A stochastic volatility model with realized measures for option pricing

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Abstract

Based on the fact that realized measures of volatility are affected by measurement errors, we introduce a new family of discrete-time stochastic volatility models having two measurement equations relating both observed returns and realized measures to the latent conditional variance. A semi-analytical option pricing framework is developed for this class of models. In addition, we provide analytical filtering and smoothing recursions for the basic specification of the model, and an effective MCMC algorithm for its richer variants. The empirical analysis shows the effectiveness of filtering and smoothing realized measures in inflating the latent volatility persistence – the crucial parameter in pricing Standard and Poor's 500 Index options.

1 Introduction

Thanks to the availability of high-frequency data, a number of realized measures of daily volatility (henceforth RVs) has been introduced so far (see Andersen and Bollerslev, 1998; Andersen et al., 2001; Barndorff-Nielsen, 2002; Barndorff-Nielsen and Shephard, 2004; Andersen et al., 2008; Hansen and Horel, 2009, to cite only a few). These measures are

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considerably more informative than the squared returns about the (true) conditional variance (henceforth CV). RVs present two crucial, although antithetic, features. On one hand, these measures lead to large economic and statistical gains when incorporated in volatility models (see, for instance, Dobrev and Szerszen, 2010; Maheu and McCurdy, 2011; Hansen et al., 2012; Christoffersen et al., 2014, 2015). On the other hand, on empirical data, they are affected by measurement errors and overnight biases. For these reasons, it is common practice in the recent literature to use RVs in the dynamic model for the latent CV (see Shephard and Sheppard (2010); Hansen et al. (2012) in a GARCH framework, Engle and Gallo (2006); Gallo and Otranto (2015) in a MEM framework, and Takahashi et al. (2009); Dobrev and Szerszen (2010); Koopman and Scharth (2013) in a stochastic volatility one). However, little work has been devoted to *semi-analytical* option pricing with stochastic volatility (henceforth SV) models incorporating RVs. We mention Khrapov and Renault (2016) who propose an affine discrete-time option pricing SV model exploiting RV which preserves the structure of the dynamics when changing measure. However, while in the theoretical part returns are assumed to be driven by a latent volatility process linked to the RV through a model-dependent relation, the empirical application focuses on the volatility factor as *observed* and proxied by RV.

In the present paper, we introduce a new family of flexible and tractable discrete-time SV models which allows for filtering the latent conditional variance process from *both* observed returns and RV-measures. Combining in an SV framework the two sources of information reduces measurement errors and permits to recover the high persistence of the CV. The proposed SV models accurately reproduce well-established stylized facts observed in financial time series, while preserving closed-form formulas for option pricing and for filtering and smoothing the latent SV process.

The first contribution of the paper is on modeling. Inspired by the above mentioned strands of literature, we employ the RV-measure in a measurement equation for the latent CV. Thus, our proposed model has two measurement densities: a Gaussian density for the daily log-returns and a gamma density for the RV-measure. In turn, the dynamics of the latent CV is modeled by borrowing from the flexible class of Heterogeneous Autoregressive Gamma process (HARG) of order p with Leverage (L) RV-LHARG(p) introduced by Majewski et al. (2015). Therefore, we label the general version of the proposed SV model as SV-LHARG(p).

As a second theoretical contribution, we show that SV-LHARG(p) belongs to the class of affine models. This allows for analytical tractability of the option pricing. Remarkably, although the option pricing literature in discrete-time traces back to Heston and Nandi (2000), analytical tractability is guaranteed only for rather specific types of models. These include: GARCH (Christoffersen et al., 2008, 2013; Bormetti et al., 2015; Huang et al., 2017) and realized volatility approaches (Stentoft, 2008; Corsi et al., 2013; Christoffersen et al., 2014; Majewski et al., 2015), later extended to separately deal with the continuous and discontinuous components of RV (Christoffersen et al., 2015; Alitab et al., 2019). Moreover, in accordance with the recent option pricing literature (see, for instance Christoffersen et al., 2013), we use a flexible pricing kernel incorporating a variance-dependent risk premium in addition to the common equity risk premium.

SV models offer increased flexibility over GARCH-type specifications since they assume separate innovation processes for the conditional mean and variance of the observables (Taylor, 1994). However, the presence of variance-specific disturbances makes inference on latent volatility more challenging and requires the adoption of suitable inference tools such as stochastic filtering and smoothing, and simulation-based inference. Filtering and smoothing recursions can be used to define a pseudo maximum likelihood estimator of the parameters (see Christoffersen et al. (2012)). Nevertheless, as argued in Durham et al. (2015), a valid statistical analysis should account for the state uncertainty. Thus, we follow a Bayesian approach where the posterior distribution of the parameters is derived jointly with the distribution of the latent states.

The third contribution relates to the state-space models and stochastic filtering literature. Takahashi et al. (2009), Shirota et al. (2014) and Bekierman and Gribisch (2016) propose to augment the state-space SV model with a RV equation and come up with flexible models for forecasting return and volatility dynamics. However, in these works neither exact filtering nor analytical pricing formulas are provided. For the SV-LHARG(p)

with one lag (i.e. $p = 1$) and no leverage components (SV-ARG henceforth) we provide exact analytical filtering and smoothing of the latent variables. In this direction, we contribute to the filtering literature by extending the results of Creal (2017) for non-linear and non-Gaussian models with one measurement equation to the case of two measurement equations. Indeed, for these type of models filtering and smoothing are usually obtained through analytical or numerical approximation techniques (Tanizaki, 1996; Doucet et al., 2001). Exceptions are Smith and Miller (1986); Shephard (1994); Ferrante and Vidoni (1998); Vidoni (1999); Deschamps (2011); de Pinho et al. (2016). The analytical recursions for SV-ARG allow to evaluate its likelihood function exactly and to develop an efficient inference procedure.

The fourth contribution of the paper is the development of an effective Bayesian inference procedure for both parameters and latent CV in the general SV-LHARG(p), and an efficient Markov Chain Monte Carlo (henceforth MCMC) procedure for posterior approximation. Specifically, we use SV-ARG as an auxiliary model in combination with the block sampling strategy introduced by Shephard and Pitt (1997) for state-space models and then successfully combined with Metropolis-Hastings (henceforth MH) for latent variable estimation (e.g., see So (2006), Casarin et al. (2011) and Billio et al. (2016)). A simulation study assesses the efficiency of the proposed MCMC algorithm.

Finally, we apply SV-LHARG on a large sample of S&P500 index options and we benchmark its performance to competitor models. In particular, we consider RV-LHARG(p) and the two-component GARCH model (henceforth CGARCH) introduced in Christoffersen et al. (2008). The comparison with RV-LHARG confirms the importance of a proper management of the RV measurement error. The effective filtering of the latent volatility in SV-LHARG translates in a better conditioning of the return process and ensures an higher persistence of the CV. Both effects improve the pricing performances across all moneyness and maturities, both in-sample and out-of-sample. Concerning CGARCH , in-sample SV-LHARG takes advantage of the RV measure economic content and fares better then the benchmark at short maturities. On the contrary, CGARCH outperforms SV-LHARG for long maturities, where the very high level of persistence of GARCH models becomes crucial.

Out-of-sample, the picture is much less clear. The superior behavior of CGARCH in the long run is less striking, whereas it may happen that – in the short run and especially for out-of-the-money calls – it over-performs SV-LHARG. However, the overall performance of SV-LHARG remains superior. Additionally, results are very much dependent on the out-of-sample period under scrutiny. For instance, SV-LHARG fares much better during the high volatility January 1, 2008–December 31, 2008 episode. In the comparison, we consider as benchmark also the (so-dubbed) RVM model by Christoffersen et al. (2015, 2016). The model exploits RV measures, filters the latent volatility, and provides a GARCH-type level of persistence. However, in the option-pricing exercise SV-LHARG fares better than RVM both in-sample and out-of-sample. In particular, the RVM performance severely deteriorate at longer horizon. The empirical analysis, along with the discussion of some mis-specification issues of RVM, is not reported in the main text, but it is extensively documented in the on-line Appendix.

The remainder of the paper proceeds as follows. Section 2 introduces the SV-LHARG(p) and presents new results on analytical filtering and smoothing for SV-ARG. Section 3 describes the Bayesian inference procedure for the general model. Section 4 benchmarks SV-LHARG against competitor models in an option pricing exercise. Section 5 concludes.

2 The model

2.1 Dynamics under physical probability $\mathbb{P}$

Consider a risky asset with closing price S_t and geometric log-return $r_t = \log(S_{t+1}/S_t)$. We indicate with h_t a continuous latent volatility process and with z_t a latent volatility state. Let $\mathcal{F}_t \doteq \sigma\left(\{r_u, \mathrm{RV}_u\}_{u\leq t}\right)$ be the filtration containing the information about observable quantities, i.e. log-return r_u and RV_u, available up to time t, $\widetilde{\mathcal{F}}_t \doteq \sigma\left(\mathcal{F}_t \vee \{h_u, z_u\}_{u\leq t}\right)$ the filtration $\mathcal{F}_t$ enlarged with latent processes up to time t, and $\widetilde{\mathcal{F}}_t^H \doteq \sigma\left(\widetilde{\mathcal{F}}_t \vee \{h_{t+1}\}\right)$ and $\widetilde{\mathcal{F}}_t^Z \doteq \sigma\left(\widetilde{\mathcal{F}}_t \vee \{z_{t+1}\}\right)$ the filtration $\widetilde{\mathcal{F}}_t$ enlarged with the latent processes in $(t+1)$. We assume the following dynamic model for the log-returns:

$$r_t = \mu + \gamma h_t + \sqrt{h_t}\varepsilon_t, \quad \varepsilon_t \overset{i.i.d.}{\sim} \mathcal{N}(0,1), \tag{2.1}$$

