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Abstract

Working within the generalised linear modelling framework, a method for modelling mortality reduction factors is described. The methodology is flexible, in the sense that it is both capable of assessing existing reduction factors, with the benefit of hindsight, and capable of projecting established data patterns in order to forecast future reduction factors. The construction of associated confidence intervals is described. Limitations of the methodology, as a means of forecasting, are identified. Three case studies, illustrating different aspects of the methodology, are presented.

Keywords: Mortality reduction factors; Generalised linear models; Pensioners; Annuitants

1. Introduction

When confronted by the need to compare two mortality experiences, matched for age, it is established practice in the actuarial literature that we should focus on the ratios of the respective mortality rates (see, for example, Benjamin and Pollard (1993)). For example, when two (or more) such mortality experiences differ only in respect of policy duration, Renshaw and Haberman (1997) and Haberman and Renshaw (1999) describe how such ratios may be studied to construct a set of well ordered mortality rates with respect to policy duration. In this paper, where the focus is on two (or more) related mortality experiences, matched for age, and which differ only in respect to calendar period, we describe how a similar approach can contribute to the analysis and construction of mortality reduction factors, for the purposes of identifying past trends and for forecasting future trends.

Specifically, let q_{xt} denote the probability of death between exact age x and $x + 1$, at time t, and focus on the ratio

$$\frac{q_{xt}}{q_{x0}}$$

based on time t ($\neq 0$) relative to time 0. Suppose, that at time 0, q_{x0} represent a set of 'standard' mortality rates. Thus t may be positive, representing calendar time subsequent to the 'standard' year, or negative, representing calendar time prior to the 'standard' year. Further, the positioning of t, with respect to the origin, has important implications for the type of analysis possible. Thus, when the data in focus are for positive t, a *retrospective* study of mortality reduction factors is possible based on the benefit of hindsight (Section 6), and when the data in focus are for negative t, a *prospective* study to suggest a new mortality reduction factor is possible (Section 7). It is thus important to appreciate the flexibility afforded by the siting of the time origin, relative to the serial nature of the data available for analysis, in order to take advantage of the full potential of the methodology.

In Section 2, mortality reduction factors are characterised and examples given. In Section 3, a method for modelling mortality reduction factors, within the generalised linear modelling framework, is suggested. In Section 4, a description is given of how the fitted values, extrapolated (forecast) values, and confidence intervals may be computed. In Section 5, a description is given of how results may be presented, on the log scale, so that the focus switches from mortality ratios, as above, to the log differences $\log q_{xt}$ - $\log q_{x0}$. In Section 6, by way of illustration, the first of three case studies is presented: a retrospective study of the UK pensioner lives' experience for 1983 to 1996. In Section 7, a second case study is presented: a prospective study of the UK pensioner lives experience for 1983 to 1994, thereby bringing it into phase with the regular, published updating of mortality reduction factors, as carried out in the UK by the CMI Committee. In Section 8, the third case study is presented, based on the UK annuitants' mortality experience over the period 1967-94, for which there are sufficient data to illustrate and assess further aspects of the potential of the methodology. In Section 9, for completeness, a description is given of how the methodology may be reformulated in terms of the force of mortality μ_{xt} - however, in our examples, we have chosen to conform with UK practice, as set down by the CMI Committee, and to concentrate on the probability of death q_{xt}, rather than the force of mortality. In Section 10, some remarkable similarities are identified between the structure underpinning this methodology and the structure underpinning that of the mortality forecasting methods of Lee and Carter (1992), which are currently under consideration by American actuaries. In Section 11, the various strands are drawn together in a summary overview.

The practical implementation of the methodology described in this paper may be fully automated, using a standard interactive generalised linear modelling package, such as GLIM (Francis *et al.* (1993)).

2. Mortality reduction factors

Under current collective UK actuarial practice, reduction factors are applied to standard mortality rates, valid at the centre of a given quadrennium, in order to adjust the standard mortality rates for future years, for both pensioners and annuitants. Let

t denote time, with origin $t = 0$ located at the centre of a so-called *base* quadrennium

q_{xt} denote the probability of death between age x and age x+1, at time $t \geq 0$,

then a mortality reduction factor RF is characterised by the equation

$$q_{xt} = q_{x0}\,\mathrm{RF}(x, t), \quad t \geq 0 \qquad (2.1)$$

subject to the contraint

$$\mathrm{RF}(x, 0) = 1 \ \forall \ x. \qquad (2.2)$$

The further constraints, $1 \geq \mathrm{RF}(x, t) > 0 \ \forall \ x, \ t > 0$, together with non-increasing monotonicity for increasing t, are implicit.

Conventionally, q_{x0} is determined by graduation, using the crude mortality rates of the base quadrennium. Then, a judgement is made about the specific nature of RF(x, t), and the projected mortality rates q_{xt} computed, using equation (2.1). While the determination of q_{x0} by graduation is essentially objective, the specification of RF(x, t) is inherently subjective, given the prospective nature of the exercise. This process is described in detail in CMI Committee (1990) for pensioners and annuitants, with reference to the base quadrennium 1979-82. Thus, as an example, a common reduction factor

$$\mathrm{RF}(x,\ t) = \alpha(x) + \{1 - \alpha(x)\}(0.4)^{t/20} \tag{2.3}$$

with

$$\alpha(x) = 0.5; \qquad x < 60$$

$$\alpha(x) = \frac{(x - 10)}{100}; \qquad 60 \leq x \leq 110$$

designed with reference to $t = 20$, and with $t = 0$ mapping to the centre of the quadrennium 1979-82, is suggested for use in estimating future mortality rates of both male and female pensioners, and for both male and female annuitants.

In the USA, a similar approach has been common. For example, the Group Annuity Valuation Table (GAVT) Task Force (1995) have suggested the following equation for projecting mortality rates:

$$\mathrm{RF}(x,\ t) = (1 - \mathrm{AA}_x)^t \tag{2.4}$$

where AA_x is termed the age specific "annual improvement factor".

3. Modelling reduction factors

Let

A_{xt} denote the actual number of deaths at individual ages x and times $t = t_1, t_1+1, ..., t_{max}$

r_{xt} denote the matching *initial* exposures,

where the upper case letter denotes a random variable. The value of t_1 will depend on the context and may be either negative or positive. Define $A_{xt} \sim \mathrm{bin}(r_{xt},\ q_{xt})$ to be statistically independent, with provision for an over-dispersion parameter ϕ. These assumptions define a generalised linear model (GLM), whose independent responses may be written as

$$Y_{xt} = \frac{A_{xt}}{r_{xt}},$$

for which

$$\mathrm{E}(Y_{xt}) = q_{xt},\ \mathrm{Var}(Y_{xt}) = \phi\,\frac{q_{xt}(1 - q_{xt})}{r_{xt}};$$

together with a linear predictor η_{xt} and a monotonic differentiable link function g, such that

$$\eta_{xt} = g(q_{xt}) \ \Leftrightarrow\ q_{xt} = g^{-1}(\eta_{xt}). \tag{3.1}$$

The choice of binomial response model is motivated by its use in the graduation of q_x values (for fixed t), based on *initial* exposures, as described by Forfar *et al.* (1988) and Renshaw (1991).

It follows from equations (2.1) & (3.1), that

$$\mathrm{RF}(x,t) = \frac{g^{-1}(\eta_{xt})}{q_{x0}}, \tag{3.2}$$

defines a mortality reduction factor, subject to the constraint (2.2). To accommodate this constraint, we decompose the linear predictor additively as

$$\eta_{xt} = \eta_{x0} + \eta'_{xt} \text{ such that } \eta'_{x0} = 0.$$

Hence

$$\mathrm{RF}(x,t) = \frac{g^{-1}(\eta_{x0} + \eta'_{xt})}{q_{x0}},$$

so that the constraint (2.2) implies

$$g^{-1}(\eta_{x0}) = q_{x0},$$

and hence

$$\eta_{x0} = g(q_{x0}).$$

This defines the term $g(q_{x0})$, which is described as an "offset" and is characterised as a known (non-parameterised) additive term in the linear predictor η_{xt}. Then, provided that the linear parameterised structure of η'_{xt} is chosen such that $\eta'_{x0} = 0$, it follows that

$$\mathrm{RF}(x,t) = \frac{g^{-1}(\eta_{x0} + \eta'_{xt})}{g^{-1}(\eta_{x0})} \tag{3.3}$$

satisfies the requirements of a reduction factor. The condition $\eta'_{x0} = 0$ is a requirement to be imposed on the linear parameterised portion of the predictor η_{xt}, and is satisfied provided that the pre-selected structure in t is forced through the time origin, $t = 0$. The parameterised structure

$$\eta'_{xt} = \beta_x t \quad \forall\ (x,t), \tag{3.4}$$

consisting of a pencil of lines with focus at the time origin, (and a different slope parameter β_x for each age x), is a typical case in point. Other alternative structures, of the general spline type

$$\eta'_{xt} = \sum_{i=1}^{r}\sum_{j=0}^{k} \beta_{ij}(x - x_j)_+^i\, t \quad \forall\ (x,t), \tag{3.5}$$

involving k+1 knots x_j, where

$$x_0 < x_1 < \dots < x_k$$

have the potential to induce smoothing in the reduction factor with respect to x. The special case of a cubic spline in x, with $r = 3$ and $\beta_{ij} = 0\ \forall\ i = 1, 2$ (any j), is particularly successful.

Link functions of possible interest include the following

link function	*link g*	*inverse link g^{-1}*
log-log	$\eta_{xt} = -\log\{-\log(q_{xt})\}$	$q_{xt} = \exp\{-\exp(-\eta_{xt})\}$
complementary log-log	$\eta_{xt} = \log\{-\log(1 - q_{xt})\}$	$q_{xt} = 1 - \exp(-\exp\eta_{xt})$
log odds	$\eta_{xt} = \log\left\{\frac{q_{xt}}{1 - q_{xt}}\right\}$	$q_{xt} = \frac{\exp\eta_{xt}}{1 + \exp\eta_{xt}}$
probit	$\eta_{xt} = \Phi^{-1}(q_{xt})$	$q_{xt} = \Phi(\eta_{xt})$

All four links are depicted in Fig. 1. For a graphical comparison of the log-log, complementary log-log and probit links with the log odds link, see Fig. 4.1, p 109, McCullagh and Nelder (1989).

4. Fitted values, confidence intervals, extrapolation

To fit the pencil of lines, defined by equation (3.4), the linear predictor is

$$\eta_{xt} = g(q_{x0}) + \beta_x t,$$

where the offset $g(q_{x0})$ is computed by applying the link transformation to the graduated probabilities of the base quadrennium. Denoting the parameter estimates as $\hat{\beta}_x$, the fitted values of the linear predictor are calculated as

$$\hat{\eta}_{xt} = g(q_{x0}) + \hat{\beta}_x t,$$

with standard errors

$$\text{se}(\hat{\eta}_{xt}) = \text{se}(\hat{\beta}_x)\, t.$$

Hence, on inverting the link (3.1)

$$\hat{q}_{xt} = g^{-1}(\hat{\eta}_{xt}) = g^{-1}\{g(q_{x0}) + \hat{\beta}_x t\},$$

while equation (3.2) implies that the fitted values of the mortality reduction factor are calculated as

$$\hat{\text{RF}}(x, t) = \hat{q}_{xt}/q_{x0} = g^{-1}\{g(q_{x0}) + \hat{\beta}_x t\}/q_{x0}. \qquad (4.1)$$

Hence

$$\log \hat{\text{RF}}(x, t) = \log[g^{-1}\{g(q_{x0}) + \hat{\beta}_x t\}] - \log q_{x0} \qquad (4.1a)$$

on the log scale.

