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# Entry Deterrence in Dynamic Second-Price Auctions\*

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## Abstract

We examine a dynamic second-price auction with independent private values and sequential costly entry. We show that *delayed revelation equilibria* exist in which some buyers with sufficiently high valuations place coordinated low early bids, and bid their true valuations just prior to the end of the auction. Compared to the benchmark *immediate revelation equilibrium*, in which buyers bid their true values immediately after entry, fewer high-value bidders enter on expectation in some delayed revelation equilibria. We show that delayed revelation of buyer values decreases social welfare, but is necessary for bidders to have a strict participation incentive. Computations suggest that the welfare effect of delayed revelation consists primarily of transfer of surplus from the seller to bidders, while efficiency losses are relatively small.

**Keywords:** Dynamic second-price auctions, costly entry, sequential entry, bidder collusion, entry deterrence, early bidding, late bidding, incremental bidding.

**JEL codes:** D44, D83

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# 1 Introduction

In dynamic auctions, a buyer's decision to participate often depends on earlier bids submitted by other buyers. For example, auctions on online marketplaces such as eBay typically run for many days, and when a potential buyer discovers an auction for an item of interest several bids may have already been placed. This buyer must then decide whether to participate and place his own bid, or walk away. Participation is not costless, as a buyer needs to assess the condition of the item, shipping charges to the buyer's location, and so on, all of which may influence his willingness to pay. On the other hand, information summarizing past bidding activity in the auction—the current auction price, the price history, or the number of bids—is often readily available and can be used in the potential buyer's decision to participate.

To examine the interaction between bidding strategies and participation decisions in such settings, we consider a single-object, second-price auction with independent private values and costly entry. The auction is open during a fixed number of time periods. In each period one risk-neutral potential bidder arrives, observes the price history, and decides whether to participate in the auction. If the bidder decides to participate, he incurs a non-refundable entry cost, learns his private valuation, and is then free to bid in every period thereafter. At the end of each period, the price is updated to the second-highest bid thus far submitted. At the end of the final period, the highest bidder wins and pays the ending price.

We show that multiple bidding equilibria exist that differ in terms of the expected number of entering bidders, expected final price, and expected payoffs obtained by the buyers and the seller. In all equilibria, buyers enter at sufficiently low prices, and every participating buyer eventually bids his valuation. What happens before that, however, depends on which equilibrium is being played. Let us begin by describing two equilibria:

1. In the first, *immediate revelation equilibrium*, a bidder enters if and only if the current price is below some cutoff price  $p_1^*$ . Then, if the bidder's value exceeds the current auction price, he submits exactly one bid equal to his valuation immediately after entry. Once the auction price reaches  $p_1^*$ , entry ceases. This will be the case after two bidders with valuations above  $p_1^*$  have entered; the one with the higher valuation eventually wins and pays a price equal to the next highest valuation.

2. In the second, *simple delayed revelation equilibrium*, a bidder enters if and only if the current price is below a different cutoff price,  $p_2^* < p_1^*$ . The entering bidder then bids as in the first equilibrium, with one exception: if the bidder's valuation exceeds  $p_2^*$ , he submits a bid equal to  $p_2^*$ . Once the auction price reaches  $p_2^*$  all further entry is deterred; this will be the case after two bidders with valuations above  $p_2^*$  have entered. These two bidders then submit a pair of truthful late bids in the final period, and the one with the higher valuation wins and pays the next highest valuation. Because  $p_2^* < p_1^*$ , fewer bidders enter on average in this equilibrium than in the first.

The second equilibrium is tacitly collusive, in the following sense: By coordinating their opening bids on the same low value, two of the early arriving bidders effectively deter participation by potential rivals who would have entered in the first equilibrium (we will explain in a moment why this is the case). These two bidders use the final period to compete against each other in a single Vickrey auction to allocate the object among themselves. Because some potential competition is eliminated, the expected surplus of these bidders is larger in the second equilibrium than in the first. On the other hand, the expected price the seller obtains is lower in the second equilibrium.

How does “entry deterrence by bidding low” work? Consider an arriving bidder's decision to enter, which depends on the expectation this bidder holds about the valuations of competing bidders, conditional on observed prices. This expectation, in turn, depends on the bidding strategies used by competing bidders. In both equilibria discussed above, the auction price at any time provides a lower bound on the highest valuation among currently participating bidders. The conditional distribution of this valuation above its lower bound is different across the two equilibria, however. In the first (immediate revelation) equilibrium, for any observed price exactly one participant has a valuation above that price. The threshold price  $p_1^*$  is then the price at which a potential entrant becomes indifferent between walking away from the auction and paying the entry cost for the right to compete against one bidder with valuation above  $p_1^*$ . In the second (delayed revelation) equilibrium, the same is true except when the price is exactly  $p_2^*$ . In this case, the arriving bidder must believe that there are *two* participants with valuations above  $p_2^*$ . Now set  $p_2^*$  equal to the price at which a potential entrant becomes indifferent between walking away from the auction and competing against two bidders with valuation above  $p_2^*$ . Because it is harder to compete against two rivals instead of

one,  $p_2^* < p_1^*$ . Thus, two bidders who coordinate their bids on  $p_2^*$  can stop entry earlier than they would be able to if they revealed their valuations immediately after entry.<sup>1</sup>

After having established the basic mechanics of entry deterrence, we proceed to construct a larger class of delayed revelation equilibria, in which more than two bidders coordinate their bids. This is not a trivial extension of the two-bidder case. For example, one may wish to construct an equilibrium in which three bidders coordinate their bids on  $p_3^*$ —the price at which a new entrant becomes indifferent between walking away and competing against three rivals with valuations above that price. But since the auction is still a second-price format, the communication possibilities that the price mechanism provides are limited. In particular, the auction price will equal  $p_3^*$  once the first *two* bidders have submitted bids equal to  $p_3^*$ . This difficulty is solvable, provided the history of past bids (and not only the price sequence) is observable. Under this assumption, a third type of equilibrium exists in which three or more bidders delay revelation of their values in order to deter entry by later arriving bidders. These equilibria involve a phase during which bidding proceeds in small, gradual increments:

3. In an *incremental bidding equilibrium* with  $k$  coordinating bidders, the first two buyers with valuations above  $p_k^*$  bid  $p_k^*$ . These bidders, along with any additional bidders whose valuations are sufficiently high, then bid the previous period's price plus a small increment  $\varepsilon$  in every period (provided this does not exceed a bidder's true valuation). Once  $k$  bidders have done so in a single period, it is common knowledge that the valuations of  $k$  participants are above  $p_k^*$ . From this moment on, entry is deterred. The  $k$  bidders whose valuations are above  $p_k^*$  now wait until the final period, at which time they reveal their valuations to allocate the object among themselves.

Bidding behavior in our delayed revelation equilibria is consistent with several empirical regularities observed in online auctions. First, on expectation, there is a concentration of bidding activity in both the early periods of the auction and in the final period. Second, some participating buyers submit more than one bid, and, in some equilibria, submit many small incremental bids. These patterns have been documented in a large empirical literature on internet auctions, which we review in the next section along with existing models that can explain these observations. Our results provide a different and, to our knowledge, new explanation. We show that early bidding,

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<sup>1</sup>Even though bidders care about the distribution of their opponents' valuations, and learn about this distribution from previously submitted bids, our results do not rely on bidders' risk aversion, nor on an assumption of correlated or common values. We assume risk neutrality and independent private values throughout.

late bidding, and multiple and incremental bidding, emerge together as equilibria of a dynamic second-price auction with costly entry.

Finally, we characterize some welfare properties of our equilibria. We show that social welfare is highest in the immediate revelation equilibrium and then decreases as one moves through the set of delayed revelation equilibria in which  $k = 2, 3, \dots$  bidders coordinate their bids. We also show that buyers obtain a zero expected surplus in the immediate revelation equilibrium, and a positive surplus in all delayed revelation equilibria. This implies that delayed value revelation is necessary for buyers to have a strict participation incentive in dynamic second-price auctions with costly entry. As far as the seller's ranking is concerned, expected revenue is higher in the immediate revelation equilibrium than in the simple delayed revelation equilibrium, and if buyer valuations are uniform the immediate revelation equilibrium maximizes seller's revenue across all equilibria.

The remainder of the paper is organized as follows. In Section 2 we review previous research on bidder collusion, auctions with endogenous entry, and online auctions, and locate our paper within this literature. In Section 3 we introduce our model. In Section 4 we characterize bidders' optimal entry decisions, given their beliefs and the bidding strategies of others. In Section 5, we construct the immediate revelation equilibrium and the delayed revelation equilibrium, and in Section 6 we construct incremental bidding equilibria. Section 7 contains our comparative results. Section 8 concludes with a discussion of some implications of our results for auction design. Most proofs are in the Appendix.

## 2 Relation to the Literature

Our paper is related to three strands of previous work: The literature on auctions with endogenous entry, the literature on bidder collusion, and the literature on bidding behavior in online auctions. Below, we review previous research in each of these areas and place our model in relation to the existing literature.

*Auctions with endogenous participation.* The literature on endogenous participation in auctions can be divided into two main branches. The first branch originated in work by McAfee and McMillan (1987) and Levin and Smith (1994) and assumes that potential bidders do not know their valuations when making their entry decisions. Bidders learn their types after paying an entry cost, which represents the costs associated with evaluating one's private willingness to pay (e.g., the mental

cost of introspection, or the monetary or time cost of due diligence). The second branch originated in Samuelson (1985) and assumes that potential bidders know their valuations before deciding to participate. In this case, the cost of entry can be interpreted as a participation fee charged by the seller, or as the cost associated with preparing and delivering a formal bid.

Our model of bidder entry is based on the first branch. We assume that bidding commences immediately after entry of the first bidder. In this context, Crémer, Spiegel, and Zheng (2009) show that the seller can implement *ex ante* welfare maximizing allocations if she can control how many buyers acquire information and set period-specific reserve prices; furthermore, through lump-sum participation fees she can extract the entire social surplus. This implies that the seller ultimately bears the information acquisition costs: When contemplating whether to invite one more buyer to participate, it is the seller who needs to weigh the cost of having an additional buyer acquire information against the social value created by having an additional bidder compete. Our focus, on the other hand, is on auction environments where these features are not available to the seller.<sup>2</sup> This implies that the buyers (instead of the seller) must weigh the cost of acquiring information against the expected surplus from entering the auction. As we show, the number of buyers who enter—and, therefore, social welfare and seller revenue—now depends entirely on which of several possible equilibria the buyers coordinate on.

Fishman (1988), Bulow and Klemperer (2009), and Roberts and Sweeting (2012) compare the outcomes of a sequential bidding and entry procedure to that of a simultaneous procedure. If all bids must be submitted simultaneously, entry decisions cannot be conditioned on prices. With sequential bidding, on the other hand, entry depends on the current price, and stops once the price becomes high enough that further entry is not profitable on expectation. The latter effect is present in our model as well. However, we compare different equilibria of the *same* sequential auction format, instead of comparing equilibria across different formats.

*Bidder collusion.* Starting with seminal work by Graham and Marshall (1987), Mailath and Zemsky (1991), and McAfee and McMillan (1992), the literature has examined the formation and effects of bidding rings. Bidding rings are explicit agreements to limit competition in an auction, in order to reduce the price for which the object is sold. The object is then reallocated among the ring members in an internal auction. Aoyagi (2003), Athey, Bagwell, and Sanchirico (2004), and Skrzypacz and Hopenhayn (2004) examine tacit bidder collusion in repeated auctions, where pre-play

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<sup>2</sup>We discuss the effects of a universal (i.e., period-independent) reserve price briefly in Section 8.



communication and side payments can be replaced by an appropriately chosen intertemporal payoff allocation. Garrat, Tröger, and Zheng (2009) examine tacit collusion in a single English auction without pre-play communication but with a resale stage. Finally, Campbell (1998), Peters and Severinov (2006), Dequiedt (2007), Pavlov (2008), and Che and Kim (2009) examine collusion and optimal auction design when bidders can coordinate their participation decisions.

Our delayed revelation equilibria resemble, in certain aspects, tacitly collusive behavior: Early entrants coordinate their bidding strategies in order to deter participation by later potential entrants; this limits competition and reduces the expected price received by the seller. Unlike in existing models, however, reduced competition is achieved by preventing participation by potential bidders who do *not* benefit from the scheme. Furthermore, our equilibria do not rely on either resale or repeated interaction to divide surpluses. Instead, the “colluding” bidders use the final period of the original auction to allocate the object among themselves.<sup>3</sup>

*Bidding behavior in online auctions.* Our paper is also related to the literature on online auctions, surveyed in detail in Bajari and Hortaçsu (2004), Ockenfels, Reiley, and Sadrieh (2006), Hasker and Sickles (2010), and Levin (2011). In these auctions, three types of bidding behavior have been empirically documented: *Early bidding*, *incremental bidding*, and *late bidding*. Early bidding occurs when bids are placed shortly after opening of an auction, while late bidding occurs when bids are placed in the final seconds. Incremental bidding occurs when a bidder places multiple bids over the course of the auction, with most of these bids resulting in only small price increases.<sup>4</sup>

To explain these bidding patterns, Ockenfels and Roth (2006) show that the presence of naive players, who submit low early bids which they increase when outbid, can induce rational participants to adopt a strategy of late bidding as a best response. Hossain (2008) constructs a model in which some bidders are not completely aware of their private valuations and adopt the strategy of “learning by bidding.” In this setting, bidders who know their valuations will bid late to prevent learning by unaware bidders. Ely and Hossain (2009) examine competing second-price auctions with sequential bidder arrival. In their model, sophisticated bidders can use an incremental bidding approach to

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<sup>3</sup>In particular, the winner still pays the second highest bid among those submitted by bidders who have entered, which, in equilibrium, will coincide with the second-highest valuation among the participating bidders.

