CAS_2019_410

Free flexural vibration of tapered beams

J.R. Banerjee and A. Ananthapuvirajah

Reply to the reviewers' comments:

The authors are grateful to the reviewers for their careful and studied assessment of the paper. All three reviewers have recommended publication of the paper more or less in its existing form without the need for any major revision. The authors have nevertheless revised the paper by taking into account all of the reviewers' mild and non-insistent comments. The reviewers' comments, subtle though they were, helped the authors to improve the paper. The paper is without doubt much improved as a result of the reviewer's comments. Some details of the revision work are given below.

Reviewer 1

The reviewer has made a very important point by implying that the bending moment and shear force at a point on the tapered beam should be given in explicit form as a function of V and x. This is reasonable and justified. The authors have taken this point on board in revising the paper, see Eqs. (10)-(12) of the revised paper. Based on the reviewer's comment the authors also felt that it was instructive to give a new figure showing the sign convention for bending moment and shear force (Fig. 2 of the revised paper). This is necessary to improve the clarity of the paper.

Reviewer 2

The reviewer's comment to provide a literature review explaining the contributions made by earlier investigators is perfectly valid and legitimate. The authors have now added an additional paragraph in their introduction, giving a commentary on earlier research. Following the other comment made by the reviewer, a new reference [Ref 21] has been added to the paper. The comments made are all helpful and much appreciated.

Reviewer 3

The reviewer has made complimentary remarks in all respect of the authors' work. The authors are deeply grateful for and appreciative of the comments.

Free flexural vibration of tapered beams

J.R. Banerjee[*], A. Ananthapuvirajah
Department of Mechanical Engineering and Aeronautics
City, University of London, Northampton Square
London EC1V 0HB, United Kingdom.

Abstract

The free flexural vibration behaviour of a range of tapered beams is investigated by making use of the exact solutions of the governing differential equations and then imposing the necessary boundary conditions. This research is prompted by a recently published paper in Computers and Structures which used the Frobenius method of series solution but needed 130 to 300 terms in the series when solving the free vibration problem of tapered beams using the transfer matrix method. This paper investigates the same problem using a direct approach, but with a twofold purpose. The first is to show that an exact solution for the problem is possible by using Bessel functions rather than relying on a series solution which is somehow unnecessary and from a computational standpoint inefficient. The second reason for writing this paper is to broaden the scope of the recently published paper by not focusing on only one type of cross-section, but on a range of cross-sections and yet retaining the exactness of the results. The application of Bessel functions is demonstrated when developing the theory and computing the numerical results. The results presented can be used as an aid to validate the finite element and other approximate methods.

Keywords: Free vibration; Tapered beam; Bessel function.

---

[*]Corresponding author, email: j.r.banerjee@city.ac.uk

1. Introduction

The free vibration of rotating tapered beams using the dynamic stiffness method was robustly investigated by Banerjee et al. [1] more than a decade ago. They presented a comprehensive set of results, including those that apply to non-rotating tapered beams which are computed as a degenerate case of the rotating beam by setting the rotational speed to zero. In a recently published paper Lee and Lee [2] studied the free vibration behaviour of a non-rotating tapered beam with cantilever boundary conditions and compared their results with those of the degenerate case of [1], showing excellent agreement. The authors of [2] used [1] as the key reference when validating their results. The methodology used in both [1] and [2] was based on the Frobenius method of series solution when seeking the solution of the governing differential equations. Of course, the obvious difference between papers [1] and [2] is that the former deals with rotating tapered beams whereas the latter is focused solely on non-rotating beams. For both sets of problems, the governing differential equations contain variable coefficients which are functions of the independent variable defined as the spatial coordinate along the length of the beam. Such differential equations with variable coefficients can nevertheless be regarded as linear for which a series solution can be applied whenever needed, and of course preferably when closed form exact analytical solutions are not possible. To this end, the use of the Frobenius method of series solution as a last resort was justified by the authors of [1] because a closed form analytical solution for the rotating tapered beam was not achievable. However, this is clearly not the case with [2] which deals with a non-rotating tapered beam and relies on the Frobenius method of series solution rather unnecessarily because a closed form solution for the problem using Bessel functions exists. The investigators of [2] required between 130 to 300 terms in their series solution for satisfactory convergence of results. This is computationally expensive and certainly avoidable. The current paper addresses this issue and provides an exact solution for the problem described in [2]. Furthermore, the tapered beam considered in [2] is restrictive in its choice of beam cross-sections in that the area variation was assumed linear whereas the second moment of area variation was considered cubic in terms of the beam length parameter. The current paper broadens the scope of [2] by extending such variation of cross-sectional properties to allow another set of cross-sectional variation so that the area of cross-section follows a square law whereas the second moment of area follows a fourth power variation along the length of the beam. When both these types of variations in cross-sectional properties are considered together, as in the present paper, the analysis will cover a substantial majority of all practical cross-sections. Although the analytical development of this paper is no doubt in marked contrast to that of Lee and Lee [2], it is not the primary object of this paper to criticise or amend their work. However, it is necessary to highlight that closed form analytical solution of governing differential equations representing a physical system offers a better alternative to achieve both computational efficiency and accuracy. Furthermore, some of the inaccuracies in [2] are rectified in this paper. Within the context of free vibration analysis of tapered beams, there is indeed, a wealth of literature, and for interested readers who wish to develop the subject further, a carefully selected sample of papers [3-20] is appended to this paper in chronological order. Some pertinent contributions made in these publications are briefly reviewed next.

Rao [3] determined the fundamental natural frequency of a uniformly tapered cantilever beam by using the Galerkin method whereas Conway and Dubil [4] utilised approximate polynomials to represent Bessel functions when formulating the frequency determinant of a truncated-cone and wedge with simply supported, clamped and free boundary conditions. Gaines and Volterra [6] on the other hand established the upper and lower bounds of the first three natural frequencies of a cantilever beam of variable cross-sections representing cones, truncated cones, wedges and truncated wedges by using Rayleigh-Ritz method. They included the effects of shear deformation and rotatory inertia and compared their results with those obtained using the classical Bernoulli-Euler theory. Later, Sanger [7] studied the transverse vibration of a class of non-uniform beams for classical boundary conditions, focusing on receptances, frequency equations, mode shapes and natural frequencies. Subsequently, Mabie and Rogers [8] made useful contributions when investigating the transverse vibrations of double tapered cantilevered beams by applying the Bernoulli-Euler theory. In contrast, Thomas and Dokumaci [9] used the finite element method to determine the natural frequencies of tapered beams for various boundary conditions and they compared their results with exact analytical solutions. Klein [10] made an interesting contribution by combining the advantages of the finite element and Rayleigh-Ritz methods when investigating the transverse vibration of non-uniform beams. An interesting feature of Klein's work is that experimental results were used to corroborate the theoretical predictions. Goel [12] took the research on transverse vibration of tapered beams a step further by elastically restraining the ends of the beam with spring attachments. He also studied the effect of an attached concentrated mass on the natural frequencies of the tapered beam. Downs [13] published a comprehensive set of results for natural frequencies of non-uniform beams with different boundary conditions using closed form analytical solution. To [14] performed the free vibration analysis of tapered beams by developing a higher order finite element. Zhou and Cheung [17] used Rayleigh-Ritz method by choosing a set of admissible functions and provided results for the natural frequencies of tapered beams with various boundary conditions. Later, they extended their work to cover tapered Timoshenko beams [18]. Interestingly Wu and Chen [19] investigated the free vibration behaviour of a wedge beam for various boundary conditions by placing point masses on the beam at arbitrary positions and compared their results with those obtained by using the finite element method. A different, but related contribution which partly motivated this work is that of Naprstek and Fischer [21].

