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Abstract

Valuation methods that smooth the short-term fluctuations in the market values of pension plan assets are regularly used when actuarial valuations of defined benefit pension plans are performed and contribution rates are determined. The "Moving Average of Market", "Deferred Recognition", "Adjusted Market" and "Write-up" methods using arithmetic averaging and allowing for cash flows and the time value of money are shown to be equivalent under the appropriate definitions. Stability and moment properties of the pension system are studied when random investment returns are made on plan assets and the resultant asset gains and losses are amortized. It is demonstrated that there is a limit to the total amount of smoothing, through both asset valuation and gain or loss amortization, if the process of funding for pension benefits is to remain stable. Typical averaging periods and amortization periods of up to 5 years appear to be efficient in terms of minimizing both the variability of plan funding levels and contribution rates. Finally, it is shown that the actuarial asset valuation methods do generate a smooth and unbiased estimate of market values.

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1 Introduction

Defined benefit pension plans are valued at regular intervals. One purpose of an actuarial valuation is to determine a suitable rate of contribution to the pension fund in the following year. Both plan assets and liabilities are valued and compared. For such funding or management valuations, assets are not always measured at their pure market values. An actuarial asset value is used in order to smooth out short-term market fluctuations. The asset value should be consistent with the actuarial value of long-term retirement liabilities. When pension plans are valued for accounting or other statutory purposes, assets may be valued at market, or according to some prescribed method.

Only actuarial valuations with the objective of setting contribution rates are considered here. Certain methods of valuing the assets of defined benefit plans are investigated. The methods are described in general terms by the Committee on Retirement Systems Research (1998). They are the "Moving Average of Market" (or "Average Value"), "Deferred Recognition", "Adjusted Market" and "Write-up" methods.

2 A Simplified Model

A simple defined benefit pension plan model is postulated in order to study the effect of using asset valuation methods. The plan provides only a straightforward pension at a specified retirement age based on final salary. The benefit rules are taken to be fixed and no discretionary benefit enhancement (save for prespecified benefit indexation) is allowed.

Projections of the experience of the pension plan must be made. Demographic experience is not a source of considerable uncertainty for large pension plans. The plan population is assumed to be constant and mortality and other decrements are assumed to be in accordance with a life table $\{l'_x\}$ which may incorporate a promotional salary scale $\{s_x\}$ such that $l_x = s_x l'_x$ (where age is indexed by x). Economic experience is more variable. Inflation on plan benefits and returns on plan assets are not independent and are not easy to model. Furthermore, the liability pertaining to active plan members is based on (projected) final salary whereas the retirees' pension liability is usually either fixed nominally or partially indexed with consumer prices inflation. As a first approximation, we assume away inflation. Salaries are subject to general economic wage inflation, as distinct from promotional, merit-based or longevity-based wage increases in $\{s_x\}$. It is assumed that pensions in payment are fully indexed with wage inflation. The actuarial liability for both retirees and actives thus increases in line with wage inflation. All monetary quantities (including values of liabilities and assets, asset returns, payroll etc.) are therefore measured net of wage inflation. (We may alternatively ignore inflation altogether and consider nominal quantities.) The *real* rate of return on plan assets (i.e. net of wage inflation) is assumed to be a sequence of independent and identically distributed random variables. This assumption is made for the sake of simplicity. It reflects market efficiency but is oversimplified as plan assets are not continuously traded but may be held to match certain liability cash flows.

Actuarial valuations are carried out at regular intervals, say at the beginning of each year. The set of actuarial assumptions used in each valuation is assumed to be time-invariant, in line with the stationary nature of plan experience as projected above. An 'individual' actuarial cost method is used, generating an actuarial liability AL and a normal cost NC (deflated by wage inflation). These are constant, given the assumptions made above. The actuarial assumption as to mortality and other decrements is based precisely on life table $\{l_x\}$. The experience of the pension plan therefore deviates from the valuation basis only as a result of variable asset returns. Asset gains and losses therefore occur.

Some notation may be introduced at this stage. The *market value* of pension plan assets at time t is $f(t)$. Time is discretized and we assume that all cash flows occur at the beginning of the year. A contribution payment of $c(t)$ is made at the beginning of year

$(t,\, t+1)$. The total pension benefit is also paid out at the beginning of the year and is constant (say B) given the assumptions made above. (Recall that all quantities are net of wage inflation.) Let the real rate of return on plan assets during year $(t-1,\, t)$ be $r(t)$ so that, for $t \geq 1$,

$$f(t) = (1 + r(t))[f(t-1) + c(t-1) - B]. \tag{1}$$

Pension liabilities are valued by discounting at a rate i and Trowbridge (1952) shows that

$$AL = (1+i)(AL + NC - B). \tag{2}$$

The model described above is a very simplified representation of reality but has the advantage of being mathematically tractable. It is similar to the models used by Trowbridge (1952), Bowers *et al.* (1979) and Dufresne (1988, 1989). One key difference is that an 'actuarial' or smoothed value $F(t)$ is placed on the assets of the pension plan at time t.

The unfunded liability based on the market value of plan assets at time $t \geq 0$ is defined to be:

$$ul(t) = AL - f(t). \tag{3}$$

It is natural to define a smoothed unfunded liability based on the smoothed actuarial value of assets at time $t \geq 0$ as follows:

$$UL(t) = AL - F(t). \tag{4}$$

Suppose that the actuarial assumption as to future returns on plan assets is not different from the valuation discount rate, as in a traditional actuarial valuation. Asset gains and losses emerge as the investment experience $\{r(t)\}$ differs from the actuarial assumption i. The *anticipated* market value of the pension fund, if experience agrees with valuation assumptions during year $(t-1,\, t)$, is

$$f^A(t) = (1+i)[f(t-1) + c(t-1) - B], \tag{5}$$

while the anticipated actuarial value is

$$F^A(t) = (1+i)[F(t-1) + c(t-1) - B], \tag{6}$$

for $t \geq 1$. The *anticipated* unfunded liabilities are correspondingly $ul^A(t) = AL - f^A(t)$ and $UL^A(t) = AL - F^A(t)$.

Losses may be measured based on the market value of assets (unsmoothed losses $l(t)$) or based on the actuarial value of assets (smoothed losses $L(t)$). In either case, an intervaluation loss emerging owing to experience in year $(t-1,\, t)$ is defined as the difference between the unfunded liability at t and the anticipated unfunded liability had valuation assumptions been realized during $(t-1,\, t)$. The unsmoothed and smoothed loss are respectively, for $t \geq 1$,

$$l(t) = ul(t) - ul^A(t) = f^A(t) - f(t), \tag{7}$$

$$L(t) = UL(t) - UL^A(t) = F^A(t) - F(t). \tag{8}$$

(Gains are, of course, negative losses.)

At time $t = 0$, pension plan assets have market value $f(0) = f_0$, with probability one. An initial unfunded liability based on market value of $ul_0 = AL - f_0$ therefore exists. The plan may have been initiated or significantly amended (i.e. the benefit rules or the asset valuation method or the actuarial cost method may have been changed) at $t = 0$ and an initial unfunded liability may have arisen. It is assumed that for $t \leq 0$, $l(t) = L(t) = 0$.

The contribution paid into the pension fund at the beginning of year $(t,\, t+1)$ is

$$c(t) = NC + adj(t). \tag{9}$$

The supplementary contribution (or contribution adjustment) $adj(t)$ should amortize, over finite periods, the initial unfunded liability as well as the smoothed values of the intervaluation losses.

