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Arthur Renshaw and Steven Haberman

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Cass Business School
106 Bunhill Row
London EC1Y 8TZ
T +44 (0)20 7040 8470
www.cass.city.ac.uk

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Mortality reduction factors incorporating cohort effects

Arthur Renshaw and Steven Haberman
Cass Business School, City University, London, UK

Summary. The Lee-Carter modelling framework is extended through the introduction of a wider class of generalised, parametric, non-linear models. This permits the modelling and extrapolation of age-specific cohort effects as well as the more familiar age-specific period effects. The choice of error distribution is generalised. The potential for incorporating prescribed non-zero asymptotes, under strict limitations, is investigated.

Keywords: Mortality projections; Mortality reduction factors; Generalised non-linear models; Time series modelling.

1. Introduction

Mortality reduction factors (CMI (1999)) are seen as an important means of capturing and projecting historic mortality trends (Renshaw and Haberman (2003b)). They are central to the extrapolation of mortality trends using the Lee-Carter methodology (Lee (2000)) and its Poisson based equivalent (Brouhns *et al* (2002)). These methods, as originally constituted, are restricted to the capture and projection of age-period effects, to the exclusion of possibly noteworthy cohort effects. Our objective is to investigate the feasibility of extending the methodology to the modelling and projection of historic age-period-cohort effects.

We begin (Section 2) by giving the Lee-Carter approach a wider setting. Given the central role played by the mortality reduction factor in generating mortality projections, which is the main object of the exercise, we emphasis the targeting of the mortality reduction factor, as opposed to the force of mortality. Thus, while the parametric structure is expanded to allow for age-cohort as well as the familiar age-period effects, the error structure is imposed by specifying the second moment properties of the model. This allows for a range of options including Poisson, both with and without dispersion, as well as Gaussian, as used in the original Lee-Carter approach. Methods of fitting such models are reviewed and expanded. Extrapolation is conducted on the familiar basis of parametric time series forecasting. The proposed age-period-cohort methodology is applied to the 1961-2000 UK population mortality experience for each gender and the 1947-2002 UK male assured lives' experience (Section 3). Here, both the patterns in the parameter estimates and the resulting projections are particularly informative and differ appreciably from the results obtained using the standard Lee-Carter approach. It further transpires that this wider approach to model specification lends itself to the imposition of prescribed non-zero asymptotes, under certain strict conditions. This is formulated and illustrated in Section 4. A critical appraisal of the methods is presented in Section 5. This includes a study of age specific secular trends (actual and projected), in the rates of change of the logarithms of the force of mortality.

2. Methodology

2.1. Formulation

Let the random variable D_{xt} denote the number of deaths in a population at age x and time t. A rectangular data array (d_{xt}, e_{xt}) is available for analysis where

d_{xt} is the actual number of deaths and e_{xt} is the matching exposure to the risk of death. The force of mortality and empirical mortality rates are denoted by μ_{xt} and $\hat{m}_{xt}$ $(= d_{xt} / e_{xt})$ respectively. Cross-classification is by individual calendar year $t \in [t_1, t_n]$ (range n) and by age $x \in [x_1, x_k]$, either grouped into k (ordered) categories, or by individual year (range k), in which case year-of-birth or cohort year $z = t - x \in [t_1 - x_k, t_n - x_1]$ (range $n + k - 1$) is defined. See Fig 1. The purpose of the analysis is to capture and then project age specific secular mortality trends, incorporating cohort effects when the data permit. To do this, we model the mortality reduction factor

$$F(x,t) = \exp\left(\beta_x^{(0)}\iota_{t-x} + \sum_{i=1}^{c}\beta_x^{(i)}\kappa_t^{(i)}\right)$$

where

$$\mu_{xt} = \exp(\alpha_x + \log F(x,t))$$

by formulating the generalised non-linear model

$$Y_{xt} = D_{xt},\ \mathrm{E}(Y_{xt}) = e_{xt}\mu_{xt} = e_{xt}\exp(\alpha_x)F(x,t),\ \mathrm{Var}(Y_{xt}) = \phi\mathrm{E}(Y_{xt})\left(= \phi\frac{\mathrm{V}(Y_{xt})}{w_{xt}}\right) \quad (1)$$

with scale parameter ϕ, variance function $\mathrm{V}(Y_{xt}) = \mathrm{E}(Y_{xt})$ and prior weights $w_{xt} = 1$. Under the log link, the non-linear predictor η_{xt} is defined as

$$(\log \mathrm{E}(Y_{xt}) =)\ \eta_{xt} = \log e_{xt} + \alpha_x + \log F(x,t).$$

Then projected mortality rates

$$\dot{m}_{x,t_n+s} = \hat{m}_{x,t_n}\dot{F}(x,t_n+s), s > 0 \quad (2)$$

are computed by alignment with the latest available mortality rates $\hat{m}_{x,t_n}$. Here,

$$\dot{F}(x,t_n+s) = \exp\left\{\hat{\beta}_x^{(0)}\left(\tilde{\iota}_{t_n-x+s} - \hat{\iota}_{t_n-x}\right) + \sum_{i=1}^{c}\hat{\beta}_x^{(i)}\left(\dot{\kappa}_{t_n+s}^{(i)} - \hat{\kappa}_{t_n}^{(i)}\right)\right\}, s > 0$$

is based on the parameter estimates $\hat{\beta}_x^{(i)}$, $\hat{\iota}_z$, $\hat{\kappa}_t^{(i)}$ and the time series forecasts

$$\{\hat{\iota}_z : z \in [t_1 - x_k, t_n - x_1]\} \mapsto \{\dot{\iota}_{t_n-x_1+s} : s > 0\}$$

$$\{(\hat{\kappa}_t^{(1)}, \hat{\kappa}_t^{(2)}, ..., \hat{\kappa}_t^{(c)}) : t \in [t_1, t_n]\} \mapsto \{(\dot{\kappa}_{t_n+s}^{(1)}, \dot{\kappa}_{t_n+s}^{(2)}, ..., \dot{\kappa}_{t_n+s}^{(c)}) : s > 0\}$$

where

$$\tilde{\iota}_{t_n-x+s} = \begin{cases} \hat{\iota}_{t_n-x+s}, & 0 < s \le x - x_1 \\ \iota_{t_n-x+s}, & s > x - x_1 \end{cases}.$$

2.2. Discussion

In formulating the model, we partition the force of mortality

$$\mu_{xt} = \exp(\alpha_x) F(x,t) \tag{3}$$

into the product of a static life-table incorporating the main age effects α_x and a parameterised mortality reduction factor F incorporating age-specific multiple period effects $\kappa_t^{(i)}$ and a single cohort effect ι_{t-x}. Under this formulation, we target F, rather than μ_{xt}, as the primary means of modelling and projecting mortality rates. In formulating the Poisson error structure, provision for over-dispersion is possible, typically in the context of life insurance data which may involve duplicate insurance policies for each individual studied, by incorporating the free standing scale parameter ϕ. The possibility of structuring ϕ when modelling either policy counts or policy amounts is under active review.

The age-period Lee-Carter type structures

$$\mathrm{LC}(c) : \beta_x^{(0)} = 0$$

and the age-period-cohort structures

$$\mathrm{M} : c = 1 \ \left(\kappa_t^{(1)} \equiv \kappa_t\right)$$

together with the substructures

$$\mathrm{H}_0 : \beta_x^{(0)} = 1, \beta_x^{(1)} = 1; \ \ \mathrm{H}_1 : \beta_x^{(0)} = 1; \ \ \mathrm{H}_2 : \beta_x^{(1)} = 1$$

for which

$$\mathrm{H}_0 \subset \mathrm{H}_1, \mathrm{H}_2 \subset \mathrm{M},$$

are of special interest. We shall also have occasion to refer to the age-cohort structure

$$\mathrm{AC} : c = 0 \ \ \left(\beta_x^{(0)} \equiv \beta_x\right).$$

The case LC(1) $\left(\beta_x^{(1)} \equiv \beta_x\right)$ implies the original Lee-Carter bilinear structure, while the original Gaussian error structure is re-established on replacing (1) with

$$Y_{xt} = \log\left(\frac{D_{xt}}{e_{xt}}\right), \ \mathrm{E}(Y_{xt}) = \alpha_x + \log F(x,t), \ \mathrm{Var}(Y_{xt}) = \frac{\phi}{w_{xt}}, \tag{4}$$

comprising a free standing scale parameter ϕ, variance function $V(Y_{xt}) = 1$ and prior weights $w_{xt} = 1$. Under the identity link, the non-linear predictor is given by

$$\left(\mathrm{E}(Y_{xt}) =\right) \eta_{xt} = \alpha_x + \beta_x \kappa_t .$$

Time series forecasts are typically generated using univariate ARIMA processes, with the possible exception of LC(c), c >1. The random walk with drift (or ARIMA(0,1,0) process) features prominently in the application of LC(1). The use of bivariate time series, including co-integration time series methods in the context of LC(2), is described in Renshaw and Haberman (2003d). If no provision for alignment with the latest available mortality rates is made, (see equation (2)), as originally in Lee and Carter (1992), the extrapolated mortality rates decompose multiplicatively as

$$\dot{m}_{x,t_n+s} = \exp\left(\hat{\alpha}_x + \hat{\beta}_x^{(0)} \hat{\iota}_{t_n - x} + \sum_{i=1}^{c} \hat{\beta}_x^{(i)} \hat{\kappa}_{t_n}^{(i)} \right) \dot{F}(x, t_n + s), s > 0 ,$$

which has the same functional form as (3), and bears direct comparison with (2).

