REPRESENTATION THEORY
An Electronic Journal of the American Mathematical Society
Volume 00, Pages 000–000 (Xxxx XX, XXXX)
S 1088-4165(XX)0000-0

BALANCED SEMISIMPLE FILTRATIONS FOR TILTING MODULES

AMIT HAZI

Abstract. Let U_l be a quantum group at an lth root of unity, obtained via Lusztig's divided powers construction. Many indecomposable tilting modules for U_l have been shown to have what we call a balanced semisimple filtration, or a Loewy series whose semisimple layers are symmetric about some middle layer. The existence of such filtrations suggests a remarkably straightforward algorithm for calculating these characters if the irreducible characters are already known. We first show that the results of this algorithm agree with Soergel's character formula for the regular indecomposable tilting modules. We then show that these balanced semisimple filtrations really do exist for these tilting modules.

Introduction

Let U_l be the Lusztig form of a quantized universal enveloping algebra at an lth root of unity, corresponding to some complex semisimple Lie algebra (as described in e.g. [10, Appendix H]). A U_l-module is called a tilting module if it has a filtration by Weyl modules and a filtration by dual Weyl modules. The indecomposable tilting modules $T_l(\lambda)$ of U_l are classified according to their highest weight λ. We are interested in calculating their Loewy series and determining their structure in general.

Andersen and Kaneda showed that $T_l(\lambda)$ is rigid (i.e. has identical radical and socle series) for λ sufficiently high [3]. In particular, because of self-duality this implies that if the Loewy length of $T_l(\lambda)$ is $2N+1$, we have $\operatorname{rad}_{i+N} T_l(\lambda) \cong \operatorname{rad}_{i-N} T_l(\lambda)$ for any i. In other words, the Loewy series is symmetric about the middle layer containing $L_l(\lambda)$. We call such Loewy series *balanced*. Additionally the examples in [3] and in previous work by Bowman-Doty-Martin [6, 7] and the author [9] show that the unique Loewy series is compatible with a certain Loewy series of the Weyl module called the dual parity filtration in [3]. This filtration has Loewy layers whose composition factor multiplicities are coefficients of Kazhdan-Lusztig polynomials. We note that if the quantum analogue of Jantzen's conjecture (written as $(F, w, s)^+$ in [10, II.C.9]) holds then this filtration coincides with the Jantzen filtration described in [4].

This suggests the following algorithm for calculating the character of $T_l(\lambda)$ given the characters of the Weyl modules of weight up to λ.

Algorithm.

Received by the editors 21 February 2017.
2010 *Mathematics Subject Classification.* 20G42.

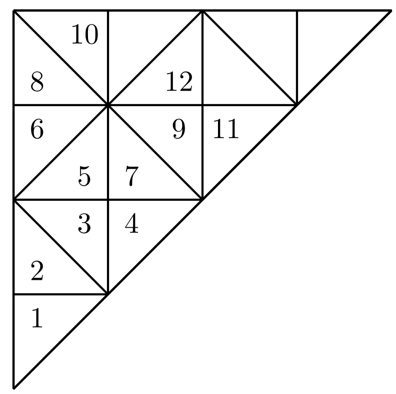

FIGURE 1. Some alcoves for the quantum group corresponding to the root system B_2. The numbering is taken from [3, Section 5.3].

(1) Write the dual parity filtration of the Weyl module $\Delta_l(\lambda)$. We view this as the bottom layers of a partial Loewy series for $T_l(\lambda)$. We will reflect Loewy layers about the "middle" Loewy layer in which $L_l(\lambda)$ appears.
(2) Find the highest "unbalanced" weight; that is, the largest $\mu < \lambda$ such that $L_l(\mu)$ appears in the series below $L_l(\lambda)$ but there is no corresponding factor $L_l(\mu)$ in the reflected layer above $L_l(\lambda)$.
(3) Add the dual parity filtration of $\Delta_l(\mu)$ to the partial Loewy series so that the head of $\Delta_l(\mu)$ is in the reflected Loewy layer above $L_l(\lambda)$.
(4) Repeat from Step 2 until the Loewy series is balanced.

As an example, we will apply the algorithm to calculate the character of an indecomposable tilting module for the quantum group corresponding to the root system B_2. We label the first few B_2 alcoves following [3, Section 5.3] (see Figure 1).

The translation principle implies that the structure of $T_l(\lambda_i)$ for some regular weight λ_i in alcove i does not depend on the choice of λ_i. Thus we may unambiguously refer to $T_l(i)$ instead of $T_l(\lambda_i)$ (and similarly $L_l(i)$ instead of $L_l(\lambda_i)$). Applying the algorithm to $T_l(9)$ yields the partial Loewy series in Figure 2. Note that in these pictures we simply write i to mean $L_l(i)$.

Another way of looking at this algorithm is through the lens of hidden gradings on various module categories. Under this philosophy, whenever there is a "Kazhdan-Lusztig-like" character formula expressing a character by evaluating certain polynomials at 1, there should be a similar graded category for which whose graded characters are given by the polynomials themselves. There have been many investigations of this behavior with respect to tilting modules, see for example [21, 5, 20].

In this paper we first prove that this naïve algorithm in fact works for all regular indecomposable tilting modules (not just rigid ones) at the level of characters. The key ingredients in this proof are Lusztig's character formula, which is true when $l > h$ (where h is the Coxeter number) in the case of quantum groups [14, 15, 16, 19] and Soergel's tilting character formula [21, 22]. In the final section we prove that the balanced semisimple filtrations alluded to above really do exist for all regular

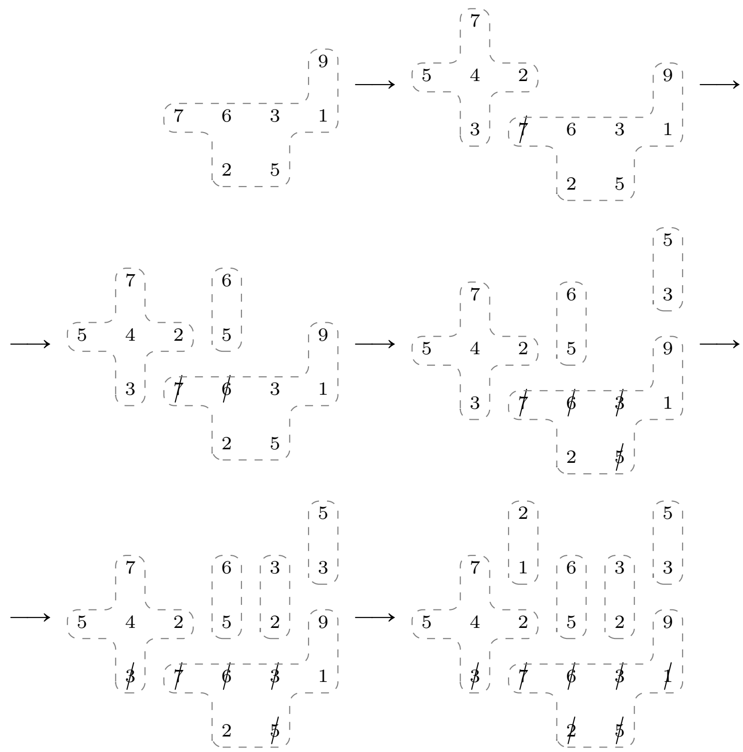

Figure 2. The partial Loewy series obtained by applying the algorithm to $T_l(9)$.

indecomposable tilting modules. In the future we hope to use similar methods to find a general character formula for the indecomposable tilting modules in the modular case.

