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Pricing general insurance with constraints

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Abstract

Deterministic control theory is used to find the optimal premium strategy for an insurer in order to maximise a given objective. The optimal strategy can be loss-leading depending on the model parameters, which may result in negative premium values. In such circumstances, it is optimal to capture as much of the market as possible before making a profit towards the end of the time horizon. In reality, the amount by which an insurer can lower premiums is constrained by borrowing restrictions and the risk inherent in building up a large exposure. Consequently, the effect of constraining the pricing problem is analysed with two forms of constraint: a bounded premium and a solvency requirement. If a lower bound is placed on the premium then an analytical solution can be found, which is not necessarily a smooth function of time. The optimal premium strategy is described in qualitative terms, without recourse to specifying particular parameter values, by considering the value of the terminal optimal premium. Solvency constraints lead to an optimisation problem which is coupled to the state equations and so there is no analytical solution. Numerical results are presented for a subset of the parameter space using control parameterisation, which turns the optimisation problem into a nonlinear programming problem.

1 Introduction

Taylor (1986) found the premium strategy for an insurer in a competitive market which maximised its terminal wealth. He found that, dependent on the model parameters, the optimal strategy could be loss-leading. Since this is not a desirable strategy for many insurers, Taylor modified the demand function of the model specifically to prohibit this phenomenon. Although there are many possible forms for the demand function in the literature (Lilien & Kotler 1983) this leads to a somewhat arbitrary parameterisation. It is also unsatisfactory in that the demand for insurance based on relative price should not depend on the preferences of the insurer. When an insurer has a management policy not to loss-lead, it is constraining the set of feasible premium strategies.

If one allows loss-leading strategies, then Emms (2006) found optimal strategies that lead to negative premiums for certain parameter sets. This is the ultimate loss-leader and reflects the idea that the optimal strategy is to generate as much demand as possible initially in order to subsequently raise prices to generate a large profit. In this work, we study the optimisation problem of insurance pricing with constraints, which are introduced to ensure positive premiums and satisfy company policy. Taylor (1986) used a discrete insurance model which yielded a recursive expression for the pricing strategy. Emms & Haberman (2005) formulated an accrued premium model, which although tractable did not yield an analytical expression for the optimal control. We use the model introduced in Emms (2006) because that model is easier to analyse with contraints.

The model in the body of Emms (2006) used the ratio of the breakeven premium to the market average premium to model claims and was unconstrained. Thus, the outstanding claims made by policyholders were not modelled directly. If we wish to constrain the model then it is more appropriate to consider a model for which one can explicitly calculate the liabilities of the insurer. Therefore, we adopt the model described in the Appendix of Emms (2006), which uses a mean claim size rate to model claims. In this model the objective of the insurer is to maximise its net wealth, that is the wealth accumulated by selling insurance and settling claims including those claims, arising from in-force policies, that have yet to be settled. The primary effect of this model formulation is to change the boundary conditions at the termination of the planning horizon.

In Section 2, we describe the model. Without constraints the optimal premium strategy is stochastic, and can be determined by solving the Bellman equation for the value function in a similar manner to Emms (2006). However, this is a difficult problem if there are constraints since the value function must be calculated in a state space of at least three dimensions and it is not smooth. Consequently, we consider a deterministic relative premium strategy. This reduces the problem to a deterministic optimal control problem if one considers the mean of the state variables. In the accrued premium model of Emms and Haberman (2005) it was found that the deterministic control problem yielded the optimal control for the stochastic problem if the market average premium was lognormally distributed and so the problem is relevant.

In Section 3, we find an analytical solution for the optimal relative premium strategy in the case that the demand function is linear in the relative premium, and the control is bounded from below. Even though an analytical solution exists a variety of possible optimal controls are possible depending on the specified parameter set. We classify the types of solution according to the equilibrium point of the control and the bounds on the premium. This analytical solution gives a good qualitative indication of the strategy, which the insurer should adopt for the more general problem. It also provides a good check on the numerical results.

Control parameterisation is used in Section 4 to calculate numerically the optimal relative premium strategy. For simplicity, the control is parameterised as a step function and this reduces the optimal control problem to a nonlinear programming problem. Two solvency constraints are considered in Section 5 and the optimal control is compared to the analytical solution. The numerical method is used to explore the sensitivity of the optimal control to the model parameters and variations in the objective of the insurer. Conclusions are contained in Section 6.

2 Model

Let us suppose the insurer sets a premium (per unit exposure) of $\mathbf{p}_t$ for insurance cover of fixed duration τ, whilst the market average premium (per unit exposure) is $\bar{\mathbf{p}}_t$. Define the relative (to market) premium as

$$\mathbf{k}_t = \frac{\mathbf{p}_t}{\bar{\mathbf{p}}_t}.$$

Further, suppose both the insurer and the market receive claims on their policies with mean claim size rate $\mathbf{u}_t$ (per unit exposure). For simplicity we suppose the mean claim size rate $\mathbf{u}_t$ is lognormally distributed with constant drift μ and volatility σ:

$$d\mathbf{u}_t = \mathbf{u}_t(\mu\, dt + \sigma\, d\mathbf{W}_t), \tag{1}$$

where $\mathbf{W}_t$ is a standard Brownian motion.

If the rest of the insurance market uses an expected value principle (Rolski *et al.* 1999) to price policies then the market average premium is

$$\bar{\mathbf{p}}_t = (1+\theta)\boldsymbol{\pi}_t, \tag{2}$$

where the breakeven premium $\boldsymbol{\pi}_t$ (per unit exposure) is related to the mean claim size rate $\mathbf{u}_t$ by

$$\boldsymbol{\pi}_t = \mathbb{E}\left[\int_t^{t+\tau} \mathbf{u}_s\, ds | \mathcal{F}_t\right]. \tag{3}$$

Here θ is the market loading and $\mathcal{F}_t$ is the filtration derived from the Brownian motion $\mathbf{W}_t$. Evaluating the conditional expectation gives the market loss rate as

$$\gamma := \frac{\mathbf{u}_t}{\bar{\mathbf{p}}_t} = \frac{\mu}{(1+\theta)(e^{\mu\tau}-1)} > 0, \tag{4}$$

which has dimension per unit time. If there is no drift in the claim size rate, $\mu = 0$, then this loss rate is just $1/(\tau(1+\theta))$.

The remaining components of the general insurance model are similar to Emms (2006) and consist of stochastic differential equations for the insurer's exposure $\mathbf{q}_t$ and wealth process $\mathbf{w}_t$:

$$d\mathbf{q}_t = \mathbf{q}_t G(\mathbf{k}_t, t)\, dt - \mathbf{q}_{t-\tau} G(\mathbf{k}_{t-\tau}, t-\tau)\, dt, \tag{5}$$
$$d\mathbf{w}_t = -\alpha \mathbf{w}_t\, dt + \mathbf{q}_t (G(\mathbf{k}_t)\mathbf{p}_t - \mathbf{u}_t)\, dt. \tag{6}$$

In (5), the rate of increase in exposure caused by new business and renewals is $\mathbf{q}_t G(\mathbf{k}_t, t)$ where $G(\mathbf{k}_t, t)$ is the demand for insurance of relative price $\mathbf{k}_t$ at time t. The rate of decrease in exposure at time t is modelled as $\mathbf{q}_{t-\tau}G(\mathbf{k}_{t-\tau}, t-\tau)$ since it is those policyholders gained at time $t-\tau$ who make the decision to renew their insurance at time t. The assumption we make here is that every policy expires after τ years, and renewals are just treated as new policies. This means that the loss of exposure at time t does not depend on those policyholders who initiated policies at $t-2\tau$, $t-3\tau\ldots$ etc. We parameterise the delay in the exposure equation by setting the loss exposure due to non-renewals as

$$\mathbf{q}_{t-\tau}G(\mathbf{k}_{t-\tau}, t-\tau) \approx -\kappa \mathbf{q}_t, \tag{7}$$

where $\kappa = \tau^{-1}$, so that the exposure decays exponentially in τ years if no more insurance is sold. This is the parameterisation adopted in Emms & Haberman (2005) and turns the stochastic differential delay equations into straightforward SDEs. In the second equation (6), α is the rate of loss of wealth from dividend payments to shareholders in the insurance company.

The demand function $G(k, t)$ is a non-negative function of the relative premium and time. A linear demand function takes the form

$$G(k) = a(b-k)^+, \tag{8}$$

where $a > \kappa > 0$ so that there are positive relative premiums which generate exposure and we ignore any explicit time dependence. In addition, a has dimension per unit time and $b \geq 1$ is dimensionless and chosen so that there is demand for insurance set below the market price. This is the simplest positive function which models the change in demand with relative premium.

For this model the wealth process $\mathbf{w}_t$ does not account for the liabilities of the insurer. If the insurer were to suspend the sale of insurance at time t then according to the state equation (5) and using (7) we have for $s > t$

$$q_s = q_t e^{-\kappa(s-t)}.$$

Therefore, though the insurer has no income, the expected loss in wealth due to the outstanding liabilities is

$$\mathbb{E}\left[\int_t^\infty q_s \mathbf{u}_s \, ds | \mathcal{F}_t\right] = \frac{q_t u_t}{\kappa - \mu},$$

where $\kappa > \mu$ in order that the expectation is finite.

The optimal relative premium can be determined by specifying an objective in terms of the state variables, and then finding the relative premium which maximises this objective. The stochastic optimisation problem can then be solved by forming a value function and then integrating the corresponding HJB equation (Fleming & Rishel 1975). However, if the optimisation problem has constraints on the control and/or the state variables, then the control is unlikely to be smooth. Rather than try and find weak solutions of the HJB equation (Fleming & Soner 1993), we restrict the form of feasible control, which yields a deterministic optimisation problem similar to that in Emms & Haberman (2005).

2.1 Deterministic relative premium

Suppose the insurer adopts the deterministic open loop strategy

$$\mathbf{k}_t = k(t). \tag{9}$$

Although the relative premium is deterministic the insurer's premium is stochastic because the market average premium is a random process. Write

$$m_u(t) = \mathbb{E}[\mathbf{u}_t|\mathcal{F}_0], \quad \text{etc.}$$

We apply this conditional expectation to the state equations (5), (6) leading to

$$dq = q(G(k) - \kappa)\, dt, \tag{10}$$

$$dm_w = -\alpha m_w\, dt + q(G(k(t))k(t)m_{\bar{p}} - m_u)\, dt. \tag{11}$$

The exposure q is a deterministic state variable, whereas $m_u(t) = \gamma m_{\bar{p}}(t)$ is a given function of time. For the lognormal process in (1)

$$m_u(t) = u_0 \exp(\mu t).$$

The optimisation problem is now deterministic and can be written in canonical form. Define the state vector by

$$\mathbf{x} = \begin{pmatrix} q \\ m_w \end{pmatrix},$$

so that the state equation is

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t), \tag{12}$$

where $\mathbf{f} = (q(G(k) - \kappa), -\alpha m_w + q(G(k)km_{\bar{p}} - m_u))^T$. The objective function can be written generally as

$$J_0(\mathbf{x}, T) = \phi_0(\mathbf{x}(T), T) + \int_0^T \mathcal{L}_0(\mathbf{x}(t), t)\, dt. \tag{13}$$

We want to find $k(t)$ to maximise the value of the objective J_0 subject to a number of constraints. For example, if the insurer wishes to maximise its expected terminal net wealth then $\mathcal{L}_0 = 0$ and

$$\phi_0(\mathbf{x}(T), T) = x_2(T) - \frac{x_1(T)m_u(T)}{\kappa - \mu}. \tag{14}$$

To maximise the total utility of net wealth with utility function U, set the salvage function $\phi_0 = 0$ and

$$\mathcal{L}_0(\mathbf{x}(t), t) = U\left(x_2(t) - \frac{x_1(t)m_u(t)}{\kappa - \mu}\right). \tag{15}$$

If the constraints are a function of the control, k, then there is an analytical solution for the deterministic optimisation problem.

