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Optimal strategies for pricing general insurance

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Abstract

Optimal premium pricing policies in a competitive insurance environment are investigated using approximation methods and simulation of sample paths. The market average premium is modelled as a diffusion process, with the premium as the control function and the maximization of the expected total utility of wealth, over a finite time horizon, as the objective. In order to simplify the optimisation problem, a linear utility function is considered and two particular premium strategies are adopted. The first premium strategy is a linear function of the market average premium, while the second is a linear combination of the break-even premium and the market average premium. The optimal strategy is determined over the free parameters of each functional form.

It is found that for both forms the optimal strategy is either to set a premium close to the break-even or not to sell insurance depending on the model parameters. If conditions are suitable for selling insurance then for the first premium strategy, in the case of no market average premium drift, the optimal premium rate is approximately $\bar{p}(0)/aT$ above break-even where $\bar{p}(0)$ is the initial market average premium, a is a constant related to the elasticity of demand and T is the time horizon. The optimal strategy for the second form of premium depends on the volatility of the market average premium. This leads to optimal strategies which generate substantial wealth since then the market average premium can be much larger than break-even leading to significant market exposure whilst simultaneously making a profit. Monte-Carlo simulation is used in order to study the parameter space in this case.

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1 Introduction

There is little in the insurance literature on modelling how insurance premiums should be determined in a competitive market and how they respond to changes in the levels of premiums being offered by competitor companies (Daykin *et al.* 1994). Despite the fact that underwriting cycles in non-life insurance are known to be present and an objective analysis is needed for properly formulating underwriting strategies rather than just following the trends, such an analysis has not been done so far. It is widely observed that for a number of years premium rates decline to a point where the market, on average, is underwriting at a considerable loss, followed by a reverse trend of large increases in premium rates to the point where the market is making a substantial profit. During these underwriting cycles, it is observed that, individual insurance companies are following the market with their premium rates declining when the market average premium rate declines and increasing when the market average premium rate increases (Cummins and Outreville 1987, Daykin *et al.*, 1994). A question arises here: What is the optimal premium strategy for an individual insurance company and how is this related to the market? This leads to the conclusion that there is a wide scope for investigation of this area.

An attempt towards this direction was carried out by Taylor (1986; 1987) who investigated the appropriate response of an insurer to the movement of market average premium rates, assuming that the management objective of the insurer is to maximize the expected present value of wealth arising over a given finite time horizon. Price elasticity of demand for the product was allowed for, as was the required rate of return on the capital supporting the insurance operation. A key factor in this investigation was the time which would elapse before market average premiums were predicted to return to profitability. The conditions that might produce "loss leaders" were also investigated. However, the optimization model developed, in Taylor (1986), involved projection of the market average premium rates over future years. The problem was approached as if such a projection could be made with certainty. Results were obtained for four different scenarios of market behaviour over a 5 year period and were varied. When the market average premium rate fell below the break-even rate, the optimal strategy involved underwriting for significant profit margins which implied a complete withdrawal from the market as demand levels in this case fell dramatically. On the other hand, when the market average premium rate was below the break-even rate, but was expected to rise and return to profitability in the near future, the optimal premium policy indicated "loss leading" in the near future which would lead to a rise in demand and hence return to profitability when the market average premium rates would recover at the end of the time period.

Taylor considered two different demand functions, the exponential demand function and a constant price elasticity demand function. We shall focus on the first of these functions. The required rate of return on the capital supporting the insurance operation is taken into consideration in the derivation of the wealth process. Ultimately, we optimize the premium policy by assuming it takes the form of:

I: a linear function of market average premium

II: a linear function of market average premium and break-even premium

and hence optimize the objective function over the free parameters of each functional form. We show that for case I that there is an analytical expression for the objective function. We then find an approximate maximum for this expression. For case II we approximate the objective function and find a maximum for this approximate expression.

The problem described herein can be viewed as a stochastic optimal control problem with the premium policy as the control variable and the maximization of the expected utility of wealth as the value function. Using the tools of stochastic optimal control theory and stochastic calculus (the dynamic programming principle and Ito's lemma), one can derive the Hamilton-Jacobi-Bellman (HJB) equation which characterizes the value function. By solving this equation, the value function and the optimal policy may be obtained in greater generality. This is the approach adopted by Emms & Haberman (2005). However, it was found that one has to increase the complexity of Taylor's model in order to avoid a bang-bang premium strategy. Specifically, Emms & Haberman assumed the premium rate is held fixed over the duration of the policy and introduced an accrued premium rate and an associated evolution equation for this state variable. They found that the optimal premium rate can be negative, which reflects a shortcoming of the model, namely that one requires sufficient capital in order to loss-lead. In order to resolve this shortcoming one must constrain the optimisation problem.

Rather than complicate Taylor's model further we adopt the alternative approach and study fixed premium strategies. We have chosen to approximate the objective function and its roots using analytical approximations. The use of approximate methods allows us to give a qualitative description of the optimal premium strategy without the need to run a simulation. Further, we are able to use the approximate solutions to study the sensitivity of the model to its parameters and in some cases we obtain good quantitative as well as qualitative agreement. We also note that numerical methods yield the optimal strategy for one point of parameter space and that, in contrast, it would require many simulations in order to determine fully the model sensitivity. The approximate analysis also informs the reader which nondimensional parameters (i.e. parameter groups) are important: in effect this reduces the dimension of the sensitivity analysis. Numerical solutions are used to confirm the validity of the asymptotic behaviour of the model, where approximate solutions exist.

In Section 2 a continuous-time model for the insurance market is constructed. We discuss appropriate values for the model parameters and adopt suitable parameterisations. The two following sections consider each strategy in turn: we find analytical and approximate forms for the optimal strategies. The sensitivity of the model to the parameters of the model is determined using these approximations. Premium Strategy II leads to a stochastic differential equation for the insurer's exposure and consequently it is more difficult to analyse. We divide the analysis into three parts corresponding to the three forms of the process describing the market average premium: constant (Section 4), drifting but deterministic (Section 4.1), and lognormal (Section 4.2). The lognormal case uses Monte-Carlo simulation to assess the parameter dependence of the model since analytical approximations have not been found. Finally we summarise these results and make suggestions for modelling improvements in Section 5.

2 Model Formulation

Following Taylor (1986), we adopt the following processes to describe an insurance market:

$\mathbf{q}(t)$ - volume of exposure at time t.

$\pi(t)$ - break-even rate (per unit of exposure) at time t (i.e. risk premium plus expenses per unit of exposure).

$\mathbf{p}(t)$ - premium rate (per unit of exposure) charged by the insurer at time t.

$\bar{\mathbf{p}}(t)$ - market average premium rate (per unit of exposure) charged by all the insurers in the market at time t.

In this paper, we extend Taylor's (1986) model by specifying a stochastic process for the market average premium $\bar{\mathbf{p}}$. Thus, we consider how uncertainty in the market affects an insurer's premium policy. We model $\bar{\mathbf{p}}$ as a *geometric Brownian motion*:

$$\frac{d\bar{\mathbf{p}}}{\bar{\mathbf{p}}} = \mu dt + \sigma d\mathbf{Z}, \tag{1}$$

where $\mathbf{Z}$ is a *Wiener process*. The drift μ and the volatility σ are both assumed to be constant. This process is widely used to model the market prices of financial instruments and is often used in option pricing (Hull 1993), and interest rate modelling (Brigo & Mercurio 2001). The future market average premium is log-normally distributed (and hence positive), i.e.

$$\log \bar{\mathbf{p}}(t) \sim \mathrm{N}\left(\log \bar{p}_0 + (\mu - \frac{1}{2}\sigma^2)t, \sigma^2 t\right).$$

A log-normal process is certainly a simple model for the process followed by the market average premium. However, it is the simplest feasible model that allows us to assess the importance of the drift and volatility of the market average premium with reference to the break-even premium. Thus, it represents a good starting point for the modelling process.

We adapt Taylor's (1986) specification of the demand function to our continuous setting. Taylor (1986) defines the demand process by

$$q_{k+1} = f(p_{k+1}, \bar{p}_{k+1}) q_k,$$

where $f(p_k, \bar{p}_k)$ is the demand function in year k. Adapting this to our continuous setting, we have

$$q_{t+\delta t} - q_t = f(p_{t+\delta t}, \bar{p}_{t+\delta t})^{\delta t} q_t - q_t.$$

Then

$$\begin{aligned}\Delta q &= (f^{\delta t} - 1)q \\ &= (\exp(\log(f^{\delta t})) - 1)q \\ &= (\log f)q\delta t + o(\delta t^2),\end{aligned}$$

using a Taylor series expansion. Therefore, the *demand process* is described by

$$\frac{d\mathbf{q}}{\mathbf{q}} = \log f(\mathbf{p}, \bar{\mathbf{p}})dt, \tag{2}$$

where $\mathbf{p} := \mathbf{p}(\bar{\mathbf{p}}, \pi, t)$ is the premium at time t.

Define the *wealth process* $\mathbf{w}$ as the insurer's capital at time t. Then $\mathbf{w}$ follows the stochastic process given by

$$d\mathbf{w} = -\alpha \mathbf{w} dt + \mathbf{q}(\mathbf{p} - \pi)dt. \tag{3}$$

where α is the excess return on capital (i.e. return on capital - risk free rate) required by the shareholders of the insurer whose strategy is under consideration. Thus, $\alpha \mathbf{w}\, dt$ is the cost of holding $\mathbf{w}$ in a small time interval dt. Notice we take the break-even premium π as constant and study the deviation of $\bar{\mathbf{p}}$ from π by allowing $\bar{\mathbf{p}}$ to drift at rate μ. For developing a model of competitive pricing in insurance it seems wise to study the simplest case first without introducing additional parameters. In reality, the break-even premium and the market average premium will be coupled, which constrains the magnitude of μ which one might reasonably set.

Our aim is to determine the strategy which maximizes the expected total utility of wealth J over a finite time horizon T. For a given utility function (of wealth) $U(\mathbf{w}, t)$ we define the value function

$$V := \max_p \left\{ J = \mathbb{E} \int_0^T U(\mathbf{w}(s), s) ds \right\}, \tag{4}$$

that is as the maximization of the objective function J over a choice of strategies p. This is similar to the objective function used by Taylor (1986) with profit replaced by total wealth at time T.

2.1 Parameterisations and numerical values

We consider two different choices for the demand function. For the exponential demand function we have

$$f(\mathbf{p}, \bar{\mathbf{p}}) = \exp\left[\frac{-a(\mathbf{p} - \bar{\mathbf{p}})}{\bar{\mathbf{p}}}\right], \tag{5}$$

for some constant $a > 0$. Whereas for the constant price elasticity demand function

$$f(\mathbf{p}, \bar{\mathbf{p}}) = \left(\frac{\mathbf{p}}{\bar{\mathbf{p}}}\right)^{-a}. \tag{6}$$

Notice that for the first of these parameterisations (5), the RHS of the demand equation (2) and the wealth equation (3) is linear in the premium $\mathbf{p}$. Thus for a purely deterministic problem the Hamiltonian formed from these expressions is linear in the control variable $\mathbf{p}$ (Sethi & Thompson 2000). Thus, if the premium is non-negative the premium strategy is degenerate (Emms & Haberman 2005): this is a consequence of the continuous version of Taylor's model where premium rates vary over the course of policies. We avoid the degeneracy here by considering fixed premium strategies so that large premium jumps are not possible.

The price-elasticity of demand (see Taylor 1986) is defined by

$$e = -\frac{\partial \log \mathbf{q}}{\partial \log \mathbf{p}}, \tag{7}$$

so that for the exponential demand function $e_{\text{exp}} = a\mathbf{p}/\bar{\mathbf{p}}$. For the constant price elasticity demand function we have $e = a$. Therefore, for the same choice of parameter a, $p \geq \bar{p} \Rightarrow e_{\text{exp}} \geq a$ and $p < \bar{p} \Rightarrow e_{\text{exp}} < a$. In general, when $p > \bar{p}$, greater elasticity has a negative impact on terminal wealth, since demand falls more rapidly. The converse also applies, so when $p < \bar{p}$, greater elasticity has a positive impact on terminal wealth.