<sup>4</sup>Roth and Ockenfels (2002) report that a bidder submits two bids on average in online auctions, and around 20% of bids are received at the end of the auction time. Bajari and Hortaçsu (2003) found that 32% of bids were received during the final 3% of time in online auctions. Shah, Joshi, Sureka, and Wurman (2003) show that early, late, and incremental bidding made up 28%, 38%, and 34% of bids on eBay auctions, respectively. Similarly, Bapna, Goes, and Gupta (2003) report that 23% of bidders place early bids, 40% submit late bids, and 37% bid incrementally, in a sample of online auctions.

discover the auction with the lowest high bid. It is then optimal for bidders who arrive early to bid early, in order to deter participation by later bidders. Ambrus, Burns, and Ishii (2013) consider a common value auction with random bidder arrival. In their model, equilibria exist in which bidders submit incremental bids in order to conceal their private information about the object’s value. Bose and Daripa (2014) show that late bidding is optimal in an eBay-like auction if buyers’ valuations are correlated and sellers cannot be prevented from submitting secret shill bids.

Our model provides an alternative explanation for these phenomena. It involves fully strategic play in a single auction with independent private values. Under these assumptions, early and incremental bidding by some buyers deters entry by potential competitors, while late bidding allocates the object efficiently among these buyers.

### 3 Sequential Second-Price Auctions with Costly Entry

A single indivisible object is sold to  $T > 2$  risk neutral potential bidders. All potential bidders are ex ante symmetric. Bidder  $t \in \{1, \dots, T\}$  has private value  $v_t$  for the object. All  $v_t$  are independent draws from a common cumulative distribution  $F$  over support  $[0, \bar{v}]$  with density  $f$ . Initially, a bidder does not know his own private value, but knows only the distribution  $F$ .

#### 3.1 Auction format

The auction format is a sequential second-price auction, or English auction, that is open over  $T$  periods. The auction price at the end of period  $t$  is denoted by  $p^t \geq 0$ ; the final ending price is  $p^T$ . The initial price at the beginning of the auction is  $p^0 = 0$ .

The bidders arrive at the auction in sequence, with bidder  $t$  arriving in period  $t$ . Upon arrival, the bidder observes the sequence of past prices  $p^0, \dots, p^{t-1}$  and then decides whether to enter the auction. We denote these entry decisions by  $e_t \in \{0, 1\}$ , where  $e_t = 1$  means “entry” and  $e_t = 0$  means “no entry” by bidder  $t$  in period  $t$ . If  $t$  enters, he pays an entry cost  $c > 0$ , learns his private value  $v_t$ , and is then free to bid in any period  $s \geq t$ .<sup>5</sup> If a bidder does not enter, he leaves the auction

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<sup>5</sup>The entry cost  $c$  has several possible interpretations. It could be the mental cost of introspection to determine one’s willingness to pay for an item. Alternatively,  $c$  may represent the opportunity cost of the time and effort a potential bidder must spend reading and processing the item description on an auction platform, in order to determine his willingness to pay. The assumption of valuations are independent and private then implies that the result of a bidder’s introspection or research effort is idiosyncratic.

and does not return.<sup>6</sup> At the onset of the auction, the pool of participating bidders is  $B^0 = \emptyset$ . After potential entry in period  $t \geq 1$ , the pool of participating bidders becomes  $B^t = \{1 \leq s \leq t : e_s = 1\}$ , so that  $B^0 \subseteq B^1 \subseteq \dots \subseteq B^T$ .

In each period  $s \in \{1, \dots, T\}$ , after potential entry, there is one round of simultaneous bidding during which all bidders in  $B^s$  submit bids simultaneously. We denote a bid submitted by bidder  $t$  in period  $s$  by  $b_t^s \in [0, \infty)$ .<sup>7</sup> For  $t \notin B^s$ , we automatically set  $b_t^s = 0$ . For  $t \in B^s$ , we require that  $b_t^s \geq b_t^{s-1}$ . That is, bidders cannot withdraw bids that they previously submitted, or revise previous bids downward. ( $b_t^s = b_t^{s-1}$  can be interpreted as  $t$  being “inactive” in period  $s$ .) We further require that  $b_t^s > p^{s-1}$  if  $b_t^s > b_t^{s-1}$ . That is, if a bidder revises his bid upward, he must bid more than the previous period’s price. Following submission of period- $s$  bids, the current high-bidder is determined and the auction price  $p^s$  is set to the second-highest bid among  $b_1^s, \dots, b_s^s$ . (If there is more than one highest bid, the second-highest bid is equal to the highest bid and the high-bidder is selected randomly among those players who submitted these bids.) Since  $b_t^s \geq b_t^{s-1} \forall t, s$ , we have  $p^0 \leq p^1 \leq \dots \leq p^T$ . Figure 1 depicts the timing of events.

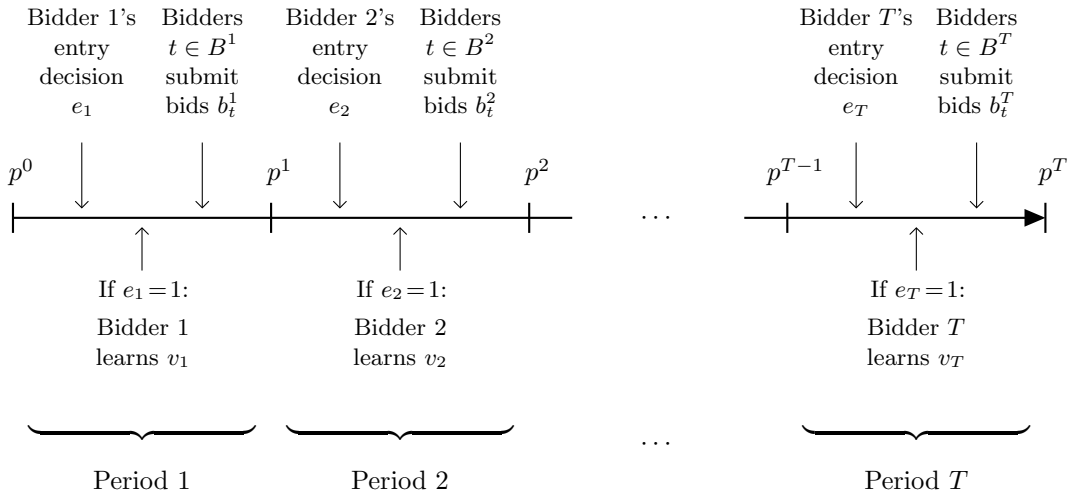


Figure 1: Timing

<sup>6</sup>This assumption is not crucial; even if a bidder who does not enter in the period he arrives were to remain in the pool of potential bidders, he would not enter in any subsequent period in any of our equilibria.

<sup>7</sup>Whenever there is a need to distinguish bidders and periods, we shall use  $t$  to denote the bidder who entered in period  $t$ , and  $s$  to denote the period under consideration. Otherwise, we use  $t$  to indicate both the bidder and the period.

At the end of the final period  $T$ , if  $B^T = \emptyset$  (i.e., if no bidders entered during the auction) the seller retains the object. If  $B^T \neq \emptyset$ , the period- $T$  high-bidder wins the object and pays  $p^T$ . Any bidder who does not win pays zero. We assume that the auction rules, the entry cost  $c$ , the distribution  $F$  of values, and the arrival sequence of bidders are common knowledge. We also assume that  $c < \int_0^{\bar{v}} (1 - F(v))F(v)dv$ . (This assumption ensures that at least two bidders can enter and obtain a positive expected surplus by bidding their valuations.)

### 3.2 Strategies and beliefs

A bidder must make two kinds of decisions: First, he must decide whether to enter the auction or not in the period he arrives; second, conditional on having entered, he must decide how much to bid in this period as well as in each subsequent period.

An entry strategy for bidder  $t$  is, in general, a mapping from the sequence of past auction prices  $p^0, p^1, \dots, p^{t-1}$  to entry decisions (either 0 or 1):

$$e_t : [0, \infty)^t \rightarrow \{0, 1\}.$$

(Since  $p^0 = p^1 = 0$ , inclusion of these variables in the entry strategy is not strictly necessary.) Note that entry never depends on the bidder's valuation, as the bidder learns his valuation only *after* having entered the auction.

A bidding strategy for bidder  $t$  prescribes, for each period  $s \in \{t, \dots, T\}$ , a bid  $b_t^s$  as a function of  $t$ 's information in period  $s$ . This information set includes  $t$ 's valuation  $v_t$ , the sequence of prices  $p^0, \dots, p^{s-1}$ , and  $t$ 's previous bids  $b_t^t, \dots, b_t^{s-1}$  (if  $s > t$ ). Thus, a bidding strategy is a mapping

$$b_t^s : [0, \bar{v}] \times [0, \infty)^s \times [0, \infty)^{s-t} \rightarrow [0, \infty)$$

that complies with the restrictions on bids imposed in Section 3.1 (that is,  $b_t^s \geq b_t^{s-1} \forall t, s$  and  $b_t^s > b_t^{s-1} \Rightarrow b_t^s > p^{s-1}$ ).

Finally, bidder  $t$  will also entertain beliefs about the distribution of opponents' valuations, conditional on observed information. The belief that will be relevant in our equilibria concerns the highest valuation among the bidders in  $B^t$ , denoted  $w^t \equiv \max\{v_j : j \in B^t\}$ . When bidder  $t > 1$  enters in period  $t$ , his belief about  $w^{t-1}$  is a conditional distribution

$$G^t(w^{t-1}|p^0, \dots, p^{t-1}) : [0, \bar{v}] \rightarrow [0, 1].$$

Note that in period 1 such a belief is not well-defined, as  $B^0 = \emptyset$ .

Our solution concept is based on the notion of sequential equilibrium (Kreps and Wilson 1982). We say that a profile of entry strategies  $(e_t)_{t=1, \dots, T}$ , bidding strategies  $(b_t)_{t=1, \dots, T}$ , and beliefs  $G^t(\cdot)$ , constitutes an *equilibrium* of the auction game if the following conditions hold for all  $t = 1, \dots, T$ :

- (i) Bidder  $t$ 's bidding strategy  $b_t$  is optimal given  $(b_j)_{j \neq t}$  and  $(e_j)_{j \neq t}$ ;
- (ii) bidder  $t$ 's entry strategy  $e_t$  is sequentially rational given bidding strategies  $(b_j)_{j \neq t}$  and, if  $t > 1$ , given beliefs  $G^t(w^{t-1}|p^0, \dots, p^{t-1})$  for all  $p^0, \dots, p^{t-1}$ ;
- (iii) there exists a sequence of perturbed strategy profiles  $(\tilde{b}_t, \tilde{e}_t) \rightarrow (b_t, e_t)$  such that any weakly increasing price sequence is possible under  $(\tilde{b}_t, \tilde{e}_t)$ , and for every  $p^0, \dots, p^{t-1} \in [0, \bar{v}]$  the belief  $G^t(\cdot|p^0, \dots, p^{t-1})$  is the limit of conditional distributions derived from Bayes' Rule under the perturbed strategies.

We further restrict our attention to equilibria that are in (weakly) undominated strategies.<sup>8</sup> Whenever a strategy or belief depends on fewer variables than the ones included above, any unnecessary arguments will be dropped.

## 4 A Look at Bidders' Entry Decisions

In this section we establish a result (Lemma 2) that relates a bidder's optimal entry decision to bidding strategies and beliefs. This result will be used to construct all equilibria throughout the paper.

For  $p \in [0, \bar{v}]$ , denote by  $F(v|p)$  the cumulative density of  $v$  conditional on  $v \geq p$ :

$$F(v|p) \equiv \frac{F(v) - F(p)}{1 - F(p)}.$$

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<sup>8</sup>This rules out many uninteresting cases that generally arise in second-price auctions. For example, a trivial equilibrium involves bidder 1 entering and bidding  $\bar{v}$ , and all subsequent bidders not entering. However, such an equilibrium is not admissible as these strategies are weakly dominated. See Blume and Heidhues (2004) for a characterization of all Nash equilibria of (static) second-price auction.

