Cass Business School
City of London

Cass means business

Faculty of Actuarial Science and Statistics

The Management of De-cumulation Risks in a Defined contribution Environment.

R.J.G. Gerrard, S. Haberman and E. Vigna

Actuarial Research Paper No. 161

February 2005

ISBN 1 901615-85-5

Cass Business School
106 Bunhill Row
London EC1Y 8TZ
T +44 (0)20 7040 8470
www.cass.city.ac.uk

The Management of De-cumulation Risks in a Defined Contribution Environment

Russell Gerrard* Steven Haberman† Elena Vigna‡

July 23, 2004

Abstract

The aim of the paper is to provide a flexible tool for finding optimal investment and consumption choices to be adopted by pensioners in a defined contribution pension scheme, when they take the income drawdown option after retirement. The investment/consumption plan is adopted until the time of compulsory annutization (if ever) or death, whichever occurs first. The mathematical tools provided by dynamic programming techniques are applied in two cases: (a) when the amount withdrawn periodically is fixed and the pensioner can only decide about the portfolio allocation, and (b) when he/she can also choose the consumption. In both cases, the effect of the bequest motive is dealt with by considering the utility attached to the wealth at the time of death, in the event that the pensioner dies before annuitization. Numerical examples are also presented.

Keywords: income drawdown option, stochastic optimal control, bequest motive.

JEL classification: C61, D91, G11, J26.

*Cass Business School, City University, London. Email: r.j.gerrard@city.ac.uk.
†Cass Business School, City University, London. Email: s.haberman@city.ac.uk.
‡Dipartimento di Statistica e Matematica Applicata, University of Turin. Email: elena.vigna@econ.unito.it.

1 Introduction

The main difference between defined benefit (DB) and defined contribution (DC) pension schemes is the way in which the financial risk is treated.
In DB plans, the financial risk is borne by the sponsors of the scheme, who do not know in advance what contribution rate will be needed to finance the benefits promised. In practice, this risk is usually borne by the employer: employees pay a fixed part of the total contribution and the employer pays the remaining, and obviously aleatory, part of the adjusted contribution rate.
In DC plans, the financial risk is borne by the member: contributions are fixed in advance and the benefits provided by the scheme depend on the investment performance experienced during the active membership and on the price of the annuity at retirement, in the case that the benefits are given in the form of an annuity. Therefore, the financial risk can be split into two parts: investment risk, during the accumulation phase, and annuity risk, focused at retirement. In order to limit the annuity risk — which is the risk that high annuity prices (driven by low bond yields) at retirement can lead to a lower than expected pension income — in many schemes the member has the possibility of deferring the annuitization of the accumulated fund. This possibility, called the income drawdown option, consists of leaving the fund invested in financial assets as in the accumulation phase, and allows for periodic withdrawals by the pensioner, until annuitization occurs (if ever).

The current actuarial literature about the financial risk in defined benefit and defined contribution pension schemes is quite rich. Papers dealing with the typical risks in DB schemes — eg solvency risk and contribution rate stability risk — are, among others, Boulier, Trussant and Florens (1995), Boulier, Michel and Wisnia (1996), Cairns (2000), Haberman and Sung (1994), Owadally and Haberman (2004). The financial risk in DC schemes in the accumulation phase is considered, among others, by Blake, Cairns and Dowd (2001), Booth and Yakoubov (2000), Boulier, Huang and Taillard (2001), Haberman and Vigna (2002), Khorasanee (1998) and Knox (1993). Arts and Vigna (2003) and Chiarolla, Longo and Stabile (2004) analyze both the accumulation and the distribution phase of a defined contribution pension scheme. The financial risk in the distribution phase of defined contribution pension schemes has been dealt with in many recent papers, including: Albrecht and Maurer (2002), Blake, Cairns and Dowd (2003), Gerrard, Haberman and Vigna (2004a), Gerrard, Haberman, Højgaard and Vigna (2004b), Kapur and Orszag (1999), Khorasanee (1996), Milevsky (2001), Milevsky and Young (2002).

The focus of this paper is on the management of risks in the de-cumulation phase of a defined contribution pension scheme, and the assumption made is that the retiree takes the income drawdown option. The pensioner who takes the drawdown option has three principal degrees of freedom:

1 he/she can decide what investment strategy to adopt in investing the fund at his/her disposal;

2 he/she can decide how much of the fund to withdraw at any time between retirement and ultimate annuitization (if any);

3 he/she can decide when to annuitize (if ever).

The three choices described may be affected (in the timing, or amounts, or both) by restrictions imposed by law or by the scheme's rules. For example, in the UK, the amount withdrawn must be between 35% and 100% of the annuity which would have been purchasable immediately on retirement, and annuitization of the fund must take place not later than the age of 75. On the other hand, there are no limitations on the asset allocation of the fund.

The first two choices represent a classical inter-temporal decision making problem, which can be dealt with using optimal control techniques in the typical Merton (1971) framework, whereas the third choice can be tackled by defining an optimal stopping time problem.

In this paper, we apply the mathematical techniques provided by the theory of dynamic programming with the aim of proposing a flexible tool that can help members of DC schemes (or DC schemes' advisors) in taking their decisions regarding the first two of the three choices outlined above. The third choice — when to annuitize, analyzed with different approaches, for example, by Blake et al. (2003), Stabile (2003), Milevsky, Moore and Young (2004) and Milevsky and Young (2004) — is the subject of ongoing research.

In particular, we consider first the more general case where the member can decide both about investment allocation and consumption. Then we consider the particular case where the income withdrawn is fixed over time. In both cases, we allow for mortality in the model, therefore the possibility of bequeathing wealth in the case of death before annuitization becomes relevant for the investment/consumption choices, which are consequently affected by the importance given to the bequest motive.

Our paper differs from most of the others in that the consumption path is considered as a control variable available to the pensioner in the post retirement phase, whereas in most of the mentioned works the amount withdrawn consists of the exact amount that a level annuity bought at retirement would provide. The effect of choosing different (optimal) consumption paths is also analyzed, in a realistic settings. Milevsky and Young (2004) also find the optimal consumption over time, solving a similar investment/consumption problem, and find also the optimal time of annuitization. However, they do not include the bequest motive in their model, use a different utility function (namely, the power utility function), and do not follow a target-based approach to the decision-making problem of the pensioner.

The remainder of the paper is organized as follows. In section 2 the investment/consumption problem is presented and solved. In section 3 the notion of "natural target" is introduced and the solution derived in section 2 is analyzed with this particular choice for the target pursued by the pensioner. Some comments on the problem with constraints are also made. In section 4 the investment problem when consumption is fixed is considered. Numerical examples are shown in section 5. Section 6 concludes.

2 The investment/consumption choice problem faced by the retiree

The retiree member of a defined contribution pension scheme acquires control of a fund which may be either used to purchase an annuity or invested in the financial market. All through the paper, the financial market will be described by the typical Black and Scholes framework: there is a risky asset and a riskless asset. The riskless asset has a constant force of interest, denoted by r. The price of the risky asset is assumed to follow a geometric Brownian motion model, i.e. evolves according to the following stochastic differential equation:

$$dQ(t) = \lambda Q(t)\, dt + \sigma Q(t)\, dW(t) \tag{2.1}$$

where $W(t)$ is a standard Brownian motion.

The pensioner withdraws from the fund an instantaneous amount of income $b(t)$ and invests a proportion of the portfolio in the risky asset equal to $y(t)$ at any time t. The stochastic differential equation that describes the evolution of the fund $X(t)$ is:

$$dX(t) = [X(t)(y(t)(\lambda - r) + r) - b(t)]dt + X(t)y(t)\sigma dW(t) \tag{2.2}$$

We assume here that the reasons that push the retiree to choose the income drawdown option are both the hope of being able to purchase in the future an annuity higher than the pension income provided by immediate annuitization at retirement and the ability of bequeathing wealth in the case of death before annuitization.

It seems reasonable to assume that the individual has a certain target in mind, pursued during the drawdown plan. In particular, we shall assume that the pensioner has two different kinds of targets: a target for the size of the fund and a desired level of income to be consumed. Deviations from the targets will result in a loss for him/her. Therefore, the loss experienced by the pensioner consists of a number of parts:

- a disutility continuously experienced when the income drawn down from the fund is below the ideal level of income; a similar disutility is experienced also if the income consumed is above the level which is considered necessary to the pensioner, in that consuming excessively may result in a high chance of failure in achieving the final target at the time of annuitization, and, even worse, may lead to a higher probability of eventual ruin;
- a disutility arising whenever the level of the fund is below or above the target level at that time; imposing a penalty for deviations above the target fund can be explained by noting that allowing the fund to exceed the target level implies that the pensioner has exposed him/herself to unnecessary risk in the past; Gerrard et al. (2004a) discuss extensively about the importance of a target-based approach to decision making in the de-cumulation phase of a defined contribution pension scheme;
- a terminal disutility engendered at time T of annuitization by any discrepancy between the level of the annuity actually purchased and the ideal level set by the investor;
- a positive utility experienced in the event of death before annuitization, due to the investor's ability to fulfil the motive of bequeathing the assets in the fund to a nominated individual.

We denote by b_0 the target level of income periodically withdrawn from the fund during the drawdown phase, by b_1 the target level of income from the annuity purchased at age T, and by $F(t)$ the running target for the level of the fund at age t. Typically, there is a relationship between the income levels b_1 and b_0, namely that the desired level b_1 after annuitization is likely to be greater than or equal to the target level of consumption b_0 during the drawdown phase (considering the fact that there may be restrictions during the de-cumulation phase and also the fact that medical expenses tend to increase at older ages). We can assume, without loss of generality, that

$$b_1 = \eta b_0.$$

The continuously experienced disutility can be written in terms of a loss function depending on the time, t, and the level of the fund, x:

$$L(t, x) = e^{-\rho t}[u(F(t) - x)^2 + v(b_0 - b(t))^2], \tag{2.3}$$

where u and v are non-negative constants, interpretable as weights given to the desire of monitoring the growth of the fund and the daily consumption respectively, and ρ is the usual subjective inter-temporal discount factor.

The chosen disutility function, quadratic in the fund and in the consumption, is not new in the literature of stochastic optimal control problems. In fact, it is a particular case of (stochastic) linear quadratic optimal control problems – problems with applications mainly in the engineering context – where the cost functional is quadratic in both the state variable and the control (see, for instance, Yong and Zhou (1999) for a detailed description of linear quadratic optimal control problems, with examples of applications).

The terminal cost which comes into operation in the event of survival to time T takes the form

$$K(x) = we^{-\rho T}(b_1 - kx)^2, \tag{2.4}$$

and the utility of bequeathing assets of x on death at age t is

$$M(t,x) = e^{-\rho t} nx. \tag{2.5}$$

The positive constant k may be seen as the amount of annuity provided by the insurance company at age T for 1 unit of capital, w and n are non-negative constants, interpretable as weights given to the achievement of the final annuity level of b_1 and to the importance given to the ability to leave a bequest. In particular, the constant n is associated with the size of the fund at the time of death.

We notice that the utility associated with the bequest is linear in the wealth, whereas the other kinds of loss are quadratic. One reason for this choice is that while it is natural for the pensioner to pursue targets for the level of consumption and the size of the fund, and the choice of a quadratic loss function is intended to penalize any deviation from the targets (see discussion above), there seems to be no natural target for the size of bequest to be left to heirs, and in case of death before annuitization, the higher the fund the better.

The total expected loss from age t onwards is

$$H_{t,x}(y(\cdot), b(\cdot)) = \mathbb{E}[\int_t^{T \wedge T_D} L(s, X(s))\, ds + K(X(T))\mathbf{1}_{T_D > T} - M(T_D, X(T_D))\mathbf{1}_{T_D < T} \,|\, X(t) = x] \tag{2.6}$$

where T_D is the random time of death.

The objective is to minimize over possible investment and consumption choices the expected discounted future loss from retirement until time $T \wedge T_D$, find the optimal value function:

$$V(t,x) = \min_{y(\cdot), b(\cdot)} H_{t,x}(y(\cdot), b(\cdot)) \tag{2.7}$$

and the optimal couple $(y^*(\cdot), b^*(\cdot))$ that satisfies $V(t,x) = H_{t,x}(y^*(\cdot), b^*(\cdot))$.