1. Quantum groups at roots of unity

Let R be a root system for a Euclidean space E of dimension n, and let A_R be the Cartan matrix associated to this root system. Let q be an indeterminate in the ring $\mathcal{A} = \mathbb{Z}[q^{\pm 1}]$. Write $U_{\mathcal{A}}$ for the Lusztig integral form quantum group associated to the Cartan matrix A_R, as described in [10, H.5]. This quantum group is a Hopf algebra over $\mathcal{A}$ with algebra generators $E_i^{(r)}, F_i^{(r)}, K_i^{\pm 1}$ ranging over $i = 1, \ldots, n$ and $r \in \mathbb{N}$.

Now let $l \in \mathbb{N}$ be an odd positive integer (with l coprime to 3 if R has a G_2-component). Set $\zeta = e^{2\pi i/l} \in \mathbb{C}$, a primitive lth root of unity. We can make $\mathcal{A}$ into a commutative $\mathbb{C}$-algebra by specializing q to ζ. This leads to a specialization $U_l = \mathbb{C} \otimes_{\mathcal{A}} U_{\mathcal{A}}$ of our quantum group at ζ.

We will restrict ourselves to the study of finite-dimensional U_l-modules of type $\mathbf{1}$ (see [10, H.10] for a precise definition). When l is prime, the representation theory of U_l-modules is analogous to the representation theory of an algebraic group G with root system R over a field of characteristic l. In particular, if R^+ denotes the set of positive roots, and we define

$$X = \{\lambda \in E : \langle \lambda, \alpha^\vee \rangle \in \mathbb{Z} \text{ for all } \alpha \in R^+\},$$
$$X^+ = \{\lambda \in E : \langle \lambda, \alpha^\vee \rangle \in \mathbb{Z}_{\geqslant 0} \text{ for all } \alpha \in R^+\}$$

to be the sets of integral and dominant integral weights respectively, then for each $\lambda \in X^+$ we have U_l-modules $\nabla_l(\lambda)$ and $\Delta_l(\lambda)$, called the dual Weyl module and Weyl module respectively, defined in a similar way to the eponymous constructions for G (in [10, H.11-H.12], these are referred to as $H^0_q(\lambda)$ and $H^n_q(w_0 \cdot \lambda)$ respectively). The module $L_l(\lambda) = \operatorname{soc} \nabla_l(\lambda) \cong \Delta_l(\lambda)/\operatorname{rad} \Delta_l(\lambda)$ is simple, and all simple modules are of this form. Moreover, familiar results from the theory of algebraic groups (including Kempf's vanishing theorem) carry over for these U_l-modules. This means that we can define the indecomposable tilting module $T_l(\lambda)$ (in a manner completely analogous to the G-modules case) as the unique indecomposable module with a Δ-filtration and a ∇-filtration with highest weight λ. If V is a U_l-module, then we can define the contravariant dual module ${}^\tau V$ analogously to the modular case (see [10, II.2.12]). The modules $L_l(\lambda)$ and $T_l(\lambda)$ are self-dual as expected, and $\Delta_l(\lambda) \cong {}^\tau \nabla(\lambda)$.

The affine Weyl group $\mathcal{W}$ is defined to be $l\mathbb{Z}R \rtimes W$, i.e. the group of Euclidean isometries of E generated by translations by the scaled root lattice $l\mathbb{Z}R$ and the Weyl group W of R. It acts on weights via the dot action, which shifts the origin to $-\rho$:

$$w \cdot \lambda = w(\lambda + \rho) - \rho.$$

An alcove in the Euclidean space E is a connected component in the complement of the union of the reflection hyperplanes for all reflections in $\mathcal{W}$. When R is irreducible the closure of an alcove is a simplex of dimension n; in general the closure of an alcove is a product of simplices. The affine Weyl group $\mathcal{W}$ acts simply transitively on the set of all alcoves. The fundamental alcove A_0 is defined to be the sets of weights

$$A_0 = \{\lambda \in E : 0 < \langle \lambda + \rho, \alpha^\vee \rangle < l \text{ for all } \alpha \in R^+\}.$$

The other alcoves can be obtained by taking the image of A_0 under some isometry from the affine Weyl group $\mathcal{W}$. The dominant alcoves are those which intersect the dominant region X^+ non-trivially.

The affine Weyl group $\mathcal{W}$ can be viewed as a Coxeter group, with $(n+1)$ generators $\mathcal{S}$ corresponding to reflections in the walls of $\overline{A}_0$ (the closure of the fundamental alcove A_0). It comes equipped with the Bruhat order and the length function $\ell : \mathcal{W} \to \mathbb{Z}_{\geqslant 0}$. Let S denote the n generators in $\mathcal{S}$ corresponding to reflections contained in the Weyl group W. By simple transitivity, any alcove A can be written as $x \cdot A_0$ for some $x \in \mathcal{W}$. As x acts on E by a Euclidean isometry, x also provides a bijection between the walls of A and the walls of A_0, which are labelled by $\mathcal{S}$.

From now on, all weights will be dominant integral weights unless otherwise stated. The linkage principle states that if $L_l(\lambda)$ and $L_l(\lambda')$ are in the same block, then $\lambda' \in \mathcal{W} \cdot \lambda$ [10, II.6.17]. We write $\mathcal{B}_\lambda$ for the full subcategory of modules whose composition factors have highest weights lying in $\mathcal{W} \cdot \lambda$, and $\mathrm{pr}_\lambda : U_l\text{−mod} \to \mathcal{B}_\lambda$ for the projection functor onto this subcategory. For a dominant alcove A and $\lambda, \mu \in \overline{A}$ the translation functor is defined by

$$T^\mu_\lambda(V) = \mathrm{pr}_\mu \left(\mathrm{pr}_\lambda(V) \otimes L_l(w(\mu - \lambda)) \right), \tag{1}$$

where $w \in W$ is chosen so that $w(\mu - \lambda) \in X^+$. Note that T_λ^μ is always exact because it is the composition of several exact functors. The translation principle states that $T_\lambda^\mu, T_\mu^\lambda : \mathcal{B}_\lambda \leftrightarrows \mathcal{B}_\mu$ are adjoint and mutually inverse if λ and μ belong to the same set of alcoves.

Suppose $\lambda, \lambda' \in X^+$ are in the same orbit of $\mathcal{W}$ and belong to adjacent alcoves A, A' with $\lambda < \lambda'$ (i.e. $\lambda' - \lambda \in R^+$). Suppose the wall between them is labelled by $s \in \mathcal{S}$, and let μ be a weight on this wall. The wall-crossing functor is defined to be $\theta_s = T_\mu^{\lambda'} \circ T_\lambda^\mu$, which is self-adjoint and exact. It is well-known that $\theta_s \Delta(\lambda) \cong \theta_s \Delta(\lambda')$, and we have an exact sequence

$$0 \to \Delta_l(\lambda') \to \theta_s \Delta_l(\lambda) \to \Delta_l(\lambda) \to 0. \tag{2}$$

2. Kazhdan-Lusztig combinatorics

Notation. We use notation from [21] and [10, C.1] for various Kazhdan-Lusztig polynomials, which we will summarize.