Consider the final period of the auction,  $t = T$ . Fix some  $k$  and assume that bidder  $T$  believes that exactly  $k$  currently participating bidders have valuations above the current price  $p^{T-1}$ :

$$G^T(w^{T-1}|p^0, \dots, p^{T-1}) = F(w^{T-1}|p^{T-1})^k.$$

Suppose further that each of these  $k$  bidders submits a bid equal to his valuation in period  $T$ . If bidder  $T$  enters and bids his own valuation, his expected continuation payoff is

$$U_T(p^{T-1}) = \int_{p^{T-1}}^{\bar{v}} \int_{p^{T-1}}^{v_t} (v_T - w^{T-1}) dF(w^{T-1}|p^{T-1})^k dF(v_T) = \int_{p^{T-1}}^{\bar{v}} F(v|p^{T-1})^k (1 - F(v)) dv. \quad (1)$$

Note that  $U_T(p^{T-1})$  is strictly decreasing in  $p^{T-1}$ , and  $U_T(\bar{v}) = 0$ . Thus, if  $U_T(0) > c$  a unique threshold price  $p_k^* \in (0, \bar{v})$  exists such that

$$\int_{p_k^*}^{\bar{v}} F(v|p_k^*)^k (1 - F(v)) dv = c. \quad (2)$$

If  $p^{T-1} < p_k^*$  bidder  $T$  enters, and if  $p^{T-1} \geq p_k^*$  he does not enter. The following result characterizes the threshold  $p_k^*$ .

**Lemma 1.** *There exists  $K \in \mathbb{N}$  such that a solution  $p_k^*$  to (2) exists if and only if  $k \in \{1, \dots, K\}$ . Furthermore,  $p_k^* > p_{k+1}^*$  for all  $k = 1, \dots, K - 1$ , and  $K \rightarrow \infty$  as  $c \rightarrow 0$ .*

Now go back one period and consider the entry decision of bidder  $T - 1$  with the same belief. Suppose that  $p^{T-2} \geq p_k^*$ . Since  $p^{T-1} \geq p^{T-2}$ , there will be no entry in period  $T$ , as shown above. But this means that bidder  $T - 1$  is the new “final bidder” and should not enter, for the same reason that prevents bidder  $T$  from entering when  $p^{T-1} \geq p_k^*$ . In fact, in the Appendix we show that  $p_k^*$  is the entry threshold used by all arriving buyers who believe that  $k$  currently participating bidders have valuations above the current price, regardless of their position in the arrival sequence.<sup>9</sup> Formally, the result is the following:

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<sup>9</sup>The formal proof is less straightforward than the intuitive argument given here: One also needs to show that, for all  $t$ ,  $p^{t-1} < p_k^*$  implies that bidder  $t$  enters.

**Lemma 2.** Fix  $k \in \{1, \dots, K\}$  and suppose bidding strategies have the following properties: A bidder never bids above his valuation, bids at least  $p_k^*$  upon entry if his value is  $p_k^*$  or higher, and bids his valuation in the last period (if it exceeds the current price). That is,

$$b_t^s \leq v_t \quad \forall s \geq t, \quad v_t \geq p_k^* > p^{t-1} \Rightarrow b_t^t \geq p_k^*, \quad v_t > p^{T-1} \Rightarrow b_t^T = v_t \quad (3)$$

for all  $t$ . Then the following is true: If in period  $t$  bidder  $t$  believes that the highest valuation among bidders in  $B^{t-1}$  has distribution  $F(w^{t-1}|p^{t-1})^k$ , an optimal entry strategy for  $t$  is to enter the auction in period  $t$  if and only if  $p^{t-1} < p_k^*$ .

Lemma 2 describes a best response entry strategy for a particular set of beliefs and bidding strategies, and is at the core of all results to follow. Loosely speaking, we will show that for each  $k \geq 2$  an equilibrium exists in which  $k$  bidders with valuations above  $p_k^*$  submit coordinated early bids equal to, or close to,  $p_k^*$ , in a way that will be made precise in the following sections. The goal of this coordination is to manipulate the beliefs of potential entrants about the distribution of the highest valuation among currently participating bidders, in order to deter these potential competitors from entering. For  $k = 1$ , on the other hand, the equilibrium boils down to that of a static second-price auction, with each participating buyer simply bidding his valuation upon entry.

## 5 Equilibrium: Two Simple Cases

In this section, we will describe two equilibria of the model—a benchmark equilibrium without bid coordination (i.e.,  $k = 1$ ), and an equilibrium in which exactly two bidders coordinate their bids (i.e.,  $k = 2$ ). We will assume that  $c$  is small enough for both equilibria to exist,

### 5.1 Immediate revelation equilibrium

We call a bidding strategy an *immediate revelation strategy* if, after entry, each bidder immediately submits a bid equal to his private valuation (if it exceeds the current price) and never revises his bid thereafter: For bidder  $t \geq 1$  and period  $s \in \{t, \dots, T\}$ ,

$$b_t^s(v_t, p^{s-1}, b_t^{s-1}) = \begin{cases} 0 & \text{if } [s = t \text{ and } v_t \leq p^{s-1}], \\ v_t & \text{if } [s = t \text{ and } v_t > p^{s-1}], \\ b_t^{s-1} & \text{otherwise.} \end{cases} \quad (4)$$

This strategy closely mirrors equilibrium bidding behavior in a static second-price auction, and we will show that it is also part of an equilibrium in our dynamic game.

Conditional on all other bidders following strategy (4), and conditional on a fixed set of participants, a single bidder who enters the auction clearly cannot do better than bid his true valuation at some point before the end of the auction. But since potential bidders enter if and only if the observed price is sufficiently low (as will be shown below), it is optimal for every bidder who has already entered the auction to bid his valuation immediately after entry—doing so results in a weakly higher price path than delaying a truthful bid, and thus reduces the likelihood of entry by competitors.

Let us now turn to the buyers' participation decisions. Given bidding strategy (4) and observed price  $p^{t-1}$ , there is (a.s.) exactly one bidder in  $B^{t-1}$  whose valuation is above  $p^{t-1}$ . Thus, the conditional distribution of  $w^{t-1}$  in period  $t$  is

$$G^t(w^{t-1}|p^{t-1}) = F(w^{t-1}|p^{t-1}). \quad (5)$$

(Note that under the presumed bidding strategy, all weakly increasing price paths such that  $p^0 = p^1 = 0$  are possible. Hence, all beliefs in this equilibrium will be Bayesian.) By Lemma 2 bidder  $t$ 's entry strategy is

$$e_t(p^{t-1}) = \begin{cases} 1 & \text{if } p^{t-1} < p_1^*, \\ 0 & \text{if } p^{t-1} \geq p_1^*. \end{cases} \quad (6)$$

This establishes the following result:

**Proposition 3. (*Immediate Revelation Equilibrium*)** *There exists an equilibrium of the auction game in which*

- (i) *Each bidder uses the immediate revelation bidding strategy (4);*
- (ii) *bidder  $t$  enters in period  $t$  if and only if  $p^{t-1} < p_1^*$ ;*
- (iii) *in every period  $t > 1$ , the arriving bidder's belief about the distribution of the highest value among bidders  $1, \dots, t-1$  is given by (5).*

The outcome resulting from the equilibrium characterized in Proposition 3 is interim efficient, in that the participating bidder with the highest valuation wins. However, the outcome is not necessarily ex-post efficient. Once the auction price reaches  $p_1^*$ , entry ceases, and since only participating



bidders learn their valuations it is possible that some non-participating bidder would have had higher valuation than the winning bidder.

## 5.2 Delayed revelation and entry deterrence

We now demonstrate that early entrants can deter entry by later bidders via what we call a *delayed revelation strategy*: After entry, a bidder submits a bid below his valuation but later revises this bid to reflect his true valuation.

We will focus on the following bidding strategy: For bidder  $t \geq 1$  and period  $s \in \{t, \dots, T\}$ ,

$$b_t^s(v_t, p^{s-1}, b_t^{s-1}) = \begin{cases} 0 & \text{if } [s = t \text{ and } v_t \leq p^{s-1}] \\ & \text{or } [s = t < T \text{ and } p^{s-1} \geq p_2^*], \\ v_t & \text{if } [s = t < T \text{ and } p_2^* > v_t > p^{s-1}] \\ & \text{or } [s = T \text{ and } v_t > p^{T-1}], \\ p_2^* & \text{if } [s = t < T \text{ and } v_t \geq p_2^* > p^{s-1}], \\ b_t^{s-1} & \text{otherwise.} \end{cases} \quad (7)$$

where  $p_2^*$  is defined via (2). Delayed revelation strategy (7) is identical to the immediate revelation strategy (4), with two exceptions: First, a bidder whose valuation is above the threshold  $p_2^*$  does not bid his valuation upon entry if the current price is below  $p_2^*$ . Instead, this bidder submits  $p_2^*$  after entry, but revises his bid to reflect his true valuation in the final period. Second, a bidder who enters at price  $p_2^*$  or above (an out-of-equilibrium event) does not bid until the final period, at which time he bids his valuation. Note that strategy (7) satisfies condition (3) in Lemma 2.

We will show that an equilibrium exists in which all bidders follow the delayed revelation strategy. If all participating bidders adopt this strategy, we have  $p^t \leq p_2^*$  for all  $t < T$ . Furthermore, in period  $t$  the entering bidder's belief about  $w^{t-1}$  is

$$G^t(w^{t-1}|p^{t-1}) = \begin{cases} F(w^{t-1}|p^{t-1}) & \text{if } p^{t-1} \neq p_2^*, \\ F(w^{t-1}|p^{t-1})^2 & \text{if } p^{t-1} = p_2^*. \end{cases} \quad (8)$$

This is so because the only possibility that price  $p_2^*$  is observed—under the presumed bidding strategy—is for exactly two bidders to have submitted a bid of  $p_2^*$ . In this case, there must be exactly two bidders with valuations above  $p_2^*$  in  $B^{t-1}$ .<sup>10</sup>

By Lemma 2, bidder  $t$  does not enter if  $p^{t-1} = p_2^*$ . If all participating bidders continue to follow strategy (7), the auction price will remain at  $p_2^*$  until the final round of bidding, so that entry is deterred in all subsequent periods as well. On the other hand, if  $p^{t-1} < p_2^*$ , then—given the presumed bidding strategy—exactly one bidder’s valuation exceeds  $p^{t-1}$ . Because  $p_1^* > p_2^* > p^{t-1}$ , Lemma 2 implies that bidder  $t$  should enter in period  $t$ . Our equilibrium entry strategy is hence the same as (6), except when the price equals  $p_2^*$ :

$$e_t(p^{t-1}) = \begin{cases} 1 & \text{if } p^{t-1} < p_2^* \text{ or } p_2^* < p^{t-1} < p_1^*, \\ 0 & \text{if } p^{t-1} = p_2^* \text{ or } p^{t-1} \geq p_1^*. \end{cases} \quad (9)$$

We now have the following result (fully proven in the Appendix):

**Proposition 4. (*Simple Delayed Revelation Equilibrium*)** *There exists an equilibrium of the auction game in which*

- (i) *Each bidder uses the delayed revelation strategy (7);*
- (ii) *bidder  $t$  enters in period  $t$  if and only if  $p^{t-1} < p_2^*$  or  $p_2^* < p^{t-1} < p_1^*$ ;*
- (iii) *in every period  $t > 1$ , the arriving bidder’s belief along the equilibrium price path about the distribution of the highest value among bidders  $1, \dots, t-1$  is given by (8).*

By delaying the revelation of their true valuations until the final period, the first two bidders with valuations above  $p_2^*$  deter entry by potential rival bidders. These two bidders then compete against one another in a single Vickrey auction in the final period.

As was the case in the immediate revelation equilibrium, the object is awarded to the participating bidder with the highest valuation, and this bidder pays the second-highest valuation among the participants. However, the set of participants is different across the two equilibria: In the simple delayed revelation equilibrium entry ceases once the auction price reaches  $p_2^*$ , and because  $p_2^* < p_1^*$  there is a positive probability that the participants with the highest and second-highest valuations in

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<sup>10</sup>Note that prices  $p^{t-1} > p_2^*$  cannot be observed under the prescribed bidding strategy; however, the equilibrium requires beliefs at such information sets as well. The perturbations in entry and bidding strategies that generate these out-of-equilibrium beliefs are specified in the proof of Proposition 4 in the Appendix.

the immediate revelation equilibrium do not participate in the simple delayed revelation equilibrium. In this case, the final allocation and price are different across the two equilibria. More precisely, the bidding and entry strategies characterized above imply that every bidder who enters in the delayed revelation equilibrium also enters in the immediate revelation equilibrium, but not necessarily the other way around. It follows that the seller’s revenue in the immediate revelation equilibrium first-order stochastically dominates that in the delayed revelation equilibrium.<sup>11</sup>

### 5.3 An example

To illustrate the different outcomes across the equilibria characterized in Proposition 3 and Proposition 4, we turn to an example with twenty periods/bidders, shown in Figure 2. The top panel contains the buyers’ valuations depicted as shaded vertical bars, along with dots representing their bids. Bids in the immediate revelation (IR) equilibrium are blue, and bids in the simple delayed revelation (DR) equilibrium are red. Bids submitted by the entering bidders are shown as dots with a solid center; dots with a white center represent revisions of earlier bids. The middle and bottom panel display the price paths and entry decisions across equilibria, using the same color scheme. (The yellow dots and lines are bids and prices in an incremental bidding (IB) equilibrium, which will be introduced and discussed in the next section.)