2.3. Model fitting

Setting aside fitting by single value decomposition (SVD), which is well documented, it is informative to survey other likelihood fitting methods. We focus on two potential procedures: Method A based on an unpublished technical report by Wilmoth (1993), and Method B based on a method of mortality analysis incorporating age-year interaction with applications to medical statistics by James and Segal (1982), which predates Lee and Carter (1992).

LC(1)

Method A.

Following Wilmoth (1993), Brouhns *et al.* (2002), maximum likelihood estimates are obtained under the original Lee-Carter Gaussian error structure (4) using an iterative process, which can be re-expressed as follows:

Set starting values $\hat{\alpha}_x, \hat{\beta}_x, \hat{\kappa}_t$ *; compute* $\hat{y}_{xt}$

↓

update $\hat{\alpha}_x$ *; compute* $\hat{y}_{xt}$

update $\hat{\kappa}_t$ *, adjust s.t.* $\sum_{t=t_1}^{t_n} \kappa_t = 0$ *; compute* $\hat{y}_{xt}$

update $\hat{\beta}_x$ *; compute* $\hat{y}_{xt}$

compute $D(y_{xt}, \hat{y}_{xt})$

↓

Repeat the updating cycle; stop when $D(y_{xt}, \hat{y}_{xt})$ *converges*

where

$$y_{xt} = \log \hat{m}_{xt}, \hat{y}_{xt} = \hat{\alpha}_x + \hat{\beta}_x \hat{\kappa}_t ,$$

$$D(y_{xt},\hat{y}_{xt}) = \sum_{x,t} \mathrm{dev}(x,t) = \sum_{x,t} 2w_{xt} \int_{\hat{y}_{xt}}^{y_{xt}} \frac{y_{xt}-u}{\mathrm{V}(u)}\,\mathrm{d}u = \sum_{x,t} w_{xt}(y_{xt}-\hat{y}_{xt})^2$$

with weights

$$w_{xt} = \begin{cases} 1, & e_{xt} > 0 \\ 0, & e_{xt} = 0 \end{cases}. \tag{5}$$

The updating of a typical parameter θ proceeds according to

$$updated(\theta) = \theta - \frac{\partial D}{\partial \theta} \Big/ \frac{\partial^2 D}{\partial \theta^2}$$

where D is the deviance of the current model. See Table 1 for details. Effective starting values, conforming to the standard Lee-Carter constraints

$$\sum_{t=t_1}^{t_n} \kappa_t = 0, \sum_x \beta_x = 1 \tag{6}$$

are $\hat{\kappa}_t = 0$, $\hat{\beta}_x = 1/k$, coupled with the SVD estimate

$$\hat{\alpha}_x = \log \prod_{t=t_1}^{t_n} \hat{m}_{xt}^{1/n} \tag{7}$$

so that α_x is estimated by the logarithm of the geometric mean of the empirical mortality rates. The model has $\nu = (k-1)(n-2)$ degrees of freedom. This iterative fitting process generates maximum likelihood estimates under the Poisson error structure (1), (Alho (2000), Brouhns *et al.* (2002), Wilmoth (1993)) on setting

$$y_{xt} = d_{xt}, \hat{y}_{xt} = \hat{d}_{xt} = e_{xt} \exp(\hat{\alpha}_x + \hat{\beta}_x \hat{\kappa}_t)$$

$$D(y_{xt},\hat{y}_{xt}) = \sum_{x,t} \mathrm{dev}(x,t) = \sum_{x,t} 2w_{xt} \int_{\hat{y}_{xt}}^{y_{xt}} \frac{y_{xt}-\mathrm{u}}{\mathrm{V}(u)}\,\mathrm{d}u = \sum_{x,t} 2w_{xt}\{y_{xt}\log(y_{xt}/\hat{y}_{xt}) - (y_{xt}-\hat{y}_{xt})\}$$

Brouhns *et al.* (2002) attribute the iterative method for estimating log-linear models with bilinear terms to Goodman (1979). See Table 1 for details of the parameter updating relationships.

Method B.
Following James and Segal (1982), we use the iterative procedure:

Set starting values $\hat{\beta}_x$
↓
given $\hat{\beta}_x$, *update* $\hat{\alpha}_x, \hat{\kappa}_t$
given $\hat{\kappa}_t$, *update* $\hat{\alpha}_x, \hat{\beta}_x$
compute $D(y_{xt}, \hat{y}_{xt})$
↓
Repeat the updating cycle; stop when $D(y_{xt}, \hat{y}_{xt})$ *converges*

Given $\hat{\beta}_x$ or $\hat{\kappa}_t$, updating is by selecting the desired generalised linear model and fitting the predictor, which is linear in the respective remaining parameters. Thus, log-link Poisson responses $y_{xt} = d_{xt}$ with offsets $\log e_{xt}$ are declared in order to generate the same results as the Brouhns *et al.* (2002) iterative fitting process. The respective predictors are declared by accessing the model formulae (design matrices), a feature available in GLIM (Francis *et al.* (1993)), for example. In specifying the model formulae, we impose the constraints

$$\kappa_{t_1} = 0, \sum_x \beta_x = 1,$$

reverting back to the Lee-Carter constraints (6) once convergence is attained.

LC(*c*)
Model identification requires the specification of the requisite number of parameter constraints, typically

$$\sum_{t=t_1}^{t_n} \kappa_t^{(i)} = 0, \ \sum_x \beta_x^{(i)} = 1, \forall\, i \geq 1, \text{ or } \kappa_{t_1}^{(i)} = 0, \ \sum_x \beta_x^{(i)} = 1, \forall\, i \geq 1.$$

The model has $\nu = k(n-c-1) - c(n-2)$ degrees of freedom. Then estimation can proceed on the bases of modifications to the first of the iterative fitting cycles above (Method A), based on the extended definitions of $\hat{y}_{xt}$. Thus the core of the iterative fitting cycle is expanded to update $\hat{\kappa}_t^{(i)}$ and $\hat{\beta}_x^{(i)}$ in sequence (i = 1, 2, ..., c), according to the LC(1) updating relationships of Table 1, for each i.

M (H_2, H_1, H_0)
It is well known that age-period-cohort modelling is problematic, since the three factors are constrained by the relationship

cohort = period – age.

In order to ensure a unique set of parameter estimates, we resort to a two-stage fitting strategy in which α_x is estimated first, typically as in (7) and the original Lee-Carter SVD approach. Then, the remaining parameters, those of F, may be estimated by

suitably adapting Method B by declaring log-link Poisson responses $y_{xt} = d_{xt}$ and the augmented offsets $\log e_{xt} + \hat{\alpha}_x$ and adapting the design matrices, incorporating the constraints

$$\iota_{t_1 - x_k} = 0, \sum_x \beta_x^{(0)} = 1; \kappa_{t_1} = 0, \sum_x \beta_x^{(1)} = 1.$$

Obvious simplifications to the design matrices are needed when fitting the associated sub-models H_1 and H_2, while the iterative element in the fitting procedure is redundant when fitting H_0. We note that the model M has $\nu = (k-2)(n-2)-3$ degrees of freedom. Effective starting values are $\beta_x^{(0)} = \beta_x^{(1)} = 1/k$. Fitting is also possible under Method A, once α_x has been estimated, using the extended definitions of $\hat{y}_{xt}$ and adapting the core of the iterative cycle in accordance with the relevant updating relationships (Table 1). Effective starting values are obtained on setting $\beta_x^{(0)} = \beta_x^{(1)} = 1$ and fitting H_0 to generate starting values for ι_z and κ_t.