2. Theory

Fig. 1 shows in a right-handed Cartesian co-ordinate system, the isometric, front and plan views of two types of linearly tapered beam of solid rectangular cross-section with the X-axis coinciding with the axis of the beam. In Fig. 1(a), the depth of the beam is varying linearly whereas its width remains constant along the length so that the cross-sectional area $A(x)$ varies linearly, while the second moment of area $I(x)$ follows a cubic variation. By contrast, in Fig. 1(b) both the depth and width of the beam vary linearly so that the area of cross-section and second moment of area follow a square and a fourth power variations of the length variable, respectively. These two types of variation of the cross-sectional properties can be described by substituting $n = 1$ and $n = 2$ in Eq. (1) below where A_g and I_g are respectively the area and the second moment of area of the cross-section at the left-hand end, c is the taper ratio and L is the length of the tapered beam. It should be noted that a substantial majority of practical cross-sections are covered by $n = 1$ and $n = 2$ and the rectangular cross-section shown here in Fig. 1 is only for convenience. For instance, for a tapered beam with a thin walled circular cross-section of constant thickness and linearly varying diameter, the value of n will be 1 whereas if both the thickness and the diameter vary linearly the value of n will be 2. Thus, a large number of other cross-sections can be constructed by using $n = 1$ and $n = 2$ in Eq. (1).

$$A(x) = A_g\left(1 + c\frac{x}{L}\right)^n \;;\; I(x) = I_g\left(1 + c\frac{x}{L}\right)^{n+2} \tag{1}$$

The characterisation of a tapered beam using Eq. (1) clearly indicates that for positive values of c the beam tapers upward from the thin end (g) at the left and to the thick end (h) at the right so that the area and second of area at the right-hand end are $A_h = A_g(1 + c)^n$ and $I_h = I_g(1 + c)^{n+2}$, respectively. If the beam tapers downward with the thick end (h) at the left and thin end (g) on the right, the alternative form of Eq. (1) is given by

$$A(x) = A_h\left(1 - \bar{c}\frac{x}{L}\right)^n \;;\; I(x) = I_h\left(1 - \bar{c}\frac{x}{L}\right)^{n+2} \tag{2}$$

where $\bar{c}$ is a new taper ratio of the tapering downward beam which has a positive value.

Clearly, the area and second moment of area at the right-hand end of the tapered beam corresponding to Eq. (2) are respectively $A_g = A_h(1 - \bar{c})^n$ and $I_g = I_h(1 - \bar{c})^{n+2}$. It can be shown with the help of Eqs. (1) and (2) that the taper ratios c and $\bar{c}$ for the tapering upward and tapering downward beams can be related as follows.

$$\bar{c} = \frac{c}{1 + c}; \quad c = \frac{\bar{c}}{1 - \bar{c}} \tag{3}$$

The introduction of Eqs. (1) and (2) showing both tapering upward and tapering downward beams was necessitated by the fact that in the published literature both forms of the representation of the tapered beam have been used. For instance, some investigators [5, 15, 16] have used the form of Eq. (1) to describe the taper whereas some others [4, 8, 14] have used the form of Eq. (2). This paper provides comparative results using both values of taper ratios c and $\bar{c}$. This will assist future investigators to validate their results, regardless of their definition of taper. Banerjee et al. [1] used the form of Eq. (2) in their earlier work, but in the current paper, the form of Eq. (1) is chosen for diversity. The essential point is that so long as the equivalency between c and $\bar{c}$ exists, (see Eq. (3)), it does not matter which expression is used. However, it needs to be emphasized that in the analysis that follows both c and $\bar{c}$ must be taken as positive, otherwise numerical problems will be encountered because the exact solutions obtained from the theory developed later are based on Bessel functions for which negative arguments, particularly for the Bessel functions of second kind, will result in complex quantities. Clearly the limits of c and $\bar{c}$ are within the bounds $0 < c < \infty$ and $0 < \bar{c} < 1$, respectively, as is evident from Eqs. (1) and (2). It should be noted that the investigation carried out by Lee and Lee [2] is based on Eq. (2) to represent the tapered beam and their investigation covers only the case when $n = 1$.

The equation of motion of the tapered beam in flexural vibration can be obtained in the usual notation as [2, 22]

$$\frac{\partial^2}{\partial x^2}\left\{EI(x)\frac{\partial^2 v}{\partial x^2}\right\} + \rho A(x)\frac{\partial^2 v}{\partial t^2} = 0 \tag{4}$$

where v is the flexural displacement, t is time, E and ρ are the Young's modulus and density of the beam material and $I(x)$ and $A(x)$ are defined in Eq. (1).

Substituting for $A(x)$ and $I(x)$ from Eq. (1), setting $\xi = 1 + c\left(\frac{x}{L}\right)$, and noting that for harmonic vibration

$$v(\xi,t) = V(\xi)\sin\omega t \tag{5}$$

gives

$$\xi^2\frac{d^4V}{d\xi^4} + 2(n+2)\xi\frac{d^3V}{d\xi^3} + (n+2)(n+1)\frac{d^2V}{d\xi^2} - \frac{\mu_g^4}{c^4}V = 0 \tag{6}$$

where

$$\mu_g = \sqrt[4]{\frac{\rho A_g \omega^2 L^4}{EI_g}} \tag{7}$$

The solution of Eq. (6) is given by [22]

$$V(\phi) = \frac{1}{\phi^n}\{C_1J_n(\phi) + C_2Y_n(\phi) + C_3I_n(\phi) + C_4K_n(\phi)\} \tag{8}$$

where

$$\phi = \left(\frac{2\mu_g}{c}\right)\sqrt{\xi} \tag{9}$$

and J_n, Y_n, I_n and K_n are Bessel functions of, respectively, the first, second, modified first and modified second kinds. Their argument is ϕ and so they are functions of ξ (see Eq. (9)). It should be noted that Bessel functions can be computed as accurately as the intrinsic functions such as the trigonometric and hyperbolic functions. There are standard subroutines available to accomplish such tasks.

With the help of Eq. (8) and following the sign convention of Fig. 2, the expressions for the amplitudes of the anticlockwise (tangential) bending rotation $\psi(x)$, hogging bending moment $M(x)$ and clockwise shear force $S(x)$ can be obtained after some mathematical manipulation of the Bessel functions

$$\psi(x) = \frac{dV}{dx} = -\frac{\mu_g}{L\phi^n\sqrt{\xi}}\{C_1J_{n+1} + C_2Y_{n+1} - C_3I_{n+1} + C_4K_{n+1}\} \tag{10}$$

$$M(x) = -EI(x)\frac{d^2V}{dx^2} = -\frac{EI_g\xi^{(n+1)}\mu_g^2}{L^2\phi^n}\{C_1J_{n+2} + C_2Y_{n+2} + C_3I_{n+2} + C_4K_{n+2}\} \tag{11}$$

$$S(x) = \frac{d}{dx}\left[EI(x)\frac{d^2V}{dx^2}\right] = \frac{EI_g\xi^{(n+1/2)}\mu_g^3}{L^3\phi^n}\{C_1J_{n+1} + C_2Y_{n+1} + C_3I_{n+1} - C_4K_{n+1}\} \quad (12)$$