3 Asset Valuation Methods

The actuarial or smoothed asset value $F(t)$ must now be defined. Let $u = 1+i$, $v = 1/(1+i)$. The present value of plan assets at time t written up over $j \geq 1$ years (allowing for cash flows) is

$$F_j(t) = u^j f(t-j) + \sum_{k=1}^{j} u^k [c(t-k) - B]. \tag{10}$$

By virtue of the definition of $l(t)$, i.e. using equations (5) and (7),

$$f(t) + l(t) = u[f(t-1) + c(t-1) - B]. \tag{11}$$

By recursion, it follows that

$$F_j(t) = f(t) + \sum_{k=0}^{j-1} u^k l(t-k), \tag{12}$$

$$F_j(t) = u[F_j(t-1) + c(t-1) - B] - u^j l(t-j). \tag{13}$$

The smoothed actuarial value of plan assets at time t is defined as:

$$F(t) = \frac{1}{n} \left\{ f(t) + \sum_{j=1}^{n-1} F_j(t) \right\}. \tag{14}$$

Replacing $F_j(t)$ from equation (10) into equation (14) yields the "Moving Average of Market" or "Average Value" method:

$$F(t) = \frac{1}{n} \left\{ \sum_{j=0}^{n-1} u^j f(t-j) + \sum_{j=1}^{n-1} (n-j) u^j [c(t-j) - B] \right\}, \tag{15}$$

for $t \geq 1$ and an averaging period $n > 1$. The smoothed value is an arithmetic average of the market values of plan assets over the past n years, allowing for cash flows and the time value of money.

If $F_j(t)$ from equation (12) is substituted into equation (14), the "Deferred Recognition" asset valuation method is obtained:

$$F(t) = f(t) + \frac{1}{n}\sum_{j=1}^{n-1}\sum_{k=0}^{j-1} u^k l(t-k) \tag{16}$$

$$= f(t) + \sum_{j=0}^{n-2} \frac{n-1-j}{n} u^j l(t-j), \tag{17}$$

for $t \geq 1$. A fraction $1/n$ of each (unsmoothed) intervaluation loss over the past $n-1$ years is recognized and amortized, while the rest is deferred. The market value of plan assets is adjusted by adding the deferred portions of each loss (with interest) and the method is also known as the "Adjusted Market" method. See also Winklevoss (1993, p. 173).

Summing both sides of equation (13) and dividing by n gives

$$\frac{1}{n}\sum_{j=1}^{n-1} F_j(t) = \frac{u}{n}\sum_{j=1}^{n-1} F_j(t-1) + \frac{(n-1)u}{n}[c(t-1) - B] - \frac{1}{n}\sum_{j=1}^{n-1} u^j l(t-j), \tag{18}$$

and after adding $\frac{1}{n}f(t)$ on both sides and using equation (14),

$$F(t) = \frac{u}{n}\left[f(t-1) + \sum_{j=1}^{n-1} F_j(t-1)\right] - \frac{1}{n}\sum_{j=0}^{n-1} u^j l(t-j)$$
$$+ \frac{1}{n}\left[f(t) - uf(t-1) + (n-1)u(c(t-1) - B) + l(t)\right]. \tag{19}$$

We obtain another form for $F(t)$ upon using equation (11):

$$F(t) = u[F(t-1) + c(t-1) - B] - \frac{1}{n}\sum_{j=0}^{n-1} u^j l(t-j) \tag{20}$$

for $t \geq 1$. This is one variant of the "Write-up" method, described by the Committee on Retirement Systems Research (1998). The actuarial asset value is the anticipated actuarial asset value, based on the valuation basis and allowing for new cash, adjusted downwards by the sum of recognized portions of previous (unsmoothed) losses. See Peat Marwick (1986, p. 25) for an explicit example where such a method is used in conjunction with accounting valuations under Financial Accounting Standards No. 87.

It is clear that the asset valuation methods described in equations (15), (17) and (20) are identical, if initial conditions are ignored. For our purposes, these initial conditions may be arbitrarily defined.

Assume that the pension fund is marked-to-market at time $t = 0$ and that $F(0) = f_0$. As regards the methods defined by equations (17) and (20), note that we assumed previously that for $t \leq 0$, $l(t) = 0$. In order that equation (15) satisfies $F(0) = f_0$, we arbitrarily choose that, for $-(n-1) \leq t \leq 0$, $c(t) = NC$ and $f(t-1) = vf(t) - NC + B$ (given $f(0) = f_0$).

4 Intervaluation Losses

A recurrence relation for the unfunded liability $ul(t)$ may be obtained in terms of the (unsmoothed) loss $l(t)$, following Dufresne (1989), by replacing $f(t)$ and $f(t-1)$ using equation (3) and also replacing $c(t-1)$ using equation (9) into equation (11):

$$ul(t) - u \cdot ul(t-1) = l(t) - u \cdot adj(t-1) \tag{21}$$

for $t \geq 1$.

The smoothed intervaluation loss $L(t)$, based on the smoothed actuarial asset value, is defined in equation (8), and using equations (6) and (20), it follows that

$$L(t) = \frac{1}{n} \sum_{j=0}^{n-1} u^j l(t-j) \tag{22}$$

for $t \geq 1$. It is clear that $L(t)$ is an arithmetic average of the present value of the unsmoothed losses in the past n years. It is also clear that the unsmoothed losses are not being immediately recognized and that portions of the losses are being deferred over up to n years.

A recurrence relation for $UL(t)$ in terms of $L(t)$ may also be obtained. Upon replacing

$F(t)$ and $F(t-1)$ from equation (4) into

$$F(t) = u[F(t-1) + c(t-1) - B] - L(t), \tag{23}$$

it is readily found that, for $t \geq 1$,

$$UL(t) - u \cdot UL(t-1) = L(t) - u \cdot adj(t-1). \tag{24}$$

Compare with equation (21).

5 Supplementary Contributions

The supplementary contribution consists of:

1. amortization payments over an initial period of M years to liquidate the initial unfunded liability ul_0;

2. amortization payments for losses over a finite period of m years—where the losses $L(t)$ are smoothed, i.e. measured in terms of the actuarial smoothed asset value.

For $t \geq 0$,

$$adj(t) = (ul_0/\ddot{a}_{\overline{M|}})1_{t \leq M-1} + \sum_{j=0}^{m-1} L(t-j)/\ddot{a}_{\overline{m|}}, \tag{25}$$

where 1_X is an indicator function such that $1_X = 1$ when X is true and $1_X = 0$ when X is false.

The supplementary contribution may be expressed directly in terms of unsmoothed losses $\{l(t)\}$, by substituting equation (22) in equation (25) and employing the following elementary identities, depending on the relative lengths of the amortization and averaging periods. If $m = n$,

$$\sum_{j=0}^{n}\sum_{k=0}^{n} a_k b_{j+k} = \sum_{j=0}^{n} b_j \sum_{k=0}^{j} a_k + \sum_{j=n+1}^{2n} b_j \sum_{k=j-n}^{n} a_k. \tag{26}$$

If $m > n$,

$$\sum_{j=0}^{m}\sum_{k=0}^{n} a_k b_{j+k} = \sum_{j=0}^{n} b_j \sum_{k=0}^{j} a_k + \sum_{j=n+1}^{m} b_j \sum_{k=0}^{n} a_k + \sum_{j=m+1}^{m+n} b_j \sum_{k=j-m}^{n} a_k. \tag{27}$$

If $m < n$,

$$\sum_{j=0}^{m}\sum_{k=0}^{n} a_k b_{j+k} = \sum_{j=0}^{m} b_j \sum_{k=0}^{j} a_k + \sum_{j=m+1}^{n} b_j \sum_{k=j-m}^{j} a_k + \sum_{j=n+1}^{m+n} b_j \sum_{k=j-m}^{n} a_k. \tag{28}$$