2.1 Solution of the problem

Minimizing equation (2.7) is a stochastic optimal control problem. In solving the problem, we follow the method used in Gerrard et al. (2004b), so the reader should refer to this and references therein for a detailed explanation of the derivation of the HJB equation. For a more general derivation of a Bellman system of differential equations in a problem when benefits are triggered by any transition between different states in a finite Markov chain, applied to an insurance policy, see Steffensen (2004).

The optimal value function $V(t,x)$ satisfies the following Hamilton-Jacobi-Bellman (HJB) equation:

$$\begin{aligned} 0 &= \min_{y,b} \Big\{ e^{-\rho t}\left[u(F(t)-x)^2 + v(b_0-b)^2\right] + V_x\left[-(b-rx)+(\lambda-r)xy\right] \\ &\qquad + V_t + \tfrac{1}{2}\sigma^2x^2y^2V_{xx} - \delta(t)V + \delta(t)e^{-\rho t}nx \Big\} \end{aligned}$$

with the boundary condition

$$V(T,x) = K(x) = we^{-\rho T}(b_1-kx)^2, \tag{2.8}$$

and where V_t, V_x and V_{xx} represent $\frac{\partial V}{\partial t}$, $\frac{\partial V}{\partial x}$ and $\frac{\partial^2 V}{\partial t^2}$ respectively, and $\delta(t)$ is the force of mortality of the individual at age t.

Minimizing over y and b we find that the optimal proportion of wealth to invest in the risky asset and the optimal income to draw down from the fund are given by the following optimal control functions:

$$y^*(t,x) = -\frac{(\lambda-r)}{\sigma^2 x}\frac{V_x}{V_{xx}} \tag{2.9}$$

$$b^*(t,x) = b_0 + \frac{1}{2v}e^{\rho t}V_x. \tag{2.10}$$

Substituting these values into the HJB equation we obtain

$$\frac{\beta^2V_x^2}{2V_{xx}} + \frac{e^{\rho t}}{4v}V_x^2 + (b_0-rx)V_x - V_t + \delta(t)V = e^{-\rho t}\left[u(F(t)-x)^2 + \delta(t)nx\right],$$

where $\beta = (\lambda-r)/\sigma$, the Sharpe ratio of the risky asset.

As in previous work, we seek a solution of the form

$$V(t,x) = e^{-\rho t}(A(t)x^2 + B(t)x + C(t)). \tag{2.11}$$

This formulation works as long as $A(t)$, $B(t)$ and $C(t)$ satisfy the following system of differential equations

$$\left\{ \begin{aligned} A'(t) &= \frac{1}{v}A(t)^2 + (\rho - 2r + \beta^2 + \delta(t))A(t) - u, \\ B'(t) &= \left(\rho + \beta^2 - r + \delta(t) + \frac{A(t)}{v}\right)B(t) + 2A(t)b_0 + 2uF(t) - n\delta(t) \\ C'(t) &= \frac{\beta^2B(t)^2}{4A(t)} + \frac{B(t)^2}{4v} + b_0B(t) + (\rho+\delta(t))C(t) - uF(t)^2 \end{aligned} \right. \tag{2.12}$$

with boundary conditions given by

$$A(T) = wk^2, \qquad B(T) = -2wkb_1, \qquad C(T) = wb_1^2. \tag{2.13}$$

Note that one requirement for the proposed solution to be optimal is that $V_{xx} > 0$, or in other words that $A(t) > 0$ for all $t \leq T$. The differential equation for $A(t)$ is of Riccati type and can be easily solved if one particular solution is known. The difficulty in finding an analytical solution comes from the presence of the time dependent term $\delta(t)$ in the coefficient of $A(t)$. Therefore, we will consider different cases for the specification of $\delta(t)$.

The difficulty of the task (solving a Riccati differential equation) arises when finding the solution in the linear stochastic regulator problem (see also Øksendal (1998)). This is due to the quadratic term in the control variable (here, the consumption).

2.2 On the solution of the differential equation for $A(t)$: constant mortality

In the special case where $\delta(t) = \delta$ for all t, we are able to find closed form solutions. In this case we have to solve the following problem:

$$\begin{cases} A'(t) = \frac{1}{v}A(t)^2 + \phi A(t) - u \\ A(T) = wk^2, \end{cases} \tag{2.14}$$

with $\phi = \rho - 2r + \beta^2 + \delta$.

The solution is:

$$A(t) = \frac{f_1(wk^2 - f_2)e^{R(T-t)} - f_2(wk^2 - f_1)}{(wk^2 - f_2)e^{R(T-t)} - (wk^2 - f_1)},$$

using the notation:

$$R = \sqrt{\phi^2 + 4\frac{u}{v}}, \qquad f_1 = \frac{v}{2}(R - \phi), \qquad f_2 = -\frac{v}{2}(R + \phi)$$

We observe that

1. $f_1 \geq 0 > f_2$;
2. $\lim_{t \to -\infty} A(t) = f_1$;
3. $A'(t)$ is positive if $A(t) > f_1$, negative if $f_2 < A(t) < f_1$. If $A(t_0) > f_1$ then $A(t) > f_1$ for all t, and in particular $A(T) > f_1$. Conversely, if $A(t_0) < f_1$ then $A(t) < f_1$ for all t, and in particular $A(T) < f_1$.
4. We can restate the same remarks from a different viewpoint. If $A(T) > f_1$ then it must have been the case that $A(t_0) > f_1$ and A increases over the range (t_0, T). If, on the other hand, $f_2 < A(T) < f_1$ then it must be the case that $A(t_0) < f_1$ and A is a decreasing function over the range (t_0, T), so it must everywhere be greater than $A(T)$, which is equal to wk^2, and therefore strictly positive.

As a consequence, $A(t) > 0$ for all $t_0 \leq t \leq T$.

2.3 On the solution of the differential equation for $A(t)$: age dependent mortality

In the more realistic case of non-constant mortality, the problem to solve is:

$$\begin{cases} A'(t) = \frac{1}{v}A(t)^2 + \phi(t)A(t) - u \\ A(T) = wk^2, \end{cases} \tag{2.15}$$

where $\phi(t) = \rho - 2r + \beta^2 + \delta(t)$.

Equation (2.15) is a Riccati differential equation, meaning that it may not be possible to write down a solution in explicit form for an arbitrarily chosen force of mortality $\delta(t)$. Some common forms for $\delta(t)$ are $\delta(t) = \phi + \psi t$ (linear), $\delta(t) = \phi e^{\omega t}$ (Gompertz) and $\delta(t) = \zeta + \phi e^{\omega t}$ (Makeham). In all of these cases the solution $A(t)$ exists, but can be written as a non-linear combination of Whittaker functions (which are combinations of Kummer functions), therefore it cannot be easily treated in the numerical applications.

However, it is possible to investigate the most important property of $A(t)$ — whether it is positive — without solving explicitly. In fact, it is possible to prove the following fact:

Proposition 1 *If the force of mortality is bounded over the range $t_0 < t < T$ then $A(t) > 0$ for all $t_0 < t < T$.*

The proof is in Appendix A.

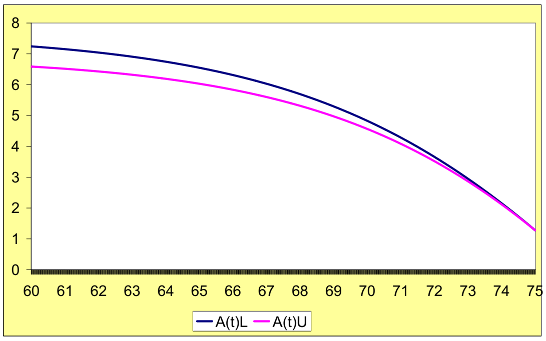
Remark. The force of mortality may be assumed to be a non-decreasing function of time over the range $t_0 < t < T$ (this is true given that we are considering post retirement ages, eg $t_0 = 60$ and $T = 75$; it would not be necessarily true for other ranges of age, like the range 15-35 for the male population), and therefore may be assumed to be bounded above and below.

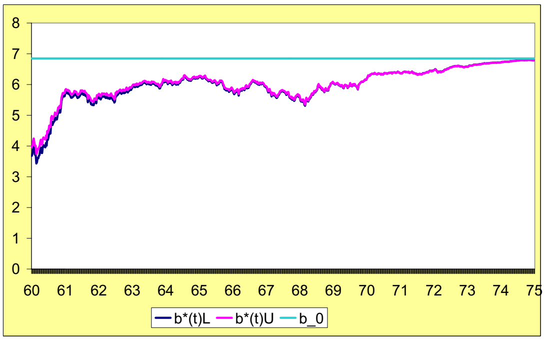
Furthermore, it is also possible to approximate rather well the solution in the applications. In fact, we prove the following lemma (the proof, which is needed in the proof of the proposition, is also in Appendix A):

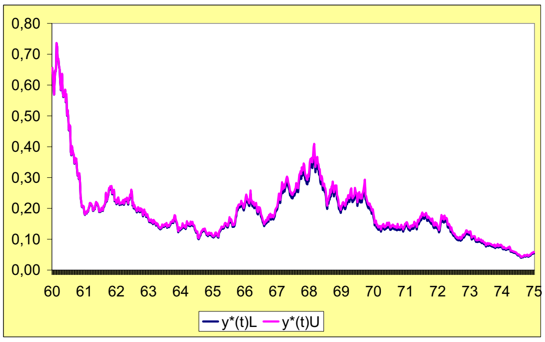
Lemma 2 *Suppose that $A_0(t)$ solves (2.15) in the case where $\phi(t) = \phi_0$, a constant, and that $A(t)$ is a solution to (2.15) in the general case. If $\phi(t) \geq \phi_0$ for all $t_0 \leq t \leq T$, then $A(t) \leq A_0(t)$ for all $t_0 \leq t \leq T$. Conversely, if $\phi(t) \leq \phi_0$ for all $t_0 \leq t \leq T$, then $A(t) \geq A_0(t)$ for all $t_0 \leq t \leq T$.*

Therefore, if the real force of mortality $\delta(t)$ is bounded between a lower constant force of mortality δ_L and a higher constant force of mortality δ_U, the behaviour of the true $A(t)$ will be bounded between the corresponding $A_L(t)$ and $A_U(t)$, solutions of the differential equation with δ_L and δ_U respectively, and can be well approximated by these boundaries if these are close enough.

In the numerical applications of the model (presented in section 5), the plot of the functions $A_L(t)$ and $A_U(t)$, corresponding respectively to $\delta_L = \delta_{t_0}$ and $\delta_L = \delta_T$ (with t_0 age at retirement eg 60 and T age of compulsory annuitization, eg $T = 75$), shows that these functions are very close to each other, as are the corresponding optimal controls. An example is given by the pictures in Figure 1, which illustrate these functions, together with the corresponding optimal consumption and investment choices and the resulting optimal growth of the fund, in one particular scenario of market returns. The forces of mortality at age 60 and at age 75 are calculated according to the Italian projected mortality table RG48 (males) data.







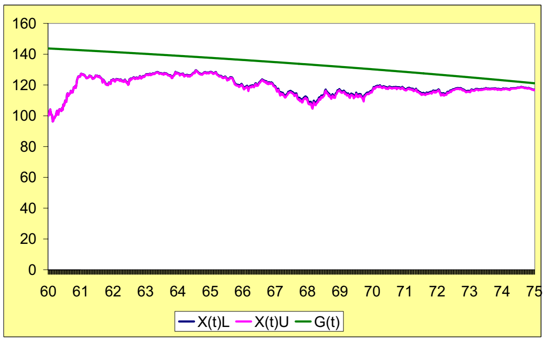


Figure 1: Comparison between optimal controls and growth of the fund with two different constant forces of mortality (at age 60 and at age 75).

In the graphs reporting the optimal controls and the evolution of the fund under optimal control, the two curves corresponding to the different forces of mortality cannot be clearly distinguished, because they are almost coincident. This underlines the negligible effect of the value of the force of mortality in the practical applications considered here.

3 The "natural" target function

We now introduce a new function, which will turn out to be useful throughout the paper:

$$G(t) = -\frac{B(t)}{2A(t)}.$$

The optimal controls can be expressed in function of $A(t)$ and $G(t)$:

$$b^*(t,x) = b_0 - \frac{A(t)}{v}(G(t) - x), \qquad y^*(t,x) = \frac{\lambda - r}{\sigma^2}\,\frac{G(t) - x}{x}. \tag{3.1}$$

Here we consider the properties of the optimally controlled process on the assumption that there is no restriction on the possible values taken by the controls b^* and y^*. The effect of imposing restrictions on the controls will be considered in a later section.