3 Analytical solution

Let us suppose that the premium is bounded from below so that $k(t) \geq k_0$ where k_0 is a constant. We must have $0 < k_0 < b$ for a positive premium that generates demand for insurance. This form of constraint can be used to prevent loss-leading premiums. The breakeven premium is according to (3)

$$\boldsymbol{\pi}_t = \left(\frac{e^{\mu\tau} - 1}{\mu} \right) \mathbf{u}_t,$$

and the market average premium $\bar{\mathbf{p}}_t = (1 + \theta)\boldsymbol{\pi}_t$. Consequently, the condition for no loss-leading ($\mathbf{p}_t > \boldsymbol{\pi}_t$) is

$$k \geq \frac{1}{1 + \theta} = k_0,$$

where θ is the insurance loading.

Suppose the insurer aims to maximise its expected terminal net wealth so that the objective is given by (14) and we adopt the linear demand function (8). The Hamiltonian is defined by

$$H(\mathbf{x}, k, \boldsymbol{\lambda}, t) = \lambda_1 x_1 (G(k) - \kappa) + \lambda_2(-\alpha x_2 + x_1(G(k) k m_{\bar{p}} - m_u)), \tag{16}$$

where $\boldsymbol{\lambda} = (\lambda_1, \lambda_2)^T$ is the adjoint vector. The Hamiltonian is linear in the state variables which means that the necessary conditions for an optimal control (The Maximum Principle) are also sufficient (Sethi & Thompson 2000). Pontryagin's maximum principle requires that the optimal control k^* maximises the Hamiltonian:

$$H(\mathbf{x}^*(t), k^*(t), \boldsymbol{\lambda}(t), t) = \max_{k \geq k_0} H(\mathbf{x}^*(t), k, \boldsymbol{\lambda}(t), t), \tag{17}$$

for $t \in [0, T]$. If this maximum is given by the first order condition then we say the control is interior and is given by

$$k^i := \frac{1}{2} \left(b - \frac{\lambda_1}{m_{\bar{p}} \lambda_2} \right). \tag{18}$$

Notice that the Hamiltonian is quadratic in the control k when $k < b$, and that the coefficient of the k^2 term is negative if $\lambda_2 > 0$ as is shown shortly. Thus, the first order condition does yield the maximum of the Hamiltonian if $k_0 < k^i < b$. If $k^i \geq b$ then $G(k^i) = 0$ and the Hamiltonian H is independent of k. Therefore the maximum of H is undefined because no insurance is sold. For continuity, we set the optimal relative premium to be $k^* = b$ in the case that $k^i \geq b$. If $k^i \leq k_0$ then the constraint gives the optimal control. Consequently, the optimal control is given by

$$k^*(t) = \max\{\min\{k^i(t), b\}, k_0\}.$$

The adjoint equations are

$$\frac{d\lambda_1}{dt} = -H_{x_1} = -\lambda_1(G(k) - \kappa) - \lambda_2(kG(k)m_{\bar{p}} - m_u), \tag{19}$$

$$\frac{d\lambda_2}{dt} = -H_{x_2} = \alpha\lambda_2, \tag{20}$$

with transversality conditions

$$\lambda_1(T) = J_{0x_1} = -\frac{m_u(T)}{\kappa - \mu},$$
$$\lambda_2(T) = J_{0x_2} = 1.$$

Equation (20) can be integrated immediately to give

$$\lambda_2 = e^{\alpha(t-T)} > 0.$$

Both adjoint variables and the interior control are independent of the current state, which means that the problem uncouples and we need only integrate backwards from $t = T$ in order to find the optimal control.

The equation for λ_1 takes on three different forms depending on the value of the control:

$$\frac{d\lambda_1}{dt} = \begin{cases} -\lambda_1(a(b-k_0)-\kappa) - e^{\alpha(t-T)}(ak_0(b-k_0)m_{\bar{p}} - m_u) & \text{if } k^* = k_0, \\ -\dfrac{a}{4m_{\bar{p}}e^{\alpha(t-T)}}\lambda_1^2 + (\kappa - \frac{1}{2}ab)\lambda_1 + e^{\alpha(t-T)}m_{\bar{p}}(\gamma - \frac{1}{4}ab^2) & \text{if } k_0 < k^* = k^i < b, \\ m_u e^{\alpha(t-T)} + \kappa\lambda_1 & \text{if } k^* = b. \end{cases}$$

We write $\omega = \lambda_1 e^{\alpha(T-t)}/m_{\bar{p}}$ so that the interior relative premium is just

$$k^i = \tfrac{1}{2}(b - \omega), \tag{21}$$

and the adjoint equation becomes

$$\frac{d\omega}{dt} = \begin{cases} \omega(\kappa - \alpha - \mu - a(b-k_0)) - ak_0(b-k_0) + \gamma & \text{if } k^* = k_0, \\ -A\omega^2 - B\omega - C & \text{if } k_0 < k^* < b, \\ \omega(\kappa - \mu - \alpha) + \gamma & \text{if } k^* = b, \end{cases} \tag{22}$$

with boundary condition

$$\omega(T) = -\frac{\gamma}{\kappa - \mu}, \tag{23}$$

and where we have defined

$$A = \tfrac{1}{4}a, \quad B = \tfrac{1}{2}ab + \alpha + \mu - \kappa, \quad C = \tfrac{1}{4}ab^2 - \gamma,$$

and used the fact that the mean claim size rate drifts at constant rate μ so that

$$dm_{\bar{p}} = \mu m_{\bar{p}}\, dt.$$

The first and third cases of (22) are easily integrated to give

$$\omega_1(t) = K_1 e^{(\kappa-\alpha-\mu-a(b-k_0))t} + \frac{\gamma - ak_0(b-k_0)}{a(b-k_0)+\alpha+\mu-\kappa}, \tag{24}$$
$$\omega_3(t) = K_3 e^{(\kappa-\mu-\alpha)t} + \frac{\gamma}{\mu+\alpha-\kappa}. \tag{25}$$

The second equation can also be integrated explicitly and the form of the solution depends on the discriminant $\Delta = B^2 - 4AC$ of the quadratic on the RHS of (22). It is convenient to introduce the notation

$$D_{\pm}^2 = \pm\Delta = \pm(ab(\alpha+\mu-\kappa) + (\alpha+\mu-\kappa)^2 + a\gamma).$$

The explicit solution can then be written

$$\omega_2(t) = \begin{cases} \dfrac{\tan(\frac{1}{2}(K_2 - t)D_-))}{2A} & \text{if } \Delta < 0, \\ \dfrac{B - D_+ - (D_+ + B)e^{(K_2-t)D_+}}{2A(e^{(K_2-t)D_+} - 1)} & \text{if } \Delta > 0, \\ \dfrac{1}{A}\left(\dfrac{1}{K_2 - t} - \frac{1}{2}B\right) & \text{if } \Delta = 0. \end{cases} \tag{26}$$

Notice that the discriminant Δ is independent of the time horizon T, so that as we vary this parameter we do not change the qualitative form of the optimal control.

We can use these expressions to piece together the optimal control. At termination

$$k^i(T) = \frac{1}{2}\left(b + \frac{\gamma}{\kappa - \mu}\right).$$

Depending on the values of b, γ, κ, and μ the optimal control at termination is either k_0, $k^i(T)$ or b. We consider each of these cases in turn.

3.1 Case $k_0 < k^*(T) < b$

If the optimal control is interior at termination then we can use the boundary condition (23) to calculate

$$K_2 = \begin{cases} T - \dfrac{2}{D_-}\tan^{-1}\left(\dfrac{2\gamma A}{D_-(\kappa-\mu)} - \dfrac{B}{D_-}\right) & \text{if } \Delta < 0, \\ T + \dfrac{1}{D_+}\log\left(\dfrac{B - D_+ - \dfrac{2\gamma A}{\kappa-\mu}}{B + D_+ - \dfrac{2\gamma A}{\kappa-\mu}}\right) & \text{if } \Delta > 0, \\ T + \dfrac{1}{\frac{1}{2}B - \dfrac{\gamma A}{\kappa-\mu}} & \text{if } \Delta = 0. \end{cases} \tag{27}$$

Here, we allow for a complex value of K_2 if $\Delta > 0$, which then leads to a real value of ω_2 when substituted into (26) since we require $e^{K_2 D_+}$.

We classify the control as Type 1 or 2 depending on the form that the *interior* control takes from (26) and its behaviour as $t \to -\infty$. The two principal cases are given by the first two forms of (26) and they are illustrated in Figure 1. If $\Delta < 0$ then as $t \to -\infty$, $\omega_2 \to \infty$ and $k^i \to -\infty$. We call this a Type 1 control if the optimal control exists. There is no optimal control if $\omega_2 = \infty$ for $t \in [0, T]$. If $\Delta > 0$ then $\omega_2 \to (-B - D_+)/2A$; the stable equilibrium point. We call this a Type 2 control. Figure 2 shows the qualitative form of the Type 1 and Type 2 controls.

If the interior control does not hit the boundaries $k^i = k_0$ or $k^i = b$ then the optimal state trajectory $\mathbf{x}^*$ can be calculated directly from the state equations for certain cases. However, the expressions rapidly become unwieldy. For example, if $\Delta > 0$ then substituting (27) into (26) yields

$$\omega_2(t) = \frac{B - D_+ - (D_+ + B)\rho e^{(T-t)D_+}}{2A\left(\rho e^{(T-t)D_+} - 1\right)},$$

where we have written

$$\rho = \frac{B - D_+ - \dfrac{2\gamma A}{\kappa - \mu}}{B + D_+ - \dfrac{2\gamma A}{\kappa - \mu}}.$$

The first state equation can be rewritten as

$$\frac{d \log x_1^*}{dt} = \tfrac{1}{2}a(b + \omega_2(t)) - \kappa,$$

and substituting the expression for $\omega_2(t)$ yields

$$x_1^*(t) = x_1^*(0)\left(\frac{\rho e^{D_+(T-t)} - 1}{\rho e^{D_+ T} - 1}\right)^2 \exp\left[(D_+ - \alpha - \mu)\, t\right] \tag{28}$$

after some simplification.

A Type 1 control is a wealth generating strategy and can be loss-leading or lead to negative premiums if the optimisation problem is unconstrained. By setting a low initial premium the insurer builds up exposure, and as the premium is increased over the planning horizon, wealth is generated from the existing customer base. A Type 2 control is often associated with the insurer withdrawing from the market. This is the optimal way to exploit the insurer's existing customers in order to maximise its objective.

