Supplementary Material to "On Double-Boundary Non-Crossing Probability for a Class of Compound Processes with Applications"

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A. Proofs of the results from Section 4.1, Inventory Management Optimization

In order to prove Proposition 4.1, we will need the following Lemmas.

Lemma A.1. Suppose $\tau[0,t]$ is a (compound) process specified by Equation (1) of the paper and $g(t)$ is a given lower boundary. For a fixed $r > 1$ and total number of units W, let $h(t) \in \mathcal{H}_r$ and $h^{'}(t) \in \mathcal{H}_r$ have jumps at the instants $t_1, ..., t_r$, with jump sizes, correspondingly W_1, ..., W_{i-1}, W_i, W_{i+1}, ..., W_{j-1}, W_j, W_{j+1}, ..., W_r and W_1, ..., W_{i-1}, $W_i^{'}$, W_{i+1}, ..., W_{j-1}, $W_j^{'}$, W_{j+1}, ..., W_r, $i = 1, ..., r-1$, $j = 2, ..., r$, such that $W_1 + ... + W_i + ... + W_j + ... + W_r = W$ and $W_1 + ... + W_i^{'} + ... + W_j^{'} + ... + W_r = W$.

(i). If $W_i \geq W_i^{'}$ and $W_j \leq W_j^{'}$, then

$$P(g(t) \leq \tau[0,t] \leq h(t),\ 0 \leq t \leq z) \geq P(g(t) \leq \tau[0,t] \leq h^{'}(t),\ 0 \leq t \leq z). \quad \text{(A.1)}$$

(ii). If $W_i \leq W_i^{'}$ and $W_j \geq W_j^{'}$, then

$$P(g(t) \leq \tau[0,t] \leq h(t),\ 0 \leq t \leq z) \leq P(g(t) \leq \tau[0,t] \leq h^{'}(t),\ 0 \leq t \leq z). \quad \text{(A.2)}$$

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Proof. Proof of Lemma A.1: Let us introduce the function

$$
\begin{aligned}
&S(W_1, ..., W_r) \\
&= W_1(t_2 - t_1) + (W_1 + W_2)(t_3 - t_2) + ... + (W_1 + ... + W_{r-1})(t_r - t_{r-1}) \\
&+ (W_1 + ... + W_r)(z - t_r) \\
&= zW - W_2 t_2 - ... - W_r t_r,
\end{aligned}
\tag{A.3}
$$

which defines the area under the function $h(t)$. Clearly, if $W_i \geq W_i^{'}$ and $W_j \leq W_j^{'}$, then

$$S(W_1, ..., W_i^{'}, ..., W_j^{'}, ..., W_r) \leq S(W_1, ..., W_i, ..., W_j, ..., W_r)$$

from which (A.1) follows directly, since the set of non-crossing trajectories of the process $\tau[0, t]$ increases as the area function $S(\cdot)$ increases. Similarly, (A.2) holds since

$$S(W_1, ..., W_i^{'}, ..., W_j^{'}, ..., W_r) \geq S(W_1, ..., W_i, ..., W_j, ..., W_r),$$

if $W_i \leq W_i^{'}$ and $W_j \geq W_j^{'}$. □

Lemma A.2. Suppose $\tau[0, t]$ is a (compound) process specified by Equation (1) of the paper and $g(t)$ is a given lower boundary. For a fixed $r > 1$ and total number of units W, let $h(t) \in \mathcal{H}_r$ and $h^{'}(t) \in \mathcal{H}_r$ have jumps of sizes W_1, ..., W_r (with $W = W_1 + ... + W_r$) at the instants, correspondingly t_1, ..., t_{i-1}, t_i, t_{i+1}, ..., t_r and t_1, ..., t_{i-1}, $t_i^{'}$, t_{i+1}, ..., t_r, $i = 1, ..., r$, such that $0 = t_1 < ... < t_{i-1} < t_i < t_{i+1} < ... < t_r$ and $0 = t_1 < ... < t_{i-1} < t_i^{'} < t_{i+1} < ... < t_r$.