The IR equilibrium is very simple: Entering participants bid their valuations if they exceed the current price, and maintain these bids to the end. The price under this bidding strategy (the blue line in the middle panel) first exceeds the entry threshold  $p_1^*$  at the end of period 16, and no further bidders enter from period 17 onward. Bidder 10 eventually wins and pays  $v_{16}$ .

Things are quite different in the DR equilibrium: Bidders 7 and 9 are the first two bidders with valuations above  $p_2^*$ . These bidders submit bids equal to  $p_2^*$  after entering in periods 7 and 9, respectively. Thus, the price equals  $p_2^*$  at the end of period 9 and entry is deterred starting in period 10. Nothing changes from this moment on, until period 20 is reached. In the final round, the two bidders reveal their valuations; bidder 9 wins and pays  $v_7$ .

Figure 2 also illustrates the role of Lemma 2 for the DR equilibrium: If exactly two bidders have valuations above the current price, and that price is at least  $p_2^*$ , entry is not profitable on expectation. But since valuations and bids are private, potential entrants can only know that this is

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<sup>11</sup>A more detailed comparison of surpluses across equilibria is in Section 7.

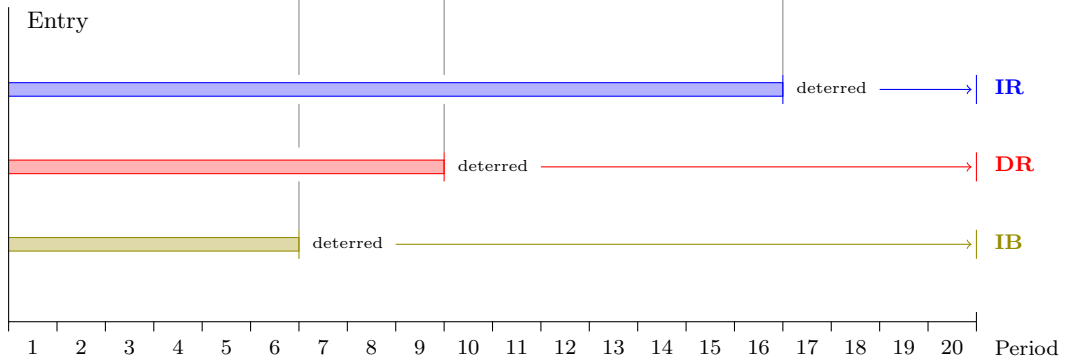
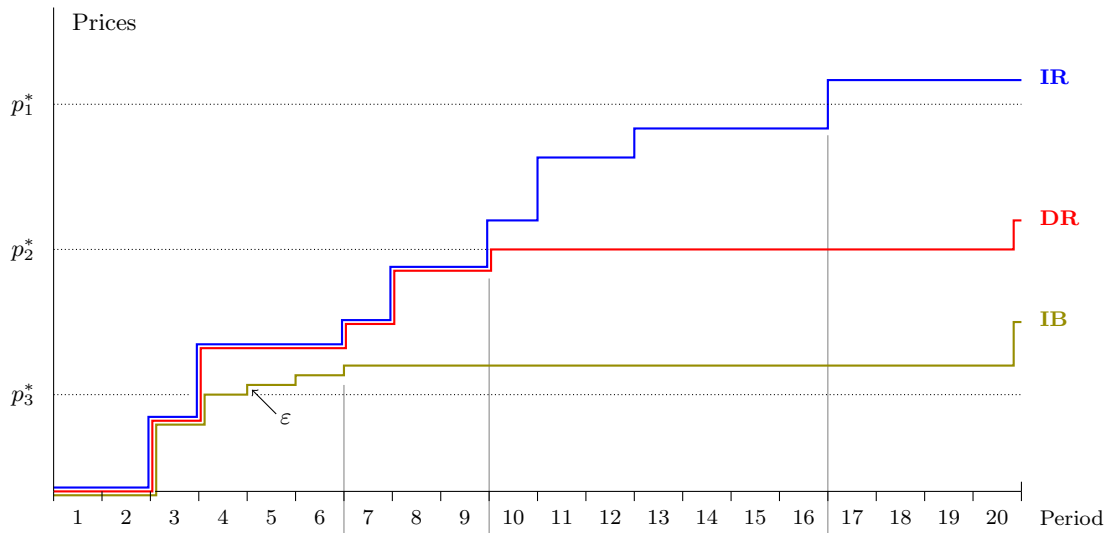
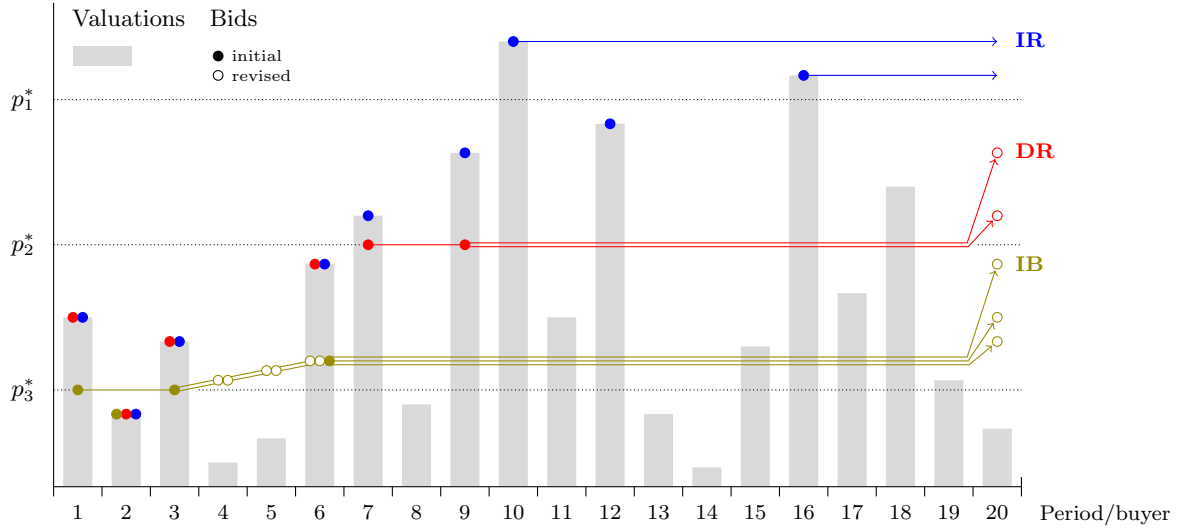


Figure 2: Bids, entry decisions, and prices in three equilibria

the case by observing patterns in the price path that reveal, in effect, the same information. In the DR equilibrium, a price equal to  $p_2^*$  is thus a signal that it is now better to “stay out.”

## 6 Entry Deterrence By More Than Two Bidders

The logic behind the simple delayed revelation equilibrium suggests that similar equilibria exist that are based on entry deterring prices  $p_k^*$  for  $k > 2$ . For example, by Lemma 2, a potential entrant who believed that three bidders’ valuations are above the current price would stay out of the auction at price  $p_3^*$ . Thus, three bidders with sufficiently high valuations could submit post-entry bids equal to  $p_3^*$  and thereby deter further entry. The problem with this argument is that the auction still follows a second-price format: Once the first two bidders in the hypothesized scheme submit bids equal to  $p_3^*$ , the auction price jumps to  $p_3^*$ . Therefore, price  $p_3^*$  does *not* signal that three bidders with valuations above  $p_3^*$  have entered.

It is nonetheless possible for  $k > 2$  bidders to deter entry at a price close to  $p_k^*$ . However, this requires a more complex bidding strategy, along with the assumption that the history of past bids is available to all players. More precisely, we assume that, at any moment, each bidder observes not only the sequence of past auction prices, but the value of all previously submitted bids except for the bid submitted by the current high-bidder. This is the information available to buyers on most eBay auctions, for example.

### 6.1 Incremental bidding

Our construction works through *incremental bidding*. The first two bidders with valuations above  $p_k^*$  bid exactly  $p_k^*$ . Once the second such bidder participates, the price equals  $p_k^*$ . These two bidders then begin to simultaneously raise their bids, period-by-period, from  $p_k^*$ , to  $p_k^* + \varepsilon$ , to  $p_k^* + 2\varepsilon$ , and so on. Every new buyer who enters during this process, and draws a valuation larger than the next  $\varepsilon$ -increment, bids in the same fashion. (If a bidder’s valuation is reached during the process, he bids his valuation and does nothing thereafter.) As long as the bid history is observable, all players can deduce the number of participating bidders whose valuation is above the current price, and will hence know when  $k$  such bidders are present. The valuations of these bidders must be above the original deterrent price  $p_k^*$ , so that entry is deterred from this period onward. As before, the bidders now wait until the final period to reveal their valuations. The role of incremental bidding, therefore, is to give each new entrant an opportunity to join the group of “colluding” bidders while

at the same time giving each existing bidder an opportunity to signal that they are still in this group themselves, until the group reaches the required size  $k$ .

Formalizing this idea requires some additional notation, however. Fix an integer  $k$  such that  $p_k^* > 0$ , and a bidding increment  $\varepsilon > 0$ . Both will be equilibrium objects.  $\varepsilon$  can be chosen arbitrarily small, and to simplify the analysis we assume that  $0 < \varepsilon < [p_{k-1}^* - p_k^*]/T$ .<sup>12</sup> This assumption ensures that the incremental bidding process that starts at  $p_k^*$  does not surpass the next largest threshold  $p_{k-1}^*$ . Now let

$$k^t \equiv |\{j \in B^t : b_j^t \geq p^t\}| \geq 1$$

denote the number of bidders who submitted period- $t$  bids that are equal to, or greater than, the period- $t$  price. The variable  $k^t$  can be computed from the observable bid histories and thus becomes part of the information set of all bidders in period  $t + 1$ . Next, for  $p \geq p_k^*$ , let

$$\iota(p) \equiv \min_{m=1,2,3,\dots} \{p_k^* + m\varepsilon : p_k^* + m\varepsilon > p\}$$

be the smallest  $\varepsilon$ -increment over  $p_k^*$  that strictly exceeds price  $p$ .

The incremental bidding strategy can now be formally stated as follows: For each  $t$  and  $s \geq t$ , bidder  $t$  in period  $s$  submits bid

$$b_t^s(v_t, p^{s-1}, b_t^{s-1}, k^{s-1}) = \begin{cases} 0 & \text{if } [s = t \text{ and } v_t \leq p^{s-1}] \\ & \text{or } [s = t < T \text{ and } k^{s-1} \geq k \text{ and } p^{s-1} \geq p_k^*], \\ v_t & \text{if } [s = t < T \text{ and } p^{s-1} < v_t < p_k^*] \\ & \text{or } [s < T \text{ and } p^{s-1} \geq p_k^* \text{ and } k^{s-1} < k \text{ and } v_t < \iota(p^{s-1})] \\ & \text{or } [s = T \text{ and } v_t > p^{T-1}], \\ p_k^* & \text{if } [s = t < T \text{ and } v_t \geq p_k^* > p^{s-1}], \\ \iota(p^{s-1}) & \text{if } [s < T \text{ and } p^{s-1} \geq p_k^* \text{ and } k^{s-1} < k \text{ and } v_t \geq \iota(p^{s-1})], \\ b_t^{s-1} & \text{otherwise.} \end{cases} \quad (10)$$

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<sup>12</sup>In real life, the minimum currency unit always provides a lower bound for the increment and thus serves as a focal point for the value of  $\varepsilon$ . Moreover, if the auction rules feature a mandatory increment, then this becomes a lower bound for  $\varepsilon$ .

Strategy (10) looks unwieldy, but it is in fact the strategy verbally described earlier.<sup>13</sup> Note also that the delayed revelation strategy (7) is a special case of (10), with  $k = 2$ .