If the interior control hits the upper control boundary at $k = b$ then we call the control a Type 3 control, while if the interior control hits the lower boundary at $k = k_0$ then we classify the control as Type 4. Once the control hits a boundary then the optimal control is governed by (24) or (25). Therefore, it seems likely that the control can enter and then leave the region $k_0 \le k \le b$. We shall ignore these possibilities in our classification of the controls. Typical Type 3 and Type 4 controls are drawn in Figure 2. It is clear that the control even for this simple model can be non-smooth and have a complex structure. Rather than give an exhaustive description of every possibility, we highlight the major cases and confirm the structure with numerical solutions.

3.2 Case $k^*(T) = k_0$

The constraint gives the optimal control at termination if

$$k^i(T) = \frac{1}{2}\left(b + \frac{\gamma}{\kappa - \mu}\right) \le k_0,$$

Since $\kappa > \mu$ and $\gamma > 0$, this forces $k_0 \geq \frac{1}{2}b$. We can use the termination condition to calculate K_1 from (23) and (24):

$$K_1 = \left(\frac{ak_0(b-k_0) - \gamma}{a(b-k_0) + \alpha + \mu - \kappa} - \frac{\gamma}{\kappa - \mu} \right) e^{(a(b-k_0)+\alpha+\mu-\kappa)T}.$$

Substituting into (24) and using (21) gives the interior premium as

$$k^i(t) = \frac{1}{2} \left(b + \left(\frac{\gamma}{\kappa - \mu} + \frac{\gamma - ak_0(b-k_0)}{a(b-k_0) + \alpha + \mu - \kappa} \right) e^{(\kappa-\alpha-\mu-a(b-k_0))(t-T)} + \frac{ak_0(b-k_0) - \gamma}{a(b-k_0) + \alpha + \mu - \kappa} \right).$$

The optimal control remains k_0 as long as $k^i(t) \leq k_0$ and that depends on the behaviour of the exponential as t is decreased from the termination time T. We can rewrite the second term in the above as

$$\frac{a(b-k_0)(\gamma - k_0(\kappa - \mu)) + \alpha\gamma}{(\kappa - \mu)\psi} e^{\psi(T-t)}, \tag{29}$$

where

$$\psi = a(b-k_0) + \alpha + \mu - \kappa.$$

If $\psi \neq 0$ then $e^{\psi(T-t)}/\psi \geq 1/\psi$ irrespective of the sign of ψ. Consequently the interior control varies according to the sign of the numerator in (29). If $a(b-k_0)(\gamma - k_0(\kappa - \mu)) + \alpha\gamma < 0$ then

$$k^i(t) \leq k_0,$$

for $t \in [0, T]$ so the optimal control is degenerate and is the constrained premium over the entire time horizon: $k^*(t) \equiv k_0$. If $a(b-k_0)(\gamma - k_0(\kappa-\mu)) + \alpha\gamma > 0$ then the interior relative premium can increase as t decreases from T and may rise above the constraint in $[0, T]$. We call this a Type 5 control as illustrated in Figure 2. If $\psi = 0$ then the behaviour of the optimal control is similar.

3.3 Case $k^*(T) = b$

In this case it is optimal not to sell insurance at termination:

$$k^i(T) = \frac{1}{2} \left(b + \frac{\gamma}{\kappa - \mu} \right) = \frac{1}{2} \left(b + \frac{\mu}{(1+\theta)(e^{\mu\tau} - 1)(\kappa - \mu)} \right) \geq b,$$

using (4). If we suppose $\mu\tau$ is small and expand the exponential we require

$$\frac{\kappa}{\kappa - \mu} \geq \frac{\kappa}{(1+\theta)(\kappa - \mu)} \geq b \geq 1.$$

Consequently, this case only applies if these is a positive drift in the claims rate over the course of the planning horizon.

Assuming the parameters do lead to $k^*(T) = b$ then the interior premium is given by

$$k^i(t) = \frac{1}{2}\left(b - \frac{\gamma}{\phi} + \frac{\alpha\gamma e^{\phi(T-t)}}{\phi(\kappa-\mu)}\right),$$

where we have set

$$\phi = \alpha + \mu - \kappa.$$

If $\phi = 0$ then $d\omega/dt = \gamma$ from (22) so ω is always increasing with t, which means that k^i is always decreasing towards its terminal value. Therefore

$$k^i(t) = k^*(T) = b,$$

for $t \in [0, T]$. Consequently it is optimal for the insurer to withdraw from the insurance market. If $\phi \neq 0$ then

$$\frac{e^{\phi(T-t)}}{\phi} \geq \frac{1}{\phi}.$$

Therefore

$$k^i(t) \geq \frac{1}{2}\left(b - \frac{\gamma}{\phi} + \frac{\alpha\gamma}{\phi(\kappa-\mu)}\right) = k^*(T) = b,$$

since $\kappa > \mu$. So again, it is optimal for the insurer to withdraw from the insurance market.

3.4 Alternative objective functions

The preceding analytical solution applies to the case that the insurer maximises its expected net terminal wealth given by (14). If the insurer wishes to maximise its expected utility of net wealth using a given utility function U then $\phi_0 = U(w^{net}(T))$ where the net wealth is

$$w^{net}(t) = x_2(t) - \frac{x_1(t)m_u(t)}{\kappa - \mu}.$$

The form of the interior control (18) and the adjoint equations (19),(20) is identical for this problem, whereas the transversality conditions become

$$\lambda_1(T) = J_{0x_1} = -U'(w^{net}(T))\frac{m_u(T)}{\kappa-\mu},$$
$$\lambda_2(T) = J_{0x_2} = U'(w^{net}(T)).$$

If we rescale λ_1 and λ_2 by $U'(w^{net}(T)) \neq 0$ then the adjoint equations are unchanged since they are homogeneous and linear in the adjoint variables. Moreover, the interior control (18) is also unchanged since it is a function of λ_1/λ_2. Consequently, the optimal control k^* is independent of the utility function irrespective of the constraints, providing that $U'(w^{net}(T)) \neq 0$. If the insurer has a power utility function of the first kind (Gerber & Pafumi 1998) and the net wealth is greater than the level of saturation then $U'(w^{net}(T)) = 0$ and the utility function does affect the optimal strategy, which may not be unique.

Suppose the insurer wishes to maximise its *total* expected utility of net wealth given in (15). For this case, the Hamiltonian is modified by incorporating the utility function:

$$H(\mathbf{x}, k, \boldsymbol{\lambda}, t) = U(w^{net}) + \lambda_1 x_1 (G(k) - \kappa) + \lambda_2(-\alpha x_2 + x_1(G(k)km_{\bar{p}} - m_u)),$$

and the first order condition for a maximum is given by (18). The adjoint equations are

$$\frac{d\lambda_1}{dt} = -H_{x_1} = \frac{U'(w^{net})m_u(t)}{\kappa - \mu} - \lambda_1(G(k) - \kappa) - \lambda_2(kG(k)m_{\bar{p}} - m_u), \qquad (30)$$

$$\frac{d\lambda_2}{dt} = -H_{x_2} = -U'(w^{net}) + \alpha\lambda_2, \qquad (31)$$

and the transversality conditions are $\lambda_1(T) = \lambda_2(T) = 0$.

Consider the case that the utility function is linear in the net wealth so that $U'(w^{net}) = 1$. The adjoint equation (30) is not homogeneous in the adjoint variable and cannot be integrated analytically except for special cases (see Emms (2006) for a similar problem). Consequently, the optimal control cannot be determined analytically and differs from that in Section 3. We expect a more conservative premium strategy with diminished loss-leading because the wealth over the entire time horizon affects the objective including any initial loss-leading. Notice that we cannot rescale the adjoint variables λ_1 and λ_2 and retrieve the adjoint equations for the net wealth problem because $U'(w^{net})$ depends on the current state.

4 Numerical solution

Here we outline the numerical technique of control parameterisation as applied to this problem. Full details can be found in Teo, Goh & Wong (1991). For simplicity we split the time domain $[0, T]$ up into n equal intervals of size Δt. Let us denote each interval by $I_i = [i\Delta t, (i+1)\Delta t]$ for $i = 0, 1, \ldots, n-1$. The control k is approximated by the step function

$$k(t) = \sum_{i=0}^{n-1} c_i \chi_{I_i},$$

where the n constants c_i are the unknowns over which we wish to maximise the objective J_0 given by (13).

The objective can be approximated using this step function so that the objective is approximately

$$\mathcal{J}_0(\mathbf{c}) = \phi_0(\mathbf{x}(T|\mathbf{c})) + \int_0^T \tilde{\mathcal{L}}_0(t, \mathbf{x}(t|\mathbf{c}), \mathbf{c})\, dt, \qquad (32)$$

where the notation $\mathbf{x}(t, \mathbf{c})$ indicates a forward integration of the state equations (12) using the step function approximation as the control and $\tilde{\mathcal{L}}_0$ is the approximate functional. Control constraints can be naturally expressed in terms of the vector $\mathbf{c}$. Consequently (32) is a nonlinear optimisation problem and the control constraints are either equality or inequality constraints on the vector of unknowns. When constraints involve the state variables the problem is more complicated.

Suppose there are n_e equality constraints involving the state variables and n_{in} inequality constraints. These constraints can be written in canonical form in a similar manner to the objective function. For the equality constraints

$$\mathcal{J}_i(\mathbf{c}) = \phi_i(\mathbf{x}(T|\mathbf{c})) + \int_0^T \tilde{\mathcal{L}}_i(t, \mathbf{x}(t|\mathbf{c}), \mathbf{c})\, dt = 0,$$

for $i = 1, \ldots, n_e$ and for the inequality constraints

$$\mathcal{J}_i(\mathbf{c}) = \phi_i(\mathbf{x}(T|\mathbf{c})) + \int_0^T \tilde{\mathcal{L}}_i(t, \mathbf{x}(t|\mathbf{c}), \mathbf{c})\, dt \geq 0,$$

for $i = n_e+1, \ldots, n_e+n_{in}+1$. The pure state constraints considered here can be written in a particularly economical form using a constraint transcription. Suppose the pure state constraint is

$$h(\mathbf{x}, t) \geq 0,$$

then set $\phi_1 = 0$ and

$$\mathcal{L}_1 \min(h, 0) = 0.$$

This converts the n inequality constraints applied at each grid point to one equality constraint. However, the transcription can lead to numerical oscillations. These can be removed by smoothing the jump in the gradient of the function $\min(h, 0)$ at $h = 0$.

We adopt the simplest numerical procedure to solve this optimisation problem without loss of generality. Given a function for the control, the state equations are integrated forwards in time using Euler's Method. This allows the integral in $\mathcal{J}_i(\mathbf{c})$ to be calculated using a simple step function integration. The objective is maximised using quadratic programming and so derivatives of the objective and the constraints are also required. Usually this requires the calculation of an adjoint variable corresponding to each unknown.

We have adopted the nonlinear optimiser written by Spellucci (1998a), which uses a sequential equality constrained quadratic programming method. Further details can be found in Spelluci (1998c), (1998b). This routine also allows numerical differentiation of the objective and constraints and so the calculation of the adjoint variables can be avoided at the expense of extra computation. This technique is analogous to replacing Newton's method by a secant rule. Control parameterisation allows us to solve this problem, and other similar premium pricing problems, as long as we specify the premium strategy (9) and the state equations, objective and constraints are linear in the state variables.

4.1 Validation of the numerical solution

First, we reproduce the analytical results we found in Section 3 and verify the convergence of the control parameterisation as the number of steps n is increased. The advantage of the analytical solution is that it conveys the sensitivity of the optimal control to the parameters in the model. Using a numerical method we must solve the optimisation problem for many parameter sets in order to determine the sensitivity of the model.

