INDECOMPOSABLE TILTING MODULES FOR THE BLOB ALGEBRA

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ABSTRACT. The blob algebra is a finite-dimensional quotient of the Hecke algebra of type B which is almost always quasi-hereditary. We construct the indecomposable tilting modules for the blob algebra over a field of characteristic 0 in the doubly critical case. Every indecomposable tilting module of maximal highest weight is either a projective module or an extension of a simple module by a projective module. Moreover, every indecomposable tilting module is a submodule of an indecomposable tilting module of maximal highest weight. We conclude that the graded Weyl multiplicities of the indecomposable tilting modules in this case are given by inverse Kazhdan–Lusztig polynomials of type $\tilde{A}_1$.

Key words and phrases: blob algebra, tilting modules, KLR algebra, Soergel bimodules.

INTRODUCTION

The blob algebra is an extension of the ordinary Temperley–Lieb algebra introduced by the second author and Saleur in [14]. It can be thought of as the Temperley–Lieb algebra of type B, as it is a quotient of the type B Hecke algebra in much the same way as the ordinary Temperley–Lieb algebra is a quotient of the Hecke algebra of type A. Originally motivated by the need to control lattice boundary conditions in lattice models in statistical mechanics, the blob algebra and its generalizations remain an active topic of research in both physics (e.g. [9, 8, 7]) and representation theory (e.g. [18, 19, 1]).

Like the ordinary Temperley–Lieb algebra, the representation theory of the blob algebra is controlled by the values of its parameters. Generically the blob algebra is semisimple, with certain integral representations $\Delta(\boldsymbol{\lambda})$ called *Weyl modules* giving a complete set of simple modules. Yet for some critical parameter values, the blob algebra is only quasi-hereditary, and the Weyl modules are no longer simple. In this paper we focus on the *doubly critical* case, when the representation theory is the most interesting (e.g. with blocks of arbitrary size, no known quiver-and-relations presentation, etc.). In this case, the block structure is controlled by a linkage principle in terms of an affine Weyl group W of type $\tilde{A}_1$.

Recall that a *tilting module* for a quasi-hereditary algebra is a representation with a filtration by Weyl modules as well as a filtration by dual Weyl modules. For each weight $\boldsymbol{\lambda}$, there is an indecomposable tilting module $T(\boldsymbol{\lambda})$ of highest weight $\boldsymbol{\lambda}$, and all indecomposable tilting modules are of this form. Our main result in this paper is a construction of $T(\boldsymbol{\lambda})$ for the doubly critical blob algebra B_n^κ over a field of characteristic 0. The construction closely depends on the quasi-hereditary partial order $\trianglelefteq$ on weights, defined in §1.3. The W-orbit of $\boldsymbol{\lambda}$ has one maximal weight $\boldsymbol{\lambda}_{\max}$ and at most two minimal weights with respect to $\trianglelefteq$. We write $L(\boldsymbol{\lambda})$ for the simple head of $\Delta(\boldsymbol{\lambda})$, $P(\boldsymbol{\lambda})$ for the projective indecomposable cover of $L(\boldsymbol{\lambda})$, and $O_{\trianglelefteq\boldsymbol{\lambda}}(M)$ for the maximal submodule of a module M whose composition factors lie in $\{L(\boldsymbol{\mu}) : \boldsymbol{\mu} \trianglelefteq \boldsymbol{\lambda}\}$. Using this notation, our construction is as follows (see also Theorems 5.3 and 5.4).

Theorem. *Suppose $\boldsymbol{\lambda}$ is a weight for B_n^κ. Let $\boldsymbol{\lambda}_{\min}$ be a minimal weight in the W-orbit of $\boldsymbol{\lambda}$. Then $T(\boldsymbol{\lambda}) = O_{\trianglelefteq\boldsymbol{\lambda}}(T(\boldsymbol{\lambda}_{\max}))$. The maximal highest weight tilting module $T(\boldsymbol{\lambda}_{\max})$ is constructed from $P(\boldsymbol{\lambda}_{\min})$ as follows.*

(i) *If $\boldsymbol{\lambda}_{\min}$ is the only minimal weight in the W-orbit of $\boldsymbol{\lambda}$, then $T(\boldsymbol{\lambda}_{\max}) = P(\boldsymbol{\lambda}_{\min})$.*

(ii) *If there is another minimal weight $\boldsymbol{\lambda}'_{\min}$ in the W-orbit of $\boldsymbol{\lambda}$, then $T(\boldsymbol{\lambda}_{\max})$ is the unique extension of the form*

$$0 \to P(\boldsymbol{\lambda}_{\min}) \to T(\boldsymbol{\lambda}_{\max}) \to \Delta(\boldsymbol{\lambda}'_{\min}) \to 0.$$
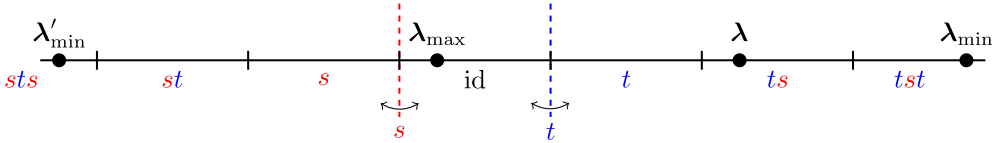
FIGURE 1. The (classical) weights $\boldsymbol{\lambda}_{\max}$, $\boldsymbol{\lambda}_{\min}$, and $\boldsymbol{\lambda}'_{\min}$, with alcoves labelled by $w_{\boldsymbol{\lambda}}$.

For $x, y \in W$, write

$$h^{x,y}(v) = \begin{cases} v^{\ell(x)-\ell(y)} & \text{if } y \leq x, \\ 0 & \text{otherwise,} \end{cases}$$

which is the inverse Kazhdan–Lusztig polynomial of type $\tilde{A}_1$. Using the decomposition numbers for B_n^κ (first calculated in [16]), our construction implies the following Weyl multiplicities for the regular indecomposable tilting modules (see also Corollary 5.5). Here for each regular weight $\boldsymbol{\lambda}$, let $w_{\boldsymbol{\lambda}} \in W$ such that $w_{\boldsymbol{\lambda}}(\boldsymbol{\lambda}_{\max}) = \boldsymbol{\lambda}$.

Theorem. *Let $\boldsymbol{\lambda}, \boldsymbol{\mu}$ be regular weights for B_n^κ. Then*

$$(T(\boldsymbol{\mu}) : \Delta(\boldsymbol{\lambda})) = h^{w_{\boldsymbol{\lambda}}, w_{\boldsymbol{\mu}}}(1).$$

See Figure 1 for an example depicting the weight and alcove labels used in these theorems.

Our proofs depends in a crucial way not only on the decomposition numbers and structure of the Weyl modules from [16], but also on the *graded* representation theory of the blob algebra. The existence of a non-trivial 'hidden' grading on the blob algebra is a consequence of the Brundan–Kleshchev isomorphism [2] between cyclotomic Hecke algebras and KLR algebras, which are graded. (This explains why previous work such as [15, 20] on full tilting modules did not get very close to determining the indecomposable tilting modules.) As a bonus we obtain the graded Weyl multiplicities of the graded indecomposable tilting modules with no extra work. Our result is perhaps the first example of how the hidden grading on the blob algebra can be used to solve problems which a priori are not graded at all.

We also make extensive use of KLR diagrammatics for the KLR presentation of the blob algebra, as described in [12]. The classical diagrammatic calculus for the blob algebra in terms of 'Temperley–Lieb diagrams with blobs' gives a cellular basis which is integral and multiplicative. However, it is difficult in general to describe the simple modules in terms of this basis. By contrast, KLR algebras have a complicated diagram calculus reflecting the KLR presentation, in which certain fixed parameter values are 'built-in' and cannot be changed. On the other hand, KLR diagrams give more information about the structure of projective modules, in particular whether certain composition factors (or extensions between composition factors) are present. Fortunately for us, we will only need a simplified (but still complicated) version of the KLR diagram calculus.

Much of this machinery applies, at least in principle, to the generalised blob algebras (cf. e.g. [1], [17], [12]). For example, the level l generalised blob algebras are controlled by an affine Weyl group W_l of type $\tilde{A}_{l-1}$, and there is a corresponding KLR presentation. For $\boldsymbol{\lambda}$ a regular weight for the level l generalised blob algebra and $\boldsymbol{\lambda}_{\max}$ maximal in the W_l-orbit of $\boldsymbol{\lambda}$, let $w_{\boldsymbol{\lambda}} \in W_l$ to be the unique element in the affine Weyl group such that $w_{\boldsymbol{\lambda}}(\boldsymbol{\lambda}_{\max}) = \boldsymbol{\lambda}$. For $x, y \in W_l$, write $h^{x,y}$ for the inverse Kazhdan–Lusztig polynomial of type $\tilde{A}_{l-1}$. The following conjecture is the natural extension of our Weyl multiplicities result.

Conjecture. *Let $\boldsymbol{\lambda}, \boldsymbol{\mu}$ be weights for the level l generalised blob algebra over a field of characteristic 0. Then*

$$(T(\boldsymbol{\mu}) : \Delta(\boldsymbol{\lambda})) = h^{w_{\boldsymbol{\lambda}}, w_{\boldsymbol{\mu}}}(1).$$

The biggest obstacle to proving this conjecture is the lack of knowledge about the (graded) structure of the Weyl modules and the projective modules in higher levels. In the modular setting, it is not immediately obvious what should replace the inverse Kazhdan–Lusztig polynomials $h^{x,y}$ above, although we have some ideas (see Remark 5.7) based on the 'Blob vs Soergel' conjecture of Libedinsky–Plaza [12].

The layout of the paper is as follows. In §1 we define the doubly critical blob algebra B_n^κ using the KLR presentation and describe the corresponding weight combinatorics. In §2 we summarise the quasi-hereditary representation theory of B_n^κ. In §3 we exploit the KLR presentation to obtain bases for the indecomposable projective modules and their composition factors. In §4 we get to work with KLR diagrammatic calculations which give the main result in the case of singular weights. Finally in §5 we use the singular version to prove the main result for all weights.

1. PRELIMINARIES: THE BLOB ALGEBRA B_n^κ

Suppose $e > 1$ is an integer and let $I = \mathbb{Z}/e\mathbb{Z}$. An *adjacency-free bicharge* is an ordered pair $\kappa = (\kappa_1, \kappa_2) \in I^2$ such that $\kappa_1 \neq \kappa_2, \kappa_2 \pm 1$ (this implicitly requires $e \geq 4$). For $i \in I$ define

$$\langle i | \kappa \rangle = \begin{cases} 1 & \text{if } i = \kappa_1 \text{ or } i = \kappa_2, \\ 0 & \text{otherwise.} \end{cases}$$

For any $n \in \mathbb{N}$, the symmetric group S_n acts on the set of tuples I^n by permutation. We write s_r for the simple transposition $(r\ r+1)$ in the symmetric group S_n.

Definition 1.1. Let $\Bbbk$ be a field, $n, e \in \mathbb{N}$, and κ be an adjacency-free bicharge. The (doubly critical) *blob algebra* B_n^κ over $\Bbbk$ is the $\mathbb{Z}$-graded $\Bbbk$-algebra generated by

(1) $\qquad\qquad\qquad \psi_r \qquad\qquad\qquad\qquad\qquad$ for $1 \leq r \leq n - 1$,

(2) $\qquad\qquad\qquad y_r \qquad\qquad\qquad\qquad\qquad$ for $1 \leq r \leq n$,

(3) $\qquad\qquad\qquad e(\boldsymbol{i}) \qquad\qquad\qquad\qquad\qquad$ for $\boldsymbol{i} \in I^n$,

subject to relations

(4) $\qquad e(\boldsymbol{i})e(\boldsymbol{j}) = \delta_{\boldsymbol{i},\boldsymbol{j}}e(\boldsymbol{i}) \qquad\qquad\qquad\qquad\qquad$ for all $\boldsymbol{i}, \boldsymbol{j} \in I^n$

(5) $\qquad \sum_{\boldsymbol{i} \in I^n} e(\boldsymbol{i}) = 1$

(6) $\qquad y_r e(\boldsymbol{i}) = e(\boldsymbol{i})y_r$

(7) $\qquad \psi_r e(\boldsymbol{i}) = e(s_r\boldsymbol{i})\psi_r$

(8) $\qquad y_r y_s = y_s y_r$

(9) $\qquad \psi_r y_s = y_s \psi_r \qquad\qquad\qquad\qquad\qquad\qquad$ when $s \neq r, r + 1$

(10) $\qquad \psi_r \psi_s = \psi_s \psi_r \qquad\qquad\qquad\qquad\qquad\qquad$ when $|r - s| > 1$

(11) $\qquad \psi_r y_{r+1} e(\boldsymbol{i}) = (y_r \psi_r - \delta_{i_r, i_{r+1}})e(\boldsymbol{i})$

(12) $\qquad y_{r+1}\psi_r e(\boldsymbol{i}) = (\psi_r y_r - \delta_{i_r, i_{r+1}})e(\boldsymbol{i})$

(13) $\qquad\quad \psi_r^2 e(\boldsymbol{i}) = \begin{cases} e(\boldsymbol{i}) & \text{if } i_{r+1} \neq i_r, i_r \pm 1 \\ 0 & \text{if } i_{r+1} = i_r \\ (y_{r+1} - y_r)e(\boldsymbol{i}) & \text{if } i_{r+1} = i_r + 1 \\ (y_r - y_{r+1})e(\boldsymbol{i}) & \text{if } i_{r+1} = i_r - 1 \end{cases}$

(14) $\quad \psi_r \psi_{r+1} \psi_r e(\boldsymbol{i}) = \begin{cases} (\psi_{r+1}\psi_r\psi_{r+1} - 1)e(\boldsymbol{i}) & \text{if } i_{r+2} = i_r = i_{r+1} - 1 \\ (\psi_{r+1}\psi_r\psi_{r+1} + 1)e(\boldsymbol{i}) & \text{if } i_{r+2} = i_r = i_{r+1} + 1 \\ \psi_{r+1}\psi_r\psi_{r+1}e(\boldsymbol{i}) & \text{otherwise} \end{cases}$

(15) $\qquad y_1^{\langle i_1 | \kappa \rangle} e(\boldsymbol{i}) = 0$

(16) $\qquad\quad e(\boldsymbol{i}) = 0 \qquad\qquad\qquad\qquad\qquad\qquad$ when $i_2 = i_1 + 1$

and a grading defined by

$$\deg e(\boldsymbol{i}) = 0, \qquad \deg y_r e(\boldsymbol{i}) = 2, \qquad \deg \psi_r e(\boldsymbol{i}) = \begin{cases} 1 & \text{if } i_{r+1} = i_r \pm 1, \\ -2 & \text{if } i_{r+1} = i_r, \\ 0 & \text{otherwise.} \end{cases}$$

In the presentation in Definition 1.1, each $e(\boldsymbol{i})$ is a (non-central) idempotent, each ψ_r is analogous to the simple transposition s_r in the symmetric group S_n, and each y_r is akin to the nilpotent part of the corresponding Jucys–Murphy element in the symmetric group algebra $\Bbbk S_n$.

There is also a presentation of this algebra in terms of *KLR diagrams* [12, § 3.2]. A KLR diagram with n strings consists of n paths of the form $p : [0,1] \to \mathbb{R} \times [0,1]$ satisfying the following properties:

- for each path p we have $p(0) = (x, 0)$ and $p_r(1) = (x', 1)$ for some $x, x' \in \mathbb{R}$;
- all intersections are transversal;
- there are no triple intersections;
- each path may be decorated with a finite number of dots at non-intersection points.

Each path p is also labelled with a residue $i \in I$.

We consider KLR diagrams up to isotopy; in other words, we are allowed to move these paths continuously as long as the properties above still hold and no intersections are added or removed. The *bottom* (resp. *top*) of a KLR diagram is the sequence of residues labelling the paths, ordered by the relation $p \prec p'$ if $p(0) < p'(0)$ (resp. $p \prec p'$ if $p(1) < p'(1)$). The product of two diagrams D and D' is defined to be their vertical concatenation (with D on top of D') whenever the bottom of D equals the top of D'. Otherwise the product is defined to be 0. The diagrammatic blob algebra B_n^κ is then the set of all $\Bbbk$-linear combinations of KLR diagrams with n strings, with a diagrammatic product defined by $\Bbbk$-linear extension, subject to the following relations:

in all regions of a KLR diagram, where $\alpha = 1$ when $i = k = j - 1$, $\alpha = -1$ when $i = k = j + 1$, and $\alpha = 0$ otherwise, as well as the relations

$$\vcenter{\hbox{(diagram with filled dot on first string)}} \quad \cdots \quad = 0, \qquad \text{if } i_1 = \kappa_j \text{ for some } j,$$
$$i_1 \quad i_2 \qquad i_n$$

$$\vcenter{\hbox{(diagram)}} \quad \cdots \quad = 0, \qquad \text{if } i_1 \neq \kappa_j \text{ for all } j,$$
$$i_1 \quad i_2 \qquad i_n$$

$$\vcenter{\hbox{(diagram)}} \quad \cdots \quad = 0, \qquad \text{if } i_2 = i_1 + 1.$$
$$i_1 \quad i_2 \qquad i_n$$

If $\underline{w} = s_{r_1} s_{r_2} \cdots s_{r_k}$ is a reduced expression in S_n, we write $\psi_{\underline{w}} = \psi_{r_1} \psi_{r_2} \cdots \psi_{r_k}$ for the product of the corresponding ψ-generators. Diagrammatically $\psi_{\underline{w}}$ (or more precisely, $\psi_{\underline{w}} e(\boldsymbol{i})$ for some $\boldsymbol{i} \in I^n$) looks like the wiring diagram for $\underline{w}$. We also write $(\bar{\ })$ for the unique anti-involution which fixes each of the generators ψ_r, y_r, and $e(\boldsymbol{i})$.

1.1. Locality.

We call a relation in the generators of B_n^κ *local* if the relation still holds when the indices of the generators are shifted by some amount. All the relations in Definition 1.1 above are local except for (15) and (16). The relation (15) is also the only one in which κ appears. Incidentally it is immediately clear that all other relations do not depend on precise values of sequences $\boldsymbol{i} \in I^n$ indexing the idempotents, but only on relative differences $i_{r+1} - i_r$ for some integer $1 \leq r \leq n$. In fact for any $i \in I$, if $\kappa' = (\kappa_1 + i, \kappa_2 + i)$ then we have $B_n^\kappa \cong B_n^{\kappa'}$, and this isomorphism maps $e(\boldsymbol{i}) \mapsto e(\boldsymbol{i} + (i, \ldots, i))$. Thus B_n^κ only depends on the difference $\kappa_1 - \kappa_2 \in I$ up to isomorphism.

When simplifying KLR diagrams we adopt the convention of circling regions in some colour wherever we apply a local relation only involving ψ-generators. These circles are only a helpful annotation and should *not* be considered an intrinsic part of the diagram. Similarly whenever we apply relations (11) or (12) in the distinct residue case, we will draw a coloured arrow parallel to the string to indicate how the y-generator 'slides' along the string. The most important non-local relation which we will use takes the following form.