Let us denote by $X^*(t)$ the process of the fund under optimal control. Then

$$dX^*(t) = \left[-b_0 + \left(\frac{A(t)}{v} + \beta^2\right)(G(t) - X^*(t)) + rX^*(t)\right] dt + \beta(G(t) - X^*(t))\, dW(t),$$

As it appears that a pivotal quantity is the shortfall $G(t) - X^*(t)$, we shall denote this process by $S(t)$. We observe that

$$dS(t) = \left[G'(t) + b_0 - rG(t) - \left(\frac{A(t)}{v} + \beta^2 - r\right) S(t)\right] dt - \beta S(t)\, dW(t).$$

Although we have no explicit solution for $A(t)$, we can nevertheless write

$$G'(t) = -\frac{B'(t)}{2A(t)} + \frac{B(t)A'(t)}{2A(t)^2} = \left(r + \frac{u}{A(t)}\right) G(t) - b_0 - \frac{2uF(t) - n\delta(t)}{2A(t)}. \tag{3.2}$$

Substituting this back in, we obtain

$$dS(t) = \left(r - \beta^2 - \frac{A(t)}{v}\right) S(t)\,dt - \beta S(t)\,dW(t) + \frac{2u(G(t) - F(t)) + n\delta(t)}{2A(t)}\,dt. \tag{3.3}$$

The stochastic differential equation (3.3) satisfied by $S(t)$ suggests a natural form for the target function $F(t)$: if it is the case that

$$F(t) - G(t) = \frac{n}{2u}\delta(t) \tag{3.4}$$

then the logarithm of $S(t)$ is a Wiener process with a time-dependent drift, and therefore $S(t)$ will always remain non-negative, under the assumption that $S(t_0) > 0$. In other words, if $X(t_0) < G(t_0)$, then $X(t)$ will always remain below the function $G(t)$.

This property was also observed for the model discussed in Gerrard et al. (2004a), in a problem where consumption was fixed. In this case, it was discovered that there was a natural explanation for the target $F(t)$ which arose out of the equations, namely that it consisted of precisely the amount of money required to fund consumption at the fixed level until the time of compulsory annuitization and then to achieve the final target pursued. Here we find a similar explanation, which includes also the bequest motive. In fact, it is easy to prove that the functions $F(t)$ and $G(t)$ which satisfy (3.2) and (3.4) are

$$G(t) = \frac{b_0}{r}(1 - e^{-r(T-t)}) + \frac{b_1}{k}e^{-r(T-t)}, \qquad F(t) = \frac{b_0}{r}(1 - e^{-r(T-t)}) + \frac{b_1}{k}e^{-r(T-t)} + \frac{n}{2u}\delta(t) \tag{3.5}$$

Even when F and G do not satisfy (3.4), it is still possible to obtain a solution to (3.2), at least in the form

$$G(t) = \frac{b_1}{k}\exp\left(-r(T-t) - u\int_t^T \frac{ds}{A(s)}\right) + \int_t^T \left(b_0 + \frac{2uF(z) - n\delta(z)}{2A(z)}\right)\exp\left(-r(z-t) - u\int_t^z \frac{ds}{A(s)}\right)dz. \tag{3.6}$$

The (reasonable) interpretation for the choice of the natural target $F(t)$ is the following. If a sum $G(t)$ were invested at time t in the risk-free asset, then the interest payments would cover consumption at rate b_0 until the age of compulsory annuitization, and thereafter would permit the purchase of an annuity paying the required amount b_1 per unit time. Therefore, the level $G(t)$ can be considered to be a sort of "safety level" for the personal needs of the pensioner (see Gerrard et al. (2004a)), and would in effect coincide with his/her target, should he/she have no bequest motive (i.e. $n = 0$). If the pensioner has a bequest motive, his/her target would be chosen accordingly to the importance given to it, and we may think of the quantity $\frac{n}{2u}\delta(t)$ as an extra amount added to the pensioner's needs to allow for the bequest desire. This amount increases also if the force of mortality increases: if the pensioner thinks he/she has a high probability of imminent death, the importance given to the amount left to the heirs in the case of death before annuitization is likely to increase.

Rather strangely, the optimal controls chosen by the individual with this choice of the target function $F(t)$ do not depend on the weight given to the bequest motive, as we can see by observing that neither $A(t)$ nor $G(t)$ depend on the value of n (see 2.12, 3.1 and 3.5), while they do depend on the force of mortality. It seems that if the pensioner does take into account the bequest motive and the subjective probability of death when selecting the target pursued (by appropriate choice of $F(t)$), his/her optimal behaviour is then the same as if the bequest motive were absent. It must be said that the pensioner can choose a target other than the natural one, and in this case the optimal control would depend on n via $G(t)$ (see 3.6). We shall ignore this case in the rest of the paper and defer the analysis of other choices of the target to future research.

3.1 Some features of the optimal controls when natural targets are chosen

The optimal control policies at time t, with a fund of $X^*(t)$, are the following:

$$b^*(t, X^*(t)) = b_0 - \frac{A(t)}{v}(G(t) - X^*(t)), \qquad y^*(t, X^*(t)) = \frac{\lambda - r}{\sigma^2}\frac{G(t) - X^*(t)}{X^*(t)}$$

When the natural targets are chosen, the shortfall of the fund from $G(t)$ is always strictly positive. Therefore, the optimal consumption never exceeds the level b_0, and the amount invested in the risky asset is always positive. This property can be of considerable importance, given the fact that consumption can be limited by regulation (in UK the amount withdrawn must lie between 35% and 100% of the amount of annuity provided by the fund at retirement), and short selling is likely to be forbidden, and given also the fact that adding constraints to the optimization problem would considerably increase the difficulty of finding and treating the solution (see below). Examples of works where an optimization problem with constraints has been solved are Di Giacinto and Gozzi (2004), in the context of a defined contribution pension scheme, and Browne (1995), who minimizes the probability of ruin when borrowing is not allowed.

We notice that the optimal amount invested in the risky asset $y^*(t)X^*(t)$ is proportional to the shortfall $S(t)$, which is the difference between the safety level and the fund level. This result is similar to a result found by Browne (1997): solving two "survival problems" (maximizing the probability of reaching a "safe region" before occurrence of ruin and minimizing the discounted penalty paid upon going bankrupt) he finds that in both problems the optimal policy implies investing in the risky asset a proportion of the (positive) difference between the amount needed for being in the safe region and the fund level.

The function $A(t)$ depends on the value of the parameters v, u and w, but it can be seen from the form of the controls that what counts most is the ratio $\frac{A(t)}{v}$. From a detailed study of the solution $A(t)$ of the Riccati differential equation in the case of constant mortality, we see that:

$$\lim_{v \to +\infty} \frac{A(t)}{v} = 0, \tag{3.7}$$

a result that can be naturally extended to the general case. This feature is also intuitive: when the importance attached to the monitoring of the consumption is high, the optimal consumption tends to coincide with the desired level b_0.

From numerical examples which will be shown later, we also see that this ratio depends heavily on the ratio $\frac{u}{v}$, rather than on the individual values of the parameters. Mathematically, this could be explained by observing that for points in time sufficiently far from time T (i.e. for $T-t$ sufficiently large), we have:

$$\frac{A(t)}{v} \sim \frac{1}{2}(R - \phi)$$

recalling that $R = \sqrt{\phi^2 + 4\frac{u}{v}}$ and that ϕ does not depend on the parameters v, u and w. When time T approaches, the ratio $\frac{w}{v}$ becomes more important (since $\frac{A(T)}{v} = \frac{w}{v}k^2$). In the numerical applications, we have observed that the most significant factor driving the controls is the ratio $\frac{u}{v}$, that gives an indication of the relative importance attached to the monitoring of the fund and of the running consumption.

3.2 Imposing restrictions on the controls

Governments (or regulators) may introduce regulations in order to restrict the freedom of the investor with regard to either the income they draw down or the proportion of the fund which may be invested in risky assets. In order to increase the generality of the treatment, however, we will consider only the natural restrictions which follow from the situation being modelled. These are

1. $b \geq 0$. Although it is not impossible to imagine negative consumption — rather than withdrawing money from the fund for day-to-day expenses, the investor pays additional sums into the fund — the protected status of pension funds is likely to rule this out as a possibility.

2. $y \leq 1$. Again, in normal circumstances an investor may well have the option of borrowing at a fixed rate of interest in order to invest in the risky asset, but in the particular context of a pension fund it is unlikely that this will be permitted.

3. $x \geq 0$. If the assets of the fund drop to zero, the rules of the fund will presumably require that the investor stops trading.

If the evolution of the fund is governed by the optimal controls derived with natural targets, then

$$
\begin{aligned}
b^*(t, X^*(t)) < 0 \quad &\Longleftrightarrow \quad S(t) > \frac{vb_0}{A(t)}, \\
y^*(t, X^*(t)) > 1 \quad &\Longleftrightarrow \quad S(t) > \frac{G(t)}{1 + \frac{\lambda - r}{\sigma^2}}, \\
X^*(t) < 0 \quad &\Longleftrightarrow \quad S(t) > G(t).
\end{aligned}
$$

It is clear that $X^*(t) < 0$ can only occur if $y^*(t, X^*(t)) > 1$, so the second restriction will take effect more frequently than the third. No such relationship can be proved for the first two restrictions, or for the first and the third restrictions, however: which one takes effect first depends on the relative values of the weighting parameters u, v and w. Namely, the critical level at which consumption becomes negative does depend on the choice of the parameters v, u and w, whereas the critical levels for ruin and borrowing money from the bank do not. This means that, by choosing appropriate values of the parameters, negative consumption can in practice be avoided (it would be sufficient, for example, to choose values of the parameters for which ruin occurs before negative consumption). In fact, increasing the value of v will result in a higher level for the barrier for negative consumption (recalling the limit 3.7). Unfortunately, there are no values of the parameters which would help in avoiding ruin or borrowing money from the bank.

Figure 2 shows how the change in the ratio $\frac{v}{u}$ can affect the level of the barrier for negative consumption. In particular, it can be seen that with a high enough $\frac{v}{u}$ (eg 160) the barrier for negative consumption is higher than the barrier for ruin, that is, ruin occurs before negative consumption.

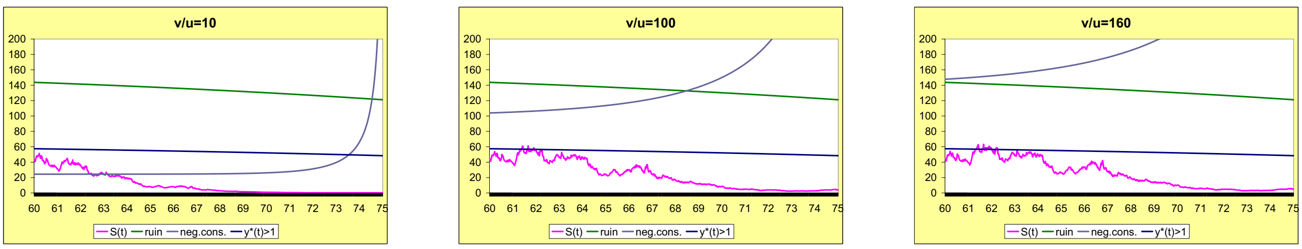


Figure 2: Barriers for the shortfall $S(t)$ for ruin, negative consumption and borrowing money from the bank, with different values of the ratio $\frac{v}{u}$. The path of $S(t)$ in one particular scenario of market returns is also reported.

3.2.1 The case $b^* < 0$

If $S(t) > \frac{vb_0}{A(t)}$, the optimal choice of b is negative. If restriction 1 is in force, it would be natural to suggest that we choose zero consumption in such cases. It should be noted that such a strategy does not lead to the optimal control of the process subject to restriction 1, but the difference may be small.

If we are constrained to choose $b = 0$, the control problem becomes one of choosing y. This is related to the situation considered in section 4, where the income rate is required to take a fixed value b_0. The remarks made in the following section apply to this case, by taking $b_0 = 0$.