(i). If $t_i \leq t_i^{'}$, then

$$P(g(t) \leq \tau[0, t] \leq h(t),\ 0 \leq t \leq z) \geq P(g(t) \leq \tau[0, t] \leq h^{'}(t),\ 0 \leq t \leq z). \tag{A.4}$$

(ii). If $t_i \geq t_i^{'}$, then

$$P(g(t) \leq \tau[0, t] \leq h(t),\ 0 \leq t \leq z) \leq P(g(t) \leq \tau[0, t] \leq h^{'}(t),\ 0 \leq t \leq z). \tag{A.5}$$

Proof. Proof of Lemma A.2: Similarly as in (A.3), we define the function

$$
\begin{aligned}
&S(t_1, ..., t_r) \\
&= W_1(t_2 - t_1) + (W_1 + W_2)(t_3 - t_2) + ... + (W_1 + ... + W_{r-1})(t_r - t_{r-1}) \\
&+ (W_1 + ... + W_r)(z - t_r) \\
&= zW - W_2 t_2 - ... - W_r t_r,
\end{aligned}
$$

which gives the area under the function $h(t)$. Clearly, if $t_i \leq t_i^{'}$, then

$$S(t_1, ..., t_i^{'}, ..., t_r) \leq S(t_1, ..., t_i, ..., t_r)$$

from which (A.4) follows directly, since the set of non-crossing trajectories of the process $\tau[0, t]$ decreases as the area function $S(\cdot)$ decreases. Similarly, (A.5) holds since

$$S(t_1, ..., t_i^{'}, ..., t_r) \geq S(t_1, ..., t_i, ..., t_r)$$

if $t_i \geq t_i^{'}$. □

Lemma A.3. Suppose $\tau[0, t]$ is a (compound) process specified by Equation (1) of the paper and $g(t)$ is a given lower boundary. For a fixed $r > 1$ and total number of units W and for an appropriately chosen $0 < \epsilon \ll 1$, there exists a (closed) domain $D(t_2, ..., t_r, W_2, ..., W_r)$, such that for the functions $\bar{h}(t) \in \mathcal{H}_r$ with $\{t_2, ..., t_r, W_2, ..., W_r\} \in D(t_2, ..., t_r, W_2, ..., W_r)$, and $\tilde{h}(t) \in \mathcal{H}_r$ with $\{t_2, ..., t_r, W_2, ..., W_r\} \notin D(t_2, ..., t_r, W_2, ..., W_r)$, $0 = t_1 < t_2 < ... < t_r < z$, we have

$$P(g(t) \leq \tau[0, t] \leq \bar{h}(t),\ 0 \leq t \leq z) \geq 1 - \epsilon > P(g(t) \leq \tau[0, t] \leq \tilde{h}(t),\ 0 \leq t \leq z). \quad \text{(A.6)}$$

Proof. Proof of Lemma A.3: From Lemmas A.1 and A.2, it is not difficult to see that $P(g(t) \leq \tau[0, t] \leq h(t), 0 \leq t \leq z)$ is a monotone increasing/decreasing function of W_i, $i \in \{1, ..., r\}$, and of t_i, $i \in \{2, ..., r\}$. Therefore, there exists a hyper-surface in the space $(t_2, ..., t_r, W_2, ..., W_r)$ that will determine a domain $D(t_2, ..., t_r, W_2, ..., W_r)$ for which (A.6) holds. □

The result of Lemma A.3 is illustrated in Figure 1 given in Section 4.1.1 of the paper. We are now in a position to prove Proposition 4.1 of the paper.

Proof. Proof of Proposition 4.1 of the paper: From Lemma A.3 it follows that there exist sequences $W_1, W_2, ..., W_r$, with $W_1 + W_2 + ... + W_r = W$, and $t_2, ..., t_r$, such that the constraint (19) is satisfied. Moreover, from Lemmas A.1, A.2 and definition (17) of the paper (compare with (A.3)), it can be seen that $TC(z, r)$ is a monotone decreasing function of t_i, $i \in \{2, ..., r\}$ and also a monotone increasing/decreasing function of W_i, $i \in \{1, ..., r\}$. Hence, a unique solution to Problem 4.1 of the paper is attained. In fact, the latter is at the boundary of the domain, $D(t_2, ..., t_r, W_2, ..., W_r)$ defined by constraint (19), (cf., Lemma A.3) which restricts the $2(r-1)$-dimensional surface $TC(z, r)$. □