We write $\mathcal{L} = \mathbb{Z}[v^{\pm 1}]$ for the ring of Laurent polynomials in v. Let $\mathcal{H} = \mathcal{H}(\mathcal{W}, \mathcal{S})$ be the Hecke algebra associated to the Coxeter system $(\mathcal{W}, \mathcal{S})$, an associative $\mathcal{L}$-algebra with generators $\{H_s\}_{s \in \mathcal{S}}$ and relations

$$H_s^2 = 1 + (v^{-1} - v) H_s \quad \text{for all } s \in \mathcal{S}, \tag{3}$$

$$\overbrace{H_s H_t H_s \cdots}^{r \text{ terms}} = \overbrace{H_t H_s H_t \cdots}^{r \text{ terms}} \qquad \text{for all } s, t \in \mathcal{S}, \text{ where } r \text{ is the order of } st. \tag{4}$$

For any reduced word $x = stu \cdots \in \mathcal{W}$, the element $H_x = H_s H_t H_u \cdots$ is well defined, and the set $\{H_x\}_{x \in \mathcal{W}}$ forms an $\mathcal{L}$-basis for $\mathcal{H}$. Each generator H_s is invertible, with $H_s^{-1} = H_s + v - v^{-1}$, so each basis element H_x is also invertible. Define the ring homomorphism

$$\begin{aligned} d : \mathcal{H} &\longrightarrow \mathcal{H} \\ v &\longmapsto v^{-1} \\ H_x &\longmapsto (H_{x^{-1}})^{-1} \end{aligned} \tag{5}$$

which extends an obvious involution on $\mathcal{L}$. We call this involution dualization, and we write $\overline{H}$ for $d(H)$. For $s \in \mathcal{S}$ we define $\underline{H}_s = H_s + v$ and $\underline{\tilde{H}}_s = H_s - v^{-1}$. Notice that $\overline{H}_s = H_s + v - v^{-1}$ so both $\underline{H}_s$ and $\underline{\tilde{H}}_s$ are self-dual, i.e. fixed by d. The sets $\{\underline{H}_s\}_{s \in \mathcal{S}}$ and $\{\underline{\tilde{H}}_s\}_{s \in \mathcal{S}}$ each generate $\mathcal{H}$ as an $\mathcal{L}$-algebra.

Now let $\mathcal{H}_W = \mathcal{H}(W, S) \leqslant \mathcal{H}$ be the Hecke algebra obtained from the finite Weyl group $W < \mathcal{W}$. Since $(H_s - v^{-1})(H_s + v) = 0$ for each generator $s \in S$, for each $u \in \{-v, v^{-1}\}$ there is a homomorphism of $\mathcal{L}$-algebras $\varphi_u : \mathcal{H}_W \to \mathcal{L}$, defined by mapping $H_s \mapsto u$. This turns $\mathcal{L}$ into a right $\mathcal{H}_W$-module which we call $\mathcal{L}(u)$. These modules are analogues of the sign/trivial representations for W. Now define two right $\mathcal{H}$-modules

$$\begin{aligned} \mathcal{M} &= \mathcal{L}(v^{-1}) \otimes_{\mathcal{H}_W} \mathcal{H}, \\ \mathcal{N} &= \mathcal{L}(-v) \otimes_{\mathcal{H}_W} \mathcal{H}. \end{aligned}$$

We can obtain an $\mathcal{L}$-basis for $\mathcal{M}$ via a set of representatives for the right cosets $W \backslash \mathcal{W}$. A natural choice for such representatives comes from the dominant alcoves, namely, the set $\mathcal{W}^+ = \{x \in \mathcal{W} : (x \cdot A_0) \cap X^+ \neq \varnothing\}$, or in other words the affine Weyl group elements which map A_0 to another dominant alcove. The elements in $\mathcal{W}^+$ are in fact precisely the minimal length representatives for the cosets $W \backslash \mathcal{W}$.

Defining M_x to be $1 \otimes H_x$ in $\mathcal{M}$, we get the $\mathcal{L}$-basis $\{M_x\}_{x \in \mathcal{W}^+}$ (and similarly for $\mathcal{N}$). The action of $\underline{H}_s$ on these bases is

$$M_x \underline{H}_s = \begin{cases} M_{xs} + vM_x & \text{if } xs \in \mathcal{W}^+ \text{ and } xs > x, \\ M_{xs} + v^{-1}M_x & \text{if } xs \in \mathcal{W}^+ \text{ and } xs < x, \\ (v + v^{-1})M_x & \text{if } xs \notin \mathcal{W}^+, \end{cases} \tag{6}$$

$$N_x \underline{H}_s = \begin{cases} N_{xs} + vN_x & \text{if } xs \in \mathcal{W}^+ \text{ and } xs > x, \\ N_{xs} + v^{-1}N_x & \text{if } xs \in \mathcal{W}^+ \text{ and } xs < x, \\ 0 & \text{if } xs \notin \mathcal{W}^+. \end{cases} \tag{7}$$

The dualization map $d : \mathcal{H} \to \mathcal{H}$ extends to a dualization map of $\mathcal{M}$ by mapping $a \otimes H \mapsto \overline{a} \otimes \overline{H}$. To see this, note that for all $s \in \mathcal{S}$

$$\phi_u(\underline{H}_s) = \begin{cases} v + v^{-1} & \text{if } u = v^{-1}, \\ 0 & \text{if } u = -v, \end{cases}$$

so $\phi_u(\underline{H}_s)$ is self-dual. This means that for $s \in S$,

$$\begin{aligned} d(a \otimes (\underline{H}_s H)) &= \overline{a} \otimes \overline{\underline{H}_s H} \\ &= \overline{a} \otimes \underline{H}_s \overline{H} \\ &= \overline{a}\phi_u(\underline{H}_s) \otimes \overline{H} \\ &= \overline{a\phi_u(\underline{H}_s)} \otimes \overline{H} \\ &= d(a\phi_u(\underline{H}_s) \otimes H). \end{aligned}$$

As $\{\underline{H}_s\}_{s \in S}$ generates $\mathcal{H}_W$ this shows that the map above is well-defined.

The following theorem describes Soergel's version of the parabolic Kazhdan-Lusztig basis for the modules $\mathcal{M}$ and $\mathcal{N}$. We reproduce the proof here as an aid to the reader and for later use of the notation therein.

Theorem 2.1 ([21, Theorem 3.1])**.** *There is a unique set of polynomials $\{m_{y,x}\}_{x,y \in \mathcal{W}^+}$ in $\mathbb{Z}[v]$ such that*

(i) *if $m_{y,x} \neq 0$, then either $y = x$ and $m_{x,x} = 1$ or $y < x$ and $m_{y,x} \in v\mathbb{Z}[v]$;*
(ii) *the element $\underline{M}_x = \sum_y m_{y,x} M_y$ is self-dual.*

There are also analogous polynomials $\{n_{y,x}\}_{x,y \in \mathcal{W}^+}$ for $\mathcal{N}$.