Using the expressions for bending displacement, bending rotation, bending moment and shear force given by Eqs. (8), (10), (11) and (12), respectively and applying appropriate boundary conditions, the natural frequencies and mode shapes of a tapered beam can be computed in a straightforward manner. For example, for a cantilever tapered beam the bending moment and shear force are zero at the free end whereas bending displacement and bending rotation are zero at the built-in end. Referring to Fig. 1 and considering the built-in end to be the thick end located at the right-hand side and the free end to be the thin end located at the left-hand side, the following boundary conditions will apply

$$\text{At } x = 0 \text{ (i.e. at } \xi = 1\text{)},\ M = 0 \text{ and } S = 0 \quad (13)$$

$$\text{At } x = L \text{ (i.e. at } \xi = 1 + c\text{)},\ V = 0 \text{ and } \psi = 0 \quad (14)$$

Substituting Eq. (13) into Eqs. (11) and (12), and Eq. (14) into Eqs. (8) and (10), the constants $C_1 - C_4$ can be eliminated to give the frequency equation yielding the natural frequencies of the cantilevered tapered beam. The mode shapes corresponding to each frequency can be computed in the usual way by setting one of the constants to an arbitrary value and expressing the remaining constants in terms of the chosen one. Results for other boundary conditions can be obtained in a similar manner by imposing appropriate boundary conditions for displacements and forces at the ends of the tapered beam.

3. Numerical results and discussion

Using the theory developed above, numerical results for natural frequencies and mode shapes are computed for a selective range of tapered beams for both cases, represented by n = 1 and n = 2. In order to make the results universal, the natural frequencies are non-dimensionalised with respect to the length (L), and bending rigidity (EI_h) and mass per unit length (ρA_h) at the thick end of the tapered beam so that

$$\lambda_i = \omega_i\sqrt{\frac{\rho A_h L^4}{EI_h}} \quad (15)$$

where ω_i is the i^{th} computed natural frequency in rad/s and λ_i is the corresponding i^{th} non-dimensional natural frequency of the tapered beam.

Note that the non-dimensional natural frequency $\overline{\omega}$ defined by Lee and Lee [2] and the corresponding results reported in their Tables 1 and 2 are somewhat inconsistent and seem to be in error. This assertion by the current authors is made for two reasons. First, the negative sign in Eq. (11) of [2] will make the non-dimensional natural frequency $\overline{\omega}$ complex which is absurd. Secondly, the taper ratio c should not have been included in Eq. (11) of [2] when defining the non-dimensional parameter $\overline{\omega}$ and then validating the results against those reported by Banerjee et al. [1], see Tables 1 and 2 of [2].Within this context and for precise numerical comparison of results, it is to be noted that Banerjee et al. [1] did not use the taper ratio c when defining their non-dimensional natural frequency parameter, (see their Eq. (51)).

When presenting the results for the natural frequencies and mode shapes of a tapered beam, the authors have carefully chosen selected values of the taper ratio c (and $\bar{c}$) for both $n = 1$ and $n = 2$. Three sets of classical boundary conditions of the tapered beam have been considered. These are Clamped-Free (C-F), Pinned-Pinned (P-P) and Clamped-Clamped (C-C), respectively so that interested readers can validate their results. For the C-F boundary condition, the thick end (end h of Fig. 1) of the tapered beam has been cantilevered. For each of the three sets of boundary conditions, Table 1 shows the first five natural frequencies of the tapered beam computed using three different values of the taper ratios c (and $\bar{c}$) when $n = 1$. The corresponding results for $n = 2$ are given in Table 2. The computed results shown in Tables 1 and 2 provide six figure accuracy as benchmark solutions so that interested readers can validate their results. Within the given limits of the computational accuracy, the results shown in Tables 1 and 2 agree very well (almost completely) with those reported in the literature [13]. Representative mode shapes of the tapered beam for the above three classical boundary conditions are shown in Figs. 3 and 4 for $n = 1$ and $n = 2$, respectively when the taper ratio c was set to 1.

4. Conclusions

Starting from the solution of the governing differential equations using Bessel functions, an exact analytical procedure is given to compute the natural frequencies and mode shapes of tapered beams for classical boundary conditions. The scope of a recently published paper which inspired this work but was based on an approximate method of series solution is substantially extended to cover a wide range of beam cross sections. Some of the inaccuracies of the published paper are addressed and errors are corrected. The results presented can be used to validate finite element and other approximate methods.

Acknowledgements

The authors are grateful to EPSRC (UK) for funding a related project which helped this work. They also wish to thank Professor David Kennedy for many stimulating discussions on the properties of Bessel functions.

References

[1] Banerjee JR, Su H, Jackson DR. Free vibration of rotating tapered beams using the dynamic stiffness method. J Sound Vib 2006;298:1034-54.

[2] Lee JW, Lee JY. Free vibration analysis using the transfer-matrix method on a tapered beam. Comput Struct 2016;164:75-82.

[3] Rao JS. The fundamental flexural vibration of a cantilever beam of rectangular cross section with uniform taper. Aero Quart 1965;16:139–44. doi:10.1017/s000192590000336x.

[4] Conway HD, Dubil JF. Vibration frequencies of truncated-cone and wedge beams. J Appl Mech 1965;32:932-4. doi:10.1115/1.3627338.

[5] Wang HC, Worley WJ. Tables of natural frequencies and nodes for transverse vibration of tapered beams. NASA CR-443 1966.

[6] Gaines JH, Volterra E. Transverse vibrations of cantilever bars of variable cross section. J Acoust Soc Amer 1966;39:674-9.

[7] Sanger DJ. Transverse vibration of a class of non-uniform beams. J Mech Eng Sci 1968;10:111-20.

[8] Mabie HH, Rogers CB. Transverse vibrations of double-tapered cantilever beams. J Acoust Soc Amer 1972;51:1771-4.

[9] Thomas J, Dokumaci E. Improved finite elements for vibration analysis of tapered beams. Aero Quart 1973;24:39–46.

[10] Klein L. Transverse vibrations of non-uniform beams. J Sound Vib 1974;37:491–505. doi:10.1016/S0022-460X(74)80029-5.

[11] Avakian A, Beskos DE. Use of dynamic stiffness influence coefficients in vibrations of non-uniform beams. J Sound Vib 1976;47:292–5. doi:10.1016/0022-460X(76)90725-2.

[12] Goel RP. Transverse vibrations of tapered beams. J Sound Vib 1976;47:1–7. doi:10.1016/0022-460X(76)90403-X.

[13] Downs B. Reference frequencies for the validation of numerical solutions of transverse vibrations of non-uniform beams. J Sound Vib 1978;61:71–8. doi:10.1016/0022-460X(78)90042-1.

[14] To CWS. Higher order tapered beam finite elements for vibration analysis. J Sound Vib 1979;63:33–50. doi:10.1016/0022-460X(79)90375-4.

[15] Rutledge WD, Beskos DE. Dynamic analysis of linearly tapered beams. J Sound Vib 1981;79:457–62. doi:10.1016/0022-460X(81)90323-0.

[16] Chaudhari TD, Maiti SK. Modelling of transverse vibration of beam of linearly variable depth with edge crack. Eng Fract Mech 1999;63:425–45. doi:10.1016/s0013-7944(99)00029-6.

[17] Zhou D, Cheung YK. The free vibration of a type of tapered beams. Comput Methods Appl Mech Eng 2000;188:203–16. doi:10.1016/S0045-7825(99)00148-6.

[18] Zhou D, Cheung YK. Vibrations of tapered Timoshenko beams in terms of static Timoshenko beam functions. J Appl Mech 2001;68:596-602. doi:10.1115/1.1357164.

[19] Wu JS, Chen DW. Bending vibrations of wedge beams with any number of point masses. J Sound Vib 2003;262:1073–90. doi:10.1016/S0022-460X(02)01084-2.