It is straightforward to establish that the supplementary contribution may be written as

$$adj(t) = (ul_0/\ddot{a}_{\overline{M|}})1_{t \le M-1} + \sum_{j=0}^{m-1}\sum_{k=0}^{n-1} u^k l(t-j-k)/(n\ddot{a}_{\overline{m|}}) \tag{29}$$

$$= (ul_0/\ddot{a}_{\overline{M|}})1_{t \le M-1} + \sum_{j=0}^{m+n-2} \pi_j l(t-j), \tag{30}$$

where for $m = n$,

$$\pi_j = \begin{cases} s_{\overline{j+1|}}/n\ddot{a}_{\overline{n|}} & 0 \le j \le n-1, \\ (s_{\overline{n|}} - s_{\overline{j-n+1|}})/n\ddot{a}_{\overline{n|}} & n \le j \le 2n-2, \\ 0 & \text{otherwise,} \end{cases} \tag{31}$$

and for $m > n$,

$$\pi_j = \begin{cases} s_{\overline{j+1|}}/n\ddot{a}_{\overline{m|}} & 0 \le j \le n-1, \\ s_{\overline{n|}}/n\ddot{a}_{\overline{m|}} & n \le j \le m-1, \\ (s_{\overline{n|}} - s_{\overline{j-m+1|}})/n\ddot{a}_{\overline{m|}} & m \le j \le m+n-2, \\ 0 & \text{otherwise,} \end{cases} \tag{32}$$

and for $n > m$,

$$\pi_j = \begin{cases} s_{\overline{j+1|}}/n\ddot{a}_{\overline{m|}} & 0 \le j \le m-1, \\ (s_{\overline{j+1|}} - s_{\overline{j-m+1|}})/n\ddot{a}_{\overline{m|}} & m \le j \le n-1, \\ (s_{\overline{n|}} - s_{\overline{j-m+1|}})/n\ddot{a}_{\overline{m|}} & n \le j \le m+n-2, \\ 0 & \text{otherwise.} \end{cases} \tag{33}$$

From equation (30), it is evident that, at time t, a fraction π_j of the unsmoothed loss $l(t-j)$ is paid into the pension fund. In fact, any loss l is liquidated by payments in successive years (starting from the year in which the loss emerged) of

$$\{\pi_0 l,\ \pi_1 l,\ \pi_2 l,\ \ldots,\ \pi_{m+n-2} l\}.$$

π_j therefore represents the fraction of a loss that is paid j years after the loss emerged.

The loss is *completely* liquidated by these payments, i.e.

$$\sum_{j=0}^{m+n-2} v^j \pi_j = 1, \tag{34}$$

since we may substitute $l(t-j)$ by v^j in the identity

$$\sum_{j=0}^{m+n-2} \pi_j l(t-j) = \frac{1}{n\ddot{a}_{\overline{m|}}} \sum_{j=0}^{m-1} \sum_{k=0}^{n-1} u^k l(t-j-k) \tag{35}$$

(cf. equations (29) and (30)) and equality (34) follows immediately.

It may also be observed that equation (34) holds when π_j from equations (31), (32) and (33) is explicitly substituted into it. For example, when $m = n$, $\sum_{j=0}^{n-1} v^j \pi_j \times n\ddot{a}_{\overline{n|}}$ represents a linearly *decreasing* annuity of term n with payments in advance of n, $n-1$, $\ldots$, 1, whereas $\sum_{j=n}^{2n-2} v^j \pi_j \times n\ddot{a}_{\overline{n|}}$ represents a linearly *increasing* annuity of term $n-1$ with payments in advance of 1, 2, $\ldots$, $n-1$, and the sum of the two yields a level annuity of term n and annual payments of n. When $m > n$ or $m < n$, an additional term intervenes. For example, when $m > n$, $\sum_{j=n}^{m-1} v^j \pi_j \times n\ddot{a}_{\overline{m|}}$ represents a sum of level annuities of term n, deferred by 1, 2, ... , $m-n$ years. (If unit payments are stacked along a time-line, these level annuities represent the extra 'diagonals' in the rectangular geometry of the overall annuity of term m paying n each year.)

Finally, note the following:

1. When pure market value is used ($n=1$), it is not difficult to see that $\pi_j = 1/\ddot{a}_{\overline{m|}}$ for $0 \le j \le m-1$, and $\pi_j = 0$ otherwise. This corresponds to the results of Dufresne (1989). Level payments (comprising both principal and interest) are made over m years in respect of a loss: the loss is amortized over m years.

2. When asset values are averaged ($n > 1$), but the resultant smoothed gains and losses are not amortized and are paid off rightaway ($m = 1$), then $\pi_j = u^j/n$ for $0 \le j \le n-1$ and $\pi_j = 0$ otherwise. The principal component of the loss in any given year is paid off in equal tranches over n years, along with interest on each tranche.

6 Decomposition of the Unfunded Liability $ul(t)$

The unfunded liability $ul(t)$ at market value may also be expressed in terms of unsmoothed losses $\{l(t)\}$. Replace $adj(t)$ from equation (30) into the recurrence relation (21) to obtain, for $t \ge 1$,

$$ul(t) - u \cdot ul(t-1) = l(t) - u \sum_{j=1}^{m+n-1} \pi_{j-1} l(t-j) - u(ul_0/\ddot{a}_{\overline{M|}}) 1_{t \le M}. \tag{36}$$

It is easy to verify that the solution to the above is, for $t \ge 1$,

$$ul(t) = (ul_0 \ddot{a}_{\overline{M-t|}}/\ddot{a}_{\overline{M|}}) 1_{t \le M-1} + \sum_{j=0}^{m+n-2} \lambda_j l(t-j), \tag{37}$$

where

$$\lambda_j = \begin{cases} 1 & j = 0, \\ u^j - \sum_{k=0}^{j-1} u^{j-k}\pi_k & 1 \le j \le m+n-2, \\ 0 & \text{otherwise.} \end{cases} \tag{38}$$

Equations (37) and (38) make sense. $u^j l(t-j)$ is the present value (at time t) of a loss that emerged j years ago. $u^{j-k}\pi_k l(t-j)$ is the present value (at time t) of the fraction of loss $l(t-j)$ that was paid k years after the loss emerged ($j-k$ years ago). Hence, $l(t-j)\sum_{k=0}^{j-1} u^{j-k}\pi_k$ represents the present value of payments that have been made in the past in respect of a loss $l(t-j)$ that is yet to be entirely liquidated as at time t. Therefore, $\lambda_j l(t-j)$ is the present value of payments that remain to be made from time t onwards in respect of loss $l(t-j)$.

We may now sum over all unpaid-off losses. $\sum_{j=0}^{m+n-2} u^j l(t-j)$ thus represents the present value of all past losses that have yet to be entirely paid off as at time t, and $\sum_{j=0}^{m+n-2}\sum_{k=0}^{j-1} u^{j-k}\pi_k l(t-j)$ represents the present value of payments that have been made in the past in respect of these losses.

Hence, the second term on the right hand side of equation (37) is the present value of payments that remain to be made from time t onwards to liquidate past losses. The unfunded liability at any time is the sum of this term and the unamortized part of the initial unfunded liability.

Remarks:

1. It may be observed that, when pure market value is used ($n = 1$), $\lambda_j = \ddot{a}_{\overline{m-j|}}/\ddot{a}_{\overline{m|}}$ for $0 \le j \le m-1$, as obtained by Dufresne (1989). λ_j is the present value of the unamortized part of a unit loss j years after it emerged.

2. When the actuarial asset value is used ($n \ge 1$) but the resultant gains/losses are not amortized ($m = 1$), then $\lambda_j = u^j(n-j)/n$ for $0 \le j \le n-1$.

7 Decomposition of the Smoothed Unfunded Liability $UL(t)$

The smoothed unfunded liability $UL(t)$ based on the actuarial asset value may also be decomposed into either smoothed losses $\{L(t)\}$ or unsmoothed losses $\{l(t)\}$. Replacing $adj(t-1)$ into equation (24) by equation (25), we find that

$$UL(t) - u \cdot UL(t-1) = L(t) - u\sum_{j=0}^{m-1} L(t-1-j)/\ddot{a}_{\overline{m}|} - u(ul_0/\ddot{a}_{\overline{M}|})1_{t\leq M}, \qquad (39)$$

which may be written as (Dufresne, 1989):

$$UL(t) = \sum_{j=0}^{m-1}(\ddot{a}_{\overline{m-j}|}/\ddot{a}_{\overline{m}|})L(t-j) + (ul_0\ddot{a}_{\overline{M-t}|}/\ddot{a}_{\overline{M}|})1_{t\leq M-1}, \qquad (40)$$

for $t \geq 1$. The smoothed unfunded liability at any time consists of the unamortized part of present and previous smoothed losses, as well as the unamortized part of the initial unfunded liability.