AC

Model identification is conveniently achieved by means of the parameter constraints

$$\iota_{t_1 - x_k} = 0, \sum_x \beta_x = 1.$$

Model fitting is then possible by reformulation of Method A in terms of α_x, β_x and ι_{t-x}. Thus, ι_{t-x} instead of κ_t is updated in the core of the iterative cycle (subject to the adjustment $\iota_{t_1 - x_k} = 0$), using the replacement updating relationships of Table 1. Fitting is also possible using Method B by replacing κ_t with ι_{t-x} and modifying the design matrices accordingly. A possible strategy for generating starting values is to set $\hat{\beta}_x = 1$ and additionally fit the main effects structure $\alpha_x + \iota_{t-x}$ in accordance with the distributional assumptions under Method A. There are $\nu = (k-1)(n-3)$ degrees of freedom in this model.

2.4. Discussion

By specifying the second moment distributional properties when defining the model error structure, the choice of distribution is not restricted to the Poisson and Gaussian distributions, and may be expanded by selecting different variance functions. Empirical evidence suggests that, for all practical purposes, maximum likelihood estimates obtained for LC(1) using the iterative fitting processes under the Gaussian error structure (4), are the same as those obtained under fitting by SVD. Unlike modelling by SVD, however, the choice of weights (5) means that estimation can proceed, in the presence of empty data cells, under the Gaussian, Poisson and any other of the viable error settings. Wilmoth (1993) uses weights $w_{xt} = d_{xt}$ in combination with the Gaussian error setting. Empirical studies reveal that this has the effect of bringing the parameter estimates into close agreement with Poisson response based estimates. When cross-referencing results obtained under both modelling approaches (with identical model structures), we have found that the same number of

iterations is required in order to induce convergence. The adaptation of Method B does not appear to be appropriate for LC(c), $c > 1$ modelling because of aliasing under the expanded model formulae. Convergence is slow when fitting M.

Diagnostic checks on the fitted model are conducted by plotting the standardised deviance residuals

$$r_{xt} = \text{sign}(y_{xt} - \hat{y}_{xt})\sqrt{\text{dev}(x,t)/\hat{\phi}}\ ,\ \hat{\phi} = \text{D}(y_{xt}, \hat{y}_{xt})/\nu\ .$$

The sole use of the proportion of the total temporal variance, as measured by the ratio of the first singular value to the sum of singular values under SVD, is not a satisfactory diagnostic indicator, in our experience. However, this index is widely quoted in the demographic literature: see, for example, Tuljapurkar *et al.* (2000).

It is possible to capture important (secondary) data features, which are otherwise lost under standard Lee-Carter modelling, by increasing the number of bilinear terms in the model structure and fitting LC(c), $c > 1$. One such application, for $c = 2$, in which the data are grouped by age, is discussed in Renshaw and Haberman (2003a), (2003c). While it is possible to capture cohort effects using LC(c) and increasing c when the data are available by individual year of age, in common with Booth *et al* (2002), we have concluded that such effects are not reflected in the resulting projections based on the associated multivariate time series. Hence, we have introduced the model structure M (and its substructures).

The parameters α_x are estimated simultaneously with the parameters of F in LC(c) and AC modelling. A two-stage estimation process is necessary, however, in which α_x is estimated separately in order to condition for the estimation of F, when fitting M (and its substructures). This two-stage approach can also be applied when fitting LC(c) and AC. For LC(c) modelling, empirical studies show that this has little practical material effect, due to the robust nature of the α_x estimate (7).

3. UK mortality projections

In order to explore the potential of the age-period-cohort model M, we present results for the UK 1961-2000 mortality experiences by gender, with cross-classification by individual year of age 0 to 99, and for the UK 1947-2002 male permanent assured lives experience (policy duration 2+ years), in the age range 22 to 89 by individual year. The Poisson error setting (1) is chosen throughout and a direct comparison is made with the standard age-period Lee-Carter model LC(1) and the age-cohort model AC.

The implications of the choice of model structure are immediately apparent from the respective residual plots, illustrated for the UK female experience (Fig 2). Here the distinctive ripple effects in the year-of-birth residual plots under age-period modelling (Fig 2a, RH frame), signifying a failure to capture cohort effects, is transferred to the calendar-year residual plots under age-cohort modelling (Fig 2b, LH frame), signifying a reciprocal failure to capture period effects. However, the distinctive ripple effects are largely removed under age-period-cohort modelling (Fig 2c), indicating the relatively successful capture of all three main effects and representing a significant improvement over the fitted Lee-Carter model. Similar patterns are observed in the residual plots for the two UK male experiences (not reproduced here but the details are available from the authors).

The implications for mortality projections are illustrated (Fig 3) by plotting the latest available log crude mortality rates (computed by averaging over the last three years) against age for each separate study, coupled with the superimposed 2020 or 2025 projections under the respective age-period and age-cohort modelling formulations (Fig 3a), and under the respective age-period and age-period-cohort modelling formulations (Fig 3b). The social and financial implications of these diverse sets of projections are considerable: we attempt to quantify the latter shortly.

The parameter estimates for the age-period-cohort modelling approach are also revealing. Focusing first on the population studies for each sex (Fig 4), it is informative to compare matching frames. Thus the main age effect plots ($\hat{\alpha}_x$ vs x) display the familiar characteristics, including 'accident' humps, of static life-tables (log scale). Recall that these effects are estimated separately, by averaging crude mortality rates over t for each x, in order to condition for both period and cohort effects. The main period effect plot ($\hat{\kappa}_t$ vs t) is linear for females but exhibits mild curvature for males, which can be characterised as piece-wise linear with a knot or hinge positioned in the first half of the 1970's. This effect is also present in the separate Lee-Carter analysis of mortality data of the G7 countries (Tuljaparker *et al.* (2000)) and is discussed further (Renshaw and Haberman (2003a)), in the context of the England and Wales mortality data. The forecasts for κ_t are based on the auto-regressive time series

$$y_t = a_0 + a_1 t + \sum_{i=1}^{p} \phi_i y_{t-i} + \varepsilon_t, \; y_t = \kappa_t - \kappa_{t-1} \tag{8}$$

with $p = 2$, and with $a_1 = 0$ in the female study, that is the ARIMA(2,1,0) process. The inclusion of the term in t (male experience) is broadly equivalent to modelling second differences, that is the ARIMA(2,2,0) process. There are noteworthy differences in the $\hat{\beta}_x^{(1)}$ patterns, which control the age specific rates of mortality decline by period in the projections. In particular, the trough in the male $\hat{\beta}_x^{(1)}$ pattern in the 20 to 40 age range is consistent with similar findings in the reanalysis of the male England and Wales mortality rates (Renshaw and Haberman (2003a)).

The main cohort effect plots ($\hat{\iota}_z$ vs $z = t - x$) are particularly revealing. Thus, noteworthy discontinuities occur corresponding to the ending of hostilities in World War I and II. While it is possible to identify the first of these with the 1919 influenza epidemic, we are not aware of the likely cause of the second discontinuity. (The 1887 discontinuity can be traced to a set of outliers, and is possibly due to mis-stated exposures in this particular cohort.) The pronounced decline in the $\hat{\iota}_z$ profile in the inter-war years is consistent with the reported rapid mortality improvements experienced by generations born between 1925 and 1945 (for both sexes) and reported by Willets (2004). The seemingly stable linear trends in the $\hat{\iota}_z$ profiles, established since the late 1940's, form the basis of the depicted time series forecasts, generated using ARIMA(1,1,0) processes. The $\hat{\beta}_x^{(0)}$ patterns, which control the age specific cohort contributions to the mortality projections, are similar, for both sexes, for ages up to 65.

For the male assured lives experience, we present age-period-cohort parameter estimates (Fig 5), under both M and H_1 modelling. Since the assured lives experience

comprises a small select subset of the wider UK male population experience, it is informative to contrast these parameter estimates with those of the wider population (Fig 4b), recognising, however, that the time domains of the two data sets are not completely over-lapping. Thus, while the main period effect ($\hat{\kappa}_t$ plot) exhibits the curvature associated with the wider male population, the main cohort effect ($\hat{\iota}_z$ plot) is markedly different. Thus, while the selection process has ensured that the pattern has remained largely constant up to the late 1940's, the trend has subsequently been upward. The forecasts depicted are all based on the auto-regressive time series (8) with $p = 1$. Forecasting is started from year of birth 1972 (instead of the outer limit of the data set 1979) in the case of $\hat{\iota}_z$, because of the paucity of exposure in the lower RH corner of the data array (Fig 1). Recall that the static life tables $\hat{\alpha}_x$ are identical under M and H_1 (Fig 5). The projections in the lower RH frame in Fig 3 are based on H_1, rather than M, for the reasons given in Section 5.