We assume the Utility function takes the linear form

$$U(w,t) = e^{-\beta t} w, \tag{8}$$

where β is the (subjective) intertemporal discount rate. A linear utility function has the advantage of simplicity and may be appropriate for an insurer which is well-capitalized and has access to the financial markets in order to diversify its risk. Other utility functions are possible but they introduce more parameters into the model. We do not expect qualitative changes to the optimal strategy for different utility parameterisations; instead we expect changes only to the terminal wealth that a different utility function would produce.

Following Taylor (1986), we assume that underwriting at an average premium rate 20% higher than the market average will produce only 60% of the volume of the exposure which would be underwritten if underwriting were carried out at the market average i.e.

$$f(1.2\bar{p}, \bar{p}) = 0.6.$$

Consequently for the exponential demand function (5) we have $a = 2.55$ p.a. and for the constant price elasticity demand function we obtain $a = 2.80$ p.a.

Typical values for the remaining parameters are taken as follows:

$$\begin{aligned} \text{market average premium drift} \quad & \mu = 0.05 \text{ p.a.}, \\ \text{a finite time horizon} \quad & T = 10 \text{ years}, \\ \text{excess return on capital} \quad & \alpha = 0.06 \text{ p.a.}, \\ \text{discount rate} \quad & \beta = 0.06 \text{ p.a.}, \\ \text{a constant break-even premium} \quad & \pi = 4\ [q]^{-1} \text{ p.a.}, \\ \text{initial wealth} \quad & w(0) = 50, \\ \text{initial exposure} \quad & q(0) = 5\ [q], \end{aligned} \tag{9}$$

where $[q]$ denotes the units for the volume of exposure, which will vary according to the type of insurance under consideration. For the Monte-Carlo Simulation we shall use $\bar{p}(0) = 1\,[q]^{-1}$ p.a. ... $10\,[q]^{-1}$ p.a. – the break-even premium rate π has been chosen to lie in this range since it is the relative value of these two quantities which is important. The value of μ represents an upper bound on the drift of $\bar{\mathbf{p}}$ that is considered herein. We shall use these data to suggest suitable approximations and also in the numerical work to follow.

3 Premium Strategy I

In order to investigate the optimal premium, we set a premium strategy of the form

$$\mathbf{p}(t) = k\bar{\mathbf{p}}(t), \tag{10}$$

where k is a constant. Thus, we assume that the premium set by the insurer is a linear function of the market average premium and calculate the objective function for a range of values of k. For an exponential demand function (5), the demand process (2) can be integrated to obtain

$$q(t) = q(0)\exp(a(1-k)t), \tag{11}$$

while the constant elasticity demand function (6) integrates to

$$q(t) = q(0)k^{-at}. \tag{12}$$

For $k \approx 1$ both forms of demand function lead to the same exposure since $\log k \approx k - 1$. This suggests that a similar qualitative strategy will arise for both parameterisations, and so henceforth we shall consider only the exponential demand function. Quantitative wealth differences do arise, since for $k = 0$, (11) predicts a finite exposure whilst (12) predicts an infinite exposure. This leads to greater wealth generation if the insurer is operating at a profit i.e. $p > \pi$ for a constant elasticity demand function. However, numerical experiments do not indicate a qualitative change to the optimal premium strategy (details not included but are available from the authors).

The exposure as described by (11) is a deterministic function of time, which yields an analytical expression for the objective function. Substituting (11) into (3) gives

$$\mathbf{w}(t) = \exp(-\alpha t)\left[w(0) + \int_0^t \exp(\alpha s)q(0)\exp(a(1-k)s)(k\bar{\mathbf{p}}(s) - \pi)ds\right]. \tag{13}$$

The mean wealth is easily found by interchanging expectation and integration to obtain

$$\mathbb{E}(\mathbf{w}(t)) = \exp(-\alpha t)\left[w(0) + \int_0^t \exp(\alpha s + a(1-k)s)q(0)(k\bar{p}(0)\exp(\mu s) - \pi)ds\right], \tag{14}$$

where we have used

$$\mathbb{E}(\bar{\mathbf{p}}(t)) = \bar{p}(0)\exp(\mu t).$$

Next taking the linear utility function (8) and again exchanging expectation and integration operators we find

$$J = \mathbb{E}\left(\int_0^T U(\mathbf{w}(t), t)dt\right) = \int_0^T e^{-\beta t}\mathbb{E}(\mathbf{w}(t))dt. \tag{15}$$

Substituting (14) into (15) finally yields

$$\begin{aligned} J(k) = w(0)\frac{1-e^{-(\alpha+\beta)T}}{\alpha+\beta} &+ \frac{kq(0)\bar{p}(0)}{\alpha+a(1-k)+\mu}\left(\frac{e^{(a(1-k)+\mu-\beta)T}-1}{a(1-k)+\mu-\beta} - \frac{1-e^{-(\alpha+\beta)T}}{\alpha+\beta}\right) \\ &- \frac{q(0)\pi}{\alpha+a(1-k)}\left(\frac{e^{(a(1-k)-\beta)T}-1}{a(1-k)-\beta} - \frac{1-e^{-(\alpha+\beta)T}}{\alpha+\beta}\right). \end{aligned} \tag{16}$$

Figure 1 shows plots of J against k for different values of $\bar{p}(0)$ using the data set (9) with $\mu = 0$ and $\pi = 4$. For approximately $\bar{p}(0) > \pi$ there is a global maximum value for J which appears to gives an optimal strategy $k^* \approx \pi/\bar{p}(0)$. Note asterisks are used throughout to denote optimal values. Another numerical simulation is shown in Figure 2 with $\mu = 0$, $\pi = 6$ and is consistent with this hypothesis. For example, if $\bar{p}(0) = 8$ then $\pi/\bar{p}(0) \approx 3/4$ which is just below the maximum at $k^* = 0.8$ in Figure 2(ii). We can show by approximation that this numerical result continues to hold as the model parameters are changed. For $\bar{p}(0)$ much larger than the break-even premium such a strategy leads to significant exposure, which generates very large values for V as shown in the second panel of Figure 1 (noting that the ordinate axis in the second panel is on the logarithmic scale).

In order to obtain the value function V, we maximize J with respect to k. Note that (16) is independent of σ, due to the assumption of a utility function which is linear in wealth. The first order condition for a maximum of J is

$$\frac{dJ}{dk} = t_1 + t_2 + t_3 + t_4 + t_5 = 0, \tag{17}$$

where

$$\begin{aligned} t_1 &= \frac{\bar{p}(0)}{\mu+\alpha+a(1-k)}\left(\frac{e^{(\mu-\beta+a(1-k))T}-1}{\mu-\beta+a(1-k)} + \frac{e^{-(\alpha+\beta)T}-1}{\alpha+\beta}\right), \\ t_2 &= \frac{ak\bar{p}(0)}{(\mu+\alpha+a(1-k))^2}\left(\frac{e^{(\mu-\beta+a(1-k))T}-1}{\mu-\beta+a(1-k)} + \frac{e^{-(\alpha+\beta)T}-1}{\alpha+\beta}\right), \\ t_3 &= \frac{ak\bar{p}(0)}{\mu+\alpha+a(1-k)}\left(\frac{e^{(\mu-\beta+a(1-k))T}(1-T(\mu-\beta+a(1-k)))-1}{(\mu-\beta+a(1-k))^2}\right), \\ t_4 &= -\frac{a\pi}{(\alpha+a(1-k))^2}\left(\frac{e^{(a(1-k)-\beta)T}-1}{a(1-k)-\beta} + \frac{e^{-(\alpha+\beta)T}-1}{\alpha+\beta}\right), \\ t_5 &= -\frac{a\pi}{\alpha+a(1-k)}\left(\frac{(e^{(a(1-k)-\beta)T}(1-T(a(1-k)-\beta))-1}{(a(1-k)-\beta)^2}\right), \end{aligned} \tag{18}$$

and we have dropped $q(0)$ without loss of generality.

Consider first the simple case of $\alpha = \mu = 0$. Figure 3 shows how each of the terms t_1—t_5 varies as a function of k where the remaining numerical parameters are chosen as in (9). It appears that a dominant balance (Bender & Orszag 1978) exists for $k < 1$ between t_3 and t_5. Balancing these terms we find the approximate optimal strategy is

$$k^* = \gamma = \frac{\pi}{\bar{p}(0)}, \tag{19}$$

which is the ratio of the break-even premium to the initial market average premium. The optimal strategy is to set a premium which has an approximately constant mean equal to π for $\gamma < 1$ (in the case $\mu = 0$). For $\gamma > 1$, numerical experiments suggest that there is no maximum objective function so that the optimal strategy is to set $p = \infty$ which yields the finite value function

$$V = \left(w(0) + \frac{q(0)\bar{p}(0)}{a} \right) \left(\frac{1 - e^{-\beta T}}{\beta} \right), \tag{20}$$

from (16). This is the asymptote in Figures 1 & 2 as $k \to \infty$.

Consequently the optimal premium strategy has two modes depending on the model parameters: either set an infinite premium and accumulate wealth from the existing customer base or set the premium at just above break-even in order the maximise market exposure whilst at the same time making a profit. The existence of a finite optimal premium strategy for $\gamma < 1$ arises from two competing forces: the desire to set as low a premium as possible in order to gain new business, and the requirement to generate a profit by setting a high premium. In reality, an infinite premium rate will correspond to not selling insurance at all since no-one will buy insurance at such a price.

For this second mode we note that it is optimal to sell insurance at just above break-even. We can determine by just how much the premium should be above the break-even level by formalising the above approximation. Set

$$\varepsilon = \frac{1}{aT} \ll 1. \tag{21}$$

This is a non-dimensional quantity expressing how fast demand grows as we change the relative premium price i.e. it is inversely related to the elasticity of demand. Put

$$k \sim k_0 + \varepsilon k_1 + \ldots \tag{22}$$

and substitute into (17) and (18). Collecting together powers of ε and neglecting exponentially small terms (which requires $k < 1$), we obtain

$$k_0 = \gamma, \quad k_1 = 1, \tag{23}$$

so that the optimal premium price is

$$\mathbf{p}^* = (\gamma + \frac{1}{aT} + \ldots)\bar{\mathbf{p}} \tag{24}$$

for $\varepsilon \ll 1$. A detailed derivation of this result appears in Appendix A. The optimal strategy is to set the premium initially at $\bar{p}(0)/aT$ above the break-even level, π.

3.1 Drifting market average premium

Next we relax the assumption that $\alpha = \mu = 0$. If we suppose that the dominant balance $t_3 \sim t_5$ is maintained as we vary the parameters of the model then we can determine the concomitant change in the optimal strategy. For this to be true, the $O(\varepsilon)$ terms in (17) must remain $O(\varepsilon)$ as the parameters are varied. Numerical experiments verify that for reasonable values of the model parameters, this is a valid assumption. Again balancing t_3 and t_5 and dropping exponentially small terms, we obtain the following implicit relationship:

$$\begin{aligned} k &\sim \gamma e^{-\mu T} \left(\frac{\alpha + a(1-k) + \mu}{\alpha + a(1-k)}\right) \left(\frac{a(1-k) - \beta + \mu}{a(1-k) - \beta}\right)^2 \left(\frac{1 - T(a(1-k) - \beta)}{1 - T(\mu + a(1-k) - \beta)}\right), \\ &= \gamma e^{-\mu T} \left(1 + \frac{\mu}{\alpha + a(1-k)}\right) \left(1 + \frac{\mu}{a(1-k) - \beta}\right)^2 \left(1 - \frac{\mu T}{1 - T(a(1-k) - \beta)}\right)^{-1}. \end{aligned} \tag{25}$$

Figure 4 shows a plot of the left-hand side (LHS) and the right-hand side (RHS) of (25) for the data set (9). Only the lower value root is an approximate solution to (17): the other root arises since we have neglected t_1, t_2 and t_4. By examining how the expression on the RHS of (25) varies as a function of the model parameters we can determine the model sensitivity. It is straightforward to see from (25) that the root is strongly dependent on the parameters γ, μ, T, a. The root is weakly dependent on α and β since $\alpha \ll a$ and $\beta \ll a$. Thus the root is a function of three parameters $k^* = k^*(\nu, \varepsilon, \gamma)$ where $\nu = \mu T$.

