MATRIX RECURSION FOR POSITIVE CHARACTERISTIC DIAGRAMMATIC SOERGEL BIMODULES FOR AFFINE WEYL GROUPS

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Abstract. Let W be an affine Weyl group, and let $\Bbbk$ be a field of characteristic $p > 0$. The diagrammatic Hecke category $\mathcal{D}$ for W over $\Bbbk$ is a categorification of the Hecke algebra for W with rich connections to modular representation theory. We explicitly construct a functor from $\mathcal{D}$ to a matrix category which categorifies a recursive representation $\xi : \mathbb{Z}W \to M_{p^r}(\mathbb{Z}W)$, where r is the rank of the underlying finite root system. This functor gives a method for understanding diagrammatic Soergel bimodules in terms of other diagrammatic Soergel bimodules which are "smaller" by a factor of p. It also explains the presence of self-similarity in the p-canonical basis, which has been observed in small examples. By decategorifying we obtain a new lower bound on the p-canonical basis, which corresponds to new lower bounds on the characters of the indecomposable tilting modules by the recent p-canonical tilting character formula due to Achar–Makisumi–Riche–Williamson.

Introduction

Diagrammatic Soergel bimodules. Let (W, S) be a Coxeter system, and let $\mathbb{H}$ be the Hecke algebra associated to W. The *diagrammatic Hecke category* $\mathcal{D}$ of W over a field $\Bbbk$ is an entirely algebraic construction of a categorification of $\mathbb{H}$ [8]. More precisely, $\mathcal{D}$ is a $\Bbbk$-linear additive graded monoidal category defined by a diagrammatic presentation, whose split Grothendieck ring $[\mathcal{D}]$ (generated by isomorphism classes of indecomposable objects) is isomorphic to $\mathbb{H}$. For each $x \in W$ there is an indecomposable object B_x labeled by x, and all indecomposable objects are of this form up to grade shift. The set $\{[B_x] : x \in W\}$ is a basis of the split Grothendieck ring $[\mathcal{D}]$ and thus corresponds to a basis of $\mathbb{H}$.

When W is finite crystallographic and $\Bbbk$ is of characteristic 0, $\mathcal{D}$ is equivalent to the category of Soergel bimodules; for this reason objects in $\mathcal{D}$ are called *(diagrammatic) Soergel bimodules*, and $\mathcal{D}$ is sometimes called the *diagrammatic category of Soergel bimodules*. The diagrammatic category $\mathcal{D}$ provides a suitable replacement for the category of Soergel bimodules in situations where Soergel bimodules are less well behaved, such as when W is infinite or $\Bbbk$ has positive characteristic. For W crystallographic, the *p-canonical basis* $\{{}^p\underline{H}_x : x \in W\}$ for W is the basis for $\mathbb{H}$ corresponding to the isomorphism classes of indecomposable objects in $\mathcal{D}$ defined over a field $\Bbbk$ of characteristic $p > 0$ [14]. Equality of the p-canonical basis and the Kazhdan–Lusztig basis $\{\underline{H}_x : x \in W\}$ is associated with several classical conjectures in modular representation theory, the most important of which is Lusztig's conjecture on the characters of the simple modules for a reductive algebraic group [15, 22]. It is now known that for any fixed $x \in W$, $\underline{H}_x = {}^p\underline{H}_x$ only when p is extremely large, i.e. exponential in the rank of W [24]. Finding efficient ways to compute the p-canonical basis for smaller p is an important open problem.

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Main results. Keeping applications to modular representation theory in mind, let Φ_{f} be a finite root system with finite Weyl group W_{f} and let $W = W_{\mathrm{f}} \ltimes \mathbb{Z}\Phi_{\mathrm{f}}$ be the affine Weyl group generated by W_{f} and translations $t_\alpha : \lambda \mapsto \lambda + \alpha$ for $\alpha \in \mathbb{Z}\Phi_{\mathrm{f}}$. For p a positive integer we write $W_p = W \ltimes p\mathbb{Z}\Phi_{\mathrm{f}}$ for the p-affine subgroup of W generated by W_{f} and $t_{p\alpha}$ for $\alpha \in \mathbb{Z}\Phi_{\mathrm{f}}$. The homomorphism

$$\begin{aligned} F : W &\longrightarrow W \\ w &\longmapsto w \qquad \text{for } w \in W_{\mathrm{f}}, \\ t_\alpha &\longmapsto t_{p\alpha} \qquad \text{for } \alpha \in \mathbb{Z}\Phi_{\mathrm{f}}, \end{aligned}$$

which fixes W_{f} and scales translations by a factor of p, induces an isomorphism $W \cong W_p$. Consider the group ring $\mathbb{Z}W$ as a $(\mathbb{Z}W_p, \mathbb{Z}W)$-bimodule. As a left $\mathbb{Z}W_p$-module $\mathbb{Z}W$ is free, with a basis ${}^pW \subset W$ of minimal length representatives for the right cosets $W_p \backslash W$. The right W-action on $\mathbb{Z}W$ induces a faithful homomorphism

$$\mathbb{Z}W \longrightarrow \mathrm{End}_{\mathbb{Z}W_p}(\mathbb{Z}W) \xrightarrow{\sim} M_{|{}^pW|}(\mathbb{Z}W_p) \xrightarrow{\sim} M_{|{}^pW|}(\mathbb{Z}W).$$

Here the first isomorphism is obtained by writing $\mathbb{Z}W_p$-endomorphisms of $\mathbb{Z}W$ as matrices with respect to the basis ${}^pW$. The second isomorphism is just F^{-1} applied to the entries of the matrices. Following [9] we call the composition $\xi : \mathbb{Z}W \to M_{|{}^pW|}(\mathbb{Z}W)$ the *matrix recursion representation* of $\mathbb{Z}W$.

Our most important result is a categorification of the matrix recursion representation. More precisely, suppose $\Bbbk$ is a field of characteristic $p > 2$. The construction of the diagrammatic category uses the data of a $\Bbbk$-*realization* of W, i.e. a reflection representation of W over $\Bbbk$. We consider two different $\Bbbk$-realizations of W: the *universal realization* of W with respect to the affine simple roots (see Definition 1.3) and the F-twist of this realization (see Proposition 1.13). Let $\mathcal{D}$ and $\mathcal{D}^F$ respectively denote the diagrammatic Hecke categories constructed from these two realizations of W. The categorification of ξ is as follows (see also Theorem 6.10).

Theorem. *There is a faithful monoidal functor*

$$\Psi : \mathcal{D}^{\mathrm{deg}} \longrightarrow \mathcal{M}_{|{}^pW|}(\hat{R} \otimes \mathcal{D}^{F,\mathrm{deg}}),$$

which induces the matrix recursion representation on Grothendieck rings.

The source of the functor is the "degrading" of $\mathcal{D}$, i.e. the ungraded category obtained by forgetting the grading on Hom-spaces in $\mathcal{D}$. The target of the functor is a certain "matrix category" constructed as follows. The objects and morphisms in the matrix category $\mathcal{M}_{|{}^pW|}(\hat{R} \otimes \mathcal{D}^{F,\mathrm{deg}})$ are just $|{}^pW| \times |{}^pW|$ matrices whose entries are objects and morphisms respectively in the category $\hat{R} \otimes \mathcal{D}^{F,\mathrm{deg}}$. Composition of matrices of morphisms is just entrywise composition (see (6.3)), while the tensor product in the matrix category is given by the categorical analogue of the ordinary matrix product (see (6.2)). Here $\hat{R} \otimes \mathcal{D}^{F,\mathrm{deg}}$ is a scalar extension of $\mathcal{D}^{F,\mathrm{deg}}$ (the degrading of $\mathcal{D}^F$) with respect to a certain complete discrete valuation ring $\hat{R}$ (see Definition 4.6).

The matrix recursion representation ξ characterizes the elements of W in terms of "smaller" elements in the following sense. For each $y \in W$ and each entry x in $\xi(y)$, the Euclidean norms of $x(0)$ and $y(0)$ satisfy the approximate inequality $p|x(0)| \lessapprox |y(0)|$. The functor behaves similarly, with direct summands B_x of entries of $\Psi(B_y)$ satisfying the same approximate inequality. Since smaller Soergel bimodules are easier to understand than larger ones, the functor is a useful tool for computations in $\mathcal{D}$. For example, by reading off the first row of the matrix and decategorifying we obtain the following new lower bound on the p-canonical basis (see also Corollary 6.18). Recall that $\mathbb{H}/(v-1) \cong \mathbb{Z}W$, and for $H \in \mathbb{H}$ write ${}_{v=1}H$ for the image of H in $\mathbb{Z}W$.

Theorem. *Let $x \in W$. Then*
$$
{}_{v=1}{}^{p}\underline{H}_x \in \sum_{\substack{y \in W \\ w \in {}^pW \\ F(y)w \leq x}} \mathbb{Z}_{\geq 0} F({}_{v=1}{}^{p}\underline{H}_y)\, {}_{v=1}H_w.
$$

This lower bound can be applied recursively to each of the p-canonical basis elements on the right-hand side to obtain an even better bound. However, there is no similar recursion for the functor, the main obstruction being that the categories $\mathcal{D}$ and $\mathcal{D}^F$ are not precisely equivalent even though their indecomposable objects both give the p-canonical basis.

Amazingly, although both of the main results above are inherently ungraded, nearly all of the algebraic and categorical constructions used to prove them extend in a natural way to a graded setting! For example, in §2.2 we construct an algebra $\mathbb{H}_*$ which is a $\mathbb{Z}[v^{\pm 1}]$-module extension of $\mathbb{Z}W$, and a $(\mathbb{H}_p, \mathbb{H}_*)$-bimodule $\mathbb{H}_{p|*}$ which is a $\mathbb{Z}[v^{\pm 1}]$-module extension of the $(\mathbb{Z}W_p, \mathbb{Z}W)$-bimodule $\mathbb{Z}W$. This extends in an interesting way to the categorical level as well through the notion of a *valuation* on a category (see Definition 4.8). We hope to study the extra structure arising from these extensions in future work.

Tilting modules. We will comment on the connection between our results and modular representation theory. Let G be a semisimple, simply connected algebraic group with root system Φ_{f} over an algebraically closed field of characteristic p. We write $\cdot_p$ for the p-dilated dot action, i.e. for $w \in W$ and λ a weight,
$$
w \cdot_p \lambda = w(\lambda + \rho) - \rho
$$
where ρ is the half-root sum. We also write A_0 for the dominant weights in the interior of the fundamental (ρ-shifted) p-alcove. We call $w \in W$ dominant if $w \cdot_p \lambda$ is dominant for any $\lambda \in A_0$.

A *tilting module* is a G-module with a filtration by Weyl modules and a filtration by dual Weyl modules. For each dominant weight λ there is exactly one indecomposable tilting module $T(\lambda)$ with highest weight λ [19, 6]. When $p \geq 2h - 2$ (where h is the Coxeter number of Φ_{f}), there is a formula for the character of the simple G-modules in terms of the characters of the indecomposable tilting modules. In the analogous setting of quantum groups at an ℓth root of unity, Soergel proved a character formula for the indecomposable tilting modules $T_\ell(\lambda)$ in terms of antispherical Kazhdan–Lusztig polynomials [20, 21]. By a lifting argument, Andersen observed in [3, §4.2] that quantum tilting characters form a lower bound for modular tilting characters in the following sense.

Observation (Andersen)**.** *Let $\lambda \in A_0$. For any $r \in \mathbb{N}$ and any dominant $x \in W$,*
$$
\operatorname{ch} T(x \cdot_p \lambda) \in \sum_{y \leq x} \mathbb{Z}_{\geq 0} \operatorname{ch} T_{p^r}(y \cdot_p \lambda).
$$

Andersen's observation is notable for being one of the few results on tilting characters in the classical modular representation theory of algebraic groups (i.e. using methods similar to those in [12]) which is valid for all weights. It was used extensively in [13] to calculate characters of indecomposable tilting modules for $G = \mathrm{SL}_3$. Andersen also made the following conjecture related to the $r = 1$ case [2, Remark 3.6(i)].

Conjecture (Andersen)**.** *Suppose Lusztig's conjecture holds for G in characteristic p. For $\lambda \in A_0$ and $x \in {}^pW$ dominant, we have* $\operatorname{ch} T(x \cdot_p \lambda) = \operatorname{ch} T_p(x \cdot_p \lambda)$.

This is essentially the strongest possible conjecture of this nature; in particular Lusztig's conjecture is necessary by [20, Theorem 5.1]. Note that weights of the form $x \cdot_p \lambda$ for $\lambda \in A_0$ and $x \in {}^pW$ are just the dominant weights in the fundamental p^2-alcove.

To extend this further, we use the fact that Soergel's character formula scales in a particular way. For

$$\chi = \sum_{\substack{x \in W \\ \lambda \in A_0}} a_{x,\lambda} e^{x \cdot_p \lambda}$$

a character and $w \in {}^pW$, write

$$F^w(\chi) = \sum_{\substack{x \in W \\ \lambda \in A_0}} a_{x,\lambda} e^{F(x) w \cdot_p \lambda}.$$

The characters of $T_{p^2}(\lambda)$ are related to those of $T_p(\mu)$ in the following manner.

Corollary (Soergel's character formula scaling)**.** *For $\lambda \in A_0$ and $x \in W$ dominant, we have*

$$\operatorname{ch} T_{p^2}(x \cdot_p \lambda) = F^w(\operatorname{ch} T_p(y \cdot_p 0)),$$

where $y \in W$ and $w \in {}^pW$ such that $x = F(y)w$.

Combining this with Andersen's observation and Andersen's conjecture gives a lower bound on the characters of indecomposable tilting modules for G.

Theorem (Andersen's tilting character lower bound)**.** *Assume Andersen's conjecture holds. For any $\lambda \in A_0$ and $x, w \in {}^pW$, we have*

$$\operatorname{ch} T(F(x) w \cdot_p \lambda) \in \sum_{\substack{y \in W \\ z \in {}^pW \\ F(y)z \leq F(x)w}} \mathbb{Z}_{\geq 0} F^z(\operatorname{ch} T(y \cdot_p \lambda)).$$

Here we note that weights of the form $F(x)w \cdot_p \lambda$ for $x, w \in {}^pW$ are just the regular dominant weights in the fundamental p^3-alcove. By contrast, our lower bound on the p-canonical basis immediately gives a far stronger result using the p-canonical tilting character formula from [1, Theorem 7.6].

Theorem. *Suppose $p > h$, and let $\lambda \in A_0$. For any $x \in W$ dominant we have*

$$\operatorname{ch} T(x \cdot_p \lambda) \in \sum_{\substack{y \in W \\ z \in {}^pW \\ F(y)z \leq x}} \mathbb{Z}_{\geq 0} F^z(\operatorname{ch} T(y \cdot_p \lambda)).$$

As with the p-canonical basis lower bound, we can apply this tilting character lower bound recursively as there are no restrictions on x.

The p-canonical tilting character formula in [1] is the "combinatorial shadow" of a conjectured equivalence between the antispherical quotient category of $\mathcal{D}$ and the full subcategory of tilting modules lying in some regular block [18]. We believe that this equivalence could lead to a more directly representation-theoretic interpretation of the functor Ψ.

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1. Affine Weyl groups

1.1. **Root systems.** Let Φ_{f} be an irreducible finite root system for a real inner product space E, with a choice of simple roots $\Sigma_{\mathrm{f}} \subset E$. In this paper we consider the reflection group W generated by reflections of the form

$$\begin{aligned} s_{\alpha,k} : E &\longrightarrow E \\ \lambda &\longmapsto \lambda - (\langle \lambda, \alpha^\vee \rangle - k)\alpha \end{aligned} \tag{1.1}$$

for all $\alpha \in \Phi_{\mathrm{f}}$ and $k \in \mathbb{Z}$. In other words, W is the affine Weyl group for the *dual* or *twisted* affine root system $\Phi = \widetilde{\Phi_{\mathrm{f}}^\vee}^\vee$. The group W is isomorphic as a reflection group to $W_{\mathrm{f}} \ltimes \mathbb{Z}\Phi_{\mathrm{f}}$, where W_{f} denotes the Weyl group of Φ_{f}, and $\mathbb{Z}\Phi_{\mathrm{f}} = \mathbb{Z}\Sigma_{\mathrm{f}}$ acts on E by translation.

An *alcove* is a connected component of

$$E \setminus \bigcap_{\substack{\alpha \in \Phi_{\mathrm{f}} \\ k \in \mathbb{Z}}} \{\lambda \in E : \langle \lambda, \alpha^\vee \rangle = k\},$$

which is the complement of the hyperplanes fixed by the reflections above. The closure of an alcove is a simplex of dimension $|\Sigma_{\mathrm{f}}|$. The affine Weyl group W acts simply transitively on the set of all alcoves $\mathcal{A}$, so fixing an alcove A_0 gives a bijection $x \mapsto xA_0$ between W and $\mathcal{A}$. We will set A_0 to be the fundamental alcove, which is

$$A_0 = \{\lambda \in E : 0 < \langle \lambda, \alpha^\vee \rangle < 1 \text{ for all } \alpha \in \Phi_{\mathrm{f}}^+\}, \tag{1.2}$$

where Φ_{f}^+ is the set of positive roots induced by the simple roots Σ_{f}. The alcove A_0 is the unique dominant alcove containing 0 in its closure.

From the alcove geometry one can show that W has a presentation as a Coxeter group, which we describe below. Let S be the set of reflections in the walls of A_0. For all distinct $s, t \in S$ take $m_{st} \in \mathbb{Z} \cup \{\infty\}$ such that the angle between the reflection hyperplanes of s and t is π/m_{st}. Then W is isomorphic to the free group on S subject to the relations

$$s^2 = 1 \qquad \text{for all } s \in S, \tag{1.3}$$

$$\underbrace{sts\cdots}_{m_{st}} = \underbrace{tst\cdots}_{m_{st}} \qquad \text{for all distinct } s, t \in S, \tag{1.4}$$

where the final relation is omitted when $m_{st} = \infty$.

In terms of the root system Φ_{f}, we have $S = S_{\mathrm{f}} \cup \{s_{\alpha_{\mathrm{h}},1}\}$, where $S_{\mathrm{f}} = \{s_{\alpha,0} : \alpha \in \Sigma_{\mathrm{f}}\}$ is the set of reflections in the simple roots and α_{h} is the highest short root in Φ_{f}. For brevity we write $\tilde{s} = s_{\alpha_{\mathrm{h}},1}$. We call the generators in $S_{\mathrm{f}} \subset S$ the *finite generators* and $\tilde{s}$ the *affine generator*. For convenience we assign a root to every generator in S. For every finite generator $s_{\beta,0} \in S_{\mathrm{f}}$ where $\beta \in \Sigma_{\mathrm{f}}$, we set $\alpha_{s_{\beta,0}} = \beta$. We also set $\alpha_{\tilde{s}} = -\alpha_{\mathrm{h}}$. Finally we write $\Sigma_{\mathrm{f}}, \Sigma = \Sigma_{\mathrm{f}} \cup \{-\alpha_{\mathrm{h}}\}$ for the set of simple roots of the affine root system.