[20] Firouz-Abadi RD, Rahmanian M, Amabili M. Exact solutions for free vibrations and buckling of double tapered columns with elastic foundation and tip mass. J Vib Acoust 2013;135:051017-1-10. doi:10.1115/1.4023991.

[21] Náprstek J, Fischer C. Static and dynamic analysis of a beam assemblies using a differential system on an oriented graph. Comput Struct 2015;155:28-41.

[22] Kloušek V. Dynamics in engineering structures. Newnes-Butterworth; 1973.

Highlights:

- Exact theory for free vibration of tapered beams using Bessel functions is developed
- The investigation extends earlier research and covers a wide range of tapered beams
- Benchmark numerical results are provided to validate other methods
- Some of the inaccuracies in recently published research are corrected

Free flexural vibration of tapered beams

J.R. Banerjee*, A. Ananthapuvirajah
Department of Mechanical Engineering and Aeronautics
City, University of London, Northampton Square
London EC1V 0HB, United Kingdom.

Abstract

The free flexural vibration behaviour of a range of tapered beams is investigated by making use of the exact solutions of the governing differential equations and then imposing the necessary boundary conditions. This research is prompted by a recently published paper in Computers and Structures which used the Frobenius method of series solution but needed 130 to 300 terms in the series when solving the free vibration problem of tapered beams using the transfer matrix method. This paper investigates the same problem using a direct approach, but with a twofold purpose. The first is to show that an exact solution for the problem is possible by using Bessel functions rather than relying on a series solution which is somehow unnecessary and from a computational standpoint inefficient. The second reason for writing this paper is to broaden the scope of the recently published paper by not focusing on only one type of cross-section, but on a range of cross-sections and yet retaining the exactness of the results. The application of Bessel functions is demonstrated when developing the theory and computing the numerical results. The results presented can be used as an aid to validate the finite element and other approximate methods.

Keywords: Free vibration; Tapered beam; Bessel function.

---

*Corresponding author, email: j.r.banerjee@city.ac.uk

1. Introduction

The free vibration of rotating tapered beams using the dynamic stiffness method was robustly investigated by Banerjee et al. [1] more than a decade ago. They presented a comprehensive set of results, including those that apply to non-rotating tapered beams which are computed as a degenerate case of the rotating beam by setting the rotational speed to zero. In a recently published paper Lee and Lee [2] studied the free vibration behaviour of a non-rotating tapered beam with cantilever boundary conditions and compared their results with those of the degenerate case of [1], showing excellent agreement. The authors of [2] used [1] as the key reference when validating their results. The methodology used in both [1] and [2] was based on the Frobenius method of series solution when seeking the solution of the governing differential equations. Of course, the obvious difference between papers [1] and [2] is that the former deals with rotating tapered beams whereas the latter is focused solely on non-rotating beams. For both sets of problems, the governing differential equations contain variable coefficients which are functions of the independent variable defined as the spatial coordinate along the length of the beam. Such differential equations with variable coefficients can nevertheless be regarded as linear for which a series solution can be applied whenever needed, and of course preferably when closed form exact analytical solutions are not possible. To this end, the use of the Frobenius method of series solution as a last resort was justified by the authors of [1] because a closed form analytical solution for the rotating tapered beam was not achievable. However, this is clearly not the case with [2] which deals with a non-rotating tapered beam and relies on the Frobenius method of series solution rather unnecessarily because a closed form solution for the problem using Bessel functions exists. The investigators of [2] required between 130 to 300 terms in their series solution for satisfactory convergence of results. This is computationally expensive and certainly avoidable. The current paper addresses this issue and provides an exact solution for the problem described in [2]. Furthermore, the tapered beam considered in [2] is restrictive in its choice of beam cross-sections in that the area variation was assumed linear whereas the second moment of area variation was considered cubic in terms of the beam length parameter. The current paper broadens the scope of [2] by extending such variation of cross-sectional properties to allow another set of cross-sectional variation so that the area of cross-section follows a square law whereas the second moment of area follows a fourth power variation along the length of the beam. When both these types of variations in cross-sectional properties are considered together, as in the present paper, the analysis will cover a substantial majority of all practical cross-sections. Although the analytical development of this paper is no doubt in marked contrast to that of Lee and Lee [2], it is not the primary object of this paper to criticise or amend their work. However, it is necessary to highlight that closed form analytical solution of governing differential equations representing a physical system offers a better alternative to achieve both computational efficiency and accuracy. Furthermore, some of the inaccuracies in [2] are rectified in this paper. Within the context of free vibration analysis of tapered beams, there is indeed, a wealth of literature, and for interested readers who wish to develop the subject further, a carefully selected sample of papers [3-20] is appended to this paper in chronological order. Some pertinent contributions made in these publications are briefly reviewed next.

Rao [3] determined the fundamental natural frequency of a uniformly tapered cantilever beam by using the Galerkin method whereas Conway and Dubil [4] utilised approximate polynomials to represent Bessel functions when formulating the frequency determinant of a truncated-cone and wedge with simply supported, clamped and free boundary conditions. Gaines and Volterra [6] on the other hand established the upper and lower bounds of the first three natural frequencies of a cantilever beam of variable cross-sections representing cones, truncated cones, wedges and truncated wedges by using Rayleigh-Ritz method. They included the effects of shear deformation and rotatory inertia and compared their results with those obtained using the classical Bernoulli-Euler theory. Later, Sanger [7] studied the transverse vibration of a class of non-uniform beams for classical boundary conditions, focusing on receptances, frequency equations, mode shapes and natural frequencies. Subsequently, Mabie and Rogers [8] made useful contributions when investigating the transverse vibrations of double tapered cantilevered beams by applying the Bernoulli-Euler theory. In contrast, Thomas and Dokumaci [9] used the finite element method to determine the natural frequencies of tapered beams for various boundary conditions and they compared their results with exact analytical solutions. Klein [10] made an interesting contribution by combining the advantages of the finite element and Rayleigh-Ritz methods when investigating the transverse vibration of non-uniform beams. An interesting feature of Klein's work is that experimental results were used to corroborate the theoretical predictions. Goel [12] took the research on transverse vibration of tapered beams a step further by elastically restraining the ends of the beam with spring attachments. He also studied the effect of an attached concentrated mass on the natural frequencies of the tapered beam. Downs [13] published a comprehensive set of results for natural frequencies of non-uniform beams with different boundary conditions using closed form analytical solution. To [14] performed the free vibration analysis of tapered beams by developing a higher order finite element. Zhou and Cheung [17] used Rayleigh-Ritz method by choosing a set of admissible functions and provided results for the natural frequencies of tapered beams with various boundary conditions. Later, they extended their work to cover tapered Timoshenko beams [18]. Interestingly Wu and Chen [19] investigated the free vibration behaviour of a wedge beam for various boundary conditions by placing point masses on the beam at arbitrary positions and compared their results with those obtained by using the finite element method. A different, but related contribution which partly motivated this work is that of Naprstek and Fischer [21].