Proof. We prove the result for $\mathcal{M}$ as the proof for $\mathcal{N}$ is identical. Induct on the length of x. Suppose for some $x \in \mathcal{W}^+$ we have already defined $\underline{M}_x$ and all $\underline{M}_u$ with $\ell(u) < \ell(x)$. Suppose $s \in \mathcal{S}$ such that $xs \in \mathcal{W}^+$ and $xs > x$. Write

$$\underline{M}_x \underline{H}_s = M_{xs} + \sum_{y < xs} m^s_{y,x} M_y. \tag{8}$$

From the action of $\underline{H}_s$ on the basis above we have (for $x, y \in \mathcal{W}^+$)

$$m^s_{y,x} = \begin{cases} m_{ys,x} + v m_{y,x} & \text{if } ys > y \text{ and } ys \in \mathcal{W}^+, \\ m_{ys,x} + v^{-1} m_{y,x} & \text{if } ys < y \text{ and } ys \in \mathcal{W}^+, \\ (v + v^{-1}) m_{y,x} & \text{if } ys \notin \mathcal{W}^+. \end{cases} \tag{9}$$

Clearly $\underline{M}_x \underline{H}_s$ is self-dual, so the element

$$\underline{M}_{xs} = \underline{M}_x \underline{H}_s - \sum_{y < xs} m^s_{y,x}(0) \underline{M}_y = M_{xs} + \sum_{y < xs} m_{y,xs} M_y,$$

whose coefficients we have labelled $m_{y,xs}$, is also self-dual with the property that $m_{y,xs}$ has zero constant coefficient. $\square$

The following theorem provides a similar basis when the coefficients are restricted to being Laurent polynomials in negative degree instead of positive degree.

Theorem 2.2 ([8]). *There is a unique set of polynomials $\{\tilde{m}_{y,x}\}_{x,y\in\mathcal{W}^+}$ in $\mathbb{Z}[v^{-1}]$ such that*

(i) *if $\tilde{m}_{y,x} \neq 0$, then either $y = x$ and $\tilde{m}_{x,x} = 1$ or $y < x$ and $\tilde{m}_{y,x} \in v^{-1}\mathbb{Z}[v^{-1}]$;*
(ii) *the element $\underline{\tilde{M}}_x = \sum_y \tilde{m}_{y,x} M_y$ is self-dual.*

Moreover, we have $\tilde{m}_{y,x} = (-1)^{\ell(x)+\ell(y)}\overline{n_{y,x}}$.

Proof. The proof of existence and uniqueness is entirely analogous to the previous case, using $\underline{\tilde{H}}_s$ instead of $\underline{H}_s$. For the final result, see e.g. [21, Theorem 3.5]. $\square$

We can now define the inverse polynomials $\{m^{y,x}\}$ for $y, x \in \mathcal{W}^+$ and $y \geqslant x$ such that the following formula holds:

$$\sum_z (-1)^{\ell(z)+\ell(x)} m^{z,x} m_{z,y} = \delta_{x,y}. \tag{10}$$

These polynomials arise as the coefficients of some element of a module related to $\mathcal{M}$ with respect to a certain basis [21, Theorem 3.6]. However, this will not matter for the sequel.

Character formulae. Let A be a dominant alcove. The structure of the module $\Delta_l(\lambda)$ for any λ contained in A only depends on A and not on the exact weight λ by the translation principle. So we may abuse notation and write $\Delta_l(A)$ instead of $\Delta_l(\lambda)$. We can even reconstruct character formulae written in this way using the linkage principle. We will also freely use the bijection between dominant alcoves and elements of $\mathcal{W}^+$ for the indices of the various Kazhdan-Lusztig polynomials. Finally, if $A = x \cdot A_0$ and $s \in \mathcal{S}$ is a simple reflection we write As for $xs \cdot A_0$.

With this notation, Lusztig's character formula can be written as follows.

Theorem 2.3 (Lusztig's character formula, [17]). *Suppose $l > h$, where h denotes the Coxeter number of the root system R. Let A be a dominant alcove. Then the following character formula*

$$[\Delta_l(A)] = \sum_B m^{A,B}(1)[L_l(B)] \tag{11}$$

holds, where the sum is over all dominant alcoves B.

This result is analogous to Lusztig's conjecture on the irreducible characters of reductive algebraic group in positive characteristic [18]. Lusztig's character formula for quantum groups was first proved in a series of papers by Kazhdan and Lusztig [14, 15, 15, 16, 19] and Kashiwara and Tanisaki [11, 12]. For the rest of this paper we will assume that $l > h$ so that Lusztig's character formula holds. An important corollary (which is in fact equivalent) is the Vogan conjecture.

Corollary 2.4 (Vogan conjecture, [1]). *Let A be a dominant alcove, and $s \in \mathcal{S}$ a simple reflection such that $s \cdot A > A$. Then $\theta_s(L_l(A))$ has socle and head isomorphic to $L_l(A)$, and the module*

$$\beta_s(L_l(A)) = \operatorname{rad}\theta_s(L_l(A))/\operatorname{soc}\theta_s(L_l(A)) \tag{12}$$

is semisimple.

If the corollary holds one can show that $[\beta_s(L_l(A)) : L_l(B)] = m^s_{A,B}(0)$. In addition it follows that for any module M, if M has Loewy length m then $\theta_s(M)$ has Loewy length at most $m+2$ (for a proof see [10, D.2]).

For indecomposable tilting modules, Soergel proved the following character formula [21, 22].

Theorem 2.5 (Soergel's tilting character formula)**.** *Let A be a dominant alcove. Then the following character formula*

$$[T_l(A)] = \sum_B n_{B,A}(1)[\Delta_l(B)] \tag{13}$$

holds, where the sum is over all dominant alcoves B.

The following combinatorial property of Kazhdan-Lusztig polynomials is the basis for the balancing property of filtrations in the next section.

Lemma 2.6. *The Laurent polynomial*

$$t_{B,A} = \sum_C n_{C,A}\overline{m^{C,B}} \tag{14}$$

is self-dual.