Lemma 1.2. *Let $\boldsymbol{i} \in I^n$ and $1 \leq r \leq n - 1$ be an integer such that $|i_r - i_{r+1}| = 1$ but $e(s_r \boldsymbol{i}) = 0$ in B_n^κ. Then $y_{r+1} e(\boldsymbol{i}) = y_r e(\boldsymbol{i})$.*

Proof. Apply (13) to obtain

$$y_{r+1} e(\boldsymbol{i}) = y_r e(\boldsymbol{i}) \pm \psi_r^2 e(\boldsymbol{i}) = y_r e(\boldsymbol{i}) \pm \psi_r e(s_r \boldsymbol{i}) \psi_r = y_r e(\boldsymbol{i}).$$

□

When applying Lemma 1.2 to a KLR diagram, we will draw a dashed coloured line transverse to the strings to indicate which idempotent $e(\boldsymbol{i})$ we are using, and a coloured arrow to show where the y-generator 'jumps' to a different string.

1.2. The classical blob algebra.

Definition 1.1 presents the blob algebra as a quotient of a cyclotomic KLR algebra as in [19], with the same generators and all the same relations plus the extra relation (16). This does not correspond to the original definition of the blob algebra in [14] as an extension of the Temperley–Lieb algebra. However, our definition is equivalent in many cases due to the Brundan–Kleshchev isomorphism [2, Theorem 1.1] between cyclotomic KLR algebras and cyclotomic Hecke algebras.

Theorem 1.3 ([19, Corollary 3.6]). *Suppose $e > 1$ is an integer which is not a multiple of the characteristic of $\Bbbk$. Let m be an integer with $1 < m < e - 1$. Set $\kappa = (0, m)$, an adjacency-free bicharge. Then B_n^κ has a presentation as an ungraded algebra over $\Bbbk$, with generators U_r for $0 \leq r \leq n - 1$ subject to the following relations:*

$$U_r^2 = -[2]U_r \qquad\qquad \textit{if } 1 \leq r \leq n-1,$$
$$U_r U_s U_r = U_r \qquad\qquad \textit{if } |r - s| = 1 \textit{ and } 1 \leq r, s \leq n-1,$$
$$U_r U_s = U_s U_r \qquad\qquad \textit{if } |r - s| > 1 \textit{ and } 0 \leq r, s \leq n-1,$$
$$U_1 U_0 U_1 = [m+1]U_1,$$
$$U_0^2 = -[m]U_0,$$

where $[k] = [k]_q = q^{-k+1} + q^{-k+3} + \cdots + q^{k-1}$, q is an e'th primitive root of unity in $\Bbbk$, and

$$e' = \begin{cases} 2e & \textit{if } e \textit{ is even}, \\ e & \textit{otherwise}. \end{cases}$$

Remark 1.4.

(1) The statement of [19, Corollary 3.6] uses the bicharge $\kappa = (k, -k)$ (where $k \in I$ such that $2k \equiv m \pmod{n}$) and a 'negative variant' form of (16). To transform this into Theorem 1.3 it is necessary to shift the residues by $-k$ (as mentioned in §1.1) and apply the isomorphism

$$R_n^{(0,-m)} \longrightarrow R_n^{(0,m)}$$
$$\psi_r \longmapsto -\psi_r$$
$$y_r \longmapsto -y_r$$
$$e(\boldsymbol{i}) \longmapsto e(-\boldsymbol{i})$$

of cyclotomic KLR algebras with bicharges $(0, -m)$ and $(0, m)$.

(2) Theorem 1.3 is the most general version of what is commonly stated in the literature, but it can probably be extended to other cases as well. For example, when e equals the characteristic of $\Bbbk$, B_n^κ behaves like the classical blob algebra over $\Bbbk$ with $q = 1$. In addition, adjacency-freeness of κ and the condition that $1 < m < e-1$ can potentially be relaxed, at the cost of modifying relation (16) (this is similar to what happens for the Temperley–Lieb algebra [19, Remark 3.7]).

1.3. **Weights and multipartitions.** In general the representation theory of KLR algebras is governed by the combinatorics of multipartitions, while that of the blob algebra is naturally governed by the geometry of a suitable weight lattice [17]. To understand the blob algebra in KLR terms it is enough to focus on one-column bipartitions.

A *one-column bipartition* of n is an ordered pair $\boldsymbol{\lambda} = (1^{\lambda_1}, 1^{\lambda_2})$ with $\lambda_1, \lambda_2 \in \mathbb{Z}_{\geq 0}$ and $\lambda_1 + \lambda_2 = n$. We write $\Lambda(n)$ for the set of all one-column bipartitions of n. The mapping

$$\Lambda(n) \longrightarrow \{-n, -n+2, \ldots, n-2, n\}$$
$$\boldsymbol{\lambda} \longmapsto \lambda_1 - \lambda_2$$

is a bijection between one-column bipartitions and the classical weight set for the blob algebra. For this reason we will usually call one-column bipartitions *weights* when working in a representation-theoretic context. For two weights $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \Lambda(n)$ we write $\boldsymbol{\lambda} \lhd \boldsymbol{\mu}$ (and say $\boldsymbol{\mu}$ dominates $\boldsymbol{\lambda}$) if $|\lambda_1 - \lambda_2| > |\mu_1 - \mu_2|$ (following [16]).

The *Young diagram* for $\boldsymbol{\lambda} \in \Lambda(n)$ is defined to be the set

$$[\boldsymbol{\lambda}] = \{(r, 1) : 1 \leq r \leq \lambda_1\} \cup \{(r, 2) : 1 \leq r \leq \lambda_2\}$$

Elements of this set are usually called *boxes*, because the traditional way to depict Young diagrams is as a collection of boxes, e.g.

$$[(1^4, 1^5)] = \left(\begin{array}{c} \square \\ \square \\ \square \\ \square \end{array}, \begin{array}{c} \square \\ \square \\ \square \\ \square \\ \square \end{array} \right)$$

A *tableau* of shape $\boldsymbol{\lambda}$ is a bijection $[\boldsymbol{\lambda}] \to \{1, 2, \ldots, n\}$, which is usually depicted by writing each assignment inside the corresponding box, e.g.

$$\left(\begin{array}{c} \boxed{1} \\ \boxed{3} \\ \boxed{4} \\ \boxed{9} \end{array}, \begin{array}{c} \boxed{2} \\ \boxed{5} \\ \boxed{6} \\ \boxed{7} \\ \boxed{8} \end{array} \right)$$

A tableau is called *standard* if the entries in the boxes increase going down each column. A standard tableau $\mathfrak{t}$ corresponds in a natural way to a sequence $\mathfrak{t}|_k \in \Lambda(k)$ of Young diagrams obtained by adding exactly one box at each stage. Such sequences are in bijection with paths of length n on the global lattice of weights $\mathbb{Z}$, where a path is just a function $p : \{0, 1, 2, \ldots, n\} \to \mathbb{Z}$ with $p(0) = 0$ and $p(k+1) - p(k) = \pm 1$ for all integers $0 \le k \le n-1$. Adding a box in the first column corresponds to a rightward $(+1)$ step and vice versa.

We write $\mathfrak{t}^{\boldsymbol{\lambda}}$ for the standard tableau of shape $\boldsymbol{\lambda}$ obtained by labelling the boxes of $[\boldsymbol{\lambda}]$ with increasing entries ordered from left to right and from top to bottom like a book, e.g.

$$\mathfrak{t}^{(1^4, 1^5)} = \left(\begin{array}{c} \boxed{1} \\ \boxed{3} \\ \boxed{5} \\ \boxed{7} \end{array}, \begin{array}{c} \boxed{2} \\ \boxed{4} \\ \boxed{6} \\ \boxed{8} \\ \boxed{9} \end{array} \right)$$

The $(\kappa$-$)$*residue* of a box with coordinates (r, m) is defined to be $\kappa_m + 1 - r \in I$. The *residue sequence* $\boldsymbol{i}^{\mathfrak{t}}$ of a tableau $\mathfrak{t}$ is the sequence of residues of the boxes $(\mathfrak{t}^{-1}(1), \mathfrak{t}^{-1}(2), \ldots, \mathfrak{t}^{-1}(n))$. We write $\boldsymbol{i}^{\boldsymbol{\lambda}}$ instead of $\boldsymbol{i}^{\mathfrak{t}^{\boldsymbol{\lambda}}}$ for the residue sequence of the dominant tableau $\mathfrak{t}^{\boldsymbol{\lambda}}$.

2. Cellularity of B_n^κ

Suppose $\mathfrak{t}$ is a standard tableau of shape $\boldsymbol{\lambda}$. Let $d_{\mathfrak{t}} \in S_n$ be the permutation such that $d_{\mathfrak{t}} \mathfrak{t}^{\boldsymbol{\lambda}} = \mathfrak{t}$.

Theorem 2.1 ([19, Theorem 6.8]). *Fix a reduced expression $\underline{d}_{\mathfrak{t}}$ for each $d_{\mathfrak{t}}$ over all $\boldsymbol{\lambda} \in \Lambda(n)$ and $\mathfrak{t} \in \mathrm{Std}(\boldsymbol{\lambda})$. The elements*

$$\psi_{\mathfrak{s}\mathfrak{t}} = \psi_{\underline{d}_{\mathfrak{s}}} e(\boldsymbol{i}^{\boldsymbol{\lambda}}) \psi_{\underline{d}_{\mathfrak{t}}^{-1}} \in B_n^\kappa$$

over all $\boldsymbol{\lambda} \in \Lambda(n)$ and all $\mathfrak{s}, \mathfrak{t} \in \mathrm{Std}(\boldsymbol{\lambda})$ form a graded cellular basis for B_n^κ with respect to the partial order $\trianglelefteq$ on weights and the anti-involution $\psi \mapsto \overline{\psi}$. In other words

For the precise definition of a graded cellular basis see [10, Definition 2.1]. An important corollary, especially in conjunction with Lemma 1.2, is the following.

Corollary 2.2. *Let $\boldsymbol{i} \in I^n$. If there is no standard tableau $\mathfrak{t}$ with $(\kappa$-$)$residue $\boldsymbol{i}$, then $e(\boldsymbol{i}) = 0$ in B_n^κ.*

Remark 2.3.
(1) The degree of $\psi_{\mathfrak{s}\mathfrak{t}}$ does not depend on the choices of $\underline{d}_{\mathfrak{s}}$ or $\underline{d}_{\mathfrak{t}}$, and has a combinatorial definition based on $\mathfrak{s}$ and $\mathfrak{t}$ (see Theorem 2.7 below).
(2) The graded cellular structure on B_n^κ is in fact graded quasi-hereditary, which we will use frequently from now on. The idempotent-truncated algebras $e(\boldsymbol{i}^{\boldsymbol{\lambda}}) B_n^\kappa e(\boldsymbol{i}^{\boldsymbol{\lambda}})$, studied extensively in [18, 12] are also graded cellular but are not quasi-hereditary.

2.1. Graded cellular and quasi-hereditary algebras.
We fix some notation for graded modules. If $M = \bigoplus_j M^j$ is a graded vector space, we define the grade shift $M\langle k \rangle$ for $k \in \mathbb{Z}$ by $M\langle k \rangle^j = M^{j-k}$. For M, N graded B_n^κ-modules, we call a degree-preserving homomorphism $M \to N$ *homogeneous* of degree 0. When we write $\mathrm{Hom}_{B_n^\kappa}(M, N)$ we always mean the space of *ungraded* homomorphisms. By convention any homomorphism we write with a grade shifted object is homogeneous of degree 0, but homomorphisms without grade shifts may be ungraded.

We recall some facts about graded cellular and quasi-hereditary algebras [10]. Let $\boldsymbol{\lambda} \in \Lambda(n)$, and write $B_n^{\kappa, \rhd \boldsymbol{\lambda}}$ for the subspace spanned by all basis elements indexed by standard tableaux for weights $\boldsymbol{\mu} \rhd \boldsymbol{\lambda}$. Cellularity essentially means that for any standard tableaux $\mathfrak{s}, \mathfrak{t} \in \mathrm{Std}(\boldsymbol{\lambda})$, we can write the action of B_n^κ on the basis vector $\psi_{\mathfrak{s}, \mathfrak{t}}$ modulo the subspace $B_n^{\kappa, \rhd \boldsymbol{\lambda}}$ as

$$a\psi_{\mathfrak{s}\mathfrak{t}} = \sum_{\mathfrak{v} \in \mathrm{Std}(\boldsymbol{\lambda})} r_{\mathfrak{s}\mathfrak{v}}(a) \psi_{\mathfrak{v}\mathfrak{t}} \pmod{B_n^{\kappa, \rhd \boldsymbol{\lambda}}}$$

where the scalars $r_{\mathfrak{s}\mathfrak{v}}(a)$ don't depend on $\mathfrak{t}$. We can use these scalars to define a module $\Delta(\boldsymbol{\lambda})$ with basis $\psi_{\mathfrak{s}}$ indexed by $\mathrm{Std}(\boldsymbol{\lambda})$, namely

$$a\psi_{\mathfrak{s}} = \sum_{\mathfrak{v} \in \mathrm{Std}(\boldsymbol{\lambda})} r_{\mathfrak{s}\mathfrak{v}}(a)\psi_{\mathfrak{v}}$$

We call such modules *cell modules* or *Weyl modules*. Graded cellularity means that there is a degree function on tableaux (see Theorem 2.7) which makes the basis $\{\psi_{\mathfrak{s}}\}$ a homogeneous basis.

For any fixed standard tableaux $\mathfrak{a}, \mathfrak{b} \in \mathrm{Std}(\boldsymbol{\lambda})$, we can define a contravariant bilinear form on $\Delta(\boldsymbol{\lambda})$ by

$$\psi_{\mathfrak{a}\mathfrak{s}}\psi_{\mathfrak{t}\mathfrak{b}} = \langle \psi_{\mathfrak{s}}, \psi_{\mathfrak{t}} \rangle \psi_{\mathfrak{a}\mathfrak{b}} \pmod{B_n^{\kappa, \rhd\boldsymbol{\lambda}}}$$

In fact this bilinear form does not depend on $\mathfrak{a}$ or $\mathfrak{b}$. For a general cellular algebra the quotient $\Delta(\boldsymbol{\lambda})/\mathrm{rad}\langle -, - \rangle$ is either a simple module, which we call $L(\boldsymbol{\lambda})$, or 0. The non-zero quotients give a complete list of non-isomorphic simple modules up to grade shift. In our case, none of the quotients are zero because B_n^{κ} is quasi-hereditary. We write $P(\boldsymbol{\lambda})$ for the graded projective cover of $L(\boldsymbol{\lambda})$. For M a graded B_n^{κ}, we define the graded composition factor multiplicities

$$[M : L(\boldsymbol{\lambda})]_v = \sum_k [M : L(\boldsymbol{\lambda})\langle k \rangle]v^k \in \mathbb{Z}_{\geq 0}[v^{\pm 1}],$$

where $[M : L(\boldsymbol{\lambda})\langle k \rangle]$ denotes the number of composition factors in a graded composition series isomorphic to $L(\boldsymbol{\lambda})\langle k \rangle$. Similarly if M has a graded Weyl filtration, we define

$$(M : \Delta(\boldsymbol{\lambda}))_v = \sum_k (M : \Delta(\boldsymbol{\lambda})\langle k \rangle)v^k \in \mathbb{Z}_{\geq 0}[v^{\pm 1}],$$

where $(M : \Delta(\boldsymbol{\lambda})\langle k \rangle)$ denotes the number of subquotients in a graded Weyl filtration isomorphic to $\Delta(\boldsymbol{\lambda})\langle k \rangle$. For the ungraded counterparts of these multiplicities we use the same notation but without the subscript v.

As B_n^{κ} is quasi-hereditary, we also have the notion of a *tilting module*. A tilting module for B_n^{κ} is a module with a filtration by Weyl modules as well as a filtration by dual Weyl modules. For each weight $\boldsymbol{\lambda}$, there is an indecomposable tilting module $T(\boldsymbol{\lambda})$ of highest weight $\boldsymbol{\lambda}$, and all indecomposable tilting modules are of this form [21]. In the graded setting this classification only gives a grading on $T(\boldsymbol{\lambda})$ up to grade shift. We will fix the grading so that $(T(\boldsymbol{\lambda}) : \Delta(\boldsymbol{\lambda}))_v = 1$.

The anti-involution gives rise to a duality functor on B_n^{κ}-modules which reverses grade shift. The unshifted simple module $L(\boldsymbol{\lambda})$ is self-dual, so the dual Weyl module $\nabla(\boldsymbol{\lambda})$ has socle isomorphic to $L(\boldsymbol{\lambda})$. Similarly the unshifted injective envelope $I(\boldsymbol{\lambda})$ is isomorphic to the dual of $P(\boldsymbol{\lambda})$. By highest weight considerations $T(\boldsymbol{\lambda})$ is self-dual. For $h \in \mathbb{Z}_{\geq 0}[v^{\pm 1}]$, we write $\overline{h} = h(v^{-1})$.

2.2. Tower of recollement.

For fixed m, e and varying n, the family of classical blob algebras (with presentation as in Theorem 1.3) has the structure of a *tower of recollement* [4, Example 1.2(ii)]. A tower of recollement consists of a collection of algebras and idempotents in these algebras which satisfy certain axioms, giving rise to several functors between module categories which pass representation-theoretic information between the algebras. Constructing the functors and verifying the axioms are both more easily accomplished in the classical presentation of the blob algebra. For this reason we will assume for the moment that Theorem 1.3 holds so that the tower of recollement structure transfers to $\{B_n^{\kappa}\}_{n \in \mathbb{N}}$. For the basic definitions and some examples see [4, Section 1], and [15, Section 3] for applications.

For each $n \in \mathbb{N}$ we have a pair of adjoint functors

$$\mathrm{ind} : B_n^{\kappa}-\mathrm{mod} \longrightarrow B_{n+1}^{\kappa}-\mathrm{mod}, \qquad\qquad \mathrm{res} : B_{n+1}^{\kappa}-\mathrm{mod} \longrightarrow B_n^{\kappa}-\mathrm{mod}$$

called *induction* and *restriction* respectively. As a right adjoint functor, restriction is left exact, and similarly induction is right exact. However, restriction also happens to be right exact as well. For $\boldsymbol{\lambda} \in \Lambda(n+1)$ write $\boldsymbol{\lambda} = (1^{\lambda_1}, 1^{\lambda_2})$. If $\lambda_1 \geq \lambda_2 > 0$ we have a short exact sequence

$$0 \longrightarrow \Delta(1^{\lambda_1-1}, 1^{\lambda_2}) \longrightarrow \mathrm{res}\,\Delta(1^{\lambda_1}, 1^{\lambda_2}) \longrightarrow \Delta(1^{\lambda_1}, 1^{\lambda_2-1}) \longrightarrow 0$$

while res $\Delta(1^{n+1}, \emptyset) = \Delta(1^n, \emptyset)$. When $0 < \lambda_1 \leq \lambda_2$ there are similar exact sequences with the two outer terms switched. Induction on Weyl modules also produces exact sequences in this way, but without a boundary exception.