2. Theory

Fig. 1 shows in a right-handed Cartesian co-ordinate system, the isometric, front and plan views of two types of linearly tapered beam of solid rectangular cross-section with the X-axis coinciding with the axis of the beam. In Fig. 1(a), the depth of the beam is varying linearly whereas its width remains constant along the length so that the cross-sectional area $A(x)$ varies linearly, while the second moment of area $I(x)$ follows a cubic variation. By contrast, in Fig. 1(b) both the depth and width of the beam vary linearly so that the area of cross-section and second moment of area follow a square and a fourth power variations of the length variable, respectively. These two types of variation of the cross-sectional properties can be described by substituting $n = 1$ and $n = 2$ in Eq. (1) below where A_g and I_g are respectively the area and the second moment of area of the cross-section at the left-hand end, c is the taper ratio and L is the length of the tapered beam. It should be noted that a substantial majority of practical cross-sections are covered by $n = 1$ and $n = 2$ and the rectangular cross-section shown here in Fig. 1 is only for convenience. For instance, for a tapered beam with a thin walled circular cross-section of constant thickness and linearly varying diameter, the value of n will be 1 whereas if both the thickness and the diameter vary linearly the value of n will be 2. Thus, a large number of other cross-sections can be constructed by using $n = 1$ and $n = 2$ in Eq. (1).

$$A(x) = A_g\left(1 + c\frac{x}{L}\right)^n \; ; \; I(x) = I_g\left(1 + c\frac{x}{L}\right)^{n+2} \tag{1}$$

The characterisation of a tapered beam using Eq. (1) clearly indicates that for positive values of c the beam tapers upward from the thin end (g) at the left and to the thick end (h) at the right so that the area and second of area at the right-hand end are $A_h = A_g(1 + c)^n$ and $I_h = I_g (1 + c)^{n+2}$, respectively. If the beam tapers downward with the thick end (h) at the left and thin end (g) on the right, the alternative form of Eq. (1) is given by

$$A(x) = A_h\left(1 - \bar{c}\frac{x}{L}\right)^n \; ; \; I(x) = I_h\left(1 - \bar{c}\frac{x}{L}\right)^{n+2} \tag{2}$$

where $\bar{c}$ is a new taper ratio of the tapering downward beam which has a positive value.

Clearly, the area and second moment of area at the right-hand end of the tapered beam corresponding to Eq. (2) are respectively $A_g = A_h(1 - \bar{c})^n$ and $I_g = I_h(1 - \bar{c})^{n+2}$. It can be shown with the help of Eqs. (1) and (2) that the taper ratios c and $\bar{c}$ for the tapering upward and tapering downward beams can be related as follows.

$$\bar{c} = \frac{c}{1 + c}; \quad c = \frac{\bar{c}}{1 - \bar{c}} \tag{3}$$

The introduction of Eqs. (1) and (2) showing both tapering upward and tapering downward beams was necessitated by the fact that in the published literature both forms of the representation of the tapered beam have been used. For instance, some investigators [5, 15, 16] have used the form of Eq. (1) to describe the taper whereas some others [4, 8, 14] have used the form of Eq. (2). This paper provides comparative results using both values of taper ratios c and $\bar{c}$. This will assist future investigators to validate their results, regardless of their definition of taper. Banerjee et al. [1] used the form of Eq. (2) in their earlier work, but in the current paper, the form of Eq. (1) is chosen for diversity. The essential point is that so long as the equivalency between c and $\bar{c}$ exists, (see Eq. (3)), it does not matter which expression is used. However, it needs to be emphasized that in the analysis that follows both c and $\bar{c}$ must be taken as positive, otherwise numerical problems will be encountered because the exact solutions obtained from the theory developed later are based on Bessel functions for which negative arguments, particularly for the Bessel functions of second kind, will result in complex quantities. Clearly the limits of c and $\bar{c}$ are within the bounds $0 < c < \infty$ and $0 < \bar{c} < 1$, respectively, as is evident from Eqs. (1) and (2). It should be noted that the investigation carried out by Lee and Lee [2] is based on Eq. (2) to represent the tapered beam and their investigation covers only the case when $n = 1$.

The equation of motion of the tapered beam in flexural vibration can be obtained in the usual notation as [2, 22]

$$\frac{\partial^2}{\partial x^2}\left\{EI(x)\frac{\partial^2 v}{\partial x^2}\right\} + \rho A(x)\frac{\partial^2 v}{\partial t^2} = 0 \tag{4}$$

where v is the flexural displacement, t is time, E and ρ are the Young's modulus and density of the beam material and $I(x)$ and $A(x)$ are defined in Eq. (1).

Substituting for $A(x)$ and $I(x)$ from Eq. (1), setting $\xi = 1 + c\left(\frac{x}{L}\right)$, and noting that for harmonic vibration

$$v(\xi,t) = V(\xi)\sin\omega t \tag{5}$$

gives

$$\xi^2\frac{d^4V}{d\xi^4} + 2(n+2)\xi\frac{d^3V}{d\xi^3} + (n+2)(n+1)\frac{d^2V}{d\xi^2} - \frac{\mu_g^4}{c^4}V = 0 \tag{6}$$

where

$$\mu_g = \sqrt[4]{\frac{\rho A_g \omega^2 L^4}{EI_g}} \tag{7}$$

The solution of Eq. (6) is given by [22]

$$V(\phi) = \frac{1}{\phi^n}\{C_1 J_n(\phi) + C_2 Y_n(\phi) + C_3 I_n(\phi) + C_4 K_n(\phi)\} \tag{8}$$

where

$$\phi = \left(\frac{2\mu_g}{c}\right)\sqrt{\xi} \tag{9}$$

and J_n, Y_n, I_n and K_n are Bessel functions of, respectively, the first, second, modified first and modified second kinds. Their argument is ϕ and so they are functions of ξ (see Eq. (9)). It should be noted that Bessel functions can be computed as accurately as the intrinsic functions such as the trigonometric and hyperbolic functions. There are standard subroutines available to accomplish such tasks.

With the help of Eq. (8) and following the sign convention of Fig. 2, the expressions for the amplitudes of the anticlockwise (tangential) bending rotation $\psi(x)$, hogging bending moment $M(x)$ and clockwise shear force $S(x)$ can be obtained after some mathematical manipulation of the Bessel functions

$$\psi(x) = \frac{dV}{dx} = -\frac{\mu_g}{L\phi^n\sqrt{\xi}}\{C_1 J_{n+1} + C_2 Y_{n+1} - C_3 I_{n+1} + C_4 K_{n+1}\} \tag{10}$$

$$M(x) = -EI(x)\frac{d^2V}{dx^2} = -\frac{EI_g \xi^{(n+1)} \mu_g^2}{L^2\phi^n}\{C_1 J_{n+2} + C_2 Y_{n+2} + C_3 I_{n+2} + C_4 K_{n+2}\} \tag{11}$$

$$S(x) = \frac{d}{dx}\left[EI(x)\frac{d^2V}{dx^2}\right] = \frac{EI_g \xi^{(n+1/2)} \mu_g^3}{L^3 \phi^n}\{C_1 J_{n+1} + C_2 Y_{n+1} + C_3 I_{n+1} - C_4 K_{n+1}\} \quad (12)$$

Using the expressions for bending displacement, bending rotation, bending moment and shear force given by Eqs. (8), (10), (11) and (12), respectively and applying appropriate boundary conditions, the natural frequencies and mode shapes of a tapered beam can be computed in a straightforward manner. For example, for a cantilever tapered beam the bending moment and shear force are zero at the free end whereas bending displacement and bending rotation are zero at the built-in end. Referring to Fig. 1 and considering the built-in end to be the thick end located at the right-hand side and the free end to be the thin end located at the left-hand side, the following boundary conditions will apply

$$\text{At } x = 0 \text{ (i.e. at } \xi = 1\text{)}, \ M = 0 \text{ and } S = 0 \quad (13)$$

$$\text{At } x = L \text{ (i.e. at } \xi = 1 + c\text{)}, \ V = 0 \text{ and } \psi = 0 \quad (14)$$

Substituting Eq. (13) into Eqs. (11) and (12), and Eq. (14) into Eqs. (8) and (10), the constants $C_1 - C_4$ can be eliminated to give the frequency equation yielding the natural frequencies of the cantilevered tapered beam. The mode shapes corresponding to each frequency can be computed in the usual way by setting one of the constants to an arbitrary value and expressing the remaining constants in terms of the chosen one. Results for other boundary conditions can be obtained in a similar manner by imposing appropriate boundary conditions for displacements and forces at the ends of the tapered beam.