To express $UL(t)$ directly in terms of the unsmoothed losses $\{l(t)\}$, substitute in equation (39) using equation (22), to yield the following recurrence relation, for $t \geq 1$:

$$\begin{aligned} &UL(t) - u \cdot UL(t-1) \\ &= \frac{1}{n}l(t) + \sum_{j=1}^{n-1}(u^j/n - u\pi_{j-1})l(t-j) - \sum_{j=n}^{m+n-1} u\pi_{j-1}l(t-j) - u(ul_0/\ddot{a}_{\overline{M}|})1_{t\leq M}. \qquad (41)\end{aligned}$$

It is easy to verify that the solution to the above is

$$UL(t) = (ul_0\ddot{a}_{\overline{M-t}|}/\ddot{a}_{\overline{M}|})1_{t\leq M-1} + \sum_{j=0}^{m+n-2} \nu_j l(t-j), \qquad (42)$$

for $t \geq 1$, where

$$\nu_j = \begin{cases} \frac{1}{n} & j = 0, \\ \frac{j+1}{n}u^j - \sum_{k=0}^{j-1} u^{j-k}\pi_k & 1 \leq j \leq n-1, \\ u^j - \sum_{k=0}^{j-1} u^{j-k}\pi_k & n \leq j \leq m+n-2, \\ 0 & \text{otherwise.} \end{cases} \qquad (43)$$

Again, equations (42) and (43) make sense. Under the asset valuation method chosen, only a fraction $1/n$ of the current loss is recognized while the rest is deferred. A fraction $(j+1)/n$ of a loss that occurred j years ago (i.e. $l(t-j)$) is recognized by time t. After n years, the loss is fully recognized, albeit not fully amortized. $\nu_j l(t-j)$ therefore represents the present value of payments that remain to be made from time t onwards in respect of the *recognized* (not deferred) portion of loss $l(t-j)$. (Compare with $\lambda_j l(t-j)$ in equation (38).) Hence, the second term on the right hand side of equation (42) is the present value of payments that remain to be made from time t onwards to liquidate the recognized (not deferred) portions of past losses.

Remarks:

1. If pure market value is used ($n = 1$), $UL(t) = ul(t)$ and $\nu_j = \lambda_j = \ddot{a}_{\overline{m-j|}}/\ddot{a}_{\overline{m|}}$ for $0 \leq j \leq m-1$, as obtained by Dufresne (1989).

2. If the actuarial asset value is used ($n \geq 1$) but the resultant gains/losses are not amortized ($m = 1$), then $adj(t) = L(t)$ and $\nu_j = \pi_j = u^j/n$ for $0 \leq j \leq n-1$.

8 Recurrence Relation for the Unsmoothed Loss

The unsmoothed intervaluation loss $l(t)$ is defined in equation (7) and, using equations (1) and (5), it is clear that

$$l(t) = (i - r(t))[f(t-1) + c(t-1) - B] \tag{44}$$

for $t \geq 1$. Upon substitution into equation (44) of equations (3) and (9), we obtain, for $t \geq 1$,

$$l(t) = (r(t) - i)[ul(t-1) - adj(t-1) - vAL]. \tag{45}$$

Now, both $adj(t)$ in equation (30) and $ul(t)$ in equation (37) have been expressed directly in terms of $\{l(t)\}$. Replacing into equation (45) gives

$$l(t+1) = [r(t+1) - i]\left\{ (ul_0 \ddot{a}_{\overline{M-t-1}|}/\ddot{a}_{\overline{M}|}) 1_{t \leq M-1} + \sum_{j=0}^{m+n-2} \beta_j l(t-j) - vAL \right\}, \tag{46}$$

for $t \geq 0$, where $\beta_j = \lambda_j - \pi_j$, i.e.

$$\beta_j = \begin{cases} u^j - \sum_{k=0}^{j} u^{j-k}\pi_k & 0 \leq j \leq m+n-2, \\ 0 & \text{otherwise.} \end{cases} \tag{47}$$

$\beta_j l(t-j)$ is clearly the present value of payments that remain to be made after time t in respect of loss $l(t-j)$.

Remarks:

1. When $n = 1$ and $m > 1$, $\beta_j = a_{\overline{m-j-1}|}/\ddot{a}_{\overline{m}|}$ for $0 \leq j \leq m-2$, and is zero otherwise. This is similar to the result of Dufresne (1989), except for slightly different time indexation.

2. When $m = 1$ and $n > 1$, $\beta_j = u^j(n-j-1)/n$ for $0 \leq j \leq n-2$, and is zero otherwise.

3. When $m = n = 1$, and a loss is entirely recognized and paid off immediately, then $\beta_j = 0 \ \forall j$.

9 First Moments

We are primarily interested in the moments of the pension system in its stationary state (i.e. ignoring initial conditions). All terms involving the initial unfunded liability ul_0 are zero for $t \geq M$. Also, define $\mathrm{E}r(t) = r$. Mathematical expectation is taken on both sides of recurrence relation (46): note that the rate of return $r(t+1)$ is independent of $r(s)$ and hence of $l(s)$, for $s \leq t$. The limit as $t \to \infty$ is then taken. A sufficient stability condition is given in Proposition 1 of Dufresne (1989). In the following proposition, $\{\beta_j\}$, $\{\lambda_j\}$ and

$\{\nu_j\}$ are summed over $j \in [0, m+n-2]$ and $s_{\overline{n|}}$ is the accumulation of an annuity in arrears of term n at rate i.

PROPOSITION 1 *If* $|r-i|\sum\beta_j < 1$, *then*

$$\lim_{t\to\infty} \mathrm{E}l(t) = \frac{-(r-i)vAL}{1-(r-i)\sum\beta_j}$$
$$= M_\infty \quad \textit{(say)}, \tag{48}$$

$$\lim_{t\to\infty} \mathrm{E}L(t) = M_\infty s_{\overline{n|}}/n, \tag{49}$$

$$\lim_{t\to\infty} \mathrm{E}ul(t) = M_\infty \sum\lambda_j, \tag{50}$$

$$\lim_{t\to\infty} \mathrm{E}f(t) = AL - M_\infty \sum\lambda_j, \tag{51}$$

$$\lim_{t\to\infty} \mathrm{E}c(t) = NC + M_\infty(ms_{\overline{n|}})/(n\ddot{a}_{\overline{m|}}), \tag{52}$$

$$\lim_{t\to\infty} \mathrm{E}UL(t) = M_\infty \sum\nu_j, \tag{53}$$

$$\lim_{t\to\infty} \mathrm{E}F(t) = AL - M_\infty \sum\nu_j. \tag{54}$$

In the above proposition, $\lim \mathrm{E}L(t)$ is obtained from equation (22); $\lim \mathrm{E}ul(t)$ is derived from the decomposition of $ul(t)$, i.e. equation (37); $\lim \mathrm{E}f(t)$ follows by virtue of equation (3); $\lim \mathrm{E}c(t)$ is obtained from equations (9), (25) and (49); $\lim \mathrm{E}UL(t)$ is derived from the decomposition of $UL(t)$, i.e. from equation (42); and $\lim \mathrm{E}F(t)$ follows from equation (4).