$t = 1, \ldots, T$, where μ is the risk-free rate, $(\gamma + 1/2)$ is the market price of risk, and $\mathcal{N}(m, \sigma^2)$ indicates the univariate normal distribution with mean m and variance σ^2. We refer to Eq. (2.1) as *return equation.* The dynamics in Eq. (2.1) differ from that proposed in Corsi et al. (2013); Majewski et al. (2015) for daily log-returns since the authors consider RV as driving process for returns. Specifically, they employ the continuous part of the realized variance (see Andersen et al. (2001); Barndorff-Nielsen (2002) for the definition), hereafter RV_t. Since RV contains information on the latent volatility process, we follow Hansen and Lunde (2006); Engle and Gallo (2006); Shephard and Sheppard (2010); Takahashi et al. (2009) and introduce another measurement equation termed *realized variance equation* which relates RV to the latent volatility process. Specifically, we assume that RV_t is sampled from a gamma distribution:

$$\mathrm{RV}_t | \widetilde{\mathcal{F}}_{t-1}^H \overset{ind}{\sim} \mathcal{G}\left(\varphi\, e^{-\kappa_2}, h_t\, e^{\kappa_2}\right),$$

where $\varphi \in \mathbb{R}_+$ and $\kappa_2 \in \mathbb{R}$ are two constants, and $\mathcal{G}(k, \vartheta)$ denotes a (central) gamma distribution with positive shape, k, and scale parameter, ϑ, respectively. In this way, we ensures a non-negative definite RV-measure and, in contrast to a log-normal specification, it preserves the analytical tractability of filtering and smoothing recursions and of derivative pricing. Under above assumptions, the first two conditional moments of return and realized variance are readily derived:

$$\mathbb{E}^{\mathbb{P}}\left[r_t | \widetilde{\mathcal{F}}_{t-1}^H\right] = \mu + \gamma h_t,\ \mathbb{V}^{\mathbb{P}}\left[r_t | \widetilde{\mathcal{F}}_{t-1}^H\right] = h_t,\ \mathbb{E}^{\mathbb{P}}\left[\mathrm{RV}_t | \widetilde{\mathcal{F}}_{t-1}^H\right] = \varphi h_t,\ \mathbb{V}^{\mathbb{P}}\left[\mathrm{RV}_t | \widetilde{\mathcal{F}}_{t-1}^H\right] = \varphi e^{\kappa_2} h_t^2.$$

The expression of the conditional variance of r_t suggests that h_t can be interpreted as the latent conditional variance of the daily log-return. The computation of $\mathbb{E}^{\mathbb{P}}\left[e^{r_t} | \widetilde{\mathcal{F}}_{t-1}^H\right] = e^{\mu + (\gamma + 1/2) h_t}$ confirms the interpretation of $(\gamma + 1/2)$ as the market price of risk. Concerning the RV conditional moments, we recall that the RV estimator has two potential drawbacks.

First, it is biased by the market micro-structure noise (induced by infrequent trading, bid-ask spread, and rounding effects). Various methods are available in the literature to mitigate this bias (Hansen and Lunde, 2005; Zhang et al., 2005; Bandi and Russell, 2006, 2008; Barndorff-Nielsen et al., 2008). Here, we adopt the Two-Scale estimator of Zhang et al. (2005). Second, the absence of trading during the overnight periods prevents us to compute the whole day RV. Therefore, the computation of the RV only from available intra-day returns results in the downward bias named overnight bias. The RV conditional mean indicates that the parameter φ is intended to adjust the overnight bias in the RV estimator. In our financial application we use data for the S&P500 index which is quoted from 9:30 AM to 4:00 PM Eastern time, thus we should expect $\varphi < 1$. A similar approach to bias correction has been proposed by Takahashi et al. (2009), Dobrev and Szerszen (2010), Koopman and Scharth (2013) and Hansen et al. (2012).

Regarding the dynamics of the volatility process, we assume that h_t depends on the past realizations $\mathbf{h}_{t-1} = (h_{t-1}, \cdots, h_{t-p})^{'}$ and on the past leverage components $\mathbf{l}_{t-1} = (l_{t-1}, \cdots, l_{t-p})^{'}$, with $l_{t-i} = \left(\varepsilon_{t-i} - \lambda\sqrt{h_{t-i}}\right)^2$ and $\lambda \in \mathbb{R}_+$. It follows an autoregressive gamma process with transition distribution (see Gouriéroux and Jasiak, 2006):

$$h_t|\widetilde{\mathcal{F}}_{t-1} \overset{d}{\sim} \bar{\mathcal{G}}(\delta, \Theta(\mathbf{h}_{t-1}, \mathbf{l}_{t-1}), c), \tag{2.2}$$

where $\bar{\mathcal{G}}(\delta, \Theta, c)$ denotes the non-central gamma distribution with shape $\delta \in \mathbb{R}_+$, scale $c \in \mathbb{R}_+$, and non-centrality parameter $\Theta \in \mathbb{R}_+$ (more details are provided in the on-line Appendix). The non-centrality parameter is given by:

$$\Theta(\mathbf{h}_{t-1}, \mathbf{l}_{t-1}) = \sum_{i=1}^{p} \beta_i h_{t-i} + \sum_{i=1}^{p} \alpha_i l_{t-i},$$

where $(\beta_1, \cdots, \beta_p)$ and $(\alpha_1, \cdots, \alpha_p)$ are the autoregressive and leverage coefficients, respectively. Note that the non-central gamma distribution arises as a Poisson mixture of a (central) gamma distributions (see Gouriéroux and Jasiak, 2006, for more details). Precisely, we can rewrite Eq. (2.2) as:

$$\begin{aligned} h_t|\widetilde{\mathcal{F}}_{t-1}^{Z} &\overset{ind}{\sim} \mathcal{G}\left(\delta + z_t, c\right), \\ z_t|\widetilde{\mathcal{F}}_{t-1} &\overset{ind}{\sim} \mathcal{P}o\left(\Theta(\mathbf{h}_{t-1}, \mathbf{l}_{t-1})\right), \end{aligned} \tag{2.3}$$

where $\mathcal{P}o(v)$ indicates the Poisson distribution with intensity parameter $v > 0$. The latter representation will be used later on in this paper for the characterization of h_t and in the inference procedure. The conditional mean and variance of the process h_t, i.e.

$$\begin{aligned} \mathbb{E}^{\mathbb{P}}\left[h_t|\widetilde{\mathcal{F}}_{t-1}\right] &= c\delta + c\left(\sum_{i=1}^{p}\beta_i h_{t-i} + \sum_{i=1}^{p}\alpha_i l_{t-i}\right) \\ \mathbb{V}^{\mathbb{P}}\left[h_t|\widetilde{\mathcal{F}}_{t-1}\right] &= c^2\delta + 2c^2\left(\sum_{i=1}^{p}\beta_i h_{t-i} + \sum_{i=1}^{p}\alpha_i l_{t-i}\right), \end{aligned}$$

are affine functions of the lagged values of the volatility and leverage processes. Summarising, SV-LHARG(p) has the following distributional state-space representation:

$$\begin{aligned} r_t|\widetilde{\mathcal{F}}_{t-1}^{H} &\overset{d}{\sim} \mathcal{N}(\mu + \gamma h_t, h_t), \\ \mathrm{RV}_t|\widetilde{\mathcal{F}}_{t-1}^{H} &\overset{d}{\sim} \mathcal{G}\left(\varphi e^{-\kappa_2}, h_t e^{\kappa_2}\right), \\ h_t|\widetilde{\mathcal{F}}_{t-1}^{Z} &\overset{d}{\sim} \mathcal{G}\left(\delta + z_t, c\right), \\ z_t|\widetilde{\mathcal{F}}_{t-1} &\overset{d}{\sim} \mathcal{P}o\left(\Theta(\mathbf{h}_{t-1}, \mathbf{l}_{t-1})\right). \end{aligned} \tag{2.4}$$

It is worth mentioning that our framework allows for the inclusion of exogenous variables in the log-return and RV dynamics, such overnight corrections, jump components, and possible structural changes of some model parameters.

We conclude this section by presenting some properties of SV-LHARG(p). First, in the following proposition we give the stationary condition for our model.

Proposition 1. *The* SV-LHARG(p) *in Eq.* (2.4) *is stationary if the following condition*

$$c\left(\sum_{i=1}^{p}\beta_i + \lambda^2\sum_{i=1}^{p}\alpha_i\right) < 1 \tag{2.5}$$

is satisfied.

Proof. See on-line Appendix. □

Additionally, from Eq.(2.4), it is possible to compute analytically the invariant unconditional mean and variance of both the latent and the observable variables:

$$\mathbb{E}^{\mathbb{P}}[h_1] = \frac{c\left(\delta + \alpha^{(1)}\right)}{1 - c\left(\beta^{(1)} + \lambda^2\alpha^{(1)}\right)},$$

$$\mathbb{V}^{\mathbb{P}}[h_1] = \frac{c^2\left(\delta + 2\alpha^{(1)} + 2\alpha^{(2)}\right) + 2c^2\mathbb{E}^{\mathbb{P}}[h_1]\left(\beta^{(1)} + \lambda^2\alpha^{(1)} + 4\lambda^2\alpha^{(2)}\right)}{1 - c^2\left(\beta^{(2)} + \lambda^4\alpha^{(2)}\right)},$$

$$\mathbb{E}^{\mathbb{P}}[r_1] = \mu + \gamma\mathbb{E}^{\mathbb{P}}[h_1], \qquad \mathbb{V}^{\mathbb{P}}[r_1] = \mathbb{E}^{\mathbb{P}}[h_1] + \gamma^2\mathbb{V}^{\mathbb{P}}[h_1],$$

$$\mathbb{E}^{\mathbb{P}}[\mathrm{RV}_1] = \varphi\mathbb{E}^{\mathbb{P}}[h_1], \qquad \mathbb{V}^{\mathbb{P}}[\mathrm{RV}_1] = \left(\varphi e^{\kappa_2} + \varphi^2\right)\mathbb{V}^{\mathbb{P}}[h_1] + \varphi e^{\kappa_2}\left(\mathbb{E}^{\mathbb{P}}[h_1]\right)^2.$$

where $\alpha^{(1)} = \sum_{i=1}^{p}\alpha_i$, $\alpha^{(2)} = \sum_{i=1}^{p}\alpha_i^2$, $\beta^{(1)} = \sum_{i=1}^{p}\beta_i$, $\beta^{(2)} = \sum_{i=1}^{p}\beta_i^2$. Finally, SV-LHARG($p$) satisfies the affine property, whose importance has been acknowledged in many studies (see, for instance Duffie et al., 2000; Darolles et al., 2006; Majewski et al., 2015). It enables us to provide an exhaustive probabilistic description of the log-return and conditional variance dynamics, and to obtain a closed-form expression for the conditional moment generating function of the SV-LHARG(p) under the physical and risk-neutral measures. Also, it allows us to derive an explicit one-to-one mapping between the parameters of SV-LHARG(p) under the measures $\mathbb{P}$ and $\mathbb{Q}$. Results linked to the affine property largely follow from Majewski et al. (2015). Therefore, hereafter, we present only the essential materials needed to understand the rest of the paper and refer to the on-line Appendix for further details.

2.2 Dynamics under the risk-neutral probability $\mathbb{Q}$

We risk-neutralize the model by employing a Stochastic Discount Factor (SDF) within the exponential affine family. Such SDF has been extensively used in the literature (see Bertholon et al., 2008; Gagliardini et al., 2011; Corsi et al., 2013; Majewski et al., 2015;

Alitab et al., 2019, among others). We assume the following form:

$$M_{t,t+1} = \frac{e^{-\nu_1 h_{t+1} - \nu_2 r_{t+1}}}{\mathbb{E}^{\mathbb{P}}\left[e^{-\nu_1 h_{t+1} - \nu_2 r_{t+1}} | \widetilde{\mathcal{F}}_t\right]}, \tag{2.6}$$

which represents the Esscher transform from the physical log-return density to the risk-neutral one (see, for instance Gerber et al., 1994; Bühlmann et al., 1996). The main advantage in using the SDF in (2.6) is the clear identification of the sources of risk and their explicit compensation by means of specific risk premia. This functional form allows for both a conditional variance premium, ν_1, and the usual equity premium, ν_2. The parameter ν_2 is fixed by no-arbitrage conditions (see the on-line Appendix), whereas ν_1 is a free parameter which is calibrated from option prices (see the on-line Appendix for a precise description of the calibration procedure). Besides, it is possible to show that under the risk-neutral measure $\mathbb{Q}$ the conditional variance follows an SV-LHARG(p) process, i.e. SV-LHARG(p) admits a structure-preserving change of measure.