Suppose first that we allow the control to be unconstrained. We adopt the following values as a base parameter set:

$$\mu = 0, \quad a = 3, \quad b = 1.5, \quad \kappa = 1, \quad \alpha = 0.05, \quad \theta = 0.1, \quad T = 3. \tag{33}$$

This data set is comparable to that used in Emms & Haberman (2005). Using these figures yields the discriminant $\Delta = -0.65$ which corresponds to the first case in (26), whereas if we reset $b = 1$ then we obtain $\Delta = 0.78$ corresponding to the second case. Figures 3(a) and (b) show the calculated step function which maximises the expected terminal net wealth for these two parameter sets. The analytical expressions from (26) are also superimposed on the graphs. The number of steps is quadrupled to $n = 80$ in the second set of graphs, which confirms that the numerical solution is converging towards the analytical result. The evolution of the state equations is also shown where the initial conditions are

$$x_1(0) = x_2(0) = 1.$$

It is clear that graph (a) shows a Type 1 control which generates wealth up until near the end of the time horizon where the premium is well above market average. Graph (b) shows a Type 2 control which loses wealth and leads to market withdrawal. Notice the rapid decay in the insurer's exposure in the second graph in (b).

In Figure 4 we show the optimal premium strategy for a parameter set which leads to a singularity in the analytical expression (26) over $[0, T]$. If the problem is unconstrained then there is no optimal premium strategy because unbounded wealth can be generated. If we attempt to obtain a numerical solution the nonlinear optimiser fails to converge.

Suppose we introduce the control constraint which prevents negative premiums

$$k \geq k_0 \geq 0. \tag{34}$$

Consider the parameter set which leads to Figure 4 if the control is unconstrained. If we now impose the constraint with $k_0 = 0$ an optimal numerical premium strategy does exist which is shown by the thin line and is a Type 4 control. It is clear that the constrained control is dependent on the unconstrained problem and so we expect parameteric dependences to be similar. In Figure 4 the optimal control is tight (i.e. it is given by the constraint) up until approximately the singularity of the analytical expression (26). Considerable wealth is generated by this strategy since we allow the insurer to give insurance away for nothing in order to build up a customer base.

Figure 5 shows a parameter set which leads to a Type 5 control when $k_0 = 0.96$. Here it is optimal to leave the insurance market and that involves gradually decreasing the insurer's premium. Towards the end of the planning horizon the relative premium hits the imposed lower bound on the control. This example demonstrates that the optimal control can be non-smooth even if the insurer leaves the insurance market: it is not just loss-leading strategies that lead to non-smooth controls. We now have confidence that the numerical optimiser can calculate the optimal control for more general demand functions, objectives and constraints.

5 Solvency constraints

A solvency constraint is expressed in terms of the current wealth of the insurer so that this leads to a state constraint. Analytical results are not available in general because the adjoint equations do not uncouple from the state equations. One is required to integrate the state equations forward in time at the same time as integrating the adjoint equations backwards from termination T in order to check that the constraint is satisfied at time t. However, if the constraint is linear and homogeneous in the state variables then the control can be determined when the constraint is tight. We develop the theory for this special case first.

5.1 Maximum principle for a pure state constraint

Let us constrain the optimal control so that the insurer has sufficient funds to pay off the expected outstanding claims should no more insurance be sold. We require

$$h(\mathbf{x}, t) = x_2 - \frac{x_1 m_u(t)}{\kappa - \mu} \geq 0. \tag{35}$$

This is a first order constraint so we define

$$h^1(\mathbf{x}, t) = \frac{dh}{dt} = -\alpha x_2 + x_1(kG(k)m_{\bar{p}} - m_u) - \frac{x_1}{\kappa - \mu}(m_u' + m_u(G(k) - \kappa)),$$

using the state equations (12). Sethi & Thompson (2000) give the Maximum Principle for constraints of this form.

Define the Lagrangian

$$L(\mathbf{x}, k, \lambda, t) = H(\mathbf{x}, k, \lambda, t) + \eta h^1(\mathbf{x}, t),$$

where H is defined by (16) and η satisfies the complementary slackness conditions

$$\eta \geq 0, \quad \eta h(\mathbf{x}, t) = 0, \quad \frac{d\eta}{dt} \leq 0.$$

The Hamiltonian maximising condition is that

$$H(\mathbf{x}^*(t), k^*(t), \lambda(t), t) \geq H(\mathbf{x}^*(t), k, \lambda(t), t),$$

for all k satisfying

$$h^1(\mathbf{x}^*(t), k, t) \geq 0 \text{ whenever } h(\mathbf{x}^*(t), t) = 0.$$

The first order condition for a maximum of the Hamiltonian gives the interior relative premium as before

$$k^i = \frac{1}{2}\left(b - \frac{\lambda_1}{m_{\bar{p}}\lambda_2}\right) = \tfrac{1}{2}(b - \Omega).$$

where we set

$$\Omega = \frac{\lambda_1}{m_{\bar{p}}\lambda_2}.$$

The adjoint equations are

$$\frac{d\lambda_1}{dt} = -L_{x_1} = -\lambda_1(G(k) - \kappa) - \lambda_2(kG(k)m_{\bar{p}} - m_u) + \frac{\eta m_u}{\kappa - \mu},$$
$$\frac{d\lambda_2}{dt} = -L_{x_2} = \alpha\lambda_2 - \eta,$$

with transversality conditions

$$\lambda_1(T-) = J_{x_1}(s^*(T), T) + \zeta h_{x_1}(s^*(T), T) = -\frac{m_u(T)(1+\zeta)}{\kappa - \mu},$$
$$\lambda_2(T-) = J_{x_2}(s^*(T), T) + \zeta h_{x_2}(s^*(T), T) = 1 + \zeta,$$

where

$$\zeta \geq 0, \quad \zeta h(\mathbf{x}^*(T), T) = 0.$$

The second adjoint equation can be integrated immediately to give

$$\lambda_2 = \frac{\eta}{\alpha} + \left(1 + \zeta - \frac{\eta}{\alpha}\right) e^{\alpha(T-t)}.$$

Using the definition of Ω we can rewrite the remaining adjoint equation as

$$\frac{d\Omega}{dt} = \Omega\left(\frac{\eta}{\lambda_2} - \alpha - \mu + \kappa - G(k)\right) - kG(k) + \gamma + \frac{\eta\gamma}{\kappa - \mu}, \tag{36}$$

which has terminal condition

$$\Omega(T) = -\frac{\gamma}{\kappa - \mu}.$$

If $\eta = 0$, that is the constraint is not tight as Ω is integrated backwards from $t = T$, then the optimal control follows the trajectory given by (26) providing the demand function is given by (8). Unfortunately, the corresponding state trajectory is a complicated analytical expression and the determination of those parameters which lead to a constraint violation in $[0, T]$ is not particularly illuminating (see (28)). However, if the constraint does become tight we know that

$$x_2 = \frac{x_1 m_u(t)}{\kappa - \mu},$$

and assuming the state trajectory is differentiable with respect to time we find

$$\dot{x}_2 = \frac{\gamma m_{\bar{p}}(\dot{x}_1 + \mu x_1)}{\kappa - \mu}.$$

Substituting the state equations (12) into this expression and simplifying then yields

$$G(k)\left(k - \frac{\gamma}{\kappa - \mu}\right) = \frac{\alpha\gamma}{\kappa - \mu}.$$

If the control is interior then G is linear in k and this expression yields a quadratic for k. Therefore the control is constant and equal to

$$k_c = \frac{1}{2}\left(b + \frac{\gamma}{\kappa - \mu}\right) - \frac{1}{2}\left(\left(b + \frac{\gamma}{\kappa - \mu}\right)^2 - \frac{4\gamma}{\kappa - \mu}\left(b + \frac{\alpha}{a}\right)\right)^{1/2}. \tag{37}$$

Notice that this argument applies irrespective of the form of the objective function.

The numerical solution of this constrained optimisation problem is shown Figure 6. Here we have set the initial state as

$$x_1(0) = 1, \quad x_2(0) = \frac{m_u(0)}{\kappa - \mu}.$$

so that the state constraint is satisfied at $t = 0$. It is clear that initially the constraint is tight and the optimal control is the constant given by (37). The parameter set is appropriate for a loss-leading Type 1 control: the unconstrained optimal control is superimposed on the first graph in the figure.

We illustrate the sensitivity of the optimal control to the parameters which directly affect the constraint in Figure 7. In the first graph we vary the drift in the mean claim size rate μ. If the claims rate is expected to decrease over time then the optimal premium is lowered because the constraint is less restrictive: the insurer expects fewer claims on its current exposure and so needs to maintain less capital to pay off those claims. If instead the claims rate is expected to increase the insurer must maintain sufficient wealth to pay off these claims. The second graph in Figure 7 shows the variation of the control with the length of policies. Longer policies (corresponding to smaller κ) lead to a longer time over which the constraint is binding because the insurer must have sufficient wealth to pay off the greater amount of claims over that period. Notice that k_c is independent of κ if the drift $\mu = 0$ as has been assumed in the figure.

Figure 8 shows how the optimal strategy changes if we replace the objective of maximising expected net terminal wealth by maximising expected total net wealth given by (15). The first graph (a) uses a linear utility function $U(w^{net}) = w^{net}$. As predicted this leads to a more conservative premium strategy if we compare the case $a = 3$ with Figure 6. As a is increased the insurer can gain more exposure for fixed relative premium and then the constraint again becomes tight in the interior of the planning region. The second graph (b) uses an exponential utility function

$$U(w^{net}) = \frac{1}{c}(1 - \exp(-cw^{net})), \tag{38}$$

with constant risk aversion c (Gerber & Pafumi 1998). It is clear that the utility parameterisation does change the optimal strategy. Loss-leading eventually generates increased wealth for the insurer and as the demand parameter a is increased more loss-leading is optimal. Greater wealth is not as desirable for the insurer using an exponential utility function and so overall the optimal control is larger than for a linear utility function.

5.2 Pure state constraint based on variation about the mean

The model has been developed with reference to just the mean claim size rate per unit exposure, $\mathbf{u}_t$, experienced by the market and the insurer. There is no account of the variation of claims about this mean rate and this is an important measure of the risk which the insurer takes in selling insurance. Daykin *et al.* (1994) (Chapter 6) give simple expressions for the capital at risk from selling insurance for a variety of claim size distributions. They gave a general upper bound for the capital at risk as a function of the premium income over one year, independent of the distribution of the claim size distribution. In terms of our variables the constraint takes the form that the insurer's current wealth $\mathbf{w}_t \geq f\mathbf{B}_t^{1/2}$ where the accrued premium is defined by

$$\mathbf{B}_t = \int_{t-\tau}^{t} G(k)\mathbf{p}_s\mathbf{q}_s\, ds.$$

and f is a constant. EU directives have a similar solvency (margin) requirement with a piecewise linear function replacing the parabola.