We also have another pair of adjoint functors

$$G : B_n^\kappa-\text{mod} \longrightarrow B_{n+2}^\kappa-\text{mod}, \qquad\qquad F : B_{n+2}^\kappa-\text{mod} \longrightarrow B_n^\kappa-\text{mod}$$

called *globalisation* and *localisation* respectively. Again localisation is right exact as well as being left exact. For $\boldsymbol{\lambda} = (1^{\lambda_1}, 1^{\lambda_2}) \in \Lambda(n+2)$ we have

$$F\Delta(1^{\lambda_1}, 1^{\lambda_2}) = \begin{cases} \Delta(1^{\lambda_1-1}, 1^{\lambda_2-1}) & \text{if } \lambda_1, \lambda_2 \geq 1, \\ 0 & \text{otherwise.} \end{cases}$$

and similarly for $\nabla(\boldsymbol{\lambda})$ and $L(\boldsymbol{\lambda})$ [15, Proposition 3]. Moreover, as long as $\lambda_1, \lambda_2 \geq 1$ we also have $P(\boldsymbol{\lambda})$, $\nabla(\boldsymbol{\lambda})$, $I(\boldsymbol{\lambda})$, and $T(\boldsymbol{\lambda})$ by [5, A1(4)], [5, Proposition A3.11], and [5, Lemma A4.5]. This implies the *stability* of decomposition numbers and tilting multiplicities across all n. In other words, for all $n \in \mathbb{N}$ and $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \Lambda(n)$ with $\boldsymbol{\lambda} = (1^{\lambda_1}, 1^{\lambda_2})$ and $\boldsymbol{\mu} = (1^{\mu_1}, 1^{\mu_2})$, the decomposition number $[\Delta(\boldsymbol{\mu}) : L(\boldsymbol{\lambda})]$ only depends on $\lambda_1 - \lambda_2$ and $\mu_1 - \mu_2$ but not on n.

For $\boldsymbol{\lambda} = (1^{\lambda_1}, 1^{\lambda_2}) \in \Lambda(n)$ globalisation behaves similarly for Weyl modules and projective modules, with

$$G\Delta(1^{\lambda_1}, 1^{\lambda_2}) = \Delta(1^{\lambda_1+1}, 1^{\lambda_2+1}), \qquad\qquad GP(1^{\lambda_1}, 1^{\lambda_2}) = P(1^{\lambda_1+1}, 1^{\lambda_2+1})$$

but *not* for simple modules, dual Weyl modules, injective modules, or tilting modules. Globalisation is exact on the full subcategory of Δ-filtered modules [15, Proposition 4]. It also acts as a right inverse for localisation, i.e. $F \circ G$ is naturally isomorphic to the identity.

Finally we have the key relationship between induction/restriction and localisation/globalisation, which is the natural isomorphism

$$\text{ind} \cong \text{res} \circ G.$$

In the case of B_n^κ, the tower of recollement structure behaves well with the anti-involution so the dual statement

$$\text{res} \cong F \circ \text{ind}$$

also holds.

2.3. **Linkage principle.** There is a linkage principle for the blob algebra, in terms of the following alcove geometry. Let W be the infinite dihedral group acting on $\mathbb{Z}$ generated by reflections s_k about the integers $(\kappa_1 - \kappa_2) + ke$ for any $k \in \mathbb{Z}$. Each alcove consists of the integers $(\kappa_1 - \kappa_2) + ke < j < (\kappa_1 - \kappa_2) + (k+1)e$ lying between two adjacent reflection points. Weights lying inside an alcove are called regular, while those on a reflection point are singular. The fundamental alcove is the unique alcove containing the integer 0. Two integers are called linked if they are in the same W-orbit. The group W also acts partially on paths in $\mathbb{Z}$. For a path p, if $p(k)$ is the reflection point $(\kappa_1 - \kappa_2) + je$, then we write

$$s_j^k p(r) = \begin{cases} p(r) & \text{if } r \leq k, \\ s_j p(r) & \text{if } r > k. \end{cases}$$

In other words, $s_j^k p$ is the path obtained by reflecting p after the kth point. We say that two paths are linked if one can be obtained by a sequence of reflections of the other.

Write $\text{Std}_{\boldsymbol{\lambda}}(\boldsymbol{\mu})$ for the set of standard tableaux of shape $\boldsymbol{\mu}$ with residue sequence $\boldsymbol{i}^{\boldsymbol{\lambda}}$. It turns out that this set can be described entirely in terms of the alcove geometry above, using the fact that weights and tableaux correspond to points in $\mathbb{Z}$ and paths in $\mathbb{Z}$ respectively.

Proposition 2.4 ([18, Lemma 4.7]). *Let $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \Lambda(n)$. Under the tableau-path bijection, the set $\text{Std}_{\boldsymbol{\lambda}}(\boldsymbol{\mu})$ corresponds to paths which end at $\boldsymbol{\mu}$ in the same linkage class as $\mathfrak{t}^{\boldsymbol{\lambda}}$.*

Example 2.5. Suppose $e = 4$, $\kappa = (0, 2)$, and $n = 9$. Let $\boldsymbol{\lambda} = (1^8, 1)$. The tableau $\mathfrak{t}^{\boldsymbol{\lambda}}$ corresponds to the path in red. This path crosses 2 alcove walls, so there are $2^2 = 4$ different paths in the

linkage class of $\mathfrak{t}^{\boldsymbol{\lambda}}$. The other 3 paths in this linkage class are illustrated in black from the point where they diverge from $\mathfrak{t}^{\boldsymbol{\lambda}}$.
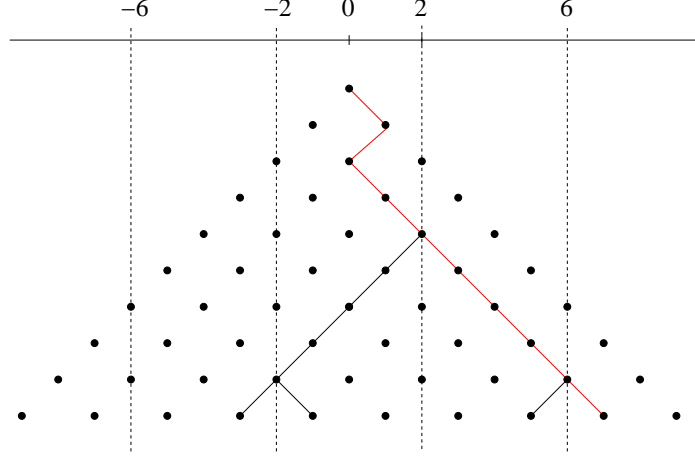
These paths correspond to the tableaux

1		2		1		2		1		2		1		2
3		5		3		5		3		9		3		
4		6		4		6		4				4		
7				9		7		5				5		
8						8		6				6		
9								7				7		
								8				8		
												9		

Corollary 2.6. *If $[\Delta(\boldsymbol{\mu}) : L(\boldsymbol{\lambda})] \neq 0$ then $\boldsymbol{\mu}$ and $\boldsymbol{\lambda}$ are in the same linkage class.*

A consequence of the above result is that if $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \Lambda(n)$ are in different linkage classes, then they are also in different blocks. We write $\mathrm{pr}_{\boldsymbol{\lambda}}$ for the functor which projects modules and homomorphisms onto the block(s) of simple modules parametrised by weights in the linkage class of $\boldsymbol{\mu}$.

The degrees of tableaux in $\mathrm{Std}_{\boldsymbol{\lambda}}(\boldsymbol{\mu})$ can also be calculated from their corresponding path. We call a subsequence of e consecutive steps in a path a *wall-to-wall step* if the steps start from a wall (i.e. a reflection point) and continue in a single direction until they reach another wall. For $\mathfrak{t} \in \mathrm{Std}_{\boldsymbol{\lambda}}(\boldsymbol{\mu})$ a standard tableau write $w(\mathfrak{t})$ for the number of wall-to-wall steps across the fundamental alcove.

Theorem 2.7 ([18, Theorem 4.9]). *Let $\mathfrak{t} \in \mathrm{Std}_{\boldsymbol{\lambda}}(\boldsymbol{\mu})$. Let $\delta(\mathfrak{t})$ be 1 if the first step after all wall-to-wall steps points toward the origin, and 0 otherwise. Then $\deg \mathfrak{t} = w(\mathfrak{t}) + \delta(\mathfrak{t})$.*

Finally we describe the decomposition numbers in characteristic 0 in terms of the alcove geometry. For any regular weight $\boldsymbol{\lambda}$, there exists a unique weight $\boldsymbol{\lambda}_{\mathrm{fund}}$ in the fundamental alcove and $w_{\boldsymbol{\lambda}} \in W$ such that $w_{\boldsymbol{\lambda}}(\boldsymbol{\lambda}_{\mathrm{fund}}) = \boldsymbol{\lambda}$. For $x, y \in W$, define $h_{y,x}$ by

$$h_{y,x}(v) = \begin{cases} v^{\ell(x) - \ell(y)} & \text{if } y \leq x, \\ 0 & \text{otherwise.} \end{cases}$$

This is the Kazhdan–Lusztig polynomial associated to W (in the notation of [22]).

Theorem 2.8 ([18, Theorem 5.11]). *Suppose $\Bbbk$ is a field of characteristic 0. Let $\boldsymbol{\lambda}, \boldsymbol{\mu}$ be two regular weights lying in the same linkage class. Then we have*

$$[\Delta(\boldsymbol{\mu}) : L(\boldsymbol{\lambda})]_v = h_{w_{\boldsymbol{\mu}}, w_{\boldsymbol{\lambda}}}(v).$$

There is also a singular version of this result. If $\boldsymbol{\lambda}$ is a singular weight, we label the weights in the linkage class of $\boldsymbol{\lambda}$ following [18, Example 5.5]. First set $\boldsymbol{\lambda}_0 = \boldsymbol{\lambda}$. Suppose that $\boldsymbol{\lambda}$ corresponds

to a positive classical weight (i.e. a weight on the right side of the origin in our pictures). Working inductively, for k even (resp. odd) we define $\boldsymbol{\lambda}_{k+1}$ to be the rightmost (resp. leftmost) weight in the linkage class distinct from $\boldsymbol{\lambda}_0, \boldsymbol{\lambda}_1, \ldots, \boldsymbol{\lambda}_k$. Similarly, when $\boldsymbol{\lambda}$ corresponds to a negative classical weight, for k even (resp. odd) we define $\boldsymbol{\lambda}_{k+1}$ to be the leftmost (resp. rightmost) weight in the linkage class distinct from $\boldsymbol{\lambda}_0, \boldsymbol{\lambda}_1, \ldots, \boldsymbol{\lambda}_k$.

Theorem 2.9 ([18, Theorem 5.14]). *Suppose $\Bbbk$ is a field of characteristic 0. Let $\boldsymbol{\lambda}$ be a singular weight. Then if $\boldsymbol{\lambda}_k$ is defined we have*

$$[\Delta(\boldsymbol{\lambda}_k) : L(\boldsymbol{\lambda})]_v = v^k.$$

Remark 2.10. In general, it is easier to use tableaux when working with permutations of the form d_{t} for some tableau t of shape $\boldsymbol{\lambda}$, as one can read off d_{t} directly from the two tableaux t and $\mathsf{t}^{\boldsymbol{\lambda}}$. By contrast, it is easier to use paths in order to apply Proposition 2.4. We will mostly use tableaux in the arguments below, but the careful reader may use the tableau-path bijection in order to translate our arguments into the language of paths if necessary.

3. Bases for projective indecomposable modules

For the rest of this paper, we will assume that $\Bbbk$ is a field of characteristic 0. Most of the previous results are known to hold in some form for the classical blob algebra. To proceed further we must make use of the KLR-style presentation of B_n^{κ}, and in particular the grading.

3.1. **A Temperley–Lieb subalgebra.** As B_n^{κ} is graded, it has a subalgebra of degree 0 elements. This subalgebra was classified in [12, § 5.4–5.5]. We summarise their results below.

Definition 3.1. Let $\boldsymbol{\lambda} = (1^{\lambda_1}, 1^{\lambda_2}) \in \Lambda(n)$. Suppose the weight $\boldsymbol{\lambda}$ does not lie in the interior of the fundamental alcove. We define $f_{\boldsymbol{\lambda}}$ to be the minimal positive integer such that the $f_{\boldsymbol{\lambda}}$th point of the path corresponding to $\mathsf{t}^{\boldsymbol{\lambda}}$ lies on a wall of the fundamental alcove. In other words,

$$(17) \qquad f_{\boldsymbol{\lambda}} = \begin{cases} \min(\{2\lambda_2 + (\kappa_1 - \kappa_2) + je : j \in \mathbb{Z}\} \cap \mathbb{N}) & \text{if } \lambda_1 \geq \lambda_2, \\ \min(\{2\lambda_1 - (\kappa_1 - \kappa_2) + je : j \in \mathbb{Z}\} \cap \mathbb{N}) & \text{if } \lambda_1 < \lambda_2. \end{cases}$$

For $j \in \mathbb{N}$ write $f(j) = f_{\boldsymbol{\lambda}} + je$. For all $j \in \mathbb{N}$ such that $f(j) \leq n - e$ we define the *diamond* of $\boldsymbol{\lambda}$ at position $f(j)$ to be

$$(18) \quad U_j^{\boldsymbol{\lambda}} = (\psi_{f(j)})(\psi_{f(j)-1}\psi_{f(j)+1})(\psi_{f(j)-2}\psi_{f(j)}\psi_{f(j)+2})\cdots$$
$$\cdots(\psi_{f(j)-e+1}\psi_{f(j)-e+3}\cdots\psi_{f(j)+e-3}\psi_{f(j)+e-1})\cdots$$
$$\cdots(\psi_{f(j)-2}\psi_{f(j)}\psi_{f(j)+2})(\psi_{f(j)-1}\psi_{f(j)+1})(\psi_{f(j)})e(\boldsymbol{i}^{\boldsymbol{\lambda}}).$$

The name 'diamond' comes from the corresponding KLR diagram for this element, e.g.
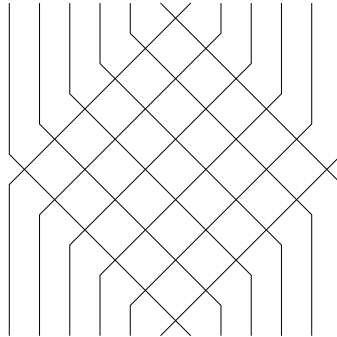
for $e = 6$. The cyclotomic KLR algebra versions of these elements previously appeared in [11, (4.2)], while the effect of similar permutations on paths was seen even earlier, e.g. [13, Figure 4].

Theorem 3.2 ([12, Theorem 5.24]). *Let $\boldsymbol{\lambda} \in \Lambda(n)$. The diamonds of weight $\boldsymbol{\lambda}$ generate the degree 0 subalgebra of $e(\boldsymbol{i}^{\boldsymbol{\lambda}})B_n^{\kappa}e(\boldsymbol{i}^{\boldsymbol{\lambda}})$. This subalgebra is isomorphic to a Temperley–Lieb algebra with loop parameter $2(-1)^{e-1}$, with the diamond at position $f_{\boldsymbol{\lambda}} + je$ corresponding to the standard*

Temperley–Lieb diagrammatic generator at index j. In other words, the diamonds of weight $\boldsymbol{\lambda}$ satisfy the relations

$$
\begin{aligned}
U_i^{\boldsymbol{\lambda}} U_j^{\boldsymbol{\lambda}} &= U_j^{\boldsymbol{\lambda}} U_i^{\boldsymbol{\lambda}} && \text{when } |i - j| > 1, \\
U_i^{\boldsymbol{\lambda}} U_j^{\boldsymbol{\lambda}} U_i^{\boldsymbol{\lambda}} &= U_i^{\boldsymbol{\lambda}} && \text{when } |i - j| = 1, \\
(U_i^{\boldsymbol{\lambda}})^2 &= 2(-1)^{e-1} U_i^{\boldsymbol{\lambda}} && \text{for all } i,
\end{aligned}
$$

and this gives a complete presentation of the subalgebra generated by them.

Recall that in quantum characteristic 0 the Temperley–Lieb algebra is semisimple, with a unique 1-dimensional irreducible module. The central idempotent corresponding to this irreducible module is sometimes called the *Jones–Wenzl projector*. We write $\mathrm{JW}^{\boldsymbol{\lambda}}$ for the corresponding idempotent in $e(\boldsymbol{i}^{\boldsymbol{\lambda}}) B_n^\kappa e(\boldsymbol{i}^{\boldsymbol{\lambda}})$. In our notation, one of the defining properties of $\mathrm{JW}^{\boldsymbol{\lambda}}$ is that $U_j^{\boldsymbol{\lambda}} \mathrm{JW}^{\boldsymbol{\lambda}} = 0$ for all j.

Lemma 3.3. *Let $\boldsymbol{\lambda} \in \Lambda(n)$. Then $P(\boldsymbol{\lambda}) \cong B_n^\kappa \mathrm{JW}^{\boldsymbol{\lambda}}$.*

Proof. Clearly $B_n^\kappa \mathrm{JW}^{\boldsymbol{\lambda}}$ is an indecomposable projective module, as $\mathrm{JW}^{\boldsymbol{\lambda}}$ is a primitive idempotent for the Temperley–Lieb subalgebra (and thus also for $e(\boldsymbol{i}^{\boldsymbol{\lambda}}) B_n^\kappa e(\boldsymbol{i}^{\boldsymbol{\lambda}})$ and B_n^κ). Suppose $B_n^\kappa \mathrm{JW}^{\boldsymbol{\lambda}} = P(\boldsymbol{\mu})$. Then B_n^κ maps onto $\Delta(\boldsymbol{\mu})$, which induces a surjective homomorphism

$$
e(\boldsymbol{i}^{\boldsymbol{\lambda}}) B_n^\kappa \mathrm{JW}^{\boldsymbol{\lambda}} \longrightarrow e(\boldsymbol{i}^{\boldsymbol{\lambda}}) \Delta(\boldsymbol{\mu})
$$

of $e(\boldsymbol{i}^{\boldsymbol{\lambda}}) B_n^\kappa e(\boldsymbol{i}^{\boldsymbol{\lambda}})$-modules. By the defining property of $\mathrm{JW}^{\boldsymbol{\lambda}}$, the degree 0 part of the domain is 1-dimensional. On the other hand, the degree 0 part of the codomain is the cellular module of weight $\boldsymbol{\mu}$ for the Temperley–Lieb subalgebra. (Here we use the fact that the cellular structure of the Temperley–Lieb subalgebra is compatible with that of $e(\boldsymbol{i}^{\boldsymbol{\lambda}}) B_n^\kappa e(\boldsymbol{i}^{\boldsymbol{\lambda}})$, because the latter is positively graded by Theorem 2.7.) This has dimension strictly larger than 1 unless $\boldsymbol{\mu} = \boldsymbol{\lambda}$. □

3.2. **Maximal degree tableaux.** The following key combinatorial lemma constructs maximal degree tableaux, which are of fundamental importance in the characteristic 0 representation theory of B_n^κ.