Equation (25) suggests that $k^* \approx \gamma e^{-\nu}$ since $\mu \ll a$, and in Figure 4, k^* is just above $\gamma e^{-\nu}$. A more formal analysis using a perturbation expansion is given in Appendix A for the case $\alpha = 0$, $\mu \neq 0$. We find the optimal strategy up to $O(\varepsilon^2)$ is

$$\mathbf{p}^* = \left(\gamma e^{-\nu} + \varepsilon \left(1 + \frac{2\nu\gamma e^{-\nu}}{1 - \gamma e^{-\nu}}\right) + O(\varepsilon^2)\right) \bar{\mathbf{p}}. \tag{26}$$

Since the market average premium has drift μ and noting the form of $\mathbb{E}(\bar{\mathbf{p}}(t))$, (26) states that the optimal strategy is to keep the expected premium value near the break-even premium value at $t = T$. In addition, for positive drift $\nu > 0$ then $p^*(0) < \pi$, i.e. this is a loss-leading optimal strategy. The insurer sets a low initial premium now in order to build up market exposure. Near to $t = T$, the premium becomes greater than break-even (because the $O(\varepsilon)$ term is positive) and the insurer makes a large profit from their existing exposure. If $k \geq 1$ then the asymptotic analysis breaks down because the exponential terms in (18) are not negligible. Numerical experiments reveal that this condition determines the mode of the optimal premium strategy. Consequently, if $\gamma e^{-\nu} < 1$ the insurer should enter the market and adopt strategy (26).

These approximations remain valid as long as the $t_3 \sim t_5$ balance is maintained, which requires that $k < 1$. If the drift of the market average premium μ is negative, then we would expect the optimal strategy to be $p = \infty$ if $\bar{p}(0)$ is below π. If $\bar{p}(0)$ is significantly above π then the optimal strategy is to set $k^* < 1$ so that $\mathbb{E}[\mathbf{p}(t)]$ is close to π at $t = T$ as in (26). However, if $\gamma \gtrsim 1$ then an optimal strategy *can* exist such that $k^* > 1$ and the approximations used hitherto break down. Essentially, the optimal strategy is then to hold onto market exposure by setting a premium slightly higher than the market average as long as that generates a profit which outweighs the subsequent loss.

We can assess the validity of the approximations described above by comparing how the approximate expression in (26) compares with the numerical solution of (17) as we vary each parameter in turn as shown in Figure 5. The numerical determination of k^* was carried out by using multiple Romberg Integrations to calculate J in (15) and then using Brent's Method in order to find the maximum of J (Press *et al.* 2002). It is better to calculate the integrals numerically rather than determining the root of (17) since (16) has a removable singularity at $k = 1 - \beta/a$. Figure 5(i) shows that the approximate expression (26) compares favourably with the solution of (17) with ε and γ fixed. If the market average premium drift is negative there is an optimal premium strategy which entails setting a premium above the market average in order to utilise the insurer's existing exposure.

Figure 5(ii) demonstrates the validity of asymptotics in the sense that the approximate root tends towards the exact value of k^* as $\varepsilon \to 0$. It is also clear that the approximation remains reasonable over a wide range of values of ε. As the elasticity of the demand function decreases, corresponding to increasing ε, so k^* increases. If ε is increased, the balance between setting a high premium in order to generate a profit and a low premium in order to generate exposure is shifted in favour of setting a higher premium.

The optimal value k^* depends strongly on γ. Figure 5(iii) shows the relationship between these two parameters for a loss-leading optimal strategy ($\nu = 0.5$). The almost linear increase in k^* arises since it is optimal to set k such that terminal premium is just above break-even (see (26)).

4 Premium Strategy II

We proceed with investigating a broader structure for the premium policy $\mathbf{p}$. A linear function of the form $k_1\bar{\mathbf{p}} + k_2\pi$ has been suggested by practitioners as representing an approach used in practice for pricing a range of non-life insurance products. However, incorporating this formulation into our model means that the optimization now involves two parameters k_1 and k_2 and this makes the calculations rather cumbersome. Therefore, guided by Strategy I, we represent the premium policy $\mathbf{p}$ as a function of the break-even premium π and the difference of the market average premium $\bar{\mathbf{p}}$ and the break-even premium:

$$\mathbf{p}(t) = \pi + r(\bar{\mathbf{p}}(t) - \pi) \tag{27}$$

and hence we optimize over the single parameter r. The strategy differs substantially from strategy I in that the demand function $f = f(\mathbf{p}/\bar{\mathbf{p}})$ is no longer deterministic so that the volatility of the market average premium affects the optimal strategy.

A simple analytical expression for the objective J can be found if $\mu = \sigma = 0$. In this case, the market average premium remains constant throughout the period i.e. $\bar{p}(t) = \bar{p}(0)$. From the demand process (2) we obtain

$$q(t) = q(0)(f(p, \bar{p}))^t. \tag{28}$$

The wealth equation (3) can now be written

$$d(e^{\alpha t} w(t)) = rq(0)(\bar{p} - \pi) \exp\left[(\alpha + \log f(p, \bar{p}))t\right].$$

This can be integrated immediately to give

$$w(t) = w(0)e^{-\alpha t} + \frac{r(\bar{p} - \pi)q(0)}{\alpha + \log f(p, \bar{p})} \left((f(p, \bar{p}))^t - e^{-\alpha t}\right). \tag{29}$$

Therefore the objective function is

$$J = w(0)\left(\frac{1 - e^{-(\alpha+\beta)T}}{\alpha + \beta}\right) + \frac{r(\bar{p} - \pi)q(0)}{\alpha + \log f(p, \bar{p})}\left(\frac{e^{T(\log f(p,\bar{p}) - \beta)} - 1}{\log f(p, \bar{p}) - \beta} + \frac{e^{-(\alpha+\beta)T} - 1}{\alpha + \beta}\right). \tag{30}$$

Plots of the objective function J for a number of values of $\bar{p}(0)$ are shown in Figure 6 for the exponential demand function (5). As for Strategy I, it is apparent that for $\bar{p}(0) > \pi = 4$ there is a maximum for J. Furthermore, there are two modes for the optimal strategy: $p = \infty$ or $r^* \approx 0$.

We can determine an approximate expression for the second mode in this special case using the same techniques as before. If we set $\alpha = \beta = 0$ and assume an exponential demand function then it is easy to see that the root of $dJ/dr = 0$ is given approximately by the equation

$$\frac{d}{dr}\left(\frac{re^{(1-r)(1-\gamma)/\varepsilon}}{(1-r)^2}\right) = 0,$$

for $\varepsilon \ll 1$ since this term is of order $e^{1/\varepsilon}$. Expanding this expression yields a quadratic for r whose relevant root is

$$r^* = \frac{\varepsilon}{1 - \gamma},$$

up to $O(\varepsilon)$. The discriminant of this quadratic determines the mode of the optimal strategy. Substituting into (27) gives the optimal premium for the second mode as

$$p^* = \pi + \varepsilon\bar{p}(0). \tag{31}$$

This complements the result (24) obtained for the first strategy albeit for the case $\sigma = 0$.

4.1 Approximate Objective Function

We have found an analytical expression for the objective function J if the market average premium is constant. For a *deterministic* but drifting process, $\bar{p}$, we are unable to find an analytical expression for dJ/dr. In this section, we approximate the integral J and then find the maximum value of this approximate expression for the case $\sigma = 0$. In taking this approach we assume that both J and the approximate expression for J have approximately the same extrema.

The sensitivity analysis of Strategy I suggests that α and β do not affect the optimal strategy substantially. We shall suppose this also holds for Strategy II so we set $\alpha = \beta = 0$. Consequently the objective function becomes

$$J = \int_0^T \left(\int_0^t rq(\bar{p}(0)e^{\mu s} - \pi)\, ds + w(0) \right) dt. \tag{32}$$

It is only the first term on the RHS of (32) which determines the optimal premium strategy, $w(0)$ only affects the magnitude of the wealth generated by this strategy. We rewrite the objective with just the first term and consider only the exponential demand function:

$$J_0 = \int_0^T I(t;x)\, dt, \tag{33}$$

where the integral

$$I(t;x) = \int_0^t r(e^{\mu s} - \gamma) \exp[x(\mu s + \gamma(e^{-\mu s} - 1))]\, ds \tag{34}$$

and

$$x = \frac{a(1-r)}{\mu}. \tag{35}$$

From the data set (9) we see that $x \gg 1$ as long as r is not too close to 1.

The integral I is in Laplace form

$$I(t;x) = \int_0^t f(s) e^{x\phi(s)}\, ds, \tag{36}$$

with $f(s) = r(e^{\mu s} - \gamma)$ and $\phi(s) = \mu s + \gamma(e^{-\mu s} - 1)$. If $\gamma < 1$ then $\phi(s)$ is a continuous increasing function of s and so attains its maximum at $s = t$ on $[0, t]$. Following Bender & Orszag (1978) (see pp. 258) the approximate value as $x \to \infty$ for $\mu > 0$ is

$$I(t;x) \sim \frac{f(t)e^{x\phi(t)}}{x\phi'(t)} = \frac{re^{x\phi(t)+\mu t}}{\mu x}. \tag{37}$$

Integrating this expression over $[0, T]$ yields an approximate form for the objective function as $x \to \infty$:

$$J^* = \frac{re^{\mu T + x(\mu T + \gamma(e^{-\mu T} - 1))}}{(\mu x)^2(1 - \gamma e^{-\mu T})} \tag{38}$$

valid if $\gamma < 1$. If $\mu < 0$ then a similar expression can be derived depending on where ϕ attains its maximum.

The extrema of J^* are determined by a quadratic equation whose relevant root is

$$r^* = \frac{\psi - 1 - (\psi^2 - 6\psi + 1)^{1/2}}{2\psi}, \tag{39}$$

where

$$\psi = \frac{1}{\varepsilon}\left(1 + \frac{\gamma}{\nu}(e^{-\nu} - 1)\right). \tag{40}$$

The discriminant in (39) determines the mode of the optimal strategy.

Figure 7(i)–(iii) shows the sensitivity of the root $r^* = r^*(\nu, \varepsilon, \gamma)$ to the model parameters. On each plot we show the approximate form of the optimal strategy calculated from (39) and the numerical computation of the maximum of (32) computed as before.

In Figure 7(i), we see that as the drift of the market average premium increases the optimal value r^* decreases. This behaviour is consistent with the idea that the optimal strategy is to aim for a large terminal exposure rather than a large profit per policyholder. By keeping the terminal premium relatively small, then $p/\bar{p}$ is small and so the exposure is large. As ε becomes very small then Figure 7(ii) shows that the optimal premium strategy is to set a premium just above the break-even premium.

