalent to using a quadratic utility function below L and a linear one beyond L as is demonstrated in Markowitz (1991).

7.5 The decision criteria 4 of Clarkson has some inconsistent implications for the implied utility function.

If the decision making criterion were simply to:

$$\max f(R, E(r))$$

and R is based on semi-variance measured below the expected return, this is equivalent to the investor having a quadratic utility function below $E(r)$ and a linear utility function thereafter. However, this is not the decision criterion proposed by Clarkson. Let us illustrate some of the inconsistencies in the Clarkson decision criteria by way of example. We will construct a number of investments, A to D. We will assume that:

$$L = 0.05 \text{ and } W(L-r) = (L-r)^2$$

and the level of R above which the investment is not considered is 0.00285.

Investment A: this has returns -0.04, -0.01, 0.10 and 0.15 with probabilities 0.2, 0.3, 0.25 and 0.25 respectively. In this case, $R = 0.0027$ and the expected return is 0.0515.

Investment B: this has a return of 0.05 with certainty. Investment A is chosen over investment B due to Clarkson Axiom 5 (described in 7.1 condition 4).

Investment C: this has returns -0.06, -0.01,0, 0.15 and 0.25 with probabilities 0.1, 0.3, 0.25, 0.25 and 0.1 respectively. $R = 0.0029$ $E(r) = 0.0535$. This investment is rejected in favour of both A and B due to Clarkson Axiom 4 also described in 7.1 condition 4.

There is no inconsistency, in terms of utility theory, in preferring Investment A to Investment C. The loss of utility from the possibility of a low return is not compensated by the gain in utility from the higher returns. However, now consider Investment D.

Investment D: this is a compound investment which provides Investment A with probability 0.5, and Investment C with probability 0.5. The distribution of returns is now: -0.06, -0.04, -0.01, 0, 0.1, 0.15 and 0.25, with probabilities 0.05, 0.1, 0.3, 0.125, 0.125, 0.25 and 0.05 respectively. $R = 0.0028$, $E(r) = 0.0525$. Investment D is preferred to all investments.

7.6 Thus the investor prefers Investment A to Investment C but prefers a 50% chance of either Investment C or A to both Investment A and Investment C. Using a sporting analogy developed in Clarkson (1989): the sportsman will not partake in hang-gliding because it is too risky and prefers football instead (in all circumstances). Yet the sportsman would prefer to have the decision taken out of his hands and toss a coin to see which sport he pursues. There is a clear inconsistency in any decision criterion which leads the investor to unambiguously preferring one investment to another yet preferring of 50% chance of obtaining either. The explanation for this in utility theory terms is that the investor, when looking at the two situations does not evaluate the utilities of wealth in the same way, when approaching the investment from a different perspective.

7.7 In recent years, there has been an increase in the volume of literature on downside risk measures. As has been mentioned, Markowitz (1952) put forward the possibility of using semi-variance but did not pursue it further. Clarkson (1989) proposed a general risk measure which had semi-variance as a special case. Clarkson (1994), Albrecht (1994) and Ferguson and Rom (1994) also consider downside measures of risk. It has already been noted that the use of semi-variance as a risk measure, if no assumptions are made about the distribution of investment returns implies quadratic utility functions below the threshold level of return and linear utility functions above that level. It was also shown that the Clarkson decision criterion could lead to inconsistent results. Here we certainly evaluate downside risk measures more thoroughly.

7.8 There are two classes of downside risk measures which can be considered:

a) Downside risk measures which only use information below a threshold return but where risk is still traded against return in the investment decision.

b) Decision criteria of the type used by Clarkson (1989) and Booth and Ong (1994) which reject firmly any portfolio which does not satisfy a particular constraint (for example if it has a probability of a negative return which is greater than a particular value).

7.9 Methodologically, these two characteristics bear no relationship to each other. The first characteristic can be reconciled in a utility maximisation framework, the second cannot.