In order to illustrate the impact of such diverse projections, we depict (in Fig 6) complete life expectancies at age 65 $e_{65}(t)$ and immediate annuity values at age 65 $a_{65}(t)$, calculated assuming a 5% pa fixed interest rate, up to the year 2005, based on both the age-period (LC(1)) and age-period-cohort (M or H_1) model projections. Computation proceeds according to the formulae

$$e_x(t) = \frac{\sum_{i\geq 0} l_{x+i}(t+i)\{1-\frac{1}{2}q_{x+i}(t+i)\}}{l_x(t)}, \quad a_x(t) = \frac{\sum_{i\geq 1} l_{x+i}(t+i)v^i}{l_x(t)} \tag{9}$$

where

$$q_x(t) \approx 1-\exp(-\mu_{xt}), \quad l_{x+1}(t+1) = \{1-q_x(t)\}l_x(t) \tag{10}$$

with discount factor v, requiring extrapolations up to the year 2040 for the population studies and the year 2030 for the assured lives study. (We note that the annuity values represent the expected present value of an income of 1 paid annually in arrears while the individual initially aged 65 remains alive). In addition, we have superimposed cross-sectional (by period) life expectancies and 5% annuity values (at age 65), computed by suppressing the variation in t in expressions (9) and (10). With the exception of the first few years of each projection, these involve the empirical mortality rate $\hat{\mu}_{xt} = \hat{m}_{xt}$ only (otherwise values are computed using the respective age-period-cohort model projections). Thus, the computation of the dynamic (by cohort) life expectancies and annuities requires lengthy extrapolations in comparison to the more immediate cross-sectional computations, and this contributes a source of increased uncertainty. The reserves that insurance companies selling annuities and pension funds would have to hold to meet their future contractual liabilities are directly related to terms like $a_{65}(t)$: Booth et al (2005). The financial implications of the upwards trends in Figure 6(b) are clear and significant and indicate the burden that increasing longevity may place on such institutions.

4. Prescriptive asymptotes

4.1. Formulation

Consider LC(1). Simplify the notation by omitting the prefix i (Section 2.1) so that the reduction factor is

$$F(x,t) = \exp\beta_x\kappa_t$$

with projections

$$\dot{F}(x,t_n+s) = \exp\hat{\beta}_x(\dot{\kappa}_{t_n+s} - \hat{\kappa}_{t_n}), s>0. \tag{11}$$

A criticism of the structural form of such reduction factors and the Lee-Carter method in general, concerns the nature of their long-run asymptotic behaviour: with typically a combination of positive β_x and decreasing κ_t leading to zero asymptotic values. To address this criticism, Wilmoth (pp87-88 Lee (2000)) has reportedly 'suggested that it would be a simple matter to have each age-specific death rate decline towards a lower bound greater than zero, by subtracting the bound before modelling and forecasting the death rates, and then adding it back in'. In a parallel situation, the CMI (1999) has prescribed the mortality reduction factor

$$\dot{F}_\alpha(x,t_0+s) = \alpha(x) + \{1-\alpha(x)\}\{1-f(x)\}^{s/20}, s>0. \tag{12}$$

with non-zero asymptotes $\alpha(x)$, (not to be confused with α_x), for adjusting standard life tables centred on t_0 (for populations with annuity or pensions arrangements).

With this background, we consider therefore the reduction factor

$$F_\alpha(x,t) = \alpha(x) + \{1-\alpha(x)\}\exp\beta_x\kappa_t,\ 0\le\alpha(x)<1, \tag{13}$$

with projections

$$\dot{F}_\alpha(x,t_n+s) = \alpha(x) + \{1-\alpha(x)\}\exp\{\hat{\beta}_x(\dot{\kappa}_{t_n+s} - \hat{\kappa}_{t_n})\}, s>0, \tag{14}$$

based on the re-parameterisation of (12)

$$\beta_x = \frac{1}{20}\log\{1-f(x)\}, \kappa_t = t.$$

Asymptotic convergence is then implicit. Once $\alpha(x)$ is prescribed, it is then possible to estimate the remaining structure of F_α using the two-stage estimation process of Section 3. We do this by rearranging (1)

$$Y_{xt} = \frac{D_{xt}}{e_{xt}\exp(\alpha_x)},\ \mathrm{E}(Y_{xt}) = F_\alpha(x,t),\ \mathrm{Var}(Y_{xt}) = \phi\frac{\mathrm{E}(Y_{xt})}{e_{xt}\exp(\alpha_x)}\left(=\phi\frac{\mathrm{V}(Y_{xt})}{w_{xt}}\right),$$

with prior weights $w_{xt} = e_{xt}\exp(\hat{\alpha}_x)$, having first estimated α_x, either according to (7) or as a standard life-table. Second stage estimation is then possible, using the non-linear predictor η_{xt}, incorporating the prescribed offset $\log\{1-\alpha(x)\}$, in combination with the 'own' link (in GLM terminology)

$$\log\{F_\alpha(x,t)-\alpha(x)\} = \eta_{xt} = \log\{1-\alpha(x)\} + \beta_x\kappa_t, \tag{15}$$

by adapting the second of the iterative fitting methods (Method B, Section 2.3). When $\kappa_t = t$, the predictor is linear and the iterative cycle is redundant.

4.2. Applications

The mortality reduction factor (12) was proposed by the CMI Committee as a means of adjusting UK actuarial standard life tables, based on the 1991-94 quadrennium, (and with t_0 centred in this quadrennium). Consideration was given to prescribing the magnitude of the reduction after (a prescribed) 20 year period has lapsed when specifying (12). This approach is further based on an informal data analysis and on expert opinion regarding the long run trajectory without recourse to rigorous modelling. Once $\alpha(x)$ has been specified

$$\alpha(x) = \begin{cases} 0.13, & x < 60 \\ 1 + 0.87\left(\dfrac{x-110}{50}\right), & x \geq 60 \end{cases} \tag{16}$$

and depicted (Fig 7a LH frame), it is informative to estimate $f(x)$ and to compare results with the prescribed value. This is done, by plotting (Fig 7a RH frame) the estimates

$$\hat{f}(x) = 1 - \exp(20\hat{\beta}_x)$$

based on the UK male assured lives' mortality experience for policy duration 2+ years covering the period 1947-94 and ages 22-89 years, using the standard life-table for 1991-94 as (first stage) estimates for $\exp(\hat{\alpha}_x)$ and setting $\kappa_t = t$ in (15). The details are taken from CMI (1999) and the prescribed value of $f(x)$ is plotted for comparison. On the bases of this evidence the prescribed level of mortality improvement is the more conservative: corresponding to a larger value of $f(x)$ and hence a smaller value of $\dot{F}_\alpha(x,t)$.

As a second application, it is instructive to compare the results under LC(1) modelling, both with and without prescribed asymptotes $\alpha(x)$. We do this by plotting (Fig 7b) scaled, projected log reduction factor differences (*without* $\alpha(x)$ as in (11) minus *with* $\alpha(x)$ as in (14)), or displaced log reduction factor ratios

$$\frac{\log\dot{F}(x,t) - \log\dot{F}_\alpha(x,t)}{\log\dot{F}(x,t)} = 1 - \frac{\log\dot{F}_\alpha(x,t)}{\log\dot{F}(x,t)}$$

for select $t > t_n$, using the prescribed $\alpha(x)$ of (16), for the UK 1961-2000 mortality experiences by gender, ages 0-99, and for the UK 1947-2002 males assured lives experience, policy duration 2+ years, ages 22-89. The magnitude of this displaced ratio, above the abscissa axis, is a comparative measure of the impact of the imposed asymptote on the specified projections. The patterns generally reflect the shape of $\alpha(x)$ (Fig 7a).

It is also possible to impose asymptotes under age-period-cohort modelling, provided that the pairs of age-period and age-cohort parameters exhibit decreasing mortality trends. These conditions hold (Fig 4) for the two population studies, but not for the male assured lives study (Fig 5). For completeness, the displaced ratios for the population studies, using the same asymptotes (16), are presented in Fig 7c. For the female study, values of the ratios for ages over 95 are suppressed, because the convergence criterion underpinning the projections is violated. For this application

$$F_\alpha(x,t) = \alpha(x) + \{1-\alpha(x)\}\exp\left(\beta_x^{(0)}\iota_{t-x} + \beta_x^{(1)}\kappa_t\right),\ 0 \le \alpha(x) < 1. \qquad (17)$$

5. Critical appraisal

In a reference to the possibility of complicating LC(1) by adding higher order effects (LC(c)), or cohort-effects (M), resulting in an improved fit, Lee (2000) comments as follows: 'The question is whether such additional effects would represent fundamental and enduring aspects of mortality patterns such that the forecasts would be improved'. We use this as a basis on which to conduct a critical appraisal of the methodology.