If all participating bidders adopt this strategy, the distribution of  $w^{t-1}$  conditional on  $p^{t-1}$  is

$$G^t(w^{t-1}|p^{t-1}, k^{t-1}) = F(w^{t-1}|p^{t-1})^{k^{t-1}}. \quad (11)$$

Thus, by Lemma 2, the equilibrium entry strategy is

$$e_t(p^{t-1}, k^{t-1}) = \begin{cases} 1 & \text{if } p^{t-1} < p_{k^{t-1}}^*, \\ 0 & \text{otherwise.} \end{cases} \quad (12)$$

This entry strategy is optimal assuming bidding proceeds as prescribed in (7). We thus have the following result (fully proven in the Appendix):

**Proposition 5. (Incremental Bidding Equilibrium)** *Fix  $k > 2$  such that  $p_k^* > 0$ , and  $\varepsilon$  such that  $0 < \varepsilon < [p_{k-1}^* - p_k^*]/T$ . Assume the number of high bidders is observable after every bidding round, together with the current price. There exists an equilibrium of the auction game in which the following holds for all  $t = 1, \dots, T$ :*

(i) Bidder  $t$  uses the incremental bidding strategy (10);

(ii) bidder  $t$ 's entry strategy is given by (12);

(iii) bidder  $t$ 's belief about the distribution of the highest value among bidders  $1, \dots, t-1$  is given by (11).

Figure 2 displays bids, prices, and entry decisions in an incremental bidding equilibrium (in yellow), for  $k = 3$ . The “colluding” bidders are buyers 1, 3, and 6. Incremental bidding commences after bidders 1 and 3 have submitted bids of  $p_3^*$ . These two bidders then increment their bids in periods 4, 5, and 6, at which time bidder 6 submits the same incremental bid  $p_3^* + 3\varepsilon$ . Entry is deterred from period 7 onward. In period 20, the three buyers bid their valuations; bidder 6 wins and pays  $v_1$ .

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<sup>13</sup>The only additional contingency that was not previously discussed concerns the bidders who enter when  $k$  or more coordinating bidders are already present (an out-of-equilibrium event). These bidders do not bid until the final period, at which time they reveal their valuations.

## 6.2 Remarks

We now discuss some features of our incremental bidding equilibria. First, while there is *always* weakly less entry in the delayed revelation equilibrium than in the immediate revelation equilibrium, the same is not true when comparing incremental bidding equilibria. In an equilibrium with a lower threshold price  $p_k^*$ , a larger group of bidders with valuations above  $p_k^*$  must be assembled. Thus, for some realizations of buyer valuations it is possible that entry is deterred later in this equilibrium, compared to one with a larger  $p_k^*$ .<sup>14</sup> We will prove, in the next section, that among all incremental bidding equilibria entry stops later *on expectation* when  $k$  is larger.

Second, the equilibrium bidding strategy (10) requires buyers to continue the incremental bidding process even when the required number of coordinating bidders,  $k$ , is no longer attainable (that is, when  $k^{t-1} < k - (T - t)$ ). This is an optimal bidding strategy, provided all bidders follow it. However, more complex strategy profiles could be constructed in which bidding moves to the next incremental bidding equilibrium, based on threshold price  $p_{k-1}^*$ , whenever the number of remaining periods in the auction becomes insufficient to assemble a group of  $k$  bidders.

Third, note that our construction relies on the assumption that new entrants have enough information to compute the number of bids that are weakly greater than the current price. In real-time auctions, no two bids can be timed to arrive at precisely the same time, so that the number of such bids cannot exceed two at any point in time. Instead, there may be several agents who submit similar bids in sequence. Within the context of our discrete-time model, these bidders can be regarded as players who make the same bid in the same period. What is required, then, for our construction to work in real-time auctions is not that the number of bidders who submitted a particular bid is observable, but that each bidder's identity is observable. This allows each a new entrant to estimate the number of unique bidders who placed incremental bids in the past, and thus the number of current participants whose valuations are above the current price.

## 7 Comparison Across Equilibria

In this section, we compare the equilibria of our model in terms of the number of participating buyers, revenue, buyer surplus, and overall welfare. We call an equilibrium a *k-equilibrium* if it

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<sup>14</sup>For example, with  $k = 3$  entry stops when the third bidder with valuation above  $p_3^*$  has entered. If this happens after the second bidder with valuation above  $p_2^*$  has entered, then there is more entry in the incremental bidding equilibrium with  $k = 3$  than in the delayed revelation equilibrium, where  $k = 2$ .



is based on threshold price  $p_k^*$ : The immediate revelation equilibrium is the 1-equilibrium, the simple delayed revelation equilibrium is the 2-equilibrium, and the incremental bidding equilibria are  $k$ -equilibria with  $k \geq 3$ . A  $k$ -equilibrium exists as long as  $p_k^*$  exists, and by Lemma 1 this will be the case for  $k = 1, \dots, K$ , for some  $K$ . We assume that  $c$  is small enough for  $K \geq 2$ . Furthermore, for  $k$ -equilibria with  $k > 2$ , we restrict attention to those whose endogenous bid increment  $\varepsilon$  is approximately zero.

## 7.1 Long auctions

Our analytical results in this section are asymptotic results for  $T \rightarrow \infty$ . That is, we characterize the limit of expected equilibrium outcomes in a sequence of auctions as the parameter  $T$  grows. This limit serves as an approximation of the expected outcome of finite- $T$  auctions that becomes arbitrarily precise as  $T$  increases, which can be interpreted either as lengthening the duration of the auction or, perhaps more realistically, as shortening the time unit in which each single potential bidder becomes aware of the auction.<sup>15</sup>

To derive our approximations, note that as  $T \rightarrow \infty$ , the probability that entry ceases before the final period converges to one in any given  $k$ -equilibrium. Since we focus on equilibria with bid increments of  $\varepsilon \approx 0$ , this happens at a price of approximately  $p_k^*$ . Thus, in the limit as  $T \rightarrow \infty$ , entry stops in 1-equilibrium once 2 bidders with valuation above  $p_1^*$  have arrived, which takes  $2/(1 - F(p_1^*))$  periods on average. In  $k$ -equilibrium for  $k \geq 2$ , entry stops once  $k$  bidders with valuations above  $p_k^*$  have entered; this takes  $k/(1 - F(p_k^*))$  periods on average (again in the limit as  $T \rightarrow \infty$ ). Therefore, the expected number of participating bidders in a large- $T$  auction is approximately

$$N_k \equiv \begin{cases} \frac{2}{1 - F(p_k^*)} & \text{if } k = 1, \\ \frac{k}{1 - F(p_k^*)} & \text{if } k > 1. \end{cases}$$

Let  $v_{r:m}^k$  denote the  $r$ th-smallest realization in a sample of  $m$  independent draws from  $F(\cdot | p_k^*)$ . The seller's revenue is the price of the item at the end of the auction. Since all bidders bid truthfully

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<sup>15</sup>We emphasize that we do *not* consider an auction with infinitely many time periods. Such an auction is ill-defined in our setup, as there would be no final period (i.e., the object would never be sold). Instead, we consider the limit of outcomes of auctions with  $T < \infty$  time periods, as  $T \rightarrow \infty$ .

in the final period, expected revenue for the seller in a large- $T$  auction is approximately

$$R_k \equiv \begin{cases} E[v_{1:2}^1] & \text{if } k = 1, \\ E[v_{k-1:k}^k] & \text{if } k > 1. \end{cases}$$

The buyers' aggregate surplus is the valuation of the winning bidder minus the price paid to the seller minus the entry costs of all participating bidders.<sup>16</sup> Again, since all bidders bid truthfully in the final period, expected surplus for the buyers in a large- $T$  auction is approximately

$$B_k = \begin{cases} E[v_{2:2}^1] - E[v_{1:2}^1] - cN_1 & \text{if } k = 1, \\ E[v_{k:k}^k] - E[v_{k-1:k}^k] - cN_k & \text{if } k > 1. \end{cases}$$

Finally, social welfare is the sum of seller revenue and buyer surplus; or, equivalently, the valuation of the winning bidder minus the aggregate entry costs:

$$W_k = R_k + B_k = \begin{cases} E[v_{2:2}^1] - cN_1 & \text{if } k = 1, \\ E[v_{k:k}^k] - cN_k & \text{if } k > 1. \end{cases}$$

The following result compares the immediate revelation equilibrium and the simple delayed revelation equilibrium ( $k = 1$  vs.  $k = 2$ ) in terms of the variables defined above.

**Proposition 6. (*IR equilibrium vs. DR equilibrium*)** Fix  $F$  and  $c$ , and assume  $k$ -equilibria exist for  $k = 1, 2$ . Consider the limit of these equilibria as  $T \rightarrow \infty$ .

- (a) The number of participating bidders in IR equilibrium first-order stochastically dominates the number of participating bidders in DR equilibrium (and, therefore,  $N_1 > N_2$ ).
- (b) The price received by the seller in IR equilibrium first-order stochastically dominates the price in DR equilibrium (and, therefore,  $R_1 > R_2$ ).
- (c) Expected buyer surplus is zero in IR equilibrium, and positive in DR equilibrium (i.e.,  $0 = B_1 < B_2$ ).
- (d) Expected social welfare is larger in IR equilibrium than in DR equilibrium (i.e.,  $W_1 > W_2$ ).

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<sup>16</sup>Since bidders differ by their location in the arrival sequence, the value  $B_k$  does not necessarily reflect an *individual* bidder's expected surplus in  $k$ -equilibrium. Suppose, however, that prior to the game Nature chooses an arrival sequence randomly and symmetrically from all possible such sequences. In this case, all buyers share an *ex ante* preference for an equilibrium with a larger  $B_k$ .

Note that in the IR equilibrium the buyers' surplus is exactly zero: Even though buyers collect an information rent from the seller (i.e.,  $v_{2:2}^1 - v_{1:2}^1 > 0$ ), they dissipate this rent on expectation through costly entry (i.e.,  $E[v_{2:2}^1] - E[v_{1:2}^1] = cN_1$ ). This observation is reminiscent of similar results in McAfee and McMillan (1987) and Levin and Smith (1994). On the other hand, in our DR equilibrium buyers retain a positive expected surplus. In fact, below we will show that  $B_k > 0$  for all  $k \geq 2$ . Because a buyer can guarantee a zero payoff by not participating in the auction, delayed revelation of buyer valuations is *necessary* if buyers are to have a strict participation incentive in dynamic second-price auctions with costly entry.

From the seller's perspective, the ranking among the two equilibria is the opposite of the buyers' ranking. For any realization of buyer valuations, the seller's revenue in IR equilibrium is at least equal to the revenue in DR equilibrium. This implies that the seller prefers the IR equilibrium over the DR equilibrium regardless of her risk preferences.

Establishing the welfare properties of  $k$ -equilibria for  $k \geq 2$  is considerably more complicated than ranking the IR and DR equilibria. The latter exercise is a relatively straightforward two-sample problem: Compare the *same* order statistic obtained from *different* underlying probability distributions. For instance, ranking the seller's revenue across the IR and DR equilibrium requires us to rank the expectations of the second-highest of two draws from  $F(\cdot | p_1^*)$  and  $F(\cdot | p_2^*)$ , respectively. Comparing the  $k$ -equilibria for  $k \geq 2$ , on the other hand, requires us to rank the expectations of *different* order statistics for *different* distributions. We are able to show the following:

**Proposition 7. (Comparisons across all equilibria)** *Fix  $F$  and  $c$ , and assume  $k$ -equilibria exist for  $k = 1, \dots, K$  (for  $k > 2$ , focus on equilibria where the bid increment  $\varepsilon$  is approximately zero). Consider the limit of these equilibria as  $T \rightarrow \infty$ .*

- (a) *The expected number of entering bidders,  $N_k$ , increases strictly in  $k$  for  $k \geq 2$  and may exceed  $N_1$ . At the same time, expected buyer surplus,  $B_k$ , is positive for all  $k \geq 2$ .*
- (b) *Expected social welfare,  $W_k$ , decreases strictly in  $k$  for  $k \geq 2$*

*Furthermore, if  $F$  is the uniform distribution on  $[0, \bar{v}]$ , then the following holds:*

- (c) *The seller's expected revenue,  $R_k$ , increases strictly in  $k$  for  $k \geq 2$  (but never reaches  $R_1$ ), and  $B_k$  decreases strictly in  $k$  for  $k \geq 2$  (but never falls to zero).*

Part (a) of Proposition 7 may seem counterintuitive: There can be more expected entry in some incremental bidding equilibria (relative to the immediate revelation equilibrium); yet buyers on

average are better off in the incremental bidding equilibria (recall that  $B_1 = 0$ ).<sup>17</sup> To explain this apparent contradiction, note that the two bidders who “effectively” compete in IR equilibrium have valuations in  $[p_1^*, \bar{v}]$ . This interval is narrower than the corresponding interval  $[p_k^*, \bar{v}]$  from which the valuations of the  $k$  bidders are drawn who “effectively” compete in  $k$ -equilibrium with large  $k$ . The higher aggregate entry cost that may be incurred in  $k$ -equilibrium is offset by the fact that the possible gap between the winning bidder’s valuation and the next highest valuation is also larger.