1.2. **Realizations.** A key component in the construction of $\mathcal{D}$ is the notion of a *realization*, which generalizes the concept of a reflection representation. Our definition is simpler than [8, Definition 3.1] because we restrict to working over fields.

Definition 1.1. Let $\Bbbk$ be a field of characteristic $\neq 2$. A *realization* of the affine Weyl group (W, S) (or more generally any Coxeter system) over $\Bbbk$ consists of a vector space V along with subsets $\{a_s : s \in S\} \subset V$ and $\{a_s^\vee : s \in S\} \subset V^*$ such that

(i) for all $s \in S$, we have $\langle a_s, a_s^\vee \rangle = 2$ (where $\langle \cdot, \cdot \rangle$ denotes the natural pairing);
(ii) if we set $s(v) = v - \langle v, a_s^\vee \rangle a_s$ for each $s \in S$ and all $v \in V$, then this defines a representation of W on V.

We call the matrix $a_{st} = \langle a_s, a_t^\vee \rangle$ the *Cartan matrix* of the realization V.

Definition 1.2. Suppose $(U, \{a_s\}, \{a_s^\vee\})$ and $(V, \{b_s\}, \{b_s^\vee\})$ are two realizations of (W, S) over a field $\Bbbk$. We call a linear map $\phi : U \to V$ a *homomorphism of realizations* if ϕ is a homomorphism of W-representations and $\phi(a_s) = b_s$ for all $s \in S$.

All of the realizations we will encounter in this paper are variants of the following.

Definition 1.3. The *universal realization* $(V_\Sigma, \{a_s\}, \{a_s^\vee\})$ of (W, S) with respect to $\Sigma \subseteq E$ is defined as follows. Let V_Σ be a $\Bbbk$-vector space with basis $\{a_s : s \in S\}$ and define $\{a_s^\vee\} \subseteq V_\Sigma^*$ by

$$\langle a_s, a_t^\vee \rangle = \langle \alpha_s, \alpha_t^\vee \rangle. \tag{1.5}$$

The universal realization $(V_{\Sigma_\mathrm{f}}, \{a_s\}, \{a_s^\vee\})$ of $(W_\mathrm{f}, S_\mathrm{f})$ with respect to $\Sigma_\mathrm{f} \subseteq E$ is defined similarly.

Remark 1.4. The Cartan matrix of V_Σ is just the Cartan matrix of the affine root system Φ. The universal realization has the following universal property: for any realization V of (W, S) with the same Cartan matrix, there is a unique homomorphism of realizations $V_\Sigma \to V$. If A is the Cartan matrix of some realization of any Coxeter group, one can construct in exactly the same way the universal realization for A which has the same universal property.

Write $\alpha_\mathrm{h} \in E$ as a sum

$$\alpha_\mathrm{h} = \sum_{s \in S_\mathrm{f}} c_s \alpha_s$$

of simple roots in E, and define coefficients $c_s^\vee$ similarly. Let

$$a_\mathrm{h} = \sum_{s \in S_\mathrm{f}} c_s a_s, \tag{1.6}$$

$$a_\mathrm{h}^\vee = \sum_{s \in S_\mathrm{f}} c_s^\vee a_s^\vee \tag{1.7}$$

for any realization of (W, S) or $(W_\mathrm{f}, S_\mathrm{f})$.

Definition 1.5. Let V_{Σ_f} be the universal realization of $(W_\mathrm{f}, S_\mathrm{f})$ with respect to Σ_f. The *inflated finite realization* $(V_{\Sigma_\mathrm{f}}^\pi, \{a_s^\pi\}, \{(a_s^\pi)^\vee\})$ of (W, S) with respect to Σ is defined as follows. As a W-representation, $V_{\Sigma_\mathrm{f}}^\pi$ is the inflation of V_{Σ_f} via the canonical projection $\pi : W \to W_\mathrm{f}$. Moreover, we take $a_s^\pi = a_s$ and $(a_s^\pi)^\vee = a_s^\vee$ for $s \in S_\mathrm{f}$, while we set $a_{\tilde{s}}^\pi = -a_\mathrm{h}$ and $a_{\tilde{s}}^\vee = -a_\mathrm{h}^\vee$.

Now we describe the relationship between the universal realization and the affine action of W on E, closely following [11, §6.5]. Let $V_\mathbb{R} = V_\Sigma$ be the universal realization of (W, S) over $\mathbb{R}$ with respect to Σ and set $v_\mathrm{fix} = a_{\tilde{s}} + a_\mathrm{h}$. The quotient space $V_\mathbb{R}/\mathbb{R}v_\mathrm{fix}$ has a basis $\{a_s + \mathbb{R}v_\mathrm{fix} : s \in S_\mathrm{f}\}$. The dual of $V_\mathbb{R}/\mathbb{R}v_\mathrm{fix}$ is naturally isomorphic to the annihilator subspace

$$Z = (\mathbb{R}v_\mathrm{fix})^\circ = \{b \in V_\mathbb{R}^* : \langle v_\mathrm{fix}, b \rangle = 0\} \tag{1.8}$$

of $V_\mathbb{R}^*$.

Proposition 1.6. *For all $s \in S$, we have $a_s^\vee \in Z$. The subsets*

$$\{a_s + \mathbb{R}v_\mathrm{fix} : s \in S\} \subseteq V_\mathbb{R}/\mathbb{R}v_\mathrm{fix},$$
$$\{a_s^\vee : s \in S\} \subseteq Z \cong (V_\mathbb{R}/\mathbb{R}v_\mathrm{fix})^*$$

define a realization of (W, S) on $V_\mathbb{R}/\mathbb{R}v_\mathrm{fix}$, which is isomorphic to the inflated finite realization $V_{\Sigma_\mathrm{f}}^\pi$.

Proof. For all $s \in S$ we have

$$\begin{aligned}\langle v_{\mathrm{fix}}, a_s^\vee\rangle &= \langle a_{\tilde{s}}, a_s^\vee\rangle + \langle a_{\mathrm{h}}, a_s^\vee\rangle \\ &= \langle a_{\tilde{s}}, a_s^\vee\rangle + \sum_{t\in S_{\mathrm{f}}} c_t\langle a_t, a_s^\vee\rangle \\ &= \langle \alpha_{\tilde{s}}, \alpha_s^\vee\rangle + \sum_{t\in S_{\mathrm{f}}} c_t\langle \alpha_t, \alpha_s^\vee\rangle \\ &= \langle -\alpha_{\mathrm{h}}, \alpha_s^\vee\rangle + \langle \alpha_{\mathrm{h}}, \alpha_s^\vee\rangle \\ &= 0.\end{aligned}$$

This simultaneously shows that $a_s^\vee \in Z$ and that $\mathbb{R}v_{\mathrm{fix}} \leq V_{\mathbb{R}}$ is fixed by W. Hence $V_{\mathbb{R}}/\mathbb{R}v_{\mathrm{fix}}$ has the structure of a realization, and it is easy to check that it is isomorphic to $V^{\pi}_{\Sigma_{\mathrm{f}}}$. ☐

Lemma 1.7. *The subsets*

$$\left\{\frac{(\alpha_s, \alpha_s)}{2} a_s^\vee : s \in S\right\} \subseteq Z,$$
$$\left\{\frac{2}{(\alpha_s, \alpha_s)} a_s + \mathbb{R}v_{\mathrm{fix}} : s \in S\right\} \subseteq (V_{\mathbb{R}}/\mathbb{R}v_{\mathrm{fix}}) = Z^*$$

define a realization of (W, S) *on* Z*, which is isomorphic to the inflated finite realization* $V^{\pi}_{\Sigma_{\mathrm{f}}}$*.*

Proof. From the previous proposition, the realization

$$\{a_s^\vee : s \in S\} \subseteq Z,$$
$$\{a_s + \mathbb{R}v_{\mathrm{fix}} : s \in S\} \subseteq (V_{\mathbb{R}}/\mathbb{R}v_{\mathrm{fix}}) = Z^*$$

is isomorphic to the inflated finite realization $V^{\pi}_{\Sigma_{\mathrm{f}}^\vee}$ for the dual root system $\Phi_{\mathrm{f}}^\vee$. The scaling factors above correct for changes in root length from dualizing. ☐

From the previous lemma, the linear transformation mapping

$$\begin{aligned} E &\longrightarrow Z \\ \alpha_s &\longmapsto \frac{(\alpha_s, \alpha_s)}{2} a_s^\vee \end{aligned} \tag{1.9}$$

for each $s \in S_{\mathrm{f}}$ is an isomorphism of W_{f}-representations. If we the inner product structure from E to Z using this isomorphism, then W_{f} acts orthogonally on Z because it does on E.

Now let $b_{\tilde{s}} \in V^*_{\mathbb{R}}$ such that

$$\langle b_{\tilde{s}}, a_s\rangle = \begin{cases} 1 & \text{if } s = \tilde{s}, \\ 0 & \text{otherwise,}\end{cases} \tag{1.10}$$

and let E' be the affine hyperplane

$$E' = Z + \frac{(\alpha_{\mathrm{h}}, \alpha_{\mathrm{h}})}{2} b_{\tilde{s}} = \left\{ b \in V^*_{\mathbb{R}} : \langle v_{\mathrm{fix}}, b\rangle = \frac{(\alpha_{\mathrm{h}}, \alpha_{\mathrm{h}})}{2}\right\} \tag{1.11}$$

in $V^*_{\mathbb{R}}$. Clearly E' is just a translation of Z, so it has the structure of an affine inner product space, i.e. a *Euclidean space.*

Lemma 1.8. *The affine Weyl group* W *acts on* E' *via Euclidean isometries. There is a unique isomorphism* $\epsilon : E \to E'$ *of affine* W*-spaces, and this isomorphism is a dilation.*

Proof. For any $w \in W$ and $b \in E'$ we have

$$\langle v_{\mathrm{fix}}, w(b)\rangle = \langle w^{-1}(v_{\mathrm{fix}}), b\rangle = \langle v_{\mathrm{fix}}, b\rangle = \frac{(\alpha_{\mathrm{h}}, \alpha_{\mathrm{h}})}{2},$$

so $w(b) \in E'$ and thus E' is stable under W. Now $\tilde{s}$ acts on Z like $\pi(\tilde{s}) \in W_{\mathrm{f}}$, so all of W (not just W_{f}) acts orthogonally on Z. But E' is a translation of Z, so W acts on E' as Euclidean isometries.

Note that W acts faithfully on E' because E' spans the faithful representation $V_{\mathbb{R}}^*$. For each $s \in S$ the fixed points of s in $V_{\mathbb{R}}^*$ are $(\mathbb{R}a_s)^\circ$, whose intersection with E' forms an affine hyperplane. Thus s is a Euclidean isometry which fixes an affine hyperplane in E' and has order 2, so it must be a reflection. For distinct $s, t \in S$, if $m_{st} \neq \infty$ then $m_{st} \in \{2, 3, 4, 6\}$ and faithfulness forces the angle between the reflection hyperplanes $E' \cap (\mathbb{R}a_s)^\circ$ and $E' \cap (\mathbb{R}a_t)^\circ$ to be π/m_{st}. This also holds even when $m_{st} = \infty$ (the only case where this occurs is when $\Phi_{\mathrm{f}} = A_1$, which can be analyzed separately). Thus the reflection hyperplanes in E' bound a simplex with the same dihedral angles as A_0 and there is a unique isomorphism of affine W-spaces $\epsilon : E \to E'$, which is a dilation because it is angle-preserving. □

Lemma 1.9. *The induced isomorphism* $\epsilon^* : \mathrm{Aff}(E') \to \mathrm{Aff}(E)$ *on the spaces of affine functions maps*

$$\begin{aligned} \epsilon^*\left(\frac{2}{(\alpha_s, \alpha_s)} a_s\right) &= \alpha_s^\vee && \textit{for all } s \in S_{\mathrm{f}}, \\ \epsilon^*\left(\frac{2}{(\alpha_{\mathrm{h}}, \alpha_{\mathrm{h}})} a_{\tilde{s}}\right) &= 1 - \alpha_{\mathrm{h}}^\vee. \end{aligned}$$

Proof. In terms of the Euclidean structures on E and E', ϵ is a dilation by some scale factor $c > 0$. The induced isomorphism $\epsilon^* : \mathrm{Aff}(E') \to \mathrm{Aff}(E)$ must map

$$\begin{aligned} \epsilon^*(\mathbb{R}a_s) &= \mathbb{R}\alpha_s^\vee && \text{for all } s \in S_{\mathrm{f}}, \\ \epsilon^*(\mathbb{R}a_{\tilde{s}}) &= \mathbb{R}(1 - \alpha_{\mathrm{h}}^\vee) \end{aligned}$$

because these correspond to the reflection hyperplanes in E and E'. Recall that the derivative of an affine function on E' is a linear function on Z (i.e. an element of Z^*). The inner products on E and Z induce inner products on E^* and Z^*, and since ϵ is a dilation with scale factor c, ϵ^* scales the derivatives of the affine functions by c^{-1}. But from Lemma 1.7 we know that

$$(\alpha_s^\vee, \alpha_s^\vee) = \left(\frac{2}{(\alpha_s, \alpha_s)} a_s + \mathbb{R}v_{\mathrm{fix}}, \frac{2}{(\alpha_s, \alpha_s)} a_s + \mathbb{R}v_{\mathrm{fix}}\right)$$

for all $s \in S_{\mathrm{f}}$, so we must have

$$\epsilon^*\left(\frac{2}{(\alpha_s, \alpha_s)} a_s\right) = c^{-1}\alpha_s^\vee$$

for all $s \in S_{\mathrm{f}}$. By linearity

$$\epsilon^*\left(\frac{2}{(\alpha_{\mathrm{h}}, \alpha_{\mathrm{h}})} a_{\mathrm{h}}\right) = c^{-1}\alpha_{\mathrm{h}}^\vee.$$

Combining this with $\epsilon^*\left(\frac{2}{(\alpha_{\mathrm{h}}, \alpha_{\mathrm{h}})} v_{\mathrm{fix}}\right) = 1$ we get

$$\epsilon^*\left(\frac{2}{(\alpha_{\mathrm{h}}, \alpha_{\mathrm{h}})} (v_{\mathrm{fix}} - a_{\mathrm{h}})\right) = \epsilon^*\left(\frac{2}{(\alpha_{\mathrm{h}}, \alpha_{\mathrm{h}})} a_{\tilde{s}}\right) = 1 - c^{-1}\alpha_{\mathrm{h}}^\vee$$

But this means that $c = 1$, which proves the result.

□

1.3. **The Frobenius map.** Let p be a natural number (we will later restrict to the case where p is prime).

Definition 1.10. The *Frobenius map* F on W is defined to be the group homomorphism mapping

$$F : W \longrightarrow W$$
$$s_{\alpha,k} \longmapsto s_{\alpha,pk}$$

for all $\alpha \in \Phi_{\mathrm{f}}$ and $k \in \mathbb{Z}$. The image of F is called the *p-affine Weyl group* and is denoted W_p.

The Frobenius map is well-defined because it corresponds to conjugation by the dilation map $\lambda \mapsto p\lambda$ on E. As F is injective it induces an isomorphism $W \xrightarrow{\sim} W_p$, so we can transfer the constructions in §1.1 to W_p. Thus $W_p \cong W_{\mathrm{f}} \ltimes p\mathbb{Z}\Phi_{\mathrm{f}}$, we have a set $\mathcal{A}_p$ of p-alcoves and a fixed fundamental p-alcove $A_{0,p}$, and W_p is a Coxeter group with Coxeter generators $S_p = S_{\mathrm{f}} \cup \{\tilde{s}_p\}$ which are reflections in the walls of $\overline{A}_{0,p}$. In particular the isomorphism $W \xrightarrow{\sim} W_p$ induced by F is an isomorphism of Coxeter groups, with $F(s) = s$ for all $s \in S_{\mathrm{f}}$ and $F(\tilde{s}) = \tilde{s}_p$. Let ${}^p\mathcal{A}$ denote the set of ordinary alcoves contained inside $A_{0,p}$. The bijection $W \xrightarrow{\sim} \mathcal{A}$ restricts to a bijection ${}^pW \xrightarrow{\sim} {}^p\mathcal{A}$, where ${}^pW$ is the set of minimal length representatives for the right cosets $W_p\backslash W$. This bijection induces a right action of W on ${}^p\mathcal{A}$.

Now define the following lattices

$$V_{\mathbb{Z}} = \{v \in V_{\mathbb{R}} : \langle v, a_s^\vee \rangle \in \mathbb{Z} \text{ for all } s \in S\}, \tag{1.12}$$

$$V_{\mathbb{Z}}^* = \{b \in V_{\mathbb{R}}^* : \langle a_s, b \rangle \in \mathbb{Z} \text{ for all } s \in S\}. \tag{1.13}$$

These lattices give a $\mathbb{Z}$-form of the universal realization. Let us also write $\alpha_{\tilde{s}_p} = -\alpha_{\mathrm{h}}$, $\alpha_{\tilde{s}_p}^\vee = -\alpha_{\mathrm{h}}^\vee$, and

$$a_{\tilde{s}_p} = (p-1)a_{\mathrm{h}} + pa_{\tilde{s}} \in V_{\mathbb{Z}}, \tag{1.14}$$

$$a_{\tilde{s}_p}^\vee = -a_{\mathrm{h}}^\vee \in V_{\mathbb{Z}}^*. \tag{1.15}$$

Lemma 1.11. *Let $w \in {}^pW$ and $s \in S$. The coefficient of $a_{\tilde{s}}$ in $wa_s \in V_{\mathbb{Z}}$ is a multiple of p if and only if $W_pws = W_pw$. When this happens we have $wsw^{-1} \in S_p$ and $wa_s = a_{wsw^{-1}}$.*

Proof. First, we calculate

$$\begin{aligned} \epsilon^*\left(\frac{2}{(\alpha_{\mathrm{h}},\alpha_{\mathrm{h}})}a_{\tilde{s}_p}\right) &= \epsilon^*\left(\frac{2(p-1)}{(\alpha_{\mathrm{h}},\alpha_{\mathrm{h}})}a_h + \frac{2p}{(\alpha_{\mathrm{h}},\alpha_{\mathrm{h}})}a_{\tilde{s}}\right) \\ &= \epsilon^*\left(\frac{2p}{(\alpha_{\mathrm{h}},\alpha_{\mathrm{h}})}(v_{\mathrm{fix}} - a_{\mathrm{h}})\right) \\ &= p - \alpha_{\mathrm{h}}^\vee. \end{aligned}$$

Thus $E' \cap (\mathbb{R}a_{\tilde{s}_p})$ is the reflection hyperplane of $\tilde{s}_p$.