Again, this leads to a time delayed model, so we parameterise the delay as before. If we introduce another state variable $x_3 = m_B$ the state equations are

$$dx_1 = x_1(G(k) - \kappa)\, dt, \tag{39}$$

$$dx_2 = -\alpha x_2\, dt + m_{\bar{p}} x_1(G(k)k - \gamma)\, dt, \tag{40}$$

$$dx_3 = G(k)km_{\bar{p}}x_1\, dt - \kappa x_3\, dt. \tag{41}$$

The objective is given in general by (13) and we take the constraint as

$$h(\mathbf{x}, t) = x_2 - fx_3^{1/2} \geq 0. \tag{42}$$

In order that the constraint is satisfied initially we set the initial state as

$$x_1(0) = 1, \quad x_2(0) = f, \quad x_3(0) = 1.$$

Note that we have used the mean of the constraint so that for particular values of the processes $\mathbf{w}_t$ and $\mathbf{B}_t$ the constraint may not hold. This does not seem unreasonable since the constraint is actually only applied on a discrete basis and not continuously.

A numerical example is shown in Figure 9 when the objective is to maximise the expected terminal net wealth. The constraint is binding ($h = 0$ in Figure 9(b)) up until about $t \sim 0.3$. However, when the constraint is binding the optimal relative premium is no longer constant. The sensitivity of the optimal control to μ and f is shown in Figure 10. Again, higher expected claim rates lead to a more conservative premium strategy. If f is increased then the constraint is more stringent and it becomes optimal to withdraw from the market. The computation for $f = 3$ requires smoothing of the constraint transcription and it is not easy to demonstrate convergence of the control as we increase the number of steps n. According to the numerical results the constraint is binding for the entire planning horizon, which suggests that even though the insurer is leaving the market, it has insufficient wealth to pay off its outstanding claims as the state evolves.

6 Conclusions

We have calculated the premium strategy which maximises the objective of the insurer subject to a constraint on the control or constraints on the reserve that the insurer must hold. Since the model is very simple an analytical solution can be found if the relative premium is bounded. Depending on the parameter values of the model this can lead to a non-smooth control. Specifically, a Type 1 control represents a loss-leading strategy, and the greater the loss-leading, the more likely the insurer exceeds its lower bound on the relative premium. Following Emms & Haberman (2005) it is easy to show that the premium strategy $\mathbf{k}_t = k(t)$ is the optimal relative premium if the mean claim rate process is lognormal. For other distributions of the mean claim rate process the feedback control depends on the current value of the state variables and so it is a stochastic process. If there are no constraints then the theory in Fleming & Rischel (1975) for stochastic optimisation problems can be employed.

When the insurer constrains the premium strategy the optimal control can be non-smooth. This makes it much more difficult to obtain stochastic optimal premium strategies from the HJB equation because we expect that equation to have non-smooth solutions. Consequently in this paper we have restricted the feasible controls to be deterministic, which turns the problem into a deterministic optimisation problem even though the actual premium charged is stochastic. The resulting optimisation problem has been demonstrated to be readily solved using control parameterisation. This is a general technique and allows the insurer to calculate optimal strategies for any reasonable objective or demand functions. It also permits the imposition of an arbitrary number of constraints without substantially increasing the computational time.

Premium restrictions lead to control constraints, while solvency requirements lead to state constraints. A control constraint can be used to prevent negative optimal premium values. The numerical problems show that the state constraints limit the amount of loss-leading that the insurer may experience with an optimal premium strategy. Further studies are concentrated on relaxing some of the assumptions of the model. Specifically, we have parameterised the delay in the exposure equation. If one forgoes this assumption then the state equations become a system of stochastic differential delay equations. By assuming a deterministic control the optimisation problem can again be solved by control parameterisation (Teo, Goh & Wong 1991, Chapter 12). However, now we need to specify the initial curves for the state variables in order to accommodate the delay in the state.

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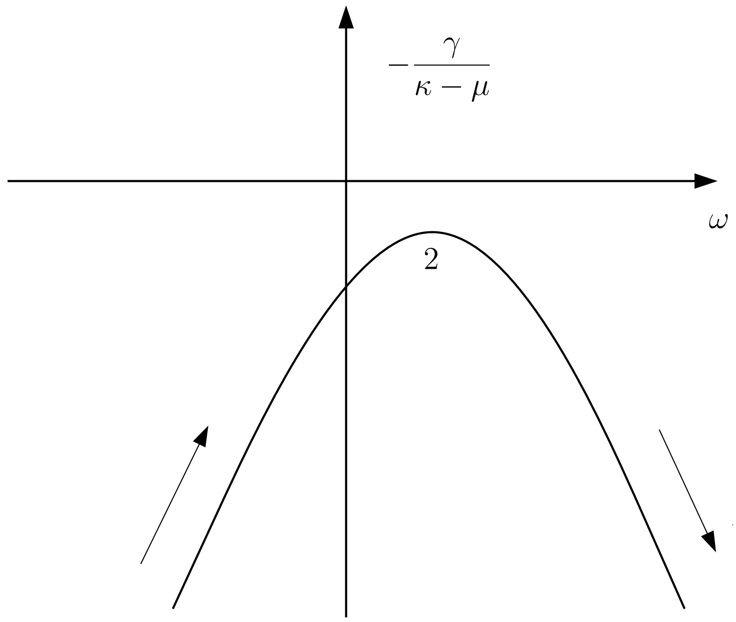
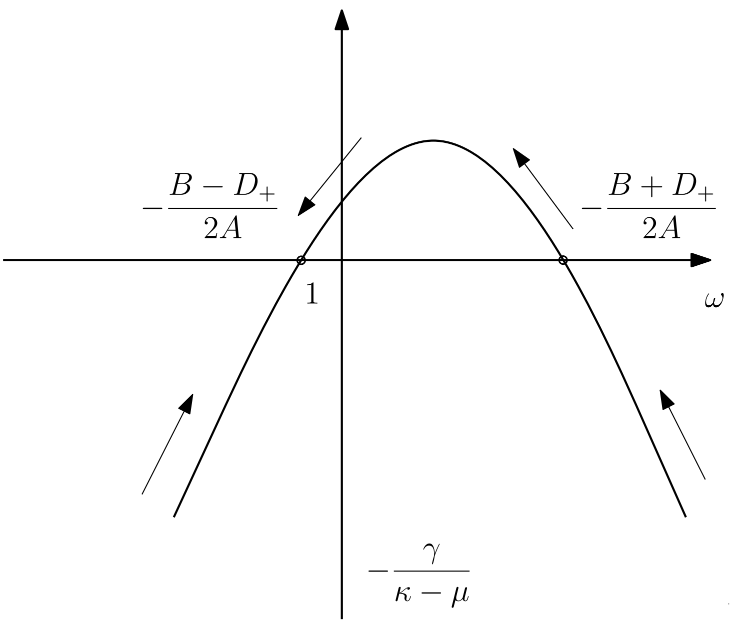
Tables and Figures



(1a)



(1b)

Figure 1: Qualitative form of the adjoint variable ω_2 if $k^*(T) = k^i(T)$ and (a) $\Delta < 0$ or (b) $\Delta > 0$. If the control is interior over $t \in [0, T]$ then the diagrams indicate whether the optimal control is Type 1 or Type 2 depending on the model parameters.