3.2.2 The case $y^* > 1$

If $S(t) > \frac{G(t)}{1+\frac{\lambda-r}{\sigma^2}}$, the optimal choice of y is greater than 1. If restriction 2 is in force, so that no bank borrowing is permitted, then the simplest suggestion is that we should take $y^* = 1$ in these cases. If we adopt this strategy then the form of the HJB equation is substantially altered, being now

$$0 = -\frac{1}{4v}e^{\rho t}V_x^2 + ue^{-\rho t}(F(t) - x)^2 + (\lambda x - b_0)V_x + V_t + \tfrac{1}{2}\sigma^2 x^2 V_{xx} - \delta_t v + \delta_t n x e^{-\rho t}.$$

It is possible to find a solution to this equation, again of the form $V(t, x) = e^{-\rho t}(A(t)x^2 + B(t)x + C(t))$, but it is different from the solution which applies in the region $y^* < 1$.

3.2.3 The case where both $b^* < 0$ and $y^* > 1$

When $X(t)$ is sufficiently low, both b^* and y^* fall outside the range permitted by the restrictions. Again a first approximation to the optimal strategy is to set $b = 0$, $y = 1$ in such cases. This implies that the entire fund is invested in the risky asset but that no income is drawn down. The fund, therefore, is a constant multiple of the price of the risky asset. As a geometric Brownian motion with positive drift, it is unable to become negative.

As a consequence, ruin is impossible when the restrictions are applied as long as X_t is such that $b^* < 0$ and $y^* > 1$.

4 The investment choice problem faced by the retiree

In this section we consider the investment allocation problem in the case that consumption is fixed over time, and equal to b_0 per unit time. Gerrard et al. (2004a) consider the same problem in the absence of mortality and the bequest motive. Not surprisingly, this situation turns out to be a natural extension of that above-mentioned work. By letting the weight v given to monitoring of the consumption tend to $+\infty$, we see that it is also a special case of the problem presented in the previous sections.

The growth of the fund is governed by the stochastic differential equation:

$$dX(t) = [X(t)(y(t)(\lambda - r) + r) - b_0]dt + X(t)y(t)\sigma dW(t) \tag{4.1}$$

The running loss function monitors the growth of the fund only:

$$L(t,x) = e^{-\rho t}[u(F(t) - x)^2] \tag{4.2}$$

The terminal cost in the case of survival at age T is:

$$K(x) = we^{-\rho T}(b_1 - kx)^2, \tag{4.3}$$

and the utility of bequeathing assets of x on death at age t is:

$$M(t,x) = e^{-\rho t}nx. \tag{4.4}$$

The aim is to find the optimal investment allocation in order to minimize expected future losses, i.e. find the optimal value function $V(t,x)$:

$$V(t,x) = \min_{y(\cdot)} H_{t,x}(y(\cdot))$$

with

$$H_{t,x}(y(\cdot)) = \mathbb{E}[\int_t^{T\wedge T_D} L(s, X(s))\, ds + K(X(T))\mathbf{1}_{T_D > T} - M(T_D, X(T_D))\mathbf{1}_{T_D < T} \,|\, X(t) = x] \tag{4.5}$$

and find the optimal control $y^*(t)$ such that:

$$V(t,x) = H_{t,x}(y^*(t)).$$

The HJB equation is now:

$$0 = \min_y \left\{ e^{-\rho t}u(F(t) - x)^2 + V_x\left[-(b_0 - rx) + (\lambda - r)xy\right] + V_t + \tfrac{1}{2}\sigma^2 x^2 y^2 V_{xx} - \delta(t)V + \delta(t)e^{-\rho t}nx \right\} \tag{4.6}$$

with the same boundary condition as in the previous problem (2.8).

By trying a value function of the same form as before (see 2.11), we obtain the following system of differential equations for $A(t), B(t)$ and $C(t)$:

$$\left\{ \begin{array}{lcl} A'(t) & = & (\rho - 2r + \beta^2 + \delta(t))A(t) - u, \\ B'(t) & = & \left(\rho + \beta^2 - r + \delta(t)\right) B(t) + 2A(t)b_0 + 2uF(t) - n\delta(t) \\ C'(t) & = & (\rho + \delta(t))C(t) + \dfrac{\beta^2 B(t)^2}{4A(t)} + b_0 B(t) - uF(t)^2 \end{array} \right. \tag{4.7}$$

with the same boundary conditions as before (2.13).

Remark. We notice that the system (4.7) is a particular case of the previous system of differential equations (2.12), when the weight v goes to infinity. Furthermore, it is an extension of the system of differential equations solved in the mentioned work (Gerrard et al. (2004a)), by adding the terms involving the force of mortality and the importance given to the bequest motive. We also notice that the difficulty inherent in solving the differential equation for $A(t)$ has disappeared, in that we have now a linear differential equation and not a Riccati one.

We proceed to solve it in the same way as before, by introducing the function $G(t)$ and the shortfall $S(t)$. The evolution of the shortfall is now:

$$dS(t) = (r - \beta^2)S(t)\,dt - \beta S(t)\,dW(t) + \frac{2u(G(t) - F(t)) + n\delta(t)}{2A(t)}\,dt \tag{4.8}$$

If the natural targets are chosen, i.e. if equation (3.4) holds, then the shortfall is always positive and we end up with the same solution as before for the functions $G(t)$ and $F(t)$ (i.e. equations (3.5)), which is not so surprising, since the parameter v was not involved in those expressions. What is probably more interesting to notice is the fact that also the optimal control $y^*(t)$, which takes the same form as before, is also the same control substantially, when the natural targets are chosen:

$$y^*(t, X^*(t)) = \frac{\lambda - r}{\sigma^2}\left(\frac{G(t) - X^*(t)}{X^*(t)}\right)$$

Obviously, the difference will be given by the path of the optimal fund $X^*(t)$, which follows a different SDE, in which consumption is fixed.

Again, the optimal amount invested in the risky asset is a proportion of the shortfall $S(t)$.

5 Numerical examples

The model outlined in the previous sections has been tested in simulated scenarios for market returns. In particular, the path of the risky asset has been simulated for 1000 scenarios via Monte Carlo simulations, and for each scenario the optimal policies (ie the optimal consumption and the optimal asset allocation) have been applied. The motivation for testing the model in simulated scenarios is to obtain extra information about many relevant issues when the optimal choices derived above are applied. These issues include the undesirable events listed above (ruin, negative consumption and borrowing money from the bank), together with some information about the final outcome of the income drawdown option, in terms of the size of the final annuity that can be bought at time T. So we are interested in investigating the following key features:

1. risk of outliving the assets before time T, called the ruin probability, and average time of ruin, when ruin occurs;
2. behaviour of the optimal consumption and the optimal investment allocation over time;
3. probability of negative consumption, average age at the time of negative consumption and average time spent consuming negative amounts;
4. probability of borrowing money from the bank for investing in the risky asset, average age at the time of borrowing money and average time spent borrowing money;
5. distribution of the final annuity that can be bought at time T;
6. probability that there is some time before T when the pensioner is able to buy a better annuity than the one that he/she could have bought at retirement, and comparison with the targeted annuity;
7. effect on the optimal controls and on the final annuity of the choice of the relative importance given to the running consumption and to the achievement of the target.

5.1 Assumptions

The assumptions made in the simulations are the following:

- retirement is at age $t_0 = 60$ and age of compulsory annuitization is $T = 75$;
- the fund at retirement is $X(t_0) = 100$;
- the amount of consumption targeted during the drawdown phase, b_0, is the level annuity that it can be bought at age 60 with a fund of 100, adopting the Italian projected mortality table RG48 males, assuming the same interest rate as used for the riskless asset and a loading factor of 5%; thus, $b_0 = 6.63$;
- the target for the final annuity is $b_1 = 1.5b_0 = 9.95$ and $b_1 = 2b_0 = 13.26$, in order to test different risk attitudes (the higher the target, the lower the risk aversion);
- parameters for the asset returns are: $r = 4\%, \lambda = 10\%, \sigma = 20\%$; the Sharpe ratio of the risky asset is therefore 0.3; the inter-temporal discount factor is given by $\rho = 4\%$;
- the boundaries chosen for the forces of mortality are $\delta_L = \delta(60) = 0.004625$ and $\delta_U = \delta(75) = 0.026254$ (derived from the survival probabilities at ages 60 and 75 of the RG48 males, assuming a constant force of mortality over the year of age);
- the price of the final annuity at age 75 is $k^{-1} = a_{75}(1 + L)$ where L is the loading factor adopted by the company, chosen to be equal to 5%; the annuity at age 75 has been calculated with the table RG48 males;
- the ratio $\frac{v}{u}$ has been chosen to be equal to 10, 50, 100 and 500, the ratio $\frac{w}{v}$ equal to 1 and 100, with a fixed $u = 1$; the weight given to the bequest motive (although it does not influence the optimal controls) has been chosen to be equal to $n = 10$.

The discretization of the process over the 15 years of the de-cumulation phase has been done on a weekly basis; for each combination of b_1, $\delta_{\tau=L,U}$, $\frac{v}{u}$ and $\frac{w}{v}$, 1000 simulations have been run, using the same 1000 streams of pseudo-random numbers for each combination (in order to allow consistent comparisons between different combination of parameters). In each simulation the Brownian motion has been simulated and hence the behaviour of the optimal controls, as well as the evolution of the fund under optimal control. The distribution over the 1000 simulations of $y^*(t)$ and $b^*(t)$, for $t = 0, 1, ..., 779$ (780 being the number of weeks in 15 years) has been analyzed through some relevant statistics, such as minimum, maximum, mean, standard deviation and some percentiles ($5^{th}, 25^{th}, 50^{th}, 75^{th}$ and 95^{th}). The distribution over the 1000 simulations of the final annuity purchasable at time T with the resulting fund has been examined using the same statistical analysis, and the characteristics listed above have been investigated by checking the path of $y^*(t)$, $b^*(t)$ and $X^*(t)$ over time.

5.2 Simulation results: optimal controls

We have found that the choice of the ratio $\frac{w}{v}$ does not significantly affect results, therefore we only show results for the case of $\frac{w}{v} = 1$ (results for $\frac{w}{v} = 100$ are available from the authors upon request). Similarly, we present results only for the higher force of mortality $\delta_U = \delta(75)$, the corresponding results for the lower force of mortality $\delta_L = \delta(60)$ (available upon request) being almost identical (see, for instance, Figure 1).

On the other hand, the dependence on the ratio $\frac{v}{u}$ is quite strong, both for the optimal controls and for the distribution of the final annuity.

The graphs in Figure 3 show the median of the optimal investment in the risky asset $y^*(t)$ and the mean of the optimal consumption $b^*(t)$ over time, with the four choices of $\frac{v}{u}$ and with the two different targets b_1.

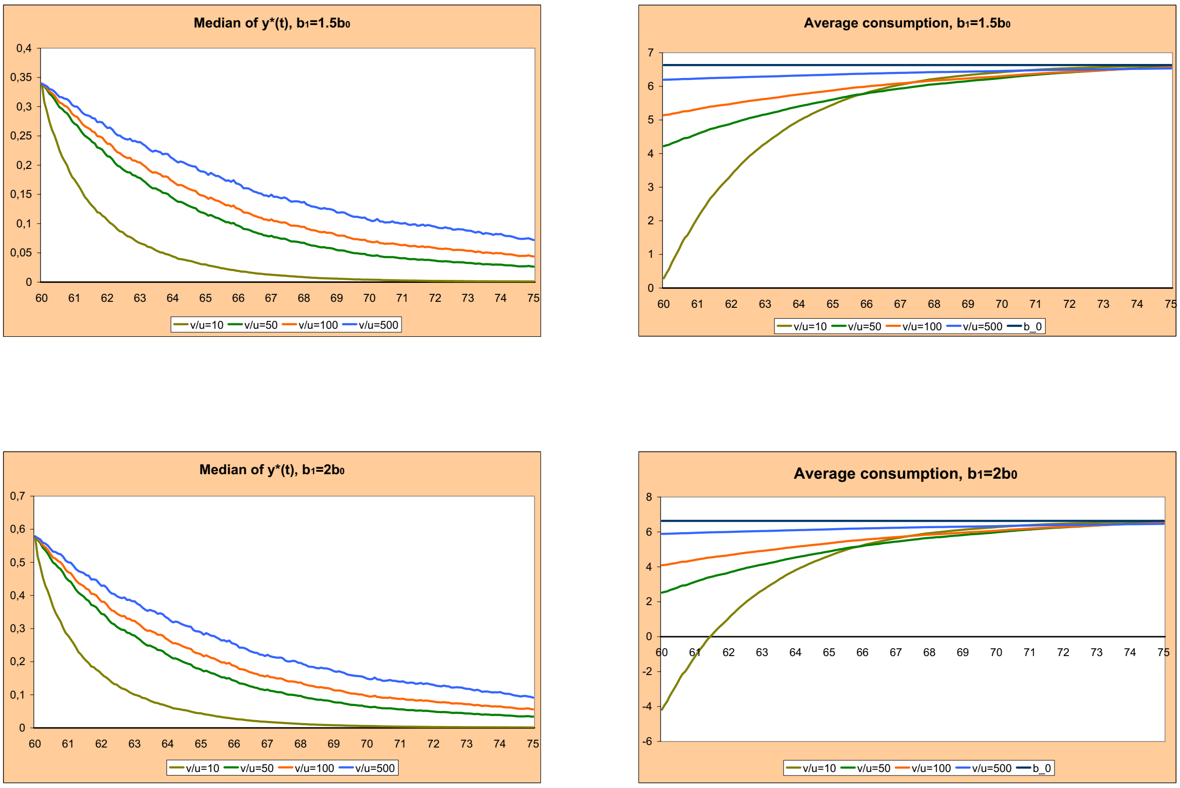


Figure 3. Median of the distribution of the optimal investment $y^*(t)$ and mean of the distribution of the optimal consumption $b^*(t)$ over time, when $\frac{v}{u}$ changes.