7.10 Using the framework defined by Albrecht (1994), if we let R be the one period return of a single financial investment or portfolio, R has a density function $f(r)$ and distribution function $F(r)$. Let m be the threshold return, above which an investment out-turn makes no contribution to the risk measure.

$$R = m + R_+(m) - R_-(m)$$

$$\text{where } R_+(m) = \max(R-m, 0) \text{ and}$$

$$R_-(m) = \max(m-R, 0)$$

The shortfall return is concerned with $R_-(m)$. The distribution function of $R_-(m)$ is:

$$\begin{cases} 0 & r < 0 \\ 1 - F(m-r) & r \leq 0 \end{cases}$$

[i.e. the shortfall return cannot take a value below zero and the probability that it takes a value less than *r*, where $r \geq 0$, is the probability that the return is greater than $(m-r)$].

$$f\_(r)\begin{cases} 0 \; r < 0 \\ 1 - F(m) \;\; r = 0 \\ f(m-r) \; r > 0 \end{cases}$$

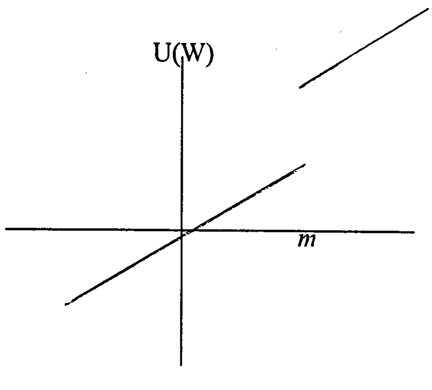
The shortfall return cannot be less than zero, and is zero if the return is greater than *m*.

7.11 A more generalised downside risk measure can be measured by $E(R^n_-)$. In terms of the notation developed in paragraph 7.10.

$$E\left(R_-^n\right) = \int_{-\infty}^{m} (m-r)^n f(r)dr$$

We will now examine further properties of downside risk measures of this form, in terms of utility theory.

7.12 If $n = 0$, the measure of risk is simply probability of shortfall or probability of loss. If the investment decision is based only on a linear combination of expected return and probability of loss the utility function is two discontinuous linear segments with a discontinuity at *m*. Any given increase in expected return, which does not affect the probability of loss benefits the investor to the same extent. Thus the discontinuous lines must have the same slope. Any increase in the probability of loss with an unchanged expected return must reduce utility, hence the discontinuity. Thus the probability of shortfall measure of risk gives rise to the following utility function.



Interestingly, this measure of risk leads to the same utility function as that implied by the revised Bailey conditions (see Section 6). If, however, the investor chooses the portfolio to limit probability of loss to a given value (say 5%) and does not "trade" probability of loss with expected return, he is not acting, in any sense, in an expected utility framework. This is similar to the "shortfall controls" mentioned in paragraphs 7.8 and 7.23.

7.13 If $n = 1$, then the measure of risk is the expected loss EL, where:

$$EL = \int_{-\infty}^{m} (m-r) f(r) dr$$

A decision based purely on expected loss and expected return leads to a combination of two linear functions, with different slopes, joining at point m. This is demonstrated by Markowitz (1952).

7.14 If $n = 2$, then the risk measure is the semi-variance (SV) measure below point m, where:

$$SV = \int_{-\infty}^{m} (m-r)^2 f(r) dr$$

As has been mentioned, Markowitz (1952) demonstrates that if the investment decision is based on expected return and semi-variance, it is equivalent to assuming a quadratic utility function below m and a linear function thereafter. A discontinuity could also be manufactured in the case of $n = 1$ and $n = 2$ by also basing the decision on the probability of shortfall.

7.15 Simple shortfall measures of risk have two major drawbacks: they all imply linear utility functions beyond the threshold return and they assume restricted forms of utility function below the threshold. However, the inclusion of "probability of shortfall", to manufacture a discontinuity, is attractive.

7.16 Ferguson and Rom (1994) suggest the use of semi-variance as a shortfall risk measure. They derive "efficient frontiers" in an expected value/semi-variance framework. Portfolios which provide a known rate of return (i.e. zero variance), where that known rate of return is below the threshold from which semi-variance is measured are not necessarily efficient (portfolios of treasury bills or other risk free assets could, for example, be below the efficient frontier). The use of semi-variance is justified on two counts: firstly, it is felt to provide a better description of risk than variance; secondly, optimisation techniques now allow the use of the semi-variance risk measure, which is difficult to deal with analytically.