When $W_pws = W_pw$, the alcoves corresponding to w and ws are separated by a wall of $\overline{A}_{0,p}$, corresponding to some $t \in S_p$. This means that $wsw^{-1} = t$ and that w maps the reflection hyperplane of s to the reflection hyperplane of t. Equivalently, in terms of the affine functions defining these hyperplanes in E' we have

$$w\left(\frac{2}{(\alpha_s,\alpha_s)}a_s\right) = \frac{2c}{(\alpha_t,\alpha_t)}a_t$$

for some scalar c using Lemma 1.9 and the above calculation. But in this situation we have $\pi(w)(\alpha_s^\vee) = c\alpha_t^\vee$ from Lemma 1.7, so we must have $c = \pm 1$ and that α_s and α_t have the same length. Finally we observe that the affine function $\epsilon^*(wa_s)$ is positive on $w(A_0) \subseteq A_{0,p}$, so c must be positive as well and thus $c = 1$. This gives $wa_s = \frac{(\alpha_s,\alpha_s)}{(\alpha_t,\alpha_t)} a_t = a_t$.

Conversely, if p divides the coefficient of $a_{\tilde{s}}$ in wa_s, then

$$\epsilon^* \left(w \left(\frac{2}{(\alpha_s, \alpha_s)} a_s \right) \right) \in p\mathbb{Z} + \mathbb{Z}\Phi_{\mathrm{f}}^\vee.$$

This means w maps the reflection hyperplane of s to a hyperplane fixed by some reflection in W_p. Geometrically, this means the alcoves corresponding to w and ws are separated by a wall of $\overline{A}_{0,p}$, so $W_p ws = W_p w$.

□

Example 1.12. Recall that $s_{\alpha_{\mathrm{h}}} \in W_{\mathrm{f}}$ denotes reflection in the highest short root α_{h}. The element $s_{\alpha_{\mathrm{h}}} \tilde{s} \in W$ acts on the Euclidean space E as translation by $(-\alpha_{\mathrm{h}})$, since it maps

$$\lambda \longmapsto \lambda - (\langle \lambda, \alpha_{\mathrm{h}}^\vee \rangle - 1)\alpha_{\mathrm{h}} \longmapsto \lambda - \alpha_{\mathrm{h}}.$$

This means that $\tilde{s} s_{\alpha_{\mathrm{h}}} = (s_{\alpha_{\mathrm{h}}} \tilde{s})^{-1}$ acts on E as translation by α_{h}. Set $w = (\tilde{s} s_{\alpha_{\mathrm{h}}})^{(p-1)/2}$ and $s = \tilde{s}$. We claim that w and s satisfy the conditions of Lemma 1.11. First we note that w acts on E as translation by $\frac{(p-1)}{2}\alpha_{\mathrm{h}}$. If λ lies in the fundamental alcove A_0, then for any $\alpha \in \Phi_{\mathrm{f}}^+$ we have

$$\langle w(\lambda), \alpha^\vee \rangle = \left\langle \lambda + \frac{(p-1)}{2}\alpha_{\mathrm{h}}, \alpha^\vee \right\rangle = \langle \lambda, \alpha^\vee \rangle + \frac{(p-1)}{2} \langle \alpha_{\mathrm{h}}, \alpha^\vee \rangle.$$

Using the inequalities

$$0 < \langle \lambda, \alpha^\vee \rangle < 1,$$
$$0 < \langle \alpha_{\mathrm{h}}, \alpha^\vee \rangle < 2$$

we conclude that $w(A_0) \subseteq A_{0,p}$ and thus $w \in {}^pW$. Finally, we check that $wsw^{-1} = \tilde{s}_p$, since $wsw^{-1} = (\tilde{s} s_{\alpha_{\mathrm{h}}})^{p-1} \tilde{s}$ maps

$$\lambda \longmapsto \lambda - (\langle \lambda, \alpha_{\mathrm{h}}^\vee \rangle - 1)\alpha_{\mathrm{h}} \longmapsto \lambda - (\langle \lambda, \alpha_{\mathrm{h}}^\vee \rangle - p)\alpha_{\mathrm{h}}.$$

Now suppose $\Bbbk$ is an arbitrary field of characteristic $\neq 2$, and let $V_\Bbbk$ denote the universal realization of (W, S) over $\Bbbk$. Observe that $V_\Bbbk = \Bbbk \otimes_{\mathbb{Z}} V_{\mathbb{Z}}$ and $V_\Bbbk^* = \Bbbk \otimes_{\mathbb{Z}} V_{\mathbb{Z}}^*$. Let $V_\Bbbk^F$ denote the F-*twist* of the W-representation $V_\Bbbk$; in other words, as a vector space $V_\Bbbk^F = V_\Bbbk$, but the W-action is given by $w \cdot_F v = F(w)v$ for all $w \in W$ and $v \in V_\Bbbk^F$. Write $a_s^F = a_{F(s)} \in V_\Bbbk^F$ and $(a_s^F)^\vee = a_{F(s)}^\vee \in (V_\Bbbk^F)^*$ for each $s \in S$.

Proposition 1.13. *For any field $\Bbbk$ of characteristic $\neq 2$, $(V_\Bbbk^F, \{a_s^F\}, \{(a_s^F)^\vee\})$ is a realization. When $p = \operatorname{char} \Bbbk$ it is isomorphic as a representation to $V_{\Sigma_{\mathrm{f}}}^\pi \oplus \Bbbk$.*

Proof. Clearly the reflections in S_{f} act as reflections on $V_\Bbbk^F$, so to show $V_\Bbbk^F$ is a realization we must show that $\tilde{s}$ does too. Choose $w \in {}^pW$ and $s \in S$ such that $wsw^{-1} = \tilde{s}_p$ (geometrically, w corresponds to an alcove whose s-wall lies on the $\tilde{s}_p$-wall of the fundamental p-alcove $\overline{A}_{0,p}$). For $v \in V^F$, we have

$$\begin{aligned}
\tilde{s} \cdot_F v &= \tilde{s}_p(v) \\
&= wsw^{-1}(v) \\
&= w(w^{-1}(v) - 2\langle w^{-1}v, a_s^\vee \rangle a_s) \\
&= v - 2\langle w^{-1}v, a_s^\vee \rangle wa_s \\
&= v - 2\langle v, wa_s^\vee \rangle a_{\tilde{s}}^F \\
&= v - 2\langle v, (a_{\tilde{s}}^F)^\vee \rangle a_{\tilde{s}}^F,
\end{aligned}$$

where the last equality follows from $\pi(w)(\alpha_s^\vee) = \alpha_t^\vee$ (see the previous proof) and Proposition 1.6. Hence $V_\Bbbk^F$ is a realization.

When $p = \operatorname{char} \Bbbk$, we have $a^F_{\tilde{s}} = -a_{\mathrm{h}}$. Thus $U = \bigoplus_{s \in S_{\mathrm{f}}} \Bbbk a_s \leq V^F_{\Bbbk}$ is a subrepresentation, which is isomorphic to $V^{\pi}_{\Sigma_{\mathrm{f}}}$. The trivial subrepresentation $\Bbbk v_{\mathrm{fix}} \leq V^F_{\Bbbk}$ is a complement to U, so $V^F \cong V^{\pi}_{\Sigma_{\mathrm{f}}} \oplus \Bbbk$. □

2. Hecke algebras

2.1. **Generators and bases.** We first recall some terminology and set up some notation for the affine Weyl group W. An *expression* in S is a finite sequence $\underline{x} = \underline{s_1 s_2 \dots s_m}$ of Coxeter generators of W written using an underline. We denote the set of all expressions in S by $\underline{S}$. We use the non-underlined counterpart of an expression to denote the product of these generators in W, i.e. $x = s_1 s_2 \cdots s_m \in W$. We write $\ell(\underline{x}) = m$ for the *length* of $\underline{x}$. As W is a Coxeter group, it is equipped with a length function $\ell : W \to \mathbb{Z}_{\geq 0}$ and a closely related partial order $\leq$ on W called the *Bruhat order*. We call $\underline{x}$ a *reduced expression* (or *rex*) for x when $\ell(\underline{x}) = \ell(x)$.

The *Hecke algebra* $\mathbb{H} = \mathbb{H}(W, S)$ of the affine Weyl group W is the $\mathbb{Z}[v^{\pm 1}]$-algebra with generators $\{H_s\}_{s \in S}$ and relations

$$H_s^2 = 1 + (v^{-1} - v) H_s \quad \text{for all } s \in S, \tag{2.1}$$

$$\overbrace{H_s H_t H_s \cdots}^{m_{st} \text{ terms}} = \overbrace{H_t H_s H_t \cdots}^{m_{st} \text{ terms}} \qquad \text{for all distinct } s, t \in S \text{ where } m_{st} \neq \infty, \tag{2.2}$$

where m_{st} is defined as in §1.1.

If $w \in W$ and $\underline{w} = \underline{s_1 s_2 \cdots s_m}$ is a rex for w, the element $H_w = H_{s_1} H_{s_2} \cdots H_{s_m}$ is well-defined, and the set $\{H_w\}_{w \in W}$ forms a $\mathbb{Z}[v^{\pm 1}]$-basis for $\mathbb{H}$. Each generator H_s is invertible, with $H_s^{-1} = H_s + v - v^{-1}$, so each basis element H_w is also invertible. The *bar involution* or *dualization map* $\overline{(\ )} : \mathbb{H} \longrightarrow \mathbb{H}$ is the ring homomorphism defined by the following action

$$\begin{aligned} \overline{v} &= v^{-1} \\ \overline{H_w} &= (H_{w^{-1}})^{-1} \end{aligned} \tag{2.3}$$

on the basis. For $s \in S$ we define $\underline{H}_s = H_s + v$, which is self-dual. The set $\{\underline{H}_s\}_{s \in S}$ forms another set of generators for $\mathbb{H}$ as a $\mathbb{Z}[v^{\pm 1}]$-algebra. For any $\underline{x} = \underline{s_1 s_2 \cdots s_m} \in \underline{S}$ we write $\underline{H}_{\underline{x}} = \underline{H}_{s_1} \underline{H}_{s_2} \cdots \underline{H}_{s_m}$.

Let $\mathbb{H}_p = \mathbb{H}(W_p, S_p)$ denote the Hecke algebra of the p-affine Weyl group. To avoid confusion with the generators of $\mathbb{H}$, we write $H_s^{(p)}$ denote the generator in $\mathbb{H}_p$ corresponding to $s \in S_p$. Note that $\mathbb{H}_p$ is isomorphic to $\mathbb{H}$ via an extension of F, mapping $H_s \mapsto H^{(p)}_{F(s)}$ for all $s \in S$. Thus everything in this section applies equally to $\mathbb{H}_p$ as well.

Notation 2.1. Let $\underline{x} = \underline{s_1 s_2 \cdots s_m}$ be an expression. A *subsequence* for $\underline{x}$ is a sequence of the form $\mathbf{e} = (\mathbf{e}_1, \dots, \mathbf{e}_m)$, where each $\mathbf{e}_i$ is an ordered pair (s_i, t_i) with $t_i \in \{0, 1\}$. We say that $\mathbf{e}_i$ is a *term* with *generator* s_i and *type* t_i, and we refer to the type of $\mathbf{e}$ to mean the sequence of types of the terms $\mathbf{e}_i$. We write e to denote the group element $s_1^{t_1} s_2^{t_2} \cdots s_m^{t_m} \in W$. We denote the set of all subsequences for $\underline{x}$ by $[\underline{x}]$.

Every subsequence $\mathbf{e}$ can be assigned an integer $d(\mathbf{e})$ called the *defect*. The defect $d(\mathbf{e})$ is based on a sequence of elements in W called the *Bruhat stroll* for $\mathbf{e}$. We delay the definition of the Bruhat stroll and the defect until §2.3, where we provide a more general construction.

Lemma 2.2 (Deodhar's defect formula [5, Proposition 3.5])**.** *Let $\underline{x} \in \underline{S}$. Then*

$$\underline{H}_{\underline{x}} = \sum_{\mathbf{e} \in [\underline{x}]} v^{d(\mathbf{e})} H_e.$$

2.2. **$*$-Hecke algebras and $(p|*)$-Hecke bimodules.**

Definition 2.3. For each $s \in S$, write

$$\begin{aligned} u_s : {}^pW &\longrightarrow \mathbb{Z}[v^{\pm 1}] \\ w &\longmapsto \begin{cases} v & \text{if } W_p ws = W_p w, \\ 1 & \text{otherwise.} \end{cases} \end{aligned}$$

Note that the function space $\mathbb{Z}[v^{\pm 1}]^{{}^pW}$ has a left W-action arising from the right W-action on ${}^pW$. The *toral coset algebra* ${}^p\mathbb{U}$ is the $\mathbb{Z}[v^{\pm 1}]$-subalgebra of $\mathbb{Z}[v^{\pm 1}]^{{}^pW}$ generated by $\{wu_s : s \in S \text{ and } w \in W\}$.

As its name suggests, the toral coset algebra is commutative. Recall that ${}^pW$ is in bijective correspondence with ${}^p\mathcal{A}$, the set of ordinary alcoves contained inside the fundamental p-alcove $A_{0,p}$. Under this correspondence u_s is the multiplicative indicator function for the subset of alcoves whose s-wall lies on one of the walls of $A_{0,p}$. Write

$$ {}^pW(s,*) = \{w \in {}^pW : W_p ws = W_p w\} \tag{2.4}$$

for the corresponding subset of coset representatives.

We can describe ${}^p\mathbb{U}$ in terms of generators and relations. Let ${}^p\mathcal{U}$ consist of all finite unions, intersections, and complements formed from the subsets ${}^pW(s,*)w \subseteq {}^pW$ for all $s \in S$ and $w \in W$. Similarly to u_s, for each $A \in {}^p\mathcal{U}$ let

$$\begin{aligned} u_A : {}^pW &\longrightarrow \mathbb{Z}[v^{\pm 1}] \\ w &\longmapsto \begin{cases} v & \text{if } w \in A, \\ 1 & \text{otherwise.} \end{cases} \end{aligned} \tag{2.5}$$

In this way $u_s = u_{{}^pW(s,*)}$.

Lemma 2.4. *The functions $\{u_A : A \in {}^p\mathcal{U}\} \subseteq {}^p\mathbb{U}$ satisfy the relations*

$$u_\emptyset = 1, \tag{2.6}$$

$$u_{{}^pW} = v, \tag{2.7}$$

$$u_A^2 = (v+1)u_A - v \qquad \text{for all } A \in {}^p\mathcal{U}, \tag{2.8}$$

$$u_A + u_B = u_{A\cup B} + u_{A\cap B} \qquad \text{for all } A, B \in {}^p\mathcal{U}, \tag{2.9}$$

$$u_A u_B = u_{A\cup B} u_{A\cap B} \qquad \text{for all } A, B \in {}^p\mathcal{U}, \tag{2.10}$$

where 1 *and* v *denote constant functions. This gives a presentation for* ${}^p\mathbb{U}$ *as a* $\mathbb{Z}[v^{\pm 1}]$*-algebra.*

Proof. Let U be the $\mathbb{Z}[v^{\pm 1}]$-algebra with the above presentation. It is easy to check that these relations hold in ${}^p\mathbb{U}$, so there is a homomorphism $U \to {}^p\mathbb{U}$. Now observe that ${}^p\mathcal{U}$ is a Boolean algebra under unions, intersections, and complements. By the Stone representation theorem [23], ${}^pW$ is partitioned by minimal subsets or *atoms* in ${}^p\mathcal{U}$, and every subset $A \in {}^p\mathcal{U}$ can be written as a disjoint union of atoms in a unique manner. As a consequence, U has a $\mathbb{Z}[v^{\pm 1}]$-basis $\{u_A : A \text{ an atom}\}$, and this maps to a $\mathbb{Z}[v^{\pm 1}]$-basis of ${}^p\mathbb{U}$, so U is isomorphic to ${}^p\mathbb{U}$. ☐

Definition 2.5. The *$*$-Hecke algebra* $\mathbb{H}_*$ is the $\mathbb{Z}[v^{\pm 1}]$-algebra with generators

$$u_A \qquad \text{for each } A \in {}^p\mathcal{U}, \tag{2.11}$$

$$H_s^{(*)} \qquad \text{for each } s \in S, \tag{2.12}$$

satisfying the relations (2.6)–(2.10) as well as

$$(H_s^{(*)})^2 = 1 + (u_s^{-1} - u_s)H_s \qquad \text{for all } s \in S, \tag{2.13}$$

$$\overbrace{H_s^{(*)}H_t^{(*)}H_s^{(*)}\cdots}^{m_{st}\text{ terms}} = \overbrace{H_t^{(*)}H_s^{(*)}H_t^{(*)}\cdots}^{m_{st}\text{ terms}} \qquad \text{for all } s,t \in S, \tag{2.14}$$

$$H_s^{(*)}u_A(H_s^{(*)})^{-1} = u_{As} \qquad \text{for all } s \in S \text{ and } A \in {}^p\mathcal{U}. \tag{2.15}$$

We follow many of the same notational conventions with $\mathbb{H}_*$ as we do with $\mathbb{H}$. For example, for any $w \in W$ with rex $\underline{w} = \underline{s_1 s_2 \cdots s_m}$ we write

$$H_w^{(*)} = H_{s_1}^{(*)}H_{s_2}^{(*)}\cdots H_{s_m}^{(*)},$$

which does not depend on the rex $\underline{w}$. There is a ring homomorphism $(\overline{}) : \mathbb{H}_* \to \mathbb{H}_*$ defined by

$$\begin{aligned} \overline{v} &= v^{-1} \\ \overline{u_s} &= u_s^{-1} \\ \overline{H_w^{(*)}} &= (H_{w^{-1}}^{(*)})^{-1} \end{aligned} \tag{2.16}$$

which we also call the bar involution. We also set $\underline{H}_s^{(*)} = H_s^{(*)} + u_s$, and for any $\underline{x} = \underline{s_1 s_2 \cdots s_m} \in \underline{S}$ write $\underline{H}_{\underline{x}}^{(*)} = \underline{H}_{s_1}^{(*)}\underline{H}_{s_2}^{(*)}\cdots\underline{H}_{s_m}^{(*)}$.

Definition 2.6. The $(p|*)$-*Hecke bimodule* $\mathbb{H}_{p|*}$ is the following $(\mathbb{H}_p \otimes_{\mathbb{Z}[v^{\pm 1}]} \mathbb{H}_*^{\mathrm{op}})$-module

$$\mathbb{H}_{p|*} = (\mathbb{H}_p \otimes_{\mathbb{Z}[v^{\pm 1}]} \mathbb{H}_*^{\mathrm{op}})/(1 \otimes H_{xw}^{(*)} - H_x^{(p)} \otimes H_w^{(*)} : x \in W,\ w \in {}^pW)$$

viewed as a $(\mathbb{H}_p, \mathbb{H}_*)$-bimodule.

The map

$$\begin{aligned} W_p \times {}^pW &\longrightarrow W \\ (x,w) &\longmapsto xw, \end{aligned}$$

is a bijection. Thus any objects indexed over $W_p \times {}^pW$ can just as easily be indexed over W. To emphasize this correspondence we will sometimes write such products in W with a bar in the form $x|w$, in order to emphasize that $x \in W_p$ and $w \in {}^pW$.