By analogy with Section 4.4.6 pp 122-24 of McCullagh and Nelder (1989), under the approximate normal assumption

$$\frac{\hat{\eta}_{xt} - g(q_{xt})}{\text{se}(\hat{\eta}_{xt})} \sim \text{N}(0, 1),$$

two-sided confidence intervals with limits

$$g(q_{xt}) = \hat{\eta}_{xt} \pm z_{\alpha/2}\text{se}(\hat{\eta}_{xt}) = g(q_{x0}) + \{\hat{\beta}_x \pm z_{\alpha/2}\text{se}(\hat{\beta}_x)\}t$$

are constructed, such that $\Phi(z_\alpha) = 1 - \alpha$, where Φ is the cumulative distribution function of N(0, 1). Under equation (3.2), these map into confidence limits

$$g^{-1}\{g(q_{x0}) + (\hat{\beta}_x \pm z_{\alpha/2}\text{se}(\hat{\beta}_x))t\}/q_{x0} \qquad (4.2)$$

for the mortality reduction factor, RF(x, t), or

$$\log[g^{-1}\{g(q_{x0}) + (\hat{\beta}_x \pm z_{\alpha/2}\text{se}(\hat{\beta}_x))t\}] - \log q_{x0} \qquad (4.2a)$$

on the log scale.

It is possible to extrapolate beyond the range of the observed t- values, to t_0 say, in order to project the fitted reduction factor. Predicted values and "prediction intervals" for such extrapolations are computed by substituting t_0 into expressions (4.1) and (4.2) respectively. The dangers associated with this extrapolation process are discussed in Section 4.4.6 pp 122-24 of McCullagh and Nelder

(1989). In particular, we note the potentially hazardous nature of such an extrapolation exercise on account of the possible variation induced by the choice of link function (due to differences in their curvature), and the need to compare such projections, for each of the four link functions listed in Section 3.

Similarly, the equivalent formulae incorporating equation (3.5), rather than equation (3.4) follow on substituting

$$\hat{\beta}_x = \sum_{i=1}^{r} \sum_{j=0}^{k} \hat{\beta}_{ij}(x - x_j)_+^i$$

throughout these expressions.

5. Presentation of results

It is particularly convenient to present all our results on the log scale, so we follow this practice throughout in this paper. Specifically, given that the target of any analysis is the reduction factor RF, equation (2.1) may be rewritten as

$$\frac{q_{xt}}{q_{x0}} = \mathrm{RF}(x, t)$$

and which trivially, on taking logs, implies that

$$\log q_{xt} - \log q_{x0} = \log \mathrm{RF}(x, t). \qquad (5.1)$$

Suppose, in addition, that the data available comprise

a_{xt} the observed number of deaths at individual ages x and times $t = t_1, t_1+1, \ldots, t_{max}$
r_{xt} the matching *initial* exposures.

Retrospective studies.

In any so-called retrospective study, as in Section 6, it is envisaged that the data are available for individual calendar years immediately following a given base quadrennium, so that the data occur for positive t, with typically t_1 denoting the first calendar year subsequent to the base quadrennium. Such data give rise to crude mortality rates

$$\overset{\circ}{q}_{xt} = \frac{a_{xt}}{r_{xt}}.$$

Focusing on the LHS of equation (5.1), we represent the data graphically by plotting the values of

$$z_{xt} = \log \overset{\circ}{q}_{xt} - \log q_{x0} \qquad (5.2)$$

against t, separately, for each age x. Then, having modelled the reduction factor as described in Section 3, and noting that under this modelling process, equation (3.3) implies that

$$\log \mathrm{RF}(x, t) = \log\{g^{-1}(\eta_{x0} + \eta'_{xt})\} - \log\{g^{-1}(\eta_{x0})\},$$

we superimpose the fitted versions of these curves (typically equation (4.1a)) against the background of the plotted z_{xt} values. By this means, it is possible to make a visual assessment of the quality of fit. In addition, the logarithm of any previously proposed reduction factor that is in current use, may be superimposed on such graphs, for purposes of comparison and monitoring.

Prospective studies.

In any so-called prospective study, as in Section 7, it is envisaged that the data are available for individual calendar years immediately prior to (and including) a given base quadrennium, so that the data occur for mainly negative t, with typically $t_{max} = 2$, corresponding to the final year of the given base quadrennium. Then, having fitted a viable parameterised structure, typically as specified by equation (3.4) or equation (3.5), the emphasis is on reporting extrapolations. This is done graphically by displaying the age-specific profile of $\log \mathrm{RF}(x, t_0)$ at some future date t_0, as determined by equation (4.1a), together with the 95% "prediction intervals" determined by equation (4.2a).

6. Retrospective study: pensioner experiences

To illustrate this methodology, we consider the UK pensioner lives experiences, for both males and females, separately. The data comprise the numbers of deaths with matching initial exposures, cross-classified by age and individual calendar years, ranging respectively, from age 60 years to age 100 years, and from calendar year 1983 to year 1996. The data are analysed relative to the base quadrennium 1979-82, with the associated base probabilities q_{x0} taken from Table A4, CMI Committee (1990), *viz* male pensioners 1979-82, lives (PML80 Base) and female pensioners 1979-82, lives (PFL80 Base). In accordance with the discussion in CMI Committee (1990), the time origin is set one and a half years into the base quadrennium, at 1st July, 1980. In addition, the data collected in the subsequent individual calendar years have been located half way through the respective calendar years when defining the values of the co-variate t in the linear predictor. These values consequently range from $t_1 = 3$ (for 1983), to $t_{max} = 16$ (for 1996).

The results are conveniently presented in graphical format as described in Section 5. Thus, the data based z_{xt} values (equation (5.2)) together with the log fitted reduction factor (equation (4.1a)) are each plotted against t on the same graph, using a separate frame for each age x. Given the extensive nature of such output, we reproduce the nine graphs for the single years of age 69 to 77 only, for the purpose of illustration: see Fig. 2 for males and Fig. 3 for females. In addition, the log of the current reduction factor (equation (2.3)) has been superimposed for the purpose of comparision, with the benefit of hindsight. Such a comparison is further facilitated by the construction of approximate 95% two sided confidence intervals (equation (4.2a)). Again, given the voluminous nature of such output, we illustrate the results for just two representative ages, *viz* ages 80 and 81, in Table 1. Here, in addition to tabulating the fitted log reduction factor together with its confidence limits for each year (first 3 columns, each sex), the values of the log current reduction factor together with a zero/one indicator are also listed for comparison (next 2 columns, each sex), the index taking the value one if the current reduction factor lies within the confidence limits and the value zero otherwise.

The choice of link function, as listed in Section 3, is not material to the fitted values, for the practical purposes of this analysis. In addition, the extrapolation of the fitted reduction factors to the year 2000 ($t = 20$), results in nearly identical predictions, in all four cases. Thus typically, by way of illustration, the extrapolated values for ages 80 and 81 and each sex, together with their confidence limits, are reproduced in Table 2. It must be emphasised however, that we are here interested in observing the degree of stability in the extrapolated values due to the different curvatures in the link functions, rather than in the construction of a set of well-ordered forecasts by age. We recall from Fig. 1 the greater curvature of the log-log link in the region of small probability values. This feature, coupled with the historic roles (Forfar *et al.* (1988); Renshaw (1991)) associated with both the log odds and complementary log-log links in the graduation process, encourages us to focus on either of these two latter link functions in this type of analysis. Consequently, unless otherwise stated, all results presented in this paper assume the log odds link.

Given the importance of ordering with respect to age in the construction of mortality reduction factors, it is informative to plot the fitted parameter values $\hat{\beta}_x$ against x. This is done, for all four links and each gender, in Fig. 4, for ages 60-100. For each gender, as anticipated, the resulting patterns for all four links are effectively identical. For males, there is an underlying increasing pattern in the values of $\hat{\beta}_x$ with increasing age x, with the notable exception of ages in excess of 95. Here, the precipitous decrease with increasing age may be attributable to the inadequate quality of the raw data at these extreme ages. For females, there is a greater degree of dispersion coupled with some evidence of a decreasing trend with increasing x, in the age range 70 to 90.

With the benefit of hindsight, it is possible to comment on the effectiveness of the mortality reduction factor given by equation (2.3), on the basis of this analysis. For males, the graphs displayed in Fig. 2 are representative of all ages with very few exceptions. Further, with the exception of ages 60, 64, 94 to 96, scrutiny of all the confidence intervals (as illustrated in Table 1), confirms that the reduction factor (2.3) significantly over estimates the level of mortality rates which have eventually occurred. For females, on the bases of this analysis, the performance of the reduction factor (2.3) proves to be more satisfactory, with a significant over estimation of mortality occurring at ages 71, 77 to 91 (with the exception of 81), 94 and 96. Thus, on the basis of this analysis and with the benefit of hindsight, it is possible to conclude that the reduction factor (2.3) has severely over estimated mortality in the male pensioner experience across all ages, and in the female pensioner experience, at older ages, during the period concerned (i.e. 1983 to 1996).

It is of interest to note that historically, in their pioneering work on mortality reduction factors, the CMI Committee (pp 8-11 (1978)) proposed the reduction factor

$$\mathrm{RF}(x, t) = (0.9)^{t/20} \qquad (6.1)$$

for application to the then standard pensioners' graduated mortality tables, centred on the 1967-70 base quadrennium. This structure translates, on the log scale, into a single line radiating from the time origin and matches the form proposed by the GAVT Task Force (1995) for US annuities and presented as equation (2.4). As subsequently noted by the CMI Committee (1990), apart from any other consideration, the long term asymptotic implications of a structure like (6.1) is untenable as $t \to \infty$. Hence, the CMI Committee subsequently modified its recommendations to reduction factors of the type (2.3), which tend to the non-zero limit $\alpha(x)$, in the long-run. However, given the small amount of curvature in the short term (on the log scale, say) associated with equation (2.3), coupled with the intrinsically speculative nature of $\alpha(x)$, we feel justified in fitting the pencil of lines (3.4), (without going further and imposing an additional quadratic term in t, say) for the limited short term span of this investigation. Note that the extent of the short-term curvature in $\log \mathrm{RF}(x,\, t)$ is captured visually in Fig. 2 & Fig. 3. Further, it may be quantified by computing the second partial derivative of $\log \mathrm{RF}(x,\, t)$ with respect to t, the details of which are omitted.

7. Prospective study: pensioner experiences

In the preceding case study (Section 6), the emphasis was on modelling with the benefit of hindsight. However, the methodology can be brought into phase with the periodic recalibration of standard actuarial mortality tables by graduation and used for extrapolation rather than monitoring purposes. In this case, a different prospective setting of the methodology is required. For example, the graduated pensioners lives mortality tables (PML80 and PFL80) for the 1979-82 quadrennium, together with the reduction factor (2.3), both used in Section 6, have recently been updated (CMI Committee (1999)) by the production of the graduated mortality tables (PML92 and PFL92), for the 1991-94 quadrennium, and the following accompanying mortality reduction factors have been proposed for extrapolation purposes (Section 6, pp 89-108, CMI Committee (1999)):

$$\mathrm{RF}(x,\, t) = \alpha(x) + \{1 - \alpha(x)\}\{1 - f(x)\}^{t/20} \qquad (7.1)$$

with

$$\alpha(x) = 0.13; \qquad x < 60$$

$$\alpha(x) = 1 + 0.87\frac{(x - 110)}{50}; \qquad 60 \leq x \leq 110$$

and

$$f(x) = 0.55; \qquad x < 60$$

$$f(x) = \frac{0.55(110 - x) + 0.29(x - 60)}{50}. \qquad 60 \leq x \leq 110$$

These are designed with reference to $t = 20$ and with $t = 0$ mapping to the centre of the quadrennium 1991-94, and are intended to apply to males and females (and to the pensioners' and annuitants' experiences).

It is possible to move exactly into phase with this recalibration process by setting the time origin $t = 0$ at 1st July, 1992, and weighting out any available data subsequent to 1994. That said, the observations for 1995 and 1996 may also be utilised as early indicators of the effectiveness of both the assumptions and the current methodology. Thus, in the model fitting stage of this study, the data range from t = -9 (for 1983) to $t_{max} = 2$ (for 1994), cross-classified by age, ranging from 60 to 100.