7.17 Whilst it may be felt that semi-variance is a more appropriate risk measure than variance, it is less than ideal, for the reasons discussed above. In addition, if applied generally and in market pricing models, it would require the additional, restrictive assumption that all investors were close to the point where the quadratic and linear portions of the utility functions meet. Semi-variance however may be an appropriate risk measure for fund managers whose major fear was underperforming a specific benchmark [see Booth and Sandford (1996)].

7.18 There are other properties of semi-variance which are of interest. If investment returns are normally distributed, with mean μ and variance σ^2, Albrecht (1994) shows that the shortfall semi-variance (measured from threshold m) is:

$$\left[(m-\mu)^2 + \sigma^2\right]\Phi\left(m_N\right) + \sigma(m-\mu)\varphi(m_N)$$

where Φ and φ represent the distribution and density functions of a standard normal distribution and m_N is the standardized quantity $(m-\mu)/\sigma$.

7.19 Thus, for a given threshold m, a decision based on semi-variance is ultimately based on mean and variance: the only two moments in the above expression. Therefore, if investment returns are normally distributed (and all linear combinations of investments have normally distributed returns) we are back in the mean-variance decision making framework.

7.20 The second property of interest is apparent, if we consider the behaviour of the semi-variance measure when $m = \mu$. Then we see that, in fact, that semi-variance is proportional to variance. The second term is zero and the first term is $\frac{1}{2}\sigma^2$.

7.21 If accumulations are log normally distributed then the accumulation of an investment compounded over several years will be log normally distributed. Albrecht gives expressions for the shortfall semi-variance of an investment accumulation in these circumstances. Again, given that there are only two parameters in the log normal distribution (both functions of mean and variance), the investment decision takes place in a mean-variance framework.

7.22 Where assets with different log normal distributions of returns are combined together, the resulting distribution of returns may have more than two parameters. Semi-variance may then be a function of parameters other than mean and variance and could be a more appropriate risk measure than variance and lead to different results. However semi-variance still suffers from the major weaknesses of imposing a utility function on investors which is probably not realistic, although it may not be as unrealistic as a simple quadratic. It is difficult to avoid coming to the same conclusion, as Markowitz did, that any theoretical advantages of using semi-variance alone as a risk measure are outweighed by practical disadvantages. However, other approaches which take into account downside risk may be appropriate.

7.23 Albrecht also suggested a "shortfall control", equivalent to the probability constraints of Booth and Ong (1994) and of Clarkson (1989). The nature of the shortfall controls suggested by Albrecht were as follows, firstly:

$$Pr(R \leq m) \leq E$$

That is, the probability that the return is less than or equal to a benchmark, m, must be less than a particular value. This particular criterion has all the difficulties associated with the Clarkson decision criteria, described earlier. It leads to inconsistent decisions in the utility maximisation framework.

Secondly:

$$SE_m(R) \leq c\, E(R)$$

That is, the shortfall expectation, measured below m, should not exceed a given percentage of the expected return. This criterion gives rise to:

$$\int_{-\infty}^{m} (m-r)f(r)dr \leq cE(R)$$

therefore:

$$m\Pr(R \leq m) - \int_{-\infty}^{m} rf(r)dr \leq cE(R)$$

Thus, m times the probability of shortfall minus the first partial moment of R should be less than a certain proportion of the expected return. The first term on the left hand side will simply increase with the proportion of the probability weight that is to the left of m. The second term on the left hand side will decrease if the weighting to the left of m is moved further left in the distribution. Both these eventualities will make it less likely that the criterion will be fulfilled. Whilst these characteristics are desirable characteristics of a shortfall measure, there is no clear intuitive justification for their combinations in this particular way. It is also not clear how the parameter c would be chosen. For a given expected return, the first term on the left is affected by any increased tendency to fail to meet the benchmark m. This is consistent with the idea of a discontinuity in a utility function. The second term takes account of the expected extent to which we do not meet the benchmark but only in a linear fashion. An increase in expected return, for a given shape of the distribution of returns would appear to make the shortfall criterion easier to fulfill. The right hand side would increase and the first term on the left hand side would decrease.