Increasing the weight given to the running consumption (ie the value of v) results in increasing the optimal consumption $b^*(t)$: when $\frac{v}{u} = 500$ the individual consumes very close to the ideal level b_0 (plotted in the graph to allow comparisons). This results also in riskier investment strategies, as the level of the (median of) $y^*(t)$ is higher when $\frac{v}{u}$ increases. However, with low values of $\frac{v}{u}$, the optimal policy would imply consuming small amounts of money at the beginning of the plan, or even consuming negatively (it turns out that this would happen only for a relatively short period of time after retirement, see later tables).

When increasing the final target from $b_1 = 1.5b_0$ to $b_1 = 2b_0$, the optimal investment allocation becomes riskier and the optimal consumption decreases. This behavior of the optimal controls is intuitive, in that the final target has increased whilst everything else has remained unchanged.

The choice of the weights is relevant to the optimal consumption. The graphs in Figure 4 show min, max and some percentiles of the optimal consumption over time with $\frac{v}{u} = 10$ and 100, with the two different final targets.

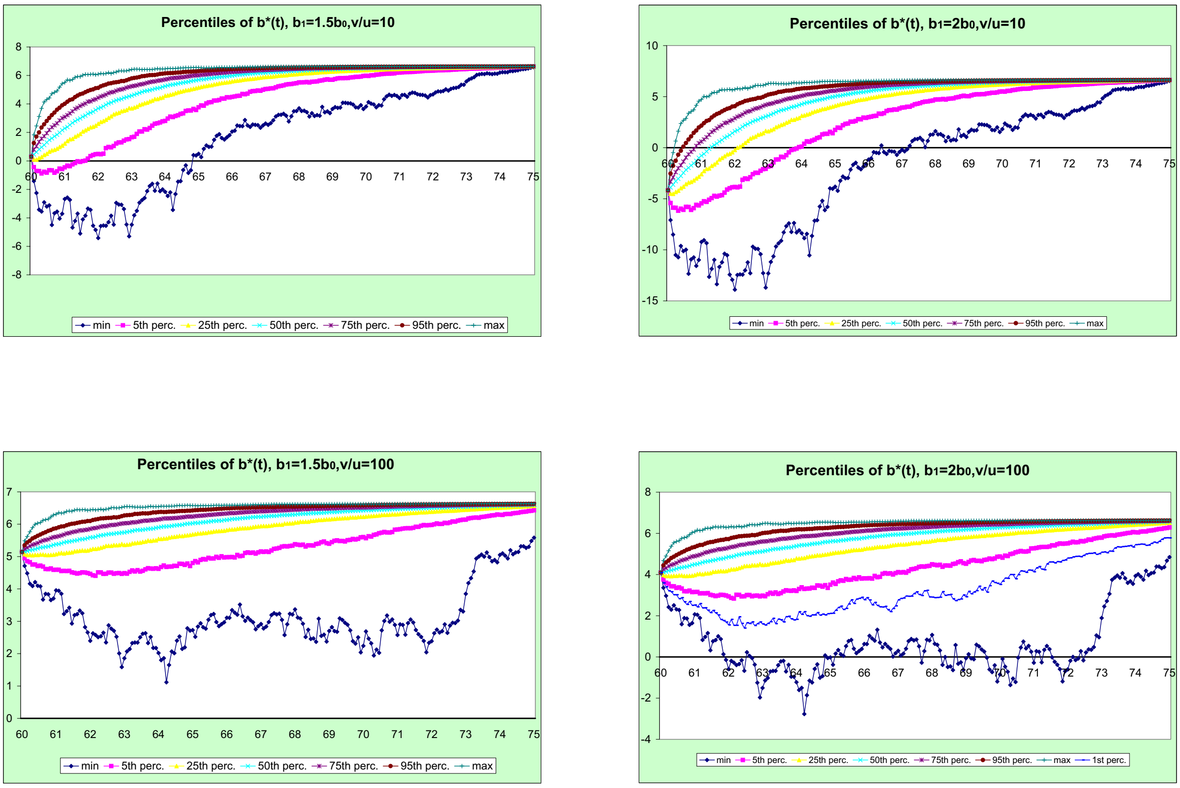


Figure 4. Percentiles of the distribution of the optimal consumption over time with $\frac{v}{u} = 10, 100$.

With $\frac{v}{u} = 10$, the initial level of optimal consumption is very close to 0 for $b_1 = 1.5b_0$ and negative for $b_1 = 2b_0$; furthermore the optimal consumption in 5% of the cases is negative for at least one and half year when $b_1 = 1.5b_0$ and for at least four years when $b_1 = 2b_0$. With $\frac{v}{u} = 100$, the optimal consumption in the simulations run is never negative for $b_1 = 1.5b_0$, and is negative in less than 1% of the cases for $b_1 = 2b_0$ (noting that in the last graph also the first percentile of the distribution of the consumption has been plotted).

5.3 Simulation results: probability of ruin, negative consumption, borrowing money from the bank and ability to purchase a better annuity than the initial one

Table 1 reports the frequency over the 1000 simulations of the undesirable events: ruin, negative consumption, borrowing money from the bank. It reports also the average age when the undesirable event occurs for the first time and the average number of weeks in which the undesirable event occurs (given that it has occurred). The mean and standard deviation of the final annuity are also reported. The ability to purchase a better annuity than the one that was possible to buy at retirement (ie b_0) has been also tested. Given that the fund cannot attain the amount needed to buy an annuity of b_1, the aim is to investigate how close to the target b_1 one can get. In particular, the growth of the fund has been monitored at any time between retirement and time T, to see if and when the fund allows the purchase of a certain level of annuity between b_0 and b_1. The four levels $b_{0.5} = b_0 + 0.5(b_1 - b_0)$, $b_{0.75} = b_0 + 0.75(b_1 - b_0)$, $b_{0.9} = b_0 + 0.9(b_1 - b_0)$ and $b_{0.95} = b_0 + 0.95(b_1 - b_0)$

have been tested. The price of the annuity used at any time has been chosen according to the age of the individual at that time, using the mortality table RG48 (males), assuming that the insurance company reviews the annuity price once a year, during the week of the pensioner's birthday. The same interest rate and loading factor adopted for the price of the annuity at age 60 and 75 have been applied.

	$b_1 = 1.5b_0 = 9.95$				$b_1 = 2b_0 = 13.26$			
	$\frac{v}{u} = 10$	$\frac{v}{u} = 50$	$\frac{v}{u} = 100$	$\frac{v}{u} = 500$	$\frac{v}{u} = 10$	$\frac{v}{u} = 50$	$\frac{v}{u} = 100$	$\frac{v}{u} = 500$
Ruin probability	0	0	0,2%	1,1%	0	0,1%	0,8%	2,8%
Mean age of ruin	-	-	70	72	-	72	69	69
Prob. (neg.cons.)	56,2%	0,2%	0	0	100%	9,7%	1,8%	0
Mean age of neg. cons.	60	63	-	-	60	62	64	-
Mean no. wks of neg. cons.	21	17	-	-	83	41	22	-
Prob. $(y^*(t) > 1)$	0	3,1%	7,6%	15,7%	3,5%	21%	28%	37,8%
Mean age of $(y^*(t) > 1)$	-	65	65	66	61	62	62	63
Mean no.wks $(y^*(t) > 1)$	-	56	71	108	16	54	88	134
Mean of final annuity	9,92	9,63	9,42	9,08	13,22	12,71	12,36	11,78
Standard deviation of final annuity	0,04	0,49	0,81	1,32	0,07	0,84	1,38	2,25
Prob(afford annuity of $b_{0.5}$)	100%	99,1%	96,7%	90,9%	100%	99,2%	97,1%	92,6%
Prob(afford annuity of $b_{0.75}$)	100%	92,7%	84,8%	71,8%	100%	94,3%	87%	75,9%
Prob(afford annuity of $b_{0.9}$)	99,6%	73,2%	54,8%	37,7%	99,8%	76,9%	60,4%	43,1%
Prob(afford annuity of $b_{0.95}$)	98,8%	48,9%	31,5%	17,7%	98,9%	54%	36,6%	21,4%
Mean age when afford ann.of $b_{0.5}$	65	67	68	69	66	68	69	69
Mean age when afford ann.of $b_{0.75}$	70	72	72	73	71	73	73	73
Mean age when afford ann.of $b_{0.9}$	73	74	74	75	74	75	75	75
Mean age when afford ann.of $b_{0.95}$	75	75	75	75	75	75	75	75

Table 1. Results of the simulations (without imposing restrictions).

The frequency of ruin is very low, ranging from 0 to around 1%, and reaching almost 3% only with a high target and a high value of $\frac{v}{u}$ (namely, 500, where consumption is very close to b_0). On average, ruin occurs after about 10 years after retirement (when it occurs).

With a low value of $\frac{v}{u}$ (ie 10) and a high target ($b_1 = 2b_0$), the optimal consumption is negative immediately after retirement, and on average remains negative for 1 and half year; with a a low target ($b_1 = 1.5b_0$) the consumption becomes negative soon after retirement in more than 50% of the cases, and remains negative on average for 5 months. After that initial period, the consumption turns positive and starts approaching the desired level b_0. On the other hand, the affordability of levels of annuity very close to the (unreachable) target before time T are almost guaranteed: the frequency of achievement of an annuity paying b_α for the rest of the life, with both values of the final target b_1, is about 99–100% for all values of α chosen: 0.5, 0.75, 0.9 and 0.95. It is clear that the initial sacrifice in terms of reduced consumption is compensated by a very high chance of getting as close as one wants to the desired target later on. This feature seems important, as there may be the opportunity for the pensioner to renounce to a few years of consumption at the beginning of the de-cumulation phase in order to be able to achieve the desired annuity almost with certainty.

With high enough $\frac{v}{u}$ (ie 100), the optimal consumption almost always remains positive. However, the frequency of being able to afford an annuity paying b_α before T is no longer close to 100% (apart from $\alpha = 0.5$) and sharply decreases when α increases, reducing to 55–60% for $\alpha = 0.9$ and dropping to values as low as about 30–35% with $\alpha = 0.95$.