Part (b) of Proposition 7, together with Proposition 6, implies that the IR equilibrium has the highest social welfare among all equilibria considered. This result is independent of the distribution from which buyer valuations are drawn. Part (c) implies that the IR equilibrium may also be the most preferred equilibrium by the seller for certain value distributions, despite the fact that more bidders may enter on expectation in some incremental bidding equilibria.<sup>18</sup>

## 7.2 Short auctions

To say more when  $T$  is relatively small, we now compare allocations, revenues, and welfare in the immediate revelation equilibrium and delayed revelation equilibrium numerically. We consider the case of uniform values on  $[0, 1]$ ,  $c \in \{.005, .01, .02\}$ , and  $T \in \{10, 20, 50\}$  as well as the limit outcomes as  $T \rightarrow \infty$ . The results are in Table 1.

The probability that the choice of equilibrium changes the allocation or the price is not negligible. However, the relative difference in revenue and welfare is less pronounced than what one might expect given the probability of different allocations and prices. This is especially true for social welfare, which in DR equilibrium is less than two percent below its value in IR equilibrium, even when the DR equilibrium results in a different allocation than the IR equilibrium with probability 29%. The percentage decrease in the seller’s revenue, while modest, is about seven times the percentage decrease in welfare. This implies that the primary consequence of delayed revelation in the example is a transfer of surplus from the seller to the buyers.

The reason why the effect on welfare is low is three-fold. First, the transfer of surplus from the seller to the buyers is welfare neutral. Second, while the DR equilibrium frequently allocates the

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<sup>17</sup>For example, if values are drawn from the uniform distribution, expected participation in  $k$ -equilibrium ( $k \geq 2$ ) is  $k/\sqrt{(k+1)(k+2)c}$ , and expected participation in 1-equilibrium is  $2/\sqrt{6c}$ . The former is less than the latter if and only if  $k \leq 6$ . At the same time,  $B_k > 0$  for all  $k > 1$ , while  $B_1 = 0$ .

<sup>18</sup>The reason is that while the seller’s revenue in  $k$ -equilibrium of a long auction is the second-largest of  $k$  valuations, conditional on these valuations being larger than  $p_k^*$ , and  $p_k^*$  decreases in  $k$ . If the seller’s revenue was the unconditional expectation of the second-highest valuation of all participating bidders, a larger number of participants would always be preferred.

$T$	$c$	$p_2^*$	$p_1^*$	Probability of		Relative change in	
				Different price	Different winner	Expected price	Expected welfare
10	.005	.7551	.8268	.2106	.1120	-2.07%	-0.31%
20				.4372	.2470	-4.46%	-0.67%
50				.4998	.2927	-5.40%	-0.79%
$\infty$				.5000	.2929	-5.41%	-0.79%
10	.01	.6536	.7551	.3442	.1882	-5.14%	-0.77%
20				.4895	.2835	-7.76%	-1.15%
50				.5000	.2929	-8.08%	-1.18%
$\infty$				.5000	.2929	-8.08%	-1.18%
10	.02	.5101	.6536	.4565	.2584	-10.83%	-1.61%
20				.4996	.2923	-12.40%	-1.82%
50				.5000	.2929	-12.44%	-1.82%
$\infty$				.5000	.2929	-12.44%	-1.82%

Table 1: Outcomes in DR vs. IR equilibrium, for  $v_t \sim U[0, 1]$

object to the “wrong” buyer, the expected difference between the respective winners’ valuations is small if both  $p_1^*$  and  $p_2^*$  are relatively close to the upper bound of bidder valuations, which is the case for low entry costs. Third, the bidders who do not enter in DR equilibrium but enter in IR equilibrium save their entry costs. If these cost savings were ignored, “welfare” in the DR equilibrium would be up to 6% lower in DR equilibrium in the cases examined in the Table 1.

## 8 Conclusion and Implications

We examined a dynamic second-price auction with costly entry and sequential and deterministic bidder arrival. We identified a novel way for some bidders to coordinate their bids in order to limit entry by bidders who arrive at the auction later. The equilibria in which this happens exhibit several bidding patterns observed in online auctions—low early bidding, late “sniping” bidding, and incremental bidding. Our results would remain intact if bidder arrival was stochastic, which is a more realistic assumption in online auctions. To see this, note that a time period in which a bidder arrives, enters, and then draws valuation  $v_t < p^t$  would “look like” a period in which no bidder arrival takes place. Thus, a straightforward way to model stochastic arrival within our framework is to extend the support of  $F$  to include some negative values.

Our analysis has some implications that can inform the design of actual auctions. First, recall that our incremental bidding equilibria relied on entering bidders observing the history of past bids. Not disclosing this information during the bidding process can thus inhibit coordination by more than three buyers, as the seller effectively restricts the communication/coordination possibilities the auction format provides to buyers. Second, in the analysis of incremental bidding equilibria we assumed, for simplicity, that the bidders were unconstrained in the choice of the increment  $\varepsilon$ . Many real-life auctions require a minimum bid increment, and a relatively large minimum increment should mitigate the effects of incremental bidding, as a bidder's private valuation is more quickly reached in the progression of increasing prices.

On the other hand, it is not clear that sellers *should* actually adopt these measures. Proposition 7 shows that the worst equilibrium for the seller can be the simple delayed revelation equilibrium with just two coordinating bidders. This equilibrium requires neither incremental bidding nor knowledge of the number of previous high bidders. The only coordination device required to implement this equilibrium is the second-price mechanism itself. Thus, the simple delayed revelation equilibrium cannot easily be prevented as long as the auction is a second-price auction that allows for multiple bids from each buyer. Furthermore, even if there was a way to enforce the immediate revelation equilibrium, doing so would leave bidders with no *ex ante* incentive to participate in the auction in the first place (Proposition 6).

Finally, sellers could use reserve prices. A reserve price  $r > 0$  can be incorporated into our model either by assuming that the initial auction price is  $p^0 = r$  or by assuming that the seller submits a publicly observable bid  $b_0^0 = r$  at time zero. A reserve price of  $r \geq p_1^*$  would then prevent the kind of bid coordination required in the delayed revelation equilibria of our model. Of course, this requires that the seller knows the distribution of valuations  $F$  as well as the entry cost  $c$ . If this assumption holds, one can show that, when  $T$  is large, the optimal reserve price results in a strictly larger expected revenue compared to that in the immediate revelation equilibrium, while leaving buyers with the same zero expected surplus (see Appendix B). Note that this conclusion runs opposite to the result that sellers cannot benefit from setting reserve prices in independent private value auction with simultaneous costly entry (Levin and Smith 1994; McAfee and McMillan 1987). With sequential costly entry, on the other hand, an appropriately determined reserve price results in a strict gain in revenue and expected welfare, relative to the welfare-optimal equilibrium without a reserve price.

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## Appendix A: Proofs

### Proof of Lemma 1

Fix  $k \in \mathbb{N}$  and define  $L_k(p) \equiv \int_p^{\bar{v}} F(v|p)^k (1 - F(v)) dv$ . Let  $p_k^*$  be the unique solution to  $L_k(p_k^*) = c$ , if it exists. The assumption  $c < \int_0^{\bar{v}} (1 - F(v)) F(v) dv$  implies that  $p_1^*$  exists. Note further that  $L_k(p_k^*) = L_{k+1}(p_{k+1}^*) = c$  and  $L_k(p) < L_{k+1}(p) \forall p < \bar{v}$ . Therefore,  $L_{k+1}(p_k^*) < c$ , and since  $L_{k+1}(p)$

is strictly decreasing in  $p$  we conclude that  $p_k^* > p_{k+1}^*$ . Finally, note that  $L_k(0) > 0 \forall k$  and  $L_k(0) \rightarrow 0$  as  $k \rightarrow \infty$ . It follows that there exists an integer  $K \geq 1$ , with  $K \rightarrow \infty$  as  $c \rightarrow 0$ , such that  $p_k^*$  exists if and only if  $k \in \{1, \dots, K\}$ .

## Proof of Lemma 2

We prove that, if bidder  $t < T$  has beliefs  $F(\cdot | p^{t-1})^k$  and bidding strategies satisfy property (3),  $p_k^*$  is the entry threshold for bidder  $t$ . We split the argument into two steps.

*Step 1.* We show that  $p^{t-1} \geq p_k^*$  implies that bidder  $t$  does not enter. This will be done by induction. Suppose  $p^{T-2} \geq p_k^*$ ; then  $p^{T-1} \geq p_k^*$  and bidder  $T$  will not enter in period  $T$  (as shown in the text). Knowing that bidder  $T$  will not enter, bidder  $T-1$  competes against the highest bidder in  $B^{T-2}$ , whose valuation is distributed by  $F(w^{T-2} | p^{T-2})^k$ . This is exactly the same problem as the one examined in the text, and since  $p^{T-2} \geq p_k^*$  it is optimal for bidder  $T-1$  not to enter in period  $T-1$ . Now suppose that  $p^{T-3} \geq p_k^*$ . Then  $p^{T-1} \geq p^{T-2} \geq p_k^*$ , so that bidders  $T$  and  $T-1$  will not enter. Bidder  $T-2$  therefore competes against the highest valuation bidder in  $B^{T-3}$ , whose valuation is distributed by  $F(w^{T-3} | p^{T-3})^k$ . Again, this is the same problem as before, and since  $p^{T-3} \geq p_k^*$  it is optimal for bidder  $T-2$  not to enter. Continuing in this fashion, we conclude that bidder  $t \in \{1, \dots, T\}$  does not enter in period  $t$  if  $p^{t-1} \geq p_k^*$ .

*Step 2.* We show that  $p^{t-1} < p_k^*$  implies that bidder  $t$  enters. To do so, we show that, when  $p^{t-1} < p_k^*$ , entering and then submitting bid  $b_t^t = v_t$  (provided  $p^{t-1} < v_t$ ) yields an expected continuation payoff that is larger than the entry cost  $c$ . Let  $z^{t+1}$  be the highest bids submitted by bidders who enters after period  $t$ , and let  $Z(z^{t+1} | w^{t-1}, v_t)$  be the distribution of  $z^{t+1}$ , conditional on  $w^{t-1}$  and  $v_t$ , under the assumed bidding strategies. If rivals' bidding strategies satisfy (3), bidder  $t$ 's continuation payoff (not including the entry cost  $c$ ) if he enters at price  $p^{t-1} < p_k^*$  and then bids  $b_t^t = v_t$  is given by

$$U_t(p^{t-1}) = \int_{p^{t-1}}^{\bar{v}} \int_{p^{t-1}}^{v_t} \int_0^{v_t} (v_t - \max\{w^{t-1}, z^{t+1}\}) dZ(z^{t+1} | w^{t-1}, v_t) dF(w^{t-1} | p^{t-1})^k dF(v_t).$$

Define

$$A(v_t) = \int_{p^{t-1}}^{p_k^*} \int_0^{v_t} (v_t - \max\{w^{t-1}, z^{t+1}\}) dZ(z^{t+1} | w^{t-1}, v_t) dF(w^{t-1} | p^{t-1})^k,$$

$$B(v_t) = \int_{p_k^*}^{v_t} \int_0^{v_t} (v_t - \max\{w^{t-1}, z^{t+1}\}) dZ(z^{t+1}|w^{t-1}, v_t) dF(w^{t-1}|p^{t-1})^k,$$

and express bidder  $t$ 's payoff from entering as follows:

$$U_t(p^{t-1}) = \int_{p^{t-1}}^{\bar{v}} [A(v_t) + B(v_t)] dF(v_t) > \int_{p_k^*}^{\bar{v}} [A(v_t) + B(v_t)] dF(v_t). \quad (13)$$

Now consider two cases:

1. First, suppose  $v_t \geq p_k^*$  and  $w^{t-1} \geq p_k^*$ . Then, by (3), the price at the end of period  $t$  must be  $p^t \geq p_k^*$ , and no entry will occur after period  $t$  as shown in Step 1. Thus, conditional on  $v_t \geq p_k^*$  and  $w^{t-1} \geq p_k^*$  we have  $z^{t+1} = 0$ , which allows us to write

$$\begin{aligned} B(v_t) &= \int_{p_k^*}^{v_t} (v_t - w^{t-1}) dF(w^{t-1}|p^{t-1}) \\ &= (1 - F(p_k^*|p^{t-1})) \int_{p_k^*}^{v_t} (v_t - w^{t-1}) dF(w^{t-1}|p_k^*). \end{aligned} \quad (14)$$

(Note that  $w^{t-1} \geq p_k^*$  implies that exactly one buyer in  $B^{t-1}$  has valuation above  $p_k^*$ , otherwise  $p^{t-1} \geq p_k^*$  by (3).)

2. Second, suppose  $v_t \geq p_k^*$  and  $w^{t-1} \leq p_k^*$ . There will be two sub-cases:
  - 2.1 If during some period  $s > t$  a bidder enters with  $v_s \geq p_k^*$ , the price at the end of period  $s$  must be  $p^s \geq p_k^*$  (by (3)), and no further entry will occur after period  $s$  as shown in Step 1. In this event,  $z^{t+1} = v_s \geq p_k^*$  with distribution  $F(z^{t+1}|p_k^*)$ .
  - 2.2 If no bidder with  $v_s \geq p_k^*$  enters during any period  $s > t$ , we have  $z^{t+1} < p_k^*$ .