Proof. Let $T_A = \sum_B \overline{m^{A,B}}\underline{\tilde{N}}_B$. Unlike $\underline{\tilde{N}}_A$, T_A is not self-dual. Now define $\underline{T}_A$ as follows:

$$\underline{T}_A = \sum_C n_{C,A}T_C.$$

We claim that this sum is self-dual. In fact, it is equal to $\underline{N}_A$:

$$\begin{aligned}
\underline{T}_A &= \sum_{B,C} n_{C,A}\overline{m^{C,B}}\underline{\tilde{N}}_B \\
&= \sum_{B,C,D} (-1)^{\ell(B)+\ell(D)} n_{C,A}\overline{m^{C,B}}\,\overline{m_{D,B}}N_D \\
&= \sum_{C,D} (-1)^{\ell(C)+\ell(D)} n_{C,A}\delta_{C,D}N_D \\
&= \sum_C n_{C,A}N_C \\
&= \underline{N}_A.
\end{aligned}$$

As the coefficient of $\underline{\tilde{N}}_B$ in $\underline{T}_A$ is $t_{B,A}$ and $\underline{\tilde{N}}_B$ is self-dual, this shows what we want. □

From the proof, we see that there is an abelian group isomorphism from the Grothendieck group of the principal block $\mathcal{B}_0$ of U_l-mod to ${}_{v=1}\mathcal{N}$ (the module $\mathcal{N}$ with v specialized to 1, sometimes called the antispherical module) mapping ${}_{v=1}\underline{N}_A \mapsto [T_l(A)]$. It also maps ${}_{v=1}N_A \mapsto [\Delta_l(A)]$ and ${}_{v=1}\underline{\tilde{N}}_A \mapsto [L_l(A)]$. Since the action of ${}_{v=1}\underline{H}_s$ on the basis $\{{}_{v=1}N_A\}$ matches the action of the wall-crossing functor θ_s on the Weyl modules on the level of characters, we have that this holds for any Δ-filtered module, so we can evaluate the character $\theta_s([T_l(A)])$ in this way.

In fact if we wait until the very end of the calculation before specializing, we obtain extra combinatorial information about the character:

$$\begin{aligned}
\underline{N}_A \underline{H}_s &= \sum_{B,C} n_{C,A} \overline{m^{C,B}} \underline{\tilde{N}}_B \underline{H}_s \\
&= \sum_{B,C} n_{C,A} \overline{m^{C,B}} \underline{\tilde{N}}_B (\underline{\tilde{H}}_s + v + v^{-1}) \\
&= \sum_{B,C} n_{C,A} \overline{m^{C,B}} \left((v + v^{-1}) \underline{\tilde{N}}_B + \underline{\tilde{N}}_{Bs} + \sum_{D<Bs} \overline{m^s_{D,B}}(0) \underline{\tilde{N}}_D \right).
\end{aligned}$$

It is clear that the unspecialized version of $\theta_s([T_l(A)])$ above respects the filtration described by Vogan's conjecture. For later use we define the following polynomials, which are $t_{B,A}$-analogues of $m^s_{B,A}$:

$$t^s_{B,A} = (v + v^{-1}) \sum_C n_{C,A} \overline{m^{C,B}} + \sum_{C,D} n_{C,A} \overline{m^{C,D}}\, \overline{m^s_{B,D}}(0). \tag{15}$$

3. Balanced semisimple filtrations

Isotropic filtrations. Let V be a self-dual U_l-module. Fix an isomorphism $\phi : V \to {}^\tau V$. This isomorphism is equivalent to a non-degenerate bilinear form $(-,-)$ on V, with the property that $(xv, v') = (v, \tau(x)v')$ for all $x \in U_l$ and $v, v' \in V$. Forms obeying this property are called contravariant [10, II.8.17]. For any contravariant form on V, we have $(V_\lambda, V_\mu) = 0$ unless $\lambda = \mu$, where V_λ and V_μ are the λ and μ weight subspaces of V. For convenience we will further assume that the form arising from ϕ is symmetric.

For a subspace U of V, recall that the orthogonal subspace is defined to be $U^\perp = \{v \in V : (u, v) = 0 \text{ for all } u \in U\}$. If the form is symmetric, $U \leqslant U^{\perp\perp}$, and by non-degeneracy the dimensions must match, so $U = U^{\perp\perp}$. If U is a submodule of V then $U^\perp$ is also a submodule of V.

Definition 3.1. Suppose U is a submodule of V. Then U is called totally isotropic (with respect to $(-,-)$) if $U \leqslant U^\perp$. Dually U is called totally coisotropic if $U^\perp \leqslant U$.

It is immediately clear that U is totally isotropic if and only if $U^\perp$ is totally coisotropic.

The translation functors T^μ_λ are exact, so the homomorphism $T^\mu_\lambda \phi : T^\mu_\lambda(V) \to T^\mu_\lambda({}^\tau V)$ is also an isomorphism. Additionally one can check that $T^\mu_\lambda({}^\tau V) \cong {}^\tau T^\mu_\lambda(V)$, so $T^\mu_\lambda \phi$ defines a non-degenerate contravariant form on $T^\mu_\lambda(V)$.

Lemma 3.2. *Let A be a dominant alcove, and suppose $\lambda, \mu \in \overline{A}$. If U is a totally isotropic submodule of V, then $T^\mu_\lambda(U)$ is a totally isotropic submodule of $T^\mu_\lambda(V)$.*

Proof. Total isotropy of U can be rephrased in terms of homomorphisms: U is totally isotropic if and only if the inclusion $U \hookrightarrow V$ factors through the inclusion $U^\perp \hookrightarrow V$:

$$\begin{tikzcd}
U \arrow[d] \arrow[dr] & \\
U^\perp \arrow[r] & V.
\end{tikzcd}$$

Applying T_λ^μ to the above triangle gives

$$\begin{tikzcd} T_\lambda^\mu(U) \arrow[d] \arrow[dr] & \\ T_\lambda^\mu(U^\perp) \arrow[r] & T_\lambda^\mu(V). \end{tikzcd}$$

Since $U^\perp \cong {}^\tau(V/U)$, we have $T_\lambda^\mu(U^\perp) \cong (T_\lambda^\mu U)^\perp$. This implies that $T_\lambda^\mu(U)$ is a totally isotropic submodule of $T_\lambda^\mu(V)$. □

Definition 3.3. A filtration $\{V_i\}$ of V is called isotropic (with respect to $(-,-)$) if it can be written in the form

$$0 = V_m^\perp \leqslant \cdots \leqslant V_1^\perp \leqslant V_1 \leqslant \cdots \leqslant V_m = V,$$

for some $m \geqslant 0$. In this situation we typically reindex so that $V_{-i} = V_i^\perp$ for $i > 0$. We call V_{-1} and V_1 the lower half and upper half of $\{V_i\}$ respectively, denoted $\operatorname{lower}\{V_i\}$ and $\operatorname{upper}\{V_i\}$. We call $\{V_i\}$ maximal isotropic if $\operatorname{lower}\{V_i\}$ is maximal, i.e. if there is no other isotropic filtration $\{V'_{i'}\}$ such that $\operatorname{lower}\{V'_{i'}\} \geqslant \operatorname{lower}\{V_i\}$. The subquotient $\operatorname{upper}\{V_i\}/\operatorname{lower}\{V_i\}$ is called the middle and is denoted $\operatorname{mid}\{V_i\}$.

We denote the layers of an isotropic filtration by

$$V^i = \begin{cases} V_{i+1}/V_i & \text{if } i > 0, \\ V_i/V_{i-1} & \text{if } i < 0, \\ V_1/V_{-1} & \text{if } i = 0. \end{cases}$$

If $\{V_i\}$ is a maximal isotropic filtration, then $\operatorname{mid}\{V_i\}$ must be semisimple. To see this, suppose otherwise. We have $(\operatorname{soc}\operatorname{mid}\{V_i\})^\perp = \operatorname{rad}\operatorname{mid}\{V_i\}$. For any non-semisimple indecomposable summand U of $\operatorname{mid}\{V_i\}$ we have $\operatorname{rad} U \geqslant \operatorname{soc} U$. From this summand we could construct a larger isotropic filtration, which is a contradiction.