2.3 Special cases

Here, we shall introduce some notations and present four instances of SV-LHARG(p) which will be used in our financial applications. We define with $\mathbf{r}_{s:t} = (r_s, \cdots, r_t)' \in \mathbb{R}^{t-s+1}$, $\mathrm{RV}_{s:t} = (\mathrm{RV}_s, \cdots, \mathrm{RV}_t)' \in \mathbb{R}_+^{t-s+1}$, $\mathbf{h}_{s:t} = (h_s, \cdots, h_t)' \in \mathbb{R}_+^{t-s+1}$, $\mathbf{z}_{s:t} = (z_s, \cdots, z_t)' \in \mathbb{R}_+^{t-s+1}$ the collections, from time s to time t, of daily log-returns, realized variances, conditional variances and state variables, respectively.

The first model considered is an SV-LHARG model with an heterogeneous autoregressive dynamics for the CV and the leverage term. It is an SV-LHARG(22) with the following restrictions: $\beta_1 = \beta^{(d)}$, $\beta_2 = \ldots = \beta_5 = 4\beta^{(w)}$, $\beta_6 = \ldots = \beta_{22} = 17\beta^{(m)}$, $\alpha_1 = \alpha^{(d)}$, $\alpha_2 = \ldots = \alpha_5 = 4\alpha^{(w)}$, $\alpha_6 = \ldots = \alpha_{22} = 17\alpha^{(m)}$. This type of parametrization was introduced in Majewski et al. (2015) to describe the dynamics of the RV, and allows us to re-write the non centrality parameter of the non-central gamma random variable in Eq.

(2.4), $\Theta\left(\mathbf{h}_{t-1}, \mathbf{l}_{t-1}\right) \doteq \Theta^{(h)}\left(\mathbf{h}_{t-1}, \mathbf{l}_{t-1}\right)$, as:

$$\Theta^{h}\left(\mathbf{h}_{t-1}, \mathbf{l}_{t-1}\right) = \beta^{(d)} h_{t-1}^{(d)} + \beta^{(w)} h_{t-1}^{(w)} + \beta^{(m)} h_{t-1}^{(m)} + \alpha^{(d)} l_{t-1}^{(d)} + \alpha^{(w)} l_{t-1}^{(w)} + \alpha^{(m)} l_{t-1}^{(m)}$$

where the quantities $h_{t-1}^{(d)}$, $h_{t-1}^{(w)}$ and $h_{t-1}^{(m)}$ are defined as:

$$h_{t-1}^{(d)} = h_{t-1}, \quad h_{t-1}^{(w)} = \frac{1}{4} \sum_{i=2}^{5} h_{t-i}, \quad h_{t-1}^{(m)} = \frac{1}{17} \sum_{i=6}^{22} h_{t-i}. \tag{2.7}$$

Analogous expressions are used to define $l_{t-1}^{(d)}$, $l_{t-1}^{(w)}$ and $l_{t-1}^{(m)}$. The previous quantities represent the heterogeneous components corresponding to the short-term or daily (d), medium-term or weekly (w) and long-term or monthly (m) conditional variance and leverage terms. This specification captures the persistence observed in financial data as well as the multi-component structure of volatility and leverage, while it preserves parameter parsimony (see Majewski et al., 2015, and reference therein). The second model derives from the SV-LHARG by setting $\beta^{(w)} = \beta^{(m)} = 0$ and $\alpha^{(w)} = \alpha^{(m)} = 0$. The third model is the SV-HARG model, which is an heterogeneous autoregressive model for the CV without leverage term, i.e. $\alpha^{(d)} = \alpha^{(w)} = \alpha^{(m)} = 0$. Finally, the last model is an SV-LHARG(p) without leverage and heterogeneous structure, thus labelled SV-ARG. This model extends the class of non-Gaussian state-space models introduced in Creal (2017) by allowing for two observation equations, while preserving analytical tractability. The distribution of the observable log-returns and the RV are normal and (central) gamma, respectively. Specifically, SV-ARG is described by Eq.(2.4) with $\Theta\left(\mathbf{h}_{t-1}, \mathbf{l}_{t-1}\right) = \beta^{(d)} c$.

A crucial feature of SV-ARG is that the likelihood function is tractable and can be used to develop efficient inference procedures for richer versions. In the following propositions, by applying similar arguments as in Creal (2017), we are able to provide analytical expressions for the conditional likelihood, the Markov transition, and the initial distribution of z_t. Also, we obtain analytical filtering and smoothing recursions for the latent processes.

Proposition 2. *For the* SV-ARG *model, the conditional likelihood,* $p(r_t, \mathrm{RV}_t | \widetilde{\mathcal{F}}_{t-1}^{Z})$*, the*

Markov transition, $p(z_t|\widetilde{\mathcal{F}}_{t-1})$*, and the initial distribution of* z_t*,* $p(z_1)$*, are given by:*

$$
\begin{aligned}
&p(r_t, \mathrm{RV}_t|\widetilde{\mathcal{F}}^Z_{t-1}) = 2\eta(z_t)K_{\lambda(z_t)}\left(\sqrt{\psi\chi^{(t)}}\right)\left(\sqrt{\frac{\chi^{(t)}}{\psi}}\right)^{\lambda(z_t)},\\
&p(z_t|\widetilde{\mathcal{F}}_{t-1}) \propto \mathcal{S}\left(\lambda(z_{t-1}), \chi^{(t-1)}\beta^{(d)}, \psi\frac{1}{\beta^{(d)}}\right),\\
&p(z_1) \propto \mathcal{NB}\left(\delta, c\beta^{(d)}\right),
\end{aligned}
$$

with

$$
\begin{aligned}
&\eta(z_t) = \frac{\exp(\gamma\mu_{1t})}{\sqrt{2\pi}}\frac{\mathrm{RV}_t^{\varphi_t-1}}{\Gamma(\varphi_t)\left(\exp(\kappa_2)\right)^{\varphi_t}}\frac{1}{\Gamma(\delta+z_t)c^{\delta+z_t}}, \quad \mu_{1t} = r_t - \mu, \quad \varphi_t = \varphi\exp(-\kappa_2)\\
&\lambda(z_t) = \delta + z_t - \varphi_t - 1/2, \quad \chi^{(t)} = \mu_{1t}^2 + 2\mu_{2t}, \quad \mu_{2t} = \frac{\mathrm{RV}_t}{\exp(\kappa_2)}, \quad \psi = \gamma^2 + \frac{2}{c},
\end{aligned}
$$

where $K_\lambda(x)$ *is the modified Bessel function of the second kind,* $\mathcal{S}(\lambda, \chi, \psi)$ *the Sichel distribution with parameters* $\lambda \in \mathbb{R}$*,* $\chi \in \mathbb{R}_+$*,* $\psi \in \mathbb{R}_+$ *and* $\mathcal{NB}(\omega, p)$ *the Negative Binomial distribution with parameters* $\omega \in \mathbb{R}_+$ *and* $p \in (0, 1)$*.*

Proof. See on-line Appendix, where definitions of the Sichel and the Negative Binomial distribution are also provided. □

Proposition 3. *Let* $p(r_t, \mathrm{RV}_t|z_t = k)$*,* $k \geq 0$*, be the joint observables density* $p(r_t, \mathrm{RV}_t|\widetilde{\mathcal{F}}^Z_{t-1})$ *of Proposition 2, conditional to* $\{z_t = k\}$ *and let* $p(z_t = k|z_{t-1} = l, r_{t-1}, \mathrm{RV}_{t-1})$*,* $k, l \geq 0$ *be the transition probabilities given by* $p(z_t|\widetilde{\mathcal{F}}_{t-1})$ *of Proposition 2 evaluated at* $\{z_t = k\}$ *and conditional to* $\{z_{t-1} = l\}$*. The predictive* $p(z_t|\mathcal{F}_{t-1})$*, filtered* $p(z_t|\mathcal{F}_t)$ *and smoothed* $p(z_t|\mathcal{F}_T)$ *distributions are defined by the recursions:*

$$
\begin{aligned}
&p(z_t = k|\mathcal{F}_{t-1}) = \sum_{l=0}^{\infty} p(z_{t-1} = l|\mathcal{F}_{t-1})p\left(z_t = k|z_{t-1} = l, r_{t-1}, \mathrm{RV}_{t-1}\right), \ t = 1, \ldots, T\\
&p(z_t = k|\mathcal{F}_t) \propto \sum_{l=0}^{\infty} p(z_{t-1} = l|\mathcal{F}_{t-1})p\left(z_t = k|z_{t-1} = l, r_{t-1}, \mathrm{RV}_{t-1}\right)p\left(r_t, \mathrm{RV}_t|z_t = k\right), \ t = 1, \ldots, T\\
&p(z_t = k|\mathcal{F}_T) \propto p(z_t = k|\mathcal{F}_t)\sum_{l=0}^{\infty} p\left(z_{t+1} = l|z_t = k, r_t, \mathrm{RV}_t\right)\frac{p(z_{t+1} = l|\mathcal{F}_T)}{p(z_{t+1} = l|\mathcal{F}_t)}, \ t = T-1, \ldots, 1
\end{aligned}
$$

Proposition 4. *Let* $\mathcal{F}^Z_t \doteq \sigma\left(\mathcal{F}_t \vee \{z_u\}_{u\leq t}\right)$ *be the filtration* $\mathcal{F}_t$ *enlarged with the hidden states* z_u *up to time* t*, and* $\lambda(z_t)$*,* $\chi^{(t)}$ *and* ψ *the quantities defined in Proposition 2. The* *marginal filtered,* $p(h_t|\mathcal{F}_t^Z)$*, and smoothed,* $p(h_t|\mathcal{F}_T^Z)$ *distributions are*

$$
\begin{aligned}
p(h_t|\mathcal{F}_t^Z) &\propto \mathcal{G}ig\left(\lambda(z_t), \chi^{(t)}, \psi\right), \quad t = 1, \dots, T \\
p(h_t|\mathcal{F}_T^Z) &\propto \mathcal{G}ig\left(\lambda(z_t) + z_{t+1}, \chi^{(t)}, \psi + 2\beta^{(d)}\right), \quad t = T-1, \dots, 1,
\end{aligned}
$$

with $\mathcal{G}ig\,(\lambda, \chi, \psi)$ *the Generalized Inverse Gaussian distribution with parameters* $\lambda \in \mathbb{R}$*,* $\chi \in \mathbb{R}_+$*,* $\psi \in \mathbb{R}_+$*.*

Proof. See on-line Appendix, where a definition of the Generalized Inverse Gaussian distribution is also provided. □

3 Bayesian Inference

In this section, we discuss the estimation procedure for SV-LHARG(p), emphasizing the role of Proposition 2 and 4. We consider heterogeneous dynamics and $p = 22$ – which corresponds to a monthly horizon – following a common specification in the recent financial econometrics literature (Corsi, 2009). However, the methodology can be easily adapted to the SV-LHARG(p) specification without restrictions.