For $x \in W$ and $w \in {}^pW$ write $H_{x|w}^{(p|*)}$ for the image of $H_x^{(p)} \otimes H_w^{(*)}$ in $\mathbb{H}_{p|*}$. It is easy to check that these elements form a $\mathbb{Z}[v^{\pm 1}]$-basis for $\mathbb{H}_{p|*}$. The subset $\{H_w^{(p|*)} : w \in {}^pW\}$ form a basis for $\mathbb{H}_{p|*}$ as a free left $\mathbb{H}_p$-module. On this subset, the right $\mathbb{H}_*$-action is given by

$$(H_w^{(p|*)})u_A = \begin{cases} vH_w^{(p|*)} & \text{if } w \in A, \\ H_w^{(p|*)} & \text{otherwise,} \end{cases} \tag{2.17}$$

$$(H_w^{(p|*)})H_s^{(*)} = \begin{cases} H_{wsw^{-1}|w}^{(p|*)} & \text{if } W_p ws = W_p w, \\ H_{|ws}^{(p|*)} & \text{otherwise,} \end{cases} \tag{2.18}$$

for all $w \in {}^pW$, $A \in {}^p\mathcal{U}$, and $s \in S$.

Later it will be more convenient to view $\mathbb{H}_*$ as an algebra of endomorphisms of the left $\mathbb{H}_p$-module $\mathbb{H}_{p|*}$.

Lemma 2.7. *The right $\mathbb{H}_*$-action on $\mathbb{H}_{p|*}$ is faithful.*

Proof. Suppose $a \in \mathbb{H}_*$ such that for all $m \in \mathbb{H}_{p|*}$, we have $ma = 0$. From the relations defining $\mathbb{H}_*$, the set

$$\{u_A H_x^{(*)} : x \in W,\ A \in {}^p\mathcal{U}\}$$

is a $\mathbb{Z}[v^{\pm 1}]$-spanning set for $\mathbb{H}_*$. Now write

$$a = \sum_{i=1}^{n} p_i H_{x_i}^{(*)}$$

where $p_i \in {}^p\mathbb{U}$ and the $x_i \in W$ are distinct. The action of $H_x^{(*)}$ on the elements $\{H_w^{(p|*)} : w \in {}^pW\}$ of $\mathbb{H}_{p|*}$ is $H_w^{(p|*)} H_x^{(*)} = H_{y|z}^{(p|*)}$ where $y \in W_p$ and $z \in {}^pW$ such that $wx = yz$. Thus we have

$$H_w^{(p|*)} a = \sum_{i=1}^{n} p_i(w) H_{y_i|z_i}^{(p|*)}$$

where for each i, $y_i \in W_p$, $z_i \in {}^pW$ such that $wx_i = y_i z_i$, and p_i as a function ${}^pW \to \mathbb{Z}[v^{\pm 1}]$ maps $w \mapsto p_i(w) \in \mathbb{Z}[v^{\pm 1}]$. Since the elements $H_{y_i|z_i}^{(p|*)}$ are linearly independent, this means that we must have $p_i(w) = 0$ for each $w \in {}^pW$. But ${}^p\mathbb{U}$ is a subalgebra of the function algebra $\mathbb{Z}[v^{\pm 1}]^{{}^pW}$, so $p_i = 0$ and the right action is faithful. $\square$

2.3. **Patterns.** The goal of this section is to define combinatorial structures which give a Deodhar-like formula for calculating products in $\mathbb{H}_*$ and $\mathbb{H}_{p|*}$. These structures necessarily generalize expressions, subsequences, and defects.

Let $\underline{S}_{p|1}$ denote the following subset

$$\underline{S}_{p|1} = \{\underline{xw} : \underline{x} \in \underline{S}_p,\ \underline{w} \in \underline{S}\}$$

of expressions involving S- and S_p-generators. We call such expressions $(p|1)$-*expressions*. As with similar products in W, we will sometimes write expressions in $\underline{S}_{p|1}$ with a bar in the form $\underline{x}|\underline{w}$ in order to emphasize that $\underline{x} \in \underline{S}_p$ and $\underline{w} \in \underline{S}$. The set $\underline{S}_{p|1}$ inherits an $(\underline{S}_p, \underline{S})$-biaction structure from the (free) monoid structures on $\underline{S}_p$ and $\underline{S}$. We set $\ell(\underline{x}|\underline{w}) = \ell(\underline{x}) + \ell(\underline{w})$.

Definition 2.8. Let $\underline{x} = \underline{s_1 s_2 \cdots s_m} \in \underline{S}_{p|1}$. A *pattern* for $\underline{x}$ is a sequence of the form $\underline{\underline{r}} = (\underline{r}_1, \underline{r}_2, \ldots, \underline{r}_m)$, where each $\underline{r}_i$ is an ordered pair (s_i, t_i) with $t_i \in \{0, 1, *\}$. As with subsequences, $\underline{r}_i$ is called a *term* with *generator* s_i and *(pattern) type* t_i, and the type of $\underline{r}$ is the sequence of types of the terms $\underline{r}_i$. We call a term of type $*$ *indeterminate*; otherwise we call it *fixed*. We write $\hat{r}$ for the product of all the generators in $\underline{r}$ of type 1.

Patterns can be viewed as generalized expressions, with fixed terms restricting possible subsequences by pre-selecting certain generators to be included or discarded. In particular, we can think of an expression as a pattern whose terms are all indeterminate. In this case, note that we always have $\hat{r} = 1$.

Definition 2.9. Let $\underline{r} = (\underline{r}_1, \underline{r}_2, \ldots, \underline{r}_m)$ be a pattern for an expression $\underline{x} \in \underline{S}_{p|1}$. A *match* for the pattern $\underline{r}$ is a sequence of the form $\mathbf{c} = (\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_m)$, where each $\mathbf{c}_i$ is an ordered pair $(\underline{r}_i, t'_i)$ with $t'_i \in \{0, 1\}$ if $\underline{r}_i$ is indeterminate and $t'_i = \emptyset$ if $\underline{r}_i$ is fixed. As above $\mathbf{c}_i$ is called a *term* with *(match) type* t'_i, and the type of $\mathbf{c}$ is the sequence of types of the terms $\mathbf{c}_i$. Let $\mathbf{e_c} \in [\underline{x}]$ be the following subsequence. For each index i, set the type of $(\mathbf{e_c})_i$ to be the type of $\underline{r}_i$ or $\mathbf{c}_i$, according to whether $\underline{r}_i$ is fixed or indeterminate respectively. We write c for the group element $e_{\mathbf{c}}\hat{r}^{-1}$ and $\underline{r}$ to denote the set of matches for $\underline{r}$.

If we think of an expression $\underline{x}$ as a pattern whose terms are all indeterminate, then matches for $\underline{x}$ correspond bijectively with subsequences for $\underline{x}$ via $\mathbf{c} \mapsto \mathbf{e_c}$. This bijection preserves the evaluation map to W, since $c = e_{\mathbf{c}}\hat{r}^{-1} = e_{\mathbf{c}}$.

Definition 2.10. Let $w \in W$, and let $\mathbf{c}$ be a match for a pattern $\underline{r} = (\underline{r}_1, \underline{r}_2, \ldots, \underline{r}_m)$ for some expression in $\underline{S}$. For integers $0 \le i \le m$, let $\underline{r}_{\le i}$ denote the pattern made up of the first i terms of $\underline{r}$ and let $\mathbf{c}_{\le i}$ be the match of $\underline{r}_{\le i}$ made up of the first i terms of $\mathbf{c}$. The *w-twisted Bruhat stroll* on $\mathbf{c}$ is the sequence $w_i = wc_{\le i}w^{-1}$. For each index i, suppose the type of $\underline{r}_i$ is t_i and the type of $\mathbf{c}_i$ is t'_i The *decorated match type* t''_i is

$$t''_i = \begin{cases} \mathrm{U}t'_i & \text{if } t_i = * \text{ and } w_{i-1}(w\hat{r}_{\le i-1})s_i(w\hat{r}_{\le i-1})^{-1} > w_{i-1}, \\ \mathrm{D}t'_i & \text{if } t_i = * \text{ and } w_{i-1}(w\hat{r}_{\le i-1})s_i(w\hat{r}_{\le i-1})^{-1} < w_{i-1}, \\ t'_i t_i & \text{if } t_i \ne *. \end{cases}$$

and the decorated type of $\mathbf{c}$ is the sequence of decorated match types. The *w-twisted defect* $d_w(\mathbf{c})$ of $\mathbf{c}$ is equal to the number of indices with decorated match type U0 minus the number of indices with decorated match type D0. When $w = 1$ we write $d(\mathbf{c})$ for $d_1(\mathbf{c})$ and call it simply the defect of $\mathbf{c}$. If $\mathbf{e}$ is a subsequence of $\underline{x} \in \underline{S}_p$ and $\mathbf{c}$ is a match for a pattern $\underline{q}$ for some expression in $\underline{S}$, we also define $d(\mathbf{e}|\mathbf{c}) = d(\mathbf{e}) + d(\mathbf{c})$, where $\mathbf{e}|\mathbf{c}$ is interpreted as a match for the pattern $\underline{x}|\underline{q}$.

If we view $\mathbf{e} \in [\underline{x}]$ for some $\underline{x} \in \underline{S}$ as a match for a pattern, then the 1-twisted Bruhat stroll and defect are the same as Deodhar's Bruhat stroll and defect from §2.1. This implies that in the last case above, the integer $d(\mathbf{e}|\mathbf{c})$ is well defined no matter how $\mathbf{e}|\mathbf{c}$ is factored.

Example 2.11. Suppose $\Phi_{\mathrm{f}} = A_2$. For succinctness we will write the generators of W as 0, 1, and 2, with $S_{\mathrm{f}} = \{1, 2\}$ and $\tilde{s} = 0$, and write the identity element as id. Consider the expression $\underline{x} = \underline{01202101} \in \underline{S}$ and the element $w = 0 \in W$. Let $\underline{r}$ be a pattern for $\underline{x}$ of type $1{*}{*}1111{*}$, and let $\mathbf{c}$ be a match for $\underline{r}$ of type $\emptyset01\emptyset\emptyset\emptyset\emptyset0$. The w-twisted Bruhat stroll for this match is

$$\mathrm{id}, \mathrm{id}, \mathrm{id}, 2, 2, 2, 2, 2, 2$$

so the decorated type of the match is

$$(\emptyset1, \mathrm{U}0, \mathrm{U}1, \emptyset1, \emptyset1, \emptyset1, \emptyset1, \mathrm{D}0)$$

Thus the w-twisted defect $d_w(\mathbf{c})$ is $1 - 1 = 0$.

Given an expression $\underline{x}$, we now construct sets of patterns whose match sets *partition* $[\underline{x}]$ under the match-subsequence correspondence.

Definition 2.12. For $\underline{x} \in \underline{S}$, let $[\underline{x}]_*$ be the function mapping every coset representative in ${}^pW$ to a set of patterns for $\underline{x}$ defined inductively in the following manner. For the empty expression we set $[]_*$ to be the function mapping every coset representative to a singleton set containing the empty pattern. Now suppose $\underline{x} = \underline{ys}$ for some $s \in S$, where $[\underline{y}]_*$ is already known. Then we set

$$[\underline{x}]_* : w \longmapsto \bigcup_{\underline{r} \in [\underline{y}]_*(w)} \underline{r}[w\hat{r}, s]_*,$$

where

$$[z, s]_* = \begin{cases} \{(s, *)\} & \text{if } zsz^{-1} \in W_p, \\ \{(s, 0), (s, 1)\} & \text{otherwise.} \end{cases}$$

For each $w \in {}^pW$, the match sets $[\underline{r}]$ for $\underline{r} \in [\underline{x}]_*(w)$ induce a partition of $[\underline{x}]$. For $\underline{x}|\underline{y} \in \underline{S}_{p|1}$ we also define $[\underline{x}|\underline{y}]_{p|*} = \underline{x}[\underline{y}]_*(1)$.

Again we note that $[\underline{x}|\underline{y}]_{p|*}$ does not depend on the factorization of $\underline{x}|\underline{y} \in \underline{S}_{p|1}$. As with $\mathbb{H}_p$ and $\mathbb{H}_*$, let $\underline{H}^{(p|*)}_{\underline{x}|\underline{y}} = \underline{H}^{(p)}_{\underline{x}} \underline{H}^{(*)}_{\underline{y}}$.

Lemma 2.13.

(i) *Let $w \in {}^pW$ and $\underline{x} \in \underline{S}$. Then*

$$H_w^{(p|*)} \underline{H}_{\underline{x}}^{(*)} = \sum_{\substack{\underline{r} \in [\underline{x}]_*(w) \\ \mathbf{e} \in [\underline{r}]}} v^{d_w(\mathbf{e})} H_{we\hat{r}}^{(p|*)}.$$

(ii) *Let $\underline{x}|\underline{y} \in \underline{S}_{p|1}$. Then*

$$\underline{H}_{\underline{x}|\underline{y}}^{(p|*)} = \sum_{\substack{\underline{r} \in [\underline{x}|\underline{y}]_{p|*} \\ \mathbf{e} \in [\underline{r}]}} v^{d(\mathbf{e})} H_{e|\hat{r}}^{(p|*)}.$$

Proof. We only prove (i), as (ii) follows from (i) and Lemma 2.2 for $\mathbb{H}_p$. Induct on the length of $\underline{x}$. When $\ell(\underline{x}) = 0$ the result holds trivially. Now suppose $\ell(\underline{x}) = m$ and that the result holds for expressions length less than m. Write $\underline{x} = \underline{zs}$ for some $\underline{z} \in \underline{S}$ and $s \in S$. Then we have

$$\begin{aligned}
H_w^{(p|*)} \underline{H}_{\underline{x}}^{(*)} &= (H_w^{(p|*)} \underline{H}_{\underline{z}}^{(*)}) \underline{H}_s^{(*)} \\
&= \left(\sum_{\substack{\underline{q} \in [\underline{z}]_*(w) \\ \mathbf{f} \in [\underline{q}]}} v^{d_w(\mathbf{f})} H_{wf\hat{q}}^{(p|*)} \right) \underline{H}_s^{(*)} \\
&= \sum_{\substack{\underline{q} \in [\underline{z}]_*(w) \\ \mathbf{f} \in [\underline{q}] \\ W_p w\hat{q}s = W_p w\hat{q}}} v^{d_w(\mathbf{f})} (H_{wfw^{-1}}^{(p)} \underline{H}_{(w\hat{q})s(w\hat{q})^{-1}}^{(p)}) H_{w\hat{q}}^{(p|*)} \\
&\quad + \sum_{\substack{\underline{q} \in [\underline{z}]_*(w) \\ \mathbf{f} \in [\underline{q}] \\ W_p w\hat{q}s \neq W_p w\hat{q}}} v^{d_w(\mathbf{f})} (H_{wf\hat{q}s}^{(p|*)} + H_{wf\hat{q}}^{(p|*)}),
\end{aligned}$$

where the last equality follows because $wf\hat{q} \in W_p w\hat{q}$. This sum is equal to

$$\sum_{\substack{\underline{r} \in [\underline{x}]_*(w) \\ \mathbf{e} \in [\underline{r}] \\ \underline{r}_m \text{ of type } *}} v^{d_w(\mathbf{e})} H_{we\hat{r}}^{(p|*)} + \sum_{\substack{\underline{r} \in [\underline{x}]_*(w) \\ \mathbf{e} \in [\underline{r}] \\ \underline{r}_m \text{ not of type } *}} v^{d_w(\mathbf{e})} H_{we\hat{r}}^{(p|*)} = \sum_{\substack{\underline{r} \in [\underline{x}]_*(w) \\ \mathbf{e} \in [\underline{r}]}} v^{d_w(\mathbf{e})} H_{e|\hat{r}}^{(p|*)}$$

which proves the result. $\square$

3. Diagrammatic Soergel bimodules

3.1. **The diagrammatic Hecke category.** Let V be a realization of (W, S) and $R = S(V)$ the symmetric algebra in V. We view R as a polynomial algebra and define a grading on R by setting $\deg(V) = 2$. For a graded vector space M and some $m \in \mathbb{Z}$, we write $M(m)$ for the degree m grade shift of M, with grading $M(m)^i = M^{i+m}$. The algebra R inherits a graded W-action from V. We also identify S with a set of colors for the purposes of drawing pictures.

Definition 3.1 ([8, Definition 5.1])**.** A *Soergel diagram* is a finite decorated graph with boundary properly embedded into $\mathbb{R} \times [0, 1]$ with the following properties:

- the edges of a Soergel diagram are colored by S;
- the planar regions are labeled with polynomials in R;
- the interior vertices are of the following types (see also Figure 1):
  - s-colored univalent vertices for $s \in S$ (degree $+1$);
  - s-colored trivalent vertices for $s \in S$ (degree -1);

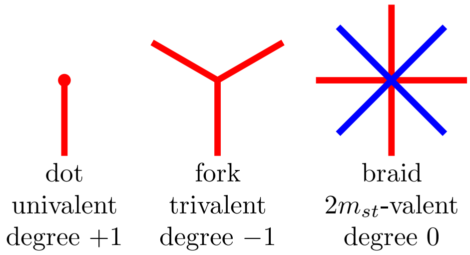


FIGURE 1. The three types of vertices in a Soergel diagram.

– s-and-t-colored $2m_{st}$-valent vertices for distinct $s, t \in S$ with $m_{st} < \infty$, with the edges alternating in color (degree 0).

The degree of a Soergel diagram is the sum of the degrees of all the vertices and the degrees of the polynomial labels. By convention we omit any labels $1 \in R$ for planar regions. The boundary points of a Soergel diagram lying in $\mathbb{R} \times \{0\}$ (resp. $\mathbb{R} \times \{1\}$) give an expression in $\underline{S}$, which we call the bottom (resp. top) boundary.

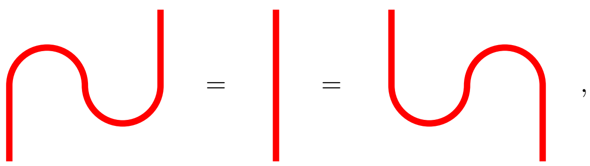
Definition 3.2 ([8, Definition 5.2])**.** The *diagrammatic Bott–Samelson category* $\mathcal{D}_{\mathrm{BS}}$ is the $\Bbbk$-linear monoidal category defined as follows.

Objects: For each expression $\underline{x} \in \underline{S}$ there is an object $B_{\underline{x}}$ in $\mathcal{D}_{\mathrm{BS}}$ called a *Bott–Samelson bimodule*. The tensor product of these objects is defined by $B_{\underline{x}} \otimes B_{\underline{y}} = B_{\underline{xy}}$.

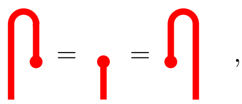
Morphisms: The morphism space $\mathrm{Hom}_{\mathcal{D}_{\mathrm{BS}}}(B_{\underline{x}}, B_{\underline{y}})$ is defined to be the set of $\Bbbk$-linear combinations of Soergel diagrams with bottom boundary $\underline{x}$ and top boundary $\underline{y}$, modulo some relations. Composition of morphisms is given by vertical concatenation, while the tensor product of morphisms is given by horizontal concatenation.