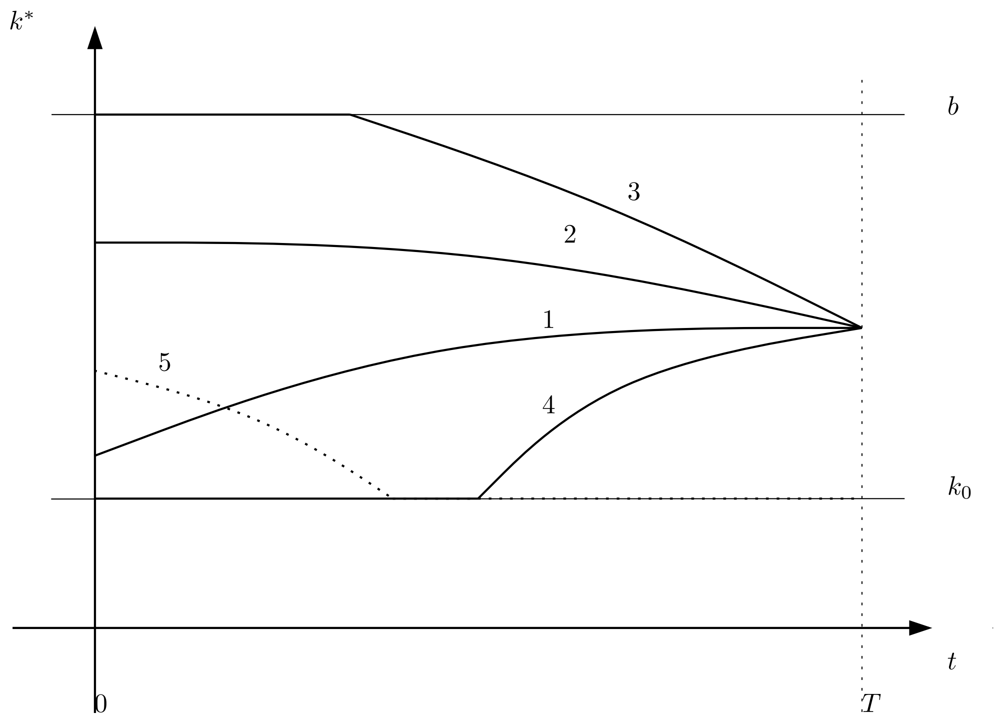
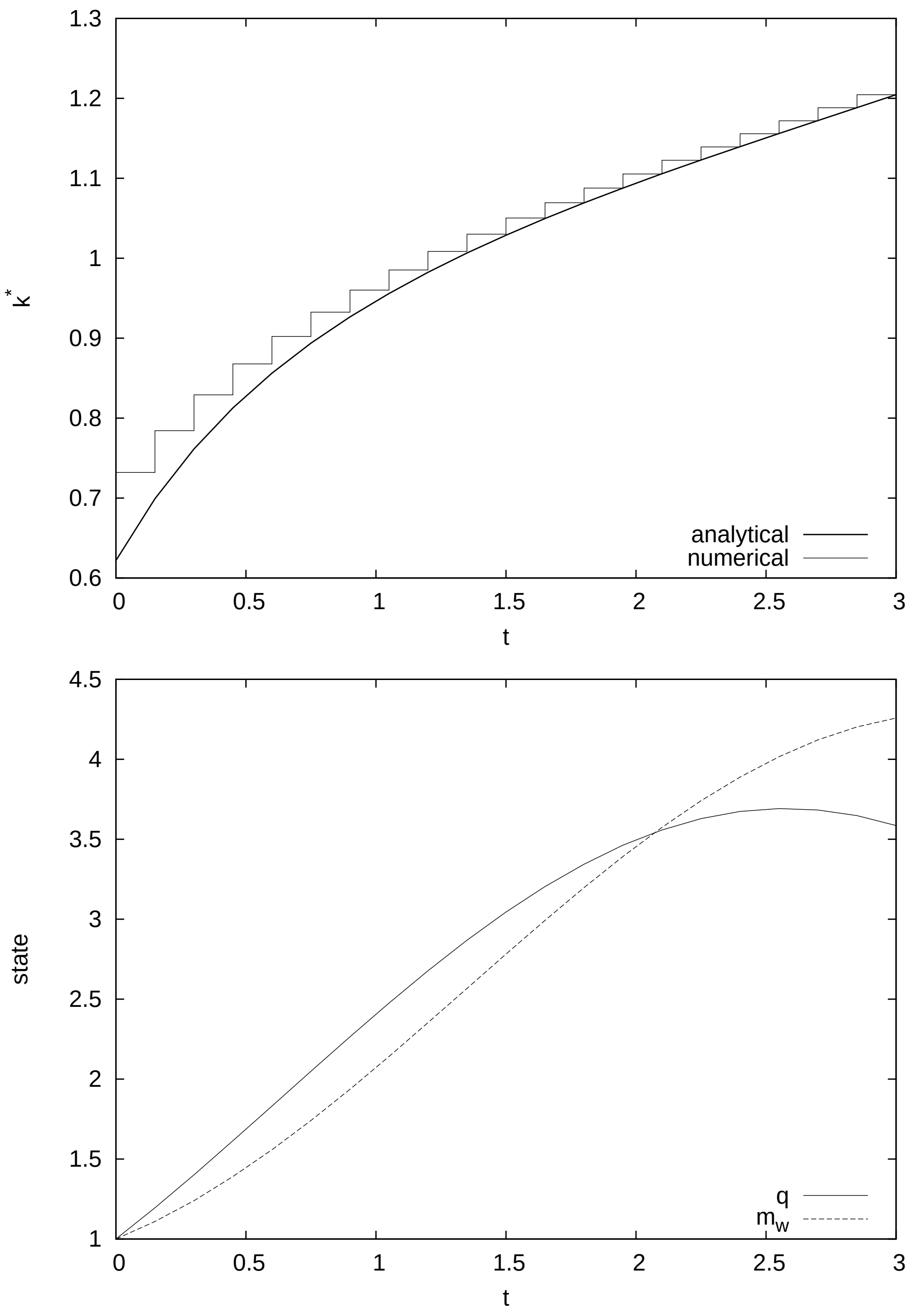
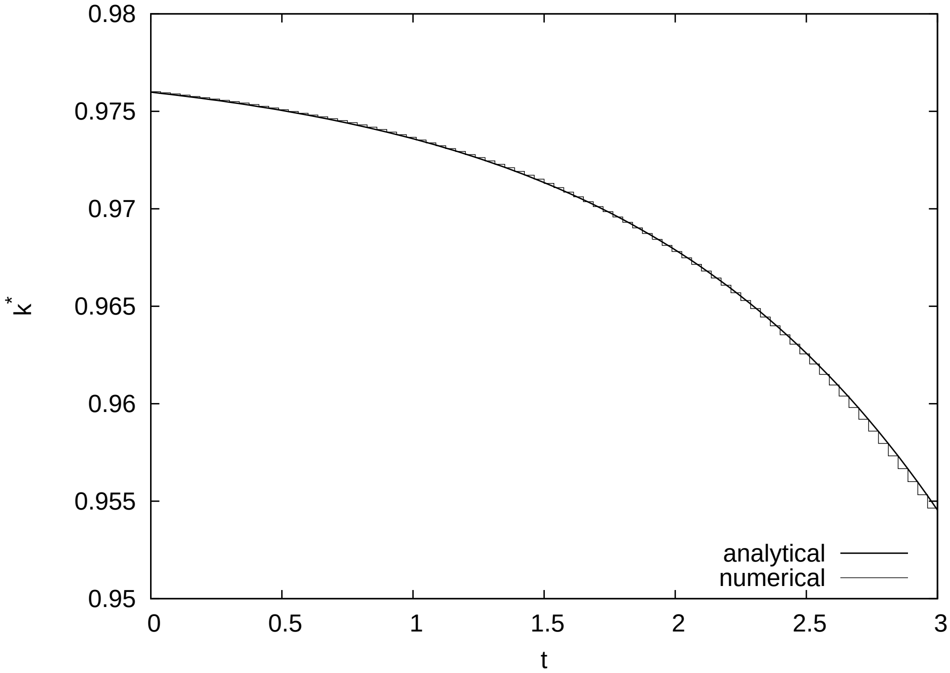
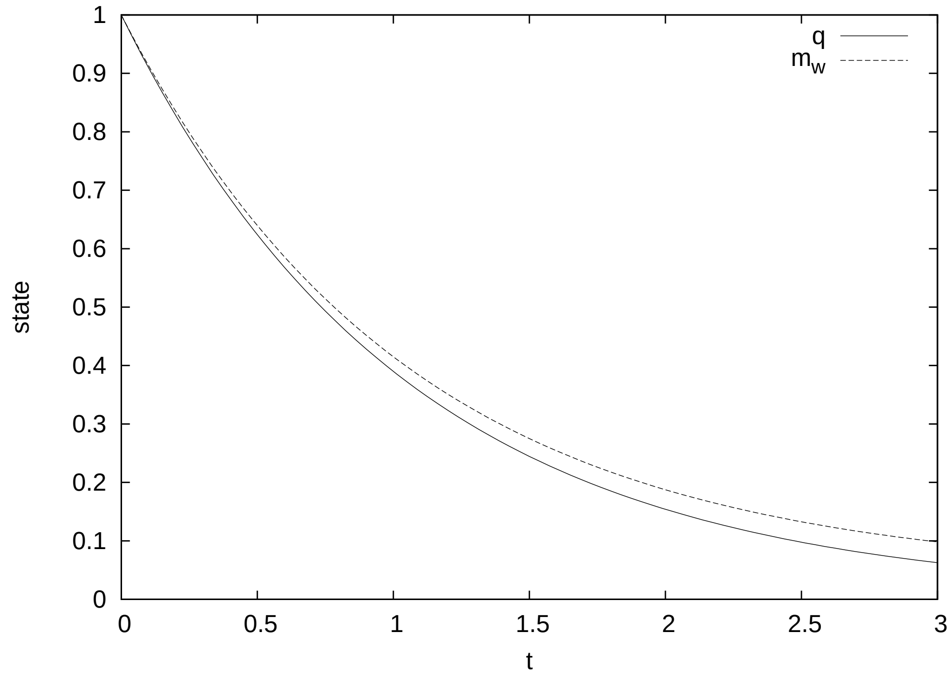


Figure 2: Classification of the the optimal relative premium k^* dependent of the value of the terminal relative premium $k^i(T)$.


1.3
1.2
1.1
1
0.9
0.8
0.7
0.6
k*
0
0.5
1
1.5
2
2.5
3
t
analytical
numerical
4.5
4
3.5
3
2.5
2
1.5
1
state
q
m_w

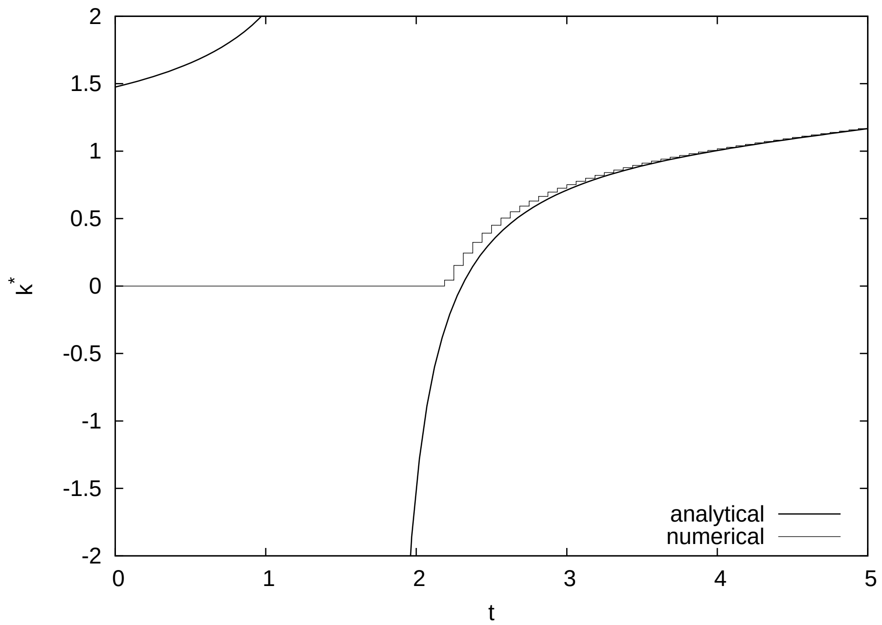
(3a)



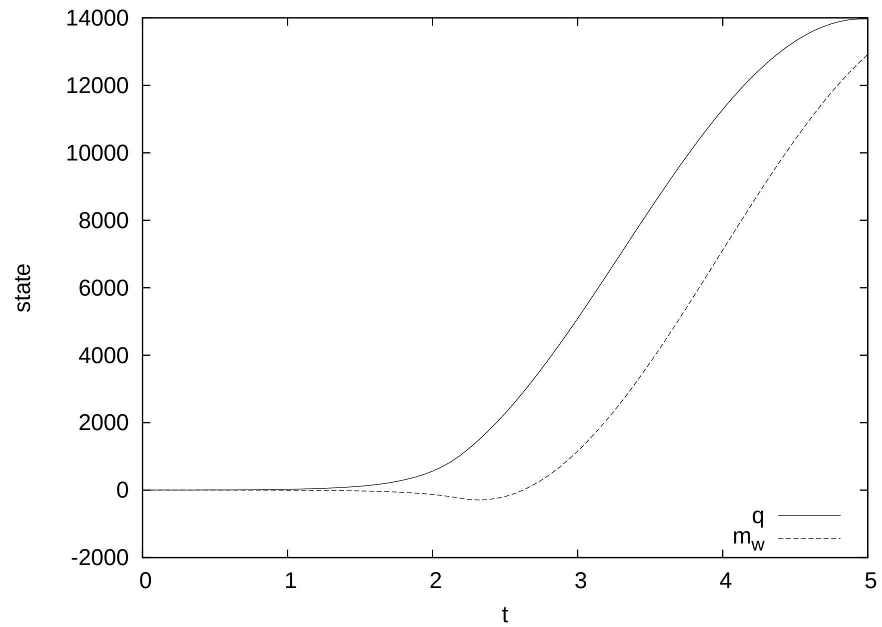


(3b)

Figure 3: Convergence of the control parameterisation to the optimal analytical control for (a) $\Delta < 0$, $n = 20$ and (b) $\Delta > 0$, $n = 80$. Graph (a) shows a Type 1 control whereas graph (b) shows a Type 2 control. The parameter set for each graph is given in (33) with for (a) $b = 1.5$ and for (b) $b = 1$.



(4a)



(4b)

Figure 4: An example illustrating the case when the unconstrained problem has no smooth solution. If there is no lower bound on the control then the nonlinear optimiser fails to converge as there are strategies which yield unbounded wealth. Numerical results are shown when the optimal control is bounded below by $k_0 = 0$. Even with this restriction considerable wealth is generated from this strategy. The parameter set (33) is modified by setting $\theta = 0.05$, $T = 5$.