As it commonly happens in latent variable modeling, the likelihood function of SV-LHARG is a high-dimensional integral with no closed-form solution. Hence, we apply a data-augmentation principle (see Tanner and Wong, 1987) and include the latent variables in the set of observations, thus obtaining a complete-data likelihood function. As regards the initial 22 values of SV-LHARG , we follow Vermaak et al. (2004) and Casarin et al. (2012) and consider a pseudo-likelihood approach by assuming that the observations start at $t = p + 1$. Denoting with $\boldsymbol{\xi}$ and $\mathbf{w}_t$ the two 2-dimensional vectors $\boldsymbol{\xi} = (\mu, \gamma)'$ and

$\mathbf{w}_t = (1, h_t)^{'}$, the complete-data pseudo-likelihood function is given by:

$$\begin{aligned}
&\mathcal{L}\left(r_{p+1:T}, \mathrm{RV}_{p+1:T}, \mathrm{h}_{p+1:T}, z_{p+1:T}|\boldsymbol{\theta}\right) \\
&= \prod_{t=p+1}^{T} \frac{1}{\sqrt{2\pi h_t}} \exp\left(-\frac{1}{2}\frac{(r_t - \boldsymbol{\xi}'\mathbf{w}_t)^2}{h_t}\right) \frac{\mathrm{RV}_t^{\varphi_t - 1}}{\Gamma(\varphi_t)(h_t \exp(\kappa_2)^{\varphi_t})} \exp\left(-\frac{\mathrm{RV}_t}{h_t \exp(\kappa_2)}\right) \\
&\cdot \prod_{t=p+1}^{T} \frac{h_t^{\delta + z_t - 1}}{\Gamma(\delta + z_t) c^{\delta + z_t}} \exp\left(-\frac{h_t}{c}\right) \frac{1}{z_t!} \left(\Theta(\mathbf{h}_{t-1}, \mathbf{l}_{t-1})\right)^{z_t} \exp\left(-\Theta(\mathbf{h}_{t-1}, \mathbf{l}_{t-1})\right).
\end{aligned} \tag{3.1}$$

The description of our Bayesian analysis is completed by the specification of the prior distribution $\pi(\boldsymbol{\theta})$ on the parameters $\boldsymbol{\theta}$. Let $\boldsymbol{\beta} = \left(\beta^{(d)}, \beta^{(w)}, \beta^{(m)}\right)$ and $\boldsymbol{\alpha} = \left(\alpha^{(d)}, \alpha^{(w)}, \alpha^{(m)}\right)$, we assume $\pi(\boldsymbol{\theta}) \propto \mathbb{I}_{\mathbb{R}^2}(\boldsymbol{\xi})\mathbb{I}(\varphi)_{\mathbb{R}_+}\mathbb{I}_{\mathbb{R}}(\kappa_{\mathbf{2}})\mathbb{I}_{\mathbb{R}_+}(\delta)\mathbb{I}_{\mathbb{R}^3_+}(\boldsymbol{\beta})\mathbb{I}_{\mathbb{R}_+}(\boldsymbol{\alpha})\mathbb{I}_{\mathbb{R}^3_+}(\lambda)\mathbb{I}_{\mathcal{A}_{\boldsymbol{\theta}}}(\boldsymbol{\theta})$, where $\mathbb{I}_{\boldsymbol{\Theta}}(\boldsymbol{\theta})$ is the indicator function which takes value 1 if $\boldsymbol{\theta} \in \boldsymbol{\Theta}$ and 0 otherwise, and $\mathcal{A}_{\boldsymbol{\theta}}$ is the set of parameter values which satisfy the stationarity condition in Proposition 1. The parameters and latent variables *joint* posterior distribution is

$$\pi\left(\boldsymbol{\theta}, \mathbf{h}_{p+1:T}, \mathbf{z}_{p+1:T}|\mathbf{r}_{p+1:T}, \mathrm{RV}_{p+1:T}\right) \propto \pi(\boldsymbol{\theta})\, \mathcal{L}\left(\mathbf{r}_{p+1:T}, \mathrm{RV}_{p+1:T}, \mathbf{h}_{p+1:T}, \mathbf{z}_{p+1:T}|\boldsymbol{\theta}\right).$$

This distribution is not tractable, thus we follow an MCMC approach and develop a Gibbs sampling algorithm to generate random draws from the posterior distribution and to approximate all posterior quantities of interest (see Casella and Robert, 2004). Note that, in a data-augmentation framework, the estimation of the parameters under the physical measure $\mathbb{P}$ involves an extra computational cost due to the estimation of the latent variables. Despite this difficulty, it is possible to use the analytical filtering and smoothing recursions derived for SV-ARG in order to design an effective MCMC algorithm for posterior approximation.

In particular, the proposed Gibbs sampler for SV-LHARG iterates over the following steps: i) Initialize $\boldsymbol{\theta}$, $\mathbf{z}_{p+1:T}$ and $\mathbf{h}_{p+1:T}$. ii) Sample $\boldsymbol{\theta}$ given $(\mathbf{r}_{p+1:T}, \mathrm{RV}_{p+1:T}, \mathbf{h}_{p+1:T}, \mathbf{z}_{p+1:T})$. iii) Sample $(\mathbf{z}_{p+1:T}, \mathbf{h}_{p+1:T})$ given $(\mathbf{r}_{p+1:T}, \mathrm{RV}_{p+1:T}, \boldsymbol{\theta})$. iv) Go to the second step.

In sampling $\boldsymbol{\theta}$, we consider the following blocks of parameters $\{\boldsymbol{\xi}, \varphi, \kappa_2, \delta, c, \boldsymbol{\beta}, \boldsymbol{\alpha}, \lambda\}$ where $\boldsymbol{\xi} = (\mu, \gamma)$, $\boldsymbol{\beta} = \left(\beta^{(d)}, \beta^{(w)}, \beta^{(m)}\right)$, $\boldsymbol{\alpha} = \left(\alpha^{(d)}, \alpha^{(w)}, \alpha^{(m)}\right)$ are sampled jointly. As emphasized in Chib et al. (2002), sampling parameters by groups is important for reducing the serial dependence in the MCMC output. In the on-line Appendix, we provide some details on the full conditional distributions of the parameters and their sampling methods.

In the third step of the Gibbs sampler we design a multi-move proposal distribution by assuming an auxiliary SV-ARG: we use the analytical relationships in Proposition 2 and 4. Indeed, to sample $(\mathbf{h}_{p+1:T}, \mathbf{z}_{p+1:T})$, we first develop an effective Forward Filtering Backward Sampling (FFBS) for the SV-ARG (see Frühwirth-Schnatter, 2006), then we use this sampler as proposal distribution for $\mathbf{h}_{p+1:T}$ and $\mathbf{z}_{p+1:T}$ in a MH step, in order to sample from $\pi(\mathbf{z}_{p+1:T}|\mathbf{r}_{p+1:T}, \mathrm{RV}_{p+1:T}, \mathbf{h}_{p+1:T}, \boldsymbol{\theta})$ and $\pi(\mathbf{h}_{p+1:T}|\mathbf{r}_{p+1:T}, \mathrm{RV}_{p+1:T}, \mathbf{z}_{p+1:T}, \boldsymbol{\theta})$. Specifically, we follow the strategy used in Creal (2017) by reformulating the original SV-ARG model as a Markov-switching model with state variable z_t. The regime $z_t \in \{0, \ldots, N-1\}$ is the outcome of an N-state inhomogeneous Markov chain with $N < \infty$. Truncating the transition distribution of Proposition 2 and the filtered and smoothed distributions of Proposition 3 on the support set $\{0, \ldots, N-1\}$, we obtain a FFBS procedure for the SV-ARG. We apply the Hamilton's filter forward in time, t from $p+1$ to T, to find $p(z_t = l|\mathcal{F}_t, \boldsymbol{\theta})$, then the Kim's algorithm backward in time, t from T to $p+1$, to find $p(z_t = l|\mathcal{F}_T, \boldsymbol{\theta})$, and draw z_t by multinomial sampling with probabilities $p(z_t = l|\mathcal{F}_T, \boldsymbol{\theta})$. Finally, given $\mathbf{z}_{p+1:T}$ a realization of $\mathbf{h}_{p+1:T}$ is obtained by sampling h_t from $p(h_t|\mathcal{F}_T)$ as in Proposition 4. We investigate the efficiency of the FFBS procedure through a simulation study.

In the MH step for the SV-LHARG latent variables, we improve the MH acceptance rate by applying a random block strategy (see Takahashi et al., 2009; Fiorentini et al., 2014; Billio et al., 2016). At the $j-th$ iteration of the Gibbs sampler we generate the proposals $\mathbf{z}^{(*)}_{\tau:\tau+\delta}$ and $\mathbf{h}^{(*)}_{\tau:\tau+\delta}$, by the FFBS procedure described here above with $\tau \in \{1, \ldots, T\}$ and $\delta \in \mathbb{N}_+$ selected randomly such that $\tau + \delta \leq T$. The proposals are either accepted or rejected with probability $\rho\left((\mathbf{z}^{(j-1)}_{\tau:\tau+\delta}, \mathbf{h}^{(j-1)}_{\tau:\tau+\delta}), (\mathbf{z}^{(*)}_{\tau:\tau+\delta}, \mathbf{h}^{(*)}_{\tau:\tau+\delta})\right)$ given by the minimum between 1 and the exponential transform of:

$$
\begin{aligned}
&\log\left(\mathcal{L}\left(r_{\tau+1:\tau+\delta}, \mathrm{RV}_{\tau+1:\tau+\delta}, h^{(*)}_{\tau+1:\tau+\delta}, z^{(*)}_{\tau+1:\tau+\delta}|\boldsymbol{\theta}\right)\right) - \log\left(\mathcal{L}\left(r_{\tau+1:\tau+\delta}, \mathrm{RV}_{\tau+1:\tau+\delta}, h^{(j-1)}_{\tau+1:\tau+\delta}, z^{(j-1)}_{\tau+1:\tau+\delta}|\boldsymbol{\theta}\right)\right)\\
&+ \sum_{t=\tau+1}^{\tau+\delta-1} \log\left(p\left(z^{(j-1)}_t|z^{(j-1)}_{t-1}, r_{t-1}, \mathrm{RV}_{t-1}, \boldsymbol{\theta}\right)\right) + \log\left(p\left(h^{(j-1)}_t|r_{p+1:T}, \mathrm{RV}_{p+1:T}, z^{(j-1)}_{p+1:T}\right)\right)\\
&- \log\left(p\left(z^{(*)}_t|z^{(*)}_{t-1}, r_{t-1}, \mathrm{RV}_{t-1}\right)\right) - \log\left(p\left(h^{(*)}_t|r_{p+1:T}, \mathrm{RV}_{p+1:T}, z^{(*)}_{p+1:T}\right)\right),
\end{aligned}
$$

where $\mathcal{L}$ is the conditional likelihood function in Eq.(3.1). The multi-move proposal allows for a rapid mixing of the MCMC chain. We show the efficiency of the MCMC through some simulation exercises. See on-line Appendices for further details.