In Fig. 5 & Fig. 6, sets of deviance residuals (standarised and Studentised: see Section 11.6.3.5 p 285, Francis *et al* (1993)) generated on fitting the 'pencil of lines' structure (3.4), for the respective male and female pensioners' experiences, are reproduced. In each figure, the left and right hand frames depict residuals plotted against time and age respectively. In the lower frames, the residuals are plotted against an index which ensures that the residuals are presented for each age serially arranged by time (left hand frame), or for each period serially arranged by age (right hand frame), as the case may be. In essence, the lower frames depict a 'stretching' of the localised detail of the upper frames, and consequently complement the corresponding upper frames. Given the detailed nature of these plots, the figures provide ample supportive evidence that the model structure has essentially captured the underlying trends inherent in both data sets. (We do not have an explanation for the outlier revealed by the male pensioner residual plots, at age 80 in year 1993. However, we have verified that the total deaths and total exposures in our data set, for 1991-94 at age 80, as well as for all other ages, are in agreement with the published figures of Table 16 p 135 of CMI Committee (1998)).

Turning next to the extrapolation of the fitted structures, it is again appropriate to follow the CMI Committee's practice and focus on $t = 20$, which implies the year 2012. As an important side issue, in parallel with the previous case study (Section 6), Table 3 may be consulted to ascertain the typical magnitudes of the differences in extrapolated values at ages 80 and 81 induced by the different link fuctions. We again find such evidence reassuring, and draw the same conclusions as before about the sensitivity, or rather the comparative lack of sensitivity, of the extrapolated values to the choice of link function.

It is informative to compare the extrapolated values to the year 2012, under this method, with the contemporary values predicted by the mortality reduction factors, given by equation (7.1). An effective way of doing this is graphically, as in Fig. 7, in which the values predicted by this method, together with their 95% confidence intervals, are plotted against age, and values predicted by formula (7.1) superimposed, as the continuous curve. This is done in separate frames, for each gender.

However, if the methodology is to be used for forecasting rather than monitoring purposes, it is necessary to smooth such model extrapolations with respect to age. It is possible to do this by switching from formula (3.4) to formula (3.5). To illustrate this, the upper frame in Fig. 8 (males) and Fig. 9 (females) depict the smoothed extrapolations of Fig. 7, based on a cubic spline (equation (3.5) with $s = 3$, and $\beta_{ij} = 0$, $i = 1, 2$, (all j)), together with the resulting 95% prediction intervals and the CMI predictions of formula (7.1). Here, the initial knot has been set at the origin, so that $x_0 = 0$, with further knots set routinely at 5 yearly ages 60, 65, ..., 95. This is equivalent to setting

$$\beta_x = \beta_0 x^3 + \sum_{j=1}^{8} \beta_j (x - (55 + 5j)_+^3)$$

in formula (3.4), prior to fitting. To further augment the implications of this smoothing process, we depict, in the respective lower frames of Fig. 8 & Fig. 9 (and again on the log scale), the graduated 1991-94 base quadrennium probabilities (PML92 or PFL92 as the case may be), together with their 20 year adjusted values using both the modelled reduction factor and the CMI reduction factor (7.1) for comparison purposes.

Thus, it is possible to conclude, on the basis of this analysis, that for the male experience, there is only statistical evidence of agreement between the mortality forecasts based on the projection of patterns present in the data (at the time the base probabilities were established), and the reduction factor (7.1), for ages in excess of 80. For ages in the range 65 to 80, there is evidence that the reduction factor (7.1) under states the likely level of projected mortality rates. For the female experience, there is the clear suggestion that, relative to the projections based on these models, equation (7.1) leads to projections with an inappropriate age-specific profile.

It is obviously important to appreciate that the extrapolated values arrived at in any such prospective study are dependent on two key assumptions, *viz* that relatively simple time trends in the data are identified and modelled, coupled with the belief that such trends continue into the future. While the first of these assumptions is verifiable, the second assumption amounts to an act of faith.

8. Case study: annuitant experiences

To investigate further aspects of the methodology, we turn to the male and female UK annuitants' experience (duration 1+), utilising data cross-classified by calendar year, from 1967 to 1994 inclusive (but with 1968, 1971 & 1975 missing), and by single year of age, from 60 to 100 years inclusive. The base quadrennium 1979-82 is chosen throughout, while the study comprises two related stages. The first stage is a prospective study, based on data up to and including the year 1982 only, in which, for the purposes of illustration, 20 year forecasts to the year 2000 are made. Then, in a "follow-up" stage, we make use of the complete data set to generate updated forecasts for the year 2000 in order to gain insight, with the benefit of hindsight, into the former set of forecasts. The base probabilities q_{x0} (IM80 Base & IF80 Base) are taken from Table A5, CMI Committee (1990), and the time origin $t = 0$ set accordingly at 1st July 1980. Hence $t = 20$ denotes the year 2000. The mortality reduction factor (2.3) is relevant to this case study and we again focus on fitting pencils of straight lines, as given by equation (3.4).

The results are presented graphically in Fig. 10 for male annuitants and Fig. 11 for female annuitants. The upper and lower frames in each figure refer respectively to the first and second stage modelling processes, described above. Thus the plotted points in the upper frames, together with their 95% confidence limits, are generated by fitting the structure (3.4) to the restricted data set, ranging from 1967 (t = -13) to 1982 (t = 2), and extrapolating to t = 20. Similarly the plotted points in the lower frames are generated by fitting the structure (3.4) to the complete data set, ranging from 1967 (t = -13) to 1994 (t = 14). In each frame, we have additionally superimposed the log(reduction factor) values determined by formula (2.3) when t = 20, and which appears as the continuous line. For reasons of economy, we reproduce the residual plots for the second stage modelling only in Fig. 12 for males and Fig. 13 for females, although the residual plots for the first stage modelling process have also been examined, and the patterns are observed to be similar in character.

When interpreting these results, we note, in particular, that the prediction intervals are wider in the upper frames compared with their counterparts in the lower frames, for the obvious reason that forecasts are made further into the future in the upper case, than in the lower case, and consequently induce greater uncertainty. On comparing the upper frames of Fig. 7 and Fig. 10, the (20 year) prediction intervals for the male pensioner experience of Fig. 7, are markedly narrower than their counter parts for the male annuitants experience of Fig. 10. However, this is not a feature when comparing the female pensioner and annuitants experiences of Fig. 7 and Fig. 11. The likely expanation of this feature can be attributed to the different quantities of data available for the various experiences. Thus, for example, the total deaths for 1991-1994, across all the ages, for the various experiences are as follows:

male pensioners lives	63,614
female pensioners lives	13,272
male annuitants lives, duration $\geq$1	2,886
female annuitants lives, duration $\geq$1	5,863

with the male comparison bringing out the sharpest distinction. By comparing the upper and lower frames of Fig. 10 and the upper and lower frames of Fig. 11, it is possible to observe how the forecast up to the year 2000 changes under this analysis as a result of acquiring the additional data for the intervening years. Obviously, no general conclusions can be drawn from any one such specific comparison. It is possible to conclude, however, that, for the period in question and on the basis of this analysis, the CMI reduction factor (2.3) has been particularly successful at predicting mortality trends for annuitants. Further, it is possible to confirm this conclusion, especially so for male annuitants, by conducting a retrospective study of the type described in Section 6, the details of which are not reproduced.

9. Reduction factors for the force of mortality

In the preceding sections, the focus has been on reduction factors for the probability of death. In this section the methodology is extended to reduction factors for the force of mortality.

Let

μ_{xt} denote the force of mortality at age x and time $t \geq 0$

and, by analogy with equation (2.1), define a reduction factor RF such that

$$\mu_{xt} = \mu_{x0}\,\mathrm{RF}(x,t), \quad t \geq 0,$$

subject to the constraint (2.2). Hence

$$\log \mu_{xt} - \log \mu_{x0} = \log \mathrm{RF}(x,t). \tag{9.1}$$

Suppose μ_{x0}, centred on the base quadrennium, is determined by graduation, using the actual deaths and matching *central* exposures of the quadrennium. Again let (a_{xt}, r_{xt}) denote the data subsequent to the base quadrennium, but where this time

r_{xt} denotes the *central* exposure at individual ages x and times t_1, t_1+1, ..., t_{max}.

Let $A_{xt} \sim \mathrm{Poi}(r_{xt}\mu_{xt})$ and define a GLM with independent responses, which may be written as

$$Y_{xt} = \frac{A_{xt}}{r_{xt}}$$

where

$$\mathrm{E}(Y_{xt}) = \mu_{xt},\ \mathrm{Var}(Y_{xt}) = \phi\frac{\mu_{xt}}{r_{xt}},$$

and the deterministic predictor-link relationship

$$\eta_{xt} = g(\mu_{xt}) \ \Leftrightarrow\ \mu_{xt} = g^{-1}(\eta_{xt}).$$

Under the additive decomposition of the linear predictor

$$\eta_{xt} = \eta_{x0} + \eta'_{xt} \text{ such that } \eta'_{x0} = 0,$$

the reduction factor

$$\mathrm{RF}(x,t) \ = \ \frac{\mu_{xt}}{\mu_{x0}} \ = \ \frac{g^{-1}(\eta_{x0} + \eta'_{xt})}{\mu_{x0}},$$

and the constraint (2.2) implies that

$$\mu_{x0} = g^{-1}(\eta_{x0}) \Leftrightarrow \eta_{x0} = g(\mu_{x0}).$$

Hence, equations (3.3) and (3.4) continue to apply, together with the detailed content of Section 4 and Section 5, subject to the replacement of q_{xt} by μ_{xt} and initial exposures by central exposures. Specific link functions of potential interest include

link function	*link g*	*inverse link* g^{-1}
log	$\eta_{xt} = \log \mu_{xt}$	$\mu_{xt} = \exp \eta_{xt}$
power (pre-specified γ)	$\eta_{xt} = \mu_{xt}^{\gamma}$	$\mu_{xt} = (\eta_{xt})^{1/\gamma}$

10. Discussion

Setting aside the methods for fitting and extrapolation (forecasting), there are some interesting parallels between the model structures in this paper and the model structures used for forecasting in the so-called Lee-Carter method (Lee and Carter (1992), Lee (2000)). Subject to the various modifications (extensions) discussed in Lee (2000), the core model structure (with the additive error term omitted) of the Lee-Carter method may be written as

$$\log m_{xt} = a_x + b_x k_t$$

where

m_{xt}	denotes the central death rate for age x at time t
a_x	describes the shape of the age profile, averaged over time
b_x	describes the pattern of deviations from the age pattern
k_t	describes the variation in the level of mortality with time t.

Then, given that a_x is typically taken to be the average of $\log m_{xt}$ over t for each x, *viz*

$$a_x = \frac{1}{h}\sum_{t=1}^{h} \log m_{xt},$$

the structure can be rewritten as

$$\log m_{xt} - \log\left(\prod_{t=1}^{h} m_{xt}\right)^{1/h} = b_x k_t. \qquad (10.1)$$

Comparison of this equation with equation equation (5.1) implies that

$$\exp(b_x k_t) \qquad (10.2)$$

may be interpreted as a type of mortality adjustment (or reduction) factor. Although different, the second term on the LHS of equation (9.1) (or equation (5.1)) and equation (10.1), involving the force of mortality μ_{x0} (or q_{x0}) and the geometric mean of the central death rate $(\prod_{t=1}^{h} m_{xt})^{1/h}$ play the same role: setting the basic shape of the age profile, in these respective cases. Further, for the log link (Section 9), equation (3.3) and equation (3.4) give rise to the reduction factor

$$\exp \eta'_{xt} = \exp \beta_x t \qquad (10.3)$$

which bares direct comparison with expression (10.2). As Lee (2000) reports, most applications of the Lee-Carter method have found that k_t can be successfully represented as a random walk with drift *viz*

$$k_t = c + k_{t-1} + e_t. \qquad (10.4)$$

The combination of (10.2) and (10.4) then leads to each age specific mortality rate changing at a constant exponential rate, as in (10.3). The Lee-Carter method allows more complex time series based models of k_t to be incorporated, in which case the pattern of secular change would be different from that represented by our approach as in (10.3).