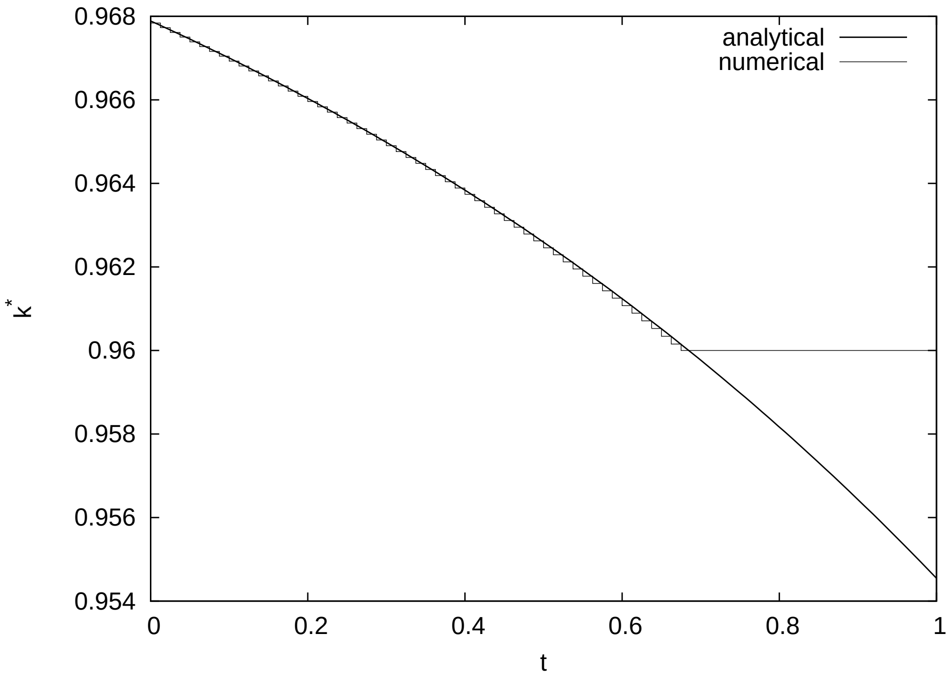


Figure 5: An example of a Type 5 control where is base parameter set is modified by $b = 1.0$, $T = 1$, $k_0 = 0.96$.

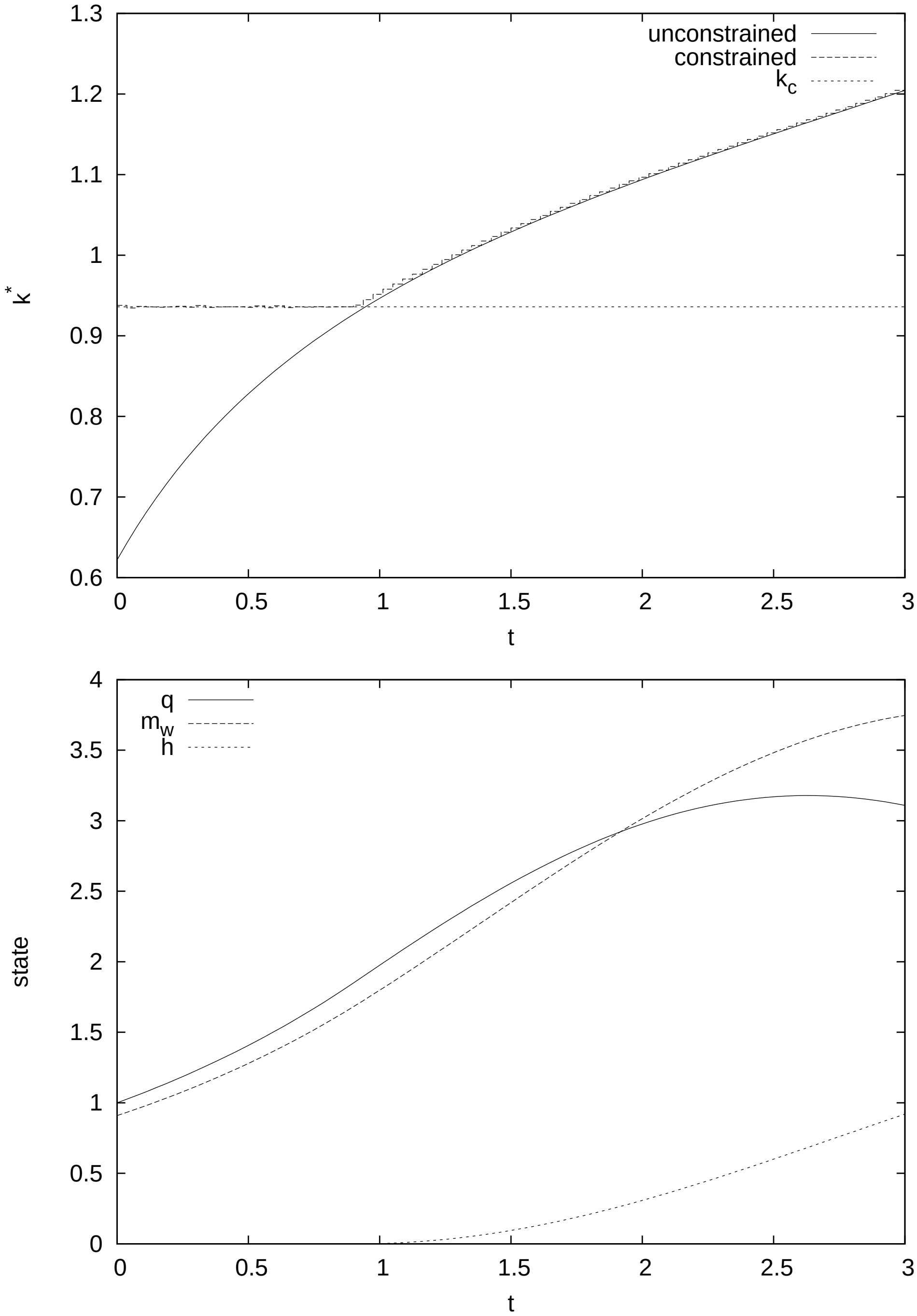
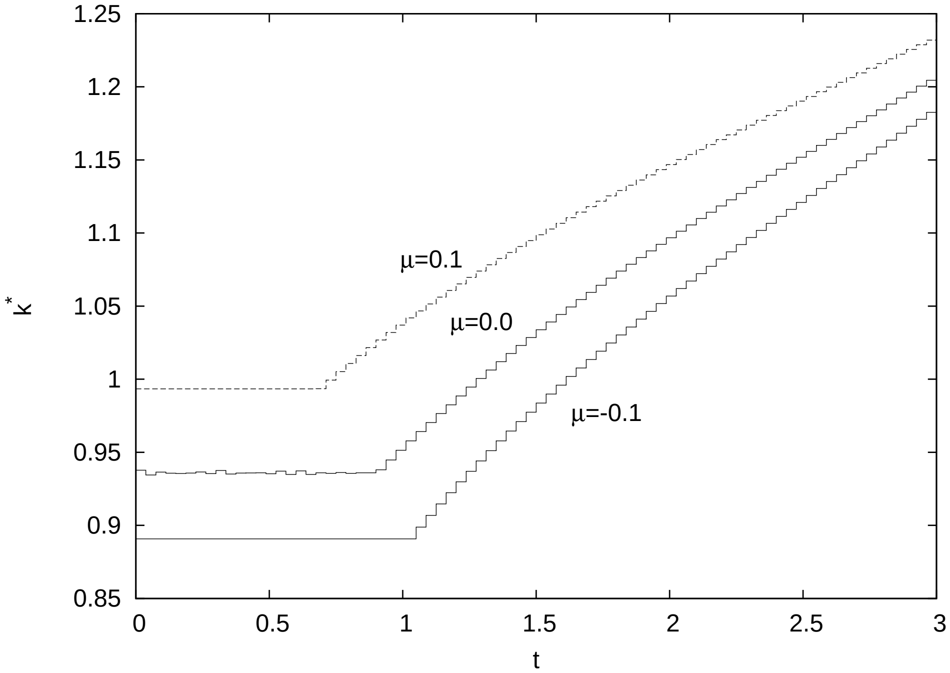
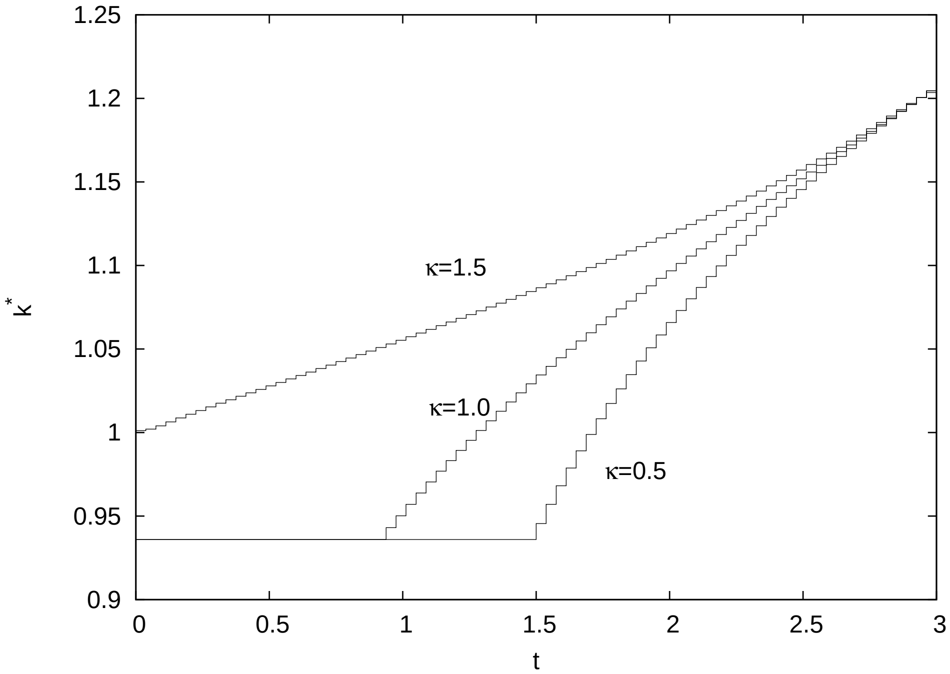


Figure 6: The optimal control with the pure state constraint (35) shown by the dashed line. The solid line shows the corresponding unconstrained optimal control given by (18). The parameter set is given by (33).