4 Financial applications

The goal of the present section is to show that the approach we have proposed so far is of substantial interest in financial applications, especially from an option pricing perspective. A comparison with two benchmark models is also presented. The first one is the RV-LHARG model introduced in Majewski et al. (2015). The second model belongs to the class of GARCH-type option pricing models, the CGARCH by Christoffersen et al. (2008). The RV-LHARG is described by the following equations:

$$\begin{aligned}
r_t &= \mu + \gamma \mathrm{RV}_t + \sqrt{\mathrm{RV}_t}\varepsilon_t \\
\mathrm{RV}_t | \mathcal{F}_{t-1} &\overset{d}{\sim} \bar{\mathcal{G}}\left(\delta, \Theta\left(\mathbf{RV}_{t-1}, \mathbf{l}_{t-1}\right), c\right),
\end{aligned}$$

where $\varepsilon_t \overset{i.i.d.}{\sim} \mathcal{N}(0,1)$ and $\mathcal{F}_t = \sigma\left(\{r_u, \mathrm{RV}_u\}_{u \leq t}\right)$ is the filtration containing the information about observable quantities available up to time t. The conditional distribution of RV_t is taken as that of h_t in SV-LHARG . Differently, the CGARCH is specified as follows:

$$\begin{aligned}
r_t &= \mu + \gamma h_t + \sqrt{h_t}\varepsilon_t \\
h_t &= q_t + b_s\left(h_{t-1} - q_{t-1}\right) + a_s\left(\left(\varepsilon_{t-1} - c_s\sqrt{h_{t-1}}\right)^2 - \left(1 + c_s^2 q_{t-1}\right)\right) \\
q_t &= \omega + \rho q_{t-1} + \varphi\left(\left(\varepsilon_{t-1}^2 - 1\right) - 2c_l\sqrt{h_{t-1}}\varepsilon_{t-1}\right),
\end{aligned}$$

where $(h_{t-1} - q_{t-1})$ and q_t represent the short- and long-run persistent volatility components, respectively. Both benchmark models are estimated by means of maximum likelihood (ML henceforth). In the RV-LHARG model all quantities are observable, so that ML estimation is straightforward. Estimation of the CGARCH proceeds as customary for GARCH models combining filtering of the latent short- and long-run conditional volatility components with ML.

4.1 Model comparison

We present our empirical results on daily log-returns and realized variances for the S&P500 Futures. Our sample spans the period from January 1, 1997 to December 31, 2009. Realized variances are computed from tick-by-tick data over the same time interval. To remove the jump component from the RV, we employ the same methodology as in Corsi et al. (2013) and Majewski et al. (2015). Specifically: i) the total variation of the log-price process is estimated using the Two-Scale estimator by Zhang et al. (2005); ii) the fraction of total variation due to jumps is identified and eliminated by means of the Threshold Bi-power variation method of Corsi et al. (2010); iii) finally, most extreme observations (seemingly due to volatility jumps) are discarded from the volatility series. Neglecting the jump component and extreme observations may introduce possible misspecification, especially in pricing options. A similar remark applies for the overnight component, for which no realized measures are available. As explained in the on-line Appendix, we account for these effects by means of the parameter φ. It is sampled MCMC, but the initial value is fixed as in Hansen and Lunde (2005). In this respect, SV-LHARG is well-designed, since the two measurement equations properly account for both trading hour and daily scale effects.

We estimate SV-ARG, SV-LARG, and SV-HARG by using the Bayesian inference procedure – more details can be found in the on-line Appendix – on the sample from January 1, 1997 to December 31, 2006. We estimate SV-LHARG on the three periods: from January 1, 1997 to December 31, 2006, from January 1, 1998 to December 31, 2007 and from January 1, 1999 to December 31 2008. RV-LHARG and CGARCH are estimated over the three time periods, too. In this way we can discuss the option pricing performance, both in-sample and out-of-sample, of the different models, in particular over the global 2007-2008 financial crisis. For sake of space, the main text reports parameter estimates for the periods January 1, 1997 – December 31, 2006 (Table 1) and January 1, 1998 – December 31, 2007 (Table 2). Further estimation results are collected in the on-line Appendix. Parameters δ for SV-LHARG and RV-LHARG, and ω for CGARCH are computed by targeting the sample mean of RV and the variance of returns, respectively (see also Christoffersen et al., 2015). Tables report the risk premium ν_1, too. Details about the option sample used to calibrate the premium are postponed to Section 4.2.

Table 1: From left to right: ML estimates with robust standard errors for the RV-LHARG, MCMC estimates with standard errors for the SV-LHARG, SV-ARG, SV-LARG, SV-HARG, ML estimates with standard errors for the CGARCH. Period: January 1, 1997 to December 31, 2006.

	RV-LHARG	SV-LHARG	SV-ARG	SV-LARG	SV-HARG		CGARCH
Parameters						Parameters	
γ	0.9	-0.03	-0.13	-0.08	-0.13	γ	1.0
	(1.7)	(0.03)	(0.03)	(0.03)	(0.03)		(0.8)
φ	–	0.60	0.62	0.62	62	b_s	0.73
		(0.01)	(0.01)	(0.01)	(0.01)		(0.05)
κ_2	–	-2.72	-2.61	-2.36	-2.65	a_s	3e-06
		(0.05)	(0.04)	(0.03)	(0.04)		(1e-06)
c	1.29e-05	4.75e-06	2.23e-06	3.19e-06	2.30e-06	c_s	4e+02
	(1e-07)	(8e-08)	(5e-08)	(5e-08)	(8e-09)		(2e+02)
δ	1.22	1.15	3.02	2.66	1.15	ω	1.30e-06
β_d	1.8e+04	2.9e+04	4.29e+05	2.79e+05	3.61e+05	c_l	1.4e+02
	(1e+03)	(5e+03)	(8e+03)	(5e+03)	(4e+03)		(2e+01)
β_w	1.5e+04	2.3e+04	—	—	3.8e+04	φ	2.4e-06
	(1e+03)	(3e+03)			(4e+03)		(4e-07)
β_m	1.0e+04	1.8e+04	—	—	2.7e+04	ρ	0.991
	(4e+03)	(4e+03)			(3e+03)		(2e-03)
α_d	0.263	0.56	—	1	—		
	(1e-03)	(0.05)		(0.1)			
α_w	0.195	0.45	—	—	—		
	(4e-03)	(0.08)					
α_m	0.03	4e-04	—	—	—		
	(0.03)	(1e-04)					
λ	214	350		92			
	(4)	(10)		(8)			
ν_1	-1424	-2439	-2537	-2938	-2327	ν_1	-12242
Persistence	0.84	0.93	0.95	0.92	0.98		0.997

Let us start by commenting on the parameter estimates within the SV-LHARG framework over the period January 1, 1997, to December 31, 2006. In the present work, we adopt the same convention used in Creal (2017) by writing the drift term of the log-return dynamics as $\mu + \gamma h_t$. In Creal (2017), the estimated γ is negative and significant, implying that the distribution of returns is negatively skewed (see the discussion in the cited paper). In our case, γ is negative and significant for SV-ARG, SV-LARG, and SV-HARG , whereas it is not significant in SV-LHARG . For all models, the estimated overnight factor φ is

Table 2: From left to right: ML estimates with robust standard errors for the RV-LHARG, MCMC estimates with standard errors for the SV-LHARG , ML estimates with standard errors for the CGARCH. Period: January 1, 1998 to December 31, 2007.

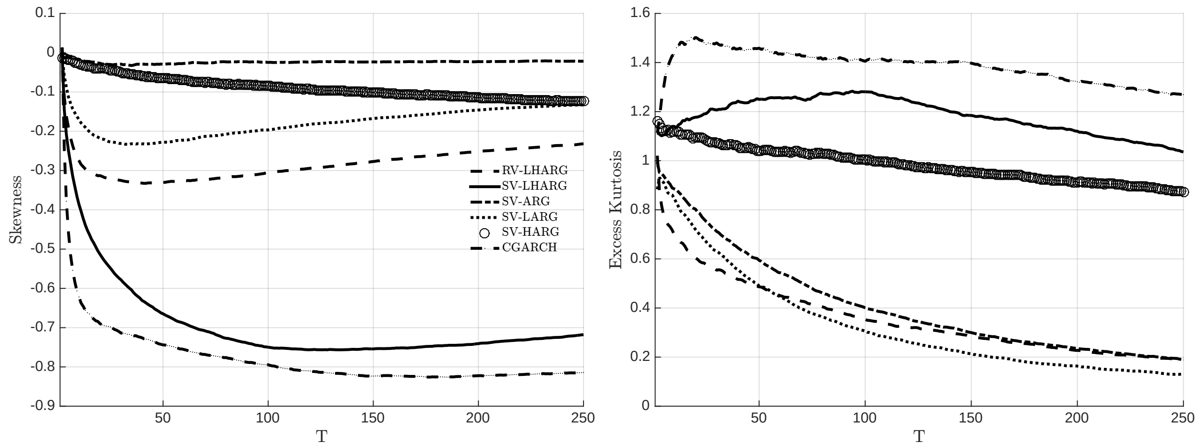
	RV-LHARG	SV-LHARG		CGARCH
Parameters			Parameters	
γ	0.1	-0.02	γ	0.9
	(1.8)	(0.03)		(0.7)
φ	–	0.60	b_s	0.75
		(0.02)		(0.04)
κ_2	–	-2.79	a_s	2e-06
		(0.06)		(1e-06)
c	1.24e-05	4.7e-06	c_s	6e+02
	(1e-07)	(1e-07)		(3e+02)
δ	1.30	1.22	c_l	1.6e+02
				(3e+01)
β_d	1.34e+04	3.2e+04	ω	1.37e-06
	(1e+02)	(5e+03)		
β_w	6.1e+03	8e+03	φ	2.2e-06
	(3e+02)	(3e+03)		(4e-07)
β_m	3.9e+03	2.2e+04	ρ	0.990
	(4e+02)	(5e+03)		(2e-03)
α_d	0.169	0.44		
	(8e-03)	(0.05)		
α_w	0.168	0.32		
	(1e-03)	(0.07)		
α_m	0.069	2.4e-04		
	(2e-03)	(9e-04)		
λ	333	411		
	(3)	(30)		
ν_1	-1584	-2308		-17339
Persistence	0.84	0.93		0.998