7.24 Overall, this particular shortfall control would appear to lose the generality of a utility theory approach without having an immediate intuitive appeal. It is also unclear why the expected return should come into a risk control criterion. If shortfall measures of risk are considered suitable, the risk control criterion should be defined only in terms of a given benchmark return, rather than the mean.

7.25 A shortfall control was also used by Stowe (1978). He suggested that the aim of investment policy should be to maximise return on assets subject to a solvency constraint (as well as institutional constraints). The shortfall control also underlines the philosophy of value at risk models used to control market risk in banking: see Allan (1995).

7.26 The use of summary downside risk measures has the disadvantage of taking a limited view of risk. However, because variance of returns looks at the upside variability in the same way as downside variability, downside risk measures may be an improvement on the use of variance. However, downside risk measures do not have the advantage of ease of mathematical manipulation that variance of return has. Given that downside risk measures are neither all encompassing nor easy to deal with mathematically, it would seem sensible to pursue methods of analysing risk which look more closely at the full distribution of investment returns, if a more generalised approach is to be used.

7.27 The second aspect of downside risk, the use of shortfall controls, has the disadvantage of sometimes producing investment decisions which are inconsistent with utility maximisation. However, they have the advantage of being simple to use, with an understandable intuitive justification. They may therefore be useful as a pragmatic device to alert the investor to potentially unsuitable investment strategies. However, the maximisation of an appropriate utility function would appear to be a better, more general method of selecting an investment strategy.

7.28 Traditional actuarial approaches have tended to use a mixture of mean-variance approach; shortfall constraints; and shortfall risk measures. Often, there has been an implied, underlying utility function. The use of an explicit utility function has the advantage of allowing a general approach to the measurement of risk to be taken. However, criticisms can be made of a utility theory approach to risk-return analysis.

8. Criticisms of the Use of Utility Theory in Actuarial Work

8.1 The use of utility theory when taking actuarial investment decisions can be criticised on at least three grounds. Firstly, the possibility that the assumptions underlying the development of the theory could be invalid. Secondly, it may be impractical to find appropriate utility functions. Thirdly, utility theory may be irrelevant to practical actuarial problems.

8.2 The first criticism was addressed by Friedman and Savage (1948) in their paper, "The Utility Analysis of Choices Involving Risk". The assumption which had been attacked was that of the "rational man", which also underlies much of modern portfolio theory. If market participants are not rational, it does not make sense to predict market outcomes using a utility theory model which assumes utility maximisation. Economists who believed that diminishing marginal utility must exist throughout the range of possible wealth outcomes could not accept that individuals could insure and gamble (a rational individual who gambles must have increasing marginal utility over a certain range of income). They therefore concluded that individuals must behave in an irrational way.

8.3 Friedman and Savage, instead, questioned the continuous diminishing marginal utility hypothesis. They suggested that investment activities could broadly be divided into three categories: absolutely secure investments (for example bonds); moderate risk investments (for example high grade shares); and high risk investments or ventures (such as certain business undertakings). Canan (1926) suggested that the highest expected monetary pay off arose from the second type of investment: this suggested that most people were risk averse in the local part of the utility function. However, some people would be happy to invest in the third group of investments, despite lower expected returns than from the second group, because they are risk loving at certain points in their utility functions.

8.4 This tendency for some people to be risk loving, regarding certain transactions, is also evident from the observation of people buying lottery tickets (which normally have a negative expected net pay off). Friedman and Savage put forward two possible and related explanations which did not involve a dismissal of the rational man concept. The first suggestion was that people have utility functions which are partially convex: this would imply that, for certain gambles, and at certain income levels, individuals can behave as if they were risk lovers. However, such individuals may also insure as the utility of wealth curve would generally be concave. The second suggestion is that individuals may attach great utility to moving to a new "socio-economic" group: men take risks to distinguish themselves. This perhaps suggests that individuals would have steeply convex utility functions (or a discontinuity) at a particular level of wealth. Having reached that new level of wealth, the utility function may then change or, at least, the individual will be operating in a different area of utility curve. The ideas of such a "status based" utility function and of a discontinuity have already been proposed as being useful in describing risk preferences in an actuarial context.