Turning first to the modelling stage and the question of whether the additional structural effects are fundamental aspects of mortality patterns, answers are provided by the analysis of residuals. Thus, on the basis of the residual plots (e.g. Fig 2), age-period-cohort models M and H_1 are demonstrated (Figs 4 & 5) to have captured characteristic mortality patterns, directly attributable to age, period and cohort effects. Then, if modelling were restricted to LC(1), distinctive patterns directly attributable to cohort effects would be lost. The question of pattern durability, however, is a matter for conjecture, while established patterns, preferably linear, in the period and cohort parameters are conducive to forecasting. For age specific declining trends, it is possible to conjecture further by imposing non-zero asymptotes (Section 4). Note, however, that in formulating (13), or the more complex version allowing of cohort terms in the exponent, (17), there are no inbuilt constraints, such as the use of the modulus sign in the exponent, for the prevention of model failure.

Turning next to the complication of the extrapolation process through the inclusion of additional age-cohort (or higher order age-period effects), there is the need to check that the dynamics of the relative contributions from the different sources, typically age-period and age-cohort effects, are maintained in the projections. To do this, we monitor the rate of change of $\log\mu_{xt}$ over t, for each age x, *viz.*

$$\frac{\partial \log\mu_{xt}}{\partial t} = \frac{\partial \log F(x,t)}{\partial t} = \beta_x^{(0)}\frac{\partial \iota_z}{\partial z} + \sum_{i=1}^{c}\beta_x^{(i)}\frac{\partial \kappa_t^{(i)}}{\partial t} \approx \beta_x^{(0)}\underset{z}{\Delta}\iota_z + \sum_{i=1}^{c}\beta_x^{(i)}\underset{t}{\Delta}\kappa_t^{(i)}$$

where Δ is the differencing operator. A representative selection of such rates of change is depicted in Figs 8 & 9. For the two population studies (Fig 8), with the

exception of a noteworthy (partially illustrated) trough, transmitted by period with increasing age in excess of 65 in the female experience, all such plots are consistently linear over the combined data and forecast periods displayed. The departure from linearity for females is associated with the relatively large projected life expectancies and annuities under M (Fig 6). For the assured lives study (Fig 9), there is a marked change of direction in the otherwise linear patterns at the interface between estimated and forecast parameters, for ages in the lower quarter of the age range, for model M. Hence we have chosen model H_1 when projecting assured lives' mortality rates (Fig 3).

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Table 1. Parameter updating relationships

	Gaussian	*Poisson*
LC(1)	$u(\hat{\alpha}_x) = \hat{\alpha}_x + \dfrac{\sum_t w_{xt}(y_{xt} - \hat{y}_{xt})}{\sum_t w_{xt}}$	$u(\hat{\alpha}_x) = \hat{\alpha}_x + \dfrac{\sum_t w_{xt}(y_{xt} - \hat{y}_{xt})}{\sum_t w_{xt}\hat{y}_{xt}}$
	$u(\hat{\kappa}_t) = \hat{\kappa}_t + \dfrac{\sum_x w_{xt}(y_{xt} - \hat{y}_{xt})\hat{\beta}_x}{\sum_x w_{xt}\hat{\beta}_x^2}$	$u(\hat{\kappa}_t) = \hat{\kappa}_t + \dfrac{\sum_x w_{xt}(y_{xt} - \hat{y}_{xt})\hat{\beta}_x}{\sum_x w_{xt}\hat{y}_{xt}\hat{\beta}_x^2}$
	$u(\hat{\beta}_x) = \hat{\beta}_x + \dfrac{\sum_t w_{xt}(y_{xt} - \hat{y}_{xt})\hat{\kappa}_t}{\sum_t w_{xt}\hat{\kappa}_t^2}$	$u(\hat{\beta}_x) = \hat{\beta}_x + \dfrac{\sum_t w_{xt}(y_{xt} - \hat{y}_{xt})\hat{\kappa}_t}{\sum_t w_{xt}\hat{y}_{xt}\hat{\kappa}_t^2}$
M	$u(\hat{\iota}_z) = \hat{\iota}_z + \dfrac{\sum_{\substack{x,t \\ t-x=z}} w_{xt}(y_{xt} - \hat{y}_{xt})\hat{\beta}_x^{(0)}}{\sum_{\substack{x,t \\ t-x=z}} w_{xt}\hat{\beta}_x^{(0)2}}$	$u(\hat{\iota}_z) = \hat{\iota}_z + \dfrac{\sum_{\substack{x,t \\ t-x=z}} w_{xt}(y_{xt} - \hat{y}_{xt})\hat{\beta}_x^{(0)}}{\sum_{\substack{x,t \\ t-x=z}} w_{xt}\hat{y}_{xt}\hat{\beta}_x^{(0)2}}$
	$u(\hat{\beta}_x^{(0)}) = \hat{\beta}_x^{(0)} + \dfrac{\sum_t w_{xt}(y_{xt} - \hat{y}_{xt})\hat{\iota}_{t-x}}{\sum_t w_{xt}\hat{\iota}_{t-x}^2}$	$u(\hat{\beta}_x^{(0)}) = \hat{\beta}_x^{(0)} + \dfrac{\sum_t w_{xt}(y_{xt} - \hat{y}_{xt})\hat{\iota}_{t-x}}{\sum_t w_{xt}\hat{y}_{xt}\hat{\iota}_{t-x}^2}$
	$u(\hat{\kappa}_t) = \hat{\kappa}_t + \dfrac{\sum_x w_{xt}(y_{xt} - \hat{y}_{xt})\hat{\beta}_x^{(1)}}{\sum_x w_{xt}\hat{\beta}_x^{(1)2}}$	$u(\hat{\kappa}_t) = \hat{\kappa}_t + \dfrac{\sum_x w_{xt}(y_{xt} - \hat{y}_{xt})\hat{\beta}_x^{(1)}}{\sum_x w_{xt}\hat{y}_{xt}\hat{\beta}_x^{(1)2}}$
	$u(\hat{\beta}_x^{(1)}) = \hat{\beta}_x^{(1)} + \dfrac{\sum_t w_{xt}(y_{xt} - \hat{y}_{xt})\hat{\kappa}_t}{\sum_t w_{xt}\hat{\kappa}_t^2}$	$u(\hat{\beta}_x^{(1)}) = \hat{\beta}_x^{(1)} + \dfrac{\sum_t w_{xt}(y_{xt} - \hat{y}_{xt})\hat{\kappa}_t}{\sum_t w_{xt}\hat{y}_{xt}\hat{\kappa}_t^2}$
AC	$u(\hat{\alpha}_x)$ computed as above	$u(\hat{\alpha}_x)$ computed as above

CAPTIONS

Fig. 1. Observed and projected age-period domains with typical cohort

Fig. 2. UK female study, residual plots- (a) model LC(1); (b) model AC; (c) model M

Fig. 3. Latest and projected $\log m_{xt}$ age profiles- (a) LC(1) and AC modelling, all three studies; (b) LC(1) and M modelling, population studies; (b) LC(1) and H_1 modelling, assured lives study

Fig. 4. UK population studies, parameter estimates, model M- (a) females; (b) males

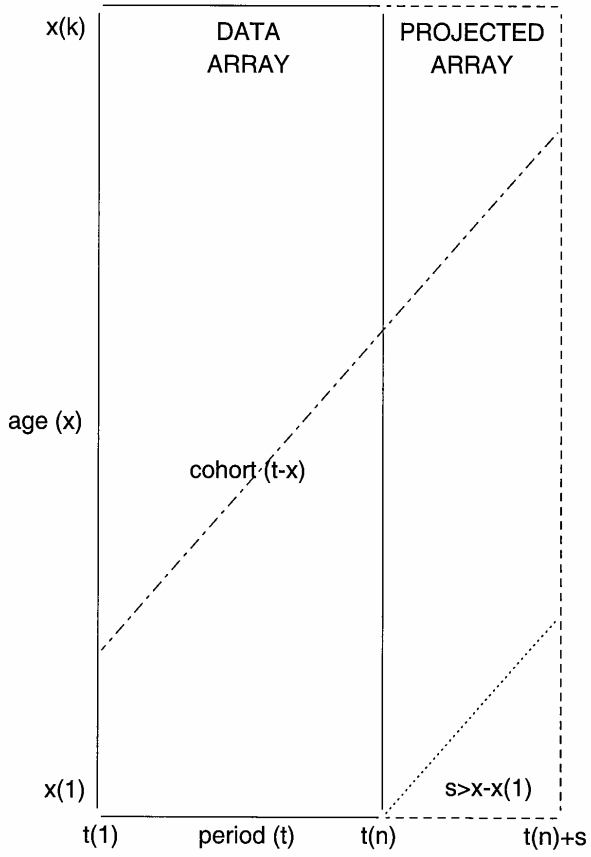
Fig. 5. UK male assured lives study, parameter estimates- (a) model M; (b) model H_1