smaller than one – as expected – and indicates that the volatility during the trading hours ranges between 60% to 62% of the total daily variation. Concerning the persistence – computed by means of the first term inequality (2.5) – its value decreases from 0.95 (SV-ARG) to 0.92 (SV-LARG), and from 0.98 (SV-HARG) to 0.93 (SV-LHARG) when the leverage effect is included into the model. This fact is in accordance with findings in Corsi et al. (2013) and Majewski et al. (2015). Within SV-LHARG , the impact of past lags on the current value of the conditional variance is determined by the partial auto-correlation coefficients. According to our estimates, the sensitivity of the conditional mean h_t on h_{t-1}, $h_{t-1}^{(w)}$

and $h_{t-1}^{(m)}$ is $c\left(\beta^{(d)}+\lambda^2\alpha^{(d)}\right)=0.46$, $c\left(\beta^{(w)}+\lambda^2\alpha^{(w)}\right)=0.38$ and $c\left(\beta^{(m)}+\lambda^2\alpha^{(m)}\right)=0.09$, respectively. We find evidence of a decreasing impact of past lags on the current value of the conditional variance (see also the estimates of the SV-HARG model). This fact has already been documented for the RV-LHARG class of models (see Corsi et al., 2013; Majewski et al., 2015). Concerning the comparison of SV-LHARG to competitors, its persistence is 0.93, while the persistence of RV-LHARG and CGARCH are equal to 0.84 and 0.997, respectively. With respect to RV-LHARG, it is evident that the introduction of the latent process mitigates the impact of the RV measurement errors and favours a more persistent conditional variance. For CGARCH, the persistence is very high, leading to a nearly integrated variance process. The level of persistence has important consequences on the term structure of skewness and kurtosis. Both are crucial ingredients in reproducing the correct shape of the implied volatility surface (Das and Sundaram, 1999). Figure 1 plots the skewness (left panel) and excess kurtosis (right panel) associated to the six models RV-LHARG, SV-LHARG, SV-ARG, SV-LARG, SV-HARG, and CGARCH under the risk-neutral measure $\mathbb{Q}$. It confirms that SV-ARG and SV-HARG are not designed to replicate the negative skewness. For RV-LHARG and for all SV models with no heterogeneous volatility structure, the level of both skewness and kurtosis is moderate and rapidly declines toward zero. The picture is significantly different for SV-LHARG and CGARCH. Among the SV class of models, SV-LHARG reaches the highest (lowest) levels of excess kurtosis (skewness). On the other hand, CGARCH is the model with the maxima level of excess kurtosis. This fact will have important consequences on the pricing performances. We expect the best performance of the CGARCH for long-term options in in-sample tests. However, the very high level of persistence can lead to systematic over-pricing of long-term options, whenever the model miss-fits the short-term level of volatility – as it may happen out-of-sample.

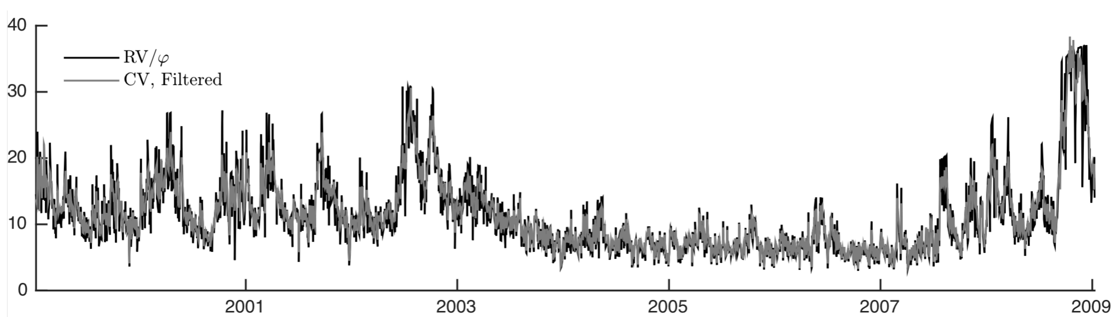
Figure 2 shows the daily realized variance annualized and in percentage terms scaled by the estimated overnight factor (*black line*) and the filtered realized variance (*gray-line*) in SV-LHARG from 1999 to 2009. The filtering procedure reduces significantly the fluctuations of the conditional latent volatility. Finally, to check for possible model miss-

Figure 1: Skewness and excess kurtosis of RV-LHARG, SV-LHARG, SV-ARG, SV-LARG, SV-HARG, and CGARCH processes under the risk-neutral measure $\mathbb{Q}$. The parameter estimates are taken from Table 1.



specifications, we consider demeaned log-returns standardized either by the square root of the RV and by the filtered CV (Andersen et al., 2010, see). Table 3 summarizes the results for the period 1999-2009, but similar conclusions hold true for the other time frames. At the 5% significance level, the Jarque-Bera test does not reject normality for SV-LHARG and rejects it for RV-LHARG. Then, SV-LHARG provides a better description of the empirical data.

Figure 2: Daily realized variance scaled by the estimated overnight factor (black line) and filtered realized variance (gray line) from 1999 to 2009.



4.2 Option pricing: Performance assessment

As for option pricing, we perform our analysis on European options, written on the S&P500 index. The option price data sample is provided by Optionmetrics for the period from January 1, 1997 to January 6, 2010 – which is the last date included in this specific dataset.

Table 3: Model misspecification tests for log-returns standardized by $\sqrt{RV_t}$ and $\sqrt{h_t}$. Lines one to four: Mean, variance, skewness, and kurtosis of standardized log-returns. Last line: Statistics and p-values (between parenthesis) for the Jarque-Bera test.

Statistics	RV-LHARG	SV-LHARG
	1999-2009	1999-2009
Mean	0.063	0.001
Variance	0.928	0.917
Skewness	0.264	-0.076
Kurtosis	3.599	3.057
Jarque-Bera	64.22 (6.9e-15)	2.75 (0.25)

As it is customary in the literature, a filter removes options with maturity less that 10 days and more that 365 days, and prices less than 5 cents (see Barone-Adesi et al., 2008; Corsi et al., 2013; Majewski et al., 2015). Following Corsi et al. (2013), we consider only Out-of-The-Money (OTM) put and call options for each Wednesday. Using K/S_t as definition of moneyness, we filter out Deep-Out-The-Money (DOTM) options with moneyness larger than 1.2 for call options and less than 0.8 for put options. Wider range of moneyness could be considered. However, for DOTM options all models should require a correction to include the impact of jumps. We investigated what happens for moneyness ranging from 0.5 to 1.5. The comparative performances among models do not change. Moreover, when moneyness varies within the interval $(0.8, 1.2)$, the fraction of neglected volume is 4% for the shortest time-to-maturities, 4.9% for the short, 6% for the medium, and 18% for the longest ones. We refer to put as DOTM options if their moneyness is between 0.8 and 0.9 and as Out-The-Money (OTM) if $0.9 < m \leq 0.98$. On the other hand, call options are termed DOTM if $1.1 < m \leq 1.2$ and OTM if $1.02 < m \leq 1.1$; options are at-the-money (ATM) if $0.98 < m \leq 1.02$. As far as the time to maturity τ is concerned, we classify options as short maturity ($\tau \leq 50$ days), short-medium maturity ($50 < \tau \leq 90$ days), long-medium maturity ($90 < \tau \leq 160$ days), and long maturity ($\tau > 160$ days).

We now analyse the option pricing performance. In our in-sample analysis, we consider the problem faced by a trader who knows the true model, but does not have the ability to see the future level of variance. We fix the parameters for each models at their full sample estimates and in the pricing kernel (i.e. the SDF) of both SV and CGARCH models we replace the latent volatility by its filtered value. In the out-of-sample exercises, we fix the parameters at the values estimated in-sample and use the filtered volatility in the pricing kernel. In order to derive the risk-neutral dynamics, the value for risk premium parameters (ν_1, ν_2) has to be specified. The value of ν_2 is reported in Tables 1 and 2 and is computed from option prices spanning the in-sample time period used for estimation. For models in the SV-LHARG family risk-neutralization is achieved by means of the SDF in Eq. (2.6) replacing $\{h_u\}_{u\leq t}$ in $\widetilde{\mathcal{F}}_t$ by their filtered values. In the recent option pricing practice, the pricing kernel employed for the risk-neutralization of CGARCH does depend on volatility (Christoffersen et al., 2013). Thus, to ensure a fair comparison among the models, we assume the following SDF for the CGARCH

$$M_{t,t+1} = \frac{e^{-\nu_1 h_{t+2} - \nu_2 \, r_{t+1}}}{\mathbb{E}^{\mathbb{P}}\left[e^{-\nu_1 h_{t+2} - \nu_2 \, r_{t+1}} | \widetilde{\mathcal{F}}_t\right]},$$

where the variance risk premium ν_1 multiplies the latent conditional variance. Since h_{t+1} is predictable, following Christoffersen et al. (2013), the pricing kernel depends on h_{t+2}. In the applications, we replace the latent conditional variance by its filtered value. In particular, the conditioning set of volatilities – $h_t, \ldots, h_{t-21}$ for SV-LHARG, h_t for CGARCH, and $\mathrm{RV}_t, \ldots, \mathrm{RV}_{t-21}$ for RV-LHARG – is fixed equal to the unconditional volatility level of each model. In order to compute option prices and the associated implied volatilities, we adopt the COS method (Fang and Oosterlee, 2008), which has been proven to be numerically efficient. The method is based on Fourier-cosine expansion and it is feasible as long as the log-return characteristic function is available in closed-form. To sum up, we proceed pricing options following four steps: i) Estimation of the model under the physical measure $\mathbb{P}$; ii) unconditional calibration of the parameter ν_1; iii) mapping of the estimated parameters into the parameters under $\mathbb{Q}$, and iv) computation of option prices via COS method using the MGF formula (see on-line Appendix 1) with risk-neutral parameters.

4.2.1 Discussion of the results

We assess both static and dynamic performances. Following previous works (Renault, 1997; Corsi et al., 2013; Majewski et al., 2015), as static performance measure we employ the

Root Mean Square Error on the percentage IV (henceforth $\mathrm{RMSE}_{\mathrm{IV}}$):

$$\mathrm{RMSE}_{\mathrm{IV}} = \sqrt{\sum_{i=1}^{\mathrm{N}} \frac{\left(\mathrm{IV}_i^{\mathrm{MOD}} - \mathrm{IV}_i^{\mathrm{MKT}}\right)^2}{\mathrm{N}}},$$

where N is the number of options, and $\mathrm{IV}^{\mathrm{MOD}}$ and $\mathrm{IV}^{\mathrm{MKT}}$ are the model and the market implied volatility, respectively. Instead, as a dynamic properties we investigate the ability of the different models to describe the time evolution of the implied volatility surface, focusing our attention on the ATM short-end of the IV surface. Importantly, we assess the option pricing performance both in-sample and out-of-sample.