In the final plot, Figure 7(iii) shows that the variation of r^* is no longer linear with γ but increases only gradually. This is due to the fact that the second strategy does not require a loss-leading strategy if $\nu > 0$. In fact we have initially $p(0) - \pi = r\bar{p}(0)(1-\gamma)$ so that loss-leading optimal strategies only occur for $\gamma > 1$. This behaviour differs markedly from Strategy I: it is apparent from the figure that for sufficiently large $\gamma > 1$ there is no finite value for r^*. This occurs when the market average premium fails to reach break-even over the time horizon.

4.2 Volatility of the Average Market Premium

The optimal value for r in Strategy II depends on the volatility σ of the market average premium $\bar{\mathbf{p}}$. This is distinct from Strategy I where $\bar{\mathbf{p}}/\mathbf{p}$ is deterministic so that the demand function and therefore the exposure are also deterministic. To determine the sensitivity of the optimal form of Strategy II to the value of σ we resort to Monte Carlo simulation. The model can be written as a system of four stochastic differential equations:

$$\begin{aligned} d\bar{\mathbf{p}} &= \bar{\mathbf{p}}(\mu\, dt + \sigma\, d\mathbf{Z}), & (41)\\ d\mathbf{q} &= \mathbf{q} \log f(\mathbf{p}, \bar{\mathbf{p}})\, dt, & (42)\\ d\mathbf{w} &= -\alpha \mathbf{w}\, dt + \mathbf{q}(\mathbf{p} - \pi)\, dt. & (43)\\ d\mathbf{H} &= U\, dt, & (44) \end{aligned}$$

with given initial conditions for $\bar{\mathbf{p}}$, $\mathbf{q}$ and $\mathbf{w}$ at $t = 0$ and $H(0) = 0$.

For certain values of r, $\mathbf{p}$ might become negative as we introduce volatility so we set a lower bound for the premium $\mathbf{p}$ equal to $\pi/5$. Consequently for the numerical simulation we set

$$\mathbf{p} = \max(\pi/5, \pi + r(\bar{\mathbf{p}} - \pi)). \tag{45}$$

We integrate (41)–(44) from $t = 0$ to $t = T$ numerically using a Mersenne Twister random number generator for $d\mathbf{Z}$. In order to minimise the number of time steps M required for each sample path, we adopt a second order simplified weakly convergent scheme (see p.466 Kloeden & Platen 1999). This integration yields $\mathbf{H}(T) =$

$\int_0^T U(\mathbf{w}, t)\, dt$. We calculate $J(T) = \mathbb{E}[\mathbf{H}(T)]$ by averaging over a large number of simulations, typically $N = 10,000$. We use antithetic variance reduction in order to decrease the number of simulations required to produce a given standard error. The value of $J(T)$ varies continuously with r if we use the same random seed for each numerical simulation. The value function $V(T) = \max_r J(T)$ is found numerically (should it exist) using Brent's Method (Press *et al.* 2002). This fits a quadratic polynomial to J and has the advantage of requiring relatively few calls to $J(T)$ and therefore few expensive Monte Carlo simulations.

We use $M = 200$ time steps for each value of r. To check the convergence of the Monte-Carlo scheme we also verify the value of r^* by calculating J at $r^* \pm 0.01$ using $N = 100,000$ simulations. Therefore the value of r^* is accurate to 2 d.p. for the number of time steps used.

For completeness we examine the effect of the volatility on the optimal premium strategy with and without market average premium drift μ. Tables 1(a)–(d) show the optimal strategy for the cases $\mu = 0$, $\sigma = 0.05$; $\mu = 0$, $\sigma = 0.1$; $\mu = 0.05$, $\sigma = 0.05$; $\mu = 0.05$, $\sigma = 0.1$. The qualitative features of the results are similar to those of Strategy I. Looking at Table 1(a),(b) we see that there are again two modes for the optimal premium determined by the value of $\gamma = \pi/\bar{p}(0)$. As the volatility σ increases, the value function increases rapidly. Moreover, as $\bar{p}(0)$ is increased, then this too leads to large values of V. The reason for this behaviour can be seen in the sample paths in Figure 8. This figure shows a path generating the value $V(T) > 10^8$ for the case $\bar{p}(0) = 10$. As σ increases so does the likelihood that $\bar{\mathbf{p}}$ will increase substantially more than π so that the optimal premium p, which is just above π, generates a large exposure. This in turn increases the wealth generated by selling insurance. This behaviour is reinforced as $\bar{p}(0)$ increases.

Tables 1(c)–(d) show the optimal strategy for market average premium with positive drift $\mu = 0.05$. This has the effect of increasing the value function V for $r = r^*$ since large exposure and wealth can now be generated. Also loss-leading strategies are possible for $\bar{p}(0) < \pi$. As σ is increased still further (say to $\sigma = 0.2$) then the Monte-Carlo simulation can take up to 10^5 simulations to converge suggesting that this problem is ill-conditioned. This is in part a restriction of the model since π is kept constant while $\bar{\mathbf{p}}$ may increase independently – in reality, competition within the insurance market would prevent this behaviour from occurring.

5 Conclusions

We have investigated optimal strategies for two particular approaches to fixing the premium. The first approach is based on a linear function of the market average premium, while the second involves a linear combination of the break-even premium and the market average premium.

The qualitative behaviour of the optimal strategy in the first case has been determined analytically. If the market average premium is driftless, then we have demonstrated that there are two optimal strategy modes: setting an infinite premium rate when the initial market average premium rate is below the break-even premium or setting the premium rate a fraction above the break-even premium when the market is underwriting at a profit. If the market average premium has upward drift then there are again two optimal strategies: an infinite premium rate or a loss-leading strategy which makes an initial loss but gains market exposure. If the market average premium has negative drift then a non-infinite optimal strategy can exist whereby the insurer sets a premium just above the market mean. This can generate enough initial wealth to offset the loss as the market average premium drifts below break-even. The important parameters which determine the optimal strategy are γ, the ratio of initial market average premium to break-even premium, ε, a measure of the inverse elasticity of the demand function, and ν, the nondimensional drift of the market average premium.

The optimal form of the strategy in the second case is similar except that the drift of the market average premium does not have such a pronounced effect on the optimal strategy. Loss leading is much less likely with this form of strategy. The second strategy is also affected by the volatility of the market average premium. However, the qualitative form of the optimal strategy remains the same. As the volatility of the market average premium increases so does the wealth generated by choosing an optimal strategy. If one views both strategies as providing constraints on $\mathbf{p}$, then the form of Strategy II allows the premium to remain closer to the break-even premium π. This means the second strategy can generate greater wealth than the first if the market average premium drifts upwards.

Some of these strategies are unrealistic in that generating a large market exposure contradicts the assumption that the insurer does not affect the rest of the market. This points to a limitation in the modelling, specifically that the market average premium should be coupled explicitly to the break-even premium. One should also relax the restriction of a constant break-even premium and this forms the basis of future modelling work. A limitation of Taylor's original model (1986) is that the change in exposure is linearly proportional to the current exposure. Clearly this is unrealistic if the current exposure is zero or if the market for policies is saturated. This is another area in which we aim to improve the pricing model.

For a large part of parameter space, the optimal strategy can be found using perturbation expansions, which indicates that this form of analysis may allow progress in respect of more general static and dynamic strategies.

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A Approximate optimal premium strategy I

In this appendix we outline the derivation of the approximate optimal strategies (24) and (26). First, we set $\alpha = \mu = 0$ in (18). Second, we write the expressions in terms of the nondimensional parameters ε, $b = \beta T$ (taken as O(1)), and γ. Third, we note that if we multiply each of (18) by a positive function of k, the value of the root is not changed. Finally, if $k < 1$ then $e^{\beta T + \varepsilon^{-1}(k-1)}$ is exponentially small i.e. it is smaller than all orders of ε if $\varepsilon \ll 1$.

With these points in mind, we multiply t_1 to t_5 by $e^{b+\varepsilon^{-1}(k-1)}/(\bar{p}(0)T^2)$ and neglect exponentially small terms to obtain

$$
\begin{aligned}
t_1 &= \frac{\varepsilon}{1-k}\left(\frac{\varepsilon}{1-k-\varepsilon b}\right),\\
t_2 &= \frac{\varepsilon k}{(1-k)^2}\left(\frac{\varepsilon}{1-k-\varepsilon b}\right),\\
t_3 &= \frac{k}{1-k}\left(\frac{1+b-\varepsilon^{-1}(1-k)}{(\varepsilon^{-1}(1-k)-b)^2}\right),\\
t_4 &= -\frac{\varepsilon\gamma}{(1-k)^2}\left(\frac{\varepsilon}{1-k-\varepsilon b}\right),\\
t_5 &= -\frac{\gamma}{1-k}\left(\frac{1-\varepsilon^{-1}(1-k)+b}{(\varepsilon^{-1}(1-k)-\beta T)^2}\right),
\end{aligned}
$$

without renaming t_1—t_5.

Next, multiply through by $(1-k)/\varepsilon$, which makes the leading order terms $O(1)$, and write the root as a perturbation expansion:

$$k \sim k_0 + \varepsilon k_1 + \dots$$

The terms determining the root k then become

$$
\begin{aligned}
t_1 &= \frac{\varepsilon}{1-k_0} + O(\varepsilon^2),\\
t_2 &= \frac{\varepsilon k_0}{(1-k_0)^2} + O(\varepsilon^2),\\
t_3 &= \frac{(k_0+\varepsilon k_1)(\varepsilon-(1-k_0-\varepsilon k_1)+\varepsilon\beta T)}{(1-k_0-\varepsilon k_1-\varepsilon\beta T)^2},\\
t_4 &= -\frac{\gamma\varepsilon}{(1-k_0)^2} + O(\varepsilon^2),\\
t_5 &= -\gamma\left(\frac{\varepsilon-(1-k_0-\varepsilon k_1)+\varepsilon\beta T}{((1-k_0-\varepsilon k_1)-\varepsilon\beta T)^2}\right).
\end{aligned}
$$

If we then do a further Taylor series expansion for t_3 and t_5 to $O(\varepsilon)$ we obtain

$$
\begin{aligned}
t_3 &= \frac{1}{(1-k_0)^2}(k_0(k_0-1)+\varepsilon(k_1(k_0-1)+(1-k_1-\beta T)k_0)),\\
t_5 &= -\frac{\gamma}{(1-k_0)^2}(k_0-1+\varepsilon(1-k_1-\beta T)).
\end{aligned}
$$

Collecting together terms of $O(1)$ yields $k_0=\gamma$, while collecting terms of $O(\varepsilon)$ gives $k_1 = 1$ as stated in equation (24).