Fig. 6. (a) Life expectancies; (b) annuities, 5% fixed rate interest- all at age 65 for a range of periods, computed by period and by cohort under age-period and age-period-cohort modelling

Fig. 7. (a) UK male assured lives study: prescribed asymptotes $\alpha(x)$, matching f(x) profiles. Displaced ratio $1-\log \dot{F}_{\alpha}/\log \dot{F}$ age profiles, fixed period (b) model LC(1); (c) model M

Fig. 8. Secular trends in $\frac{1}{\mu_{xt}}\frac{\partial \mu_{xt}}{\partial t}$ for selected ages x, incorporating model M projections- (a) UK female population study; (b) UK male population study

Fig. 9. Secular trends in $\frac{1}{\mu_{xt}}\frac{\partial \mu_{xt}}{\partial t}$ for selected ages x, UK male assured lives study, incorporating- (a) model M projections; (b) model H_1 projections



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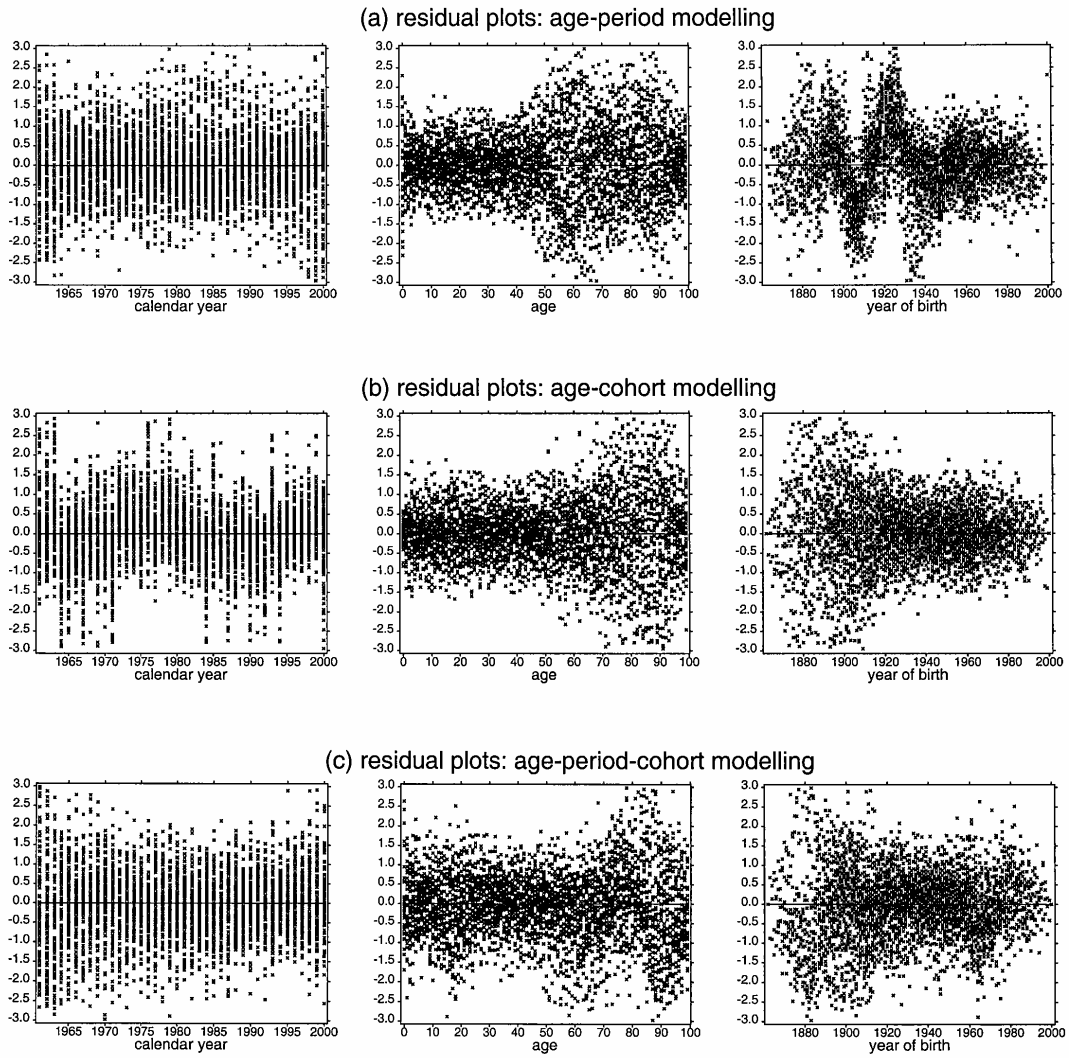

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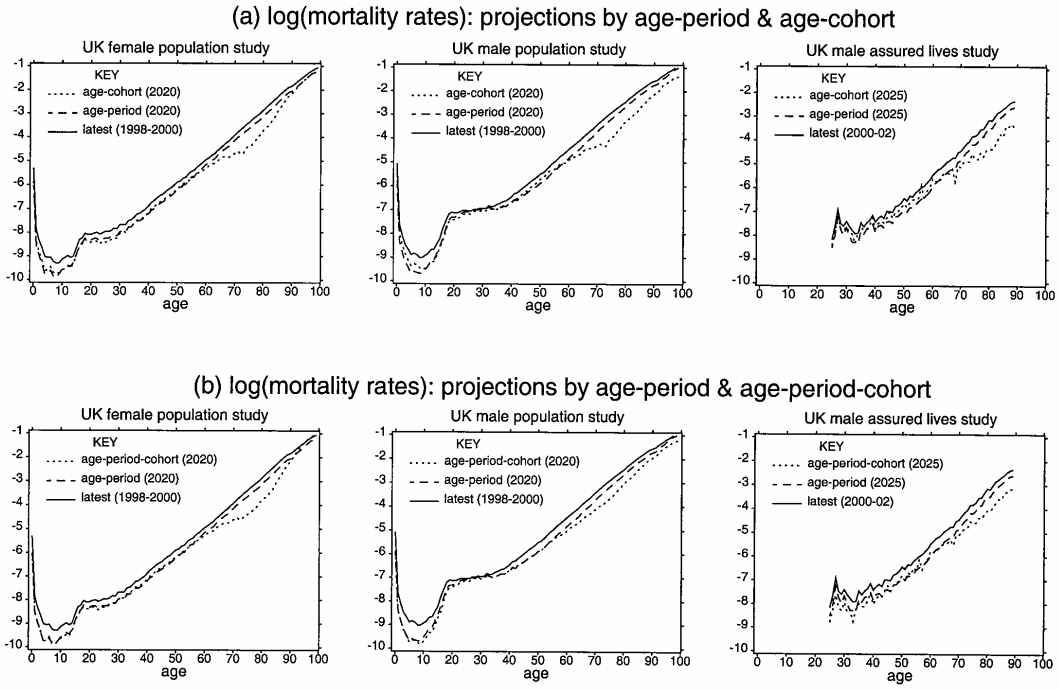