The comparison between models in the SV class are reported in the on-line Appendix. SV-LHARG consistently shows the best option pricing performance among the models and, for this reason, hereafter, only SV-LHARG will be used in the comparison with the RV-LHARG and CGARCH models. Let us focus on the estimation period January 1, 1998 – December 31, 2007. The analogous analyses for the other two periods are available in the on-line Appendix. We proceed as follows: i) estimate the three models over the selected time interval and calibrate the variance risk-premium ν_1; ii) price options over the estimation intervals (*in-sample* pricing); iii) keep the parameter values and risk premium fixed as at point i) and price options over the following year, from January 1, 2008 to December 31, 2008 (*out-of-sample* pricing). The overnight correction factor φ, the shape parameter δ for RV models, and the CGARCH parameter ω are fixed via targeting on the estimation period and kept unchanged out-of-sample.

Table 4 reports the in-sample option pricing performance. The significance of the relative performances is assessed by means of a t-test using HAC standard errors to take into account the dependence structure of RMSE that are likely to be a function of time, moneyness, time-to-maturity, and level of the market and its volatility (see also Christoffersen et al., 2016). Overall (Panel D), SV-LHARG outperforms both RV-LHARG and CGARCHin all the ranges of moneyness. Nonetheless, CGARCH fares better than RV-LHARG . This latter result deserves a more detailed discussion. Indeed, Corsi et al. (2013), Table 4 second row, reported a comparison between the HARGL model and CGARCH assessing the (global) superiority of the former over the period January 1, 1997 – December 31, 2004.

Table 4: Option pricing performance on S&P500 OTM options from January 1, 1998 to December 31, 2007. Parameters values are from Table 2. Panel A: Percentage implied volatility root mean squared error ($\mathrm{RMSE}_{\mathrm{IV}}$) of the SV-LHARG model sorted by moneyness and maturity. Panels B and C: Relative $\mathrm{RMSE}_{\mathrm{IV}}$ with statistical significance from a t-test (* : p-value < 0.05, ** : p-value < 0.01, *** : p-value < 0.001) . Panel D: global option pricing performance. $\mathrm{RMSE}_{\mathrm{IV}}$ of the SV-LHARG model for different moneyness range (first row). Second and last rows: Relative $\mathrm{RMSE}_{\mathrm{IV}}$.

Moneyness	Maturity			
	$\tau \leq 50$	$50 < \tau \leq 90$	$90 < \tau \leq 160$	$160 < \tau$
Panel A	SV-LHARG Implied Volatility RMSE			
$0.8 \leq m \leq 0.9$	13.01	7.45	6.32	4.80
$0.9 < m \leq 0.98$	5.74	3.88	4.15	3.97
$0.98 < m \leq 1.02$	3.08	2.93	3.39	3.80
$1.02 < m \leq 1.1$	3.33	3.66	3.94	6.29
$1.1 < m \leq 1.2$	7.41	2.97	2.79	5.67
Panel B	SV-LHARG/RV-LHARG Implied Volatility RMSE			
$0.8 \leq m \leq 0.9$	0.98^{***}	0.91^{***}	0.88^{***}	0.84^{***}
$0.9 < m \leq 0.98$	0.99^{***}	0.89^{***}	0.88^{***}	0.88^{***}
$0.98 < m \leq 1.02$	0.91^{***}	0.80^{***}	0.83^{**}	0.89^{***}
$1.02 < m \leq 1.1$	0.90^{***}	0.74^{***}	0.77^{***}	0.89^{***}
$1.1 < m \leq 1.2$	1.12^{***}	0.95^{***}	0.79^{***}	0.92^{***}
Panel C	SV-LHARG/CGARCH Implied Volatility RMSE			
$0.8 \leq m \leq 0.9$	1.06^{***}	1.03^{***}	1.03^{***}	1.07^{***}
$0.9 < m \leq 0.98$	0.99	0.94	0.99^{***}	1.05^{***}
$0.98 < m \leq 1.02$	0.80^{***}	0.90	0.96^{**}	0.99^{***}
$1.02 < m \leq 1.1$	0.79^{***}	1.17^{***}	1.15^{***}	1.03
$1.1 < m \leq 1.2$	0.70^{***}	0.85^{**}	1.12^{***}	1.06^{***}
Panel D				
Model	Moneyness			
	$0.9 < m < 1.1$		$0.8 < m < 1.2$	
SV-LHARG	4.48		5.65	
SV-LHARG/RV-LHARG	0.91^{***}		0.93^{***}	
SV-LHARG/CGARCH	0.94^{***}		0.97^{*}	

To dig into this result, we repeated the estimation and calibration of SV-LHARG , RV-LHARG , and CGARCH over the same period. Consistently with Corsi et al. (2013), the global pricing performance favours SV-LHARG , while RV-LHARG and CGARCH rank second and third, respectively. When including periods following 2004, the results advocate for those models with the largest persistence (CGARCH and SV-LHARG). To provide an explanation of why this happens, let us look (again) at Figure 2. When compared with the previous ten years, the years 2005–2006 are characterized by a low level of unconditional volatility. RV models struggle to capture the downward swing in volatility. To track the 2005-2006 regime, the conditional volatility should decrease. In order to do so, the conditional mean of gamma models has to decrease, without compromising the unconditional target level. This effect can be achieved by inflating the model persistence and by decreasing the level of the shape parameter δ. The latter is bounded from below by one, the so-called Feller condition. Beside, being the conditional variance of the latent volatility proportional to the shape parameter, a decrease in δ lowers the dispersion of the conditional volatility. These constraints tighten the relation between persistence, unconditional targeting, conditional mean and variance of the volatility process in RV models. The same constraints do not hold for CGARCH. Consistently, CGARCH provides the best description of the market ATM volatility during 2005-2006, when it is sizeably low. During the same period, SV-LHARG partially catches up CGARCH, while RV-LHARG is not adequately flexible. The SV dynamics describes a conditional volatility process which is less disperse than the RV time series. Then, the shape parameter can take a smaller value, thus decreasing the conditional mean and preserving the long-term level by inflating the volatility persistence. This picture is confirmed by the value of the estimated parameters, and also by looking at the dynamic side of the pricing performance. More precisely, Figure 3 reports the evolution of the IV level (i.e., the average IV of short-term, ATM options) implied by SV-LHARG, RV-LHARG, and CGARCH for the period January 1, 1999 – December 31, 2008. The degree of accuracy in tracking the short-end of the IV surface before and after 2004 varies across different models. Before 2004 and after 2006, both SV-LHARG and RV-LHARG have more reactive dynamics than CGARCH, which tends to reproduce the empirical level dynamics with some delay. Within the intermediate time window – which corresponds to the low volatility period – contrary to both SV-LHARG and CGARCH, RV-LHARG is not able to adequately track the IV level. It systematically over-estimates the volatility unconditional level. Consistently with explanation above, the persistence of RV-LHARG is moderate and never exceeds 0.85. Finally, the choice of the 1999–2008 time period allows to assess graphically the relative performance of the three models in the final part of 2008, after the rise of volatility ignited by the Lehman Brothers default. It is clear that – at variance with the CGARCH– the RV measure provides the SV-LHARG with the ability to promptly react to sudden changes of market volatility.

To gain a deeper understanding of the pricing performances, Panel B and C of Table 4 reports the results in terms of RMSEs disaggregated for different maturities and moneyness. SV-LHARG always outperforms the RV-LHARG model (Panel B). When compared to CGARCH, the performance is more balanced (Panel C). For short time-to-maturities, SV-LHARG takes advantage of the economic content of the realized measures and fares much better. On the contrary, in the long-run, the strongest persistence of the CGARCH guarantees to this model the best in-sample performance.

As far as the out-of-sample analysis is concerned, Tables 5 ans 6 confirm that, globally, SV-LHARG model outperforms RV-LHARG . When considering CGARCH , the same is approximately true only for the period January 1, 2008 – December 31, 2008. In the other two periods – from January 1, 2007 to December 31, 2007 and from January 1, 2009 to December 31, 2009 – reported for completeness, the picture is much less clear and the $\mathrm{RMSE}_{\mathrm{IV}}$ ratio is not always statistical significant. For 2007, this fact is consistent with the observation that during the preceding two years the ability of SV-LHARG to track the short term ATM IV volatility is slightly worse than that of CGARCH.

This section concludes reporting two examples of implied volatility smiles reproduced by the three classes of models explored in this paper. For the in-sample smiles, we consider time-to-maturities ranging from 50 to 90 days. For the out-of-sample exercise, we move to time-to-maturities ranging from 10 to 50 days. In both cases, the smile of empirical data is quite pronounced. The left panel in Figure 4 shows the in-sample smiles from the market together with those of SV-LHARG , RV-LHARG , and CGARCH averaged over the period January 1, 1999 – December 31, 2008. The right panel reports the smiles obtained averaging the data on the out-of-sample period January 1, 2008 – December 31, 2008. The plots confirm the ability of the SV model to adequately track the ATM level of the IV at short time-to-maturities, and to reproduce the qualitative features of the smile in a comparatively better way than competitor models.

Figure 3: *Level* dynamic from January 1, 1999 to December 31, 2008. *Level* is defined as the average implied volatility of ATM options (precisely, options with moneyness $0.95 < m < 1.05$) at the shortest available maturity on a given day. In each panel, the black line represents the data, the gray line, the model. The parameter estimates are taken from Table 10 in the on-line Appendix.