Lemma 3.4. *Let $\boldsymbol{\lambda} \in \Lambda(n)$ be a weight. For each $\boldsymbol{\mu} \in W\boldsymbol{\lambda}$ with $\boldsymbol{\lambda} \trianglelefteq \boldsymbol{\mu}$, there is a unique tableau $\mathfrak{t}_{\boldsymbol{\lambda}}^{\boldsymbol{\mu}} \in \mathrm{Std}_{\boldsymbol{\lambda}}(\boldsymbol{\mu})$ of maximal degree*

$$
\deg \mathfrak{t}_{\boldsymbol{\lambda}}^{\boldsymbol{\mu}} = \begin{cases} \ell(w_{\boldsymbol{\lambda}}) - \ell(w_{\boldsymbol{\mu}}) & \text{if } \boldsymbol{\lambda} \text{ is regular,} \\ k & \text{if } \boldsymbol{\lambda} \text{ is singular and } \boldsymbol{\mu} = \boldsymbol{\lambda}_k. \end{cases}
$$

Proof. Let $\mathfrak{t} \in \mathrm{Std}_{\boldsymbol{\lambda}}(\boldsymbol{\mu})$, and write d for $\ell(w_{\boldsymbol{\lambda}}) - \ell(w_{\boldsymbol{\mu}})$. From Theorem 2.7 recall that $\deg \mathfrak{t}$ is either $w(\mathfrak{t})$ or $w(\mathfrak{t}) + 1$, where $w(\mathfrak{t})$ is the number of wall-to-wall steps inside the fundamental alcove for the path corresponding to $\mathfrak{t}$. By Proposition 2.4 $\mathfrak{t}$ lies in the linkage class of $\mathfrak{t}^{\boldsymbol{\lambda}}$. The path corresponding to $\mathfrak{t}^{\boldsymbol{\lambda}}$ contains $\ell(w_{\boldsymbol{\lambda}}) - 1$ wall-to-wall steps, whereas any path with endpoint $\boldsymbol{\mu}$ must have at least $\ell(w_{\boldsymbol{\mu}}) - 1$ wall-to-wall steps outside the fundamental alcove to get there. Thus $w(\mathfrak{t})$ is bounded above by d.

There are four cases, according to the parity of d and whether $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ lie on the same side of the origin or not. We will focus on one of these cases; the other three are similar. Suppose d is even and that $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ both lie on the same side of the origin. First we note that since paths to $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ must eventually pass through the same wall of the fundamental alcove, $w(\mathfrak{t})$ is even for all

$\mathsf{t} \in \operatorname{Std}_{\boldsymbol{\lambda}}(\boldsymbol{\mu})$. There exists a tableau $\mathsf{t}_{\boldsymbol{\lambda}}^{\boldsymbol{\mu}} \in \operatorname{Std}_{\boldsymbol{\lambda}}(\boldsymbol{\mu})$ with $w(\mathsf{t}_{\boldsymbol{\lambda}}^{\boldsymbol{\mu}}) = d$ maximal, e.g.
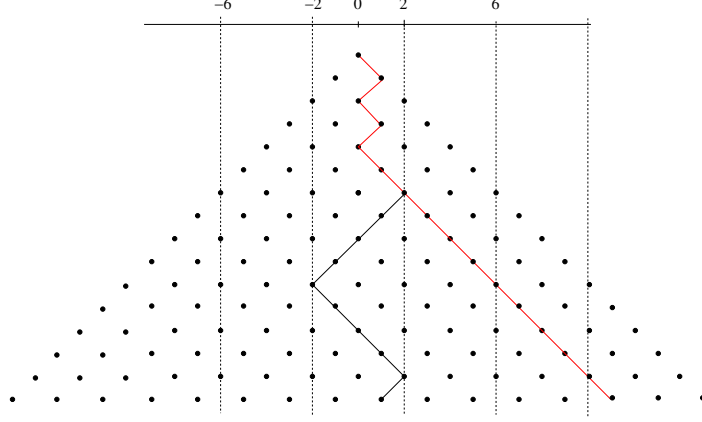
Moreover, this tableau is unique: for any such path, the wall-to-wall steps inside the fundamental alcove must occur as early as possible. If not, the path would have to leave and then return to the fundamental alcove, wasting wall-to-wall steps in the process. Finally, $\mathsf{t}_{\boldsymbol{\lambda}}^{\boldsymbol{\mu}}$ has maximal degree too. From the picture above $\deg \mathsf{t}_{\boldsymbol{\lambda}}^{\boldsymbol{\mu}} = w(\mathsf{t}_{\boldsymbol{\lambda}}^{\boldsymbol{\mu}})$, and for all other tableaux t we have

$$\deg \mathsf{t} \le w(\mathsf{t}) + 1 \le (w(\mathsf{t}_{\boldsymbol{\lambda}}^{\boldsymbol{\mu}}) - 2) + 1 < \deg \mathsf{t}_{\boldsymbol{\lambda}}^{\boldsymbol{\mu}}$$

$\square$

Remark 3.5. An alternative proof of this result uses [12, Theorem 4.9] to reduce the problem of determining graded dimensions of Weyl modules to a calculation in the Iwahori–Hecke algebra corresponding to W. The result follows from the observation that the 'Bott–Samelson' elements (i.e. products of simple Kazhdan–Lusztig generators) in this algebra are just sums of Kazhdan–Lusztig basis elements.

An interesting application of these maximal degree tableaux is the following lemma, which identifies composition factors of Weyl modules in terms of the cellular basis.

Lemma 3.6. *Let* $\boldsymbol{\lambda} \in \Lambda(n)$, *and suppose* $\boldsymbol{\mu} \in W\boldsymbol{\lambda}$ *with* $\boldsymbol{\lambda} \trianglelefteq \boldsymbol{\mu}$. *Consider the submodule* $B_n^\kappa \psi_{\mathsf{t}_{\boldsymbol{\lambda}}^{\boldsymbol{\mu}}}$ *of the Weyl module* $\Delta(\boldsymbol{\mu})$. *Then there is a homomorphism*

$$B_n^\kappa \psi_{\mathsf{t}_{\boldsymbol{\lambda}}^{\boldsymbol{\mu}}} \longrightarrow L(\boldsymbol{\lambda})\langle \deg \mathsf{t}_{\boldsymbol{\lambda}}^{\boldsymbol{\mu}} \rangle$$
$$\psi_{\mathsf{t}_{\boldsymbol{\lambda}}^{\boldsymbol{\mu}}} \longmapsto \psi_{\mathsf{t}^{\boldsymbol{\lambda}}}.$$

Proof. Let $d = \deg \mathsf{t}_{\boldsymbol{\lambda}}^{\boldsymbol{\mu}} = \ell(w_{\boldsymbol{\lambda}}) - \ell(w_{\boldsymbol{\mu}})$. From Theorem 2.8 we have

$$[\Delta(\boldsymbol{\mu}) : L(\boldsymbol{\lambda})]_v = h_{w_{\boldsymbol{\mu}}, w_{\boldsymbol{\lambda}}}(v) = v^d,$$

so the Weyl module $\Delta(\boldsymbol{\mu})$ contains exactly one subquotient isomorphic to $L(\boldsymbol{\lambda})\langle d \rangle$. Recall that $L(\boldsymbol{\lambda})$ is generated by a vector of residue $\boldsymbol{i}^{\boldsymbol{\lambda}}$ in degree 0. This means that the unique subquotient of $\Delta(\boldsymbol{\mu})$ isomorphic to $L(\boldsymbol{\lambda})\langle d \rangle$ is generated by some vector of residue $\boldsymbol{i}^{\boldsymbol{\lambda}}$ in degree d. But from Proposition 2.4 and Lemma 3.4 we have

$$\dim_v e(\boldsymbol{i}^{\boldsymbol{\lambda}})\Delta(\boldsymbol{\mu}) = \sum_{\mathsf{t} \in \operatorname{Std}_{\boldsymbol{\lambda}}(\boldsymbol{\mu})} v^{\deg \mathsf{t}} = v^d + \text{l.o.t.}$$

In other words, the subspace of vectors with the correct residue and degree is one-dimensional, spanned by $\psi_{\mathsf{t}_{\boldsymbol{\lambda}}^{\boldsymbol{\mu}}}$, so the result follows. $\square$

Applying Brauer–Humphreys reciprocity, we can also identify the Weyl subquotient isomorphic to $\Delta(\boldsymbol{\mu})$ inside $P(\boldsymbol{\lambda})$.

Corollary 3.7. *Let* $\boldsymbol{\lambda} \in \Lambda(n)$, *and suppose* $\boldsymbol{\mu} \in W\boldsymbol{\lambda}$ *with* $\boldsymbol{\lambda} \trianglelefteq \boldsymbol{\mu}$. *There is a surjective homomorphism*

$$B_n^\kappa \psi_{\mathsf{t}^{\boldsymbol{\mu}} \mathsf{t}_{\boldsymbol{\lambda}}^{\boldsymbol{\mu}}} \mathrm{JW}^{\boldsymbol{\lambda}} \longrightarrow \Delta(\boldsymbol{\mu})\langle \deg \mathsf{t}_{\boldsymbol{\lambda}}^{\boldsymbol{\mu}} \rangle$$
$$\psi_{\mathsf{t}^{\boldsymbol{\mu}} \mathsf{t}_{\boldsymbol{\lambda}}^{\boldsymbol{\mu}}} \mathrm{JW}^{\boldsymbol{\lambda}} \longmapsto \psi_{\mathsf{t}^{\boldsymbol{\mu}}},$$

where the domain is a submodule of $B_n^\kappa \mathrm{JW}^{\boldsymbol\lambda} \cong P(\boldsymbol\lambda)$.

4. SINGULAR PROJECTIVE MODULES

The aim of this section is to determine the socles of the indecomposable projective modules associated to singular weights — Theorem 4.13 and Corollary 4.14. This turns out to be enough to completely determine the structure of these modules. The result will then be used in §5.1 to address the corresponding (harder) non-singular cases.

Our general strategy is to identify possible generators for the socle in Lemma 4.7 and then to rule out all but one of them via direct computation. The computation involves the Jones–Wenzl projector, which is difficult to work with directly because in the standard basis it is a sum with many terms. Luckily nearly all of these terms combine or vanish in the computation when multiplied by certain cellular basis elements.

In this section we will assume that $n \equiv \kappa_1 - \kappa_2 \pmod{e}$, or in other words that there is a wall at n. Fix $\boldsymbol\eta = (1^n, \emptyset) \in \Lambda(n)$ and let $m \in \mathbb{N}$ such that $n = f_{\boldsymbol\eta} + me$ (see (17) for a definition of $f_{\boldsymbol\eta}$). Recall how the linkage class of $\boldsymbol\eta$ consists of the weights $\boldsymbol\eta_j$ for some non-negative integers j. The maximal weight in this linkage class is $\boldsymbol\eta_m$, which is on a wall of the fundamental alcove. Note that $f_{\boldsymbol\eta_j} = f_{\boldsymbol\eta} + je$, because the distance from $\boldsymbol\eta_j$ to the nearest fundamental alcove wall is $(m-j)e$ steps.

4.1. Cellular basis factorization.
We begin with a factorization of some of the distinguished cellular basis elements from the previous section.

Proposition 4.1. *For all integers $0 \le j \le k \le m$ we have*

$$\psi_{\mathfrak{t}^{\boldsymbol\eta_k} \mathfrak{t}^{\boldsymbol\eta_k}_{\boldsymbol\eta_j}} = x_j x_{j+1} \cdots x_{k-1} \psi_{f_{\boldsymbol\eta}+je} \psi_{f_{\boldsymbol\eta}+(j+1)e} \cdots \psi_{f_{\boldsymbol\eta}+(k-1)e} e(\boldsymbol{i}^{\boldsymbol\eta_j})$$

for some elements $x_r \in B_n^\kappa$ (with $j \le r < k$) which satisfy the following properties:

   (i) *for fixed r the element x_r does not depend on j or k;*

   (ii) *for $r \ne s$, $x_r x_s = x_s x_r$ and $x_r \psi_{f_{\boldsymbol\eta}+se} = \psi_{f_{\boldsymbol\eta}+se} x_r$;*

   (iii) *for each r we have*

$$x_r \overline{x_r} = e(\boldsymbol{i}^{\boldsymbol\eta_k}),$$
$$\overline{x_r} x_r = e(s_{f_{\boldsymbol\eta}+re} \boldsymbol{i}^{\boldsymbol\eta_j}).$$

Proof. Let $d = d_{\mathfrak{t}^{\boldsymbol\eta_k}_{\boldsymbol\eta_j}}$. Recall that d is the permutation which maps $\mathfrak{t}^{\boldsymbol\eta_k}$ to $\mathfrak{t}^{\boldsymbol\eta_k}_{\boldsymbol\eta_j}$.

For $0 \le l \le m$, write $\boldsymbol\eta_l = (1^{\eta_{l,1}}, 1^{\eta_{l,2}})$ and set $r_l = 2\min(\eta_{l,1}, \eta_{l,2})$. From (17) it is clear that

$$f_{\boldsymbol\eta_l} = f_{\boldsymbol\eta} + le = \begin{cases} r_l + f_{\boldsymbol\eta} & \text{if } l \text{ is even,} \\ r_l + (e - f_{\boldsymbol\eta}) & \text{if } l \text{ is odd.} \end{cases}$$

This means that

$$r_l = \begin{cases} le & \text{if } l \text{ is even,} \\ (l-1)e + 2f_{\boldsymbol\eta} & \text{if } l \text{ is odd.} \end{cases}$$

Thus the integers $1 \le r \le r_j$ lie in the same boxes in the tableaux $\mathfrak{t}^{\boldsymbol\eta_j}$, $\mathfrak{t}^{\boldsymbol\eta_k}_{\boldsymbol\eta_j}$, and $\mathfrak{t}^{\boldsymbol\eta_k}$ so we have $d(r) = r$. Similarly when $r_k < r \le n$, r is in the same box in both $\mathfrak{t}^{\boldsymbol\eta_k}$ and $\mathfrak{t}^{\boldsymbol\eta_k}_{\boldsymbol\eta_j}$ so $d(r) = r$ here as well.

For $j \le l < k$, the boxes in $\mathfrak{t}^{\boldsymbol\eta_k}_{\boldsymbol\eta_j}$ with labels $r_l < r \le r_{l+1}$ form the skew tableau

if l is even,

$$
\begin{array}{cc}
\boxed{\begin{array}{c} r_l + e \\ - f_\eta + 1 \end{array}} & \boxed{r_l + 1} \\[4pt]
\boxed{\begin{array}{c} r_l + e \\ - f_\eta + 2 \end{array}} & \boxed{r_l + 2} \\
\vdots & \vdots \\
\boxed{\begin{array}{c} r_l + 2e \\ - 2f_\eta \end{array}} & \boxed{r_l + e - f_\eta}
\end{array}
\qquad \text{if } l \text{ is odd,}
$$

while the same boxes in $\mathbf{t}^{\eta_k}$ form the skew tableau

$$
\begin{array}{cc}
\boxed{r_l + 1} & \boxed{r_l + 2} \\[4pt]
\boxed{r_l + 3} & \boxed{r_l + 4} \\
\vdots & \vdots \\
\boxed{r_l + 2f_\eta - 1} & \boxed{r_l + 2f_\eta}
\end{array}
\qquad \text{if } l \text{ is even,}
$$

$$
\begin{array}{cc}
\boxed{r_l + 1} & \boxed{r_l + 2} \\[4pt]
\boxed{r_l + 3} & \boxed{r_l + 4} \\
\vdots & \vdots \\
\boxed{\begin{array}{c} r_l + 2e \\ -2f_\eta - 1 \end{array}} & \boxed{\begin{array}{c} r_l + 2e \\ - 2f_\eta \end{array}}
\end{array}
\qquad \text{if } l \text{ is odd.}
$$
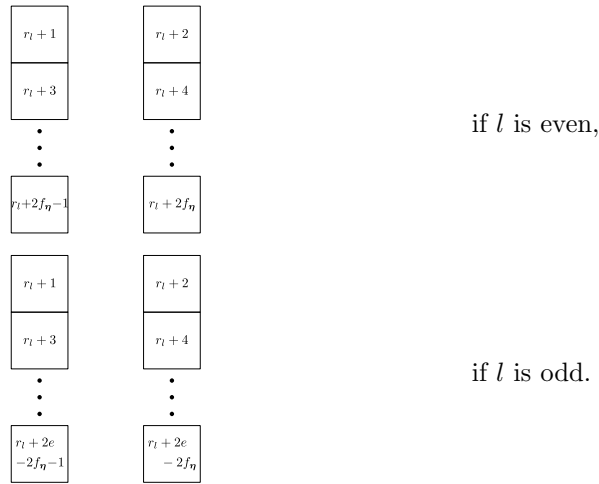
This of course means that d restricted to $r_l < r \le r_{l+1}$ is still a permutation d_l. In fact d_l corresponds to a triangular portion of the lower half of a 'diamond permutation':
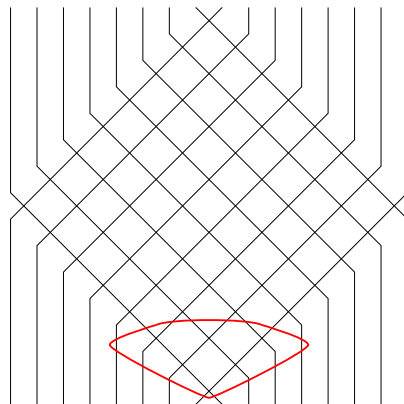
The easiest way to see this is to apply the 'layers' (each a product of several commuting transpositions) in turn to the skew tableaux above. For example, the first $(f_\eta - 1)$ layers permute the skew

tableau with $f_{\boldsymbol{\eta}}$ rows as follows:

The number of layers in the triangle is either $f_{\boldsymbol{\eta}}-1$ or $e-f_{\boldsymbol{\eta}}$ depending on parity. But $2 \leq f_{\boldsymbol{\eta}} \leq e-2$, so in both cases the corresponding diagram in 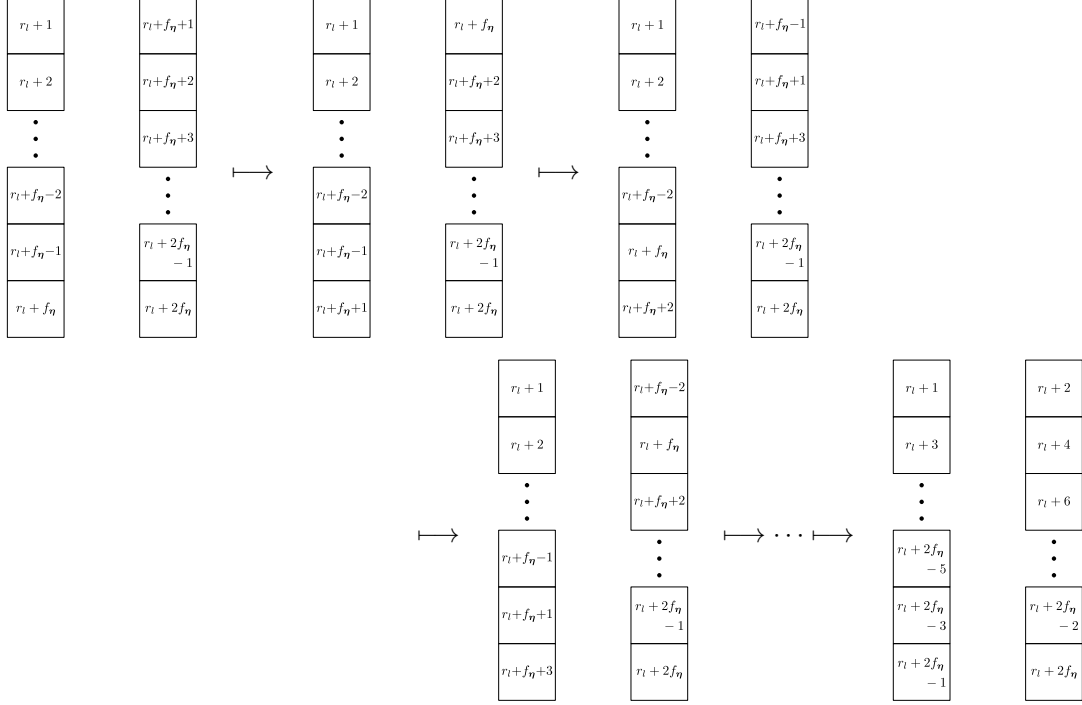the blob algebra factors as $x_l \psi_{f_{\boldsymbol{\eta}}+le}$ with x_l generated by transpositions of degree 0. Properties (i)–(iii) follow immediately. $\square$

Example 4.2. Let $n = 21$, $e = 6$ and $\kappa = (0, 3)$. The weight $\boldsymbol{\eta} = (1^{21}, \emptyset)$ is singular because $21 \equiv 3 - 0 \pmod 6$. Observe that $\boldsymbol{\eta}_1 = (1^3, 1^{15})$ and that $\boldsymbol{\eta}_3 = (1^9, 1^{12})$. Then

$$\psi_{\mathfrak{t}^{\eta_3} \mathfrak{t}^{\eta_1}} =$$

We also have

$$x_1 x_2 =$$

Some immediate consequences of Proposition 4.1 include the following corollaries.