Relations: The morphisms between two Bott–Samelson bimodules satisfy the following relations (see also [8, (5.1)–(5.11)]). The diagrams in these relations should be viewed as generators for all the relations with respect to composition and tensor products. In other words, any *region* of a diagram can be simplified using these relations.

Isotopy: We only consider Soergel diagrams up to isotopy; Informally, this means edges can be moved continuously, e.g.

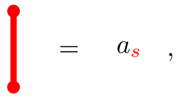


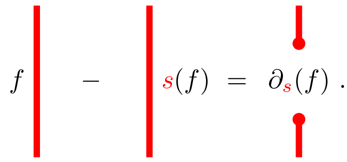
and that vertices can be twisted continuously, e.g.



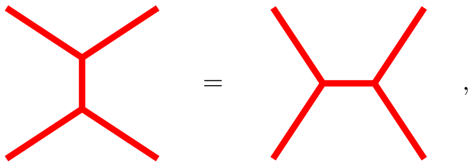
for a dot. The twisting-invariance of vertices is also called *cyclicity* [8, p. 325].

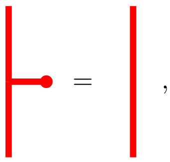
Polynomial relations: For each color (i.e. each generator $s \in S$) we have

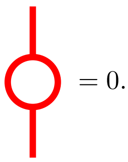
(3.1)


(3.2)


One-color relations: For each color we have

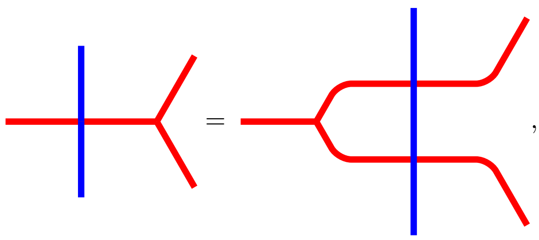
(3.3)


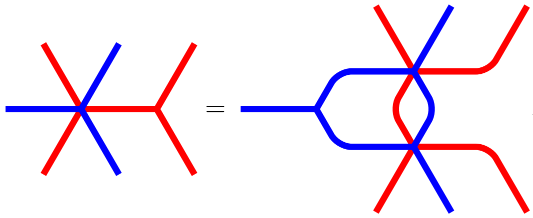
(3.4)


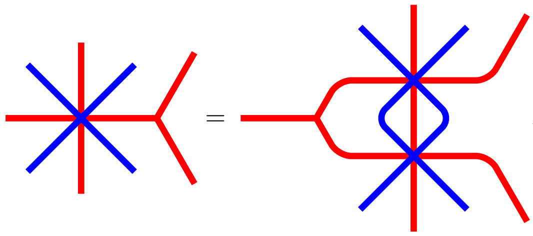
(3.5)


Two-color relations: For every finite rank 2 parabolic subgroup of W (i.e. for each $s, t \in S$ such that $m_{st} < \infty$) there are two relations called *two-color associativity* and the *Jones–Wenzl relation*. We will only depict the most common forms of these relations; for the general case, see [8, (5.6)–(5.7)] and [8, §5.2]. In the diagrams below s is colored red and t is colored blue.

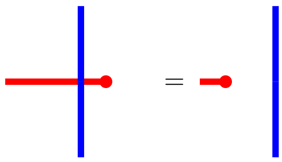
- Two-color associativity involves forks and braid vertices and does not depend on the realization, only on the order m_{st}. It has the following form for parabolics of Coxeter types $A_1 \times A_1$, A_2, and BC_2 (i.e. $m_{st} = 2, 3, 4$):

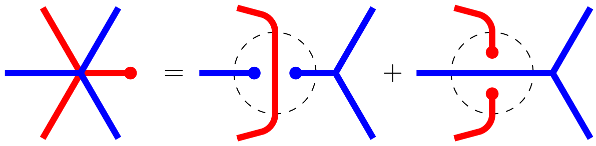
(3.6)


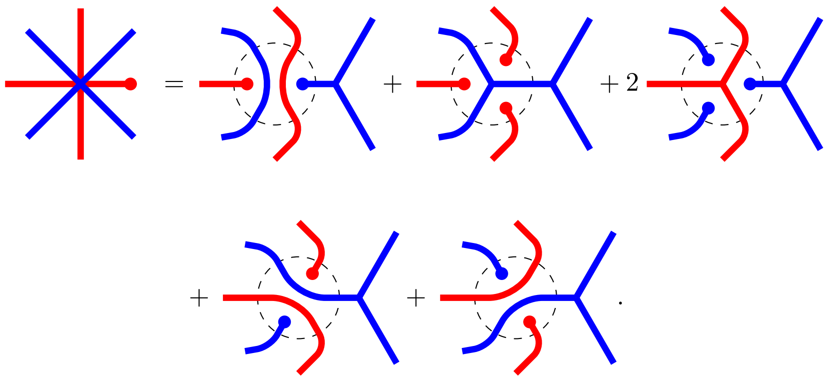
(3.7)  ,

(3.8)  .

- The Jones–Wenzl relation involves dots and braid vertices. Unlike two-color associativity it depends on the Cartan matrix of the realization. It has the following form for parabolics of Dynkin types $A_1 \times A_1$, A_2, and B_2 (for the last case, assume $a_{st} = -2$ and $a_{ts} = -1$, i.e. a_t corresponds to the short root vector):

(3.9)  ,

(3.10)  ,

(3.11)  .

For each relation, the linear combination of diagrams within the circular region is called a Jones–Wenzl morphism. It is not technically a morphism of Bott–Samelson bimodules, as the diagrams are embedded inside the disk instead of the strip $\mathbb{R} \times [0, 1]$

but they can be embedded into a disk-shaped region inside a Soergel diagram as in the relations.

Three-color relations: For each finite rank 3 parabolic subgroup of W there is a relation called the *Zamolodchikov relation*. We do not reproduce the diagrams here but instead point the reader to [8, (5.8)–(5.11)].

We write the Bott–Samelson bimodule corresponding to the empty expression as R. We will usually write Bott–Samelson bimodules corresponding to an expression of length one *without* an underline, i.e. B_s instead of $B_{\underline{s}}$ for $s \in S$. We will occasionally use the following non-diagrammatic notation for each possible vertex in a Soergel diagram. Namely, for each $s \in S$ we write

$$\mathrm{dot}_s : B_s \longrightarrow R$$
$$\mathrm{fork}_s : B_s \otimes B_s \longrightarrow B_s$$

for the morphisms corresponding to a fixed orientation of the s-colored dot and fork, and for all distinct $s, t \in S$ with $m_{st} < \infty$ we write

$$\mathrm{braid}_{s,t} : \overbrace{B_s \otimes B_t \otimes B_s \otimes \cdots}^{m_{st}} \longrightarrow \overbrace{B_t \otimes B_s \otimes B_t \otimes \cdots}^{m_{st}}$$

for the morphism corresponding to a fixed orientation of the s-and-t-colored braid.

There are left and right R-actions on each Hom-space induced by multiplication of the leftmost or rightmost label in each diagram. Thus $\mathcal{D}_{\mathrm{BS}}$ has the structure of an R-linear category. As R-modules the Hom-spaces are graded by the degree of the Soergel diagrams. We will write $\mathrm{Hom}^\bullet_{\mathcal{D}_{\mathrm{BS}}}$ for the Hom-space considered as a graded vector space, with $\mathrm{Hom}^i_{\mathcal{D}_{\mathrm{BS}}}$ denoting the morphisms of degree i.

Definition 3.3. The *diagrammatic Hecke category* $\mathcal{D}$ is the additive graded Karoubi envelope of $\mathcal{D}_{\mathrm{BS}}$. In other words $\mathcal{D}$ is the closure of $\mathcal{D}_{\mathrm{BS}}$ with respect to all finite direct sums, all direct summands, and all grade shifts of objects and morphisms in $\mathcal{D}_{\mathrm{BS}}$. Here grade shifts of objects are defined intrinsically; for objects B, B' in $\mathcal{D}$ and $m, n \in \mathbb{Z}$, the grade shifts of B and B' have the defining property that

$$\mathrm{Hom}^\bullet_{\mathcal{D}}(B(m), B'(n)) = \mathrm{Hom}^\bullet_{\mathcal{D}}(B, B')(n - m).$$

We call objects in $\mathcal{D}$ *(diagrammatic) Soergel bimodules*. We note the useful fact that the additive graded Karoubi envelope construction ensures that $\mathcal{D}_{\mathrm{BS}}$ is a *full* subcategory of $\mathcal{D}$. In other words,

$$\mathrm{Hom}^\bullet_{\mathcal{D}}(B_{\underline{x}}, B_{\underline{y}}) = \mathrm{Hom}^\bullet_{\mathcal{D}_{\mathrm{BS}}}(B_{\underline{x}}, B_{\underline{y}})$$

for all $\underline{x}, \underline{y} \in \underline{S}$, so there are no additional morphisms in $\mathcal{D}$ between two Bott–Samelson bimodules besides those in $\mathcal{D}_{\mathrm{BS}}$.

Finally, there is a degree-preserving duality functor $\overline{(\ )} : \mathcal{D}_{\mathrm{BS}} \to \mathcal{D}_{\mathrm{BS}}^{\mathrm{op}}$ on $\mathcal{D}_{\mathrm{BS}}$ defined as follows. For each $\underline{x} \in \underline{S}$ we have $\overline{B_{\underline{x}}} = B_{\underline{x}}$, and for any morphism $\phi : B \to B'$, the morphism $\overline{\phi} : \overline{B'} \to \overline{B}$ corresponds to flipping the diagrams representing ϕ upside-down. The duality functor extends to $\mathcal{D}$ and has the effect of reversing grade shift, i.e. $\overline{B(m)} = \overline{B}(-m)$.

3.2. **Light leaves and double leaves.** We briefly discuss bases for the Hom-spaces in $\mathcal{D}$, as described in [8, §6]. These bases yield a classification of the indecomposable objects in $\mathcal{D}$.

Let $\underline{x} \in \underline{S}$ and $\mathbf{e} \in [\underline{x}]$. Choose a rex $\underline{w}$ for e. A *light leaves morphism* $\mathrm{LL}_{\mathbf{e},\underline{w}} : B_{\underline{x}} \to B_{\underline{w}}$ is a morphism in $\mathcal{D}_{\mathrm{BS}}$ of degree $d(\mathbf{e})$, defined inductively in [8, Construction 6.1]. This construction depends on several non-canonical choices, and as such $\mathrm{LL}_{\mathbf{e},\underline{w}}$ is not uniquely defined, but this will not matter for what follows. For $\underline{x}, \underline{w} \in \underline{S}$ with $\underline{w}$ a rex, let $\mathrm{LL}_{[\underline{x}],\underline{w}}$ denote a complete selection of light leaves morphisms $\{\mathrm{LL}_{\mathbf{e},\underline{w}}\}$ over all subsequences $\mathbf{e} \in [\underline{x}]$ such that $e = w$.

Now suppose $\underline{x}, \underline{y} \in \underline{S}$, and that for each $w \in W$ we have fixed a corresponding rex $\underline{w}$. For subsequences $\mathbf{e} \in [\underline{x}]$ and $\mathbf{f} \in [\underline{y}]$ such that e and f are the same element $w \in W$, the *double leaves morphism* is defined to be

$$\mathbb{LL}_{\mathbf{e}}^{\mathbf{f}} = \overline{\mathrm{LL}_{\mathbf{f},\underline{w}}} \circ \mathrm{LL}_{\mathbf{e},\underline{w}} : B_{\underline{x}} \longrightarrow B_{\underline{y}}, \tag{3.12}$$

where $\underline{w}$ is the rex corresponding to w. We write $\mathbb{LL}_{[\underline{x}]}^{[\underline{y}]}$ to denote a complete selection of double leaves morphisms $B_{\underline{x}} \to B_{\underline{y}}$ over all such pairs of subsequences. Regardless of the realization of W and any choices made during the construction of the light leaves morphisms, we have the following theorem.

Theorem 3.4 ([8, Theorem 6.12])**.** *Let $\underline{x}, \underline{y} \in \underline{S}$. The double leaves morphisms $\mathbb{LL}_{[\underline{x}]}^{[\underline{y}]}$ form a basis for*

$$\mathrm{Hom}^{\bullet}_{\mathcal{D}}(B_{\underline{x}}, B_{\underline{y}}) = \mathrm{Hom}^{\bullet}_{\mathcal{D}}(B_{\underline{x}}, B_{\underline{y}})$$

as a graded left (or right) R-module.

There is also a light leaves variant of this basis result. Let $w \in W$, and take $I_{\not\geq w} = \{z \in W : z \not\geq w\} \subseteq W$ an ideal in the Bruhat order (i.e. $z \in I$ and $y \leq z$ implies $y \in I_{\not\geq w}$). Let $\mathbb{LL}_{\not\geq w}$ denote the span of the $\mathbb{LL}$ maps $\{\mathbb{LL}_{\mathbf{e}}^{\mathbf{f}} : e = f \in I_{\not\geq w}\}$. In other words, $\mathbb{LL}_{\not\geq w}$ is spanned by double leaves morphisms which factor through $I_{\not\geq w}$. By [8, Claim 6.19] $\mathbb{LL}_{\not\geq w}$ is a 2-sided ideal of morphisms in $\mathcal{D}_{\mathrm{BS}}$. We define $\mathcal{D}_{\mathrm{BS}}^{\geq w} = \mathcal{D}_{\mathrm{BS}}/\mathbb{LL}_{\not\geq w}$ and write $\mathcal{D}^{\geq w}$ for the additive graded Karoubi envelope of $\mathcal{D}_{\mathrm{BS}}^{\geq w}$. In [8, §6.5] Elias–Williamson show that the light leaves morphisms give a basis for certain Hom-spaces in $\mathcal{D}_{\mathrm{BS}}^{\geq w}$.

Theorem 3.5. *Let $\underline{x}, \underline{w} \in \underline{S}$ with $\underline{w}$ a rex. The light leaves morphisms $\mathrm{LL}_{[\underline{x}],\underline{w}}$ form a basis for $\mathrm{Hom}^{\bullet}_{\mathcal{D}_{\mathrm{BS}}^{\geq w}}(B_{\underline{x}}, B_{\underline{w}})$ as a graded left (or right) R-module.*

Another consequence of the double leaves basis is the following classification of the indecomposable Soergel bimodules.

Theorem 3.6 ([8, Theorem 6.26])**.** *Suppose $w \in W$, and let $\underline{w}$ be a rex for w. There is a unique indecomposable summand B_w of $B_{\underline{w}}$ which is not a summand of $B_{\underline{y}}$ for $\underline{y}$ a rex with $y < w$. Up to isomorphism, the object B_w does not depend on the choice of rex for w. Every indecomposable Soergel bimodule is isomorphic to a grade shift of B_w for some $w \in W$.*

Example 3.7. Let $s \in S$. By Theorem 3.4 the endomorphisms

form a left R-module basis for $\mathrm{End}_{\mathcal{D}}(B_{\underline{s}})$. It follows that $\mathrm{End}^{\bullet}_{\mathcal{D}}(B_{\underline{s}}) \cong R[x]/(x^2 - a_s x)$ as an R-module and as a $\Bbbk$-algebra, with x corresponding to the second basis element above. As x is of degree 2, this means that $\mathrm{End}^{0}_{\mathcal{D}}(B_{\underline{s}}) \cong \Bbbk$. Thus $B_{\underline{s}}$ is indecomposable, so $B_s = B_{\underline{s}}$.

For B a Soergel bimodule, write $[B]$ to denote the isomorphism class of B, and $[\mathcal{D}]$ for the Grothendieck ring of $\mathcal{D}$. Since $\mathcal{D}$ is graded, $[\mathcal{D}]$ has the structure of a $\mathbb{Z}[v^{\pm 1}]$-algebra, with $v = [R(1)]$. Theorem 3.5 tells us that for any rex $\underline{w}$ and any Soergel bimodule B, the left R-module $\mathrm{Hom}_{\mathcal{D}_{\mathrm{BS}}^{\geq w}}(B, B_{\underline{w}})$ is a graded projective R-module. Yet R is a polynomial algebra over a field $\Bbbk$, so in fact all projective

R-modules are free. The *diagrammatic character* [8, Definition 6.24] is the $\mathbb{Z}[v^{\pm 1}]$-linear map

$$\begin{aligned} \mathrm{ch} : [\mathcal{D}] &\longrightarrow \mathbb{H} \\ [B] &\longmapsto \sum_{w \in W} \operatorname{grk} \operatorname{Hom}^\bullet_{\mathcal{D}_{\geq w}}(B, B_w) H_w, \end{aligned}$$

where grk denotes the graded rank as a free R-module. A combination of Lemma 2.2, Theorem 3.5, and Theorem 3.6 then yields the following categorification result.

Corollary 3.8 ([8, Corollary 6.27])**.** *The map* $\mathrm{ch} : [\mathcal{D}] \to \mathbb{H}$ *is an isomorphism of* $\mathbb{Z}[v^{\pm 1}]$*-algebras.*

3.3. **Localization and mixed diagrams.** Let $Q = \operatorname{Frac} R$ be the fraction field of R. We write $Q \otimes_R \mathcal{D}_{\mathrm{BS}}$ for the (left) scalar extension of $\mathcal{D}_{\mathrm{BS}}$ to a Q-linear category. In $Q \otimes_R \mathcal{D}_{\mathrm{BS}}$ diagrams can have rational functions in Q as left coefficients. Since we can "push" coefficients through edges of a Soergel diagram using the relation (3.2), we can also consider rational right coefficients as well. In this paper we primarily work over a certain extension of this category which is closed under direct summands.

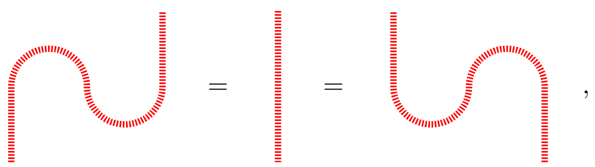
Definition 3.9 ([8, §5.4])**.** The *(diagrammatic) Bott–Samelson-standard category* $\mathcal{D}_{\mathrm{BS,std}}$, or the *mixed category* for short, is the following Q-linear monoidal extension of $Q \otimes_R \mathcal{D}_{\mathrm{BS}}$.

Objects: For each $\underline{x} \in \underline{S}$ add the object $Q_{\underline{x}}$, which is called a *(diagrammatic) standard bimodule*. As with Bott–Samelson bimodules the tensor product is defined by $Q_{\underline{x}} \otimes Q_{\underline{y}} = Q_{\underline{xy}}$.