We note also that, in particular, while expression (10.2) is a special case of the LHS of equation (10.3) (provided $k_0 = 0$), the RHS of equation (10.3) is a special case of expression (10.2). The other reduction factors described in this paper, may be viewed as special cases of the various transformations of expression (10.2). Thus, for the log-log link (Section 3), where the transformation is mathematically tractable, equation (3.3) and equation (3.4) lead to the reduction factor

$$\exp[-\exp(-\eta_{x0})\{\exp(-\beta_x t) - 1\}].$$

11. Summary

It is instructive to summarise the various forecasting processes encountered in this paper, see Fig.

14. For each case (process), time flows from left to right, with separate representations depicting the various phases of each process. Thus a base quadrennium is depicted by a rectangle, the siting (notional or actual) of an age profile (Lee (2000)) by a circle, the data used for model fitting by a continuous line, and the resulting forecasts depicted by a pair of diverging lines. Thus Case I represents a typical cycle in the recalibration of standard UK mortality tables, as currently practised by the CMI Committee. Here, each cycle involves the graduation of the crude mortality rates of a base quadrennium, with the subsequent forecasts generated by the application of a nominated mortality reduction factor. Cases II, III and IV represent the respective retrospective, prospective and second stage studies of Sections 6, 7 and 8. Case III also represents the first stage study of Section 8. Case V represents the general method of forecasting mortality rates described in Lee (2000), in which the base age profile is set by averaging over the available data, as part of the fitting process. Case VI represents a suggested (Lee (2000)) variation on this process, in which the base age profile relates to the single, most recent year, for which data are available.

By formulating the methodology within the GLM framework, it is possible to select a modelling distribution which accords with UK actuarial graduation practice, as described in Forfar *et al.* (1989). Thus, when constructing mortality reduction factors for adjusting the probability of death q_{x0}, we have selected the binomial modelling distribution. The methodology depends on establishing a (verifiable) linear relationship over time in the linear predictor and extrapolating this to obtain forecasts. The linear relationship in time may be established for each age separately, or established for a smoothed age specific profile. It is envisaged that the data available for modelling range over some 15 to 20 consecutive years and that forecasts are made over a similar time span into the future, thereby conforming to UK actuarial practice. In all three case studies presented in this paper, the individual years of age range over a 40 year span, from age 60 to age 100.

Acknowledgement

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References

Benjamin, B. and J.H. Pollard (1993). *The Analysis of Mortality and Other Actuarial Statistics.* Institute and Factulty of Actuaries, Oxford.

CMI Committee (1978). Proposed standard tables for life office pensioners and annuitants. *Continuous Mortality Investigation Reports*, Institute and Faculty of Actuaries, 3, 1-130

CMI Committee (1990). Standard tables of mortality based on the 1979-82 experiences. *Continuous Mortality Investigation Reports*, Institute and Faculty of Actuaries, 10, 1-138

CMI Committee (1998). *Continuous Mortality Investigation Reports*, Institute and Faculty of Actuaries, 16, 135

CMI Committee (1999). Standard tables of mortality based on the 1991-94 experiences. *Continuous Mortality Investigation Reports*, Institute and Faculty of Actuaries, 17, 1-227

Francis, B., M, Green and C. Payne (1993). *GLIM4 The Statistical System for Generalised Linear Interactive Modelling.* Clarendon Press, Oxford.

Forfar, D.O., J.J. McCutcheon and A.D. Wilkie (1988). On graduation by mathematical formula. *Journal of the Institute of Actuaries*, 115, 1-135.

Haberman, S. and A.E. Renshaw (1999). A simple graphical method for the comparison of tow mortality experiences. *Applied Stochastic Models in Business and Industry*, 15, 333-352.

Group Annuity Valuation Table Task Force (1995). 1994 Group annuity mortality and reserving tables. *Transactions of the Society of Actuaries*, 47, 865-913.

Lee, R.D. (2000). The Lee-Carter method for forecasting mortality, with various extensions and applications. *North American Actuarial Journal* (with discussion), 4, no 1, 80-93.

Lee, R.D. and L. Carter. (1992). Modeling and forecasting the time series of U.S. mortality. *Journal of the Americal Statistical Association* (with discussion), 87, 659-71.

McCullagh, P. and J.A. Nelder (1989). *Generalised Linear Models.* Chapman and Hall.

Renshaw, A.E. (1991). Actuarial graduation practice and generalised linear and non-linear models. *Journal of the Institute of Actuaries*, 118, 295-312.

Renshaw, A.E. and S. Haberman (1997). Dual modelling and select mortality. *Insurance: Mathematics and Economics*, 19, 105-126.

Table 1

Confidence limits, log scale

age 80	*male pensioners*					*female pensioners*				
	lower limit	*pred't'd value*	*upper limit*	*current value*	*index*	*lower limit*	*pred't'd value*	*upper limit*	*current value*	*index*
1983	-0.075	-0.070	-0.065	-0.039	0	-0.062	-0.050	-0.038	-0.039	1
1984	-0.100	-0.093	-0.087	-0.052	0	-0.083	-0.067	-0.050	-0.052	1
1985	-0.125	-0.117	-0.109	-0.063	0	-0.104	-0.084	-0.063	-0.063	1
1986	-0.150	-0.140	-0.131	-0.075	0	-0.125	-0.100	-0.076	-0.075	0
1987	-0.175	-0.164	-0.153	-0.086	0	-0.146	-0.117	-0.088	-0.086	0
1988	-0.201	-0.188	-0.175	-0.097	0	-0.167	-0.134	-0.101	-0.097	0
1989	-0.226	-0.211	-0.197	-0.107	0	-0.188	-0.151	-0.114	-0.107	0
1990	-0.251	-0.235	-0.219	-0.117	0	-0.209	-0.168	-0.127	-0.117	0
1991	-0.277	-0.259	-0.241	-0.126	0	-0.230	-0.184	-0.139	-0.126	0
1992	-0.302	-0.283	-0.263	-0.136	0	-0.251	-0.201	-0.152	-0.136	0
1993	-0.328	-0.307	-0.285	-0.145	0	-0.272	-0.218	-0.165	-0.145	0
1994	-0.354	-0.331	-0.307	-0.153	0	-0.293	-0.235	-0.177	-0.153	0
1995	-0.379	-0.355	-0.330	-0.161	0	-0.314	-0.252	-0.190	-0.161	0
1996	-0.405	-0.379	-0.352	-0.169	0	-0.335	-0.269	-0.203	-0.169	0

age 81	*male pensioners*					*female pensioners*				
	lower limit	*pred't'd value*	*upper limit*	*current value*	*index*	*lower limit*	*pred't'd value*	*upper limit*	*current value*	*index*
1983	-0.064	-0.059	-0.054	-0.038	0	-0.053	-0.041	-0.029	-0.038	1
1984	-0.085	-0.079	-0.073	-0.050	0	-0.070	-0.055	-0.039	-0.050	1
1985	-0.106	-0.099	-0.091	-0.061	0	-0.088	-0.068	-0.049	-0.061	1
1986	-0.128	-0.118	-0.109	-0.072	0	-0.105	-0.082	-0.059	-0.072	1
1987	-0.149	-0.138	-0.127	-0.083	0	-0.123	-0.096	-0.068	-0.083	1
1988	-0.171	-0.158	-0.146	-0.093	0	-0.141	-0.109	-0.078	-0.093	1
1989	-0.193	-0.178	-0.164	-0.103	0	-0.158	-0.123	-0.088	-0.103	1
1990	-0.214	-0.198	-0.182	-0.113	0	-0.176	-0.137	-0.098	-0.113	1
1991	-0.236	-0.218	-0.201	-0.122	0	-0.194	-0.151	-0.108	-0.122	1
1992	-0.258	-0.239	-0.219	-0.131	0	-0.212	-0.164	-0.117	-0.131	1
1993	-0.280	-0.259	-0.238	-0.139	0	-0.229	-0.178	-0.127	-0.139	1
1994	-0.301	-0.279	-0.256	-0.148	0	-0.247	-0.192	-0.137	-0.148	1
1995	-0.323	-0.299	-0.275	-0.156	0	-0.265	-0.206	-0.147	-0.156	1
1996	-0.345	-0.319	-0.294	-0.163	0	-0.283	-0.220	-0.157	-0.163	1

Table 2

Extrapolated predictions for the year 2000 with confidence limits, log scale

	male pensioners						*female pensioners*					
	age 80			*age 81*			*age 80*			*age 81*		
link	*lower limit*	*pr't'd value*	*upper limit*	*lower limit*	*pr'd't value*	*upper limit*	*lower limit*	*pr'd't value*	*upper limit*	*lower limit*	*pr'd't value*	*upper limit*
log-odds	-.509	-.475	-.442	-.433	-.401	-.368	-.420	-.337	-.254	-.354	-.275	-.196
c log-log	-.506	-.473	-.440	-.431	-.399	-.367	-.419	-.336	-.254	-.354	-.275	-.196
probit	-.517	-.483	-.448	-.440	-.406	-.373	-.425	-.340	-.256	-.358	-.277	-.197
log-log	-.526	-.490	-.455	-.446	-.412	-.378	-.430	-.343	-.258	-.362	-.279	-.198

Table 3
Extrapolated predictions for the year 2012 with confidence limits, log scale

	male pensioners						*female pensioners*					
		age 80			*age 81*			*age 80*			*age 81*	
link	*lower limit*	*pr't'd value*	*upper limit*	*lower limit*	*pr'd't value*	*upper limit*	*lower limit*	*pr'd't value*	*upper limit*	*lower limit*	*pr'd't value*	*upper limit*
log-odds	-.298	-.231	-.164	-.336	-.267	-.198	-.090	.093	.273	-.198	-.018	.161
c log-log	-.296	-.229	-.163	-.332	-.264	-.197	-.090	.093	.274	-.197	-.018	.161
probit	-.308	-.237	-.167	-.348	-.275	-.203	-.090	.092	.266	-.201	-.018	.159
log-log	-.319	-.243	-.170	-.362	-.283	-.207	-.090	.091	.260	-.204	-.018	.157