B. Graphical Illustrations and Sensitivity Analysis of the Solution to Problem 4.1 of Section 4.1, Inventory Management Optimization

In Figure B.1, we have illustrated graphically the solution of Problem 4.1 in the paper based on Example 4.1 therein, given by the optimal replenishment function $h_{opt}(t)$ with $r^* = 2$, $W_1^* = 22$, $W_2^* = 13$, $t_1^* = 0$, $t_2^* = 0.47$. The latter is plotted together with 500 simulated trajectories of the demand process $\tau[0, t]$. As is required by the non-exit probability constraint (19) of the paper, approximately 10% of all the trajectories cross either the (optimal) upper boundary $h_{opt}(t)$ or the lower boundary $g(t)$. As can be seen from Figure B.1 in the rest of the cases, the trajectories are non-crossing and fill in the entire corridor between the boundaries as one would expect from an optimal solution with respect to the cost function as specified in Problem 4.1 of the paper.

Next, we perform a short sensitivity study of the optimal solution of Problem 4.1 of the paper with respect to the holding costs parameter, c_h, and the shipment costs function, $c_s(r)$.

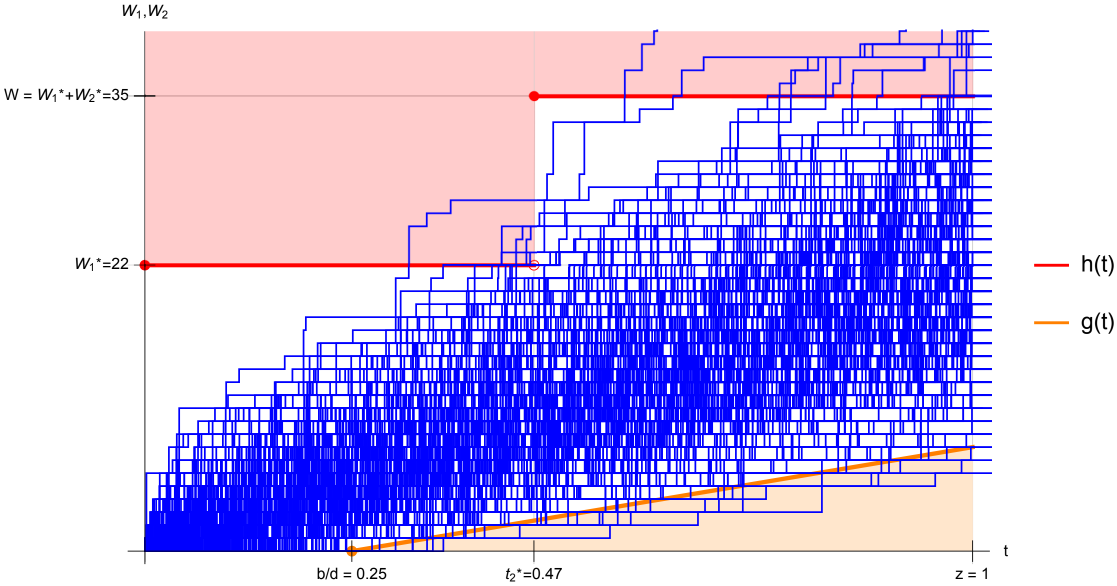

Figure B.1: Graphical Illustration of the Solution to Problem 4.1 of the Paper Based on Example 4.1 of the Paper, Given by $h_{opt}(t)$ with $r^* = 2$, $W_1^* = 22$, $W_2^* = 13$, $t_1^* = 0$, $t_2^* = 0.47$ Plotted Together with 500 Trajectories of the Poisson-Logarithmic Demand Process $\tau[0, t]$.

Example B.1. Assume all the parameters are as in Example 4.1 of the paper, except for the holding costs parameter, c_h, which is now increased to $c_h = 1.5$ (from $c_h = 0.9$ as in Example 4.1 of the paper). The optimal solutions to Problem 4.1 of the paper for fixed values of $r = 1, 2, ..., 5$ are presented in Table B.1.