3. Numerical results and discussion

Using the theory developed above, numerical results for natural frequencies and mode shapes are computed for a selective range of tapered beams for both cases, represented by n = 1 and n = 2. In order to make the results universal, the natural frequencies are non-dimensionalised with respect to the length (L), and bending rigidity (EI_h) and mass per unit length (ρA_h) at the thick end of the tapered beam so that

$$\lambda_i = \omega_i \sqrt{\frac{\rho A_h L^4}{EI_h}} \quad (15)$$

where ω_i is the i^{th} computed natural frequency in rad/s and λ_i is the corresponding i^{th} non-dimensional natural frequency of the tapered beam.

Note that the non-dimensional natural frequency $\overline{\omega}$ defined by Lee and Lee [2] and the corresponding results reported in their Tables 1 and 2 are somewhat inconsistent and seem to be in error. This assertion by the current authors is made for two reasons. First, the negative sign in Eq. (11) of [2] will make the non-dimensional natural frequency $\overline{\omega}$ complex which is absurd. Secondly, the taper ratio c should not have been included in Eq. (11) of [2] when defining the non-dimensional parameter $\overline{\omega}$ and then validating the results against those reported by Banerjee et al. [1], see Tables 1 and 2 of [2].Within this context and for precise numerical comparison of results, it is to be noted that Banerjee et al. [1] did not use the taper ratio c when defining their non-dimensional natural frequency parameter, (see their Eq. (51)).

When presenting the results for the natural frequencies and mode shapes of a tapered beam, the authors have carefully chosen selected values of the taper ratio c (and $\bar{c}$) for both n = 1 and n = 2. Three sets of classical boundary conditions of the tapered beam have been considered. These are Clamped-Free (C-F), Pinned-Pinned (P-P) and Clamped-Clamped (C-C), respectively so that interested readers can validate their results. For the C-F boundary condition, the thick end (end h of Fig. 1) of the tapered beam has been cantilevered. For each of the three sets of boundary conditions, Table 1 shows the first five natural frequencies of the tapered beam computed using three different values of the taper ratios c (and $\bar{c}$) when n = 1. The corresponding results for n = 2 are given in Table 2. The computed results shown in Tables 1 and 2 provide six figure accuracy as benchmark solutions so that interested readers can validate their results. Within the given limits of the computational accuracy, the results shown in Tables 1 and 2 agree very well (almost completely) with those reported in the literature [13]. Representative mode shapes of the tapered beam for the above three classical boundary conditions are shown in Figs. 3 and 4 for n = 1 and n = 2, respectively when the taper ratio c was set to 1.

4. Conclusions

Starting from the solution of the governing differential equations using Bessel functions, an exact analytical procedure is given to compute the natural frequencies and mode shapes of tapered beams for classical boundary conditions. The scope of a recently published paper which inspired this work but was based on an approximate method of series solution is substantially extended to cover a wide range of beam cross sections. Some of the inaccuracies of the published paper are addressed and errors are corrected. The results presented can be used to validate finite element and other approximate methods.

Acknowledgements

The authors are grateful to EPSRC (UK) for funding a related project which helped this work. They also wish to thank Professor David Kennedy for many stimulating discussions on the properties of Bessel functions.

References

[1] Banerjee JR, Su H, Jackson DR. Free vibration of rotating tapered beams using the dynamic stiffness method. J Sound Vib 2006;298:1034-54.

[2] Lee JW, Lee JY. Free vibration analysis using the transfer-matrix method on a tapered beam. Comput Struct 2016;164:75-82.

[3] Rao JS. The fundamental flexural vibration of a cantilever beam of rectangular cross section with uniform taper. Aero Quart 1965;16:139–44. doi:10.1017/s000192590000336x.

[4] Conway HD, Dubil JF. Vibration frequencies of truncated-cone and wedge beams. J Appl Mech 1965;32:932-4. doi:10.1115/1.3627338.

[5] Wang HC, Worley WJ. Tables of natural frequencies and nodes for transverse vibration of tapered beams. NASA CR-443 1966.

[6] Gaines JH, Volterra E. Transverse vibrations of cantilever bars of variable cross section. J Acoust Soc Amer 1966;39:674-9.

[7] Sanger DJ. Transverse vibration of a class of non-uniform beams. J Mech Eng Sci 1968;10:111-20.

[8] Mabie HH, Rogers CB. Transverse vibrations of double-tapered cantilever beams. J Acoust Soc Amer 1972;51:1771-4.

[9] Thomas J, Dokumaci E. Improved finite elements for vibration analysis of tapered beams. Aero Quart 1973;24:39–46.

[10] Klein L. Transverse vibrations of non-uniform beams. J Sound Vib 1974;37:491–505. doi:10.1016/S0022-460X(74)80029-5.

[11] Avakian A, Beskos DE. Use of dynamic stiffness influence coefficients in vibrations of non-uniform beams. J Sound Vib 1976;47:292–5. doi:10.1016/0022-460X(76)90725-2.

[12] Goel RP. Transverse vibrations of tapered beams. J Sound Vib 1976;47:1–7. doi:10.1016/0022-460X(76)90403-X.

[13] Downs B. Reference frequencies for the validation of numerical solutions of transverse vibrations of non-uniform beams. J Sound Vib 1978;61:71–8. doi:10.1016/0022-460X(78)90042-1.

[14] To CWS. Higher order tapered beam finite elements for vibration analysis. J Sound Vib 1979;63:33–50. doi:10.1016/0022-460X(79)90375-4.

[15] Rutledge WD, Beskos DE. Dynamic analysis of linearly tapered beams. J Sound Vib 1981;79:457–62. doi:10.1016/0022-460X(81)90323-0.

[16] Chaudhari TD, Maiti SK. Modelling of transverse vibration of beam of linearly variable depth with edge crack. Eng Fract Mech 1999;63:425–45. doi:10.1016/s0013-7944(99)00029-6.

[17] Zhou D, Cheung YK. The free vibration of a type of tapered beams. Comput Methods Appl Mech Eng 2000;188:203–16. doi:10.1016/S0045-7825(99)00148-6.

[18] Zhou D, Cheung YK. Vibrations of tapered Timoshenko beams in terms of static Timoshenko beam functions. J Appl Mech 2001;68:596-602. doi:10.1115/1.1357164.

[19] Wu JS, Chen DW. Bending vibrations of wedge beams with any number of point masses. J Sound Vib 2003;262:1073–90. doi:10.1016/S0022-460X(02)01084-2.

[20] Firouz-Abadi RD, Rahmanian M, Amabili M. Exact solutions for free vibrations and buckling of double tapered columns with elastic foundation and tip mass. J Vib Acoust 2013;135:051017-1-10. doi:10.1115/1.4023991.

[21] Náprstek J, Fischer C. Static and dynamic analysis of a beam assemblies using a differential system on an oriented graph. Comput Struct 2015;155:28-41.

[22] Koloušek V. Dynamics in engineering structures. Newnes-Butterworth; 1973.