Corollary 4.3. *For all integers $0 \leq j \leq k \leq l \leq m$ we have $\psi_{\mathfrak{t}^{\eta_l} \mathfrak{t}^{\eta_k}} \psi_{\mathfrak{t}^{\eta_k} \mathfrak{t}^{\eta_j}} = \psi_{\mathfrak{t}^{\eta_l} \mathfrak{t}^{\eta_j}}.$*

Corollary 4.4. *For all integers $0 \leq j \leq k \leq m$ we have*

$$\psi_{\mathfrak{t}^{\eta_k} \mathfrak{t}^{\eta_k}_{\eta_j \eta_j}} = \psi^2_{f_{\boldsymbol{\eta}}+je} \psi^2_{f_{\boldsymbol{\eta}}+(j+1)e} \cdots \psi^2_{f_{\boldsymbol{\eta}}+(k-1)e} e(\boldsymbol{i}^{\eta_j}).$$

4.2. **Vanishing terms.** It will be important to know later that certain products vanish in B_n^{κ}. Somewhat surprisingly this can happen even when the total degree is small.

Lemma 4.5. *We have*

$$\psi_{\mathfrak{t}^{\eta_1} \mathfrak{t}^{\eta}_{\eta}} \psi_{\mathfrak{t}^{\eta} \mathfrak{t}^{\eta_1}} = \psi^2_{f_{\boldsymbol{\eta}}} e(s_{f_{\boldsymbol{\eta}}} \boldsymbol{i}^{\eta}) = 0.$$

Proof. From Proposition 4.1 it is clear that the first product above vanishes if and only if the second product vanishes. We expand the first product by pulling apart the double transposition of degree 2 and rewriting as a difference of dotted strings. In the first term, the left string with its dot can be pulled all the way to the left, because the residues of all the strings that it passes through are distinct. In the second term, the dot on the right string can jump almost all the way to the left, slide down a string, and then make one final jump to the leftmost string. Dots on the left vanish in B_n^κ, so we are done. The diagrams below depict this process when $f_{\boldsymbol{\eta}} = 4$.

$$= \quad 0 - 0.$$

$\square$
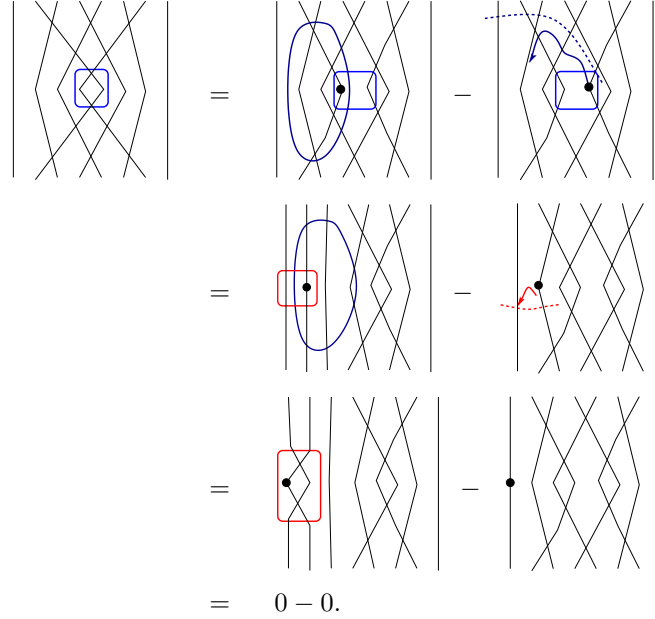
Another useful fact is that many cellular basis elements have a diamond as a factor, and thus vanish when multiplied by $\mathrm{JW}^{\boldsymbol{\eta}}$.

Proposition 4.6. *Suppose $k < l \le 2k$. Let $\mathfrak{t} \in \mathrm{Std}_{\boldsymbol{\eta}}(\boldsymbol{\eta}_l)$ with $\deg \mathfrak{t} = 2k - l$. Then $\psi_{\mathfrak{t}\mathfrak{t}_{\boldsymbol{\eta}}^{\eta_l}} \in U_j^{\boldsymbol{\eta}} B_n^\kappa$ for some $j \in \mathbb{N}$.*

Proof. Since $\deg \mathfrak{t} = 2k - l < k < l$ it is clear that $\mathfrak{t} \ne \mathfrak{t}_{\boldsymbol{\eta}}^{\boldsymbol{\eta}_l}$. This means that the path corresponding to $\mathfrak{t}$ must diverge from that of $\mathfrak{t}_{\boldsymbol{\eta}}^{\boldsymbol{\eta}_l}$, by leaving the fundamental alcove early, before turning back at some wall after $(f_{\boldsymbol{\eta}} + je)$ steps for some $j < l$. The skew tableau corresponding to the e steps before and after this turn-back point looks either like
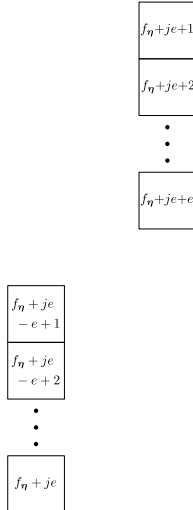
or like
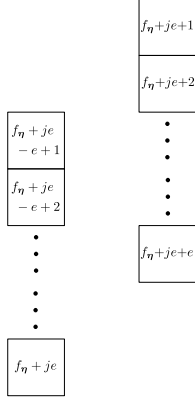
or their mirror images.

Suppose we are in the first case, e.g. with $\mathfrak{t}$ corresponding to the black path below:
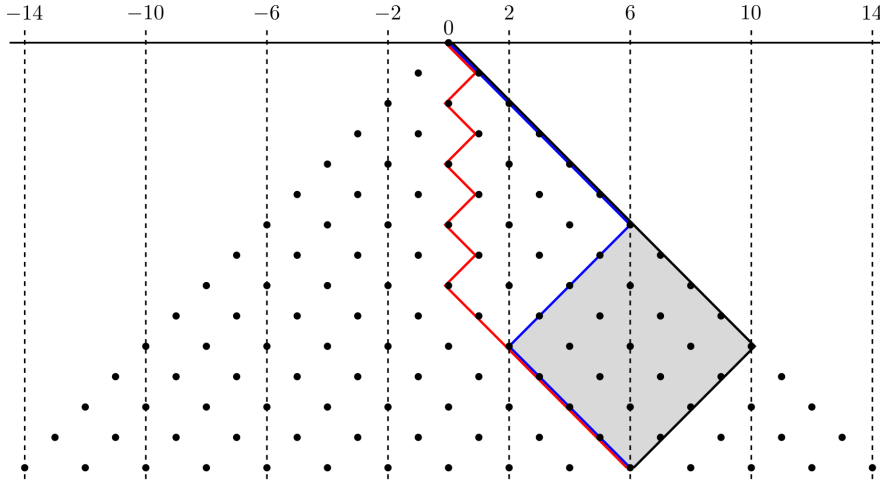
Let $u \in S_n$ be the permutation which swaps the one-column partitions in the skew tableau above and fixes everything else. As these columns contain two adjacent subsets of e entries each, u corresponds to a diamond permutation. Write $\underline{u}$ for standard the reduced expression for u coming from Definition 3.1. Applying u to $\mathfrak{t}$ yields a new tableau $\mathfrak{t}'$ with the same residue (corresponding to the blue path above). In the path picture, it is clear that the region in grey bounded by $\mathfrak{t}'$ and $\mathfrak{t}$ is entirely contained within the region bounded by $\mathfrak{t}^{\eta_l}$ (the red path) and $\mathfrak{t}$. This means that $\underline{d}_{\mathfrak{t}} = \underline{u}\underline{d}_{\mathfrak{t}'}$ as reduced expressions (see e.g. [12, Algorithm 5.2]). Thus

$$\psi_{\mathfrak{t}\mathfrak{t}_{\boldsymbol{\eta}}^{\eta_l}} = \psi_{\underline{u}}\psi_{\mathfrak{t}'\mathfrak{t}_{\boldsymbol{\eta}}^{\eta_l}} = U_j^{\boldsymbol{\eta}}\psi_{\mathfrak{t}'\mathfrak{t}_{\boldsymbol{\eta}}^{\eta_l}} \in U_j^{\boldsymbol{\eta}}B_n^{\kappa}$$

which proves the result.

Now suppose we are in the second case, e.g. with $\mathfrak{t}$ corresponding to the black path below:
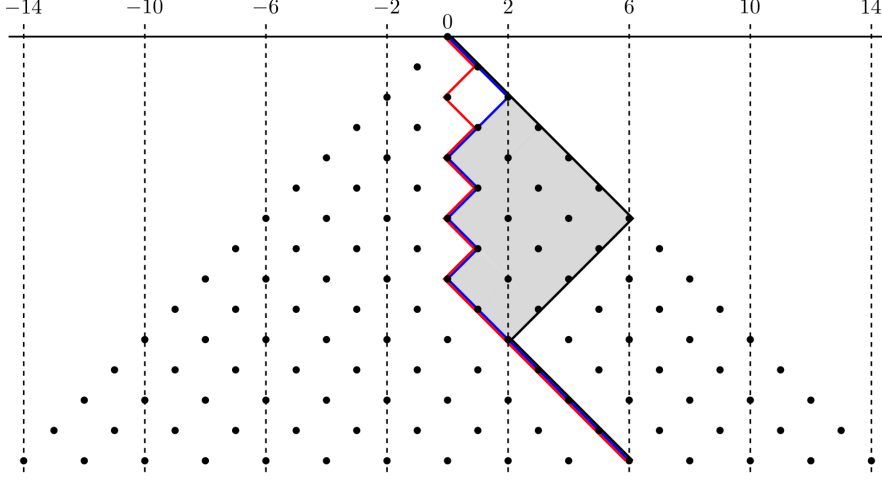
Let u' be the following permutation in blue
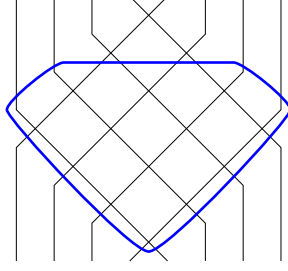
which we call a 'cut diamond' permutation, corresponding to the first

$$\begin{cases} 2e - f_{\boldsymbol{\eta}} & \text{if } j \text{ is even,} \\ e + f_{\boldsymbol{\eta}} - 1 & \text{if } j \text{ is odd} \end{cases}$$

layers of the diamond permutation centred at $f_{\boldsymbol{\eta}} + je$, and fix $\underline{u'}$ to be the corresponding reduced expression for u'. Applying u' to $\mathfrak{t}$ yields a new tableau $\mathfrak{t}'$ (corresponding to the blue path above), whose entries within the skew tableau are given by
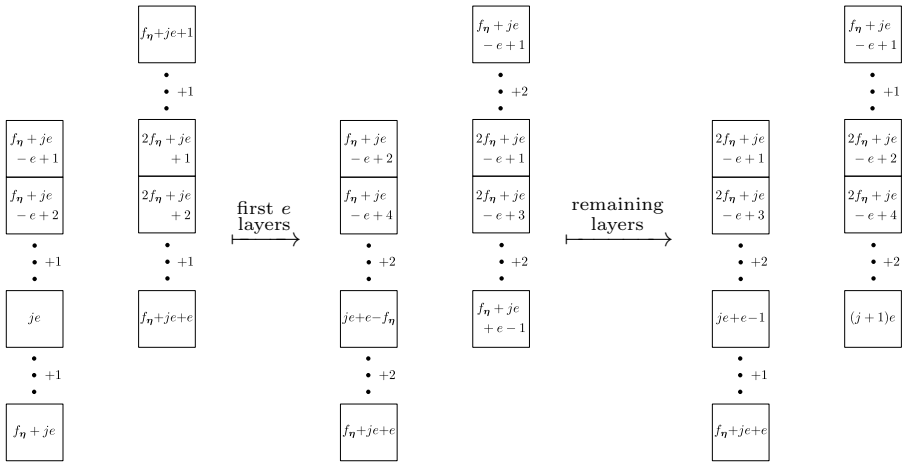
where we have written the increments in the omitted boxes with $+1$ or $+2$. Working in the path picture, we again note that the region in grey bounded by $\mathfrak{t}'$ and $\mathfrak{t}$ is entirely contained with the region bounded by $\mathfrak{t}^{\eta_l}$ (the red path) and $\mathfrak{t}$. As before this means that $\underline{d}_{\mathfrak{t}} = (\underline{u'})^{-1}\underline{d}_{\mathfrak{t}'}$ as reduced expressions. From Proposition 4.1 we also know that $x_j \psi_{f_{\boldsymbol{\eta}}+je} e(\boldsymbol{i^{\eta}})$ is a factor of $\psi_{\mathfrak{t}^{\eta_l} \mathfrak{t}^{\eta_l}_{\boldsymbol{\eta}}}$. Moreover, from the proof of this proposition, there is a reduced expression $\underline{u''}$ (which comes from the complement of the 'cut diamond' in the KLR diagram above) for which $x_j \psi_{f_{\boldsymbol{\eta}}+je} e(\boldsymbol{i^{\eta}}) = \psi_{\underline{u''}} e(\boldsymbol{i^{\eta}})$

and $\underline{u} = (\underline{u'})^{-1}\underline{u''}$. In fact, the construction of $\mathfrak{t}'$ ensures that $d_{\mathfrak{t}'}$ commutes with u'', because the support of u'' (i.e. the elements not fixed by u'') is fixed by $d_{\mathfrak{t}'}$. Combining everything together, we get

$$\psi_{\mathfrak{t}\mathfrak{t}_{\boldsymbol{\eta}}^{\eta_l}} = \psi_{(\underline{u'})^{-1}}\psi_{\mathfrak{t}'\mathfrak{t}^{\eta_l}}\psi_{\mathfrak{t}^{\eta_l}\mathfrak{t}_{\boldsymbol{\eta}}^{\eta_l}} \in \psi_{(\underline{u'})^{-1}}\psi_{\underline{d}_{\mathfrak{t}'}}\psi_{\underline{u''}}e(\boldsymbol{i^\eta})B_n^\kappa = \psi_{(\underline{u'})^{-1}}\psi_{\underline{u''}}e(\boldsymbol{i^\eta})\psi_{\underline{d}_{\mathfrak{t}'}}B_n^\kappa \subseteq U_j^{\boldsymbol{\eta}}B_n^\kappa.$$

$\square$

The next result identifies possible candidates for generators of the socle of $P(\boldsymbol{\eta})$.

Lemma 4.7. *If* $\operatorname{soc} P(\boldsymbol{\eta})$ *contains a copy of* $L(\boldsymbol{\eta})\langle 2k \rangle$ *for some* $k \geq 0$, *then it must be the subspace*

$$\Bbbk \mathrm{JW}^{\boldsymbol{\eta}}\psi_{\mathfrak{t}_{\boldsymbol{\eta}}^{\eta_k}\mathfrak{t}_{\boldsymbol{\eta}}^{\eta_k}}\mathrm{JW}^{\boldsymbol{\eta}}$$

Proof. $L(\boldsymbol{\eta})$ is 1-dimensional, so it restricts to the unique 1-dimensional irreducible representation of the Temperley–Lieb subalgebra. This means $\mathrm{JW}^{\boldsymbol{\eta}}$ acts on it as the identity, and thus any submodule isomorphic to $L(\boldsymbol{\eta})\langle 2k \rangle$ lies in the degree $2k$ part of $\mathrm{JW}^{\boldsymbol{\eta}}B_n^\kappa\mathrm{JW}^{\boldsymbol{\eta}}$. From Corollary 3.7, all cellular basis elements $\psi \neq \psi_{\mathfrak{t}_{\boldsymbol{\eta}}^{\eta_k}\mathfrak{t}_{\boldsymbol{\eta}}^{\eta_k}}$ with top and bottom residues $\boldsymbol{i^\eta}$ and degree $2k$ which do not vanish in $P(\boldsymbol{\eta}) = B_n^\kappa e(\boldsymbol{i^\eta})\mathrm{JW}^{\boldsymbol{\eta}}$ have the form $\psi = \psi_{\mathfrak{t}\mathfrak{t}_{\boldsymbol{\eta}}^{\eta_l}}$ for some integer $k < l \leq 2k$ with $\deg\mathfrak{t} = 2k - l$. By Proposition 4.6 such basis elements factor as $\psi = U_j^{\boldsymbol{\eta}}x$ for some $j \in \mathbb{N}$ and $x \in B_n^\kappa$. Since $\mathrm{JW}^{\boldsymbol{\eta}}U_j^{\boldsymbol{\eta}} = 0$ the result follows. $\square$

4.3. **Diamond simplification.** By Lemma 4.7, determining the socle of $P(\boldsymbol{\eta})$ will necessitate calculations involving $\mathrm{JW}^{\boldsymbol{\eta}}$. The next few lemmas give some methods for reducing the workload by eliminating diamonds.