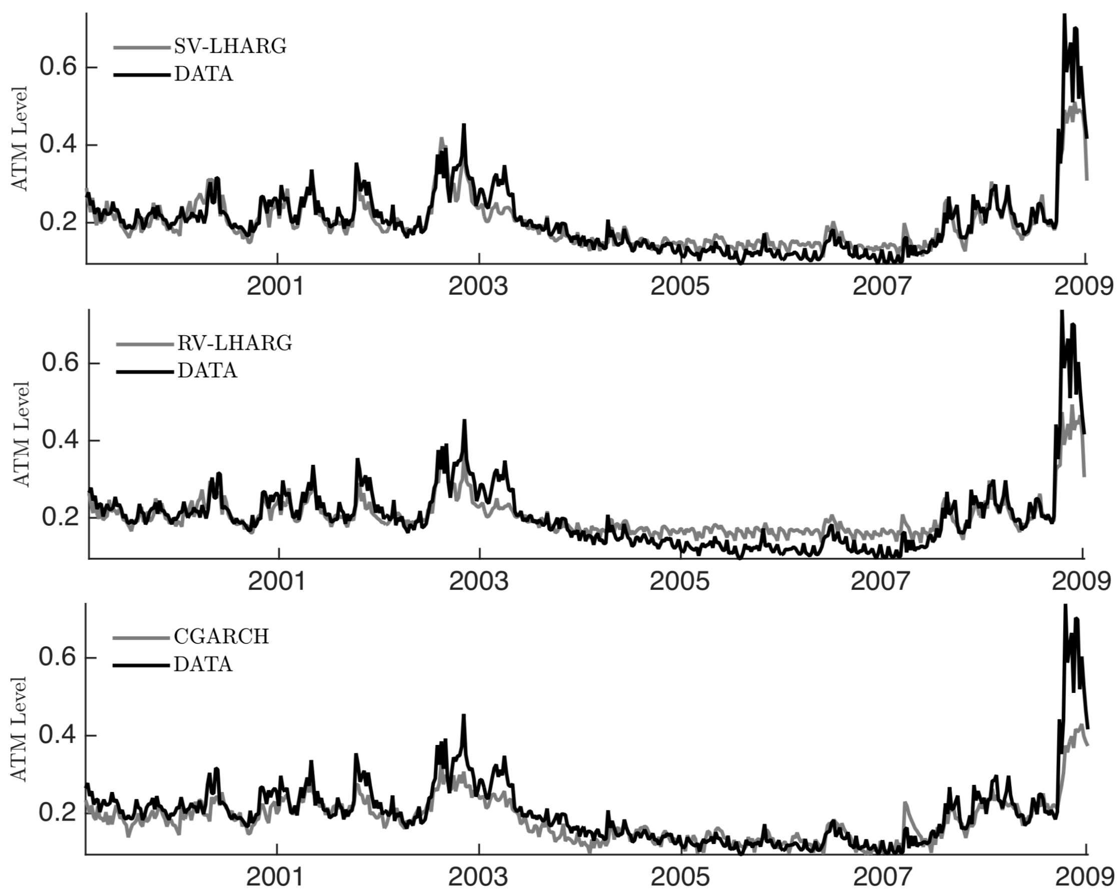


Table 5: Out-of-sample option pricing performance on S&P500 OTM options. In each panel the relative $\mathrm{RMSE}_{\mathrm{IV}}$ with significance levels, sorted by moneyness (rows) and maturity (columns).

Moneyness	Maturity			
	$\tau \leq 50$	$50 < \tau \leq 90$	$90 < \tau \leq 160$	$160 < \tau$
Out of sample period: January 1, 2007 – December 31, 2007				
Panel A1	SV-LHARG/RV-LHARG Implied Volatility RMSE			
$0.8 \leq m \leq 0.9$	0.85***	0.69***	0.70***	0.72*
$0.9 < m \leq 0.98$	0.78***	0.68***	0.43**	0.93
$0.98 < m \leq 1.02$	1.00	0.93*	0.90	1.00
$1.02 < m \leq 1.1$	1.18**	1.06	0.94	1.04***
$1.1 < m \leq 1.2$	1.14***	1.39	1.08	1.02
Panel A2	SV-LHARG/CGARCH Implied Volatility RMSE			
$0.8 \leq m \leq 0.9$	0.99**	0.81***	0.88***	0.85*
$0.9 < m \leq 0.98$	0.83***	0.67***	0.82	0.97
$0.98 < m \leq 1.02$	0.86	0.79	1.02	1.07
$1.02 < m \leq 1.1$	1.46***	1.46***	1.20***	1.10***
$1.1 < m \leq 1.2$	0.86	2.97***	1.73***	1.14***
Out of sample period: January 1, 2008 – December 31, 2008				
Panel B1	SV-LHARG/RV-LHARG Implied Volatility RMSE			
$0.8 \leq m \leq 0.9$	0.59***	0.45***	0.43***	0.49***
$0.9 < m \leq 0.98$	0.47***	0.34***	0.38***	0.44***
$0.98 < m \leq 1.02$	0.52***	0.38***	0.38***	0.41***
$1.02 < m \leq 1.1$	0.98	1.01	1.09	0.90
$1.1 < m \leq 1.2$	1.05	1.49**	1.46	1.05
Panel B2	SV-LHARG/CGARCH Implied Volatility RMSE			
$0.8 \leq m \leq 0.9$	0.55***	0.45***	0.49***	0.61***
$0.9 < m \leq 0.98$	0.39***	0.32***	0.43***	0.54***
$0.98 < m \leq 1.02$	0.39***	0.30***	0.38***	0.49***
$1.02 < m \leq 1.1$	0.64	0.78	0.99	1.03
$1.1 < m \leq 1.2$	0.57***	1.02**	1.35	1.21

5 Conclusions

Motivated by the presence of measurement errors in the empirical RV measures, we introduce a new family of discrete-time SV option pricing models, named SV-LHARG(p). The SV-LHARG(p) model is characterized by two *measurement equations* (one extracting information from the daily returns and the other from the RV measure) and a transition equation for the latent CV states described by a Heterogeneous Autoregressive Gamma process with leverage effects. SV-LHARG(p) represents the first semi-analytical option pricing framework for discrete-time SV models incorporating RV. Indeed, it is completely characterized in several respects: (i) the recursive formula of the conditional MGF un-

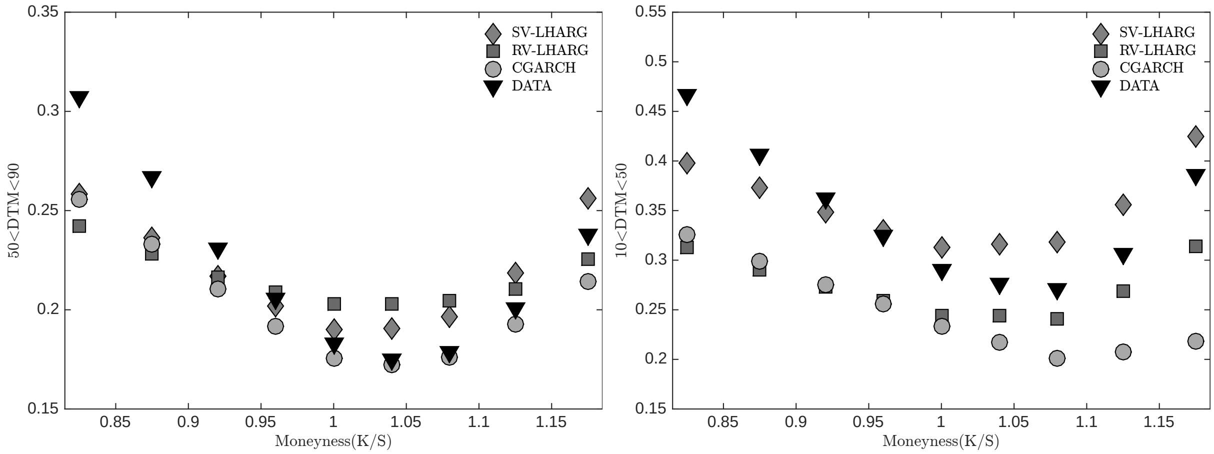
Table 6: (Continued from Table 5) Panel D: Global option pricing performance over the three out-of-sample periods.

Moneyness	Maturity $\tau \leq 50$	$50 < \tau \leq 90$	$90 < \tau \leq 160$	$160 < \tau$
Out of sample period: January 1, 2009 – December 31, 2009				
Panel C1	SV-LHARG/RV-LHARG Implied Volatility RMSE			
$0.8 \leq m \leq 0.9$	0.98	0.93	0.92	0.90
$0.9 < m \leq 0.98$	0.95**	0.91	0.94***	0.92***
$0.98 < m \leq 1.02$	0.95***	0.95***	0.96**	0.92***
$1.02 < m \leq 1.1$	1.04*	0.95***	0.95***	0.85***
$1.1 < m \leq 1.2$	1.07***	0.97**	0.91***	0.85***
Panel C2	SV-LHARG/CGARCH Implied Volatility RMSE			
$0.8 \leq m \leq 0.9$	1.05	1.06	1.05	0.99
$0.9 < m \leq 0.98$	1.04	1.05	1.04	0.97
$0.98 < m \leq 1.02$	0.99	1.03*	1.00	0.93
$1.02 < m \leq 1.1$	0.85*	0.92	0.91	0.85
$1.1 < m \leq 1.2$	0.80	0.77	0.80	0.75

Panel D Model	Moneyness $0.9 < m < 1.1$	$0.8 < m < 1.2$
Out of sample period: January 1, 2007 – December 31, 2007		
SV-LHARG/RV-LHARG	0.93	0.97***
SV-LHARG/CGARCH	1.01	1.03
Out of sample period: January 1, 2008 – December 31, 2008		
SV-LHARG/RV-LHARG	0.62***	0.59***
SV-LHARG/CGARCH	0.52***	0.49***
Out of sample period: January 1, 2009 – December 31, 2009		
SV-LHARG/RV-LHARG	0.92***	0.94***
SV-LHARG/CGARCH	0.98*	0.98*

der $\mathbb{P}$, (ii) the explicit change of measure for a general and flexible exponentially affine SDF, (iii) the no-arbitrage condition in terms of risk premia, (iv) the explicit one-to-one mapping between the parameters of the latent processes under $\mathbb{P}$ and $\mathbb{Q}$, (v) the recursive formula for the conditional MGF under $\mathbb{Q}$. In addition, building on Creal (2017), we derive the analytical filtering and smoothing for the basic specification of the SV-LHARG(p) with $p = 1$ and no leverage effect, dubbed SV-ARG. We employ these results to design an effective Bayesian inference procedure for both the parameters and the latent factor of the general model SV-LHARG(p). The estimation methodology is extensively tested on simulated data and applied to real data on the S&P 500 Future index. The financial application in the context of option pricing consent to benchmark SV-LHARG with competitor

Figure 4: IV smiles from the market (triangles), from the CGARCH (circles), from the SV-LHARG (diamonds), and from the RV-LHARG (squares). Left panel: in-sample period, January 1, 1999 to December 31, 2008. Right panel: out-of-sample smiles from January 1, 2008 to December 31, 2008.



models. Our findings shows that the SV-LHARG model tracks the dynamics of the short time-to-maturity ATM implied volatility surface with remarkable realism. Consistently with what has been documented in Corsi et al. (2013), the CGARCH tends to reproduce the empirical level with some delay (especially during periods of high volatility), whereas the SV model reacts more dynamically to changes in the volatility level. The high level of persistence of volatility in the SV-LHARG model guarantees a better pricing performance than that of the RV-LHARG model. This holds true both in-sample and out-of-sample. Concerning the CGARCH, the SV-LHARG performs better in the short-run and although its behaviour worsen comparatively for longer horizons, the overall performance remain superior. Additionally, SV-LHARG fares better than CGARCH in periods of market turmoil, in particular during the (out-of-sample) high volatility January 1, 2008–December 31, 2008 episode.

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