If the actuarial assumption as to the rate of return on plan assets is an unbiased estimate, i.e. $r = i$, then clearly

$$\mathrm{E}l(t) = \mathrm{E}L(t) = 0 \qquad \forall t \tag{55}$$

from equations (22) and (46). We *expect* no loss to emerge. Also, from equations (37) and (42),

$$\mathrm{E}ul(t) = \mathrm{E}UL(t) = \begin{cases} ul_0 \ddot{a}_{\overline{M-t|}}/\ddot{a}_{\overline{M|}} & 0 \le t \le M-1, \\ 0 & t \ge M. \end{cases} \tag{56}$$

Since no loss arises on average, the unfunded liability (smoothed and unsmoothed) is expected to equal the unamortized part of the initial unfunded liability and be zero after the initial unfunded liability is amortized. From equations (9) and (25), it also follows that

$$\mathrm{E}c(t) = \begin{cases} NC + ul_0/\ddot{a}_{\overline{M|}} & 0 \le t \le M-1, \\ NC & t \ge M. \end{cases} \tag{57}$$

An additional payment or supplementary contribution or supplemental cost is required to amortize the initial unfunded liability in the first M years.

10 Second Moments

As in Dufresne (1989), it is now assumed that $r = i$. When pure market values are used and losses are amortized, Dufresne (1989) observes that the losses are a sequence of uncorrelated zero-mean random variables when the rate of return process is independent from year to year. The same result obtains when smoothed asset values are used. It is clear from equation (45) that, for $t > s$,

$$\mathrm{E}l(t)l(s) = \mathrm{E}[r(t) - i] \times \mathrm{E}\{[ul(t-1) - adj(t-1) - vAL] \cdot l(s)\} = 0, \tag{58}$$

given again the independence of $r(t+1)$ from $r(s)$, $ul(s)$ and $adj(s)$ for $s \le t$. Define $\mathrm{Var}r(t) = \sigma^2$. Following the method of Dufresne (1989), we obtain, from equation (46),

$$\begin{aligned} \mathrm{Var}l(t+1) &= \mathrm{E}l(t+1)^2 \\ &= \sigma^2 \left\{ \left[(ul_0 a_{\overline{M-t-1|}}/\ddot{a}_{\overline{M|}}) 1_{t \le M-1} - vAL \right]^2 + \sum_{j=0}^{m+n-2} \beta_j^2 \mathrm{Var}l(t-j) \right\}. \end{aligned} \tag{59}$$

Other moments follow by similarly exploiting the serially uncorrelated structure of $\{l(t)\}$. From equation (22), we find that

$$\mathrm{Var}L(t) = \frac{1}{n^2}\sum_{j=0}^{n-1} u^{2j}\mathrm{Var}l(t-j). \tag{60}$$

And from equations (3) and (37), it follows that

$$\mathrm{Var}f(t) = \mathrm{Var}ul(t) = \sum_{j=0}^{m+n-2} \lambda_j^2 \mathrm{Var}l(t-j). \tag{61}$$

From equations (9) and (30),

$$\mathrm{Var}c(t) = \sum_{j=0}^{m+n-2} \pi_j^2 \mathrm{Var}l(t-j). \tag{62}$$

Given equations (4) and (42), it follows that

$$\mathrm{Var}F(t) = \mathrm{Var}UL(t) = \sum_{j=0}^{m+n-2} \nu_j^2 \mathrm{Var}l(t-j). \tag{63}$$

Covariances may also be found. For example, if $ul_0 = 0$,

$$\begin{aligned}
\mathrm{Cov}[f(t), F(t)] &= \mathrm{Cov}[ul(t), UL(t)] \\
&= \mathrm{Cov}\left[\sum_{j=0}^{m+n-2} \lambda_j l(t-j), \sum_{j=0}^{m+n-2} \nu_j l(t-j)\right] \\
&= \sum_{j=0}^{m+n-2} \lambda_j \nu_j \mathrm{Var}l(t).
\end{aligned} \tag{64}$$

Likewise, with $ul_0 = 0$,

$$\mathrm{Cov}[f(t), c(t)] = \sum_{j=0}^{m+n-2} \lambda_j \pi_j \mathrm{Var}l(t). \tag{65}$$

The unsmoothed losses $\{l(t)\}$ are serially uncorrelated when $r = i$, but the smoothed losses $\{L(t)\}$ *are* correlated with a cutoff after lag $n-1$ since, for $\tau \geq 0$,

$$\begin{aligned}
\mathrm{Cov}[L(t), L(t-\tau)] &= \frac{1}{n^2}\mathrm{Cov}\left[\sum_{j=0}^{n-1} u^j l(t-j), \sum_{j=0}^{n-1} u^j l(t-j-\tau)\right] \\
&= \begin{cases} \frac{1}{n^2}\sum_{j=\tau}^{n-1} u^{2j-\tau}\mathrm{Var}l(t-j) & \text{if } \tau \leq n-1, \\ 0 & \text{if } \tau > n-1. \end{cases}
\end{aligned} \tag{66}$$

Similarly, assuming $ul_0 = 0$ and $\tau \geq 0$,

$$\mathrm{Cov}[f(t), f(t-\tau)] = \begin{cases} \sum_{j=\tau}^{m+n-2} \lambda_j \lambda_{j-\tau} \mathrm{Var} l(t-j) & \text{if } \tau \leq m+n-2, \\ 0 & \text{if } \tau > m+n-2, \end{cases} \tag{67}$$

$$\mathrm{Cov}[f(t), f(t-\tau)] = \begin{cases} \sum_{j=\tau}^{m+n-2} \pi_j \pi_{j-\tau} \mathrm{Var} l(t-j) & \text{if } \tau \leq m+n-2, \\ 0 & \text{if } \tau > m+n-2. \end{cases} \tag{68}$$

We are interested in the first instance in moments in the limit as $t \to \infty$. All terms involving ul_0 are zero for $t \geq M$. A necessary and sufficient condition for the stability of difference equations such as equation (59) is given in Proposition 2 of Dufresne (1989) (note that $\sigma^2 \beta_j^2 \geq 0$). In the following proposition, $\{\beta_j^2\}$, $\{\lambda_j^2\}$ etc. are summed over $j \in [0, m+n-2]$, unless otherwise specified, and $\tilde{s}_{\overline{N}|}$ in equations (70) and (76) is the accumulation of an annuity in arrears of term N at rate $2i + i^2$.

PROPOSITION 2 *Suppose that $r = i$. If and only if $\sigma^2 \sum \beta_j^2 < 1$, then*

$$\lim_{t\to\infty} \mathrm{Var} l(t) = \frac{\sigma^2 v^2 AL^2}{1 - \sigma^2 \sum \beta_j^2}$$

$$= V_\infty \quad \textit{(say)}, \tag{69}$$

$$\lim_{t\to\infty} \mathrm{Var} L(t) = V_\infty \tilde{s}_{\overline{n}|}/n^2, \tag{70}$$

$$\lim_{t\to\infty} \mathrm{Var} f(t) = \lim_{t\to\infty} \mathrm{Var} ul(t) = V_\infty \sum \lambda_j^2, \tag{71}$$

$$\lim_{t\to\infty} \mathrm{Var} c(t) = V_\infty \sum \pi_j^2, \tag{72}$$

$$\lim_{t\to\infty} \mathrm{Var} F(t) = \lim_{t\to\infty} \mathrm{Var} UL(t) = V_\infty \sum \nu_j^2, \tag{73}$$

$$\lim_{t\to\infty} \mathrm{Cov}[f(t), F(t)] = V_\infty \sum \lambda_j \nu_j, \tag{74}$$

$$\lim_{t\to\infty} \mathrm{Cov}[f(t), c(t)] = V_\infty \sum \lambda_j \pi_j, \tag{75}$$

$$\lim_{t\to\infty} \mathrm{Cov}[L(t), L(t-\tau)] = \begin{cases} V_\infty u^\tau \tilde{s}_{\overline{n-\tau}|}/n^2 & 0 \le \tau \le n-1, \\ 0 & \tau \ge n, \end{cases} \tag{76}$$

$$\lim_{t\to\infty} \mathrm{Cov}[f(t), f(t-\tau)] = \begin{cases} V_\infty \sum_{j=\tau}^{m+n-2} \lambda_j \lambda_{j-\tau} & 0 \le \tau \le m+n-2, \\ 0 & \tau > m+n-2, \end{cases} \tag{77}$$

$$\lim_{t\to\infty} \mathrm{Cov}[c(t), c(t-\tau)] = \begin{cases} V_\infty \sum_{j=\tau}^{m+n-2} \pi_j \pi_{j-\tau} & 0 \le \tau \le m+n-2, \\ 0 & \tau > m+n-2. \end{cases} \tag{78}$$

11 Constraints on Averaging and Amortization Periods

The stability condition in Proposition 2 limits the range of periods n over which asset values may be averaged. An excessively long averaging period fails to stabilise the pension funding process and it eventually becomes non-stationary, with the contribution rates and asset values having infinite variances. The stability condition also constrains the range of asset loss amortization periods, m, with very long amortization periods being unstable.