8.5 Thus Friedman and Savage provide an explanation for behaviour which was previously thought to undermine the rational man concept. Three further points can be made, however, the first two also identified by Friedman and Savage. Firstly, utility theory is a model. It is not necessary for every man to behave rationally for it to be useful in modelling behaviour in general. It is not even necessary for any man to consult a utility function before deciding how to behave. Friedman and Savage used the analogy of a billiard player. The result of any billiard shot can be obtained from a consideration of the mechanics underlying the shot. However, a billiard player, without consideration of mechanics, plays shots as if he considered the mechanics. Mechanics still provides a good model. Investors invest to their maximum advantage without, generally, explicitly consulting a utility function. This does not rule out utility theory as a useful model of investor behaviour. This philosophical consideration of the usefulness of theoretical modelling is developed at greater length in Friedman (1953). Secondly, Friedman and Savage note that, in fact, most people do insure. This is an indication of generally rational behaviour in the face of generally diminishing marginal utility. A third point to note is that the applications of utility theory in the actuarial field generally relate to discovering how rational investors should behave. For example, how an institution should rationally allocate its assets. The results of any investigations (unlike many of the models in financial economics) need not assume that everybody else behaves rationally.

8.6 We turn now to the difficulty of finding appropriate utility functions. Von Neumann and Morgenstern (1947) made a comparison between utility theory and the theory of heat. When the theory of heat was developed, it allowed physicists to determine which of two bodies was the hotter but not by how much. Similarly, there are clear difficulties adding and subtracting utilities in the same way that we can add and subtract quantities of money. However, this is not a difficulty in principle, as long as it is possible to derive, from empirical observation, a utility function, the functional form of which describes the additional utility derived from an increment to wealth.

8.7 Friedman and Savage (1948) again deal with this question. They suggest that we can move towards the development of a cardinal utility function by observing market behaviour (of investors or people who insure) under uncertainty. For example, if investment A is preferred to investment B which is preferred to investment C, we can say that investment A is preferred to investment C. If, in turn, we are offered a 50:50 combination of investments A and C and this is preferred to investment B, this shows that the difference between the utility of investments A and B is greater than the difference between the utility of investments B and C. This does not get us very far towards postulating a utility function: particularly if the three investments provide a range of cash flows with different probabilities. However, if we know the alternative cash flows we can say something about the marginal utility of different income increments. Given an unlimited number of investments and an investor's preferences for those investments, we can certainly derive his utility function (an analogy here would be with the difficulties faced deriving a continuous spot rate yield curve from gross redemption yields on fixed interest securities). Even if deriving a precise smooth functional form is impossible, given information about how investors behave in the face of risk, we can certainly propose plausible utility functions which make useful models.

8.8 Thus, if we accept that utility theory, despite its imperfections, provides us with a good model which we can use to analyse risk, is utility theory useful in actuarial investment work? The starting point in answering this question must be to point out that any view of risk, taken by an actuary involves, implicitly, expressing the extent to which we prefer different increments to wealth. The Bailey, Markowitz, Wise, Wilkie and Clarkson approaches to risk all have an underlying (if complex or inconsistent) utility of wealth function.

8.9 If an actuary is to take a view on investment risk, it is natural that the view should be based on a prior view of the utility of different increments to wealth. But the difficulties of deriving a reasonable utility function for the individual are great enough: are the difficulties of deriving a usable utility model for a corporation so great that it is better to rely on intuition rather than some kind of formal model? A number of points can be made in response to this question.