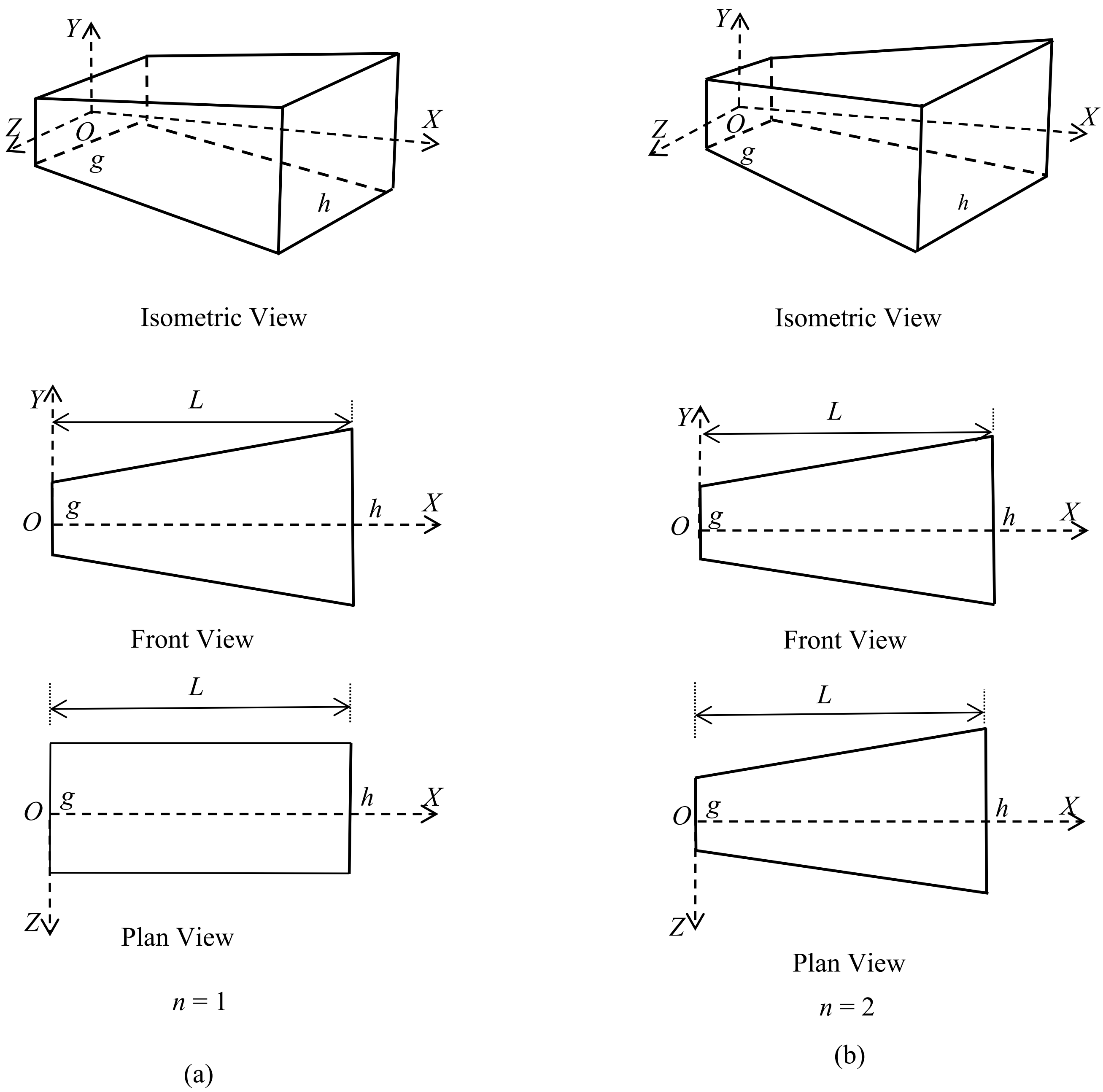

Fig. 1. A tapered beam of solid rectangular cross-section with (a) a constant width and a linearly varying depth for which the variations of the cross-sectional area and the second moment of area along the length are respectively linear and cubic (n=1) and (b) a linearly varying both width and depth for which the variations of the cross-sectional area and the second moment of area along the length are respectively second and fourth order (n=2).