If $\mu \neq 0$ we write $\nu = \mu T$, set $\alpha = 0$ and proceed as above. Multiplying (18) through by $e^{b+\varepsilon^{-1}(k-1)}/(\varepsilon\bar{p}(0)T^2)$ gives

$$
\begin{aligned}
t_1 &= \frac{\varepsilon e^{\nu}}{(\varepsilon\nu + 1 - k)(\varepsilon\nu - b\varepsilon + 1 - k)}, \\
t_2 &= \frac{\varepsilon k e^{\nu}}{(\varepsilon\nu + 1 - k)^2(\varepsilon\nu - b\varepsilon + 1 - k)}, \\
t_3 &= \frac{k e^{\nu}(\varepsilon - \nu\varepsilon + b\varepsilon + k - 1)}{(\varepsilon\nu + 1 - k)(\varepsilon\nu - b\varepsilon + 1 - k)^2}, \\
t_4 &= -\frac{\gamma\varepsilon}{(1-k)^2(1-k-b\varepsilon)}, \\
t_5 &= -\frac{\gamma(\varepsilon + k - 1 + b\varepsilon)}{(1-k)(1-k-b\varepsilon)^2}.
\end{aligned}
$$

The expansion of t_3 and t_5 to $O(\varepsilon)$ in similar to that described earlier. Thus, we write $k \sim k_0 + \varepsilon k_1 + \ldots$, group terms, and ignore those of $O(\varepsilon^2)$

$$
t_3 = \frac{e^{\nu}(k_0 + \varepsilon k_1)(k_0 - 1 + \varepsilon(1 + k_1 - \nu + b))}{(1 - k_0 + \varepsilon(\nu - k_1))(1 - k_0 + \varepsilon(\nu - b - k_1))^2},
$$

First, take out the factor $(1-k_0)^3$:

$$
t_3 = \frac{e^{\nu}(k_0 + \varepsilon k_1)}{(1-k_0)^3} \times (k_0 - 1 + \varepsilon(1 + k_1 - \nu + b)) \\
\times \left(1 + \varepsilon\left(\frac{\nu - k_1}{1 - k_0}\right)\right)^{-1} \times \left(1 + \varepsilon\left(\frac{\nu - b - k_1}{1 - k_0}\right)\right)^{-2}.
$$

Now expand the last two terms and ignore terms of $O(\varepsilon^2)$

$$
t_3 = \frac{e^{\nu}(k_0 + \varepsilon k_1)}{(1-k_0)^3} \times (k_0 - 1 + \varepsilon(1 + k_1 - \nu + b)) \\
\times \left(1 - \varepsilon\left(\frac{\nu - k_1}{1 - k_0}\right)\right) \times \left(1 - 2\varepsilon\left(\frac{\nu - b - k_1}{1 - k_0}\right)\right).
$$

Next, gather together the $O(1)$ and $O(\varepsilon)$ terms

$$
t_3 = \frac{e^{\nu}}{(1-k_0)^3}\left(k_0(k_0 - 1) + \right. \\
\left. \varepsilon\left(k_1(k_0 - 1) + k_0(1 + k_1 - \nu + b) + k_0(\nu - k_1) + 2k_0(\nu - b - k_1)\right)\right).
$$

For the term t_5 we have

$$
t_5 = -\frac{-\gamma(k_0 - 1 + \varepsilon(k_1 + 1 + b))}{(1 - k_0 - \varepsilon k_1)(1 - k_0 - \varepsilon(k_1 + b))^2}.
$$

Then take out the factor $(1-k_0)^3$:

$$t_5 = -\frac{\gamma}{(1-k_0)^3} \times (k_0 - 1 + \varepsilon(k_1 + 1 + b)) \times \left(1 - \frac{\varepsilon k_1}{1-k_0}\right)^{-1} \times \left(1 - \varepsilon\left(\frac{k_1+b}{1-k_0}\right)\right)^{-2},$$

and then collect together $O(\varepsilon)$ terms

$$t_5 = -\frac{\gamma}{(1-k_0)^3}\left(k_0 - 1 + \varepsilon(k_1 + 1 + b - k_1 - 2(k_1 + b))\right).$$

Bringing these results together we have

$$\begin{aligned} t_3 &= \frac{e^{\nu}}{(1-k_0)^3}(k_0(k_0-1) + \varepsilon(k_0 - k_0k_1 - k_1 - bk_0 + 2k_0\nu)), \\ t_5 &= -\frac{\gamma}{(1-k_0)^3}(k_0 - 1 + \varepsilon(1 - b - 2k_1)). \end{aligned}$$

Examining t_3 and t_5 we find $k_0 = \gamma e^{-\nu}$ at leading order and writing down the $O(\varepsilon)$ part of all terms gives

$$\frac{e^{\nu}}{(1-\gamma e^{-\nu})^2} + \frac{e^{\nu}}{(1-k_0)^3}(k_0 - k_0k_1 - k_1 - bk_0 + 2k_0\nu) - \frac{\gamma(1-b-2k_1)}{(1-k_0)^3} = 0,$$

since the contributions from t_2 and t_4 cancel. Simplifying, we obtain $k_1 = 1 + 2\nu\gamma e^{-\nu}/(1-\gamma e^{-\nu})$ at $O(\varepsilon)$. Therefore, the result (26) follows.

Tables and Figures

$\bar{p}(0)$	r	Initial p^*	V	Relative error (%)
1.0	$-\infty$	∞	-	-
2.0	$-\infty$	∞	-	-
3.0	$-\infty$	∞	-	-
4.0	0.41	4.00	$3.00e+002$	0.06
5.0	0.22	4.22	$6.32e+002$	0.54
6.0	0.13	4.26	$4.18e+003$	0.92
7.0	0.10	4.30	$2.70e+004$	0.81
8.0	0.09	4.35	$1.23e+005$	0.68
9.0	0.08	4.39	$4.22e+005$	0.57
10.0	0.07	4.44	$1.16e+006$	0.48

(a)

$\bar{p}(0)$	r	Initial p^*	V	Relative error (%)
1.0	$-\infty$	∞	-	-
2.0	$-\infty$	∞	-	-
3.0	$-\infty$	∞	-	-
4.0	0.15	4.00	$3.81e+002$	2.96
5.0	0.11	4.11	$2.09e+003$	6.30
6.0	0.09	4.18	$1.64e+004$	4.60
7.0	0.08	4.24	$8.63e+004$	3.27
8.0	0.07	4.30	$3.24e+005$	2.48
9.0	0.07	4.35	$9.51e+005$	1.98
10.0	0.07	4.40	$2.32e+006$	1.64

(b)

$\bar{p}(0)$	r	Initial p^*	V	Relative error (%)
1.0	$-\infty$	∞	-	-
2.0	$-\infty$	∞	-	-
3.0	$-\infty$	∞	-	-
4.0	0.20	4.00	$6.02e+002$	0.57
5.0	0.12	4.12	$7.39e+003$	0.87
6.0	0.09	4.18	$6.90e+004$	0.69
7.0	0.08	4.24	$3.76e+005$	0.55
8.0	0.07	4.29	$1.40e+006$	0.45
9.0	0.07	4.34	$4.02e+006$	0.37
10.0	0.06	4.38	$9.51e+006$	0.32

(c)

$\bar{p}(0)$	r	Initial p^*	V	Relative error (%)
1.0	$-\infty$	∞	-	-
2.0	$-\infty$	∞	-	-
3.0	0.15	3.85	$3.15e+002$	1.40
4.0	0.11	4.00	$1.95e+003$	5.87
5.0	0.09	4.09	$2.61e+004$	3.88
6.0	0.08	4.15	$1.86e+005$	2.60
7.0	0.07	4.21	$8.24e+005$	1.92
8.0	0.07	4.26	$2.64e+006$	1.52
9.0	0.06	4.31	$6.78e+006$	1.25
10.0	0.06	4.36	$1.47e+007$	1.06

(d)

Table 1: The optimal expected total utility V for Strategy II over a range of initial mean market values $\bar{p}(0)$. The parameters values are those specified in (9) but with (a) $\mu = 0$, $\sigma = 0.05$; (b) $\mu = 0$, $\sigma = 0.1$; (c) $\mu = 0.05$, $\sigma = 0.05$; (d) $\mu = 0.05$, $\sigma = 0.1$. The relative error is defined as the standard error divided by V.

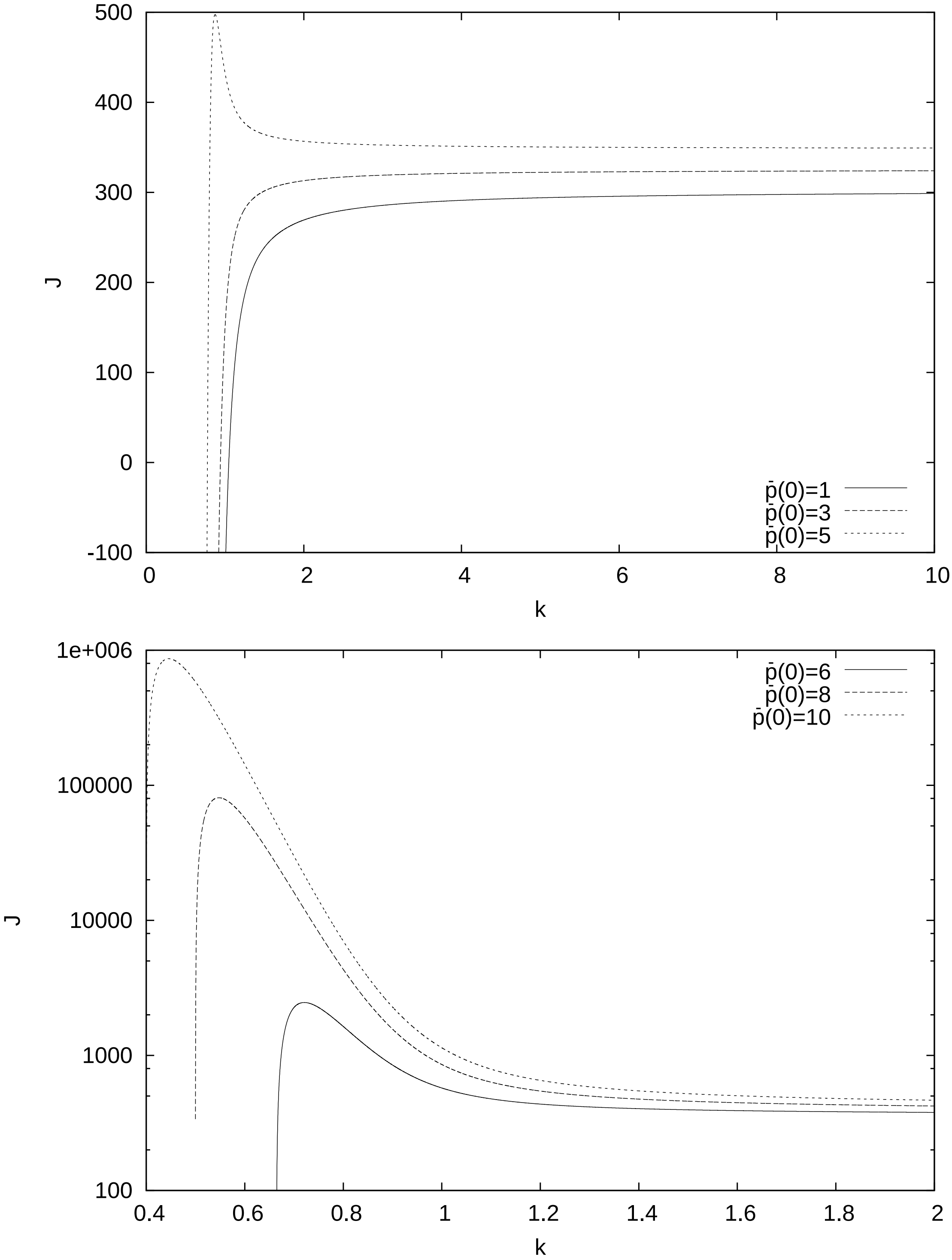


Figure 1: The objective function J of Strategy I for a number of initial market average premiums $\bar{p}(0)$. The parameters values have been set using (9) but with $\mu = 0$ and $\pi = 4$.