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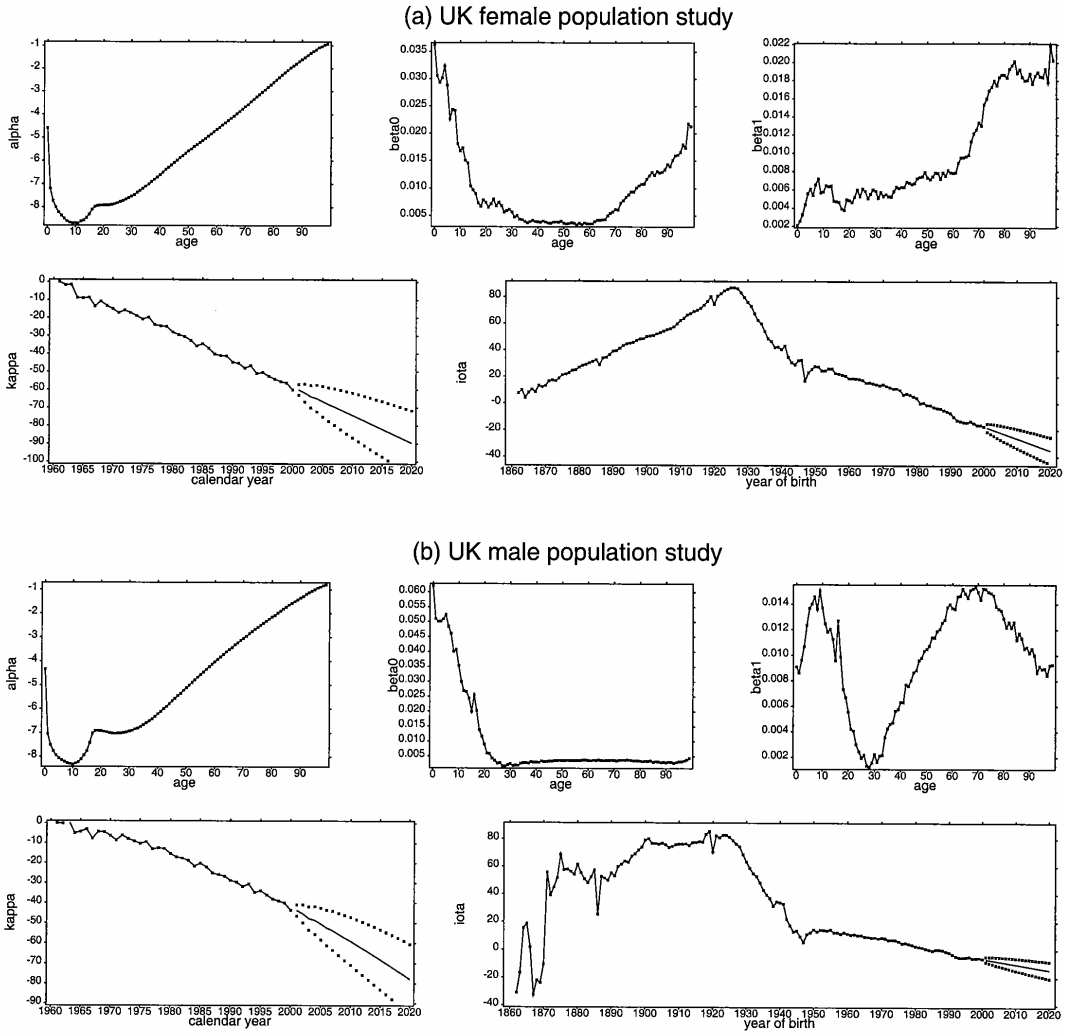

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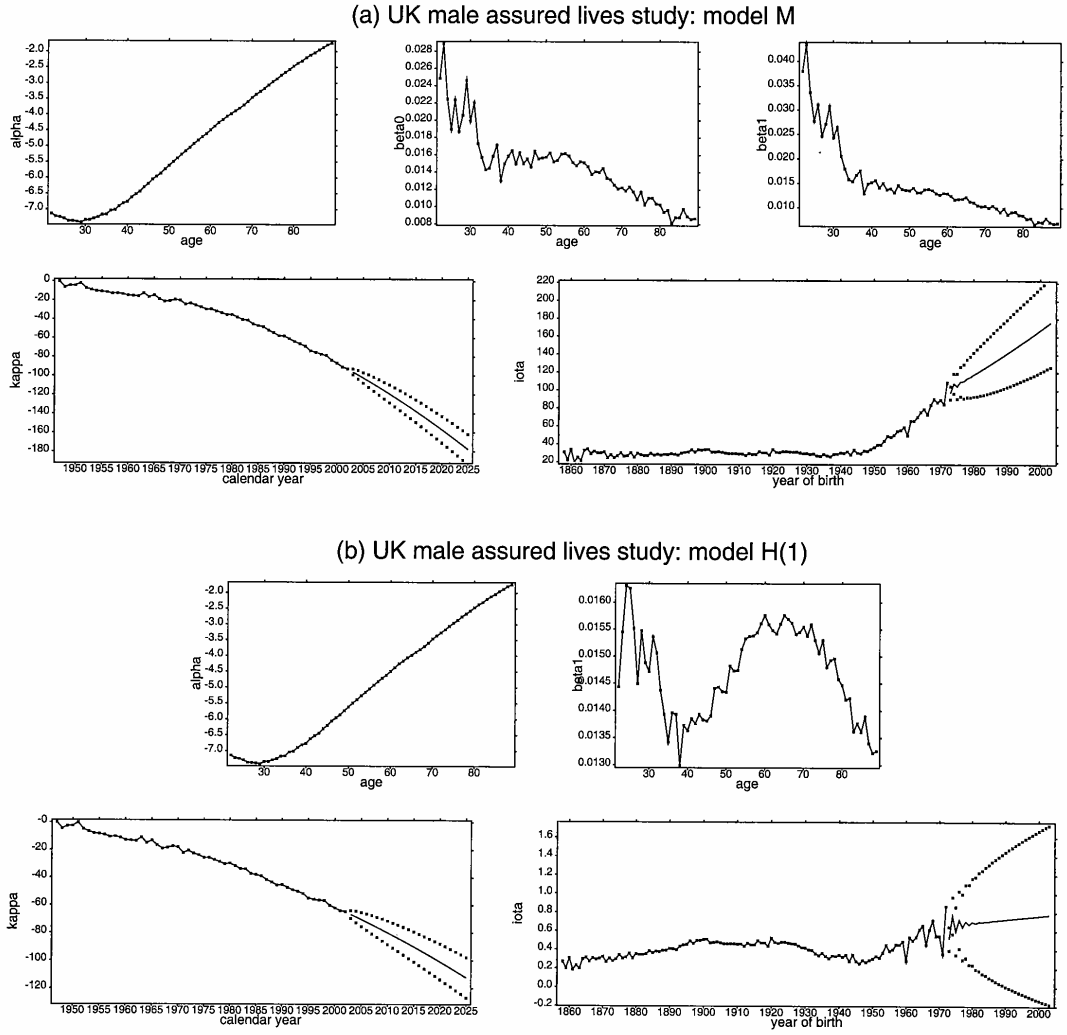
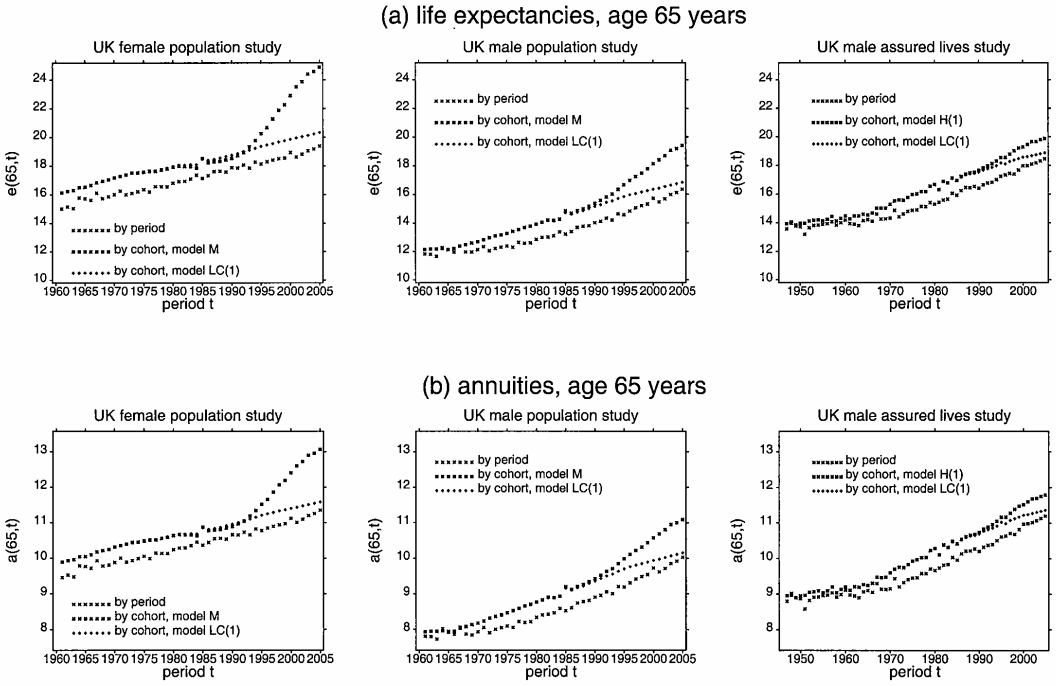


Fig. 5. UK male assured lives study, parameter estimates- (a) model M; (b) model H_1