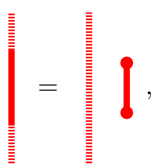
Morphisms: As in $\mathcal{D}_{\mathrm{BS}}$ the Hom-spaces are spanned by diagrams with some fixed bottom and top boundary. Here the diagrams are *mixed diagrams*, where some of the edges are dashed. Dashed edges on the top or bottom boundary denote standard bimodules in the domain or codomain respectively. There are two new morphisms between standard bimodules and Bott–Samelson bimodules, which are both of degree $+1$. These are drawn diagrammatically as *bivalent vertices*:

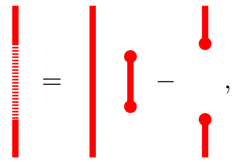
(3.13) 

Relations: In addition to isotopy of dashed edges, i.e.

(3.14) 

we add the following relations involving the bivalent vertices:

(3.15) 

(3.16) 

(3.17) 

Remark 3.10.

(1) Note that the relation (3.17) implies that the bivalent vertex is *not* cyclic! In other words, twisting a bivalent vertex by 180 degrees does not always result in the other bivalent vertex. Thankfully the failure of cyclicity is only up to a sign change. In particular our sign convention differs from that in [8] by a sign. This is to ensure that every menorah vertex in §5.1 is semi-cyclic.
(2) The "Q" in the notation $Q_{\underline{x}}$ for a standard bimodule is a holdover from non-diagrammatic Soergel bimodules, where each standard bimodule is isomorphic to Q as a left Q-module. This provides us with a mnemonic that this category is defined over Q.

For convenience we will abuse notation and generally write the Bott–Samelson object $Q \otimes_R B_{\underline{x}}$ in $\mathcal{D}_{\mathrm{BS,std}}$ as $B_{\underline{x}}$ whenever it is unambiguous, e.g. in the presence of standard bimodules $Q_{\underline{y}}$. We also write

$$\mathrm{bivalent}_s : B_s \longrightarrow Q_s$$

for the morphism corresponding to a fixed orientation of the s-colored bivalent vertex.

The mixed category also has an *R-form*, or an R-linear monoidal subcategory ${}_R\mathcal{D}_{\mathrm{BS,std}}$ which has the property that $Q \otimes_R {}_R\mathcal{D}_{\mathrm{BS,std}} \cong \mathcal{D}_{\mathrm{BS,std}}$ [8, p. 340]. Its objects are the same as in $\mathcal{D}_{\mathrm{BS,std}}$, and its Hom-spaces are spanned by mixed diagrams with coefficients in R. For consistency standard bimodules in the R-form are labeled $R_{\underline{x}}$ for $\underline{x} \in \underline{S}$.

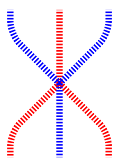
We call morphisms between standard bimodules *standard*. The most important standard morphisms in $\mathcal{D}_{\mathrm{BS,std}}$. For each $s \in S$, the *s-colored standard cap* stdcap_s and *s-colored standard cup* stdcup_s are the morphisms corresponding to the diagrams

, 

respectively. For all distinct $s, t \in S$ with $m_{st} < \infty$, the *s-and-t-colored standard braid* is a morphism

$$\mathrm{stdbraid}_{s,t} : \overbrace{Q_s \otimes Q_t \otimes Q_s \otimes \cdots}^{m_{st}} \longrightarrow \overbrace{Q_t \otimes Q_s \otimes Q_t \otimes \cdots}^{m_{st}}$$

defined in [8, (5.27)]. It is depicted as a dashed version of the braid vertex, e.g.



when s is colored red, t is colored blue, and $m_{st} = 3$. The following result about these morphisms is a consequence of [8, Proposition 5.23].

Proposition 3.11.

(i) *For every $s \in S$, $\mathrm{stdcap}_s : Q_{\underline{ss}} \to Q$ and $\mathrm{stdcup}_s : Q \to Q_{\underline{ss}}$ are mutually inverse isomorphisms in $\mathcal{D}_{\mathrm{BS,std}}$.*

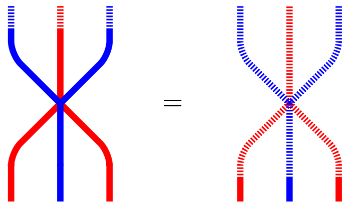
, 

(ii) *For all distinct $s, t \in S$ with $m_{st} < \infty$, we have*
- $\overline{\mathrm{stdbraid}_{s,t}} = \mathrm{stdbraid}_{t,s}$*;*
- $\mathrm{stdbraid}_{s,t}$ *and* $\mathrm{stdbraid}_{t,s}$ *are mutually inverse isomorphisms in* $\mathcal{D}_{\mathrm{BS,std}}$*;*
- *the morphism* $\mathrm{stdbraid}_{s,t}$ *is cyclic;*
- *the relation*

$$\overbrace{(\mathrm{bivalent}_t \otimes \mathrm{bivalent}_s \otimes \mathrm{bivalent}_t \otimes \cdots)}^{m_{st}} \circ \mathrm{braid}_{s,t} = \mathrm{stdbraid}_{s,t} \circ \overbrace{(\mathrm{bivalent}_s \otimes \mathrm{bivalent}_t \otimes \mathrm{bivalent}_s \otimes \cdots)}^{m_{st}} \tag{3.18}$$

holds.

Diagrammatically, (3.18) looks like



illustrated here in the case $m_{st} = 3$.

An immediate consequence of this result is that if $\underline{x}, \underline{y} \in \underline{S}$ such that $x = y$, then $Q_{\underline{x}} \cong Q_{\underline{y}}$. Thus we may label each standard bimodule Q_x up to isomorphism by an element $x \in W$ instead of an expression. In fact, *any* diagram between standard bimodules whose vertices are all standard braids and has no non-trivial coefficients is an isomorphism. We call such diagrams *basic standard*. These isomorphisms span all morphisms by the following result.

Theorem 3.12 ([8, Theorem 4.8])**.** *Let $\underline{x}, \underline{y} \in \underline{S}$.*

(i) *If $x = y$, then all basic standard diagrams with top $\underline{x}$ and bottom $\underline{y}$ are equal in $\mathcal{D}_{\mathrm{BS,std}}$, and any such diagram spans the entire* Hom*-space in $\mathcal{D}_{\mathrm{BS,std}}$ as a left or right Q-module, i.e.* $\mathrm{Hom}_{\mathcal{D}_{\mathrm{BS,std}}}(Q_{\underline{x}}, Q_{\underline{y}}) \cong Q$*.*

(ii) *If $x \neq y$, then* $\mathrm{Hom}_{\mathcal{D}_{\mathrm{BS,std}}}(Q_x, Q_y) = 0$*.*

If $\underline{x}, \underline{y} \in \underline{S}$ are rexes with $x = y$, we call a basic standard diagram with top $\underline{x}$ and bottom $\underline{y}$ a *rex move*. We write $\mathcal{D}_{\mathrm{std}}$ for the full subcategory of standard bimodules.

From (3.15) and (3.16) we observe that after rescaling by a_s^{-1}, the composition of two bivalent vertices gives an idempotent endomorphism on B_s, with a complementary idempotent formed from two dot vertices. Thus the Bott–Samelson bimodule B_s decomposes as a direct sum $Q_s \oplus Q$. By induction, for $\underline{x} \in \underline{S}$ we obtain the following decomposition (see also [8, §5.5])

$$B_{\underline{x}} \cong \bigoplus_{\mathbf{e} \in [\underline{x}]} Q_e. \tag{3.19}$$

A choice of isomorphism $B_s \to Q_s \oplus Q$ corresponds to a choice of factorizations for the bivalent- and dot-formed idempotents into inclusions composed with projections. However, the extra a_s^{-1} factor in these idempotents is *not* a square in Q, so it cannot be "shared" between the projection and the inclusion; it must associate with either the projection or the inclusion, but not both. Rather than accept asymmetry, we will introduce the following notation.

Notation 3.13. Let $f, g \in Q$. We write $\binom{f}{g}$ in a region of a mixed diagram to indicate a *bi-valued* scalar, i.e. a possible choice between the scalars f or g in this region. Choices $\binom{f_1}{g_1}, \binom{f_2}{g_2}, \ldots$ across multiple regions must be consistent, i.e. all upper scalars $f_1, f_2, \ldots$ or all lower scalars $g_1, g_2, \ldots$. In other words, a mixed diagram with bi-valued scalars in some regions is actually shorthand for *two* diagrams. We also define $\deg \binom{f}{g} = \frac{1}{2}(\deg f + \deg g)$ and set $\overline{\binom{f}{g}} = \binom{g}{f}$ for the purposes of dualizing.

Using this notation, the following diagrams

(3.20) $\binom{a_s^{-1}}{1}$,  $\binom{a_s^{-1}}{1}$,

(3.21) $\binom{1}{a_s^{-1}}$,  $\binom{1}{a_s^{-1}}$,

describe projections from B_s and the inclusions to B_s respectively after associating the scalar a_s^{-1} with the projections (upper scalars) or the inclusions (lower scalars).

Definition 3.14. Let $\underline{x}, \underline{y} \in \underline{S}$, and suppose $\phi : B_{\underline{x}} \to B_{\underline{y}}$ is a morphism in $\mathcal{D}_{\mathrm{BS}}$. The *localization matrix* of ϕ is a bi-valued $[\underline{y}] \times [\underline{x}]$ matrix of standard morphisms, whose $(\mathbf{f}, \mathbf{e})$-entry is the composition

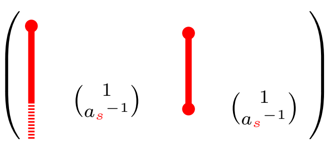
$$Q_e \xrightarrow{i} B_{\underline{x}} \xrightarrow{\phi} B_{\underline{y}} \xrightarrow{p} Q_f,$$

where i and p above are tensor products of the inclusion maps (3.21) and projection maps (3.20) respectively. When it is clear from context, we will omit the basic standard diagrams and only write the left scalar coefficients in the entries of the localization matrix.

Sometimes it will be useful to localize only some of the generators in the domain or codomain. In such situations we say that a generator has been *standardized* if a bivalent inclusion or projection map has been added to it (i.e. it corresponds to a subsequence term of type 1).

Example 3.15. The localization matrix of the dot morphism

is

 $= \left(0 \quad \binom{a_s}{1}\right)$.

The localization matrix of a diagram is well-defined, not just up to sign, since isotopy of solid colored edges is still a relation in $\mathcal{D}_{\mathrm{BS,std}}$. More importantly, localization is faithful, i.e. two morphisms in $\mathcal{D}_{\mathrm{BS}}$ are equal if and only if they have the same localization matrices [8, §5.5]. Localization matrices also satisfy the following algebraic properties.

Lemma 3.16. *Let ϕ be a morphism in $\mathcal{D}_{\mathrm{BS}}$.*

(i) *If ϕ is homogeneous, then all entries of the localization matrix of ϕ have degree equal to* $\deg \phi$.

(ii) *The localization matrix of $\overline{\phi}$ is equal to the transpose-dual of the localization matrix of ϕ.*

Remark 3.17. In [8, §5.5] Elias–Williamson use single-valued matrices for computations involving localization. These cannot be compared directly with localization matrices because we have used a different sign convention for $\mathcal{D}_{\mathrm{BS,std}}$ (see Remark 3.10), but taking upper scalar values everywhere gives a matrix more closely resembling their conventions.

4. The Frobenius functor

For the rest of this paper, we fix $\Bbbk$ to be a field of positive characteristic $p \neq 2$. Let V be the universal realization of (W, S) over $\Bbbk$ with respect to Σ, and let $\mathcal{D}_{\mathrm{BS}}$, $\mathcal{D}$, $\mathcal{D}_{\mathrm{BS,std}}$, and $\mathcal{D}_{\mathrm{std}}$ be the associated categories over this realization defined in the previous section. Recall that the F-twist of V is denoted V^F and by Proposition 1.13 is a realization too. Write $\mathcal{D}^F_{\mathrm{BS}}$, $\mathcal{D}^F$, and $\mathcal{D}^F_{\mathrm{std}}$ for the associated categories over V^F. In general we will write a superscript F to denote any Soergel bimodule construction which is applied using the realization V^F, such as B^F_s for a Bott–Samelson bimodule in $\mathcal{D}^F_{\mathrm{BS}}$.

The goal of this section is to construct a monoidal functor F which embeds $\mathcal{D}^F_{\mathrm{BS}}$ inside $\mathcal{D}_{\mathrm{BS,std}}$. The action of this functor on the degraded Grothendieck rings of the respective categories (which are both just $\mathbb{Z}W$) is the Frobenius map, so we call F the *Frobenius functor*. We first describe how this functor localizes.

4.1. Standard Frobenius diagrammatics.

The following objects are essentially "p-dilated" versions of Q_s inside $\mathcal{D}_{\mathrm{std}}$. Note from Example 1.12 that

$$(\tilde{s}s_{\alpha_{\mathrm{h}}})^{(p-1)/2}\tilde{s}(s_{\alpha_{\mathrm{h}}}\tilde{s})^{(p-1)/2} = \tilde{s}_p = F(\tilde{s}),$$

where $s_{\alpha_{\mathrm{h}}} \in W_{\mathrm{f}}$ denotes the reflection in the highest short root $\alpha_{\mathrm{h}} \in \Phi_{\mathrm{f}}$.

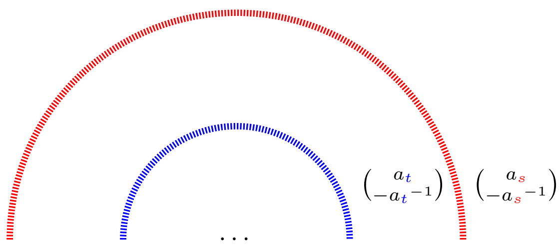
Definition 4.1. Fix a rex $\underline{s_{\alpha_{\mathrm{h}}}} \in \underline{S_{\mathrm{f}}}$ for the reflection $s_{\alpha_{\mathrm{h}}}$. For each $s \in S$ we write $Q^{(F)}_s$ to mean the following object in $\mathcal{D}_{\mathrm{std}}$:

$$Q^{(F)}_s = \begin{cases} Q_s & \text{if } s \in S_{\mathrm{f}}, \\ (Q_{\underline{\tilde{s}s_{\alpha_{\mathrm{h}}}}})^{\otimes(p-1)/2} \otimes Q_{\underline{\tilde{s}}} \otimes (Q_{\underline{s_{\alpha_{\mathrm{h}}}\tilde{s}}})^{\otimes(p-1)/2} & \text{if } s = \tilde{s}, \end{cases}$$

For $\underline{x} = \underline{s_1 s_2 \cdots s_m} \in \underline{S}$ we write $Q^{(F)}_{\underline{x}} = Q^{(F)}_{s_1} \otimes Q^{(F)}_{s_2} \otimes \cdots Q^{(F)}_{s_m}$.

We define certain morphisms on the objects $Q^{(F)}_{\underline{x}}$ which play the same role as stdcup^F_s and $\mathrm{stdbraid}^F_{s,t}$ in $\mathcal{D}^F_{\mathrm{std}}$.

Definition 4.2. For $\underline{w} = \underline{st\cdots} \in \underline{S}$, let

$\epsilon_{\underline{w}} =$ 

,

a bi-valued morphism in $\mathcal{D}_{\mathrm{std}}$.

(1) For each $s \in S$ we write $\mathrm{stdcap}_s^{(F)} : Q_{\underline{ss}}^{(F)} \to Q$ for the following bi-valued morphism in $\mathcal{D}_{\mathrm{std}}$. If $s \in S_{\mathrm{f}}$ then $\mathrm{stdcap}_s^{(F)} = \mathrm{stdcap}_s$. Otherwise if $s = \tilde{s}$ we set

$$\begin{aligned}\mathrm{stdcap}_{\tilde{s}}^{(F)} = \epsilon_{\underline{(\tilde{s}s_{\alpha_{\mathrm{h}}})}^{(p-1)/2}} \circ (\mathrm{id}_{Q_{\underline{\tilde{s}s_{\alpha_{\mathrm{h}}}}}}^{\otimes(p-1)/2} \otimes \mathrm{stdcap}_{\tilde{s}} \otimes \mathrm{id}_{Q_{\underline{s_{\alpha_{\mathrm{h}}}\tilde{s}}}}^{\otimes(p-1)/2}) \\ \circ (\mathrm{id}_{Q_{\underline{\tilde{s}s_{\alpha_{\mathrm{h}}}}}}^{\otimes(p-1)/2} \otimes \mathrm{id}_{Q_{\tilde{s}}} \otimes \epsilon_{\underline{(s_{\alpha_{\mathrm{h}}}\tilde{s})}^{(p-1)/2}} \otimes \mathrm{id}_{Q_{\tilde{s}}} \otimes \mathrm{id}_{Q_{\underline{s_{\alpha_{\mathrm{h}}}\tilde{s}}}}^{\otimes(p-1)/2}).\end{aligned}$$

We also set $\mathrm{stdcup}_s^{(F)} = \overline{\mathrm{stdcap}_s^{(F)}}$ for all $s \in S$.

(2) Suppose $s \in S_{\mathrm{f}}$ such that $m_{s\tilde{s}} < \infty$. Fix a standard diagram (with no scalar factors in any region) depicting an isomorphism

$$\overbrace{Q_{\tilde{s}}^{(F)} \otimes Q_s^{(F)} \otimes Q_{\tilde{s}}^{(F)} \otimes \cdots}^{m_{s\tilde{s}}} \longrightarrow \overbrace{Q_s^{(F)} \otimes Q_{\tilde{s}}^{(F)} \otimes Q_s^{(F)} \otimes \cdots}^{m_{s\tilde{s}}}$$

with the minimal number of standard cups, caps, and braids. This is equivalent to fixing a minimal length sequence of Coxeter relations showing that $\tilde{s}s\tilde{s}\cdots = s\tilde{s}s\cdots$. We write $\mathrm{stdbraid}_{s,\tilde{s}}^{(F)}$ for the above morphism rescaled as follows: add a scalar factor of $\left(\begin{smallmatrix} a_t \\ -a_t^{-1} \end{smallmatrix}\right)$ to the right of any t-colored standard cap, and a dual scalar factor of $\left(\begin{smallmatrix} -a_t^{-1} \\ a_t \end{smallmatrix}\right)$ to the right of any t-colored standard cup. We also set $\mathrm{stdbraid}_{\tilde{s},s}^{(F)} = \overline{\mathrm{stdbraid}_{s,\tilde{s}}^{(F)}}$. We similarly write $\mathrm{stdbraid}_{s,t}^{(F)} = \mathrm{stdbraid}_{s,t}$ for all distinct $s, t \in S_{\mathrm{f}}$.

Example 4.3. Let $p = 3$ and $\Phi_{\mathrm{f}} = A_2$, and let us follow the same convention as Example 2.11 in labeling the generators. Set $\underline{s_{\alpha_{\mathrm{h}}}} = \underline{121}$, and color the generators $0, 1, 2$ blue, red, and green respectively. Figure 2 depicts the morphisms $\mathrm{stdcap}_0^{(F)}$ and $\mathrm{stdbraid}_{0,1}^{(F)}$.