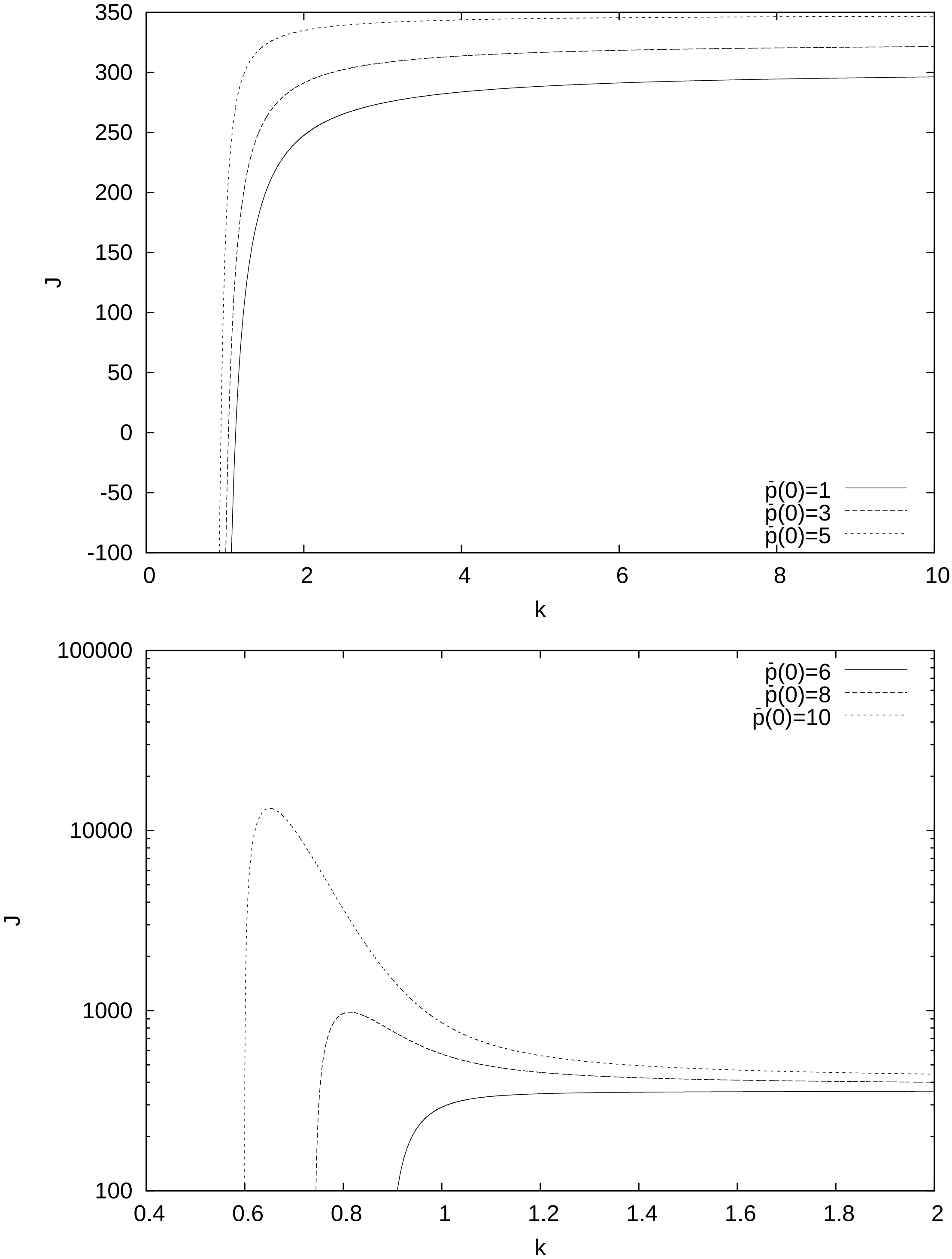


Figure 2: The objective function J of Strategy I for a number of initial market average premiums $\bar{p}(0)$. The parameters values have been set using (9) but with $\mu = 0$ and $\pi = 6$.

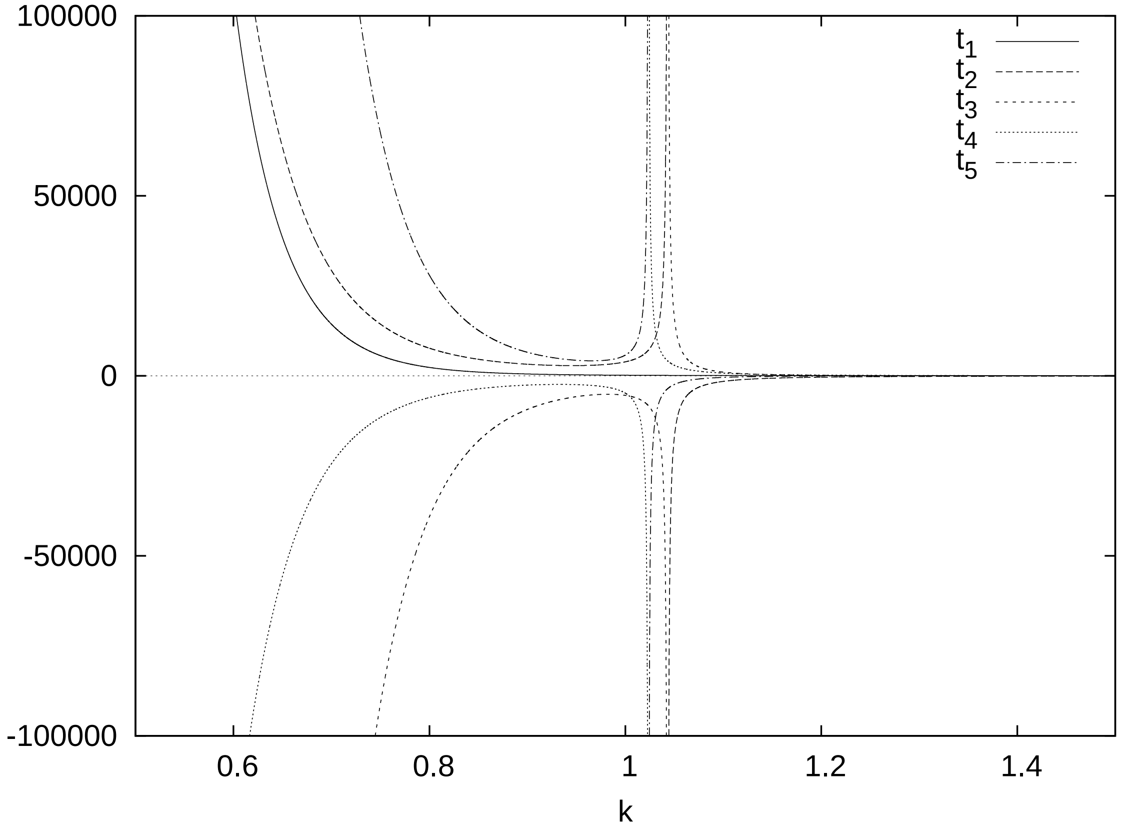


Figure 3: Plot of terms determining the optimal k for Strategy I with $\bar{p}(0) = 5$ and the remaining data as in (9).

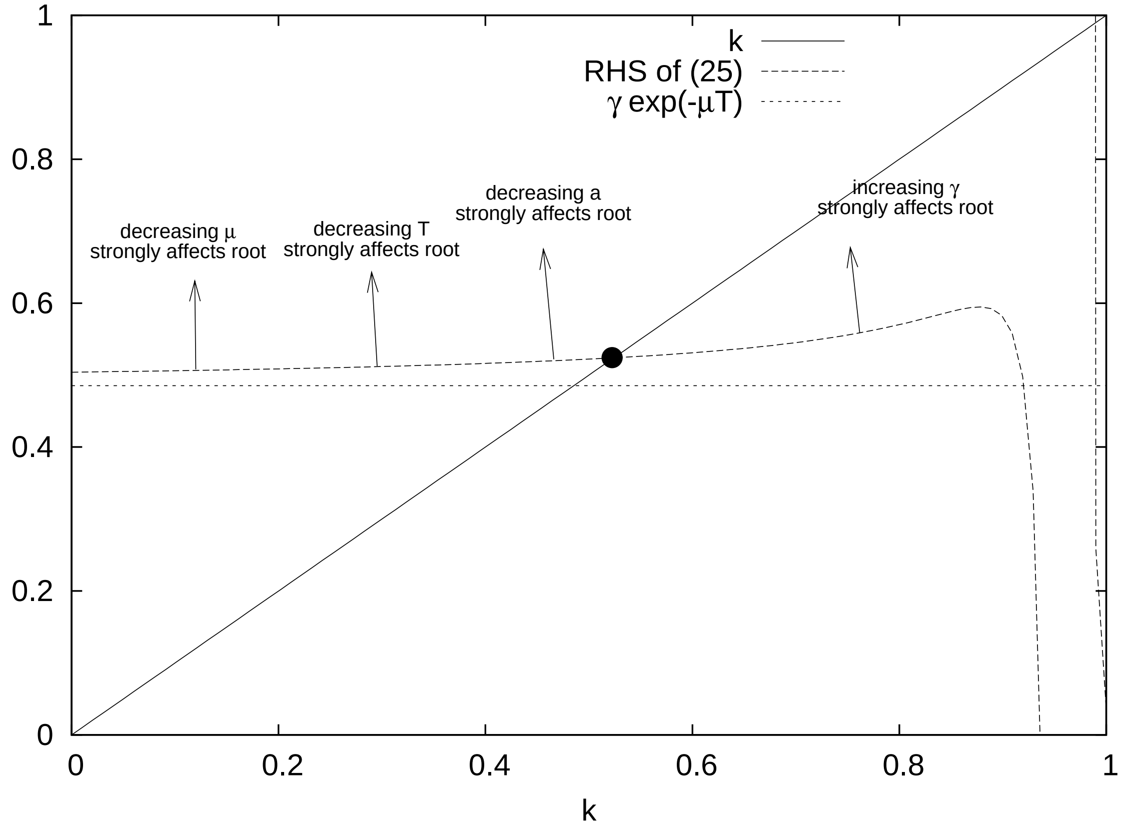
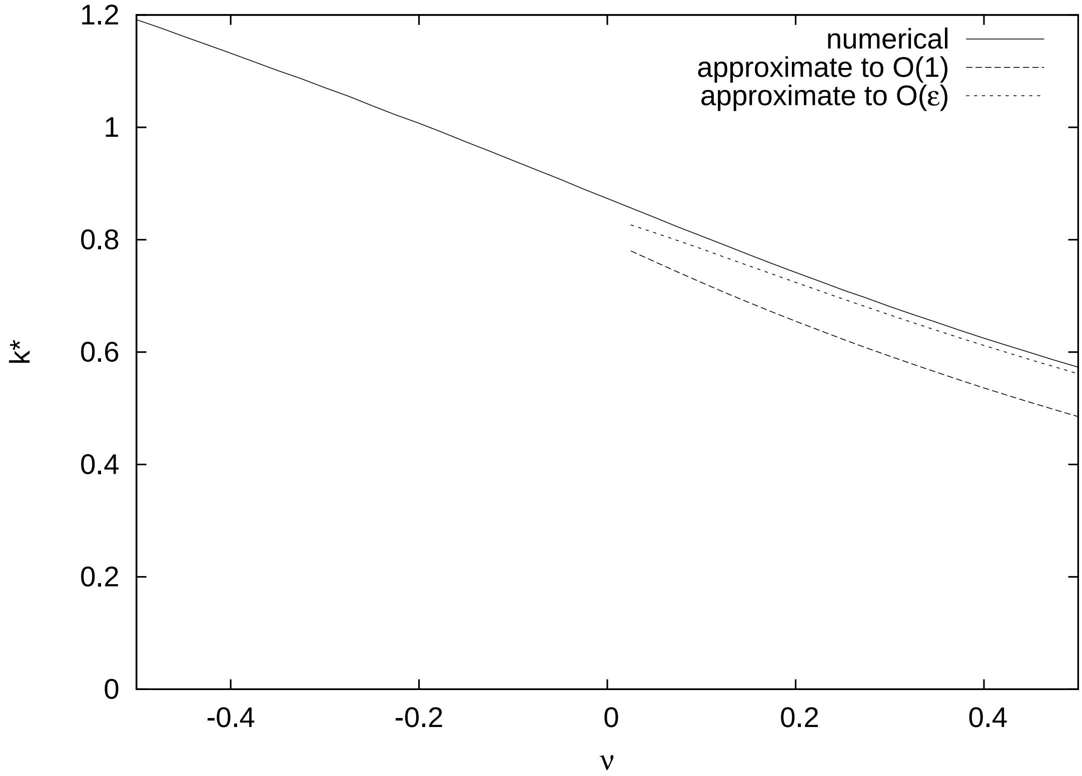
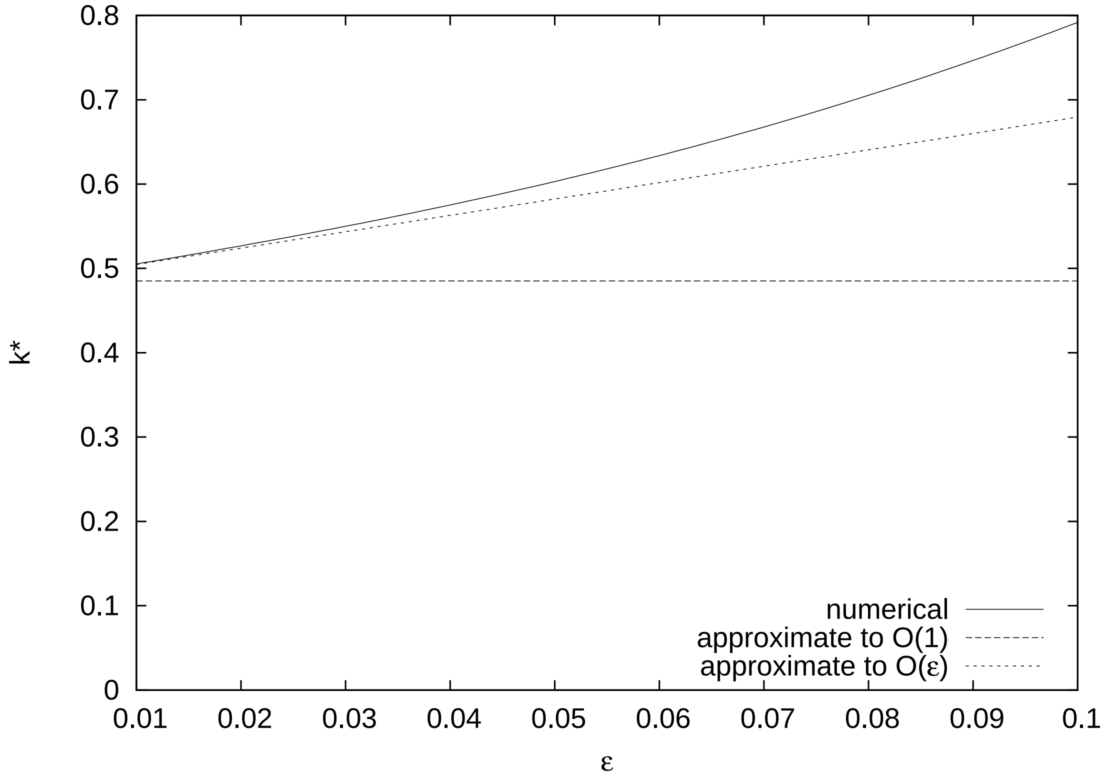