Lemma 4.8. *For all* k *we have*

$$\psi_{f_{\boldsymbol{\eta}}+(k-1)e}U_k^{\boldsymbol{\eta}}\psi_{f_{\boldsymbol{\eta}}+(k-1)e} = \pm\psi_{f_{\boldsymbol{\eta}}+ke}^2 e(s_{f_{\boldsymbol{\eta}}+(k-1)e}\boldsymbol{i^\eta}).$$

Proof. Apply [12, Lemma 5.16] several times across the diamond. The remaining transpositions are all of degree 0 except for the degree 1 transpositions at the top and bottom. The degree 0 transpositions cancel out and the result follows. The diagrams below depict what happens when $e = 6$.

$\square$
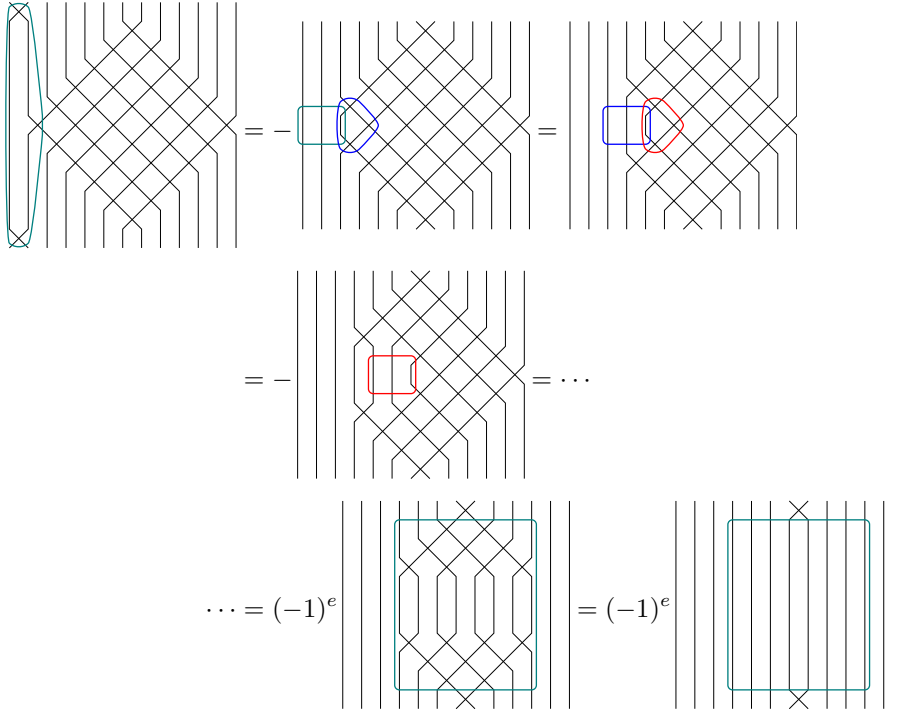
Lemma 4.9. *For all k we have*

$$U_k^{\boldsymbol{\eta}} U_{k-1}^{\boldsymbol{\eta}} \psi_{f_{\boldsymbol{\eta}+ke}}^2 = \pm U_k^{\boldsymbol{\eta}} \psi_{f_{\boldsymbol{\eta}+(k-1)e}}^2.$$

Proof. This follows immediately from a variant of Lemma 4.8, which is proved in the same way.

$\square$
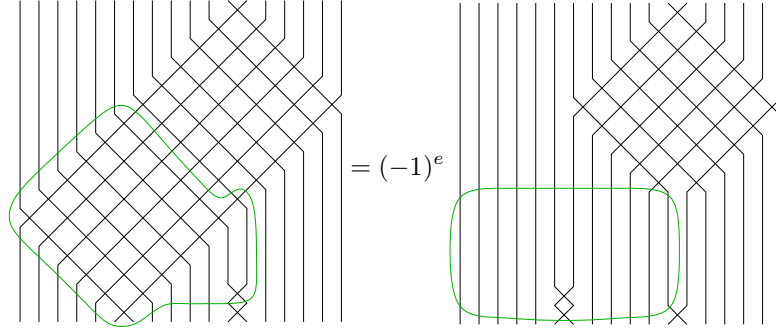
Lemma 4.10. *For all $1 < k < m$ we have*

$$U_{k-1}^{\boldsymbol{\eta}} \psi_{\mathfrak{t}_{\boldsymbol{\eta}}^{\eta_k} \mathfrak{t}_{\boldsymbol{\eta}}^{\eta_k}} = 0.$$

Proof. Use Proposition 4.1 to rewrite $\psi_{\mathfrak{t}_{\boldsymbol{\eta}}^{\eta_k} \mathfrak{t}_{\boldsymbol{\eta}}^{\eta_k}}$ as a product of double transpositions. Expand the rightmost double transposition as a difference of dotted strings. First we show that these dots can 'migrate' leftwards until they lie on top of the next pair of transpositions. In the first term, the dot on the left string can jump until it is on the right string above this double transposition. In the second term, the dot on the right string can slide along the southwest border of the diamond, jump left one string and slide until it is in place on the left string above the double transposition.
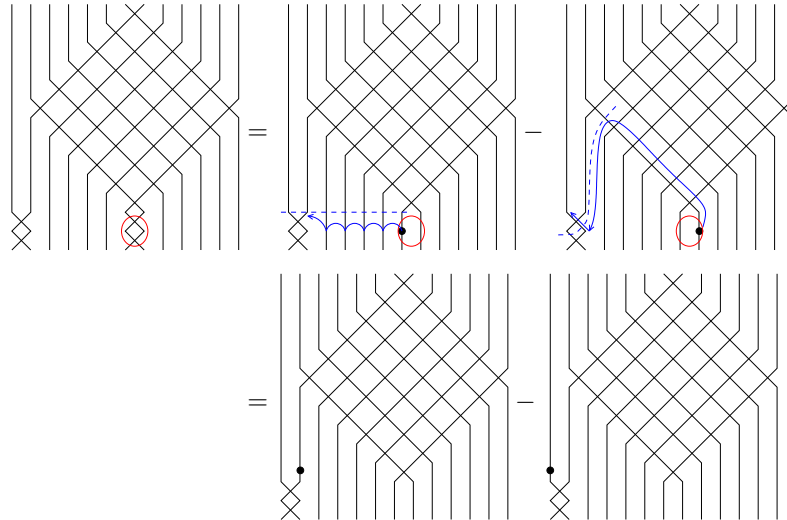
Next, we show we can continue this migration process leftwards without the diamond. As before, the dot on the left string above the double transposition can jump several strings leftwards until it is on the right string above the next double transposition. For the dot on the right string, we replace the both pairs of transpositions with pairs of maximally sized triangles, as seen in the proof of Proposition 4.1. This dot then slides southwest along its string, jumps one string, and slides

northwest until it is in the correct position.
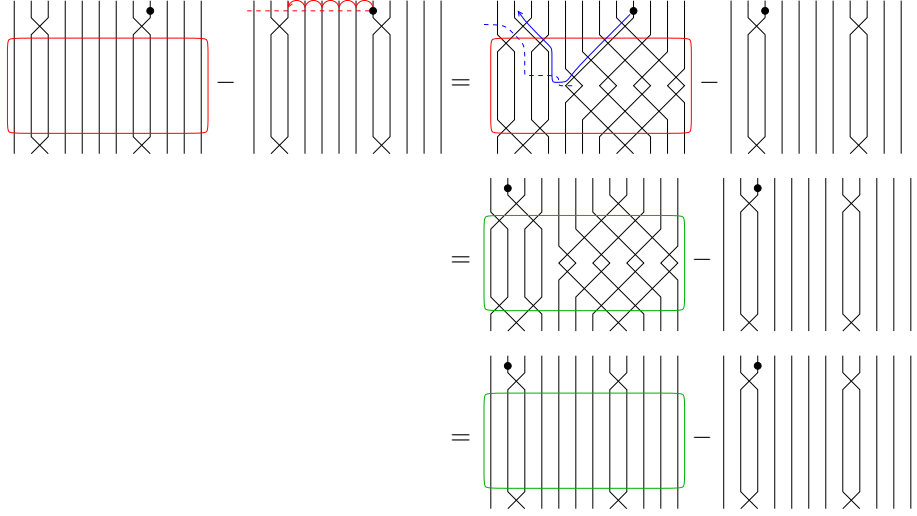
Note that in both of the figures above we are only drawing a portion of the complete diagram.

Finally we end up with a difference of dotted strings for the leftmost double transposition. But we can replace this difference with another double transposition. Applying Lemma 4.5 gives the result. □

4.4. **Socle calculation.** We pool together our previous results into one grand calculation to identify the socle of $P(\boldsymbol{\eta})$. The heart of the argument is to show that certain products of $\mathrm{JW}^{\boldsymbol{\eta}}$ with cellular basis elements do not vanish in B_n^κ. This is potentially extremely difficult, as the number of summands when $\mathrm{JW}^{\boldsymbol{\eta}}$ is written in the standard monomial basis grows very quickly. Thankfully many of these monomials end up vanishing in the product. For $r \leq s$ write $U_{r,s}^{\boldsymbol{\eta}} = U_r^{\boldsymbol{\eta}} U_{r+1}^{\boldsymbol{\eta}} \cdots U_s^{\boldsymbol{\eta}}$. First, we identify a non-vanishing monomial in the product.

Theorem 4.11. *Let $r \leq s$. If*
$$\psi_{\mathfrak{t}^{\eta_1} \mathfrak{t}_{\boldsymbol{\eta}}^{\eta_1}} U_{r,s}^{\boldsymbol{\eta}} \psi_{\mathfrak{t}_{\boldsymbol{\eta}}^{\eta_k} \mathfrak{t}_{\boldsymbol{\eta}}^{\eta_k}} \mathrm{JW}^{\boldsymbol{\eta}} \neq 0$$
then $(r,s) = (1,k)$. In this case, we have
$$\psi_{\mathfrak{t}^{\eta_1} \mathfrak{t}_{\boldsymbol{\eta}}^{\eta_1}} U_{1,k}^{\boldsymbol{\eta}} \psi_{\mathfrak{t}_{\boldsymbol{\eta}}^{\eta_k} \mathfrak{t}_{\boldsymbol{\eta}}^{\eta_k}} \mathrm{JW}^{\boldsymbol{\eta}} = \pm \psi_{\mathfrak{t}_{\boldsymbol{\eta}_1}^{\eta_{k+1}} \mathfrak{t}_{\boldsymbol{\eta}}^{\eta_{k+1}}} \mathrm{JW}^{\boldsymbol{\eta}}.$$

Proof. When $r > 1$, we have
$$\psi_{\mathfrak{t}^{\eta_1} \mathfrak{t}_{\boldsymbol{\eta}}^{\eta_1}} U_r^{\boldsymbol{\eta}} = U_{r-1}^{\boldsymbol{\eta}_1} \psi_{\mathfrak{t}^{\eta_1} \mathfrak{t}_{\boldsymbol{\eta}}^{\eta_1}} \tag{19}$$
by Proposition 4.1. Similarly when $r > k$, we have
$$U_r^{\boldsymbol{\eta}} \psi_{\mathfrak{t}_{\boldsymbol{\eta}}^{\eta_k} \mathfrak{t}_{\boldsymbol{\eta}}^{\eta_k}} = \psi_{\mathfrak{t}_{\boldsymbol{\eta}}^{\eta_k} \mathfrak{t}_{\boldsymbol{\eta}}^{\eta_k}} U_r^{\boldsymbol{\eta}}. \tag{20}$$

When $1 < r \leq s$ we have
$$\begin{aligned}
\psi_{\mathfrak{t}^{\eta_1} \mathfrak{t}_{\boldsymbol{\eta}}^{\eta_1}} U_{r,s}^{\boldsymbol{\eta}} \psi_{\mathfrak{t}_{\boldsymbol{\eta}}^{\eta_k} \mathfrak{t}_{\boldsymbol{\eta}}^{\eta_k}} &= U_{r-1,s-1}^{\boldsymbol{\eta}_1} \psi_{\mathfrak{t}^{\eta_1} \mathfrak{t}_{\boldsymbol{\eta}}^{\eta_1}} \psi_{\mathfrak{t}_{\boldsymbol{\eta}}^{\eta_k} \mathfrak{t}_{\boldsymbol{\eta}}^{\eta_k}} \\
&= U_{r-1,s-1}^{\boldsymbol{\eta}_1} \psi_{\mathfrak{t}^{\eta_1} \mathfrak{t}_{\boldsymbol{\eta}}^{\eta_1}} \psi_{f_{\boldsymbol{\eta}}+e}^2 \psi_{f_{\boldsymbol{\eta}}+2e}^2 \cdots \psi_{f_{\boldsymbol{\eta}}+(k-1)e}^2 \\
&= U_{r-1,s-1}^{\boldsymbol{\eta}_1} \psi_{\mathfrak{t}^{\eta_1} \mathfrak{t}_{\boldsymbol{\eta}}^{\eta_1}} \psi_{\mathfrak{t}^{\eta_1} \mathfrak{t}_{\boldsymbol{\eta}}^{\eta_1}} \psi_{f_{\boldsymbol{\eta}}+2e}^2 \cdots \psi_{f_{\boldsymbol{\eta}}+(k-1)e}^2 \\
&= 0
\end{aligned}$$
using (19), Corollary 4.4, and Lemma 4.5. Similarly when $r \leq s \leq k-1$ this expression vanishes by Corollary 4.4 and Lemma 4.10. Finally
$$U_{r,s}^{\boldsymbol{\eta}} \psi_{\mathfrak{t}_{\boldsymbol{\eta}}^{\eta_k} \mathfrak{t}_{\boldsymbol{\eta}}^{\eta_k}} \mathrm{JW}^{\boldsymbol{\eta}} = U_{r,s-1}^{\boldsymbol{\eta}} \psi_{\mathfrak{t}_{\boldsymbol{\eta}}^{\eta_k} \mathfrak{t}_{\boldsymbol{\eta}}^{\eta_k}} U_s^{\boldsymbol{\eta}} \mathrm{JW}^{\boldsymbol{\eta}} = 0$$
if $s > k$ by (20) and the defining property of $\mathrm{JW}^{\boldsymbol{\eta}}$. Putting this all together, if
$$\psi_{\mathfrak{t}^{\eta_1} \mathfrak{t}_{\boldsymbol{\eta}}^{\eta_1}} U_{r,s}^{\boldsymbol{\eta}} \psi_{\mathfrak{t}_{\boldsymbol{\eta}}^{\eta_k} \mathfrak{t}_{\boldsymbol{\eta}}^{\eta_k}} \mathrm{JW}^{\boldsymbol{\eta}} \neq 0$$
then $r = 1$ and $s = k$.

Using Corollary 4.4 and Lemma 4.9, we observe that

$$\psi_{\mathfrak{t}^{\eta_1}\mathfrak{t}^{\eta}_{\eta}} U^{\eta}_{1,k} \psi_{\mathfrak{t}^{\eta_k}_{\eta}\mathfrak{t}^{\eta_k}_{\eta}} \mathrm{JW}^{\eta} = \psi_{\mathfrak{t}^{\eta_1}\mathfrak{t}^{\eta}_{\eta}} U^{\eta}_{1,k-2} U^{\eta}_{k-1} U^{\eta}_{k} \psi^2_{f_{\eta}+(k-1)e} \psi^2_{f_{\eta}+(k-2)e} \cdots \psi^2_{f_{\eta}+e} \mathrm{JW}^{\eta}$$

$$= \pm \psi_{\mathfrak{t}^{\eta_1}\mathfrak{t}^{\eta}_{\eta}} U^{\eta}_{1,k-2} U^{\eta}_{k-1} U^{\eta}_{k} U^{\eta}_{k-1} \psi^2_{f_{\eta}+ke} \psi^2_{f_{\eta}+(k-2)e} \psi^2_{f_{\eta}+(k-3)e} \cdots \psi^2_{f_{\eta}+e} \mathrm{JW}^{\eta}$$

$$= \pm \psi_{\mathfrak{t}^{\eta_1}\mathfrak{t}^{\eta}_{\eta}} U^{\eta}_{1,k-1} \psi^2_{f_{\eta}+(k-2)e} \psi^2_{f_{\eta}+(k-3)e} \cdots \psi^2_{f_{\eta}+e} \psi^2_{f_{\eta}+ke} \mathrm{JW}^{\eta}.$$

Apply this several times to obtain

$$\psi_{\mathfrak{t}^{\eta_1}\mathfrak{t}^{\eta}_{\eta}} U^{\eta}_{1,k} \psi_{\mathfrak{t}^{\eta_k}_{\eta}\mathfrak{t}^{\eta_k}_{\eta}} \mathrm{JW}^{\eta} = \pm \psi_{\mathfrak{t}^{\eta_1}\mathfrak{t}^{\eta}_{\eta}} U^{\eta}_{1} \psi^2_{f_{\eta}} \psi^2_{f_{\eta}+2e} \psi^2_{f_{\eta}+3e} \cdots \psi^2_{f_{\eta}+ke} \mathrm{JW}^{\eta}.$$

Then by Lemma 4.8 and Corollaries 4.3 and 4.4 this is equal to

$$\pm \psi_{\mathfrak{t}^{\eta_1}\mathfrak{t}^{\eta}_{\eta}} \psi^2_{f_{\eta}+e} \psi^2_{f_{\eta}+2e} \psi^2_{f_{\eta}+3e} \cdots \psi^2_{f_{\eta}+ke} \mathrm{JW}^{\eta} = \pm \psi^2_{f_{\eta}+e} \psi^2_{f_{\eta}+2e} \psi^2_{f_{\eta}+3e} \cdots \psi^2_{f_{\eta}+ke} \psi_{\mathfrak{t}^{\eta_1}\mathfrak{t}^{\eta}_{\eta}} \mathrm{JW}^{\eta}$$

$$= \pm \psi_{\mathfrak{t}^{\eta_{k+1}}_{\eta_1}\mathfrak{t}^{\eta_1}} \psi_{\mathfrak{t}^{\eta_1}\mathfrak{t}^{\eta_{k+1}}_{\eta_1}} \psi_{\mathfrak{t}^{\eta_1}\mathfrak{t}^{\eta}_{\eta}} \mathrm{JW}^{\eta}$$

$$= \pm \psi_{\mathfrak{t}^{\eta_{k+1}}_{\eta_1}\mathfrak{t}^{\eta_{k+1}}_{\eta}} \mathrm{JW}^{\eta}.$$

$\square$

Next, we show that other monomials wind up in an ideal of B^{κ}_n.