Bidder  $t$ 's payoff will be lower in the first case than in the second. Thus, we can write

$$\begin{aligned} A(v_t) &> \int_{p^{t-1}}^{p_k^*} \int_{p_k^*}^{v_t} (v_t - z^{t+1}) dF(z^{t+1}|p_k^*) dF(w^{t-1}|p^{t-1}) \\ &= F(p_k^*|p^{t-1}) \int_{p_k^*}^{v_t} (v_t - z^{t+1}) dF(z^{t+1}|p_k^*). \end{aligned} \quad (15)$$

Combining (13)–(15), we have

$$U_t(p^{t-1}) > \int_{p_k^*}^{\bar{v}} \int_{p_k^*}^{v_t} (v_t - v) dF(v|p_k^*) dF(v_t) \geq \int_{p_k^*}^{\bar{v}} \int_{p_k^*}^{v_t} (v_t - v) dF(v|p_k^*)^k dF(v_t) = c.$$

(The weak inequality follows from the fact that  $(v_t - v)$  decreases in  $v$  and  $F(v|p_k^*) \geq F(v|p_k^*)^k$ , and the equality follows from the definition of  $p_k^*$ .) Thus, when  $p^{t-1} < p_k^*$ , the expected surplus for bidder  $t$  from entering the auction in period  $t$  exceeds the entry cost  $c$ , so bidder  $t$  enters.  $\square$

## Proof of Proposition 4

Most of the result was shown already in the text in Section 5.2. What is left is to establish the optimality of the equilibrium bidding strategy given the equilibrium entry strategy (Step 1), and the optimality of the entry strategy following off-equilibrium prices; that is, prices that exceed  $p_2^*$  (Step 2).

*Step 1.* Clearly, in the final period a truthful bid  $b_t^T = v_t$  is optimal for every bidder  $t$ . Let us therefore consider bidding by buyer  $t$  in periods  $t \leq s < T$ . We consider three cases.

1.  $p^{s-1} = p_2^*$ . In this case, all entry has stopped. Each bidder who participates at this point receives an expected payoff equal to the payoff he would receive in a second price auction against exactly one other bidder whose valuation was known to be  $p_2^*$  or larger. Suppose that bidder  $t$  deviates from the equilibrium strategy and bids more than  $p_2^*$  in period  $s < T$ . If this deviation does not change the price in some period  $s' \in \{s, \dots, T-1\}$ , then it has no effect on entry. In this case, bidder  $t$  effectively remains in a second price auction against one opponent with valuation above  $p_2^*$ , which means that the original bidding strategy is no worse than the deviation. If the deviation changes the price in some period  $s' \in \{s, \dots, T-1\}$  from  $p^{s'} = p_2^*$  to  $p^{s'} \neq p_2^*$ , then additional bidders enter with positive probability, strictly lowering the expected payoff to bidder  $t$ .
2.  $p^{s-1} < p_2^*$ . The bidding strategy calls for bidders with valuations  $v_t \geq p_2^*$  to bid  $p_2^*$  (case 2a), and for bidders with valuations  $v_t < p_2^*$  to bid  $v_t$  (case 2b).
  - 2a.  $v_t \geq p_2^*$ . The same argument as in case 1 applies: If bidder  $t$  deviates and bids  $b_t^s \neq p_2^*$ , and this deviation does not, in some period  $s' \geq s$ , change the price from  $p^{s'} = p_2^*$  to  $p^{s'} \neq p_2^*$ , it cannot be better than the original equilibrium bidding strategy. If the deviation changes the price in some period  $s' \geq s$  from  $p^{s'} = p_2^*$  to  $p^{s'} \neq p_2^*$ , then additional bidders will enter with positive probability, reducing the expected payoff to bidder  $t$ .

- 2b.  $v_t < p_2^*$ . If bidder  $t$  deviates by bidding  $b_t^s < v_t$  in period  $s$ , the deviation will not affect the final allocation and prices. If bidder  $t$  deviates by bidding above his valuation, he can only benefit if doing so deters entry by rival bidders sooner than it would otherwise have. For this to happen we need  $b_t^{s'} \geq p_2^*$  in some period  $s'$ , which means that some other bidder  $j \neq t$  must submit a bid  $b_j^{s''} \geq p_2^*$  in some period  $s''$ . But this implies that  $p^T \geq p_2^*$ . Thus, if the deviation by bidder  $t$  is successful in deterring entry that would have occurred otherwise, and  $t$  wins, he will pay a price larger than his valuation. In all other cases, bidder  $t$ 's payoff is exactly the same as it would have been without the deviation.
3.  $p^{s-1} > p_2^*$ . Once the price has surpassed  $p_2^*$  (an off-equilibrium event), the entry and bidding strategies are identical to those in the immediate revelation equilibrium; the bidding strategy is therefore optimal.

We therefore conclude that no entering bidder has an incentive to deviate from the equilibrium bidding strategies (7).

*Step 2.* Next, we consider the bidders' entry decisions. We need to find a sequence of strategy profiles, converging to the equilibrium strategies, such that prices above  $p_2^*$  are possible along the sequence and the equilibrium entry strategies are sequentially rational under the limit of Bayesian beliefs generated by the sequence of perturbed strategies.

To this end, let  $\delta \in (0, 1)$  and consider the following perturbed strategy for every player:

$\tilde{e}_t$ : In period  $t$ , enter with probability  $(1 - \delta)e_t(p^{t-1}) + \delta(1 - e_t(p^{t-1}))$ , where  $e_t(\cdot)$  is the equilibrium entry strategy.

$\tilde{b}_t$ : Conditional on having entered, bid as follows: In every period  $s \geq t$ , with probability  $1 - \delta$  submit the equilibrium bid prescribed by strategy (7), if the equilibrium strategy has been followed until then. With probability  $\delta$ , bid  $b_t^s = b_t^{s+1} = \dots = v_t$ .

Note that any weakly increasing sequence of prices can occur under this strategy profile, as long as  $\delta > 0$ . Furthermore, as  $\delta \rightarrow 0$  the profile converges to the equilibrium strategies.

Now suppose some potential bidder  $t$  observes out-of-equilibrium price  $p^{t-1} > p_2^*$ . This can only happen if at least two participating bidders did not play their equilibrium strategies, and submitted bids larger than  $p_2^*$ . Given the perturbed profile, these must be truthful bids. Thus, any observed

$p^{t-1} > p_2^*$  must be the second-highest valuation of bidders in  $B^{t-1}$ , which means that the resulting Bayesian posterior distribution of  $w^{t-1}$  is  $F(w^{t-1}|p^{t-1})$ . Since this distribution does not depend on  $\delta$ , the limit belief as  $\delta \rightarrow 0$  is also  $F(w^{t-1}|p^{t-1})$ . It is then optimal for bidder  $t$  to enter the auction as long as  $p^{t-1} < p_1^*$ , as prescribed by the equilibrium entry strategy (9).  $\square$

## Proof of Proposition 5

We need to establish the optimality of the equilibrium bidding strategy given the equilibrium entry strategy (Step 1), and the optimality of the entry strategy following off-equilibrium prices.

*Step 1.* Fix a bidder  $t$ . In the final period a truthful bid  $b_t^T = v_t$  is optimal for bidder  $t$ . Let us therefore consider bidding in periods  $s \in \{t, \dots, T-1\}$ . If  $p^{s-1} \geq p_k^*$  and  $k^{s-1} \geq k$ , the argument is analogous to case 1 in the proof of Proposition 4, and if  $p^{s-1} < p_k^*$  the argument is analogous to case 2 in the proof of Proposition 4.

Now consider  $p^{s-1} \geq p_k^*$  and  $k^{s-1} < k$ . In this case, the auction is in the ‘‘collusion formation phase.’’ The bidding strategy calls for bidder  $t$  in period  $s$  with valuation  $v_t \geq \iota(p^{s-1})$  to bid  $\iota(p^{s-1})$  (case a), and with valuation  $v_t < \iota(p^{s-1})$  to bid  $v_t$  (case b).

- a.  $v_t \geq \iota(p^{s-1})$ . Suppose all other bidders follow the incremental bidding strategy, and consider a (unilateral) deviation from the equilibrium bid  $b_t^s = \iota(p^{s-1})$  to a bid strictly larger than  $\iota(p^{s-1})$ . This deviation has no effect  $k^s$ , and thus leaves the expected number of entering bidders unchanged. It will hence either not change  $t$ 's expected payoff, or decrease it (if the deviation is to a bid in excess of  $t$ 's valuation  $v_t$ ). A deviation to a bid below  $\iota(p^{s-1})$  weakly decreases the variable  $k^s$ , and thus increase the expected number of bidders at the end of the auction. This decreases the expected payoff to bidder  $t$ .
- b.  $v_t < \iota(p^{s-1})$ . Consider first a deviation from the equilibrium bid to any other bid strictly below  $\iota(p^{s-1})$ . This deviation has (a.s.) no effect  $k^s$  and thus leaves the expected number of entering bidders unchanged. It will hence either not change  $t$ 's expected payoff, or decrease it (if the deviation is to a bid in excess of  $t$ 's valuation  $v_t$ ). Next, consider a deviation to a bid of  $\iota(p^{s-1})$  or larger. This deviation weakly increases the variable  $k^t$ , and thus decreases the expected number of bidders at the end of the auction. However, this cannot help bidder  $t$ , because if he were to win he would pay  $p^T \geq \iota(p^{s-1}) > v_t$ .

Thus, no entering bidder has an incentive to deviate from the equilibrium bidding strategies (10).

*Step 2.* Similar to Step 2 in the proof of Proposition 4, let  $\delta \in (0, 1)$  and consider the following perturbed strategy for every player  $t$ :

$\tilde{e}_t$ : In period  $t$ , enter with probability  $(1 - \delta)e_t(p^{t-1}, k^{t-1}) + \delta(1 - e_t(p^{t-1}, k^{t-1}))$ , where  $e_t(\cdot)$  is the equilibrium entry strategy.

$\tilde{b}_t$ : Conditional on having entered, bid as follows: In every period  $s \geq t$ , with probability  $1 - \delta$  submit the equilibrium bid prescribed by strategy (10), if the equilibrium strategy has been followed until then. With probability  $\delta$ , bid  $b_t^s = b_t^{s+1} = \dots = v_t$ .

As  $\delta \rightarrow 0$  the profile converges to the equilibrium strategies. Now suppose some potential bidder  $t$  observes either an out-of-equilibrium price, that is, some price  $p^{t-1} > \iota(p^{t-2}) > p_k^*$ . This can only happen if at least two participating bidders did not play their equilibrium strategies, and submitted bids larger than  $\iota(p^{t-2})$ . Given the perturbed profile, these deviating bidders must have submitted truthful bids. Thus, any observed  $p^{t-1} > \iota(p^{t-2}) > p_k^*$  will be interpreted as the second-highest valuation of bidders in  $B^{t-1}$ . The resulting Bayesian posterior distribution of  $w^{t-1}$  is therefore  $F(w^{t-1}|p^{t-1})$ . This distribution does not depend on  $\delta$ ; the limit belief as  $\delta \rightarrow 0$  is therefore also  $F(w^{t-1}|p^{t-1})$ . It is then optimal for bidder  $t$  to enter the auction as long as  $p^{t-1} < p_1^*$ , as prescribed by the equilibrium entry strategy (12).  $\square$

## Proof of Proposition 6

To establish parts (a) and (b), recall that, for  $k \in \{1, 2\}$ , entry stops in  $k$ -equilibrium once two bidders with valuations weakly greater than  $p_k^*$  have arrived. Since  $p_1^* > p_2^*$ , given a sequence of valuations  $(v_t)_{t=1,2,\dots}$ , every buyer who enters in 2-equilibrium also enters in 1-equilibrium, but not necessarily vice versa. Thus, the number of participating bidders in 1-equilibrium strictly dominates the number of participating bidders in 2-equilibrium (in the sense of first-order stochastic dominance). Similarly, the price paid to the seller in  $k$ -equilibrium ( $k = 1, 2$ ) is the lower one of the first two valuations in  $(v_t)$  that weakly exceed  $p_k^*$ . Again, since  $p_1^* > p_2^*$  the price in 1-equilibrium strictly dominates the price in 2-equilibrium. It then follows that  $N_1 > N_2$  and  $R_1 > R_2$ .