From [3] and [2], it follows that the dual Weyl modules have parity filtrations determined by the Kazhdan-Lusztig polynomials $m^{B,A}$. In other words, there exists a filtration $\nabla_l(A)_i$ of $\nabla_l(A)$ such that the successive subquotients $\nabla_l(A)^i = \nabla_l(A)_{i+1}/\nabla_l(A)_i$ are all semisimple, with character

$$[\nabla_l(A)^i] = \sum_B (m^{A,B})_i [L_l(B)], \tag{16}$$

where $(m^{A,B})_i$ denotes the coefficient of v^i in the polynomial $m^{A,B}$.

Definition 3.4. Suppose T is a tilting module, and we fix an isomorphism $\phi : T \to {}^\tau T$ which induces a contravariant symmetric form $(-,-)$. A semisimple isotropic filtration (with respect to $(-,-)$) $\{T_i\}$ of T is called a balanced semisimple filtration if there is a Δ-filtration

$$0 \leqslant T_{(\lambda_1,1)} \leqslant T_{(\lambda_1,2)} \leqslant \cdots \leqslant T_{(\lambda_1,n_1)} \leqslant T_{(\lambda_2,1)} \leqslant \cdots \leqslant T,$$

indexed over distinct weights and integers, such that the following conditions hold:

- $\lambda_1, \lambda_2, \ldots$ are distinct weights labelled such that if $\lambda_j < \lambda_k$ then $j < k$;
- $n_1, n_2, \ldots$ are positive integers;
- for each k and r, $T_{(\lambda_k,r)}/T_{(\lambda_k,r-1)} \cong \Delta(\lambda_k)$;

- the following induced filtration on the above subquotient (cf. [9])

$$(T_{(\lambda_k,r)}/T_{(\lambda_k,r-1)})_i = (T_{(\lambda_k,r)} \cap T_i + T_{(\lambda_k,r-1)})/T_{(\lambda_k,r-1)}$$

is a shifted version of the parity filtration, i.e.

$$(T_{(\lambda_k,r)}/T_{(\lambda_k,r-1)})_i \cong \nabla_l(\lambda_k)_{i+m(\lambda_k,r)}$$

for some integer shift $m(\lambda_k, r)$ which weakly increases as r increases.

When using alcoves instead of weights as labels, we will use Weyl filtrations labelled like $\{T_{(C_k,r)}\}$ instead of $T_{(\lambda_k,r)}$, where C_k is the alcove containing λ_k.

Proof of the main theorem. Before we state and prove the main theorem, we will need an auxiliary result regarding indecomposable tilting module endomorphisms.

Lemma 3.5. *Let T be an indecomposable tilting module with highest weight vector v. An endomorphism $\phi : T \to T$ is an isomorphism if and only if $\phi(v) \neq 0$.*

Proof. From the classification of indecomposable tilting modules the highest weight space of T is $\mathbb{C}v$. As T is indecomposable, $\mathrm{End}(T)$ is local. The subspace I of endomorphisms mapping v to 0 is clearly an ideal, and the quotient $\mathrm{End}(T)/I$ is isomorphic to $\mathbb{C}$, so I is the unique maximal ideal of all non-isomorphisms of T. □

Next we develop some language for talking about subquotients of a module. Suppose we have a flag of submodules $W < V < U$. We say that U/V lies above V/W if the extension U/W doesn't split. Otherwise there is a submodule $M \leqslant U$ with $M + V = U$ and $M \cap V = W$. Then we have $U/V = (M + V)/V \cong M/W$ and also $M \leqslant U$, so $U/M = (M + V)/M \cong V/W$, and we can switch the order of the subquotients.

Finally we introduce some convenient notation for Laurent polynomials. Suppose $p = \sum_j p_j v^j \in \mathbb{Z}_{\geqslant 0}[v, v^{-1}]$. For $i \in \mathbb{Z}$ set

- $(p)_i = p_i$;
- $(p)_{\leqslant i} = \sum_{j \leqslant i} p_i$;
- $\{p\}_i = v^j$ if $(p)_{\leqslant j-1} < i \leqslant (p)_{\leqslant j}$ and is zero otherwise;
- $\{p\}_{\leqslant i} = \sum_{j \leqslant i} \{p\}_j$.

In other words, $(-)_i$ and $(-)_{\leqslant i}$ take (sums of) coefficients of terms with degree i or degree at most i respectively, while $\{-\}_i$ and $\{\}_{\leqslant i}$ take the ith monomial or the first i monomials respectively, where the monomials are ordered by degree. For example,

$$\begin{aligned}
(v^{-1} + 2v^2 + 3v^3)_{\leqslant 1} &= 1, & \{v^{-1} + 2v^2 + 3v^3\}_{\leqslant 1} &= v^{-1},\\
(v^{-1} + 2v^2 + 3v^3)_{\leqslant 2} &= 3, & \{v^{-1} + 2v^2 + 3v^3\}_{\leqslant 2} &= v^{-1} + v^2,\\
(v^{-1} + 2v^2 + 3v^3)_{\leqslant 3} &= 6, & \{v^{-1} + 2v^2 + 3v^3\}_{\leqslant 3} &= v^{-1} + 2v^2,\\
(v^{-1} + 2v^2 + 3v^3)_{\leqslant 4} &= 6, & \{v^{-1} + 2v^2 + 3v^3\}_{\leqslant 4} &= v^{-1} + 2v^2 + 2v^3.
\end{aligned}$$

Theorem 3.6. *Let $T = T_l(A)$. There exists a balanced semisimple filtration $\{T_i\}$ of T with Weyl filtration $\{T_{(C_k,r)}\}$ such that*

$$[T^i : L_l(B)] = (t_{B,A})_i, \tag{17}$$

$$[(T_{(C_k,r)}/T_{C_k,r-1})^i : L_l(B)] = (\{n_{C_k,A}\}_r m^{C_k,B})_i. \tag{18}$$

Proof. Write $A = x \cdot A_0$ and induct on $\ell(x)$. The base case is when $A = A_0$ is the fundamental alcove and we have $T_l(A_0) \cong L_l(A_0)$. Pick an isomorphism $\phi : L_l(A_0) \to {}^\tau L_l(A_0)$, which gives a non-degenerate contravariant symmetric form $(-,-)$ [10, II.8.17]. The isotropic filtration in this case is $0 = T_1^\perp \leqslant T_1 = T_l(A)$, which has the properties we want.

For the inductive step, suppose we have shown that the claim holds for all alcoves $y \cot A_0$ where $y < x$ in the Bruhat order, and that we have chosen isomorphisms between these tilting modules and their duals which induce symmetric contravariant forms. Pick a simple reflection $s \in \mathcal{S}$ such that $As > A$ in the dominance ordering. Define $Q = \theta_s(T_l(A))$. Then Q decomposes as $T_l(As) \oplus Q'$ where Q' is a tilting module with highest weights lower than As. Fix an isomorphism $Q \xrightarrow{\sim} T_l(As) \oplus Q'$ once and for all. We will denote $T_l(A)$ by T and $T_l(As)$ by T' for simplicity.