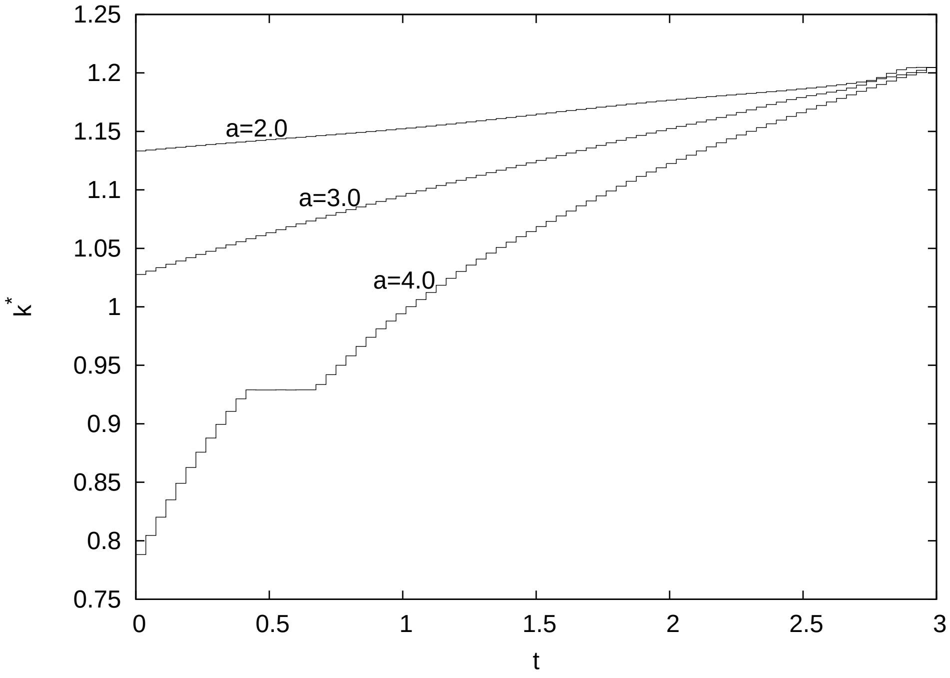


(7a)

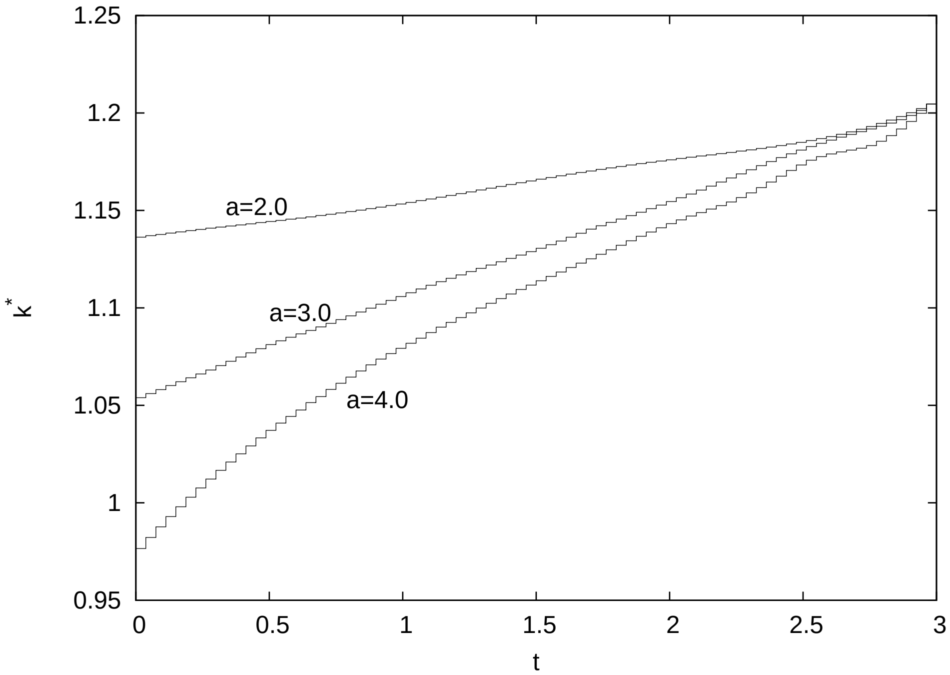


(7b)

Figure 7: Sensitivity of the optimal control to the model parameters. We use the pure state constraint (35) and vary (a) the growth in the claims rate μ and (b) the duration of the policies κ^{-1}.

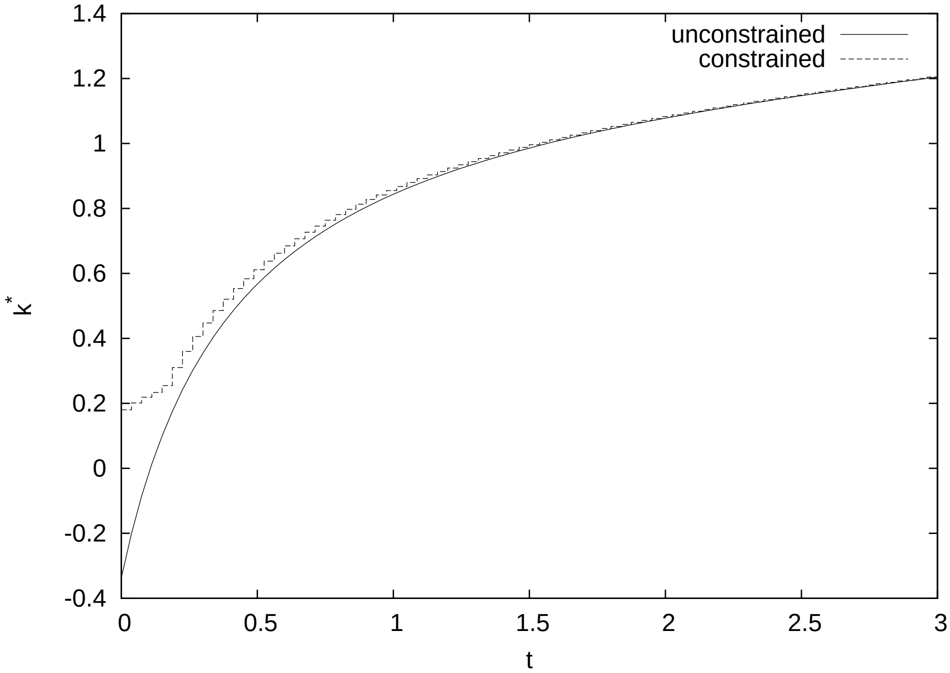


(8a)

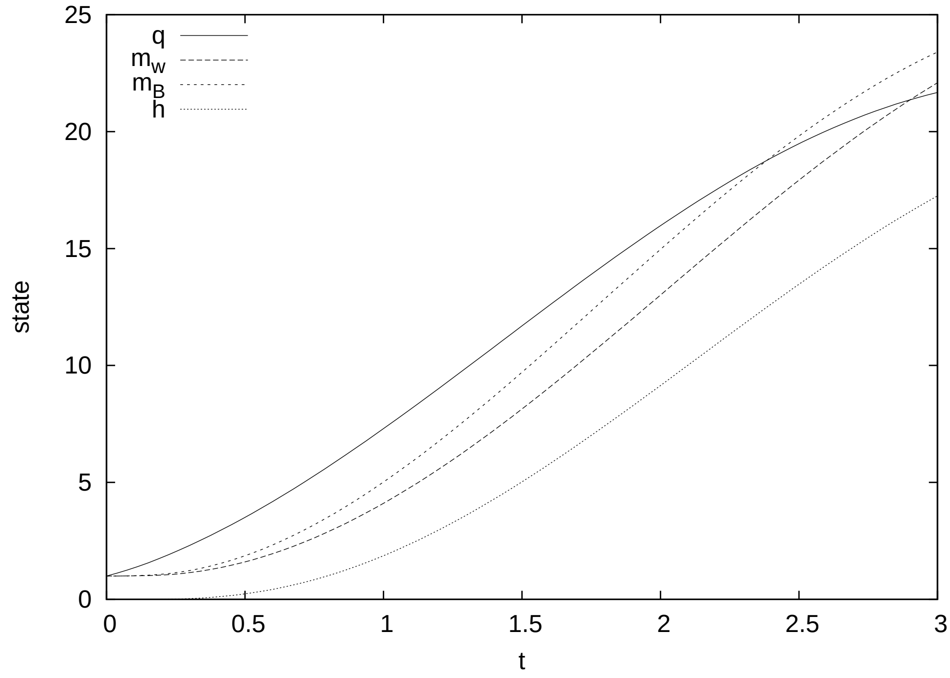


(8b)

Figure 8: Optimal control with the pure state constraint (35) but with the objective to maximise the total expected utility of wealth. Graph (a) shows the optimal control with a linear utility function while (b) shows the control with an exponential utility function (38) with $c = 1$. The parameter set is (33) but we vary the demand parameter a.



(9a)



(9b)

Figure 9: Optimal control with the pure state constraint (42). The base parameter set is modified augmented by $\kappa = 0.75$, $f = 1$.

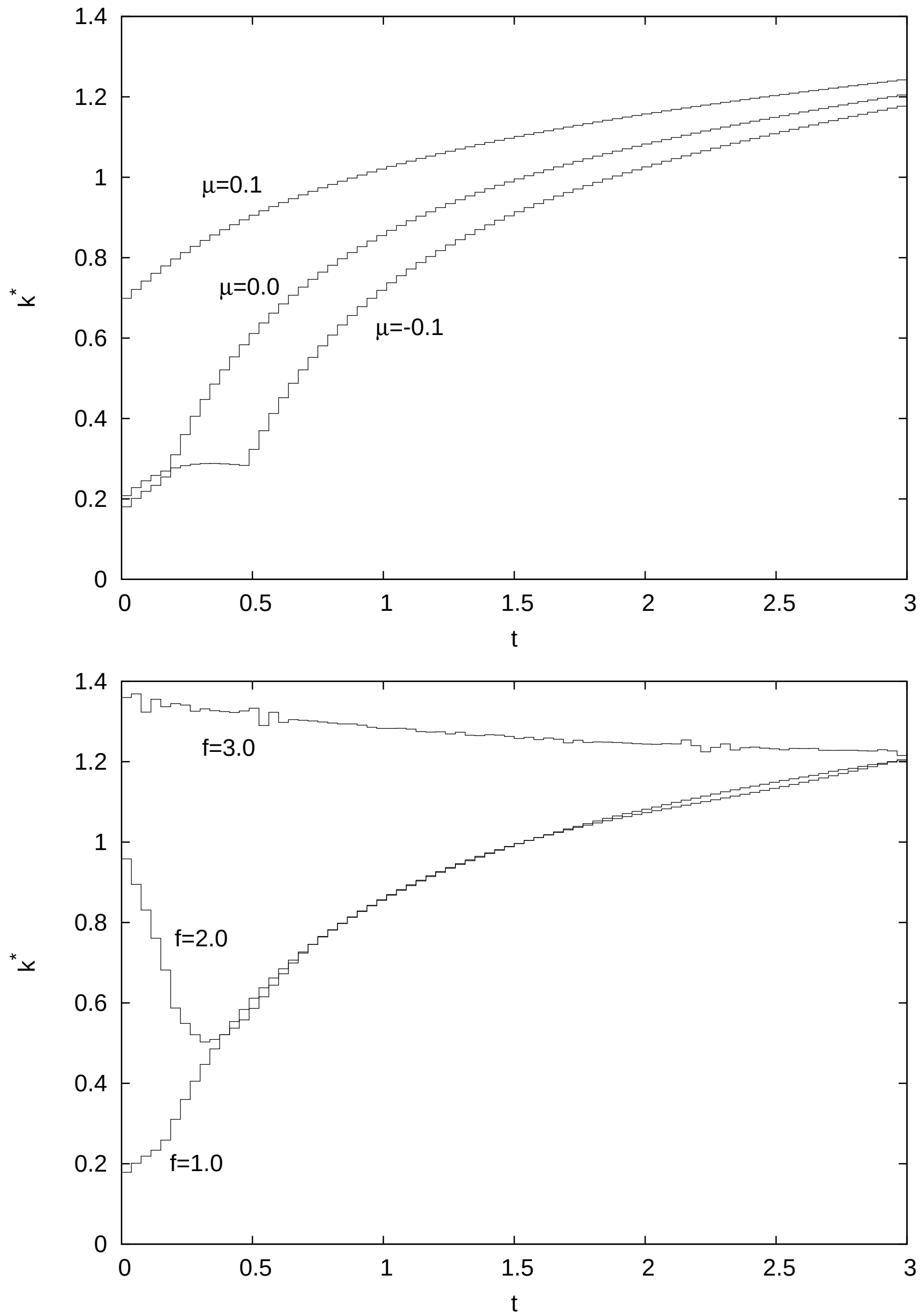


Figure 10: Sensitivity of the optimal control with the pure state constraint (42). The base parameter set applies except where the relevant parameter is varied.

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