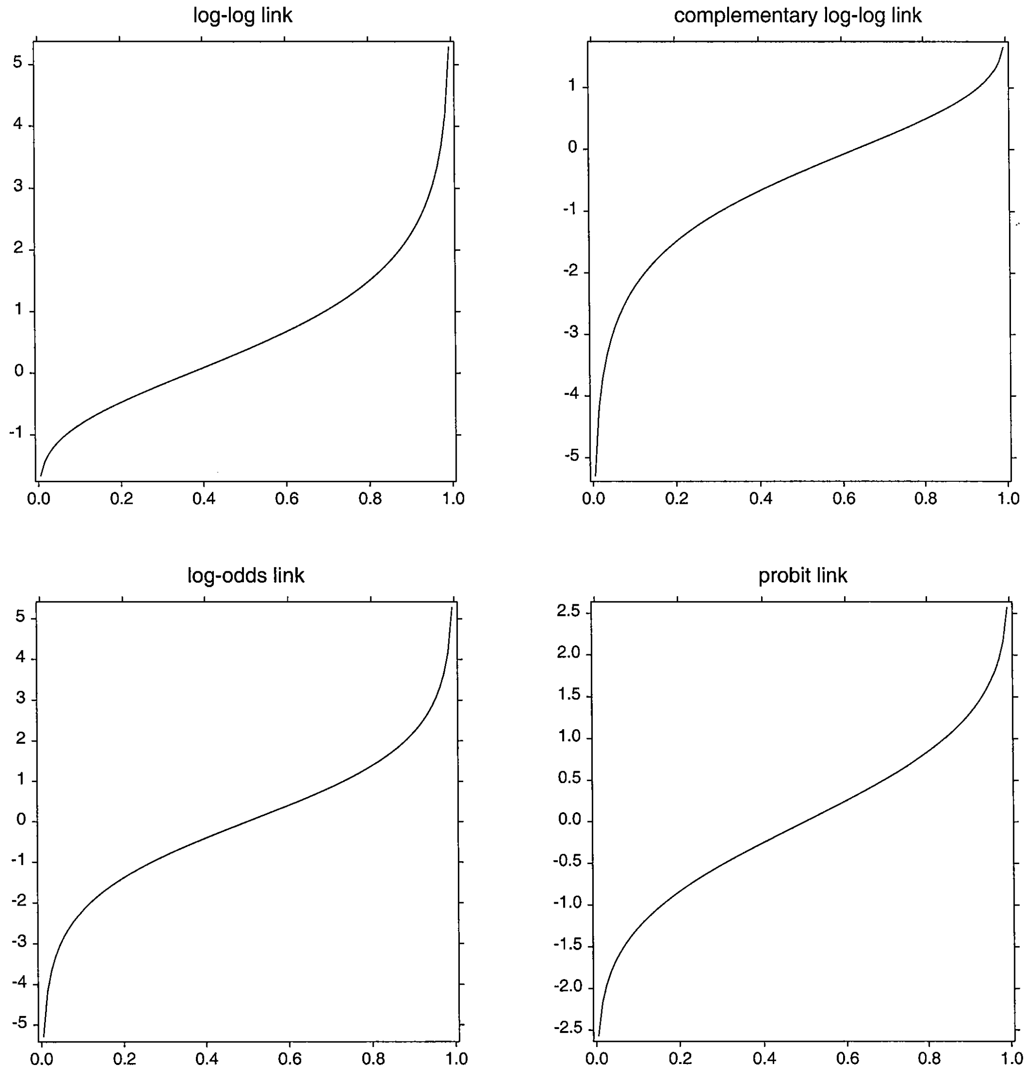


Fig. 1. Link functions $g(q)$ plotted against q

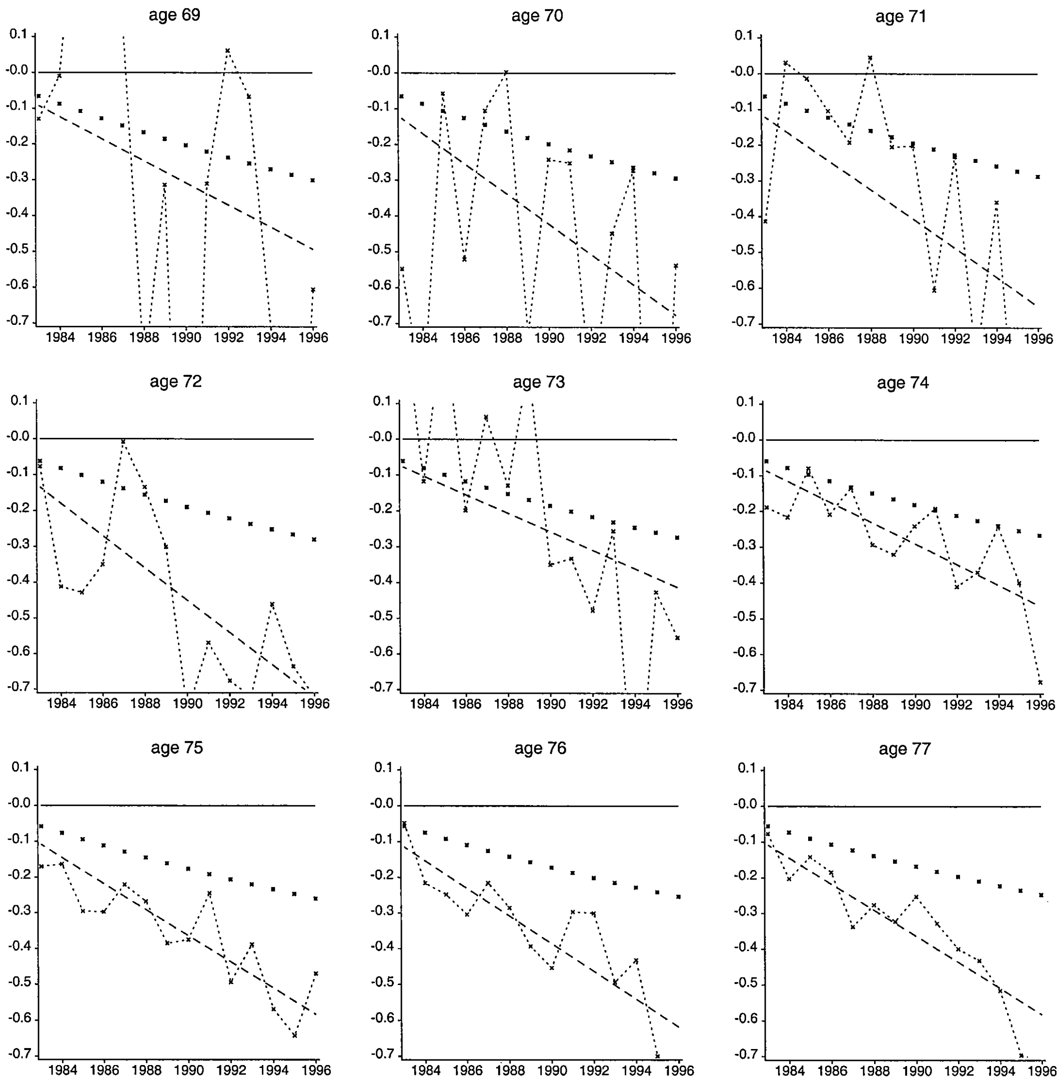


Fig. 2. Male pensioner experience: log(RF) plots against year, each age

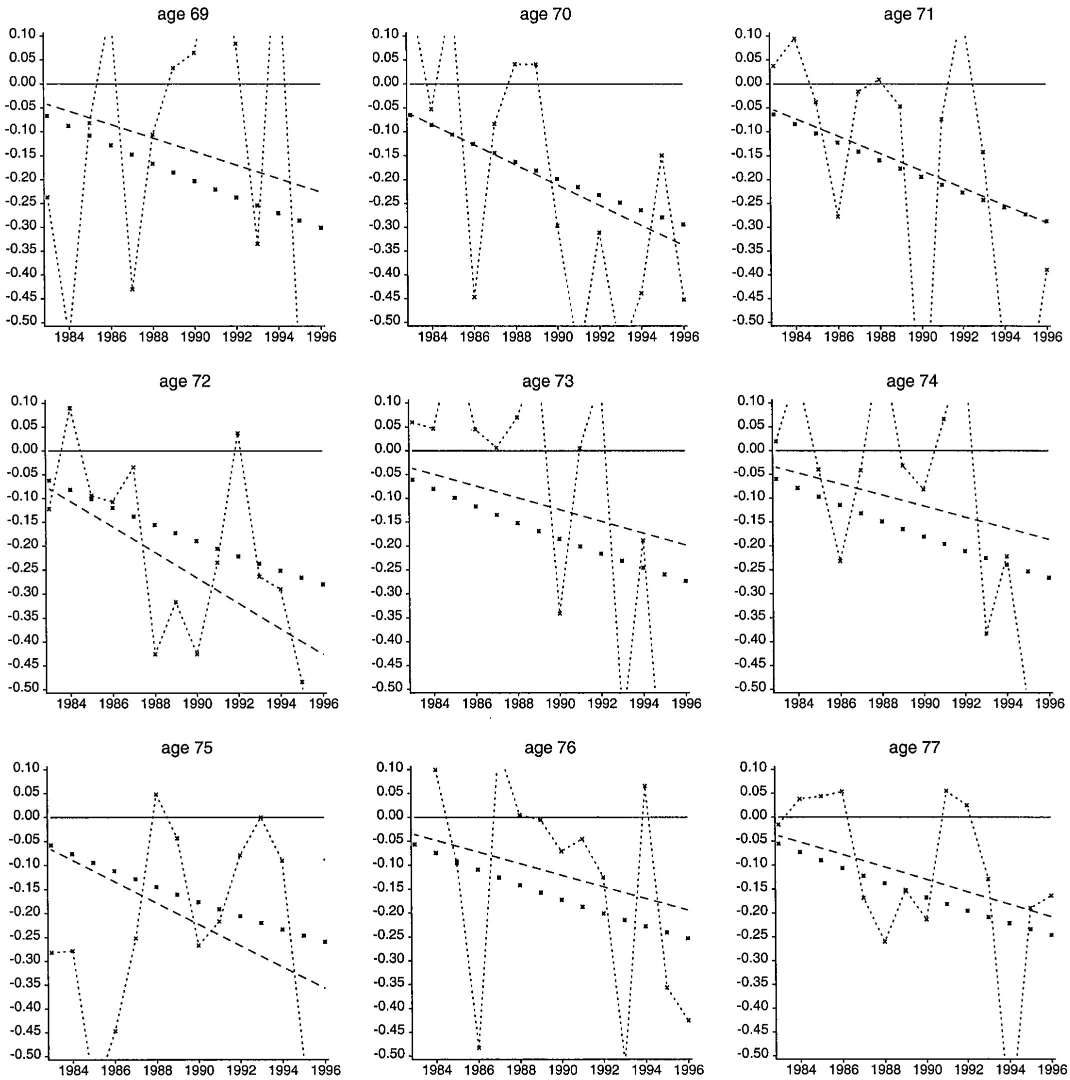


Fig. 3. Female pensioner experience: log(RF) plots against year, each age

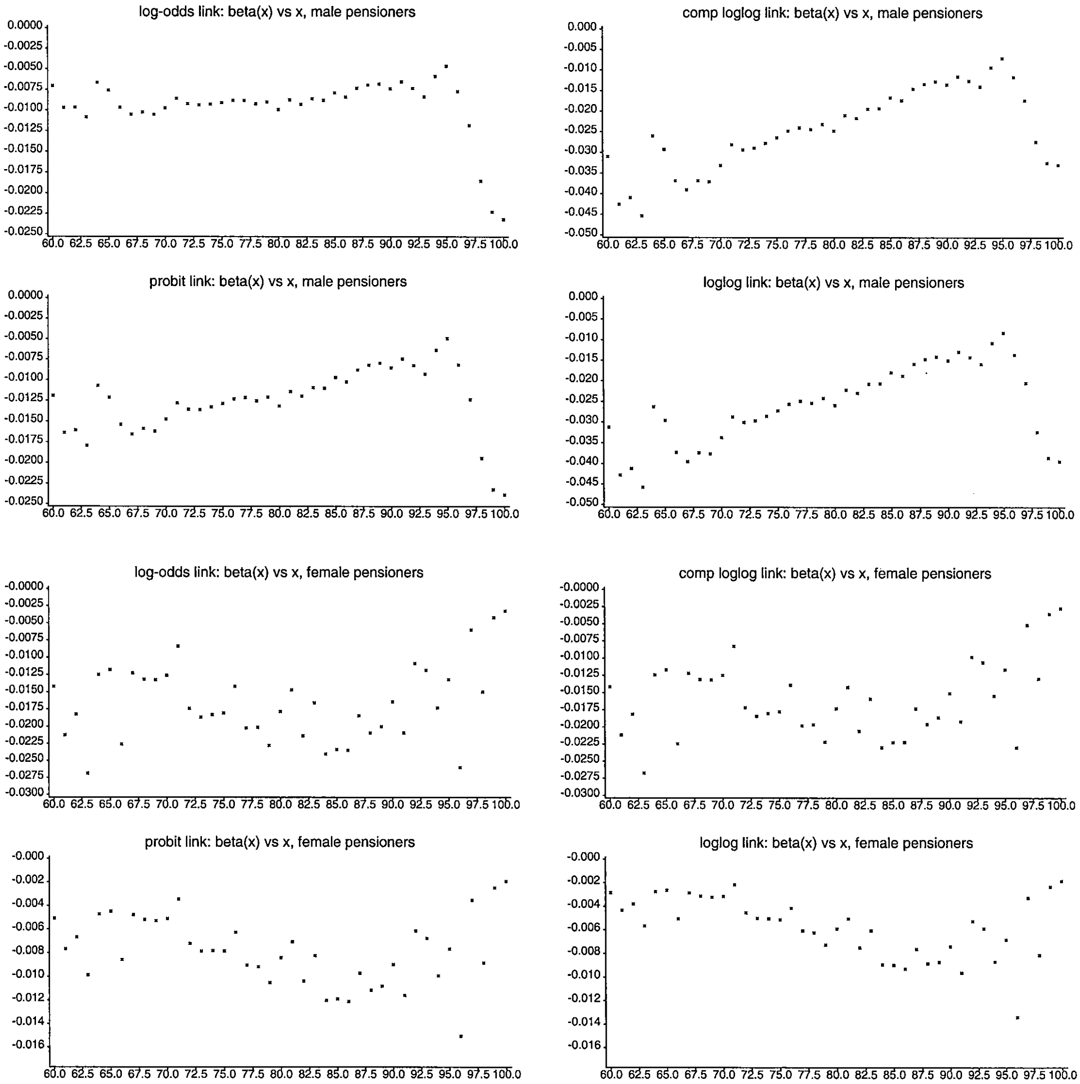


Fig. 4. Pensioner experinces: beta parameter plotted against age, various links, each sex