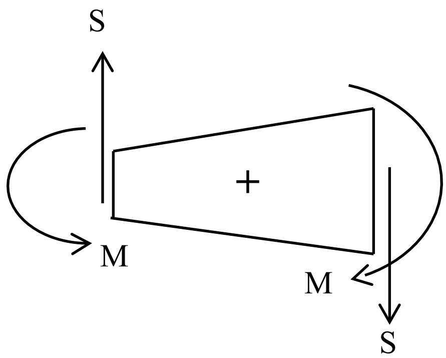

Fig. 2. Sign convention for positive bending moment and shear force

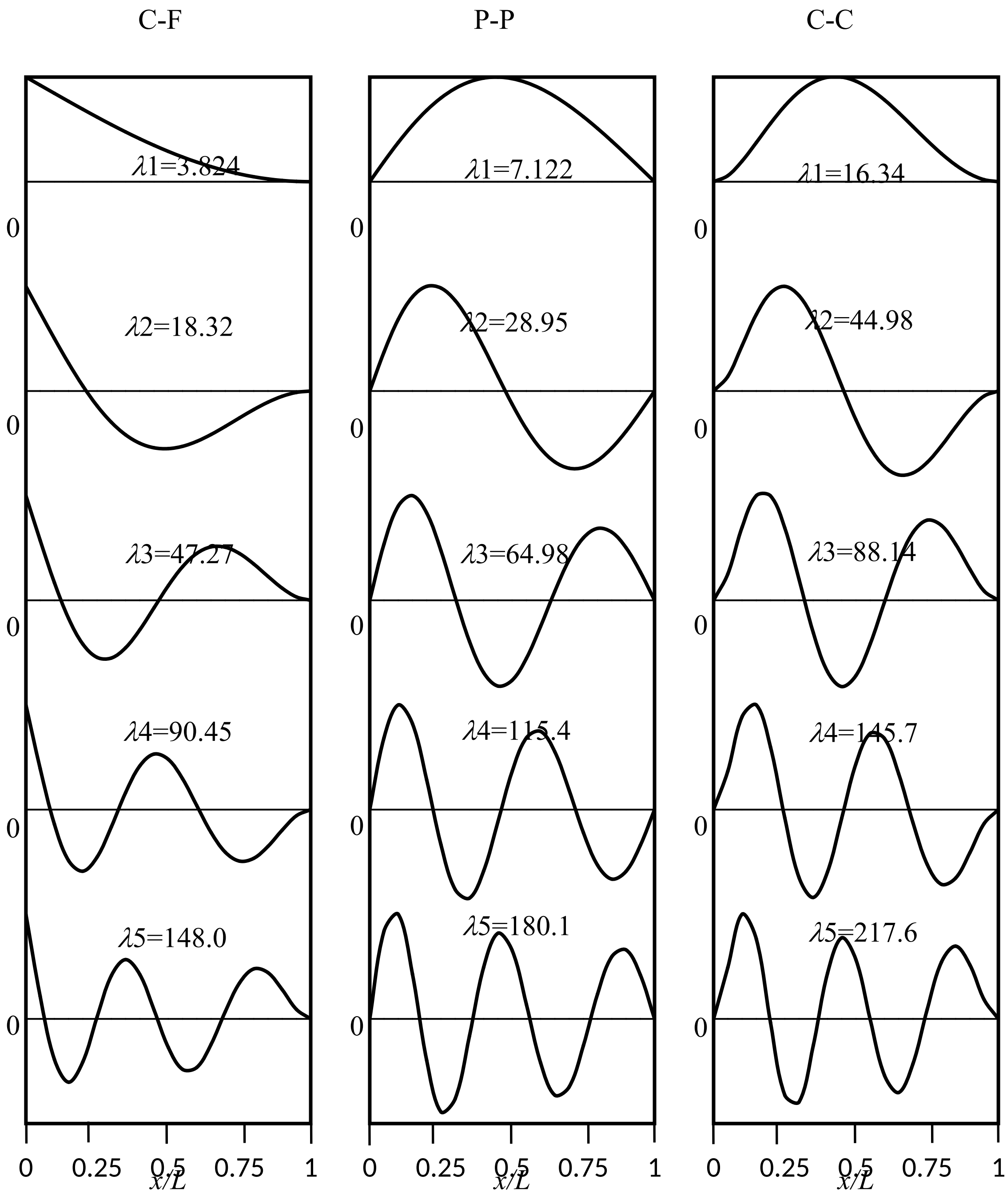

Fig. 3. The first five non-dimensional natural frequencies and mode shapes for a tapered beam with different boundary conditions when n =1 and c = 1.

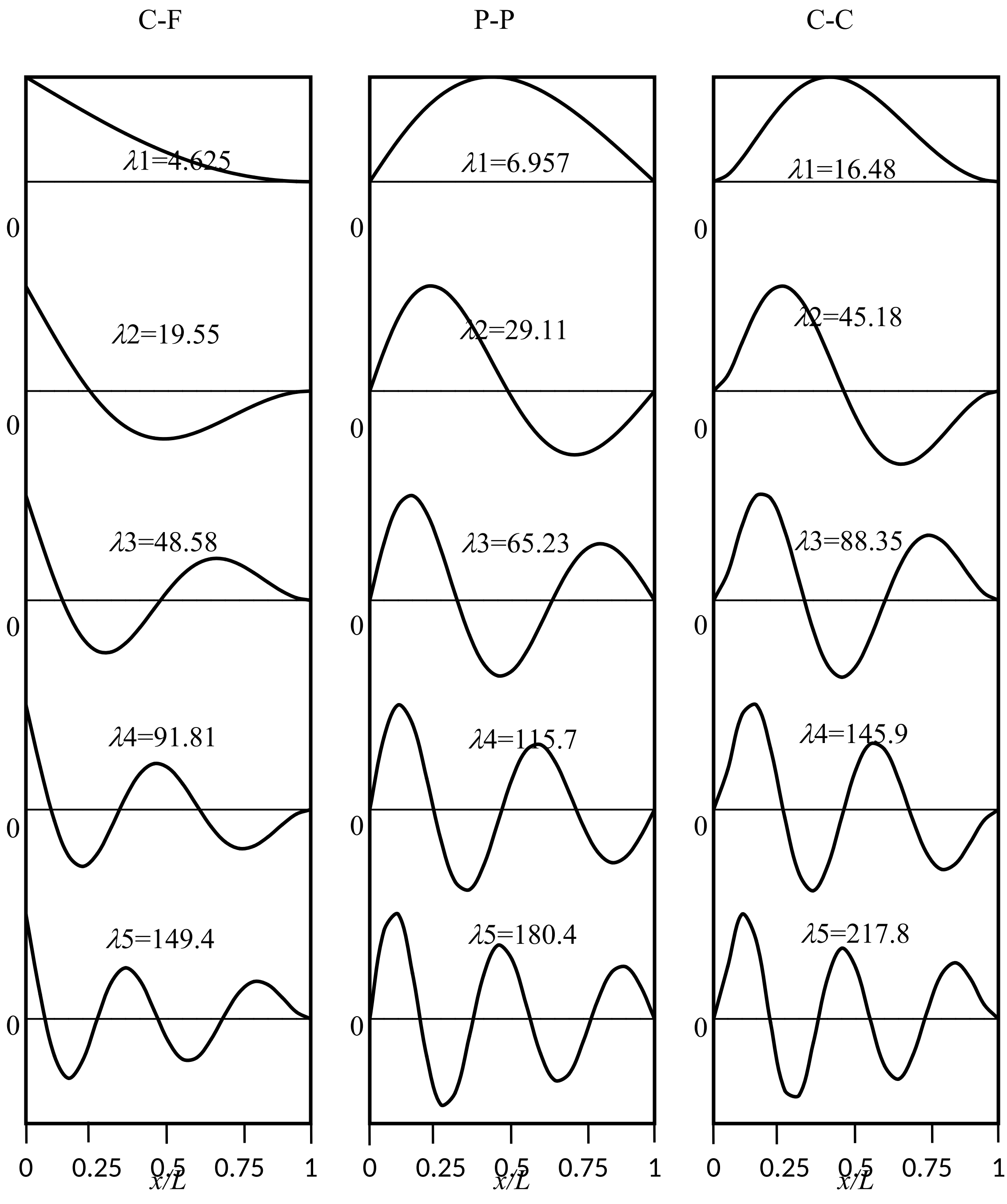

Fig. 4. The first five non-dimensional natural frequencies and mode shapes for a tapered beam with different boundary conditions when n =2 and c = 1.

Table 1. Non-dimensional natural frequencies (λ_i) of tapered beams for various boundary conditions when n = 1.

Frequency number (i)	Non-dimensional natural frequency (λ_i)								
	C-F			P-P			C-C		
	c=0.25 ($\bar{c}$=0.2)	c=1 ($\bar{c}$=0.5)	c=4 ($\bar{c}$=0.8)	c=0.25 ($\bar{c}$=0.2)	c=1 ($\bar{c}$=0.5)	c=4 ($\bar{c}$=0.8)	c=0.25 ($\bar{c}$=0.2)	c=1 ($\bar{c}$=0.5)	c=4 ($\bar{c}$=0.8)
1	3.60827	3.82379	4.29249	8.84619	7.12153	4.91976	20.0782	16.3356	11.8417
2	20.6210	18.3173	15.7427	35.4452	28.9518	21.3445	55.3399	44.9806	32.4755
3	56.1923	47.2648	36.8846	79.7303	64.9788	47.4820	108.483	88.1382	63.5118
4	109.318	90.4505	68.1164	141.721	115.351	83.8216	179.324	145.665	104.867
5	180.163	148.002	109.594	221.420	180.089	130.436	267.875	217.572	156.554

Table 2. Non-dimensional natural frequencies (λ_i) of tapered beams for various boundary conditions when n = 2.

Frequency number (i)	Non-dimensional natural frequency (λ_i)								
	C-F			P-P			C-C		
	c=0.25 ($\bar{c}$=0.2)	c=1 ($\bar{c}$=0.5)	c=4 ($\bar{c}$=0.8)	c=0.25 ($\bar{c}$=0.2)	c=1 ($\bar{c}$=0.5)	c=4 ($\bar{c}$=0.8)	c=0.25 ($\bar{c}$=0.2)	c=1 ($\bar{c}$=0.5)	c=4 ($\bar{c}$=0.8)
1	3.85512	4.62515	6.19639	8.82458	6.95659	4.35267	20.0966	16.4790	12.3819
2	21.0567	19.5476	18.3855	35.4656	29.1103	21.9379	55.3650	45.1758	33.2179
3	56.6303	48.5789	39.8336	79.7624	65.2277	48.4030	108.510	88.3528	64.3401
4	109.763	91.8128	71.2418	141.759	115.647	84.9298	179.353	145.890	105.747
5	180.611	149.390	112.828	221.462	180.413	131.666	267.905	217.805	157.467