To prove (c), we use Pearson's (1902) formula for the expected difference between consecutive order statistics to write

$$E[v_{k:k}^k - v_{k-1:k}^k] = k \int_{p_k^*}^{\bar{v}} F(v|p_k^*)^{k-1} (1 - F(v|p_k^*)) dv.$$

Thus, expected buyer surplus in 1-equilibrium can be written as

$$\begin{aligned}
B_1 &= E[v_{2:2}^1 - v_{1:2}^1] - N_1 c = 2 \int_{p_1^*}^{\bar{v}} F(v|p_1^*)(1 - F(v|p_1^*)) dv - \frac{2}{1 - F(p_1^*)} c \\
&= \frac{2}{1 - F(p_1^*)} \int_{p_1^*}^{\bar{v}} F(v|p_1^*)(1 - F(v)) dv - \frac{2}{1 - F(p_1^*)} c \\
&= \frac{2}{1 - F(p_1^*)} (c - c) = 0,
\end{aligned}$$

where the last line follows from (2). Similarly, in any  $k$ -equilibrium with  $k \geq 2$ , we have

$$\begin{aligned}
B_k &= E[v_{k:k}^k - v_{k-1:k}^k] - N_k c = k \int_{p_k^*}^{\bar{v}} F(v|p_k^*)^{k-1} (1 - F(v|p_k^*)) dv - \frac{k}{1 - F(p_k^*)} c \\
&= \frac{k}{1 - F(p_k^*)} \int_{p_k^*}^{\bar{v}} F(v|p_k^*)^{k-1} (1 - F(v)) dv - \frac{k}{1 - F(p_k^*)} c \\
&> \frac{k}{1 - F(p_k^*)} \int_{p_k^*}^{\bar{v}} F(v|p_k^*)^k (1 - F(v)) dv - \frac{k}{1 - F(p_k^*)} c \\
&= \frac{k}{1 - F(p_k^*)} (c - c) = 0.
\end{aligned}$$

To prove (d), for  $k = 1, 2$  write social welfare as follows:

$$W_k = E[v_{2:2}^k] - N_k c = \int_{p_k^*}^{\bar{v}} v dF(v|p_k^*)^2 - \frac{2c}{1 - F(p_k^*)} = \bar{v} - \int_{p_k^*}^{\bar{v}} F(v|p_k^*)^2 dv - \frac{2c}{1 - F(p_k^*)}.$$

Now treat  $k$  as a continuous variable and define  $p_k^*$  as before, through condition (2). Then  $p_k^*$  is differentiable with respect to  $k$ , with  $dp_k^*/dk < 0$ , and for  $k \in [0, 1]$  we have

$$\begin{aligned}
\frac{\partial}{\partial k} W_k &= \frac{-2f(p_k^*) \frac{dp_k^*}{dk}}{(1 - F(p_k^*))^2} \left( - \int_{p_k^*}^{\bar{v}} F(v|p_k^*) (1 - F(v)) dv + c \right) \\
&\stackrel{\text{sg}}{=} - \int_{p_k^*}^{\bar{v}} F(v|p_k^*) (1 - F(v)) dv + c \\
&< - \int_{p_k^*}^{\bar{v}} F(v|p_k^*)^k (1 - F(v)) dv + c \quad (\text{for } k > 1) \\
&= -c + c = 0,
\end{aligned}$$



where the last line, again, follows from (2). Thus,  $W_1 > W_2$ .  $\square$

### Proof of Proposition 7

To prove (a), let  $k \geq 2$  and differentiate  $N_k$  with respect to  $k$ :

$$\frac{d}{dk}N_k = \frac{1 - F(p_k^*) + kf(p_k^*)\frac{dp_k^*}{dk}}{(1 - F(p_k^*))^2} > 0 \Leftrightarrow \frac{dp_k^*}{dk} \frac{kf(p_k^*)}{1 - F(p_k^*)} > -1. \quad (16)$$

Differentiate the defining equation for  $p_k^*$ , (2), implicitly with respect to  $k$ ,

$$\int_{p_k^*}^{\bar{v}} \left[ \ln(F(v|p_k^*))F(v|p_k^*)^k + kF(v|p_k^*)^{k-1} \left( -f(p_k^*)\frac{dp_k^*}{dk} \right) \frac{1 - F(v)}{(1 - F(p_k^*))^2} \right] (1 - F(v))dv = 0,$$

and rearrange to get

$$\frac{dp_k^*}{dk} \frac{kf(p_k^*)}{1 - F(p_k^*)} = \frac{\int_{p_k^*}^{\bar{v}} \ln(F(v|p_k^*))F(v|p_k^*)^k (1 - F(v))dv}{\int_{p_k^*}^{\bar{v}} (1 - F(v|p_k^*))F(v|p_k^*)^{k-1} (1 - F(v))dv}. \quad (17)$$

Thus, for (16) to be true, we need to show that

$$\int_{p_k^*}^{\bar{v}} \ln(F(v|p_k^*))F(v|p_k^*)^k (1 - F(v))dv > - \int_{p_k^*}^{\bar{v}} (1 - F(v|p_k^*))F(v|p_k^*)^{k-1} (1 - F(v))dv.$$

We show that this inequality holds pointwise at each  $v \in (p_k^*, \bar{v})$ :

$$\begin{aligned} \ln(F(v|p_k^*))F(v|p_k^*)^k (1 - F(v)) &> -(1 - F(v|p_k^*))F(v|p_k^*)^{k-1} (1 - F(v)) \\ \Leftrightarrow \ln(F(v|p_k^*)) &> 1 - \frac{1}{F(v|p_k^*)}. \end{aligned}$$

This is indeed the case, as  $\ln x > 1 - 1/x \forall x \in (0, 1)$ , and it follows that  $N_k$  increases strictly in  $k$  for  $k \geq 2$ . The example provided in Footnote 17 in the text shows that it is possible that  $N_k > N_1$  for sufficiently large  $k$ . Finally,  $B_k > 0$  for  $k \geq 2$  is already shown in the proof of Proposition 6 (c).

To prove (b), for  $k \geq 2$  write social welfare as follows:

$$W_k = E[v_{k:k}^k] - N_k c = \int_{p_k^*}^{\bar{v}} v dF(v|p_k^*)^k - \frac{kc}{1 - F(p_k^*)} = \bar{v} - \int_{p_k^*}^{\bar{v}} F(v|p_k^*)^k dv - \frac{kc}{1 - F(p_k^*)}.$$

Differentiate this expression with respect to  $k$ :

$$\begin{aligned} \frac{d}{dk} W_k &= \frac{dp_k^*}{dk} \frac{kf(p_k^*)}{1 - F(p_k^*)} \left( \int_{p_k^*}^{\bar{v}} F(v|p_k^*)^{k-1} (1 - F(v|p_k^*)) dv - \frac{c}{1 - F(p_k^*)} \right) \\ &\quad - \int_{p_k^*}^{\bar{v}} \ln(F(v|p_k^*)) F(v|p_k^*)^k dv - \frac{c}{1 - F(p_k^*)}. \end{aligned}$$

Use (2) to express this further as

$$\begin{aligned} \frac{d}{dk} W_k &= \frac{dp_k^*}{dk} \frac{kf(p_k^*)}{1 - F(p_k^*)} \left( \int_{p_k^*}^{\bar{v}} F(v|p_k^*)^{k-1} (1 - F(v|p_k^*))^2 dv \right) \\ &\quad - \int_{p_k^*}^{\bar{v}} F(v|p_k^*)^k (\ln(F(v|p_k^*)) + 1 - F(v|p_k^*)) dv. \quad (18) \end{aligned}$$

Divide both the numerator and denominator on the right-hand side of (17) by  $1 - F(p_k^*)$ , to get

$$\frac{dp_k^*}{dk} \frac{kf(p_k^*)}{1 - F(p_k^*)} = \frac{\int_{p_k^*}^{\bar{v}} \ln(F(v|p_k^*)) F(v|p_k^*)^k (1 - F(v|p_k^*)) dv}{\int_{p_k^*}^{\bar{v}} F(v|p_k^*)^{k-1} (1 - F(v|p_k^*))^2 dv}. \quad (19)$$

Plug (19) back in (18) and simplify, to get

$$\frac{d}{dk} W_k = - \int_{p_k^*}^{\bar{v}} F(v|p_k^*)^k \left[ \ln(F(v|p_k^*)) F(v|p_k^*) + (1 - F(v|p_k^*)) \right] dv < 0$$

because for  $\ln x > 1 - 1/x \forall x \in (0, 1)$ . It follows that  $W_k$  decreases strictly in  $k$  for  $k \geq 2$ .

Finally, to prove part (d), let valuations be uniform on  $[0, \bar{v}]$ . Then  $k$ -equilibrium exists for  $k \leq K = \lfloor \frac{1}{2} \sqrt{1 + 4\bar{v}/c} - \frac{3}{2} \rfloor$ , with  $p_k^* = \bar{v} \left( 1 - \sqrt{(k+1)(k+2)c/\bar{v}} \right)$ . Note further that, for  $m, k \leq K$  and  $r \leq m$ , the expectation of the  $r$ th-smallest of  $m$  draws from the uniform distribution on  $[p_k^*, \bar{v}]$  is

$$E[v_{r:m}^k] = \frac{r}{m+1} \bar{v} + \frac{m+1-r}{m+1} p_k^*.$$

Thus, expected seller revenue in the uniform values case is given by

$$R_k = \begin{cases} \bar{v} \left( 1 - 2\sqrt{2c/(3\bar{v})} \right) & \text{if } k = 1, \\ \bar{v} \left( 1 - 2\sqrt{(k+2)c/[(k+1)\bar{v}]} \right) & \text{if } k > 1. \end{cases}$$

$R_k$  increases strictly in  $k$  for  $k \geq 2$ , but never reaches  $R_1$ . Since  $W_k = R_k + B_k$  decreases strictly in  $k$ , as shown in part (b), it follows that  $B_k$  decreases strictly in  $k$  for  $k \geq 2$ . Proposition 6 (c) implies that  $B_k > 0$  for  $k \geq 2$ .  $\square$

## Appendix B: Reserve Prices

Suppose the seller sets reserve price  $r > p_1^*$ . We think of this reserve price as a bid submitted by the seller at time zero,  $b_t^t = r$ . A construction analogous to our IR equilibrium (Proposition 3) implies that an equilibrium exists in which buyer  $t$  bids  $b_t^t = v_t$  if  $v_t > r$ , and entry stops after one buyer has arrived with  $v_t > r$ . If  $T \rightarrow \infty$ , the probability that this happens converges to one, and the winning buyer pays  $r$  in this equilibrium. An entering buyer's expected payoff (not counting the entry cost) is

$$\int_r^{\bar{v}} (v - r) dF(v) = \bar{v} - r - \int_r^{\bar{v}} F(v) dv,$$

which is strictly decreasing in  $r$ . The highest possible reserve price for which entry occurs is therefore implicitly given by

$$\bar{v} - r^* - \int_{r^*}^{\bar{v}} F(v) dv = c. \quad (20)$$

Note that  $r^* > p_1^*$ . (To see this, suppose a buyer knows that exactly one participating bidder has valuation weakly greater than  $p_1^*$ , and this valuation was *exactly* equal to  $p_1^*$ . By Lemma 2, the buyer would have a strict incentive to enter at  $p_1^*$ . Thus,  $r^*$  that makes the buyer indifferent between entering and not entering must exceed  $p_1^*$ .)

If the reserve price is set to  $r^*$ , then all buyers obtain a zero expected surplus, and the seller's revenue is  $r^*$ . This means that social welfare is  $r^*$ . We will show that this is larger than welfare in IR equilibrium. To do so, divide both sides of (20) by  $1 - F(p_1^*)$  and rearrange, to get

$$\begin{aligned} \frac{\bar{v} - r^*}{1 - F(p_1^*)} - \left[ \int_{r^*}^{\bar{v}} F(v|p_1^*) dv + \frac{F(p_1^*)(\bar{v} - r^*)}{1 - F(p_1^*)} \right] &= \frac{c}{1 - F(p_1^*)} \\ \Leftrightarrow \bar{v} - r^* - \int_{r^*}^{\bar{v}} F(v|p_1^*) dv &= \frac{c}{1 - F(p_1^*)} \\ \Leftrightarrow r^* &= \bar{v} - \int_{r^*}^{\bar{v}} F(v|p_1^*) dv - \frac{c}{1 - F(p_1^*)}. \end{aligned} \quad (21)$$

Since the right-hand side of (21) strictly decreases in  $r^*$ , and  $p_1^* < r^*$ , it is sufficient to show that

$$\begin{aligned}
W_1 &= \bar{v} - \int_{p_1^*}^{\bar{v}} F(v|p_1^*)^2 dv - \frac{2c}{1 - F(p_1^*)} \geq \bar{v} - \int_{p_1^*}^{\bar{v}} F(v|p_1^*) dv - \frac{c}{1 - F(p_1^*)}, \\
&\Leftrightarrow \int_{p_1^*}^{\bar{v}} F(v|p_1^*)(1 - F(v|p_1^*)) dv \geq \frac{c}{1 - F(p_1^*)} \\
&\Leftrightarrow \int_{p_1^*}^{\bar{v}} F(v|p_1^*)(1 - F(v)) dv \geq c.
\end{aligned}$$

This is true as an equality, by (2). Thus, compared to the IR equilibrium, the reserve price  $r^*$  strictly increases revenue, does not change buyer surpluses, and strictly increases social welfare.