This suggests that long averaging and amortization periods should not be combined as excessive smoothing may lead to an unstable pension funding process.

Numerical work shows that the stability condition becomes more constraining as σ and i increase but also that it does not appear to be very significant in practical conditions. For example, the condition holds for rates of return averaging up to 10% and with standard deviations of up to 25% when averaging and amortization periods between 1 and 10 years are used.

The following short-hand notation is now employed:

$$V_f = \lim_{t\to\infty} \mathrm{Var}f(t), \qquad V_c = \lim_{t\to\infty} \mathrm{Var}c(t).$$

The *observations* below are based on further numerical experiments on stable $\{\sigma, i, m, n\}$ in Tables 1 and 2:

1. For a given n, as m increases,
   (a) V_f increases monotonically;
   (b) V_c exhibits a minimum, except for large enough n when it increases monotonically.
2. For a given m, as n increases,
   (a) V_f increases monotonically;
   (b) V_c exhibits a minimum, except for large enough m when it increases monotonically.

The monotonic increasing nature of V_f with n and m is depicted in Figures 1 and 2 respectively. See also Figure 3. The behavior of V_c with n and m is illustrated in Figures 4 and 5 respectively. Figure 6 is a plot of V_c with both n and m. The effects of increasing n and m are similar. This is not surprising given the similar smoothing functions of asset valuation and asset gain and loss amortization. Averaging and amortization are nevertheless not identical smoothing mechanisms and the contour plots of Figure 7 are not symmetrical about $m = n$.

The observations concerning V_f suggest that both shorter asset value averaging periods and shorter amortization periods lead to more stable levels of funding and hence to increased security of pension benefits. This is reasonable as gains and losses are being recognized earlier and amortized faster.

Conversely, later recognition and slower amortization of gains and losses should imply smoother and less variable contributions. The observations regarding V_c suggest that, as the averaging and amortization periods increase, contributions do indeed become less variable—but only up to a point. Longer averaging and amortization periods beyond that point (the minimum of V_c) is counterproductive and contributions become more variable. In addition, if gains and losses are being amortized over long enough periods, then increasing the averaging period in an effort to achieve further smoothing is also counterproductive as it makes contributions more variable (V_c increases monotonically with n for large enough m). Likewise, if asset values are averaged over long enough periods, then lengthening the term of gain and loss amortization schedules does not further stabilize contributions, but makes them more variable (V_c increases monotonically with m for large enough n).

These observations encompass Proposition 6 of Owadally & Haberman (1999) about the effects on the security and stability of pension funding of varying the amortization period for gains and losses (they consider only pure market values of assets, i.e. $n = 1$). These observations are also in line with Proposition 2 of Dufresne (1988) about similar effects under a different gain/loss amortization mechanism (he also considers only pure market values).

Dufresne (1988) postulates that a reasonable actuarial objective in the long term funding of pension benefits is to maximize the security of these benefits (for example by minimizing the variability in funding levels, i.e. minimizing $\lim \mathrm{Var} f(t)$) and also to maximize the stability of contribution rates (by minimizing $\lim \mathrm{Var} c(t)$). Thus, if both V_f and V_c increase as some actuarial control parameter increases over a given range, then the smallest value of that parameter should be chosen. But if V_f increases and V_c decreases as the parameter increases, a tradeoff exists between security and stability and no unique choice of the parameter value is preferable. Furthermore, if V_c exhibits a minimum while V_f increases monotonically as the parameter increases, then selecting a parameter value in the range for which V_c increases is inefficient, since there is always a smaller parameter value outside that range that yields a lower V_f for equal V_c.

The V_c-minimizing value of n for various choices of $\{\sigma,\ i,\ m\}$ is given in Table 1 and the V_c-minimizing value of m for various choices of $\{\sigma,\ i,\ n\}$ is given in Table 2. The V_c-minimizing value of n decreases as m increases in Table 1 and for large enough m, V_c increases monotonically with n and the smallest value of V_c occurs at $n = 1$. Similarly, the V_c-minimizing value of m decreases as n increases in Table 2 and for large enough n, V_c is monotonic increasing with m with its smallest value at $m = 1$.

Under the efficiency criterion of minimizing V_f and V_c as used by Dufresne (1988) and extrapolating from the numerical observations above, we may conclude that:

1. Suppose $\{\sigma,\ i,\ m\}$ are given. Asset values should be averaged over periods ranging from 1 to the V_c-minimizing value of n in Table 1. If the amortization period m is long enough and V_c has no minimum, then pure market values ($n = 1$) should be used to value plan assets.

2. Suppose $\{\sigma,\ i,\ n\}$ are given. Asset gains and losses should be amortized over periods ranging from 1 to the V_c-minimizing value of m in Table 1. If the averaging period n is long enough and V_c has no minimum, then gains and losses should not be amortized and should be paid off immediately ($m = 1$).

These conclusions are not mathematically rigorous and are based on the restricted set of parameters in Tables 1 and 2 (and also on the simplified modeling assumptions set out earlier). They are nevertheless important because they demonstrate the following:

1. There are finite limits to the periods over which asset values may be averaged and gains and losses amortized, not just in order to maintain stability in the pension funding process, but also in order to stabilise it efficiently. There is a limit to the *total* amount of smoothing used in actuarial valuations.

2. The typical choice of between 1 and 5 years, both for the term over which asset values are averaged and for the period over which gains and losses are amortized, appears to be efficient under normal economic conditions (see Tables 1 and 2).

3. The choice of intervals over which to average asset values and over which to amortize asset gains and losses must be made *in combination.*

4. Asset valuation and asset gain or loss amortization have a complementary smoothing function in the pension funding process and cannot be meaningfully considered separately.

12 Variability of the Actuarial Asset Value

Finally, the variability of the actuarial asset value generated by the "Moving Average of Market" or "Deferred Recognition" or "Adjusted Market" or "Write-up" methods, as defined in equations (15), (17) and (20), is examined.