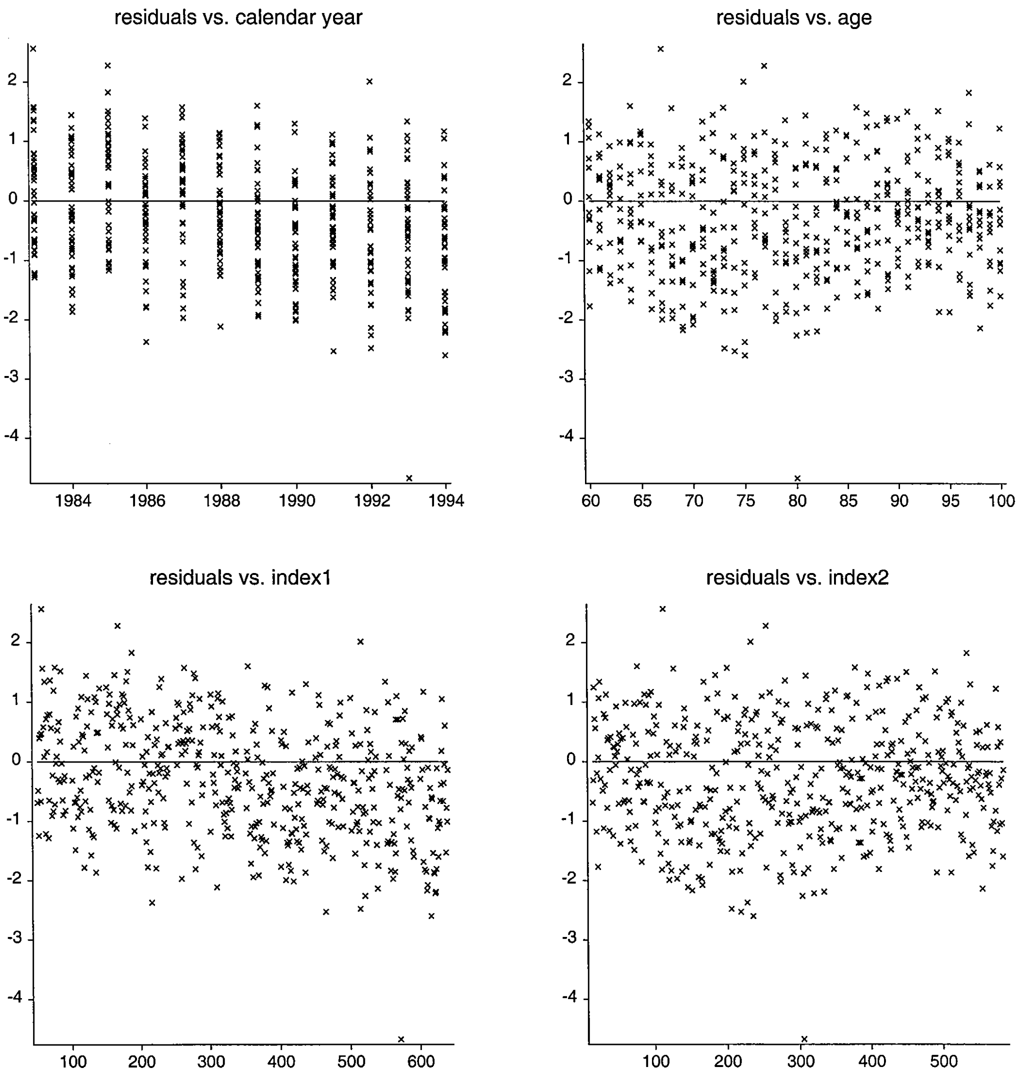


Fig. 5. Male pensioner experience: residual plots

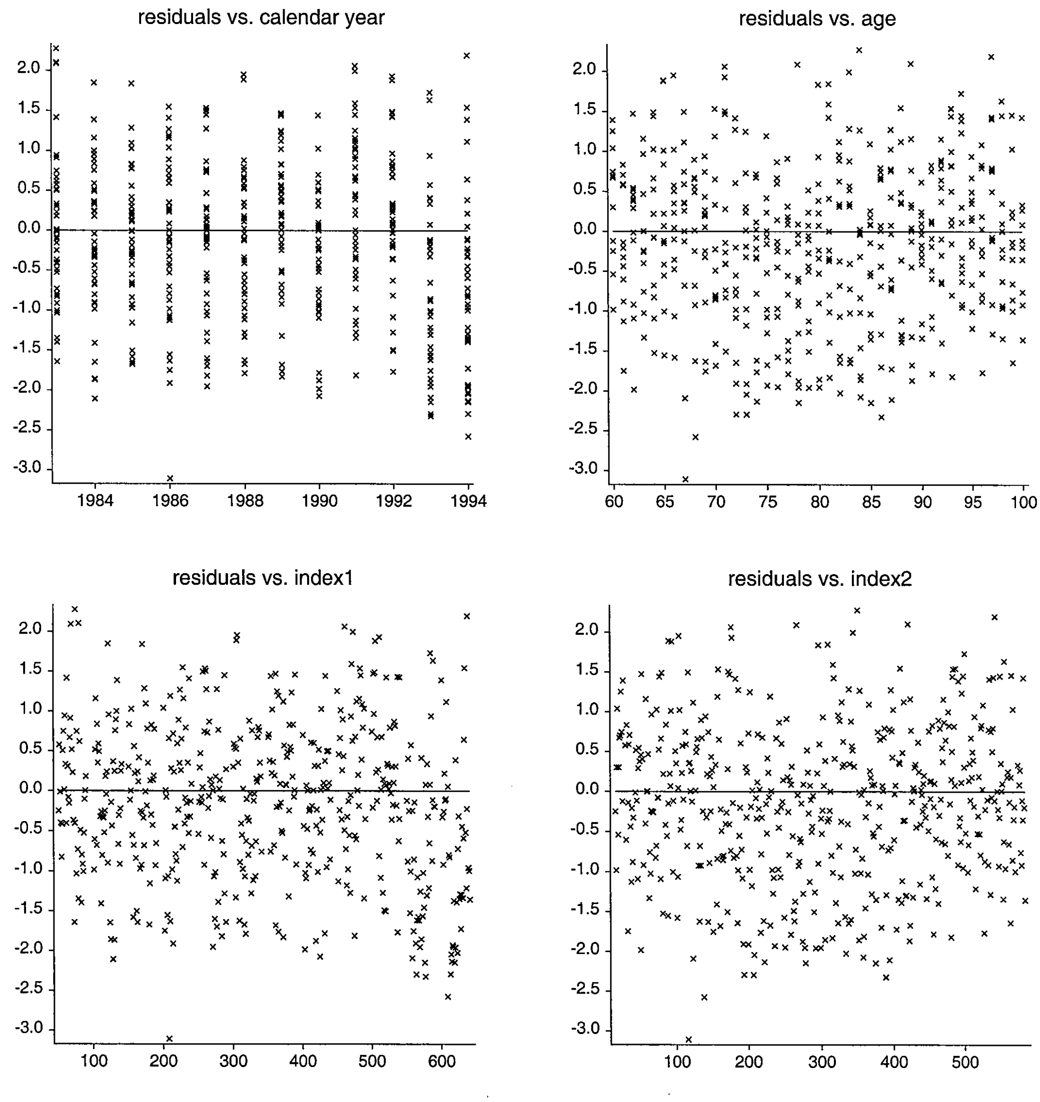


Fig. 6. Female pensioner experience: residual plots

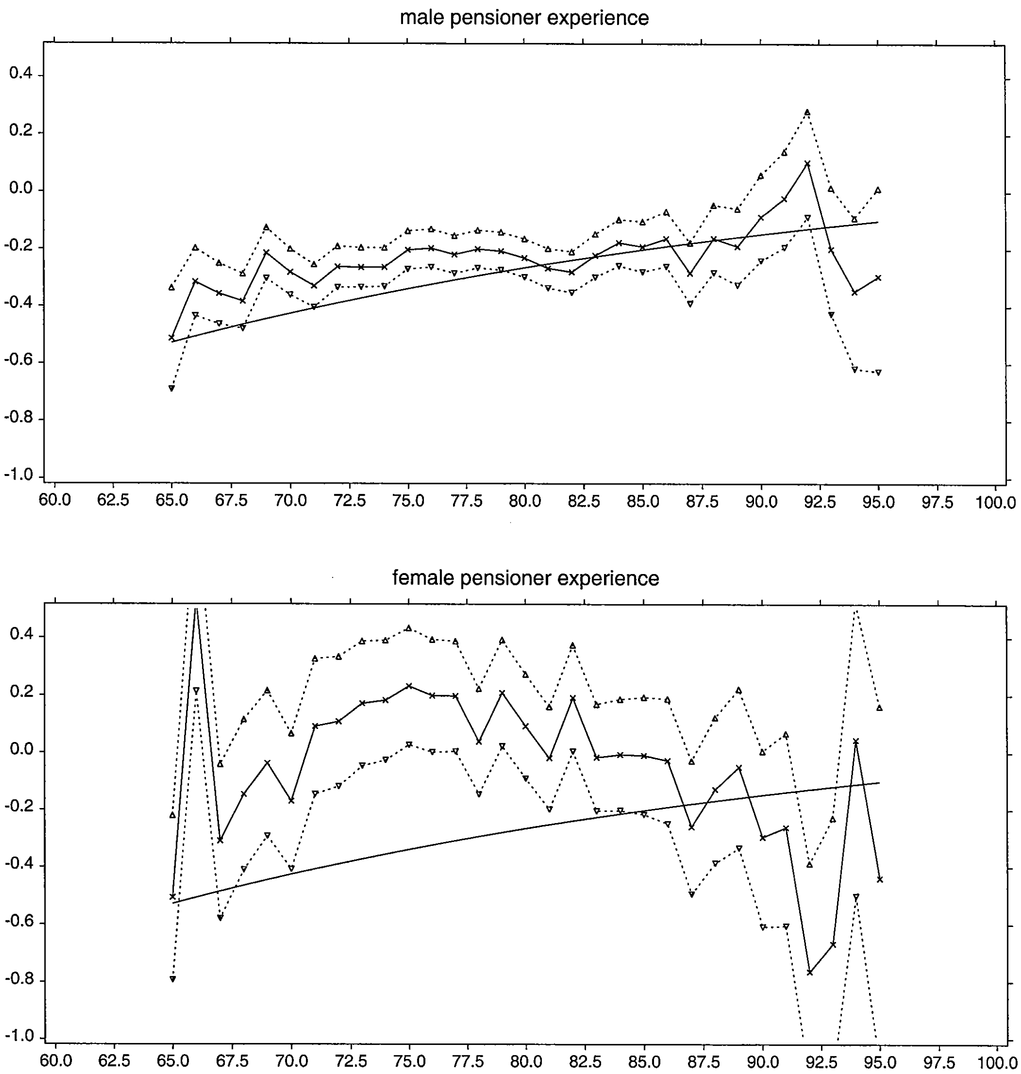


Fig. 7. Pensioner experiences: reduction factor predictions

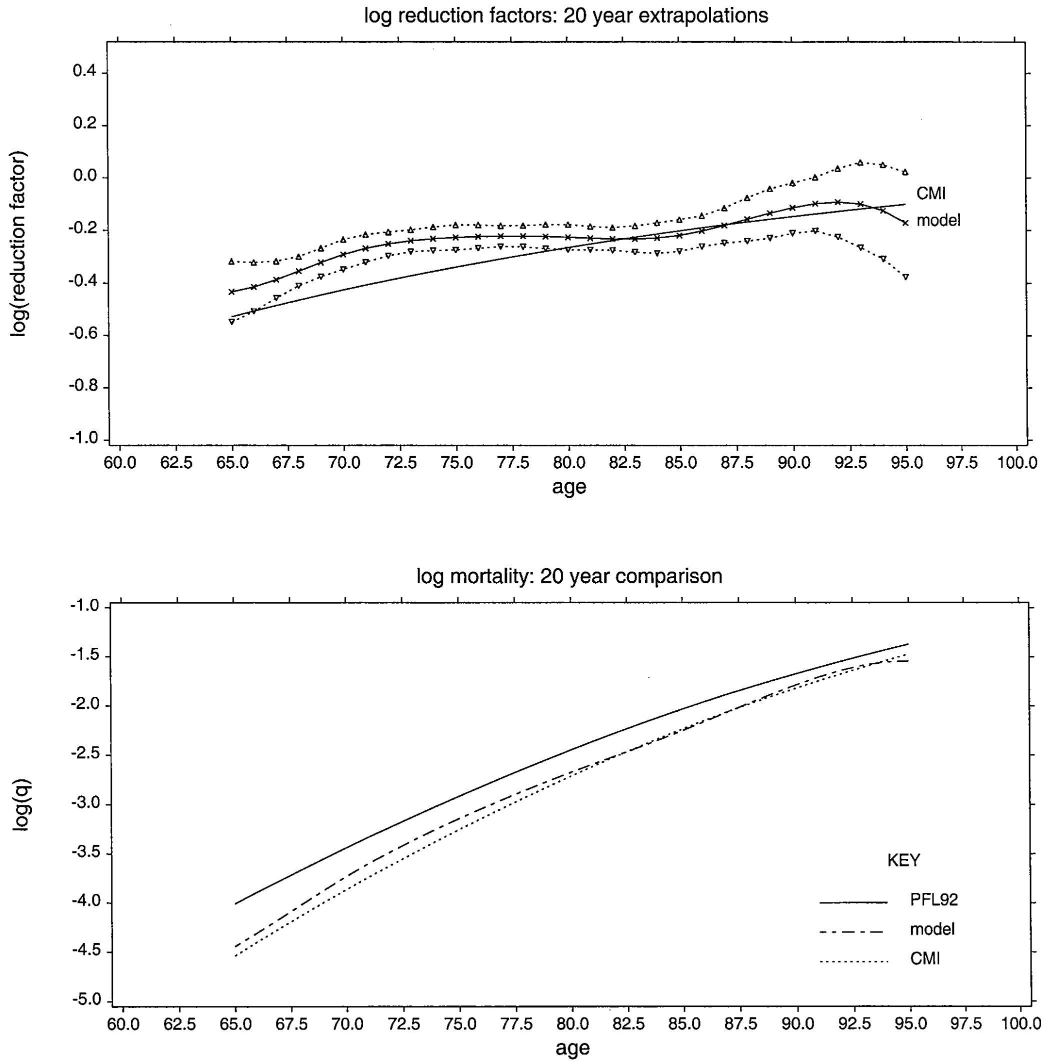
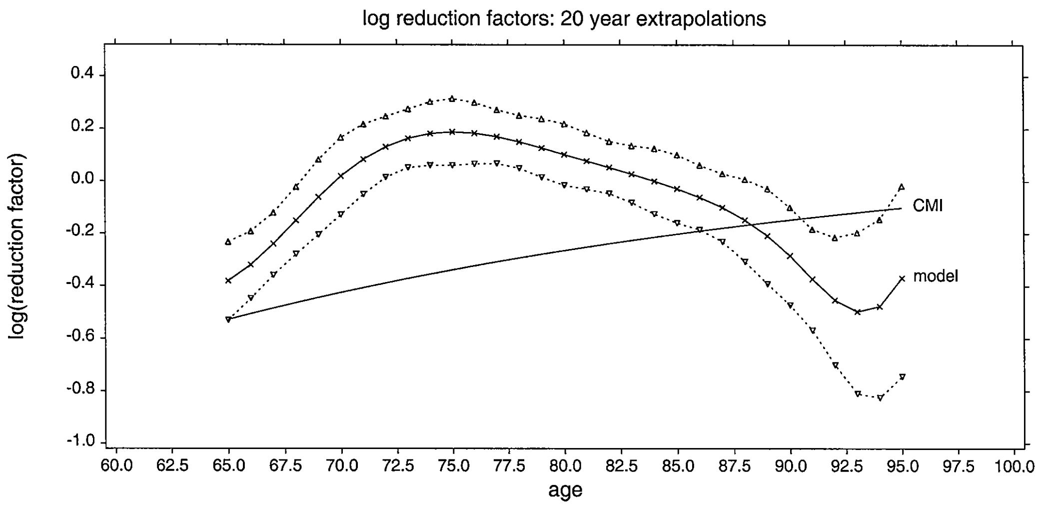