Theorem 4.12. *Let U be a monomial in the generators of the Temperley–Lieb subalgebra. If $U \neq U^{\eta}_{1,k}$ then*

$$\psi_{\mathfrak{t}^{\eta_1}\mathfrak{t}^{\eta}_{\eta}} U \psi_{\mathfrak{t}^{\eta_k}_{\eta}\mathfrak{t}^{\eta_k}_{\eta}} \mathrm{JW}^{\eta} \in B^{\kappa, \rhd \eta_{k+1}}_n.$$

Proof. Every monomial in the generators of the Temperley–Lieb subalgebra can be written in the form $U^{\eta}_{r_1,s_1} U^{\eta}_{r_2,s_2} \cdots U^{\eta}_{r_p,s_p}$ for some strictly decreasing sequences $r_1 > r_2 > \cdots > r_p$ and $s_1 > s_2 > \cdots > s_p$ of some length $p \geq 0$ with $r_j \leq s_j$ for all j. Suppose $U \neq U_{1,k}$ is a monomial of this form such that

$$\psi_{\mathfrak{t}^{\eta_1}\mathfrak{t}^{\eta}_{\eta}} U \psi_{\mathfrak{t}^{\eta_k}_{\eta}\mathfrak{t}^{\eta_k}_{\eta}} \mathrm{JW}^{\eta} \neq 0.$$

First of all we must have $p \geq 1$ by Lemma 4.5. Since $r_j > r_p \geq 1$ and $s_j > s_p \geq r_p \geq 1$ for all $1 \leq j < p$, we can apply (19) to the expression above:

$$\psi_{\mathfrak{t}^{\eta_1}\mathfrak{t}^{\eta}_{\eta}} U^{\eta}_{r_1,s_1} U^{\eta}_{r_2,s_2} \cdots U^{\eta}_{r_{p-1},s_{p-1}} U^{\eta}_{r_p,s_p} \psi_{\mathfrak{t}^{\eta_k}_{\eta}\mathfrak{t}^{\eta_k}_{\eta}} \mathrm{JW}^{\eta} =$$

$$= U^{\eta_1}_{r_1-1,s_1-1} U^{\eta_1}_{r_2-1,s_2-1} \cdots U^{\eta_1}_{r_{p-1}-1,s_{p-1}-1} \psi_{\mathfrak{t}^{\eta_1}\mathfrak{t}^{\eta}_{\eta}} U^{\eta}_{r_p,s_p} \psi_{\mathfrak{t}^{\eta_k}_{\eta}\mathfrak{t}^{\eta_k}_{\eta}} \mathrm{JW}^{\eta}.$$

Theorem 4.11 then implies that $r_p = 1$ and $s_p = k$. Assuming $U \neq U_{1,k}$, we must have $p > 1$.

Now suppose $s_{p-1} > k + 1$. Applying Theorem 4.11 again as well as (19) and (20), we observe that

$$U^{\eta_1}_{s_{p-1}-1} \psi_{\mathfrak{t}^{\eta_1}\mathfrak{t}^{\eta}_{\eta}} U^{\eta}_{1,k} \psi_{\mathfrak{t}^{\eta_k}_{\eta}\mathfrak{t}^{\eta_k}_{\eta}} \mathrm{JW}^{\eta} = \pm U^{\eta_1}_{s_{p-1}-1} \psi_{\mathfrak{t}^{\eta_{k+1}}_{\eta_1}\mathfrak{t}^{\eta_{k+1}}_{\eta}} \mathrm{JW}^{\eta}$$

$$= \pm \psi_{\mathfrak{t}^{\eta_{k+1}}_{\eta_1}\mathfrak{t}^{\eta_{k+1}}_{\eta}} U^{\eta}_{s_{p-1}} \mathrm{JW}^{\eta}$$

$$= 0.$$

This is a factor of the previous expression, so it follows that $s_{p-1} = k + 1$. Thus it is enough to show that

$$\psi_{\mathfrak{t}^{\eta_1}\mathfrak{t}^{\eta}_{\eta}} U^{\eta}_{k+1} U^{\eta}_{1,k} \psi_{\mathfrak{t}^{\eta_k}_{\eta}\mathfrak{t}^{\eta_k}_{\eta}} \mathrm{JW}^{\eta} = \pm U^{\eta_1}_{k} \psi_{\mathfrak{t}^{\eta_{k+1}}_{\eta_1}\mathfrak{t}^{\eta_{k+1}}_{\eta}} \mathrm{JW}^{\eta} \in B^{\kappa, \rhd \eta_{k+1}}_n.$$

Using Corollaries 4.3 and 4.4 this is equal to

$$\pm U^{\eta_1}_{k} \psi^2_{f_{\eta}+ke} \psi^2_{f_{\eta}+(k-1)e} \cdots \psi_{f_{\eta}+e} \psi_{\mathfrak{t}^{\eta}\mathfrak{t}^{\eta_1}_{\eta}} \mathrm{JW}^{\eta}$$

In the proof of Proposition 4.1 we showed that $U^{\eta_1}_{k} = z_{k+1} x_{k+1} \psi_{f_{\eta}+(k+1)e}$ for some $z_{k+1} \in B^{\kappa}_n$. Thus we obtain

$$\pm U^{\eta_1}_{k} \psi^2_{f_{\eta}+ke} \psi^2_{f_{\eta}+(k-1)e} \cdots \psi_{f_{\eta}+e} \psi_{\mathfrak{t}^{\eta}\mathfrak{t}^{\eta_1}_{\eta}} \mathrm{JW}^{\eta} \in B^{\kappa}_n \psi_{\mathfrak{t}^{\eta_{k+2}}\mathfrak{t}^{\eta_{k+2}}_{\eta}} \mathrm{JW}^{\eta} \leq B^{\kappa, \rhd \eta_{k+1}}_n.$$

$\square$

Finally we are in a position to calculate the socle.

Theorem 4.13. *We have $\operatorname{soc} P(\eta) = L(\eta)\langle 2m \rangle$.*

Proof. Since $P(\boldsymbol{\eta})$ has a Weyl filtration and the socle of every Weyl module is $L(\boldsymbol{\eta})$, it is clear that $\operatorname{soc} P(\boldsymbol{\eta})$ is the direct sum of copies of $L(\boldsymbol{\eta})$. The graded decomposition numbers for singular weights (from Theorem 2.9) indicate that the socle can contain at most one copy of $L(\boldsymbol{\eta})\langle 2k\rangle$ for each integer $0 \le k \le m$ and no copies of $L(\boldsymbol{\eta})$ in odd degree. The submodule $L(\boldsymbol{\eta}) \le \Delta(\boldsymbol{\eta}_m) \le P(\boldsymbol{\eta})$ gives one copy of $L(\boldsymbol{\eta})$ of degree $2m$ in the socle. By Lemma 4.7, if $\operatorname{soc} P(\boldsymbol{\eta})$ does contain a copy of $L(\boldsymbol{\eta})\langle 2k\rangle$ for some $k < m$, then it must be spanned by

$$\mathrm{JW}^{\boldsymbol{\eta}}\psi_{\mathfrak{t}_{\boldsymbol{\eta}}^{\eta_k}\mathfrak{t}_{\boldsymbol{\eta}}^{\eta_k}}\mathrm{JW}^{\boldsymbol{\eta}}.$$

We will prove that this vector does not generate a copy of $L(\boldsymbol{\eta})$ in the socle by showing that

$$\psi_{\mathfrak{t}^{\eta_1}\mathfrak{t}_{\boldsymbol{\eta}}^{\eta_1}}\mathrm{JW}^{\boldsymbol{\eta}}\psi_{\mathfrak{t}_{\boldsymbol{\eta}}^{\eta_k}\mathfrak{t}_{\boldsymbol{\eta}}^{\eta_k}}\mathrm{JW}^{\boldsymbol{\eta}} \ne 0.$$

Write $\mathrm{JW}^{\boldsymbol{\eta}}$ as a sum of monomials. It is known that the coefficient of $U_{1,k}^{\boldsymbol{\eta}}$ in $\mathrm{JW}^{\boldsymbol{\eta}}$ is non-zero (see e.g. [6, Proposition 3.10]), so we may write

$$\mathrm{JW}^{\boldsymbol{\eta}} = cU_{1,k}^{\boldsymbol{\eta}} + \sum_{\text{monomials } U \ne U_{1,k}^{\boldsymbol{\eta}}} c_U U$$

where $c, c_U \in \Bbbk$ and $c \ne 0$. Then using Theorems 4.11 and 4.12 we obtain

$$\psi_{\mathfrak{t}^{\eta_1}\mathfrak{t}_{\boldsymbol{\eta}}^{\eta_1}}\mathrm{JW}^{\boldsymbol{\eta}}\psi_{\mathfrak{t}_{\boldsymbol{\eta}}^{\eta_k}\mathfrak{t}_{\boldsymbol{\eta}}^{\eta_k}}\mathrm{JW}^{\boldsymbol{\eta}} = \psi_{\mathfrak{t}^{\eta_1}\mathfrak{t}_{\boldsymbol{\eta}}^{\eta_1}}\left(cU_{1,k}^{\boldsymbol{\eta}} + \sum_{\text{monomials } U \ne U_{1,k}^{\boldsymbol{\eta}}} c_U U\right)\psi_{\mathfrak{t}_{\boldsymbol{\eta}}^{\eta_k}\mathfrak{t}_{\boldsymbol{\eta}}^{\eta_k}}\mathrm{JW}^{\boldsymbol{\eta}}$$

$$\in \psi_{\mathfrak{t}_{\boldsymbol{\eta}_1}^{\eta_{k+1}}\mathfrak{t}_{\boldsymbol{\eta}}^{\eta_{k+1}}}\mathrm{JW}^{\boldsymbol{\eta}} + B_n^{\kappa, \rhd\boldsymbol{\eta}_{k+1}}.$$

As $\psi_{\mathfrak{t}_{\boldsymbol{\eta}_1}^{\eta_{k+1}}\mathfrak{t}_{\boldsymbol{\eta}}^{\eta_{k+1}}}\mathrm{JW}^{\boldsymbol{\eta}} \notin B_n^{\kappa, \rhd\boldsymbol{\eta}_{k+1}}$ we are done. $\square$

Applying the globalisation functor, we see that $G\Delta(\boldsymbol{\eta}) = \Delta(1^{n+1}, 1)$ and $GP(\boldsymbol{\eta}) = P(1^{n+1}, 1)$. Using adjunction we see that

$$\operatorname{Hom}_{B_{n+2}^{\kappa}}(G\Delta(\boldsymbol{\eta}), GP(\boldsymbol{\eta})) \cong \operatorname{Hom}_{B_n^{\kappa}}(\Delta(\boldsymbol{\eta}), FGP(\boldsymbol{\eta})) = \operatorname{Hom}_{B_n^{\kappa}}(L(\boldsymbol{\eta}), P(\boldsymbol{\eta}))$$

so $P(1^{n+1}, 1)$ also has a simple socle. Repeated globalisation in this manner allows us to drop our assumption on n and extend our result to all singular weights. For a singular weight $\boldsymbol{\lambda} \in \Lambda(n)$, write $\boldsymbol{\lambda}_{\min}, \boldsymbol{\lambda}_{\max} \in \Lambda(n)$ for the unique minimal and maximal weights respectively in the same linkage class.

Corollary 4.14. *Let n be arbitrary, and let $\boldsymbol{\lambda} \in \Lambda(n)$ be a singular weight. Then we have*

$$\operatorname{soc} P(\boldsymbol{\lambda}) = L(\boldsymbol{\lambda}_{\min})\langle 2\deg \mathfrak{t}_{\boldsymbol{\lambda}}^{\boldsymbol{\lambda}_{\max}} + \deg \mathfrak{t}_{\boldsymbol{\lambda}_{\min}}^{\boldsymbol{\lambda}}\rangle.$$

5. Main results

5.1. **Regular projective modules.** We introduce some useful weight terminology. Let $\boldsymbol{\lambda} \in \Lambda(n)$. If the linkage class of $\boldsymbol{\lambda}$ has a unique $\boldsymbol{\lambda}' \in \Lambda(n)$ which is incomparable to $\boldsymbol{\lambda}$ then we say that $\boldsymbol{\lambda}$ is *paired*. Otherwise we call $\boldsymbol{\lambda}$ *unpaired*. For example, every singular weight is unpaired because singular linkage classes are totally ordered. On the other hand the poset structure of a regular linkage class means that the only regular unpaired weights are either maximal (i.e. are contained in the fundamental alcove) or possibly minimal.

Lemma 5.1. *Let $\boldsymbol{\lambda} = (1^{\lambda_1}, 1^{\lambda_2}) \in \Lambda(n)$ be a regular weight. Then $\boldsymbol{\lambda}$ is unpaired if and only if $\ell(w_{\boldsymbol{\lambda}}) = 0$ or $|\lambda_1 - \lambda_2| < 2\ell(w_{\boldsymbol{\lambda}})e - n$.*

Proof. Suppose that $\boldsymbol{\lambda}$ is not contained in the fundamental alcove and that $\lambda_1 > \lambda_2$. Let $w_{\boldsymbol{\lambda}}'$ be the unique element of W such that $\ell(w_{\boldsymbol{\lambda}}') = \ell(w_{\boldsymbol{\lambda}})$ but $w_{\boldsymbol{\lambda}}' \ne w_{\boldsymbol{\lambda}}$. Since $\ell(w_{\boldsymbol{\lambda}}' w_{\boldsymbol{\lambda}}^{-1}) = 2\ell(w_{\boldsymbol{\lambda}})$, the unique incomparable *classical* weight in the global linkage class of $(\lambda_1 - \lambda_2)$ is $(\lambda_1 - \lambda_2) - 2\ell(w_{\boldsymbol{\lambda}})e$, which does not correspond to a weight in $\Lambda(n)$ if it is less than $-n$. The case where $\lambda_1 < \lambda_2$ is similar. $\square$

Generalising our singular terminology, for an arbitrary weight $\boldsymbol{\lambda} \in \Lambda(n)$ write $\boldsymbol{\lambda}_{\min} \in \Lambda(n)$ for *some* minimal weight in the linkage class of $\boldsymbol{\lambda}$ and $\boldsymbol{\lambda}_{\max} \in \Lambda(n)$ for the unique maximal weight in the same linkage class. For $\boldsymbol{\lambda}$ regular it is evident that $\boldsymbol{\lambda}_{\max} = \boldsymbol{\lambda}_{\mathrm{fund}}$. We now can extend Corollary 4.14 to all weights.

Theorem 5.2. *Let $\boldsymbol{\lambda} \in \Lambda(n)$. We have*

$$\operatorname{soc} P(\boldsymbol{\lambda}) = \begin{cases} (L(\boldsymbol{\lambda}_{\min}) \oplus L(\boldsymbol{\lambda}'_{\min}))\langle 2 \deg \mathfrak{t}_{\boldsymbol{\lambda}}^{\boldsymbol{\lambda}_{\max}} + \deg \mathfrak{t}_{\boldsymbol{\lambda}_{\min}}^{\boldsymbol{\lambda}} \rangle & \text{if } \boldsymbol{\lambda}_{\min} \text{ is paired,} \\ L(\boldsymbol{\lambda}_{\min})\langle 2 \deg \mathfrak{t}_{\boldsymbol{\lambda}}^{\boldsymbol{\lambda}_{\max}} + \deg \mathfrak{t}_{\boldsymbol{\lambda}_{\min}}^{\boldsymbol{\lambda}} \rangle & \text{if } \boldsymbol{\lambda}_{\min} \text{ is unpaired.} \end{cases}$$

Proof. We prove the ungraded result first. Note that for any $\boldsymbol{\mu} \rhd \boldsymbol{\lambda}$ in the same linkage class, the ungraded socle of $\Delta(\boldsymbol{\mu})$ is

$$\begin{cases} L(\boldsymbol{\lambda}_{\min}) \oplus L(\boldsymbol{\lambda}'_{\min}) & \text{if } \boldsymbol{\lambda}_{\min} \text{ is paired,} \\ L(\boldsymbol{\lambda}_{\min}) & \text{if } \boldsymbol{\lambda}_{\min} \text{ is unpaired.} \end{cases}$$

As $P(\boldsymbol{\lambda})$ is filtered by Weyl modules its socle may only contain copies of these simple modules.

Write $\boldsymbol{\lambda} = (1^{\lambda_1}, 1^{\lambda_2})$ and without loss of generality suppose $\lambda_1 > \lambda_2$. If $\boldsymbol{\lambda}$ lies in the fundamental alcove, then $P(\boldsymbol{\lambda}) = \Delta(\boldsymbol{\lambda})$ and the result follows by [16, Theorem 9.4], so we will assume that $\boldsymbol{\lambda}$ does not lie in the fundamental alcove. Take $k \in \mathbb{N}$ minimal such that $\boldsymbol{\mu} = (1^{\lambda_1}, 1^{\lambda_2+k}) \in \Lambda(n+k)$ lies on a wall and let $\boldsymbol{\lambda}^{(1)} = (1^{\lambda_1}, 1^{\lambda_2+k-1}) \in \Lambda(n+k-1)$. There is a minimal weight $\boldsymbol{\lambda}^{(1)}_{\min} \in \Lambda(n+k-1)$ in the linkage class of $\boldsymbol{\lambda}^{(1)}$ whose classical weight is only 1 away from $\boldsymbol{\mu}_{\min}$. We observe that

$$\operatorname{pr}_{\boldsymbol{\mu}}(\operatorname{ind} \Delta(\boldsymbol{\lambda}^{(1)}_{\min})) = \Delta(\boldsymbol{\mu}_{\min}),$$
$$\operatorname{pr}_{\boldsymbol{\mu}}(\operatorname{ind} \Delta((\boldsymbol{\lambda}^{(1)}_{\min})')) = \Delta((\boldsymbol{\mu}_{\min})_1),$$

and

$$\operatorname{res} P(\boldsymbol{\mu}) \cong F(\operatorname{ind} P(\boldsymbol{\mu})) = FP(1^{\lambda_1+1}, 1^{\lambda_2+k}) = P(\boldsymbol{\lambda}^{(1)})$$

using the tower of recollement structure on B_n^κ. Thus

$$\dim \operatorname{Hom}_{B_{n+k-1}^\kappa}(\Delta(\boldsymbol{\lambda}^{(1)}_{\min}), P(\boldsymbol{\lambda}^{(1)})) = \dim \operatorname{Hom} B_{n+k}^\kappa(\Delta(\boldsymbol{\mu}_{\min}), P(\boldsymbol{\mu})) = 1,$$
$$\dim \operatorname{Hom}_{B_{n+k-1}^\kappa}(\Delta((\boldsymbol{\lambda}^{(1)}_{\min})'), P(\boldsymbol{\lambda}^{(1)})) = \dim \operatorname{Hom}_{B_{n+k}^\kappa}(\Delta((\boldsymbol{\mu}_{\min})_1), P(\boldsymbol{\mu})) = 1$$

by Corollary 4.14. This establishes the result for $\boldsymbol{\lambda}^{(1)}$.