PROPOSITION 3 *Suppose that $r = i$. If and only if $\sigma^2 \sum \beta_j^2 < 1$, then*

$$\lim_{t\to\infty} \mathrm{E}[f(t) - F(t)]^2 < \infty, \tag{79}$$

$$\lim_{t\to\infty} \mathrm{Var}F(t) \leq \lim_{t\to\infty} \mathrm{Var}f(t). \tag{80}$$

Proof of Proposition 3. $\mathrm{E}[f(t)-F(t)]^2 = \mathrm{Var}f(t)+\mathrm{Var}F(t)-2\mathrm{Cov}[f(t),F(t)]+[\mathrm{E}f(t)-\mathrm{E}F(t)]^2$ and all the terms on the right hand side are convergent as $t \to \infty$ as shown in Propositions 1 and 2 provided the stability condition holds. As for inequality (80), we note from equations (71) and (73) that

$$\lim \mathrm{Var}f(t) - \lim \mathrm{Var}F(t) = V_\infty \sum_{j=0}^{m+n-2} (\lambda_j^2 - \nu_j^2). \tag{81}$$

Now $\lambda_j > 0$ and $\nu_j > 0$ for $0 \leq j \leq m+n-2$, and

$$\sum_{j=0}^{m+n-2} (\lambda_j - \nu_j) = \sum_{j=0}^{n-2} u^j \left(1 - \frac{j+1}{n}\right), \tag{82}$$

which is positive for $n > 1$. Equality follows when $n = 1$ and $F(t) = f(t)$. $\square$

The first part of Proposition 3 indicates that the deviation between the market value of plan assets and the actuarial smoothed value (as defined in equation (15) or (17) or (20)) remains bounded in the mean square. The actuarial smoothed value remains in the proximity of market value in the long term. It is generally understood in the context of the U.S. Employee Retirement Income Security Act, 1974 (ERISA) that the actuarial value of plan assets should reflect the current market value of plan assets and presumably not deviate excessively from it (Winklevoss, 1993, p. 172). The U.S. Internal Revenue Service also imposes a 20% corridor of market value within which the actuarial value must lie (McGill *et al.*, 1996, p. 679).

The second part of Proposition 3 indicates that the actuarial value of plan assets is less variable than the pure market value. Note also that no gain or loss emerges on average in the long term, in equation (55), when actuarial assumptions are unbiased ($r = i$). These are desirable properties of an asset valuation method. In this respect, see particularly the Standard of Practice for Valuation of Pension Plans of the Canadian Institute of Actuaries (1994, para. 5.01) as well as the Actuarial Standard of Practice No. 4 of the Actuarial Standards Board (1993, para. 5.2.6). These facts qualify the particular variants of the

"Moving Average of Market", "Deferred Recognition", "Adjusted Market" and "Write-up" methods defined in this paper as being suitable for pension plan asset valuation, given the aforementioned constraints on the combined choice of averaging and amortization periods.

13 Conclusion

The "Moving Average of Market", "Deferred Recognition", "Adjusted Market" and "Write-up" methods of valuing the assets of defined benefit pension plans were defined and shown to be identical (disregarding initial conditions). These methods involve an arithmetic average over the market values of plan assets, allowing for cash flows and the time value of money. In the context of the simple pension plan model used by Dufresne (1989) and in which only asset gains and losses are assumed to emerge, it was shown that the asset valuation methods smooth these gains and losses and defer their recognition. The smoothed value of the loss, the supplementary contribution paid to defray losses and the unfunded liabilities based both on market and actuarial asset values were analyzed in terms of the asset gains and losses (at market value).

The losses (at market value) are shown to be zero on average and uncorrelated over time when rates of return on plan assets are random and when the actuarial assumption as to the rate of return is unbiased. The stability of the pension system was explored and its first and second moments were obtained by following the method of Dufresne (1989). Numerical work appeared to confirm the conclusion of Owadally & Haberman (1999) concerning the existence of an efficient range of amortization periods (m) based on Dufresne's (1988) criterion of minimizing the variability in the pension plan funding level and contribution rate in the long term. To achieve efficient funding, the periods (n) over which asset values may be averaged are likewise subject to a maximum. Typical choices of n and m between 1 and 5 years appear to be efficient. Asset valuation and amortization of asset gains and losses have a complementary smoothing function and, if the pension funding process is to remain stable and efficient, the total amount of smoothing in the actuarial management of this process is restricted. Finally, the actuarial value of plan assets under the asset valuation methods was shown to be more stable than the market value of assets.

Various avenues for further research are possible. The simplifying assumptions at the outset of the model could be relaxed. Inflation could be explicitly modelled, along with the returns on various asset types. The pension liability valuation discount rate should be bond-based. Comparisons with other methods of asset valuation and of amortization of gains and losses are also possible. The choice of averaging and amortization periods under statutory requirements, such as the restrictions imposed by the U.S. Internal Revenue Service, on actuarial asset valuation methods should also be investigated.

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σ	i	$m = 1$	3	5	7	9	10	15	20	25
0.1	1%	53	52	51	49	48	48	45	41	35
	3%	23	22	21	20	19	18	†	†	†
	5%	15	14	13	12	2	†	†	†	†
	10%	8	7	2	†	†	†	†	†	†
	15%	6	4	†	†	†	†	†	†	†
0.15	1%	37	36	34	33	31	31	26	2	†
	3%	20	19	17	16	14	13	†	†	†
	5%	13	12	11	9	†	†	†	†	†
	10%	8	7	†	†	†	†	†	†	†
	15%	5	4	†	†	†	†	†	†	†
0.2	1%	26	25	23	21	20	18	†	†	†
	3%	16	15	14	12	2	†	†	†	†
	5%	12	11	9	2	†	†	†	†	†
	10%	7	6	†	†	†	†	†	†	†
	15%	5	2	†	†	†	†	†	†	†
0.25	1%	19	17	16	14	2	2	†	†	†
	3%	13	12	10	2	†	†	†	†	†
	5%	10	9	7	†	†	†	†	†	†
	10%	7	5	†	†	†	†	†	†	†
	15%	5	2	†	†	†	†	†	†	†

Table 1: [$\lim \mathrm{Varc}(t)$]-minimizing values of n for various choices of $\{\sigma,\ i,\ m\}$. † indicates that $\lim \mathrm{Varc}(t)$ increases monotonically with n with smallest value at $n = 1$.

σ	i	$n=1$	3	5	7	9	10	15	20	25
0.1	1%	68	66	64	63	61	60	56	51	45
	3%	33	32	30	29	27	26	†	†	†
	5%	22	21	19	17	15	2	†	†	†
	10%	13	11	9	†	†	†	†	†	†
	15%	9	7	†	†	†	†	†	†	
0.15	1%	44	42	40	38	36	35	30	2	†
	3%	26	24	23	21	19	17	†	†	†
	5%	19	17	15	13	†	†	†	†	†
	10%	11	10	2	†	†	†	†	†	†
	15%	8	6	†	†	†	†	†	†	
0.2	1%	29	27	26	24	21	20	†	†	†
	3%	20	18	17	14	2	†	†	†	†
	5%	16	14	12	2	†	†	†	†	†
	10%	10	8	†	†	†	†	†	†	
	15%	8	6	†	†	†	†	†	†	
0.25	1%	21	19	17	15	2	2	†	†	†
	3%	16	14	12	2	†	†	†	†	†
	5%	13	11	9	†	†	†	†	†	†
	10%	9	7	†	†	†	†	†		
	15%	7	5	†	†	†	†	†		

Table 2: [$\lim \mathrm{Varc}(t)$]-minimizing values of m for various choices of $\{\sigma,\ i,\ n\}$. † indicates that $\lim \mathrm{Varc}(t)$ increases monotonically with m with smallest value at $m=1$. Blanks indicate instability.

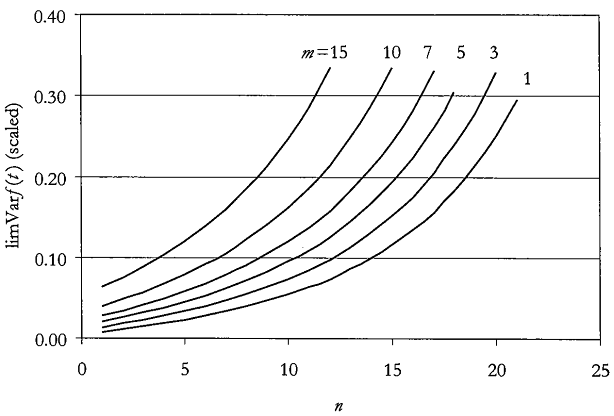

Figure 1: $\lim \mathrm{Var} f(t)$ (scaled) against n for various m. $i = 10\%$, $\sigma = 10\%$.