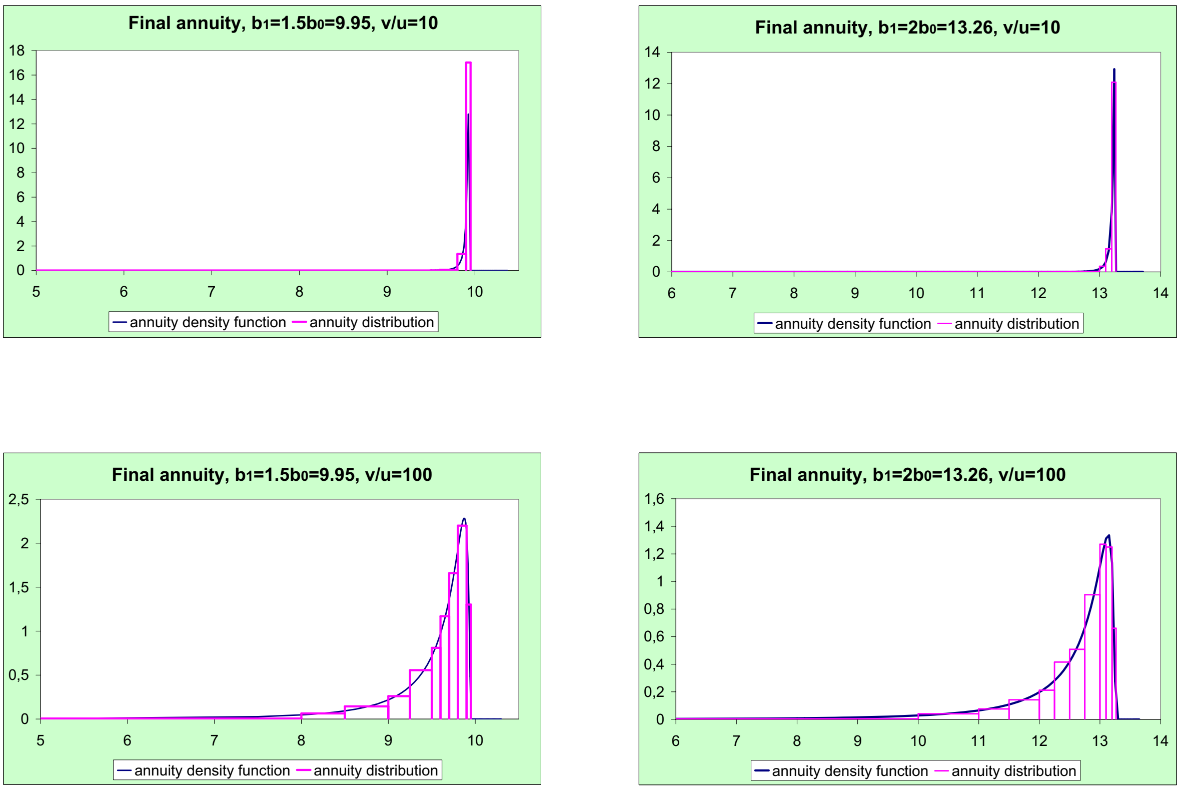
With $\frac{v}{u} = 500$, the probability of reaching the level $b_{0.95}$ before time T goes down to about 20% with both targets. The price that one has to pay for a stable consumption path very close to the desired level b_0 is a lower chance of being able to approach the final annuity target during the drawdown phase, or (which is equivalent) accepting a lower level of lifetime annuity at the time of annuitization. Quite interestingly, we notice that the chances of getting very close to the target ($\alpha = 0.9$ and 0.95) are slightly higher with $b_1 = 2b_0$ than with $b_1 = 1.5b_0$: this seems to suggest that the higher the target, the higher the reward in terms of chances of approaching it. The same feature was observed in Gerrard et al. (2004a).

The mean of the final annuity decreases and the standard deviation increases when $\frac{v}{u}$ increases, which is intuitive. Furthermore, the restriction on the investment allocation is violated more often when $\frac{v}{u}$ is raised, with increases in both the probability of borrowing money from the bank to invest in the risky asset and in the number of weeks during which this happens.

5.4 Simulation results: density function and empirical distribution of the final annuity

The density function of the final annuity $kX(T)$ that can be bought at the time of compulsory annuitization can be exactly calculated. In Figure 5 the density function is plotted together with the histogram of the empirical distribution of the final annuity from the 1000 simulations, for the two targets and for $\frac{v}{u} = 10, 100$ and 500.

It has to be said that the distribution depends heavily on the choice of the Sharpe ratio, $\frac{\lambda - r}{\sigma}$, of the risky asset (for a detailed explanation of the dependence of results on the Sharpe ratio, and for a sensitivity analysis, see Gerrard et al. (2004a)). The results here reported are relative to a Sharpe ratio of 0.3; if this value were increased (reduced), the distribution would in consequence become more (less) concentrated to the left of the target b_1.



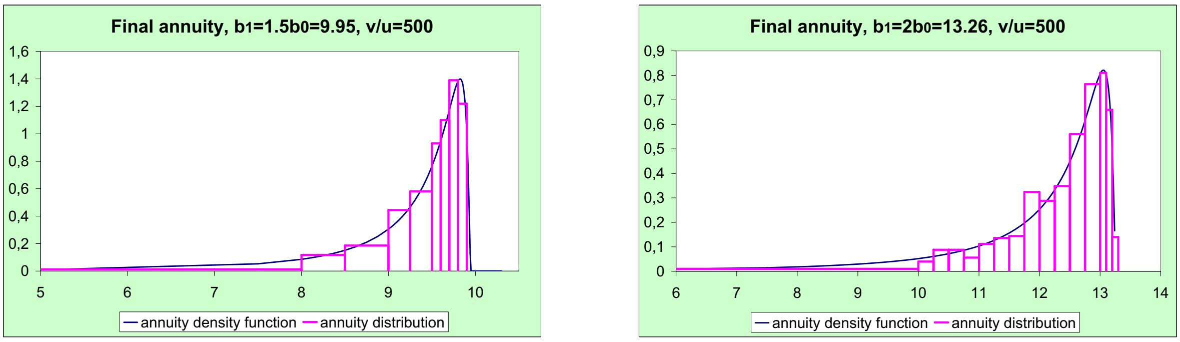


Figure 5. Density function and empirical distribution of the final annuity.

The graphs confirm the results found in the table: the higher is the weight given to the running consumption (value of $\frac{v}{u}$), the more spread out the distribution of the final annuity.

5.5 Imposing restrictions on the optimal choices: sub-optimal policies

As mentioned in section 3, the problem with constrained controls has not been solved, due to the difficulty of the task. However, it is possible to act in such a way as to avoid unacceptable situations. In fact, the pensioner can set the consumption equal to 0 whenever the optimal policy would imply negative consumption, and can invest the whole portfolio in the risky asset whenever the optimal policy would imply borrowing money from the bank to invest in the risky asset. Furthermore, the process should be stopped if and when the fund hits the barrier 0, ie when ruin occurs. The choices just described would not be optimal in the classic sense, in that they would not be the exact solution to the optimal control problem with constraints. However, the difference from the true optimal solution is likely to be small; equally importantly, these restrictions are quite easy to implement.

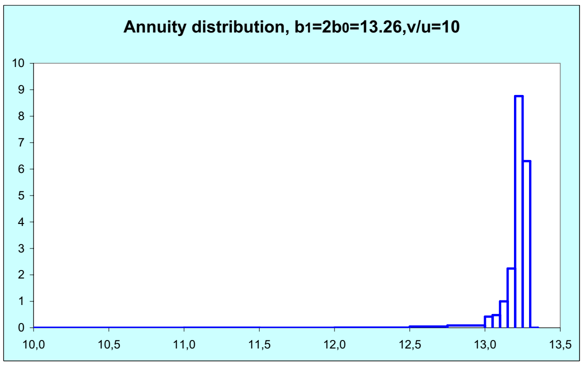
We have implemented these sub-optimal policies with constraints in a small number of cases, and investigated the difference from the results of the unrestricted problem. The scenarios chosen are those where imposing such restrictions is likely to have a significant effect on the investment/consumption choices. Therefore, we have selected $b_1 = 2b_0$, $\frac{v}{u} = 10$ (which is the case where negative consumption is most likely to appear), $b_1 = 2b_0$, $\frac{v}{u} = 100$ (which is the case where borrowing money from the bank is most likely to appear) and $b_1 = 2b_0$, $\frac{v}{u} = 500$ (which is the case where ruin is most likely to appear). Table 2 reports the results. For consistent comparisons, the stream of pseudo-random numbers generated is the same as the one used in the previous case.

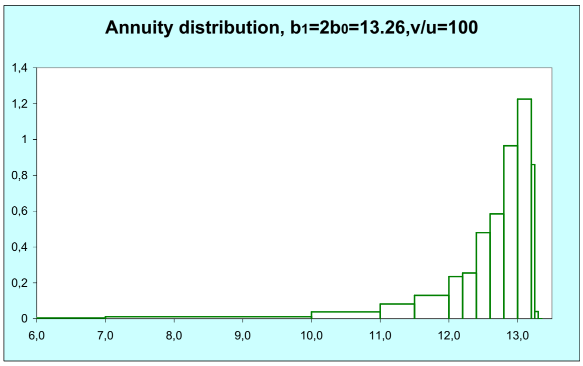
	$b_1 = 2b_0 = 13.26$		
	$\frac{v}{u} = 10$	$\frac{v}{u} = 100$	$\frac{v}{u} = 500$
Ruin probability	0	0	0,4%
Mean age of ruin	-	-	73
Prob. (neg.cons.)	0	0	0
Prob. $(y^*(t) > 1)$	0	0	0
Mean of final annuity	13,19	12,24	11,32
Standard deviation of final annuity	0,29	1,62	2,88
Prob(afford annuity of $b_{0.5}$)	99,9%	95,5%	87,4%
Prob(afford annuity of $b_{0.75}$)	99,8%	85,6%	72,1%
Prob(afford annuity of $b_{0.9}$)	98,9%	59,7%	42,2%
Prob(afford annuity of $b_{0.95}$)	97,4%	36,5%	21%
Mean age when afford annuity of $b_{0.5}$	67	69	69
Mean age when afford annuity of $b_{0.75}$	71	73	73
Mean age when afford annuity of $b_{0.9}$	74	75	75
Mean age when afford annuity of $b_{0.95}$	75	75	75

Table 2. Results of the simulations (imposing restrictions).

Imposing restrictions on the controls does not seem to have a significant effect on the results. Clearly, there are no unreasonable situations any longer (negative consumption and borrowing money from the bank). The frequency of ruin decreases in comparison with the unrestricted case, probably because the high values of $y^*(t)$ are truncated to 1. The mean and standard deviation of the final annuity are similar to the case without restrictions (slightly worse, with the former decreasing and the latter increasing). The chances of getting close to the desired annuity level slightly decrease in all cases (but only by 1—2 percentage points, apart from one case, where it decreases by 5%).

The distribution of the final annuity over the 1000 simulations is plotted in the graphs of Figure 6. The histograms are very similar to the corresponding histograms of the unrestricted case, and could not be distinguished if they were to be plotted on the same graph.