If $k = 1$, then we are done as $\boldsymbol{\lambda} = \boldsymbol{\lambda}^{(1)}$. Otherwise let $\boldsymbol{\lambda}^{(2)} = (1^{\lambda_1}, 1^{\lambda_2+k-2}) \in \Lambda(n+k-2)$. Again, there is at least one minimal weight $\boldsymbol{\lambda}^{(2)}_{\min}$ in the linkage class of $\boldsymbol{\lambda}^{(2)}$ whose classical weight is 1 away from $\boldsymbol{\lambda}^{(1)}_{\min}$ or $(\boldsymbol{\lambda}^{(1)}_{\min})'$, and if this weight is also paired then there is another minimal weight $(\boldsymbol{\lambda}^{(2)}_{\min})'$. It is clear that $\operatorname{pr}_{\boldsymbol{\lambda}^{(1)}}(\operatorname{ind} \Delta(\boldsymbol{\lambda}^{(2)}_{\min}))$ (and $\operatorname{pr}_{\boldsymbol{\lambda}^{(1)}}(\operatorname{ind} \Delta((\boldsymbol{\lambda}^{(2)}_{\min})'))$ if it exists) is a minimal weight Weyl module. We also have

$$\operatorname{pr}_{\boldsymbol{\lambda}^{(2)}}(\operatorname{res} P(\boldsymbol{\lambda}^{(1)})) \cong \operatorname{pr}_{\boldsymbol{\lambda}^{(2)}}(F(\operatorname{ind} P(\boldsymbol{\lambda}^{(1)})))$$
$$\cong F(\operatorname{pr}_{\boldsymbol{\lambda}^{(2)}}(\operatorname{ind} P(\boldsymbol{\lambda}^{(1)})))$$
$$= F(P(1^{\lambda_1+1}, 1^{\lambda_1+k-1}))$$
$$= P(\boldsymbol{\lambda}^{(2)}).$$

Thus $\dim \operatorname{Hom}_{B_{n+k-2}^\kappa}(\Delta(\boldsymbol{\lambda}^{(2)}_{\min}), P(\boldsymbol{\lambda}^{(2)})) = 1$ (and similarly for $(\boldsymbol{\lambda}^{(2)}_{\min})'$ if it exists) and the result holds for $\boldsymbol{\lambda}^{(2)}$. Continuing in this fashion, we obtain the ungraded result for $\boldsymbol{\lambda}^{(k)} = \boldsymbol{\lambda}$. The correct grade shift is apparent from the graded decomposition numbers of B_n^κ (Theorem 2.8). $\square$

5.2. Tilting modules.
We are finally in a position to present the main results of this paper.

Theorem 5.3. *Let $\boldsymbol{\lambda} \in \Lambda(n)$ be a maximal weight.*
  (i) *If $\boldsymbol{\lambda}_{\min}$ is unpaired, then $T(\boldsymbol{\lambda}) = P(\boldsymbol{\lambda}_{\min})\langle -\deg \mathfrak{t}_{\boldsymbol{\lambda}_{\min}}^{\boldsymbol{\lambda}} \rangle$.*
  (ii) *If $\boldsymbol{\lambda}_{\min}$ is paired, then $T(\boldsymbol{\lambda})$ is the unique non-split extension*

$$0 \to P(\boldsymbol{\lambda}_{\min})\langle -\deg \mathfrak{t}_{\boldsymbol{\lambda}_{\min}}^{\boldsymbol{\lambda}} \rangle \to T(\boldsymbol{\lambda}) \to \Delta(\boldsymbol{\lambda}'_{\min})\langle -\deg \mathfrak{t}_{\boldsymbol{\lambda}_{\min}}^{\boldsymbol{\lambda}} \rangle \to 0.$$

Proof. As in the previous theorem we prove the ungraded form of the result first. For the first claim, if $\boldsymbol{\lambda}_{\min}$ is unpaired then $\operatorname{soc} P(\boldsymbol{\lambda}_{\min}) = L(\boldsymbol{\lambda}_{\min})$ by Theorem 5.2. Thus $P(\boldsymbol{\lambda}_{\min})$ embeds inside $I(\boldsymbol{\lambda}_{\min})$. But both modules have the same character, so we must in fact have $P(\boldsymbol{\lambda}_{\min}) = I(\boldsymbol{\lambda}_{\min})$ is self-dual and therefore is an indecomposable tilting module. By weight considerations it must be a grade shift of $T(\boldsymbol{\lambda})$, which we reverse using the singular graded decomposition numbers.

For the second claim, we induct on n. Assume that the indecomposable tilting module in B_m^κ with the same classical weight has the structure above for all $m < n$. By stability of tilting multiplicities this implies that in B_n^κ we have

$$(T(\boldsymbol{\lambda}) : \Delta(\boldsymbol{\mu})) = 1 = (P(\boldsymbol{\lambda}_{\min}) : \Delta(\boldsymbol{\mu}))$$

whenever $\boldsymbol{\mu} \neq \boldsymbol{\lambda}_{\min}, \boldsymbol{\lambda}'_{\min}$. By [5, Lemma A4.1] and its proof $P(\boldsymbol{\lambda}_{\min})$ embeds inside $T(\boldsymbol{\lambda})$ and

$$(T(\boldsymbol{\lambda}) : \Delta(\boldsymbol{\lambda}'_{\min})) = \dim \operatorname{Ext}_{B_n^\kappa}^1(\Delta(\boldsymbol{\lambda}'_{\min}), P(\boldsymbol{\lambda}_{\min})),$$
$$(T(\boldsymbol{\lambda}) : \Delta(\boldsymbol{\lambda}_{\min})) = \dim \operatorname{Ext}_{B_n^\kappa}^1(\Delta(\boldsymbol{\lambda}_{\min}), P(\boldsymbol{\lambda}_{\min})) + 1.$$

We will calculate the dimension of the first Ext-group; the second calculation is similar.

Let $\Omega\Delta(\boldsymbol{\lambda}'_{\min})$ be the kernel of the natural map $P(\boldsymbol{\lambda}'_{\min}) \to \Delta(\boldsymbol{\lambda}'_{\min})$. We have a short exact sequence

$$0 \to \Omega\Delta(\boldsymbol{\lambda}'_{\min}) \to P(\boldsymbol{\lambda}'_{\min}) \to \Delta(\boldsymbol{\lambda}'_{\min}) \to 0$$

which induces a long exact sequence

$$0 \to \operatorname{Hom}_{B_n^\kappa}(\Delta(\boldsymbol{\lambda}'_{\min}), P(\boldsymbol{\lambda}_{\min})) \to \operatorname{Hom}_{B_n^\kappa}(P(\boldsymbol{\lambda}'_{\min}), P(\boldsymbol{\lambda}_{\min})) \longrightarrow \operatorname{Hom}_{B_n^\kappa}(\Omega\Delta(\boldsymbol{\lambda}'_{\min}), P(\boldsymbol{\lambda}_{\min}))$$
$$\to \operatorname{Ext}_{B_n^\kappa}^1(\Delta(\boldsymbol{\lambda}'_{\min}), P(\boldsymbol{\lambda}_{\min})) \to \operatorname{Ext}_{B_n^\kappa}^1(P(\boldsymbol{\lambda}'_{\min}), P(\boldsymbol{\lambda}_{\min})) = 0.$$

The first term has dimension 1 by Theorem 5.2, while the second term has dimension $[P(\boldsymbol{\lambda}_{\min}) : L(\boldsymbol{\lambda}'_{\min})]$. For the third term, we apply [5, Proposition A3.13] several times to obtain

$$\operatorname{Hom}_{B_n^\kappa}(\Omega\Delta(\boldsymbol{\lambda}'_{\min}), P(\boldsymbol{\lambda}_{\min})) \cong \operatorname{Hom}_{B_{n-2r}^\kappa}(F^r(\Omega\Delta(\boldsymbol{\lambda}'_{\min})), F^r P(\boldsymbol{\lambda}_{\min}))$$

where $r \in \mathbb{N}$ is minimal such that $F^r L(\boldsymbol{\lambda}_{\min}) = F^r L(\boldsymbol{\lambda}'_{\min}) = 0$.

Localising does not change the Δ-multiplicities in $\Omega\Delta(\boldsymbol{\lambda}'_{\min})$ because it has a Δ-filtration with subquotients labelled by weights larger than $\boldsymbol{\lambda}'_{\min}$. This means that $F^r(\Omega\Delta(\boldsymbol{\lambda}'_{\min}))$ has the same Δ-multiplicities as $T(1^{\lambda_1-r}, 1^{\lambda_2-r})$ by induction. Let $\boldsymbol{\mu} = (1^{\mu_1}, 1^{\mu_2}) \in \Lambda(n)$ be a weight dominating $\boldsymbol{\lambda}_{\min}$ and $\boldsymbol{\lambda}'_{\min}$ but no other weights, and define $\boldsymbol{\mu}' \neq \boldsymbol{\mu}$ similarly if such a weight exists. Applying [5, Proposition A3.13] again we get

$$\dim \operatorname{Hom}_{B_{n-2r}^\kappa}(\Delta(1^{\mu_1-r}, 1^{\mu_2-r}), F^r(\Omega\Delta(\boldsymbol{\lambda}'_{\min}))) = \dim \operatorname{Hom}_{B_{n-2r}^\kappa}(F^r\Delta(\boldsymbol{\mu}), F^r(\Omega\Delta(\boldsymbol{\lambda}'_{\min})))$$
$$= \dim \operatorname{Hom}_{B_n^\kappa}(\Delta(\boldsymbol{\mu}), \Omega\Delta(\boldsymbol{\lambda}'_{\min}))$$
$$= 1$$

and similarly for $\boldsymbol{\mu}'$, so $\operatorname{soc} F^r(\Omega\Delta(\boldsymbol{\lambda}'_{\min})) = \operatorname{soc} T(1^{\lambda_1-r}, 1^{\lambda_2-r})$. Another application of [5, Lemma A4.1] establishes that $F^r(\Omega\Delta(\boldsymbol{\lambda}'_{\min})) = T(1^{\lambda_1-r}, 1^{\lambda_2-r})$.

On the other hand, from the short exact sequence

$$0 \to \Omega\Delta(\boldsymbol{\lambda}_{\min}) \to P(\boldsymbol{\lambda}_{\min}) \to \Delta(\boldsymbol{\lambda}_{\min}) \to 0$$

it is clear that $F^r P(\boldsymbol{\lambda}_{\min}) = F^r \Omega\Delta(\boldsymbol{\lambda}_{\min})$. As before $F^r \Omega\Delta(\boldsymbol{\lambda}_{\min}) = T(1^{\lambda_1-r}, 1^{\lambda_2-r})$. Thus

$$\dim \operatorname{Hom}_{B_n^\kappa}(\Omega\Delta(\boldsymbol{\lambda}'_{\min}), P(\boldsymbol{\lambda}_{\min})) = \dim \operatorname{End} T(1^{\lambda_1-r}, 1^{\lambda_2-r}),$$

which by [5, Proposition A2.2(ii)] equals the number of Weyl subquotients in $T(1^{\lambda_1-r}, 1^{\lambda_2-r})$. But by induction this is just 1 less than the number of Weyl subquotients in $P(\boldsymbol{\lambda}_{\min})$, which is exactly $[P(\boldsymbol{\lambda}_{\min}) : L(\boldsymbol{\lambda}'_{\min})]$. Thus the relevant Ext-group is 1-dimensional and the ungraded result follows. The correct grade shift follows from the regular graded decomposition numbers. $\qquad\square$

To write the other tilting modules, it is useful to introduce some notation due to Donkin. For $\boldsymbol{\lambda} \in \Lambda(n)$ and M a B_n^κ-module, write $O_{\trianglelefteq\boldsymbol{\lambda}}(M)$ for the maximal submodule of M whose composition factors are all of the form $L(\boldsymbol{\mu})$ for some $\boldsymbol{\mu} \trianglelefteq \boldsymbol{\lambda}$. Dually we write $O^{\trianglelefteq\boldsymbol{\lambda}}(M)$ for the

minimal submodule of M such that $M/O^{\trianglelefteq\boldsymbol{\lambda}}(M)$ has composition factors of the form $L(\boldsymbol{\mu})$ for some $\boldsymbol{\mu} \trianglelefteq \boldsymbol{\lambda}$,

Theorem 5.4. *Let* $\boldsymbol{\lambda} \in \Lambda(n)$ *be any weight. Then* $T(\boldsymbol{\lambda}) = O_{\trianglelefteq\boldsymbol{\lambda}}(T(\boldsymbol{\lambda}_{\max}))\langle -\deg \mathsf{t}_{\boldsymbol{\lambda}}^{\boldsymbol{\lambda}_{\max}}\rangle$.

Proof. First of all, it is clear that $O_{\trianglelefteq\boldsymbol{\lambda}}(T(\boldsymbol{\lambda}_{\max}))$ has a ∇-filtration. By [5, Lemma A4.5] $O_{\trianglelefteq\boldsymbol{\lambda}}(T(\boldsymbol{\lambda}_{\max}))$ is the indecomposable tilting module of highest weight $\boldsymbol{\lambda}$ in the algebra $B_n^\kappa(\trianglelefteq \boldsymbol{\lambda}) = B_n^\kappa/O^{\trianglelefteq\boldsymbol{\lambda}}(B_n^\kappa)$. Using [5, Proposition A3.3] we have

$$\mathrm{Ext}^1_{B_n^\kappa}(O_{\trianglelefteq\boldsymbol{\lambda}}(T(\boldsymbol{\lambda}_{\max})), \nabla(\boldsymbol{\mu})) = \mathrm{Ext}^1_{B_n^\kappa(\trianglelefteq\boldsymbol{\lambda})}(O_{\trianglelefteq\boldsymbol{\lambda}}(T(\boldsymbol{\lambda}_{\max})), \nabla(\boldsymbol{\mu})) = 0$$

for any $\boldsymbol{\mu} \trianglelefteq \boldsymbol{\lambda}$. This means that $O_{\trianglelefteq\boldsymbol{\lambda}}(T(\boldsymbol{\lambda}_{\max}))$ has a Δ-filtration too, and thus must be a tilting module for B_n^κ. But the socle of $O_{\trianglelefteq\boldsymbol{\lambda}}(T(\boldsymbol{\lambda}_{\max}))$ is as small as possible by Theorem 5.2, so it must also be indecomposable, and thus $O_{\trianglelefteq\boldsymbol{\lambda}}(T(\boldsymbol{\lambda}_{\max}))$ is a grade shift of $T(\boldsymbol{\lambda})$, and we surmise the correct grade shift from knowledge of the graded decomposition numbers. $\square$

5.3. **Tilting characters.** For $x, y \in W$, define the (Laurent) polynomial $h^{x,y}$ by

$$h^{x,y}(v) = \begin{cases} v^{\ell(x)-\ell(y)} & \text{if } y \le x, \\ 0 & \text{otherwise.} \end{cases}$$

Our use of a superscript is intentional. We mean to emphasise the fact that these are the *inverse* Kazhdan–Lusztig polynomials associated to W (in the notation of [22]), which happen to coincide with ordinary Kazhdan–Lusztig polynomials in type $\tilde{A}_1$. The graded Weyl multiplicities of the regular indecomposable tilting modules are as follows.

Corollary 5.5. *Let* $\boldsymbol{\lambda}, \boldsymbol{\mu}$ *be regular weights lying in the same linkage class. Then we have*

$$(T(\boldsymbol{\mu}) : \Delta(\boldsymbol{\lambda}))_v = \overline{h^{w_{\boldsymbol{\lambda}}, w_{\boldsymbol{\mu}}}}.$$

There is also a singular version.

Corollary 5.6. *Let* $\boldsymbol{\lambda}$ *be a singular weight. Then we have*

$$(T(\boldsymbol{\lambda}_k) : \Delta(\boldsymbol{\lambda}))_v = v^{-k}.$$

We conclude with a few remarks on possible extensions of this result.

Remark 5.7.

(1) The blob algebra is the quotient of a level 2 cyclotomic Hecke algebra. The *generalised blob algebras* are analogous quasi-hereditary quotients of level l cyclotomic Hecke algebras for integers $l > 2$. These algebras have a very similar KLR presentation [12]. Moreover, the representation theory of the level l generalised blob algebra is governed by the combinatorics of one-column l-multipartitions, with a linkage principle coming from the affine Weyl group of type $\tilde{A}_{l-1}$. As a result nearly all of the notation generalises to the level l case easily. We conjecture that for two regular one-column l-multipartitions $\boldsymbol{\lambda}, \boldsymbol{\mu}$, we have

$$(T(\boldsymbol{\mu}) : \Delta(\boldsymbol{\lambda}))_v = \overline{h^{w_{\boldsymbol{\lambda}}, w_{\boldsymbol{\mu}}}}$$

in the level l generalised blob algebra over a field $\Bbbk$ of characteristic 0, where $h^{x,y}$ is the inverse Kazhdan–Lusztig polynomial of type $\tilde{A}_{l-1}$.

(2) Over a field $\Bbbk$ of characteristic $p > 0$, the graded decomposition numbers of the blob algebra coincide with the *p-Kazhdan–Lusztig polynomials* ${}^p h_{y,x}$ of type $\tilde{A}_1$ [12, Theorem 5.26] (see also [3]). We hypothesise that the graded Weyl multiplicities of the indecomposable tilting modules of the level l generalised blob algebra should be given by a p-analogue ${}^p h^{x,y}$ of inverse Kazhdan–Lusztig polynomials of type $\tilde{A}_{l-1}$. As far as we are aware, no such analogue has been constructed before. In the spherical $l = 2$ case, it is reasonable to guess that the graded Weyl multiplicities of indecomposable tilting modules for $\mathrm{TL}_n(1)$ (the n-strand Temperley–Lieb algebra with parameter 1 over $\Bbbk$) give a p-analogue $\overline{{}^p m^{x,y}}$ of the inverse spherical Kazhdan–Lusztig polynomials, truncated after weight n. Equivalently, using the Ringel duality of $\mathrm{TL}_n(1)$ and (a quotient of) the hyperalgebra of SL_2, we should have

$$[\Delta_{\mathrm{SL}_2}(x \cdot_p 0) : L_{\mathrm{SL}_2}(y \cdot_p 0)] = {}^p m^{x,y}(1),$$

where $\cdot_p$ denotes the p-dilated dot action. This can be extended to higher levels in the spherical case by replacing SL_2 with SL_l.

(3) In general, p-Kazhdan–Lusztig polynomials are defined via Soergel bimodules over a field of characteristic p. The relationship between p-Kazhdan–Lusztig polynomials in type $\tilde{A}_1$ and graded decomposition numbers of the blob algebra is the combinatorial shadow of the 'Categorical Blob vs Soergel conjecture' [12, §1.8]. This conjecture posits an equivalence between a 'blob category' (whose Hom-spaces are certain idempotent truncations of the level l generalised blob algebra) and the category of Soergel bimodules in type $\tilde{A}_{l-1}$. Such an equivalence, combined with our tilting character conjecture above, would imply that the inverse (p-)Kazhdan–Lusztig polynomials of type $\tilde{A}_{l-1}$ appear in the corresponding category of Soergel bimodules. Yet Soergel bimodules make sense for all types, so this would lead to a categorification (resp. construction) of inverse (p-)Kazhdan–Lusztig polynomials in all types. The classical relationship between Kazhdan–Lusztig polynomials and inverse Kazhdan–Lusztig polynomials could then be reinterpreted as saying something about a form of 'Ringel duality' for Soergel bimodules.

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