Table B.1: Optimal Solutions to Problem 4.1 of the Paper for Fixed Values of $r = 1, 2, ..., 5$, for Model Parameters as in Example 4.1 of the Paper, and Examples B.1 and B.2.

r	$TC_{opt} - Ex.4.1$	$TC_{opt} - Ex.B.1$	$TC_{opt} - Ex.B.2$	Optimal Solutions
1	48.5	69.5	55.5	$W = W_1^* = 35$; $t_1^* = 0$;
2	46.501	63.835	57.001	$W_1^* = 22$, $W_2^* = 13$; $t_1^* = 0$, $t_2^* = 0.47$;
3	47.188	63.58	59.788	$W_1^* = 17$, $W_2^* = 10$, $W_3^* = 8$; $t_1^* = 0$, $t_2^* = 0.28$, $t_3^* = 0.61$;
4	47.895	63.825	61.895	$W_1^* = 16$, $W_2^* = 7$, $W_3^* = 6$, $W_4^* = 6$; $t_1^* = 0$, $t_2^* = 0.23$, $t_3^* = 0.46$, $t_4^* = 0.68$;
5	48.567	64.245	63.617	$W_1^* = 14$, $W_2^* = 6$, $W_3^* = 5$, $W_4^* = 5$, $W_5^* = 5$; $t_1^* = 0$, $t_2^* = 0.17$, $t_3^* = 0.34$, $t_4^* = 0.52$, $t_5^* = 0.71$.

As can be seen from the third column in Table B.1, the optimal number of shipments is now $r^* = 3$, with optimal replenishment amounts, $W_1^* = 17$, $W_2^* = 10$, $W_3^* = 8$, at optimal times $t_1^* = 0$, $t_2^* = 0.28$, $t_3^* = 0.61$. In other words, as the holding cost per unit of time per unit of product, c_h, increases, the optimal number of shipments increases. This is natural to expect as it is now more costly to keep units in stock, so it is optimal to have an extra shipment at a later time, t_3^* (compared to Example 4.1 of the paper). Note that changes in the parameters of the costs functions do not affect the location of the optimal solution in the domain $D(t_2, ..., t_r, W_2, ..., W_r)$ but only the optimal number of shipments r^* and the value of $TC_{opt}(z = 1, r^*)$.

The sensitivity of the optimal solution with respect to the shipment costs function, $c_s(r)$, is illustrated by the following example.

Example B.2. Assume all parameters are as in Example 4.1 of the paper, except for the shipment costs function, $c_s(r)$, which is now increased to $c_1 = 0.4$, $c_2 = 0.2$, $c_3 = 0.12$, $c_4 = 0.08$, $c_i = 0.06$, $i = 5, 6, ...$, (from $c_1 = 0.2$, $c_2 = 0.1$, $c_3 = 0.06$, $c_4 = 0.04$, $c_i = 0.03$, $i = 5, 6, ...$, as in Example 4.1 of the paper). The optimal solutions to Problem 4.1 of the paper for fixed values of $r = 1, 2, ..., 5$ are presented in Table B.1 (see the last two columns therein).

As can be seen, the optimal number of shipments is now $r^* = 1$, with optimal replenishment amount $W_1^* = 35$ at optimal time $t_1^* = 0$. Again, this is natural to expect, as the optimal number of shipments has decreased to only a single shipment due to the increased shipment costs.

C. Application of the Proposed FFT-Based Method in Computing Non-Exit Probabilities for Brownian Motion and Double-Barrier Option Pricing

Computing DB(non-)C probabilities for Brownian motion has attracted considerable attention in the applied probability literature where approximation schemes have been developed for the case of (piece-wise) linear boundaries (Borovkov & Novikov 2005, Wang & Pötzelberger 2007, Ycart & Drouilhet 2016), strictly continuous boundaries (Fu

& Wu 2010) and a numerical approximation method for general boundaries based on direct convolution (Khmaladze & Shinjikashvili 2001). Our purpose in this section is to demonstrate that the proposed FFT-based method can be viewed as a significant enhancement of the approach taken by the latter authors, achieving much better efficiency in computing DB(non-)C probabilities for general, possibly discontinuous boundaries. This is illustrated in particular on three examples, Example C.1 on DBC probabilities and Examples C.2 and C.3 on double-barrier option pricing with jump discontinuities in the barriers.