In [8, Definition 4.3] Elias–Williamson give a presentation of $\mathcal{D}_{\mathrm{std}}$. The next result shows that ϵ_s satisfies the same relations as stdcap_s does in this presentation, up to sign.

Lemma 4.4.

(i) *For any $s \in S$, we have*

$$\epsilon_{\underline{s}} \circ \overline{\epsilon_{\underline{s}}} = -\mathrm{id}_Q, \qquad \overline{\epsilon_{\underline{s}}} \circ \epsilon_{\underline{s}} = -\mathrm{id}_Q, \tag{4.1}$$

$$(\epsilon_{\underline{s}} \otimes \mathrm{id}_{Q_s}) \circ (\mathrm{id}_{Q_s} \otimes \overline{\epsilon_{\underline{s}}}) = \mathrm{id}_{Q_s}, \qquad (\mathrm{id}_{Q_s} \otimes \epsilon_{\underline{s}}) \circ (\overline{\epsilon_{\underline{s}}} \otimes \mathrm{id}_{Q_s}) = \mathrm{id}_{Q_s}. \tag{4.2}$$

(ii) *Suppose $s, t \in S$ are distinct with $m_{st} < \infty$. If m_{st} is even, then*

$$(\mathrm{id} \otimes \epsilon_{\underline{s}}) \circ (\mathrm{id}_{Q_s} \otimes \mathrm{stdbraid}_{s,t} \otimes \mathrm{id}_{Q_s}) \circ (\overline{\epsilon_{\underline{s}}} \otimes \mathrm{id}) = \mathrm{stdbraid}_{t,s}, \tag{4.3}$$

$$(\epsilon_{\underline{t}} \otimes \mathrm{id}) \circ (\mathrm{id}_{Q_t} \otimes \mathrm{stdbraid}_{s,t} \otimes \mathrm{id}_{Q_t}) \circ (\mathrm{id} \otimes \overline{\epsilon_{\underline{t}}}) = \mathrm{stdbraid}_{t,s}. \tag{4.4}$$

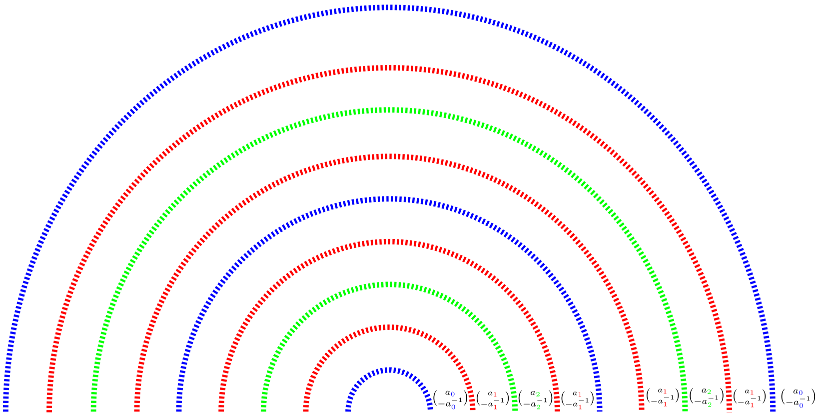
If m_{st} is odd, then

$$(\mathrm{id} \otimes \epsilon_{\underline{t}}) \circ (\mathrm{id}_{Q_s} \otimes \mathrm{stdbraid}_{s,t} \otimes \mathrm{id}_{Q_t}) \circ (\overline{\epsilon_{\underline{s}}} \otimes \mathrm{id}) = \mathrm{stdbraid}_{t,s}, \tag{4.5}$$

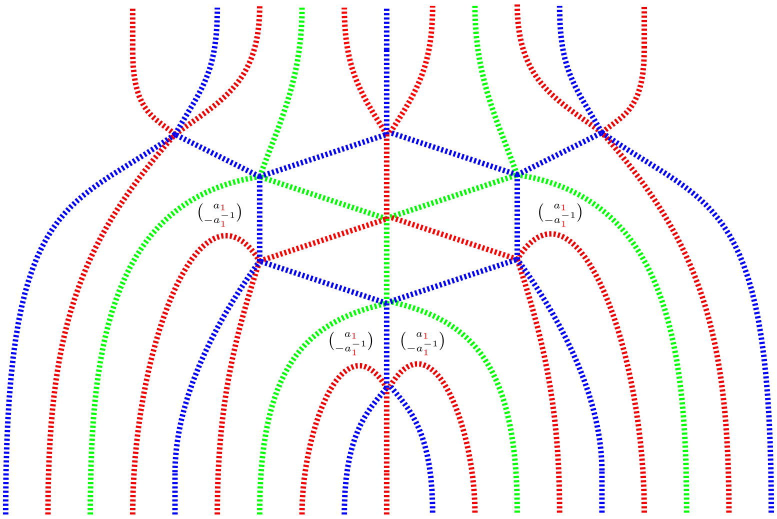
$$(\epsilon_{\underline{t}} \otimes \mathrm{id}) \circ (\mathrm{id}_{Q_t} \otimes \mathrm{stdbraid}_{s,t} \otimes \mathrm{id}_{Q_s}) \circ (\mathrm{id} \otimes \overline{\epsilon_{\underline{s}}}) = \mathrm{stdbraid}_{t,s}. \tag{4.6}$$

Proof. For any $s \in S$ we check that

$$\begin{aligned}\epsilon_{\underline{s}} \circ \overline{\epsilon_{\underline{s}}} &= \begin{pmatrix} a_s \\ -a_s^{-1} \end{pmatrix} \begin{pmatrix} -a_s^{-1} \\ a_s \end{pmatrix} \mathrm{id}_Q \\ &= -\mathrm{id}_Q,\end{aligned}$$



(A) $\mathrm{stdcap}_0^{(F)}$



(B) $\mathrm{stdbraid}_{0,1}^{(F)}$

FIGURE 2. Examples of $\mathrm{stdcap}_s^{(F)}$ and $\mathrm{stdbraid}_{s,t}^{(F)}$ morphisms for $p = 3$ and $\Phi_{\mathrm{f}} = A_2$.

and

$$\begin{aligned}\overline{\epsilon_{\underline{s}}} \circ \epsilon_{\underline{s}} &= \begin{pmatrix} -a_s^{-1} \\ a_s \end{pmatrix} \begin{pmatrix} a_s \\ -a_s^{-1} \end{pmatrix} \mathrm{id}_{Q_{\underline{ss}}} \\ &= -\mathrm{id}_{Q_{\underline{ss}}}.\end{aligned}$$

We also have

$$\begin{aligned}(\epsilon_{\underline{s}} \otimes \mathrm{id}_{Q_s}) \circ (\mathrm{id}_{Q_s} \otimes \overline{\epsilon_{\underline{s}}}) &= \begin{pmatrix} a_s \\ -a_s^{-1} \end{pmatrix} \begin{pmatrix} a_s^{-1} \\ -a_s \end{pmatrix} (\mathrm{stdcap}_s \otimes \mathrm{id}_{Q_s}) \circ (\mathrm{id}_{Q_s} \otimes \mathrm{stdcup}_s) \\ &= \mathrm{id}_{Q_s}, \\ (\mathrm{id}_{Q_s} \otimes \epsilon_{\underline{s}}) \circ (\overline{\epsilon_{\underline{s}}} \otimes \mathrm{id}_{Q_s}) &= \begin{pmatrix} -a_s \\ a_s^{-1} \end{pmatrix} \begin{pmatrix} -a_s^{-1} \\ a_s \end{pmatrix} (\mathrm{id}_{Q_s} \otimes \mathrm{stdcap}_s) \circ (\mathrm{stdcup}_s \otimes \mathrm{id}_{Q_s}) \\ &= \mathrm{id}_{Q_s}.\end{aligned}$$

If $t \in S$ with $s \neq t$ and $m_{st} < \infty$, then

$$\underbrace{\cdots tst}_{m_{st}}(a_s) = -a_s.$$

So if m_{st} is even, we have

$$\begin{aligned}&(\mathrm{id}\otimes\epsilon_{\underline{s}})\circ(\mathrm{id}_{Q_s}\otimes\mathrm{stdbraid}_{st}\otimes\mathrm{id}_{Q_s})\circ(\overline{\epsilon_{\underline{s}}}\otimes\mathrm{id})\\&=\begin{pmatrix}-a_s\\a_s^{-1}\end{pmatrix}\begin{pmatrix}-a_s^{-1}\\a_s\end{pmatrix}(\mathrm{id}\otimes\mathrm{stdcap}_s)\circ(\mathrm{id}_{Q_s}\otimes\mathrm{stdbraid}_{st}\otimes\mathrm{id}_{Q_s})\circ(\mathrm{stdcup}_s\otimes\mathrm{id})\\&=\mathrm{stdbraid}_{ts},\end{aligned}$$

as well as

$$\begin{aligned}&(\epsilon_{\underline{t}}\otimes\mathrm{id})\circ(\mathrm{id}_{Q_t}\otimes\mathrm{stdbraid}_{st}\otimes\mathrm{id}_{Q_t})\circ(\mathrm{id}\otimes\overline{\epsilon_{\underline{t}}})\\&=\begin{pmatrix}a_t\\-a_t^{-1}\end{pmatrix}\begin{pmatrix}-a_t^{-1}\\-a_t\end{pmatrix}(\mathrm{stdcap}_t\otimes\mathrm{id})\circ(\mathrm{id}_{Q_t}\otimes\mathrm{stdbraid}_{st}\otimes\mathrm{id}_{Q_t})\circ(\mathrm{id}\otimes\mathrm{stdcup}_t)\\&=\mathrm{stdbraid}_{ts}.\end{aligned}$$

The odd case is similar. □

Theorem 4.5. *There is a well-defined monoidal functor $F:\mathcal{D}^F_{\mathrm{std}}\to\mathcal{D}_{\mathrm{std}}$ which for all $s\in S$ maps*

$$\begin{aligned}Q_s^F&\longmapsto Q_s^{(F)}\\a_s^F\mathrm{id}_{Q^F}&\longmapsto a_s^F\mathrm{id}_Q\\v_{\mathrm{fix}}\mathrm{id}_{Q^F}&\longmapsto v_{\mathrm{fix}}\mathrm{id}_Q\\\mathrm{stdcap}_s^F&\longmapsto\mathrm{stdcap}_s^{(F)}\\\mathrm{stdcup}_s^F&\longmapsto\mathrm{stdcup}_s^{(F)}\end{aligned}$$

and for all distinct $s,t\in S$ such that $m_{st}<\infty$ maps

$$\mathrm{stdbraid}_{st}^F\longmapsto\mathrm{stdbraid}_{st}^{(F)}.$$

Proof. We prove that the mapping above is a well-defined functor using the generators and relations for $\mathcal{D}^F_{\mathrm{std}}$ introduced in [8, Definition 4.3]. The only non-trivial relations to check are the ones involving $Q_{\tilde{s}}^{(F)}$. The most basic of these is the biadjunction equality [8, p. 325], which follows from (4.2):

$$\begin{aligned}(\mathrm{stdcap}_{\tilde{s}}^{(F)}\otimes\mathrm{id}_{Q_{\tilde{s}}^{(F)}})\circ(\mathrm{id}_{Q_{\tilde{s}}^{(F)}}\otimes\mathrm{stdcup}_{\tilde{s}}^{(F)})&=\mathrm{id}_{Q_{\tilde{s}}^{(F)}}\\&=(\mathrm{id}_{Q_{\tilde{s}}^{(F)}}\otimes\mathrm{stdcap}_{\tilde{s}}^{(F)})\circ(\mathrm{stdcup}_{\tilde{s}}^{(F)}\otimes\mathrm{id}_{Q_{\tilde{s}}^{(F)}}).\end{aligned}$$

The polynomial relation [8, (4.1)] follows from the fact that $Q_{\tilde{s}}^{(F)}\cong Q_{\tilde{s}_p}$ and $\tilde{s}_p=F(\tilde{s})$. Relations [8, (4.2)–(4.3)] follow from (4.1):

$$\begin{aligned}\mathrm{stdcap}_{\tilde{s}}^{(F)}\circ\mathrm{stdcup}_{\tilde{s}}^{(F)}&=(-1)^{2\ell(\tilde{s}s_{\alpha_{\mathrm{h}}})}\mathrm{id}_Q\\&=\mathrm{id}_Q,\\\mathrm{stdcup}_{\tilde{s}}^{(F)}\circ\mathrm{stdcap}_{\tilde{s}}^{(F)}&=(-1)^{2\ell(\tilde{s}s_{\alpha_{\mathrm{h}}})}\mathrm{id}_{Q^{(F)}_{\underline{\tilde{s}\tilde{s}}}}\\&=\mathrm{id}_{Q^{(F)}_{\underline{\tilde{s}\tilde{s}}}}.\end{aligned}$$

Next suppose $s\in S_{\mathrm{f}}$ with $m_{\tilde{s}s}<\infty$. We check the cyclicity of the standard braid vertex and relation [8, (4.4)]. There are two cases depending on the parity of $m_{\tilde{s}s}$.

If $m_{\tilde{s}s}$ is even, then $\mathrm{stdbraid}_{\tilde{s},s}^{(F)}$ is a rex move, as the domain and codomain of this map are rexes of the same length. We obtain

$$(\mathrm{id}\otimes\mathrm{stdcap}_{\tilde{s}}^{(F)})\circ(\mathrm{id}_{Q_{\tilde{s}}^{(F)}}\otimes\mathrm{stdbraid}_{\tilde{s},s}^{(F)}\otimes\mathrm{id}_{Q_{\tilde{s}}^{(F)}})\circ(\mathrm{stdcup}_{\tilde{s}}\otimes\mathrm{id})=\mathrm{stdbraid}_{s,\tilde{s}}^{(F)}$$

using (4.3)–(4.6), while

$$(\mathrm{stdcap}_s^{(F)} \otimes \mathrm{id}) \circ (\mathrm{id}_{Q_s^{(F)}} \otimes \mathrm{stdbraid}_{\tilde{s},s}^{(F)} \otimes \mathrm{id}_{Q_s^{(F)}}) \circ (\mathrm{id} \otimes \mathrm{stdcup}_s^{(F)}) = \mathrm{stdbraid}_{s,\tilde{s}}^{(F)}$$

follows from the cyclicity of all standard braid vertices in $\mathcal{D}_{\mathrm{std}}$ with respect to s-colored standard cups and caps. Combining these gives cyclicity of $\mathrm{stdbraid}_{\tilde{s},s}^{(F)}$. Relation [8, (4.4)] follows immediately, as the inverse of a rex move equals its dual.

On the other hand suppose $m_{\tilde{s}s}$ is odd. The difference in expression length between the domain and codomain of $\mathrm{stdbraid}_{\tilde{s},s}^{(F)}$ is $2\ell(\tilde{s}s_{\alpha_{\mathrm{h}}})$, so $\mathrm{stdbraid}_{\tilde{s},s}^{(F)}$ has $\ell(\tilde{s}s_{\alpha_{\mathrm{h}}})$ standard caps. Applying (4.3)–(4.6), we can simplify

$$(\mathrm{id}_{Q_{\underline{\tilde{s}}}^{(F)}} \otimes \mathrm{stdbraid}_{\tilde{s},s}^{(F)} \otimes \mathrm{id}_{Q_s}) \circ (\mathrm{id}_{Q_{\underline{\tilde{s}s_{\alpha_{\mathrm{h}}}}}}^{\otimes(p-1)/2} \otimes \mathrm{id}_{Q_{\tilde{s}}} \otimes \overline{\epsilon_{(\underline{s_{\alpha_{\mathrm{h}}}\tilde{s}})^{(p-1)/2}}} \otimes \mathrm{id})$$

into a rex move, because the domain and codomain of this map have the same length. Similarly

$$\begin{aligned}(\mathrm{id} \otimes \mathrm{stdcap}_s) \circ (\mathrm{id}_{Q_{\underline{\tilde{s}}}^{(F)}} \otimes \mathrm{stdbraid}_{\tilde{s},s}^{(F)} \otimes \mathrm{id}_{Q_s})& \\ \circ (\mathrm{id}_{Q_{\underline{\tilde{s}s_{\alpha_{\mathrm{h}}}}}}^{\otimes(p-1)/2} \otimes \mathrm{id}_{Q_{\tilde{s}}} \otimes \overline{\epsilon_{(\underline{s_{\alpha_{\mathrm{h}}}\tilde{s}})^{(p-1)/2}}} \otimes \mathrm{id}) &\circ (\mathrm{id}_{Q_{\underline{\tilde{s}s_{\alpha_{\mathrm{h}}}}}}^{\otimes(p-1)/2} \otimes \mathrm{stdcup}_{\tilde{s}} \otimes \mathrm{id}),\end{aligned}$$

is equivalent to a rex move, this time using cyclicity of standard braid vertices in $\mathcal{D}_{\mathrm{std}}$ with respect to $\tilde{s}$-colored standard cups and caps. Finally, if we compose this with

$$\overline{\epsilon_{(\underline{\tilde{s}s_{\alpha_{\mathrm{h}}}})^{(p-1)/2}}} \otimes \mathrm{id}$$

on the right we obtain

$$(\mathrm{id} \otimes \mathrm{stdcap}_s) \circ (\mathrm{id}_{Q_{\underline{\tilde{s}}}^{(F)}} \otimes \mathrm{stdbraid}_{\tilde{s},s}^{(F)} \otimes \mathrm{id}_{Q_s}) \circ (\mathrm{stdcup}_{\tilde{s}}^{(F)} \otimes \mathrm{id})$$

and using (4.3)–(4.6) again we observe that this must in fact be equivalent to $\mathrm{stdbraid}_{s,\tilde{s}}^{(F)}$ because it has $\ell(\tilde{s}s_{\alpha_{\mathrm{h}}})$ standard cups. Similarly we can show that

$$(\mathrm{stdcap}_s \otimes \mathrm{id}) \circ (\mathrm{id}_{Q_s} \otimes \mathrm{stdbraid}_{\tilde{s},s}^{(F)} \otimes \mathrm{id}_{Q_{\underline{\tilde{s}}}^{(F)}}) \circ (\mathrm{id} \otimes \mathrm{stdcup}_{\tilde{s}}^{(F)}) = \mathrm{stdbraid}_{s,\tilde{s}}^{(F)}$$

which establishes the cyclicity of $\mathrm{stdbraid}_{\tilde{s},s}^{(F)}$. Moreover,

$$\begin{aligned}\mathrm{stdbraid}_{\tilde{s},s} \circ \mathrm{stdbraid}_{s,\tilde{s}} &= (-1)^{\ell(\tilde{s}s_{\alpha_{\mathrm{h}}})}\mathrm{id} \\ \mathrm{stdbraid}_{s,\tilde{s}} \circ \mathrm{stdbraid}_{\tilde{s},s} &= (-1)^{\ell(\tilde{s}s_{\alpha_{\mathrm{h}}})}\mathrm{id}\end{aligned}$$

using (4.1) and (4.3)–(4.6). But $\ell(\tilde{s}s_{\alpha_{\mathrm{h}}})$ is even because $\tilde{s}$ and $s_{\alpha_{\mathrm{h}}}$ both act as reflections on the Euclidean space E. This establishes relation [8, (4.4)].