Figure 4: Sensitivity analysis for Strategy I showing the approximate dependence of the root of (17) on the model parameters. The diagram uses the data in (9) with $\bar{p}(0) = 5$.



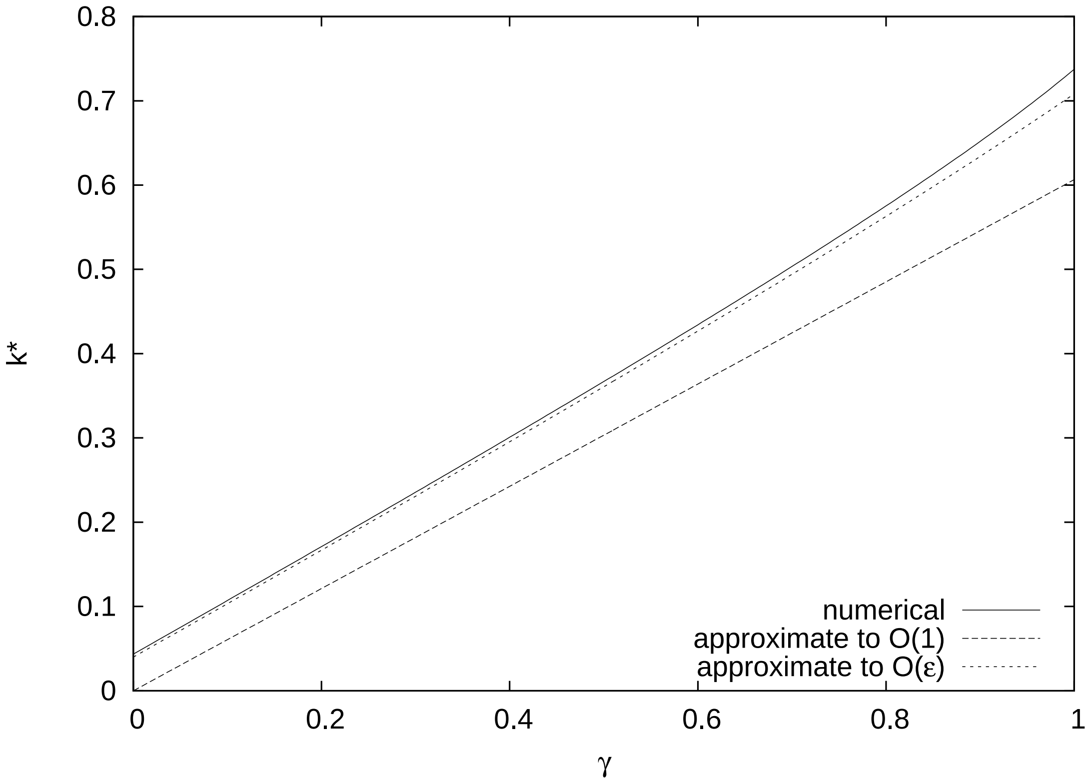
1.2
1
0.8
0.6
0.4
0.2
0
k*
-0.4
-0.2
0
0.2
0.4
ν
numerical
approximate to O(1)
approximate to O(ε)

(i)



0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
k*
0.01
0.02
0.03
0.04
0.05
0.06
0.07
0.08
0.09
0.1
ε
numerical
approximate to O(1)
approximate to O(ε)

(ii)



(iii)

Figure 5: Parameter analysis of Strategy I: (i) as a function of the (scaled) market average premium drift ν ($\varepsilon = 0.04$, $\gamma = 0.8$); (ii) as a function of the inverse market elasticity ε ($\nu = 0.5$, $\gamma = 0.8$); (iii) as a function of the ratio of initial break-even to market average premium γ ($\varepsilon = 0.04$, $\nu = 0.5$). The approximate optimal strategy (26) is shown with and without the $O(\varepsilon)$ term for comparison.

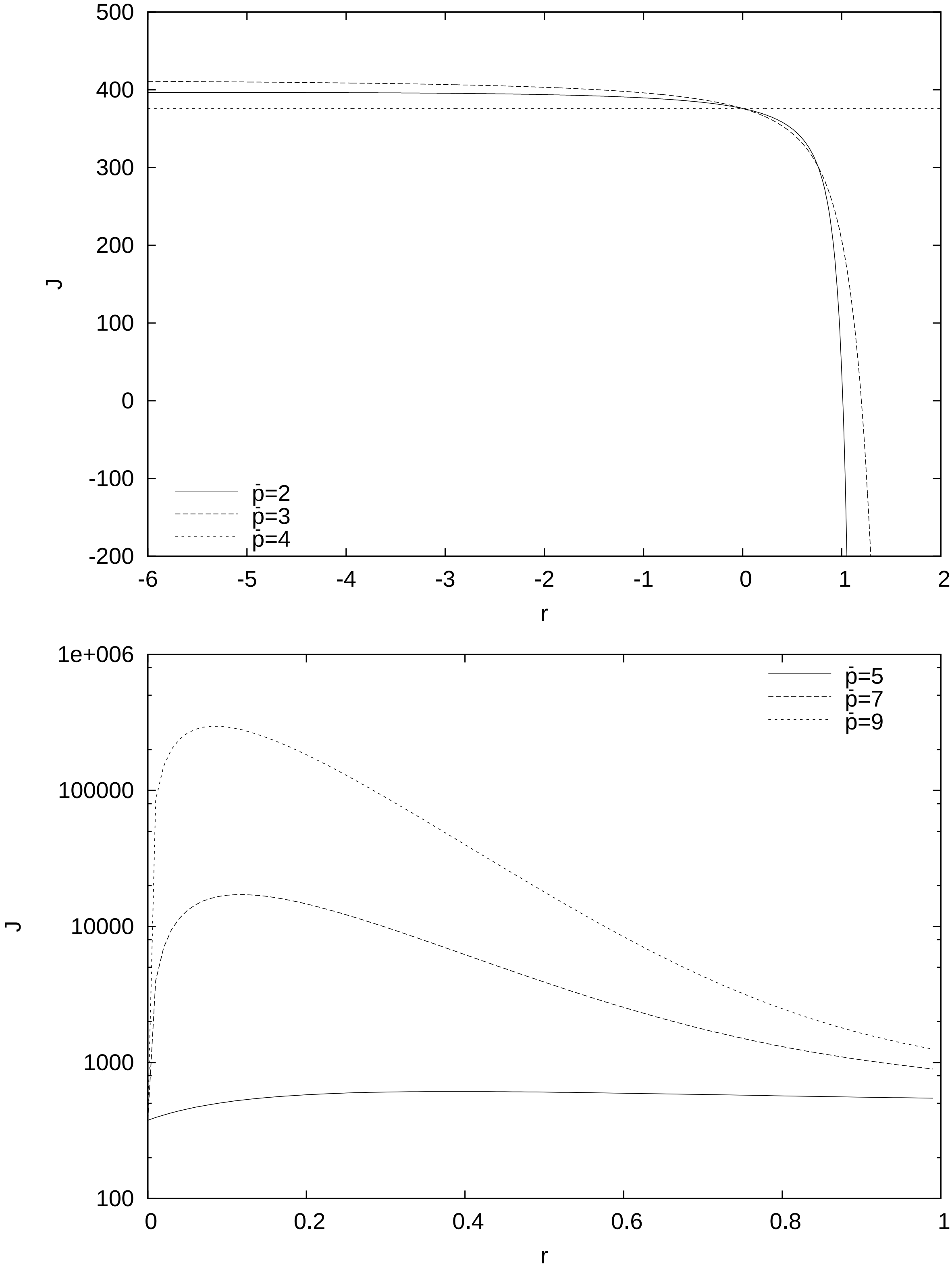
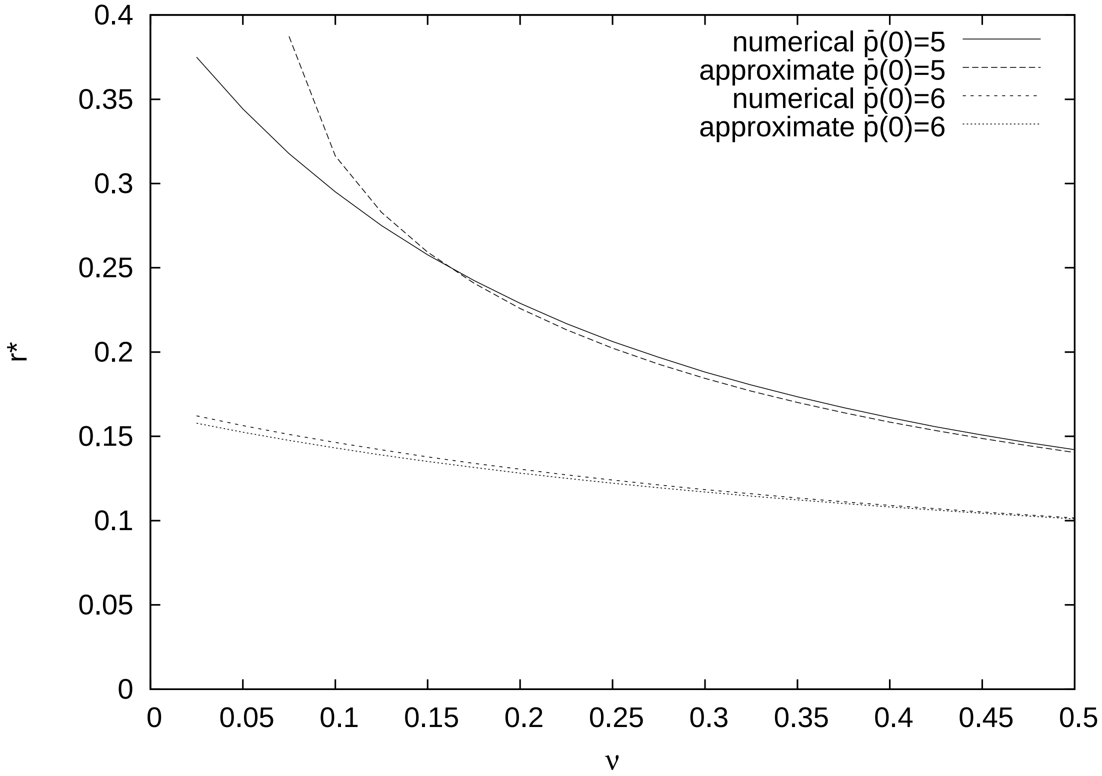
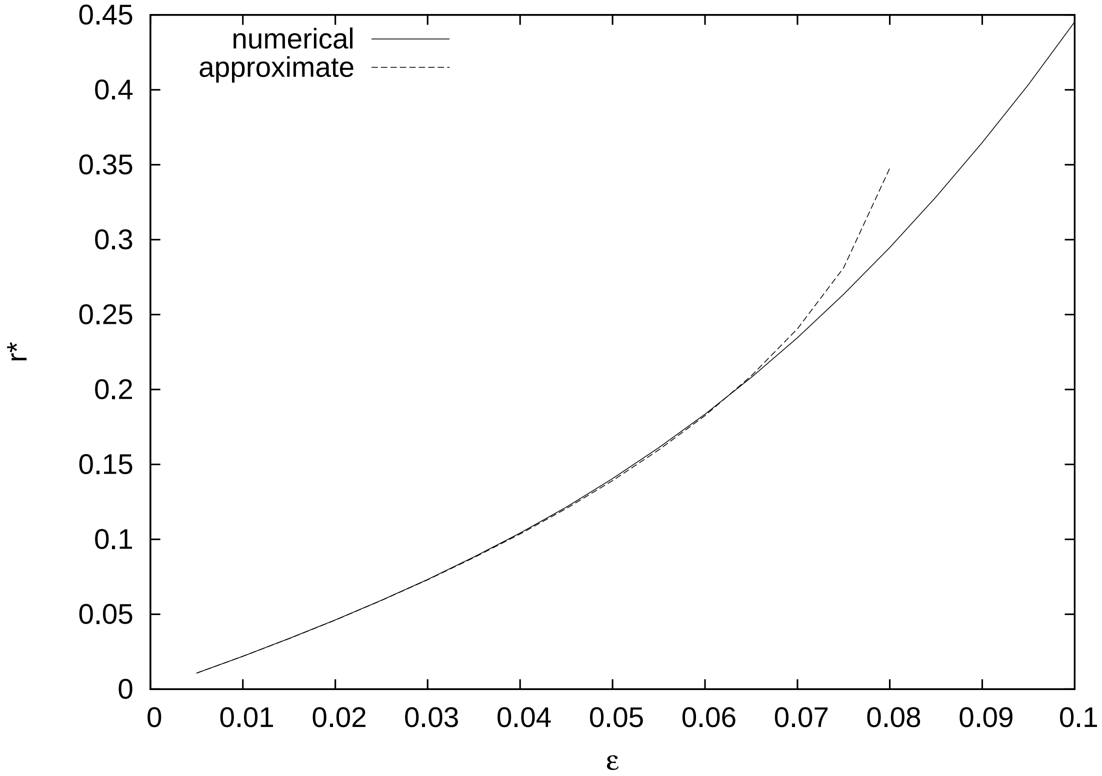