C.1. Computing DB(non-)C Probabilities for Brownian Motion for General Boundaries

We will be concerned here with computing the DB(non-)C probability for a standard Brownian motion, $\{B_t\}, t \geq 0$, of the type

$$P(\alpha(t) < B_t < \beta(t), 0 \leq t \leq T), \tag{C.1}$$

where $\alpha(t) < \beta(t)$, $\forall t$, with $\alpha(0) < 0 < \beta(0)$, are real-valued functions, possibly with jump discontinuities. The DB(non-)C probability in (C.1) can be approximated through the DB(non-)C probability in terms of a Poisson process, i.e.,

$$P\left(\alpha(t) < \frac{\xi(t) - \lambda t}{\sqrt{\lambda}} < \beta(t), 0 \leq t \leq z\right) \xrightarrow{\lambda \to \infty} P(\alpha(t) < B_t < \beta(t), 0 \leq t \leq z), \tag{C.2}$$

where $\xi(t)$ is a Poisson process with intensity rate λ and where (C.2) follows based on the fact that

$$\frac{\xi(t) - \lambda t}{\sqrt{\lambda}},\ t \in [0, +\infty)$$

converges in distribution to B_t as $\lambda \to \infty$ (see e.g., Shorack & Wellner 1986). It can directly be seen that (C.2) can be rewritten as

$$P(\alpha(t) < B_t < \beta(t), 0 \leq t \leq z) \approx P(\sqrt{\lambda}\alpha(t) + \lambda t < \xi(t) < \sqrt{\lambda}\beta(t) + \lambda t, 0 \leq t \leq z), \tag{C.3}$$

which is of the type $P(g(t) \leq \tau[0,t] \leq h(t),\ 0 \leq t \leq z)$, noting that $\tau[0,t]$ defined as in Equation (1) of the paper equals $\xi(t)$ when $P(X_k = 1) = 1$ for all $k = 1, 2, ..., \xi(t)$, i.e., $\tau[0,t] = \xi(t)$, with $g(t) = \sqrt{\lambda}\alpha(t) + \lambda t$ and $h(t) = \sqrt{\lambda}\beta(t) + \lambda t$. Therefore, we can efficiently compute the probability in (C.3) using the proposed FFT-based algorithm which will be illustrated by the following Example C.1.

Example C.1. We consider the DBC probabilities for a Brownian motion with different boundaries $\alpha(t)$ and $\beta(t)$, $t \in [0,1]$ $(z = 1)$ summarized in Table C.1, and compute the approximate probability on the right hand side of (C.3) following the proposed FFT-based method.

Table C.1: Example C.1 - DBC Probabilities for Brownian Motion, Approximated Using the FFT-Based Method with Different Values of λ and Using Fu & Wu (2010)'s Method. Numbers in () Are the Computation Times in Seconds.

$\alpha(t), \beta(t)$	$\lambda = 10000$		$\lambda = 100000$		$\lambda = 1000000$		Fu & Wu (2010)
$\pm\exp(-t)$	0.984018	(0.2)	0.984312	(6.2)	0.984405	(183)	0.984439
$\pm(1 + t - t^2)$	0.510010	(0.4)	0.510873	(10.8)	0.511156	(352)	0.511254
$\pm\sqrt{t+1}$	0.390155	(0.4)	0.391035	(11.3)	0.391312	(369)	0.391403
$\pm(1 + t)$	0.179997	(0.5)	0.180555	(14.0)	0.180731	(469)	0.180803

Our results are summarized and compared to those obtained by Fu & Wu (2010). As shown in Table C.1, as λ increases, the DBC probabilities for a Poisson process (with boundaries $g(t)$ and $h(t)$) do converge to the DBC probabilities for a Brownian motion (as suggested by (C.3)). When $\lambda = 100000$, our FFT-based algorithm can approximate the boundary-crossing probabilities for a Brownian motion up to three decimal places within 6-14 seconds, as can be seen in the fourth column of Table C.1.