8.10 Firstly, the development of a utility function is easier the less complicated are the interests in investment policy or the more easily interests can be separated. Thus, the analysis of the optimal investment strategy for a unit linked fund only requires a reasonable utility function for personal investors, this is analysed in Booth (1995). A mainly non-profit life office needs to be able to deal with the separate interests of shareholders, policyholders, professionals, managers and regulators: see Ong (1996). However the interests of all but the first group tend only to be operative when a position close to insolvency is reached. Modelling with-profit business may well be somewhat more difficult. Secondly, utility functions can be used alongside concepts with which actuaries can feel more comfortable. Clarkson (1989) for example rejects all investments which have a certain degree of risk or greater. Lipman (1990) in discussion of investment policy for Australian superannuation schemes suggests a generally smooth utility function combined with either a discontinuity around a benchmark return or a probability constraint requiring that benchmark to be achieved with a particular probability. Booth et al (1993) and Booth and Ong (1994) use a utility function combined with an insolvency constraint. It was seen in the discussion of Clarkson's risk measure that the use of probability constraints actually implies inconsistent utility functions. However, this disadvantage may be outweighed by the advantage of using concepts with which actuaries may feel comfortable.

8.11 It should also be remembered that it is not the role of the actuary to produce incontrovertible, mathematically correct answers to financial problems. The role of the actuary is to manage risk. Utility theory is proposed as a theoretically correct method of managing risk: not of producing definitive answers to questions. Utility models are a management tool. They can be used to discern the effect on optimal investment policy of changing tolerance to risk or to understand the true risks to shareholders and policyholders of following mismatching investment policies. They can aid managers of financial risk in choosing investment strategies to maximise the benefit to policyholders. There are, in fact, only two options other than using utility theory, to manage risk: one is to take no scientific view whatsoever and rely on intuition; the other is the use of some other decision criterion which does, in fact, imply some form of "hidden" utility function. Actuaries have often been critical of mean-variance analysis for not taking a sufficiently realistic view of risk. It is better and possible to build a model which removes the weaknesses of mean-variance analysis, without disposing of the sound foundations which underlie it.

8.12 Clarkson (1994) also criticises the use of the expected utility maxim, using evidence from Allais (1953). Allais provides evidence that individuals come to investment or gambling decisions which are inconsistent with the utility maximisation criteria. The Allais criticism is more serious than that which was rebutted by Friedman and Savage because Allais' evidence cannot be reconciled within any expected utility maximisation framework, even if increasing marginal utility is allowed. The use of downside measures of risk, does not help overcome the objections raised by Allais (as downside measures of risk still imply an underlying utility function). However, the use of probability constraints may be compatible with Allais' evidence as they are themselves not consistent with the utility maximisation criterion.

8.13 Allais' objections can be countered in a number of ways. Firstly, the existence of individuals who do not maximise expected utility is not incompatible with the use of a utility maximisation model by somebody who does want to maximise expected utility. Secondly, Allais' evidence relates to questions about investment decisions which are posed in a way which makes an understanding of the implications of the responses difficult. Thirdly, Allais' scenarios relate to extremes of increments to wealth: as Friedman and Savage point out, decisions relating to extreme movement in wealth will often be different from those relating to smaller movements in wealth.

8.14 It is felt that the utility maximisation criterion provides a good model in which investment risk can be understood. In addition, utility theory provides a robust justification for the use of different investment risk measures. The assumption of a continuous smooth quadratic utility function, which would enable mean-variance analysis to be used is probably inappropriate except as an approximation. Its appropriateness or otherwise can be considered from a utility theory perspective. The possibility of using simulation allows more general utility functions and risk measures to be used, including utility functions which allow for kinks and discontinuities. Despite its drawbacks, the use of a shortfall criterion as an additional tool may also be very helpful. In practice, it may well be found that shortfall criteria produce results similar to those produced by using discontinuities in the utility function, if the range of possible investment returns is limited. This is an area for further research.

9. <u>Practical Applications of an Appropriate Utility Model</u>

9.1 For utility theory to be useful in the area of actuarial investment risk management, we must be able to develop a utility of wealth function which is a practical model of reality.