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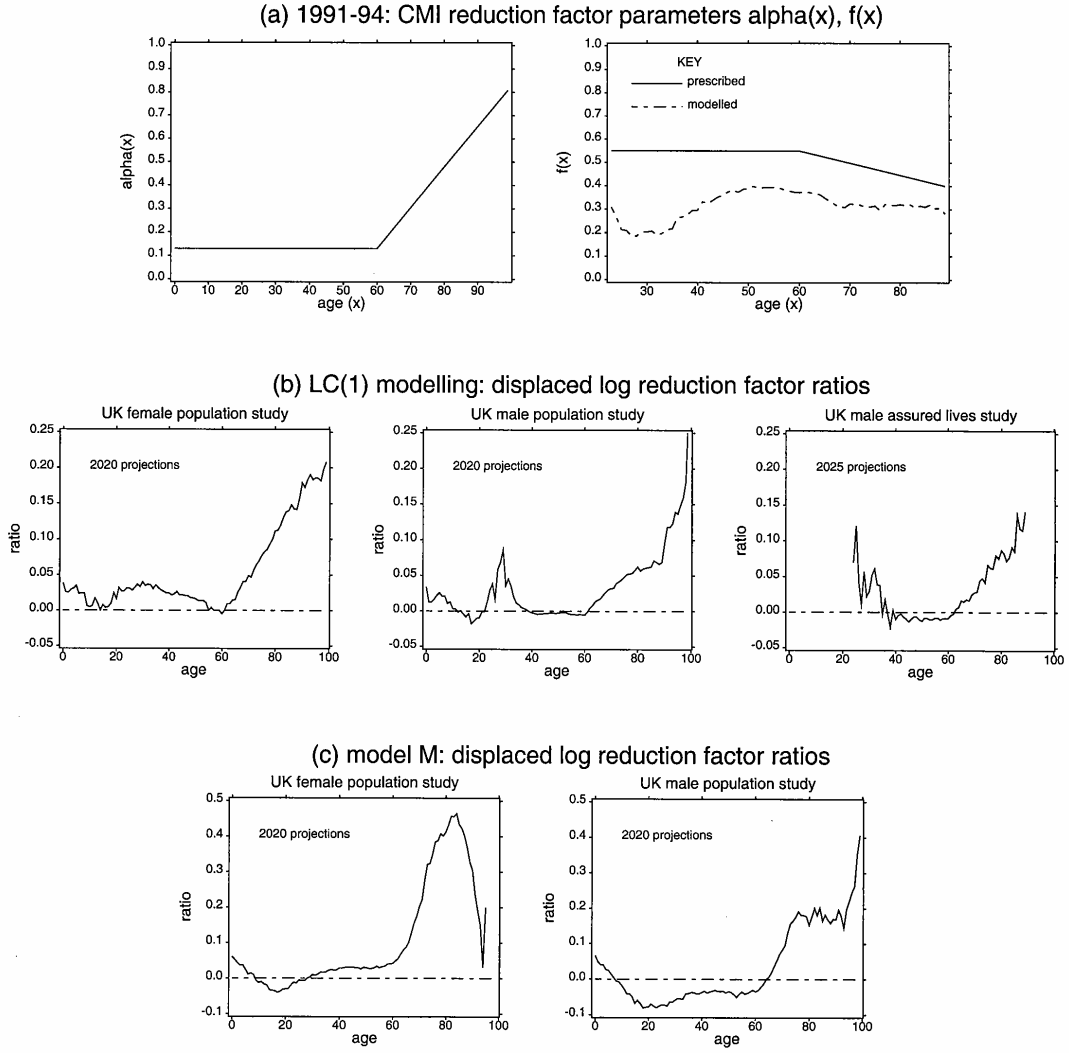

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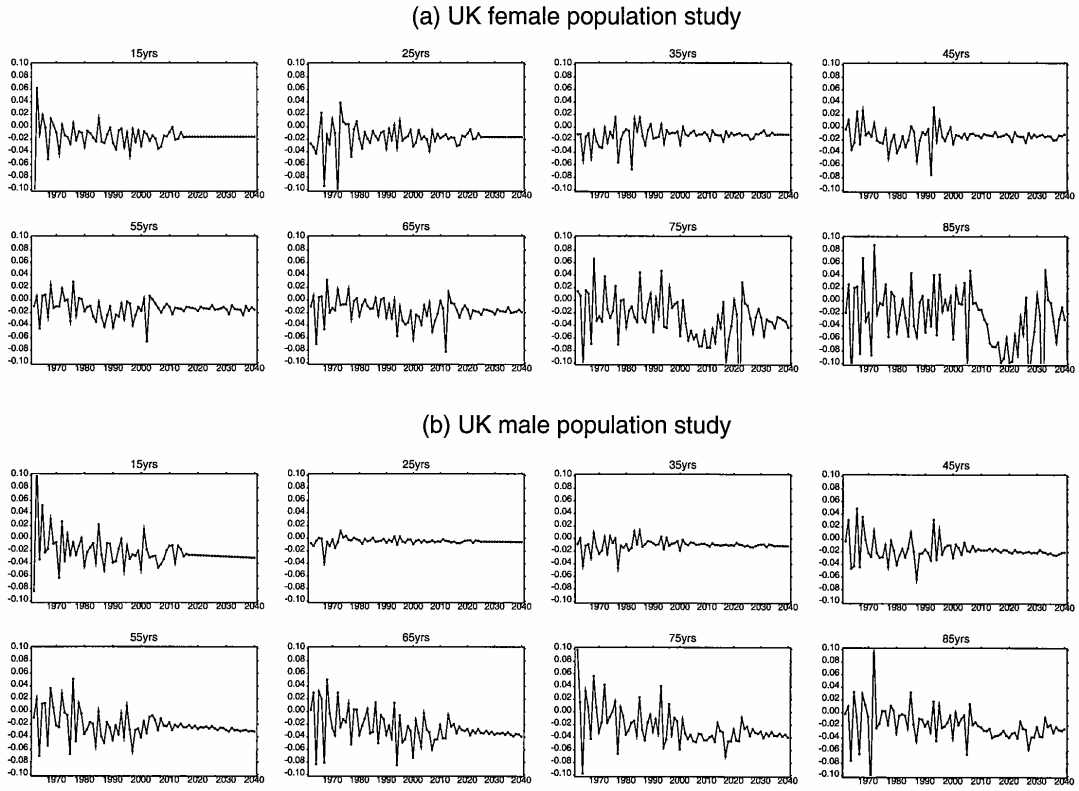

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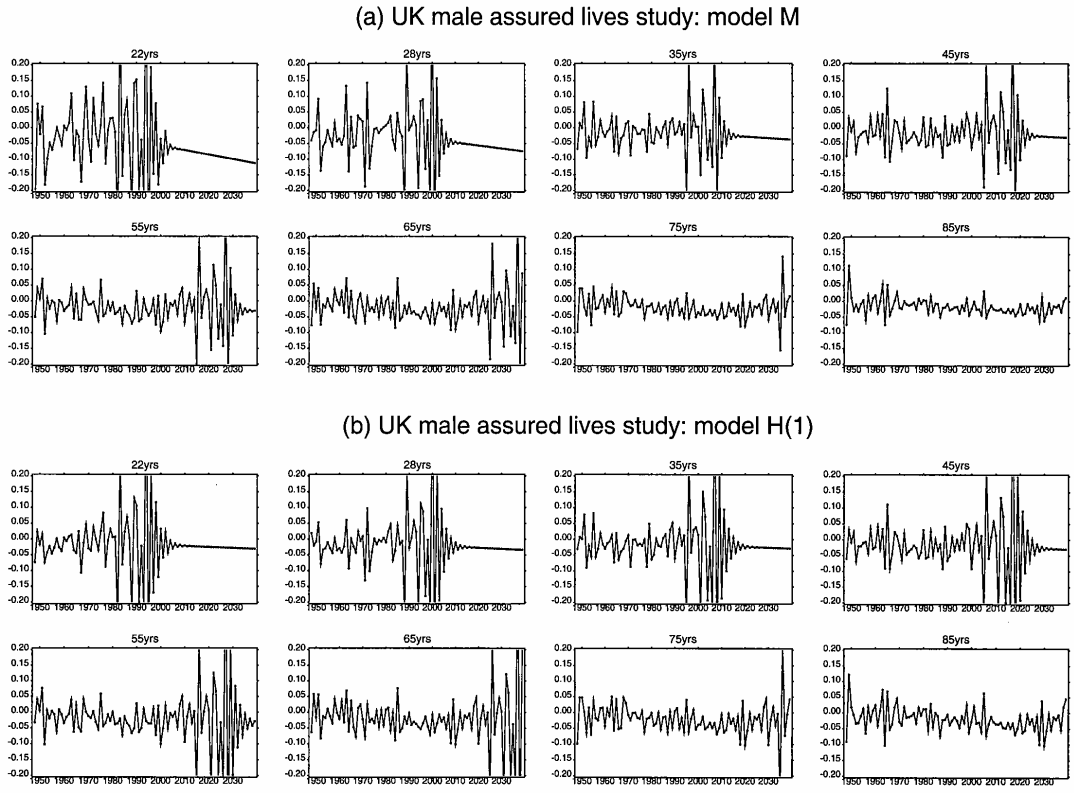

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