Fig. 8. Male pensioner experience: predictions



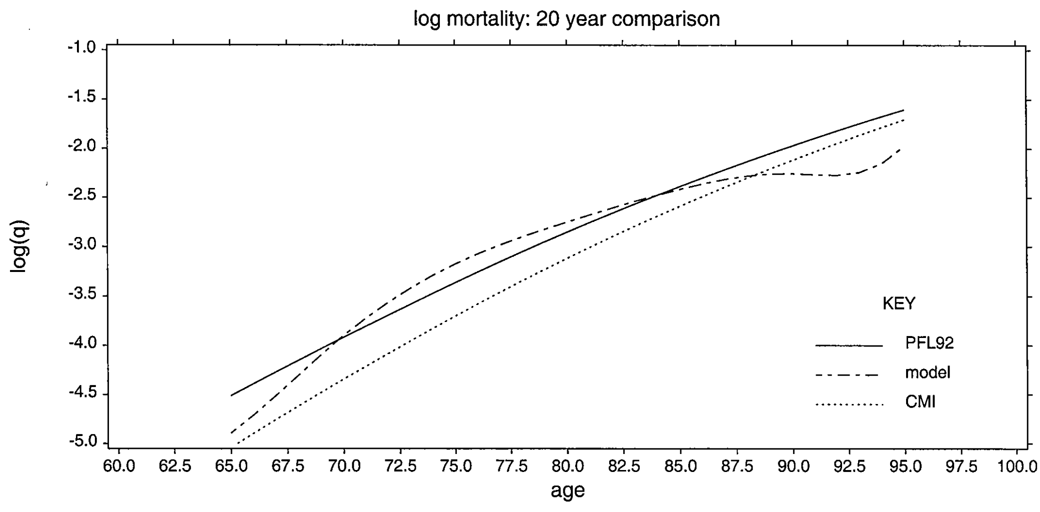


Fig. 9. Female pensioner experience: predictions

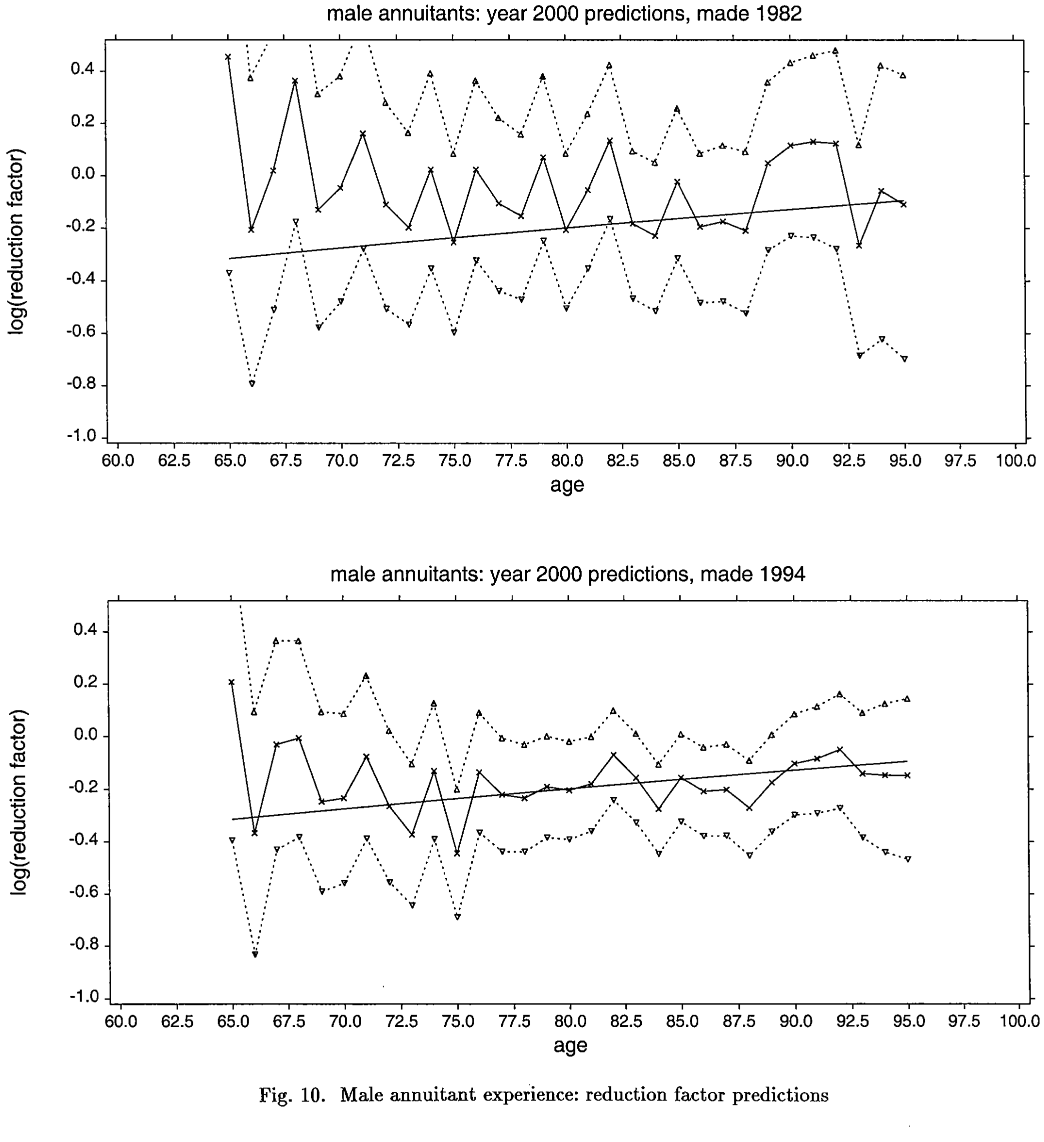


Fig. 10. Male annuitant experience: reduction factor predictions

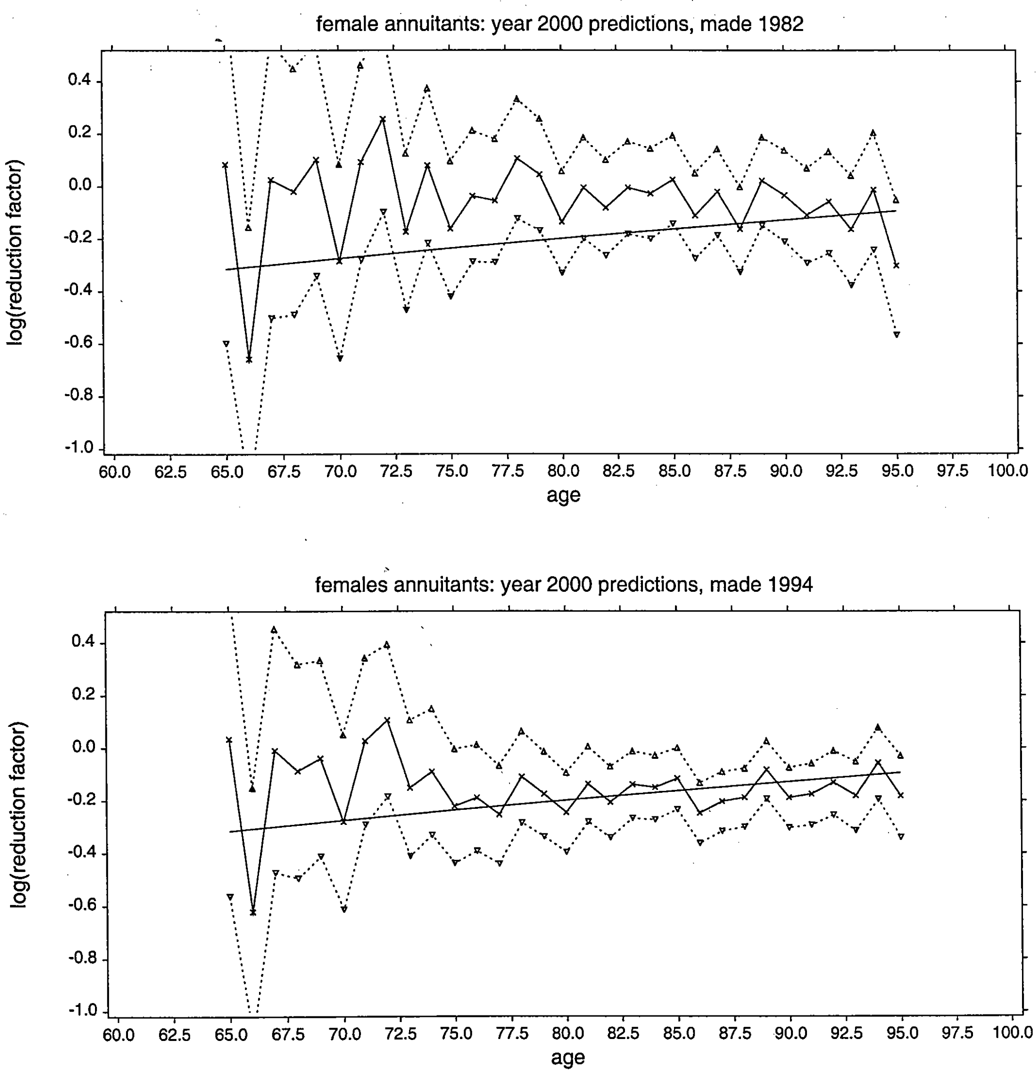


Fig. 11. Female annuitant experience: reduction factor predictions