Figure 6: The objective function J for Strategy II with $\mu = \sigma = \alpha = 0$, $\pi = 4$, and an exponential demand function.



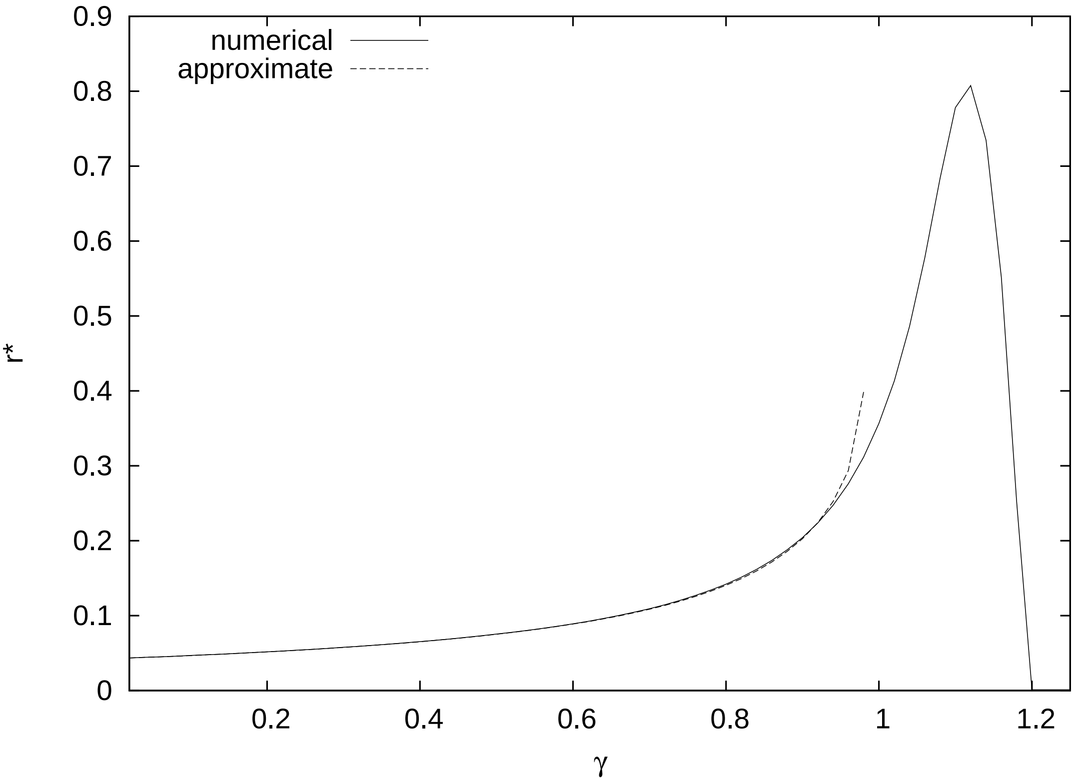
numerical p̄(0)=5
approximate p̄(0)=5
numerical p̄(0)=6
approximate p̄(0)=6
r*
ν

(i)



numerical
approximate
r*
ε

(ii)



(iii)

Figure 7: Parameter analysis of Strategy II for $\sigma = 0$: (i) as a function of the (scaled) market average premium drift $\nu = \mu T$ ($\gamma = 0.667, 0.8$, $\varepsilon = 0.04$); (ii) as a function of the inverse market elasticity ε ($\gamma = 0.8$, $\nu = 0.5$); (iii) as a function of the ratio of initial break-even to market average premium γ ($\varepsilon = 0.04$, $\nu = 0.5$).

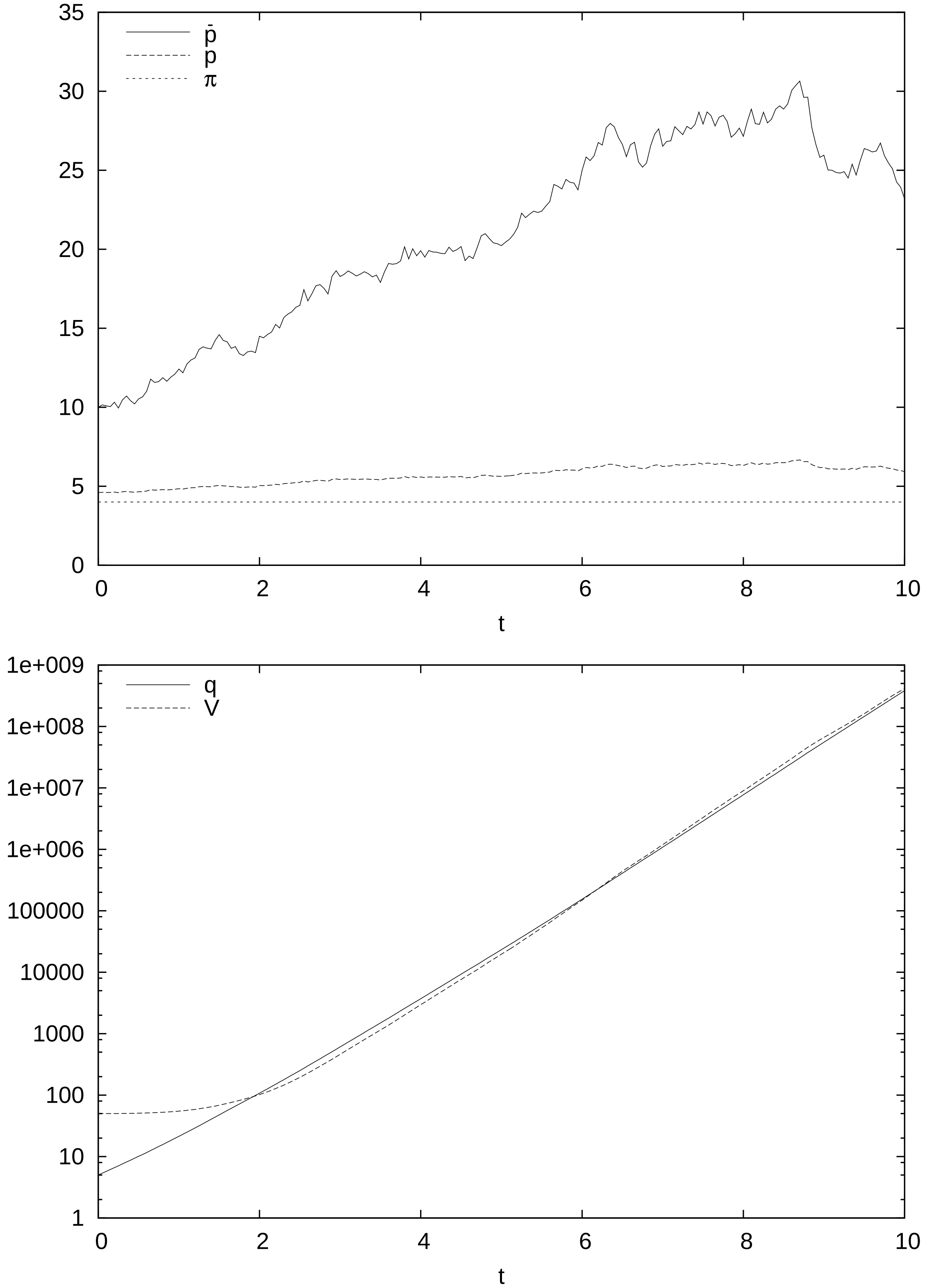


Figure 8: Sample path for Strategy II demonstrating considerable wealth generation. The parameters are taken as $r = 0.1$, $\mu = 0$, $\sigma = 0.1$, $\bar{p}(0) = 10$.

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