By induction there is a non-degenerate symmetric contravariant form on T and a balanced semisimple filtration $\{T_i\}$ satisfying the claim. Applying the functor θ_s to the form on T gives a form with the same properties on Q. By Lemma 3.2, $\{\theta_s(T_i)\}$ is an isotropic filtration of Q, which we will label $\{Q_i\}$.

Suppose the bottom layer of T is $T_m = 0$ for some $m \leqslant 0$. Consider the submodules $0 = Q_m \leqslant Q_{m+1} \leqslant Q_{m+2}$. These describe a filtration for a summand of the module $\theta_s(T_{m+2})$. Clearly T_{m+2} has Loewy length at most 2, so by Vogan's conjecture $\theta_s(T_{m+2})$ has a Loewy length of at most $2 + 2 = 4$.

Now define Q^+_{m+1} and Q^-_{m+1} such that

$$Q^+_{m+1}/Q_{m+1} \cong \operatorname{soc}(Q_{m+2}/Q_{m+1}),$$
$$Q^-_{m+1}/Q_m \cong \operatorname{rad}(Q_{m+1}/Q_m).$$

As Q^+_{m+1}/Q_{m+1} is semisimple, any composition factor can be written as U/Q_{m+1}, and similarly any composition factor of Q_{m+1}/Q^-_{m+1} can be written Q_{m+1}/W. If there is a composition factor U/Q_{m+1} which lies above Q_{m+1}/W, then the Loewy length of Q_{m+2} is at least 6, which is a contradiction. Thus all such composition factors can be switched, so there exists a module Y which does this, i.e. $Y + Q_{m+1} = Q^+_{m+1}$ and $Y \cap Q_{m+1} = Q^-_{m+1}$ (see Figure 3).

This leaves us with a semisimple filtration $0 = Q_m \leqslant Q^+_m \leqslant Q^-_{m+1} \leqslant Y \leqslant Q^+_{m+1} \leqslant Q^-_{m+2} \leqslant Q_{m+2}$, where we have continued the notation suggested above in the obvious manner. Yet $Y/Q^-_{m+1} \cong Q^+_{m+1}/Q_{m+1}$ and Q^-_{m+1}/Q^+_m have the same Kazhdan-Lusztig parity, so in fact Y/Q^+_m is semisimple. Similarly Q^-_{m+2}/Y is semisimple. With this in mind, we redefine the filtration $\{Q_i\}$ so that its first few lower layers are $0 \leqslant Q^+_m \leqslant Y \leqslant Q^-_{m+2} \leqslant Q_{m+2}$. We continue in this manner up through the lower half of Q, re-indexing as we go along so that all indices are integers. Obviously by taking orthogonal spaces this works for the upper half as well.

By induction $\operatorname{mid}\{T_i\}$ is semisimple. Thus $\operatorname{mid}\{Q_i\} = \theta_s(\operatorname{mid}\{T_i\})$, which is a self-dual module of Loewy length 3 by Vogan's conjecture. If we define V such that $V/Q_{-1} \cong \operatorname{rad}(Q_1/Q_{-1})$ then we have $Q_1/V \cong \operatorname{head}(Q_1/Q_{-1})$ so $V^\perp/Q_{-1} \cong \operatorname{soc}(Q_1/Q_{-1})$, and $V \geqslant V^\perp$. Thus $V^\perp$ is a larger totally isotropic submodule of Q, so we can redefine Q_1 and Q_{-1} to be V and $V^\perp$ respectively. The resulting filtration after all these changes has layers given by (15), i.e. $[Q^i : L_l(B)] = (t^s_{B,A})_i$ for any integer i and any alcove B.

The module Q naturally has a Weyl filtration because T does, which we label $Q_{(C_k,r)}$. Recall that if $\operatorname{Ext}^1(\Delta_l(C), \Delta_l(D)) \neq 0$ then $C < D$. This means we can

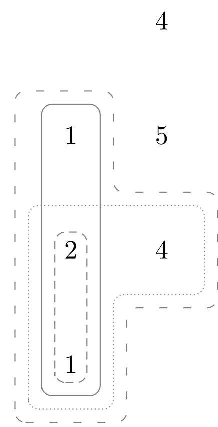

FIGURE 3. An illustration of a possible Loewy series for Q_{m+2}. As in the example in the introduction, the numbers are composition factors. The Kazhdan-Lusztig parity of a factor corresponds to the parity of the number labeling it. The submodule Q_{m+1} is circled with a solid line, while Q^-_{m+1} and Q^+_{m+1} are circled with dashed lines and the submodule Y is circled with a dotted line.

rearrange and relabel the Weyl factors (as described in the beginning of this section) so that they have the same ordering properties as in Definition 3.4. We claim that $Q_{(C_k,r)} \cap Q_i$ has the following character[1] based on a "partial" version of $t^s_{B,A}$:

$$[Q_{(C_k,r)} \cap Q_i : L_l(B)] = \tag{19}$$
$$= \left((v+v^{-1}) \sum_{j \leqslant k} \{n_{C_j,A}\}_{\leqslant r} \overline{m^{C_j,B}} + \sum_{\substack{j \leqslant k \\ D}} \{n_{C_j,A}\}_{\leqslant r} \overline{m^{C_j,D}} \overline{m^s_{B,D}}(0) \right)_{\leqslant i} .$$

To see this, note that a similar result holds for the original filtration on Q, since it was a wall-crossed version of a balanced semisimple filtration on T. The modifications made to this filtration don't change the fact that composition factors in the layers Q^i can be identified as belonging to some Weyl subquotient.

[1]We have implicitly assumed positivity of various Kazhdan-Lusztig polynomials. For Weyl groups and affine Weyl groups this follows from geometric interpretations of these polynomials first shown in [13].

The induced filtration on $Q_{(\lambda_k,r)}/Q_{(\lambda_k,r-1)}$ has ith layer

$$\begin{aligned}\frac{(Q_i \cap Q_{(\lambda_k,r)} + Q_{(\lambda_k,r-1)})/Q_{(\lambda_k,r-1)}}{(Q_{i-1} \cap Q_{(\lambda_k,r)} + Q_{(\lambda_k,r-1)})/Q_{(\lambda_k,r-1)}} &\cong \\ &\cong \frac{Q_i \cap Q_{(\lambda_k,r)} + Q_{(\lambda_k,r-1)}}{Q_{i-1} \cap Q_{(\lambda_k,r)} + Q_{(\lambda_k,r-1)}} \\ &\cong \frac{Q_i \cap Q_{(\lambda_k,r)}}{(Q_i \cap Q_{(\lambda_k,r)}) \cap (Q_{i-1} \cap Q_{(\lambda_k,r)} + Q_{(\lambda_k,r-1)})} \\ &= \frac{Q_i \cap Q_{(\lambda_k,r)}}{Q_i \cap Q_{(\lambda_k,r)} \cap Q_{(\lambda_k,r-1)} + Q_{i-1} \cap Q_{(\lambda_k,r)}} \\ &= \frac{Q_i \cap Q_{(\lambda_k,r)}}{Q_i \cap Q_{(\lambda_k,r-1)} + Q_{i-1} \cap Q_{(\lambda_k,r)}}.\end{aligned}$$