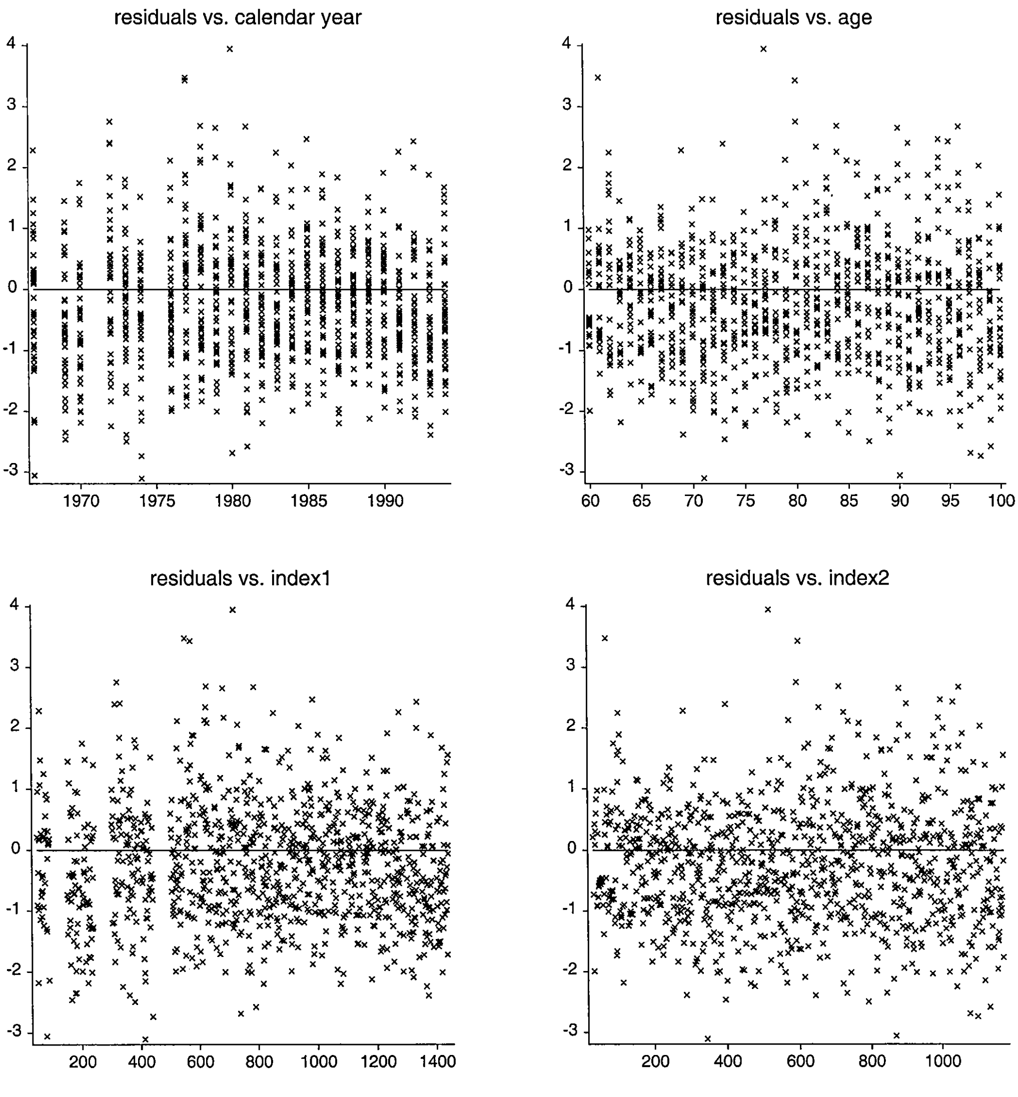


Fig. 12. Male annuitant experience: residual plots

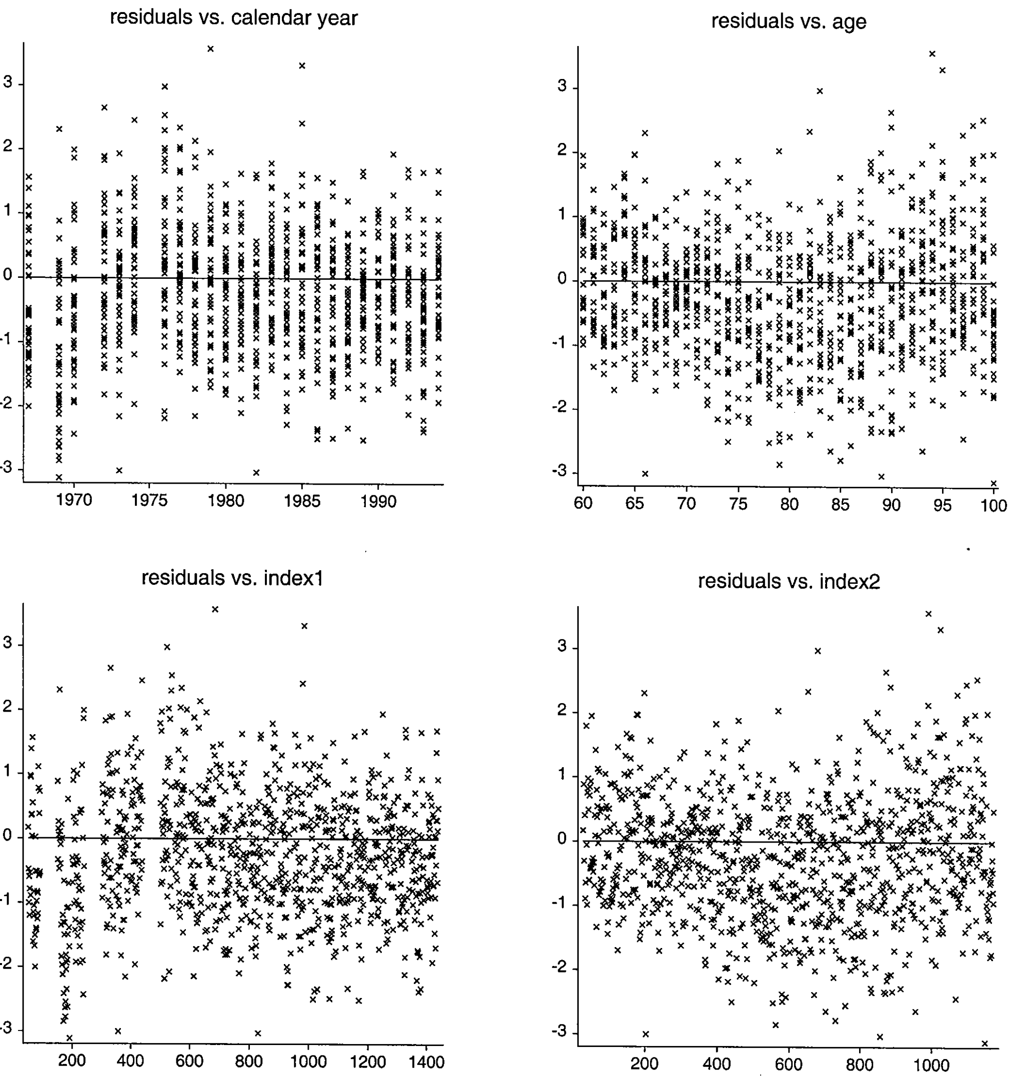


Fig. 13. Female annuitant experience: residual plots

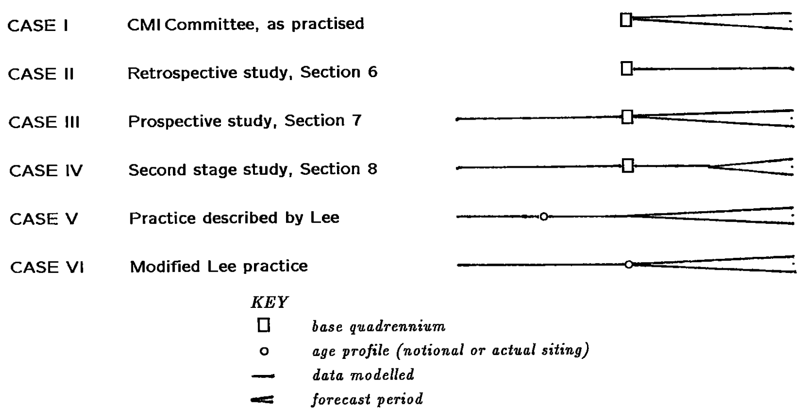


Fig. 14 Schematic representation (not to scale): evolution of various forecasting scenarios

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