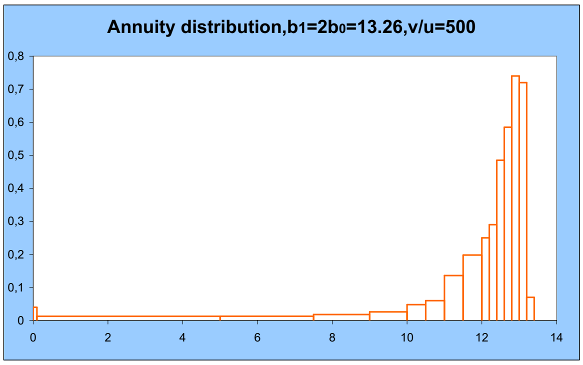
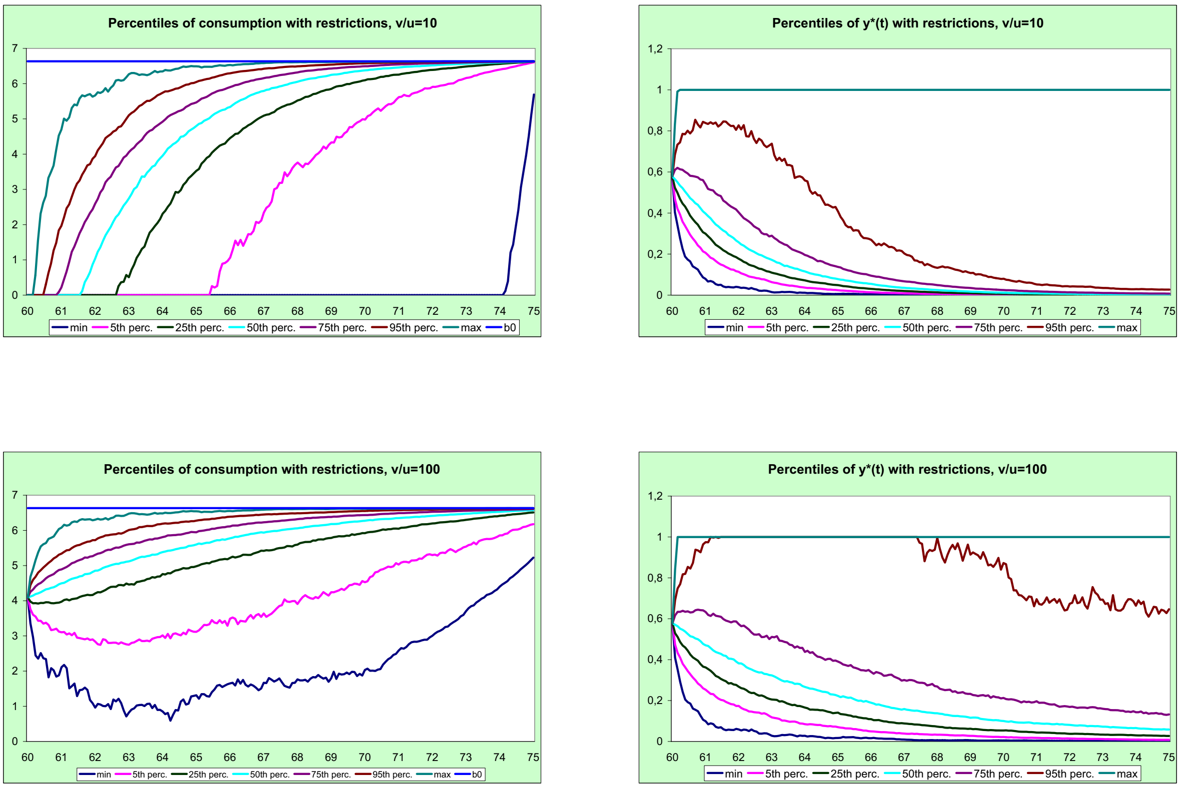


Figure 6. Distribution of the final annuity when imposing restrictions on the controls via sub-optimal policies.

It may be of interest to observe the behaviour of the sub-optimal controls when the restrictions are applied. The graphs in Figure 7 report the percentiles of the consumption and of the investment allocation in the cases analyzed.



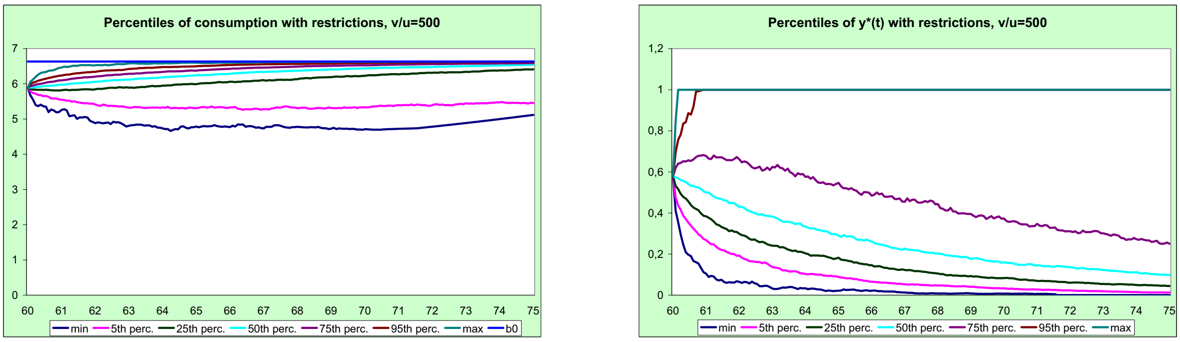


Figure 7. Sub-optimal controls when the restrictions are applied.

As expected, the paths of the percentiles of the controls are very stable over time and are more stable than the corresponding percentiles of the unrestricted case. For example, from an inspection of the case $\frac{v}{u} = 100$, we can see that in the unrestricted case the curve of the minimum consumption becomes negative many times between ages 62 and 72 (see Figure 4), whereas it lies strictly above 0 at all ages in the restricted case. This apparent contradiction can be explained by a thorough investigation of the single trajectories in the scenarios where the optimal consumption becomes negative: in the unrestricted case the negative consumption at certain points in time is due to very low values of the fund at that time, while in the restricted case the fund takes higher values, due to the fact that the investment allocation is restricted, and this leads to positive consumption. A further comparison of the single trajectories of the fund in the unrestricted and restricted case (in the same scenarios for market returns) seems to show that the path of the fund is more stable when restrictions are applied, and this, again, is to be explained by the fact that the amount invested in the risky asset cannot exceed the whole portfolio.

6 Conclusions

In this work, we propose a flexible tool that could help the member of a defined contribution pension scheme in making his/her decisions in the post retirement phase before annuitization. In particular, he/she can set a desired level of annuity to be bought when ultimate annuitization occurs, and invest and consume in the meantime, according to this target. Mortality has been included in the model, in that the pensioner runs the optimization problem until annuitization or death, whichever occurs first. Furthermore, the individual can give due importance to the ability to leave a bequest in case of death before annuitization. The problem has been tackled and solved with the techniques of stochastic optimal control theory, in a typical Black and Scholes financial market, with a riskless and a risky asset. We have solved the problem also in the case that consumption is fixed, and the only choice available to the individual is the investment allocation. The solution is found with a particular definition for the target function, which is called the “natural target”, in that it acts as a sort of safety level for the needs of the pensioner and takes into account his/her bequest motive and his/her (subjective) force of mortality. With a constant force of mortality, the optimal controls are given in closed form; with a more realistic age-dependent force of mortality, we show that the solution exists and can be well approximated by solving the problem with a constant lower level and a constant higher level for the force of mortality. An unexpected and surprising result is that by choosing the natural target, that is linked to the bequest motive, the individual acts optimally as though his/her bequest motive was null.

In the model presented, the optimal running consumption turns out to be bounded above at a certain level set by the pensioner (which may be useful, in case there are some restrictions on the amount withdrawn periodically from the fund). Similarly, the annuity target chosen can never be reached, but the density function of the final annuity that can be purchased shows that it can be approached very closely.

A sensitivity analysis with respect to the level of the final annuity target and the relative importance given to the level of running consumption and the achievement of the final target has been carried out, by running Monte Carlo simulations for the risky asset. The simulations allow the investigation of relevant issues like the probability of ruin, the occurrence of undesirable or unrealistic events, the probability of being able to buy a better annuity than the one purchasable at retirement and the distribution of the final annuity that can be bought at the time of compulsory annuitization. By undesirable events we mean negative or too low optimal consumption and optimal investment allocation that implies borrowing money from the bank in order to invest in the risky asset.

We find that the key characteristics strongly depend on the relative weights given to the level of running consumption during the drawdown phase and to the achievement of the natural target. It can be seen that by giving high enough importance to the running consumption level, the optimal consumption lies very close to the upper level throughout the drawdown phase. The price that the member has to pay is a less close approach to the natural interim and final targets, and a less favourable distribution of the final annuity. On the other hand, if more importance is attached to the achievement of the natural target, the fund approaches the interim and final targets very closely (a result confirmed by the density function of the final annuity), but the optimal consumption is very likely to be negative at the beginning of the drawdown phase and for a short period of time (up to 1–2 years) thereafter, approaching the desired level only later on. Therefore, the main conclusion seems to be that the trade-off between the different desires of the pensioner regarding consumption and final annuity target can be easily dealt with by choosing appropriate weights for these factors in the initial setting of the optimization problem.

The problem has been solved without restrictions on the optimal investment and consumption choices, due to the difficulty inherent in solving the optimal control problem with constraints. However, we have implemented sub-optimal policies by restricting the controls to reasonable boundaries in the simulations and we have compared the results between the unrestricted and the restricted case. Intuitively, the controls applied turn out to be more stable in the restricted case. Also the evolution of the fund turns out to be more stable, and the probability of ruin decreases when applying restrictions, and this is mainly due to the boundaries imposed on the investment allocation, that prevent unreasonable values. Furthermore, the distribution of the final annuity seems to be very similar to the one obtained in the unrestricted case. On the other hand, the probability of being able to buy certain levels of annuity before the time of ultimate annuitization decreases for all levels when passing from the unrestricted to the restricted case, but only slightly.

In further research, it would be of interest to solve the problem with constraints numerically, and make the comparison with the restricted sub-optimal policies applied in this work. Another interesting task would be to investigate the optimal time of annuitization between retirement and the compulsory annuitization age; this is an optimal stopping problem.

Appendix A

Before proving Proposition 1, we need to prove the Lemma 2:

Proof of Lemma 2. If there is any t such that $A(t) = A_0(t)$ then $A'(t) - A_0'(t) = (\phi(t) - \phi_0)A_0(t)$, and we know that $A_0(t) > 0$. If $\phi(t) - \phi_0$ is always positive, then the only such occurrences involve A crossing A_0 from below, whereas if $\phi(t) - \phi_0$ is negative, then A crosses A_0 from above. In both cases, therefore, there can never be more than one such crossing. But such a crossing occurs at $t = T$, and so that is the only one.

Proof of Proposition 1. Let $\phi_U = \max_{t_0 \le t \le T} \phi(t)$, $\phi_L = \min_{t_0 \le t \le T} \phi(t)$, and let $A_U(t)$, $A_L(t)$ be respectively the solutions to (2.15) in the case where $\phi \equiv \phi_U$ and the case where $\phi \equiv \phi_L$. Since ϕ is constant in both cases we may deduce that $A_L(t), A_U(t) > 0$ for all t. As a result of the Lemma, $A(t)$ remains sandwiched between the two solutions $A_L(t)$ and $A_U(t)$, and therefore is bounded away from 0.

References

Albrecht, P. and Maurer, R. (2002). Self-annuitization, consumption shortfall in retirement and asset allocation: The annuity benchmark, *Journal of Pension Economics and Finance* **1**: 269–288.

Arts, B. and Vigna, E. (2003). A switch criterion for defined contribution pension schemes, *Proceedings of the 13th AFIR Colloquium, volume 1*, pp. 261–290.

Blake, D., Cairns, A. J. G. and Dowd, K. (2001). Pensionmetrics: Stochastic pension plan design and value-at-risk during the accumulation phase, *Insurance: Mathematics and Economics* **29**: 187–215.

Blake, D., Cairns, A. J. G. and Dowd, K. (2003). Pensionmetrics 2: Stochastic pension plan design during the distribution phase, *Insurance: Mathematics and Economics* **33**: 29–47.

Booth, P. and Yakoubov, Y. (2000). Investment policy for defined contribution pension scheme members close to retirement: an analysis of the 'lifestyle' concept, *North American Actuarial Journal* **4**: 1–19.

Boulier, J.-F., Huang, S. and Taillard, G. (2001). Optimal management under stochastic interest rates: the case of a protected defined contribution pension fund, *Insurance: Mathematics and Economics* **28**: 173–189.

Boulier, J.-F., Michel, S. and Wisnia, V. (1996). Optimizing investment and contribution policies of a defined benefit pension fund, *Proceedings of the 6th AFIR Colloquium*, Vol. 1, pp. 593–607.

Boulier, J.-F., Trussant, E. and Florens, D. (1995). A dynamic model for pension funds management, *Proceedings of the 5th AFIR Colloquium, volume 1*, pp. 351–384.

Browne, S. (1995). Optimal investment policies for a firm with a random risk process: Exponential utility and minimizing the probability of ruin, *Mathematics of Operations Research* **20**: 937–958.

Browne, S. (1997). Survival and growth with a liability: Optimal portfolio strategies in continuous time, *Mathematics of Operations Research* **22**: 468–493.

Cairns, A. J. G. (2000). Some notes on the dynamics and optimal control of stochastic pension fund models in continuous time, *ASTIN Bulletin* **30**: 19–55.

Chiarolla, M. B., Longo, M. and Stabile, G. (2004). Pension planning under transactions costs, *Proceedings of the 8th International Congress on Insurance: Mathematics and Economics, Rome*.

Di Giacinto, M. and Gozzi, F. (2004). A dynamic allocation strategy for pension funds with a minimum guarantee, *Proceedings of the 8th International Congress on Insurance: Mathematics and Economics, Rome*.

Gerrard, R., Haberman, S. and Vigna, E. (2004a). Optimal investment choices post retirement in a defined contribution pension scheme, *to appear in Insurance: Mathematics and Economics*.