C.2. Multi-Step Double-Barrier Option Pricing

The problem of pricing double-barrier options is considered by many authors among which Kunitomo & Ikeda (1992), Geman & Yor (1996), Pelsser (2000), Feng & Linetsky (2008), Cai et al. (2009), Fusai et al. (2016) in the case of constant barriers, Guillaume (2010) for piece-wise constant barriers and Borovkov & Novikov (2005) for arbitrary time-dependent barriers. Similarly, here we are interested in pricing double-barrier options assuming general, possibly curvilinear, barriers allowing discontinuities. More precisely, we consider the pricing of a kick-out (double) barrier call option with maturity T, spot interest rate $r_t > 0$, strike price K_T, and general time-dependent upper and lower barriers $G_\pm(t)$, such that $G_-(t) < G_+(t), t \leq T$, where the underlying asset follows a price process $\{S_t\}, t \geq 0$ with a deterministic volatility $\sigma_t > 0$ (with $\sigma^2 := \int_0^T \sigma_t^2 dt < \infty$). Note the kick-out (double) barrier call option is exercisable when the underlying asset price S_t stays within the corridor between $G_-(t)$ and $G_+(t)$ for $0 \leq t \leq T$. Borovkov & Novikov (2005) show that the fair price of the kick-out (double) barrier call option, in a Black-Scholes setting, is expressed through certain DB(non-)C probabilities for a standard Brownian motion of the type (C.1). More precisely, according to Proposition 1 of Borovkov & Novikov (2005), if the time is suitably transformed such that the volatility is constant over $[0, T]$, the fair price of the kick-out (double) barrier call option is of the same structure as the Black-Scholes formula for the European call option, namely it is

$$S_0 p_1 - K_T \exp\left\{ -\int_0^T r_s ds \right\} p_0, \tag{C.4}$$

where

$$\begin{aligned}
p_1 &= P(f_-(t) < \sigma B_t + \sigma^2 t < f_+(t), 0 \leq t \leq T;\ \sigma B_T + \sigma^2 T > F), \\
p_0 &= P(f_-(t) < \sigma B_t < f_+(t), 0 \leq t \leq T;\ \sigma B_T > F), \\
F &= \ln(K_T/S_0) + \frac{1}{2}\sigma^2 T - \int_0^T r_s ds, \\
f_\pm(t) &= \ln(G_\pm(t)/S_0) + \frac{1}{2}\sigma^2 t - \int_0^t r_s ds,\ 0 \leq t \leq T.
\end{aligned} \tag{C.5}$$

Clearly, the DB(non-)C probabilities, p_0 and p_1 in (C.5), are of the type as in (C.1), with the boundaries $\alpha_i(t)$ and $\beta_i(t)$, $i = 0, 1$, accordingly expressed as

$$\begin{aligned}
\alpha_0(t) &= \frac{f_-(t)}{\sigma}, \quad \beta_0(t) = \frac{f_+(t)}{\sigma}, \\
\alpha_1(t) &= \frac{f_-(t) - \sigma^2 t}{\sigma}, \quad \beta_1(t) = \frac{f_+(t) - \sigma^2 t}{\sigma}.
\end{aligned} \tag{C.6}$$

Therefore, in order to price the kick-out (double) barrier call option using (C.4), it suffices to calculate the DB(non-)C probabilities for a Brownian motion, p_0 and p_1 in (C.5). The latter can be approximated following (C.3), with $g(t) = \sqrt{\lambda}\alpha_i(t) + \lambda t$ and $h(t) = \sqrt{\lambda}\beta_i(t) + \lambda t$, where $\alpha_i(t)$ and $\beta_i(t)$ for $i = 0, 1$, are defined as in (C.6).

In the following examples, we apply approximation (C.3) and the FFT-based method to estimate the probabilities p_0, p_1 in (C.5), and price kick-out barrier call options with general barriers allowing jump discontinuities, using (C.4). More precisely, we consider the so-called *(multi-) step* double-barrier options in which the barriers are piecewise constant functions with arbitrary number of jumps (i.e., steps). Such options are popular in over-the-counter markets, as noted by Guillaume (2010). Hence, the author derives a close-form pricing formula in the special case when the upper and lower barriers have a single jump discontinuity (i.e., have two steps) and notes the prohibitive difficulties in obtaining such formulas for two or more jumps in the barriers. In what follows, we consider two examples of two-step and three-step double-barrier options and highlight the applicability of our FFT-based method to pricing general multi-step double-barrier options.