Now we calculate the character of the denominator in the above quotient:

$$\begin{aligned}[Q_i \cap Q_{(\lambda_k,r-1)} + Q_{i-1} \cap Q_{(\lambda_k,r)}] &= \\ &= [Q_i \cap Q_{(\lambda_k,r-1)}] + [Q_{i-1} \cap Q_{(\lambda_k,r)}] \\ &\quad - [(Q_i \cap Q_{(\lambda_k,r-1)}) \cap (Q_{i-1} \cap Q_{(\lambda_k,r)})] \\ &= [Q_i \cap Q_{(\lambda_k,r-1)}] + [Q_{i-1} \cap Q_{(\lambda_k,r)}] - [Q_{i-1} \cap Q_{(\lambda_k,r-1)}].\end{aligned}$$

Using (19), the character of this ith layer is

$$\begin{aligned}&\left((v+v^{-1})\sum_{j\leqslant k}\{n_{C_j,A}\}_{\leqslant r}\overline{m^{C_j,B}} + \sum_{\substack{j\leqslant k\\ D}}\{n_{C_j,A}\}_{\leqslant r}\overline{m^{C_j,D}}\,\overline{m^s_{B,D}}(0)\right)_{\leqslant i} \\ &- \left((v+v^{-1})\sum_{j\leqslant k}\{n_{C_j,A}\}_{\leqslant r-1}\overline{m^{C_j,B}} + \sum_{\substack{j\leqslant k\\ D}}\{n_{C_j,A}\}_{\leqslant r-1}\overline{m^{C_j,D}}\,\overline{m^s_{B,D}}(0)\right)_{\leqslant i} \\ &- \left((v+v^{-1})\sum_{j\leqslant k}\{n_{C_j,A}\}_{\leqslant r}\overline{m^{C_j,B}} + \sum_{\substack{j\leqslant k\\ D}}\{n_{C_j,A}\}_{\leqslant r}\overline{m^{C_j,D}}\,\overline{m^s_{B,D}}(0)\right)_{\leqslant i-1} \\ &+ \left((v+v^{-1})\sum_{j\leqslant k}\{n_{C_j,A}\}_{\leqslant r-1}\overline{m^{C_j,B}} + \sum_{\substack{j\leqslant k\\ D}}\{n_{C_j,A}\}_{\leqslant r-1}\overline{m^{C_j,D}}\,\overline{m^s_{B,D}}(0)\right)_{\leqslant i-1} \\ &= (\{n^s_{C_k,A}\}_r\overline{m^{C_k,B}})_i(1),\end{aligned} \tag{20}$$

which is a shifted version of the dual parity filtration.

Now we will obtain analogous results for the direct summand T' of Q. First note that the restriction of the bilinear form on Q to T' is non-degenerate if and only if the map

$$\begin{aligned}T' &\longrightarrow {}^\tau T' \\ v &\longmapsto (v,-)\end{aligned}$$

is an isomorphism. In the case of the above map, this is readily apparent: for v_{As} a highest weight vector of T' (and therefore of Q) we have $(v_{As}, Q_\lambda) = 0$ for all λ below the highest weight, so $(v_{As}, v_{As}) \neq 0$ as the form is non-degenerate on Q. As $T' \cap T'^{\perp} = 0$, this implies that Q is isomorphic to $T' \oplus T'^{\perp}$ as a vector space. But $T'^{\perp}$ is a submodule isomorphic to $Q/T' \cong Q'$ so without loss of generality $Q' = T'^{\perp}$ and the form is non-degenerate on Q' too. Let $\pi_{T'}, \pi_{Q'}$ be the projection maps onto the two summands of Q. We say a subquotient U/V lies entirely in T' if $\pi_{T'}(U)/\pi_{T'}(V) \cong U/V$ and $\pi_{Q'}(U) = \pi_{Q'}(V)$.

We will modify each Weyl factor to lie entirely in either T' or Q'. Recall that the filtration shift of the Weyl factor $Q_{(C_k,r)}/Q_{(C_k,r-1)}$ is the smallest i such that $Q_{(C_k,r)} \leqslant Q_i$. From (20) this corresponds to the degree of some monomial term in $n^s_{C_k,A}$. These terms can be divided into those which come from $n_{C_k,As}$ and those which don't, corresponding to Weyl factors lying in T' and Q' respectively.

Consider the first Weyl factor $Q_{(C_1,1)}$. It has to be isomorphic to the highest Weyl factor $\Delta_l(As)$. From highest weight theory $\mathrm{Hom}(\Delta_l(As), Q') = 0$, so $\pi_{Q'}(Q_{(C_1,1)}) = 0$ and thus $Q_{(C_1,1)} \leqslant T'$. The quotient $Q/Q_{(C_1,1)}$ still has a Weyl filtration, and we induct on the number of Weyl factors. Suppose the quotient $Q/Q_{(C_k,r-1)}$ has bottom Weyl factor $Q_{(C_k,r)}/Q_{(C_k,r-1)}$. In general if one of T' or Q' doesn't have $\Delta_l(C_k)$ as a factor, then the same trick still works.

Otherwise, suppose this bottom Weyl factor has filtration shift i, and both T' and Q' contain copies of $\Delta_l(C_k)$ but only one of $n_{C_k,As}$ and $n^s_{C_k,A} - n_{C_k,As}$ has a non-zero coefficient of v^i. Then the Weyl factor lies entirely in T' or Q' respectively. To see this, note that the top simple factor $L_l(C_k)$ in this copy of $\Delta_l(C_k)$ corresponds to a summand in Q^i, and is dual to a summand in Q^{-i}. By induction and using Lemma 2.6 this summand in Q^{-i} lies entirely in only one of T'^{-i} or Q'^{-i}, so by non-degeneracy the top summand of the Weyl factor lies entirely in either T'^i or Q'^i, which implies that the whole Weyl factor does too.

Finally suppose both T' and Q' contain copies of $\Delta_l(C_k)$ and both $n_{C_k,As}$ and $n^s_{C_k,A} - n_{C_k,As}$ have non-zero coefficient of v^i. Pick $s > r$ maximal such that the submodule $Q_{(C_k,s)}/Q_{(C_k,r)}$ is isomorphic as a filtered module to a direct sum of copies of $\Delta_l(C_k)$ all shifted by i. Clearly all indecomposable direct summands are filtration isomorphic, so one can choose a new direct sum decomposition of this module so that each summand lies entirely in one of T' or Q'. The number of summands lying in each also corresponds to the coefficient of v^i in each of the above polynomials, using a similar argument to the previous case. Thus T' has a balanced semisimple filtration whose filtration layers are given by (17). □

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Department of Pure Mathematics and Mathematical Statistics, Centre for Mathematical Sciences, University of Cambridge, Wilberforce Road, Cambridge, CB3 0WB, United Kingdom

E-mail address: `A.Hazi@dpmms.cam.ac.uk`