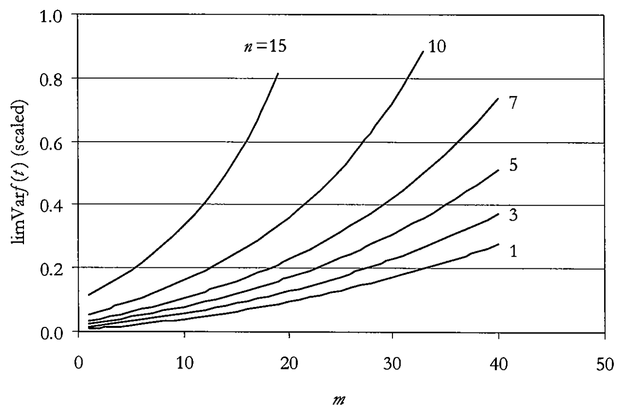

Figure 2: $\lim \mathrm{Var} f(t)$ (scaled) against m for various n. $i = 10\%$, $\sigma = 10\%$.

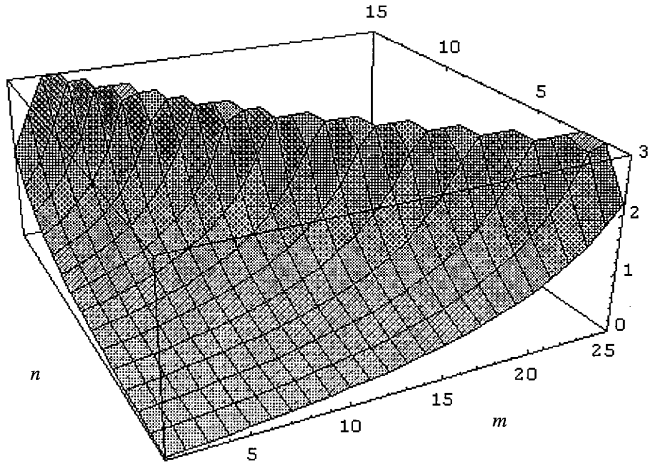

Figure 3: $\lim \mathrm{Var} f(t)$ (scaled) against n and m. $i = 10\%$, $\sigma = 25\%$.

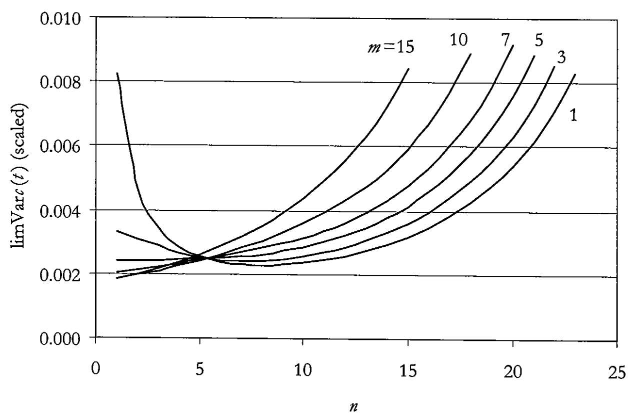

Figure 4: $\lim \mathrm{Var}c(t)$ (scaled) against n for various m. $i = 10\%$, $\sigma = 10\%$.

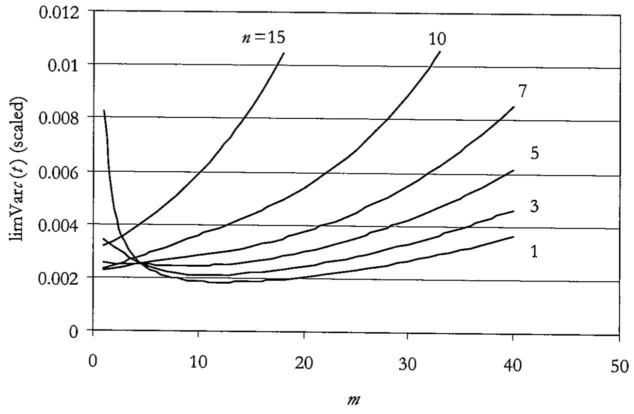

Figure 5: $\lim \mathrm{Var}c(t)$ (scaled) against m for various n. $i = 10\%$, $\sigma = 10\%$.

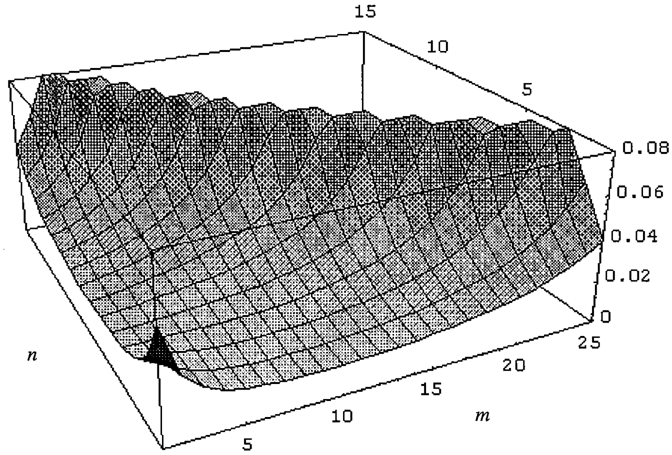

Figure 6: $\lim \mathrm{Varc}(t)$ (scaled) against n and m. $i = 10\%$, $\sigma = 25\%$.

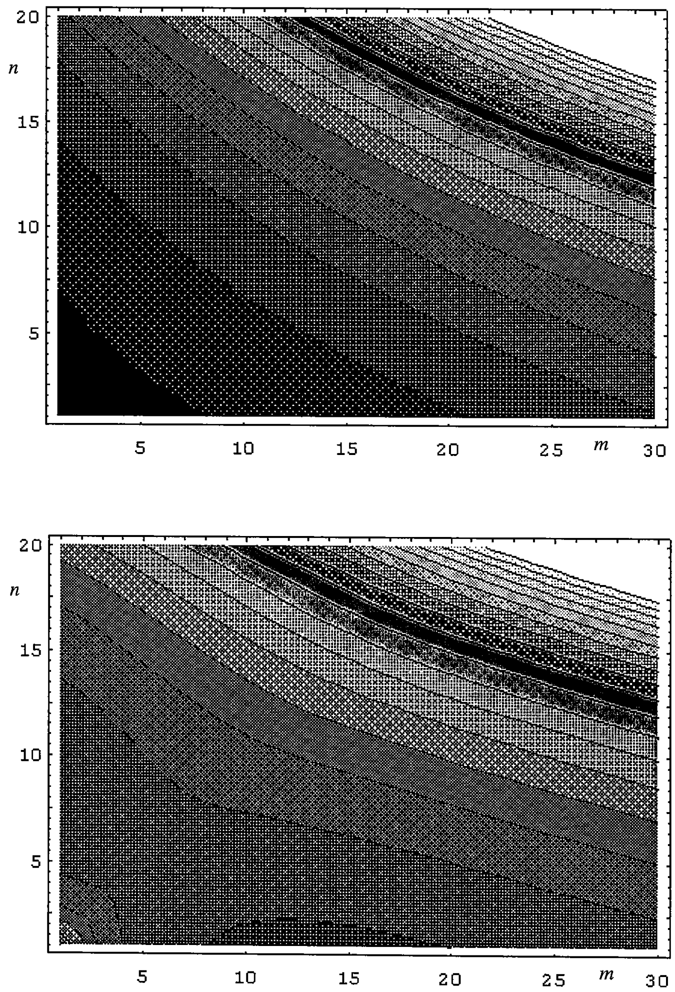

Figure 7: Contour plots of $\lim \mathrm{Var} f(t)$ (above) and $\lim \mathrm{Var} c(t)$ (below) against m and n. $i = 10\%$, $\sigma = 5\%$. Lighter shading represents higher values.

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