9.2 With regard to an individual taking an investment decision (or institution taking a decision on behalf of an individual), we can divide the proponents of different measures of investment risk into three schools: those who give upside variability the same weight as downside variability (in the main, mean-variance proponents); those who give downside variability greater weight (often mean semi-variance proponents); and those who wish to see much more weight given to significant losses (often proponents of shortfall constraints). In an ideal model all three strands should be taken in account.

9.3 An intuitively reasonable utility function for most investors, at most levels of income, would be a log function. The implication of the log function is that all investors view gives a proportionate increase in their wealth to be of equal value, regardless of their starting level of wealth. In general, all moments of a distribution of investment returns should be taken into account when considering investment risk but the log utility function could be approximated as a function of mean and variance over certain ranges. However it would seem reasonable to include in the utility function a discontinuity which would be at the wealth point below which the investor is believed to suffer particularly adverse consequences. The log utility function with a discontinuity would capture all the considerations described in paragraph 9.2.

9.4 An important advantage in applying a log utility function, in practice, is that it leads to the same investment decision being taken by all investors, at any initial level of wealth, if they invest the same proportion of their wealth. The discontinuity can then be used to model the situation faced by investors who are particularly averse to dropping below a given level of wealth. The practical effects of a log function with discontinuity would be:

(a) the investor would be averse to fluctuations in investment return above the mean (as in the mean-variance framework).

(b) the investor would be more averse to downside variability than upside variability (as with the semi-variance measure).

(c) the investor is likely to reject investments which have a high probability of a level of return which would take the investor below a particular value of wealth (as with shortfall controls). However such investments could redeem themselves if they had other desirable characteristics.

9.5 The development of mathematical aspects of the expected utility maximisation problem in these circumstances is the subject of ongoing research. However, it is of interest to note that in Booth (1995) a utility theory approach to asset allocations was compared with a mean-variance and mean semi-variance approach. The utility theory approach produced highly intuitive utility maximising portfolios. This work is being developed to allow for discontinuities in the utility function and using different investments model assumptions. Instead of discontinuities, shortfall constraints were included to try to mirror the effect of discontinuities. Possible levels of wealth at which a discontinuity of shortfall constraint could be used are:

(i) at the level of wealth necessary to purchase a pension equal to a given fraction of final salary at retirement

(ii) for an investor with a particularly low level of income, there could be a discontinuity at the level of wealth necessary to buy a "subsistence" pension

(iii) for the pension fund investment managers, there may be a discontinuity at the point when the investments underperform a competitor form of investment (see Lipman (1990)).

9.6 It is the existence of these benchmark levels of wealth which can lead to optimal investment policy being quite different for different individuals. Utility theory is a very appropriate framework in which this can be investigated. Ludvick (1994) uses a shortfall control or probability of loss, in the context of a defined contribution pension scheme, instead of utility theory. This is a useful management tool but can lead investors to take quite inconsistent decisions, if it is applied rigidly.

9.7 Mathematical aspects of the application of utility functions for institutions must also be the subject of further work. As has been mentioned, Sherris (1992) and Sharpe and Tinte (1990) have done some work in this field. It is of interest to note that Booth and Ong (1994) obtained intuitive results using a utility maximisation approach to life insurance company asset allocation. This work has been developed much further by Ong (1996): one interesting aspect of his work is the way in which classical results can be found (such as an immunised position) as limiting cases of a general utility maximisation problem. The use of utility theory, however, allows more general problems to be solved.

9.8 It is useful to conclude by drawing on some of the observations of Smith (1996). Firstly, except in the analysis of unitised personal investment funds, liabilities should always be taken into account in the consideration of risk. This is a feature of most of the approaches to risk that have been discussed. Secondly, the use of the concepts that have been discussed can be an aid to decision making and also help us to understand the world about us. However, this does not mean that the abstract pricing models developed from financial economics are necessarily appropriate. To conclude on a philosophical note: historically, the actuarial profession has used a mixture of science and art. The scientific techniques discussed in this paper are important in managing risk. However, it would be wrong to pretend that any particular scientific model or set of techniques can capture all the features of the real world. There is ample scope for judgement, dispute and for seeking an understanding of the true nature of the practical problems actuaries face as well as for scientifically modelling what we know about the world.

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