Example C.2. Consider a two-step kick-out double-barrier call option with maturity $T = 0.5$, $r_t = 0.03$, $K_T = 120$, $S_0 = 100$, $\sigma_t = 0.15$, as specified by Guillaume (2010). The upper barrier $G_+(t)$ is defined as

$$G_+(t) = U_1 + (U_2 - U_1)\mathbb{1}_{\{t_1 \le t \le T\}},$$

whereas the lower barrier $G_-(t)$ is defined as

$$G_-(t) = D_1 + (D_2 - D_1)\mathbb{1}_{\{t_1 \le t \le T\}}.$$

Let $U_1 = 125$, $D_1 = 75$, $U_2 = 130$, $D_2 = 70$, $t_1 = 0.25$. Using the approach described earlier, we estimate the price of this two-step double-barrier call option and compare it with the price obtained by Guillaume (2010) (cf., Table C.2).

A possible Brownian bridge simulation-based pricing approach, for the general case of a multi-step kick-out double-barrier option, is briefly outlined in Guillaume (2010). It can be seen that the speed and accuracy of this method depends on the number of jumps in the barriers and whether they jump at the same times. However, no numerical implementation and examples are provided by the author. In the next example, we illustrate our FFT-based approach in pricing such multi-step kick-out double-barrier call options, noting that its efficiency does not depend on the number of jumps in the barriers and their locations.

Example C.3. We consider a three-step kick-out double-barrier call option with maturity $T = 0.5$, $r_t = 0.03$, $K_T = 120$, $S_0 = 100$, $\sigma_t = 0.15$. The upper barrier $G_+(t)$ is defined as

$$G_+(t) = U_1 + (U_2 - U_1)\mathbb{1}_{\{t_1 \le t \le t_2\}} + (U_3 - U_2)\mathbb{1}_{\{t_2 \le t \le T\}},$$

whereas the lower barrier $G_-(t)$ is defined as

$$G_-(t) = D_1 + (D_2 - D_1)\mathbb{1}_{\{t_1 \le t \le t_2\}} + (D_3 - D_2)\mathbb{1}_{\{t_2 \le t \le T\}}.$$

Let $U_1 = 125$, $D_1 = 75$, $U_2 = 130$, $D_2 = 70$, $U_3 = 135$, $D_3 = 65$, $t_1 = 0.25$, $t_2 = 0.375$. We estimate the price of this double-barrier call option using the proposed FFT-based method and the results are summarized in Table C.2. For comparison, we have also estimated the price using Monte Carlo (MC) simulation.

Table C.2: Examples C.2 and C.3 - The Price of a Two/Three-Step Kick-Out Double Barrier Call Option Approximated Using the FFT-Based Method with Different Values of λ and the Prices Obtained by Guillaume (2010) and MC Simulation. Numbers in () Are the Computation Times in Seconds.

	$\lambda = 100000$		$\lambda = 1000000$		$\lambda = 4000000$		Option Price	
Example C.2	0.122126	(18)	0.108027	(612)	0.108302	(5067)	0.1086	Guillaume (2010)
Example C.3	0.18918	(20)	0.175522	(619)	0.175935	(5211)	0.180072	MC simulation

It can be seen from Table C.2 that as λ increases, the price estimated by applying the proposed FFT-based method to approximate the probabilities p_0, p_1 in (C.5) using (C.3) converges up to three digits after the decimal. We note however that the accuracy of the "true" price, 0.1086 (and the method) due to Guillaume (2010), has not been commented upon by the author. Furthermore, for the three-step double-barrier option, the simulated price 0.180072 has been obtained by simulating discretized Brownian motion trajectories with a time step, 10^{-3}. It is an average of 10 runs, each with 10^4 simulated trajectories, and is obtained at a relatively high computational cost. Its standard deviation based on the 10 runs is 0.0117, which confirms the relatively poor performance of the MC simulation.

Our purpose here has been to demonstrate that FFT-based option pricing method can achieve reasonable rate of convergence (up to three digits in this example) in reasonable time. However, its efficiency (time and accuracy) can be significantly improved by implementing multi-core processing combined with using GPUs, as well as a continuity correction in (C.3), which is outside the scope of this paper.

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