Gerrard, R., Haberman, S., Højgaard, B. and Vigna, E. (2004b). The income drawdown: quadratic loss, *Actuarial Research Paper No. 155, Cass Business School, London*.

Haberman, S. and Sung, J.-H. (1994). Dynamic approaches to pension funding, *Insurance: Mathematics and Economics*, Vol. 15, pp. 151–162.

Haberman, S. and Vigna, E. (2002). Optimal investment strategies and risk measures in defined contribution pension schemes, *Insurance: Mathematics and Economics* **31**: 35–69.

Kapur, S. and Orszag, J. M. (1999). A portfolio approach to investment and annuitization during retirement, *Proceedings of the Third International Congress on Insurance: Mathematics and Economics, London*.

Khorasanee, M. Z. (1996). Annuity choices for pensioners, *Journal of Actuarial Practice* **4**: 229–255.

Khorasanee, M. Z. (1998). Deterministic modelling of defined contribution pension funds, *North American Actuarial Journal* **1**: 83–103.

Knox, D. M. (1993). A critique of defined contribution plans using a simulation approach, *Journal of Actuarial Practice* **1**: 49–66.

Merton, R. (1971). Optimum consumption and portfolio rules in a continuous time model, *Journal of Economic Theory* **3**: 373–413.

Milevsky, M. (2001). Optimal annuitization policies: Analysis of the options, *North American Actuarial Journal* **5**: 57–69.

Milevsky, M. A. and Young, V. R. (2002). Optimal asset allocation and the real option to delay annuitization: It's not now-or-never, working paper.

Milevsky, M. A. and Young, V. R. (2004). Annuitization and asset allocation, *Proceedings of the 8th International Congress on Insurance: Mathematics and Economics, Rome*.

Milevsky, M. A., Moore, K. S. and Young, V. R. (2004). Optimal asset allocation and ruin-minimization annuitization strategies, *Proceedings of the 8th International Congress on Insurance: Mathematics and Economics, Rome*.

Øksendal, B. (1998). *Stochastic Differential Equations*, Springer Verlag, Berlin.

Owadally, M. I. and Haberman, S. (2004). Efficient gain and loss amortization and optimal funding in pension plans, **8**: 21–36.

Stabile, G. (2003). Optimal timing of annuity purchases: a combined stochastic control and optimal stopping problem, *working paper*. Universita‘ degli Studi di Roma ’La Sapienza’.

Steffensen, M. (2004). On merton’s problem for life insurers, *ASTIN Bulletin* **34**: 5–26.

Yong, J. and Zhou, X. Y. (1999). *Stochastic Controls: Hamiltonian Systems and HJB Equations*, Springer Verlag, New York.

CITY Cass Business School
City of London

FACULTY OF ACTUARIAL SCIENCE AND STATISTICS

Actuarial Research Papers since 2001

---

135. Renshaw A. E. and Haberman S. On the Forecasting of Mortality Reduction Factors. February 2001.

ISBN 1 901615 56 1

136. Haberman S., Butt Z. & Rickayzen B. D. Multiple State Models, Simulation and Insurer Insolvency. February 2001. 27 pages.

ISBN 1 901615 57 X

137. Khorasanee M.Z. A Cash-Flow Approach to Pension Funding. September 2001. 34 pages.

ISBN 1 901615 58 8

138. England P.D. Addendum to "Analytic and Bootstrap Estimates of Prediction Errors in Claims Reserving". November 2001. 17 pages.

ISBN 1 901615 59 6

139. Verrall R.J. A Bayesian Generalised Linear Model for the Bornhuetter-Ferguson Method of Claims Reserving. November 2001. 10 pages.

ISBN 1 901615 62 6

140. Renshaw A.E. and Haberman. S. Lee-Carter Mortality Forecasting, a Parallel GLM Approach, England and Wales Mortality Projections. January 2002. 38 pages.

ISBN 1 901615 63 4

141. Ballotta L. and Haberman S. Valuation of Guaranteed Annuity Conversion Options. January 2002. 25 pages.

ISBN 1 901615 64 2

142. Butt Z. and Haberman S. Application of Frailty-Based Mortality Models to Insurance Data. April 2002. 65 pages.

ISBN 1 901615 65 0

143. Gerrard R.J. and Glass C.A. Optimal Premium Pricing in Motor Insurance: A Discrete Approximation. (Will be available 2003).

144. Mayhew, L. The Neighbourhood Health Economy. A systematic approach to the examination of health and social risks at neighbourhood level. December 2002. 43 pages.

ISBN 1 901615 66 9

145. Ballotta L. and Haberman S. The Fair Valuation Problem of Guaranteed Annuity Options: The Stochastic Mortality Environment Case. January 2003. 25 pages.

ISBN 1 901615 67 7

146. Haberman S., Ballotta L. and Wang N. Modelling and Valuation of Guarantees in With-Profit and Unitised With-Profit Life Insurance Contracts. February 2003. 26 pages.

ISBN 1 901615 68 5

147. Ignatov Z.G., Kaishev V.K and Krachunov R.S. Optimal Retention Levels, Given the Joint Survival of Cedent and Reinsurer. March 2003. 36 pages.

ISBN 1 901615 69 3

148. Owadally M.I. Efficient Asset Valuation Methods for Pension Plans. March 2003. 20 pages.

ISBN 1 901615 70 7

149. Owadally M.I. Pension Funding and the Actuarial Assumption Concerning Investment Returns. March 2003. 32 pages.
ISBN 1 901615 71 5

150. Dimitrova D, Ignatov Z. and Kaishev V. Finite time Ruin Probabilities for Continuous Claims Severities. Will be available in August 2004.

151. Iyer S. Application of Stochastic Methods in the Valuation of Social Security Pension Schemes. August 2004. 40 pages.
ISBN 1 901615 72 3

152. Ballotta L., Haberman S. and Wang N. Guarantees in with-profit and Unitized with profit Life Insurance Contracts; Fair Valuation Problem in Presence of the Default Option[1]. October 2003. 28 pages.
ISBN 1-901615-73-1

153. Renshaw A. and Haberman. S. Lee-Carter Mortality Forecasting Incorporating Bivariate Time Series. December 2003. 33 pages.
ISBN 1-901615-75-8

154. Cowell R.G., Khuen Y.Y. and Verrall R.J. Modelling Operational Risk with Bayesian Networks. March 2004. 37 pages.
ISBN 1-901615-76-6

155. Gerrard R.G., Haberman S., Hojgaard B. and Vigna E. The Income Drawdown Option: Quadratic Loss. March 2004. 31 pages.
ISBN 1-901615-77-4

156. Karlsson, M., Mayhew L., Plumb R, and Rickayzen B.D**.** An International Comparison of Long-Term Care Arrangements. An Investigation into the Equity, Efficiency and sustainability of the Long-Term Care Systems in Germany, Japan, Sweden, the United Kingdom and the United States. April 2004. 131 pages.
ISBN 1 901615 78 2

157. Ballotta Laura. Alternative Framework for the Fair Valuation of Participating Life Insurance Contracts. June 2004. 33 pages.
ISBN 1-901615-79-0

158. Wang Nan. An Asset Allocation Strategy for a Risk Reserve considering both Risk and Profit. July 2004. 13 pages.
ISBN 1 901615-80-4

159. Spreeuw Jaap. Upper and Lower Bounds of Present Value Distributions of Life Insurance Contracts with Disability Related Benefits. December 2004. 35 pages.
ISBN 1 901615-83-9

160. Renshaw Arthur and Haberman Steven. Mortality Reduction Factors Incorporating Cohort Effects. January 2005. 25 pages.
ISBN 1 901615- 84- 7

161. Gerrard R G, Haberman S, Vigna E. The Management of De-cumulation Risks in a Defined Contribution Environment. February 2005. 28 pages.
ISBN 1 901615-85-5

Statistical Research Papers

1. Sebastiani P. Some Results on the Derivatives of Matrix Functions. December 1995. 17 Pages.
ISBN 1 874 770 83 2

2. Dawid A.P. and Sebastiani P. Coherent Criteria for Optimal Experimental Design. March 1996. 35 Pages.
ISBN 1 874 770 86 7

3. Sebastiani P. and Wynn H.P. Maximum Entropy Sampling and Optimal Bayesian Experimental Design. March 1996. 22 Pages.
ISBN 1 874 770 87 5

4. Sebastiani P. and Settimi R. A Note on D-optimal Designs for a Logistic Regression Model. May 1996. 12 Pages.
ISBN 1 874 770 92 1

5. Sebastiani P. and Settimi R. First-order Optimal Designs for Non Linear Models. August 1996. 28 Pages.
ISBN 1 874 770 95 6

6. Newby M. A Business Process Approach to Maintenance: Measurement, Decision and Control. September 1996. 12 Pages.
ISBN 1 874 770 96 4

7. Newby M. Moments and Generating Functions for the Absorption Distribution and its Negative Binomial Analogue. September 1996. 16 Pages.
ISBN 1 874 770 97 2

8. Cowell R.G. Mixture Reduction via Predictive Scores. November 1996. 17 Pages.
ISBN 1 874 770 98 0

9. Sebastiani P. and Ramoni M. Robust Parameter Learning in Bayesian Networks with Missing Data. March 1997. 9 Pages.
ISBN 1 901615 00 6

10. Newby M.J. and Coolen F.P.A. Guidelines for Corrective Replacement Based on Low Stochastic Structure Assumptions. March 1997. 9 Pages.
ISBN 1 901615 01 4.

11. Newby M.J. Approximations for the Absorption Distribution and its Negative Binomial Analogue. March 1997. 6 Pages.
ISBN 1 901615 02 2

12. Ramoni M. and Sebastiani P. The Use of Exogenous Knowledge to Learn Bayesian Networks from Incomplete Databases. June 1997. 11 Pages.
ISBN 1 901615 10 3

13. Ramoni M. and Sebastiani P. Learning Bayesian Networks from Incomplete Databases. June 1997. 14 Pages.
ISBN 1 901615 11 1

14. Sebastiani P. and Wynn H.P. Risk Based Optimal Designs. June 1997. 10 Pages.
ISBN 1 901615 13 8

15. Cowell R. Sampling without Replacement in Junction Trees. June 1997. 10 Pages.
ISBN 1 901615 14 6

16. Dagg R.A. and Newby M.J. Optimal Overhaul Intervals with Imperfect Inspection and Repair. July 1997. 11 Pages. ISBN 1 901615 15 4

17. Sebastiani P. and Wynn H.P. Bayesian Experimental Design and Shannon Information. October 1997. 11 Pages. ISBN 1 901615 17 0

18. Wolstenholme L.C. A Characterisation of Phase Type Distributions. November 1997. 11 Pages. ISBN 1 901615 18 9

19. Wolstenholme L.C. A Comparison of Models for Probability of Detection (POD) Curves. December 1997. 23 Pages. ISBN 1 901615 21 9

20. Cowell R.G. Parameter Learning from Incomplete Data Using Maximum Entropy I: Principles. February 1999. 19 Pages. ISBN 1 901615 37 5

21. Cowell R.G. Parameter Learning from Incomplete Data Using Maximum Entropy II: Application to Bayesian Networks. November 1999. 12 Pages ISBN 1 901615 40 5

22. Cowell R.G. FINEX : Forensic Identification by Network Expert Systems. March 2001. 10 pages.

ISBN 1 901615 60X

23. Cowell R.G. When Learning Bayesian Networks from Data, using Conditional Independence Tests is Equivalant to a Scoring Metric. March 2001. 11 pages.

ISBN 1 901615 61 8

24. Kaishev, V.K., Dimitrova, D.S., Haberman S., and Verrall R.J. Autumatic, Computer Aided Geometric Design of Free-Knot, Regression Splines. August 2004. 37 pages.

ISBN 1-901615-81-2

25. Cowell R.G., Lauritzen S.L., and Mortera, J. Identification and Separation of DNA Mixtures Using Peak Area Information. December 2004. 39 pages.

ISBN 1-901615-82-0

Faculty of Actuarial Science and Statistics

Actuarial Research Club

The support of the corporate members

CGNU Assurance
Computer Sciences Corporation
English Matthews Brockman
Government Actuary's Department
Watson Wyatt Partners

is gratefully acknowledged.

ISBN 1-901615-85-5