The last relations to check are the Zamolodchikov relations [8, (4.5)–(4.8)]. They are equivalent to the statement that all rex moves in $\mathcal{D}_{\mathrm{std}}^F$ with the same domain and codomain are equal. Thus it is enough for us to show that for any two rexes $\underline{x}, \underline{y} \in \underline{S}$ with $x = y$, if $f^F, g^F : Q_{\underline{x}}^F \to Q_{\underline{y}}^F$ are two rex moves in $\mathcal{D}_{\mathrm{std}}^F$, then the morphisms $f^{(F)}, g^{(F)} : Q_{\underline{x}}^{(F)} \to Q_{\underline{y}}^{(F)}$ obtained by substituting $\mathrm{stdbraid}_{s,t}^{(F)}$ for each $\mathrm{stdbraid}_{s,t}^F$ are equal. We already know from Theorem 3.12 applied to $\mathcal{D}_{\mathrm{std}}$ that $f^{(F)}$ and $g^{(F)}$ are equivalent up to scalars. But Lemma 4.4 show that ϵ_s and $\overline{\epsilon_s}$ satisfy the same relations in $\mathcal{D}_{\mathrm{std}}$ (up to sign) as stdcap_s and stdcup_s, so in fact $f^{(F)} = \pm g^{(F)}$. Finally the signs must match, because as previously mentioned each morphism $\mathrm{stdbraid}_{s,t}^{(F)}$ has an even number of standard cups or caps. □

4.2. **A new ring.** To define a version of F on $\mathcal{D}_{\mathrm{BS}}$ it will be necessary to work over a certain ring extension of R.

Definition 4.6. The ring $\hat{R}$ is the following iterated extension of R. First let $R' = R[a_s/a_t : s,t \in S_{\mathrm{f}}]$. This is the subring of the fraction field Q generated by R and the fractions

$$\{a_s/a_t : s,t \in S_{\mathrm{f}}\} \subset Q.$$

In R', for any $s \in S_{\mathrm{f}}$ the ideal $\mathfrak{p}' = R'a_s$ is prime and does not depend on s, so we may consider the localization $R'_{\mathfrak{p}'}$ of R' at $\mathfrak{p}'$. Finally let $\hat{R}$ be the completion of $R'_{\mathfrak{p}'}$ with respect to $\mathfrak{p}'_{\mathfrak{p}'}$.

$$R \subset R' = R[a_s/a_t : s,t \in S_{\mathrm{f}}] \subset R'_{\mathfrak{p}'} \subset \hat{R}$$

As $\mathfrak{p}'$ is a principal ideal, $R'_{\mathfrak{p}'}$ and therefore $\hat{R}$ are discrete valuation rings. Let val be the valuation on $\hat{R}$ normalized so that $\operatorname{val} a_s = 2$ for all $s \in S_{\mathrm{f}}$ and $\operatorname{val} a_{\tilde{s}} = 0$. For $f, g \in \hat{R}$ we also set $\operatorname{val}\binom{f}{g} = \frac{1}{2}(\operatorname{val} f + \operatorname{val} g)$.

The construction of $\hat{R}$ above is somewhat involved, but the only essential properties of $\hat{R}$ that we will need for later are:

($\hat{R}$1) the non-invertible elements in $V \subseteq \hat{R}$ are precisely $\bigoplus_{s \in S_{\mathrm{f}}} \Bbbk a_s$;
($\hat{R}$2) $\hat{R}$ is a complete discrete valuation ring.

Remark 4.7. The structure of the maximal ideal of $\hat{R}$ ensures that for each $s \in S$, the object $B_s^{(F)}$ (defined in the next section) is indecomposable because a_s^F is not invertible in $\hat{R}$. This is analogous to B_s being indecomposable in $\mathcal{D}$ because a_s is not invertible in R (see Example 3.7). The valuation in $\hat{R}$ gives a useful invariant, playing a role largely analogous to degree in R. To emphasize this connection (and to avoid confusion with the generator of $\mathbb{Z}[v^{\pm 1}]$) we have chosen the notation val instead of the more usual v or ν. It is also important that $\hat{R}$ is a complete local ring so that when we later define the category $\mathcal{D}_{p|*}$ over $\hat{R}$, it retains the Krull–Schmidt property (cf. [8, Lemma 6.25]).

In order for $\hat{R}$ to take the role of R in later sections, we will require a suitable replacement for the notion of a graded R-module. The valuation on $\hat{R}$ gives rise to a (non-archimedean) absolute value. It turns out that the appropriate replacement for "graded R-modules" is "normed $\hat{R}$-modules". We give a brief overview of the relevant theory, translated into the language of valuations (see also e.g. [4, §2.2]).

Definition 4.8.

(1) Let M be an $\hat{R}$-module. A *valuation* on M is a function $\operatorname{val} : M \to \mathbb{Z} \cup \{\infty\}$ which satisfies the following properties for all $m, m' \in M$ and $f \in \hat{R}$:
- $\operatorname{val} m = \infty$ if and only if $m = 0$;
- $\operatorname{val}(m + m') \geq \min(\operatorname{val} m, \operatorname{val} m')$;
- $\operatorname{val}(fm) = \operatorname{val} f + \operatorname{val} m$.

An *$\hat{R}$-valuation module* is an $\hat{R}$-module equipped with a particular valuation. We say that two $\hat{R}$-valuation modules M and N are *valuation isomorphic* if there are mutually inverse $\hat{R}$-module isomorphisms $M \to N$ and $N \to M$ which preserve the valuation.

(2) Let $\mathcal{M}$ be a category enriched in $\hat{R}$-modules. A *valuation* on the Hom-spaces of $\mathcal{M}$ is a collection of functions $\operatorname{val} : \operatorname{Hom}_{\mathcal{M}}(M,N) \to \mathbb{Z} \cup \{\infty\}$ for all objects M, N in $\mathcal{M}$ which give each Hom-space the structure of a valuation module, with the following additional properties for all morphisms β and γ in $\mathcal{M}$:
- $\operatorname{val}(\mathrm{id}_M) = 0$;

- $\operatorname{val}(\beta \circ \gamma) \geq \operatorname{val} \beta + \operatorname{val} \gamma$ if β and γ are composable.

We write $\operatorname{Hom}^\bullet_{\mathcal{M}}$ to refer to the Hom-space considered as a valuation module. We say that M and N are *valuation isomorphic* if there are two mutually inverse isomorphisms $M \to N$ and $N \to M$ of valuation 0.

Example 4.9.

(1) The ring $\hat{R}$ as a left $\hat{R}$-module is a valuation module, with valuation equal to the ring valuation. More generally, any finitely generated free $\hat{R}$-module $\hat{R}^{\oplus m}$ is a valuation module with valuation $\operatorname{val}(r_1 \oplus r_2 \oplus \cdots r_m) = \min_i \operatorname{val}(r_i)$.

(2) Consider the category $\mathcal{M}$ of all $\hat{R}$-valuation modules. For $\beta : M \to N$ a morphism in this category, let $\operatorname{val}_{\mathcal{M}} \beta = \inf_{m \in M}(\operatorname{val}_N(\beta(m)) - \operatorname{val}_M m)$. Then $\operatorname{val}_{\mathcal{M}}$ is a valuation on $\mathcal{M}$, called the *induced valuation.* Moreover, if β is an isomorphism then β preserves the valuation if and only if $\operatorname{val}_{\mathcal{M}} \beta = 0$. In other words, the two notions of valuation isomorphism above coincide.

Let M be a valuation module with valuation val_M. For $i \in \mathbb{Z}$, the *valuation shift* $M\langle i \rangle$ is the valuation module given by M with the new valuation $\operatorname{val}_{M\langle i \rangle} m = i + \operatorname{val} m$. Valuation shift plays a similar role for valuation modules as the grade shift functor does for graded modules. For valuation modules M, N with valuations $\operatorname{val}_M, \operatorname{val}_N$ respectively, the direct sum $M \oplus N$ is a valuation module with valuation $\operatorname{val}_{M \oplus N}(m \oplus n) = \min(\operatorname{val} m, \operatorname{val} n)$. This is a generalization of the valuations on free $\hat{R}$-modules from the previous example. We call a valuation module a *free valuation module* if it is valuation isomorphic to a valuation module of the form $\bigoplus_i \hat{R}\langle d_i \rangle$ for some $d_i \in \mathbb{Z}$. We call a basis (b_i) for a free valuation module M a *valuation basis* if for some $d_i \in \mathbb{Z}$ the $\hat{R}$-module homomorphism

$$\begin{aligned} \bigoplus_i \hat{R}\langle d_i \rangle &\longrightarrow M \\ e_i &\longmapsto b_i \end{aligned}$$

is a valuation isomorphism.

Remark 4.10.

(1) There is an alternative construction of $\hat{R}$ more closely based on properties ($\hat{R}1$)–($\hat{R}2$) above. Take the prime ideal $\mathfrak{p} = (a_s : s \in S_{\mathrm{f}})$ in R and consider the localization $R_{\mathfrak{p}}$. This ring is a Noetherian local domain, so it is dominated by a discrete valuation ring (e.g. [7, Exercises 11.2–11.3]). This discrete valuation ring is constructed as follows. Pick some $t \in S_{\mathrm{f}}$ and let $R'' = R_{\mathfrak{p}}[a_s/a_t : s \in S_{\mathrm{f}}]$ be the subring of Q generated by $R_{\mathfrak{p}}$ and $\{a_s/a_t : s \in S_{\mathrm{f}}\}$. Now $\mathfrak{p}'' = R'' a_t$ is a principal prime ideal lying over $\mathfrak{p}$, so the localization $R''_{\mathfrak{p}''}$ is a discrete valuation ring dominating $R_{\mathfrak{p}}$. In fact, we have $R''_{\mathfrak{p}''} = R'_{\mathfrak{p}'}$ so this ring does not depend on t. As before taking the

completion of $R''_{\mathfrak{p}''}$ gives us $\hat{R}$.

$$\begin{tikzcd}
 & & \hat{R} \arrow[d, dash] & & \\
 & & R'_{\mathfrak{p}'} = R''_{\mathfrak{p}''} \arrow[ddll, dash] \arrow[drr, dash] & & \\
 & & & & R'' \arrow[d, dash] \\
R' \arrow[drr, dash] & & & & R_{\mathfrak{p}} \arrow[dll, dash] \\
 & & R & &
\end{tikzcd}$$

(2) The ring $\hat{R}$ has the structure of a filtered $\Bbbk$-algebra, with decreasing filtration given by
$$\hat{R}^i = \{f \in \hat{R} : \operatorname{val} f \geq i\}.$$
Every $\hat{R}$-module M is filtered in a similar way, giving it the structure of a filtered $\hat{R}$-module as defined in e.g. [10, Definition 1.2]. In this context valuation isomorphisms and valuation shifts are special cases of filtered isomorphisms and filtration shifts respectively.

4.3. **Bott–Samelson Frobenius diagrammatics.** We now proceed to construct the Frobenius functor in earnest, as an "integral" version of the functor introduced in Theorem 4.5. Recall that ${}_R\mathcal{D}_{\mathrm{BS,std}}$ is the R-form of $\mathcal{D}_{\mathrm{BS,std}}$ from Remark 3.10.

Definition 4.11. Take $\underline{s_{\alpha_{\mathrm{h}}}} \in \underline{S}_{\mathrm{f}}$ to be the rex fixed in Definition 4.1 for $s_{\alpha_{\mathrm{h}}}$. For each $s \in S$ we write $B_s^{(F)}$ to mean the following object in ${}_R\mathcal{D}_{\mathrm{BS,std}}$:
$$B_s^{(F)} = \begin{cases} B_s & \text{if } s \in S_{\mathrm{f}}, \\ (R_{\underline{\tilde{s}s_{\alpha_{\mathrm{h}}}}})^{\otimes(p-1)/2} \otimes B_{\tilde{s}} \otimes (R_{\underline{s_{\alpha_{\mathrm{h}}}\tilde{s}}})^{\otimes(p-1)/2} & \text{if } s = \tilde{s}. \end{cases}$$
For $\underline{x} = \underline{s_1 s_2 \cdots s_m} \in \underline{S}$ we write $B_{\underline{x}}^{(F)} = B_{s_1}^{(F)} \otimes B_{s_2}^{(F)} \otimes \cdots B_{s_m}^{(F)}$.

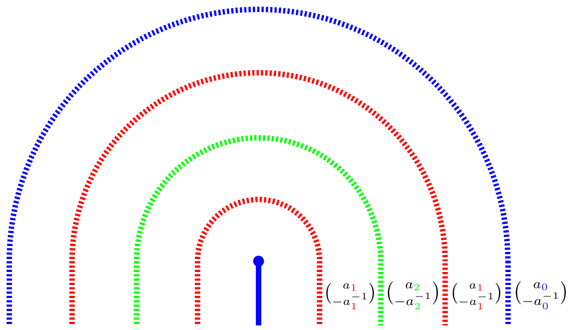
Let $\hat{R} \otimes_R {}_R\mathcal{D}_{\mathrm{BS,std}}$ denote scalar extension of ${}_R\mathcal{D}_{\mathrm{BS,std}}$ to an $\hat{R}$-linear category. As with the mixed category we will generally omit the "$\hat{R} \otimes_R (-)$" when describing objects in $\hat{R} \otimes_R {}_R\mathcal{D}_{\mathrm{BS,std}}$ whenever possible.

The following morphisms in $\hat{R} \otimes_R {}_R\mathcal{D}_{\mathrm{BS,std}}$ are analogous to dot_s^F, fork_s^F, and $\mathrm{braid}_{s,t}^F$.

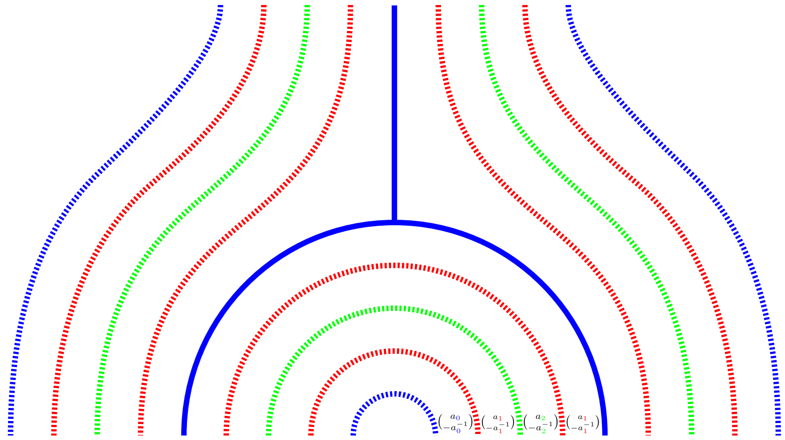
Definition 4.12.

(1) For each $s \in S$, we write $\mathrm{dot}_s^{(F)} : B_s^{(F)} \to \hat{R}$ for the following bi-valued morphism defined in $\hat{R} \otimes_R {}_R\mathcal{D}_{\mathrm{BS,std}}$. If $s \in S_{\mathrm{f}}$ then $\mathrm{dot}_s^{(F)} = \mathrm{dot}_s$. Otherwise if $s = \tilde{s}$ we set
$$\mathrm{dot}_{\tilde{s}}^{(F)} = \epsilon_{(\underline{\tilde{s}s_{\alpha_{\mathrm{h}}}})^{(p-1)/2}} \circ (\mathrm{id}^{\otimes(p-1)/2}_{\hat{R}_{\underline{\tilde{s}s_{\alpha_{\mathrm{h}}}}}} \otimes \mathrm{dot}_{\tilde{s}} \otimes \mathrm{id}^{\otimes(p-1)/2}_{\hat{R}_{\underline{s_{\alpha_{\mathrm{h}}}\tilde{s}}}}).$$

(2) For each $s \in S$, we write $\mathrm{fork}_s^{(F)} : B_s^{(F)} \otimes B_s^{(F)} \to B_s^{(F)}$ for the following bi-valued morphism defined in $\hat{R} \otimes_R {}_R\mathcal{D}_{\mathrm{BS,std}}$. If $s \in S_{\mathrm{f}}$ then $\mathrm{fork}_s^{(F)} = \mathrm{fork}_s$.



(A) $\mathrm{dot}_0^{(F)}$



(B) $\mathrm{fork}_0^{(F)}$

FIGURE 3. Examples of $\mathrm{dot}_s^{(F)}$ and $\mathrm{fork}_s^{(F)}$ morphisms for $p = 3$ and $\Phi_{\mathrm{f}} = A_2$.

Otherwise if $s = \tilde{s}$ we set

$$\begin{aligned}\mathrm{fork}_{\tilde{s}}^{(F)} = {} & (\mathrm{id}_{\hat{R}_{\underline{\tilde{s}s_{\alpha_{\mathrm{h}}}}}}^{\otimes(p-1)/2} \otimes \mathrm{fork}_{\tilde{s}} \otimes \mathrm{id}_{\hat{R}_{\underline{s_{\alpha_{\mathrm{h}}}\tilde{s}}}}^{\otimes(p-1)/2}) \\ & \circ (\mathrm{id}_{\hat{R}_{\underline{\tilde{s}s_{\alpha_{\mathrm{h}}}}}}^{\otimes(p-1)/2} \otimes \mathrm{id}_{\hat{R}_{\underline{\tilde{s}}}} \otimes \epsilon_{(\underline{s_{\alpha_{\mathrm{h}}}\tilde{s}})^{(p-1)/2}} \otimes \mathrm{id}_{\hat{R}_{\underline{\tilde{s}}}} \otimes \mathrm{id}_{\hat{R}_{\underline{s_{\alpha_{\mathrm{h}}}\tilde{s}}}}^{\otimes(p-1)/2}).\end{aligned}$$

(3) For all distinct $s, t \in S_{\mathrm{f}}$, we set $\mathrm{braid}_{s,t}^{(F)} = \mathrm{braid}_{s,t}$.

We will define a similar morphism $\mathrm{braid}_{\tilde{s},s}^{(F)}$ for each $s \in S_{\mathrm{f}}$ with $m_{\tilde{s}s} < \infty$ in Proposition 4.14.

Example 4.13. Let $p = 3$ and $\Phi_{\mathrm{f}} = A_2$. Following the same diagrammatic convention as Example 4.3, $\mathrm{dot}_0^{(F)}$ and $\mathrm{fork}_0^{(F)}$ are depicted in Figure 3.

Proposition 4.14.

(i) *The morphisms* $\mathrm{dot}_s^{(F)}$, $\mathrm{fork}_s^{(F)}$ *for all* $s \in S$ *and* $\mathrm{braid}_{s,t}^{(F)}$ *for all distinct* $s, t \in S_{\mathrm{f}}$ *satisfy all the relations in* $\mathcal{D}_{\mathrm{BS}}^F$ *which do not involve* $\mathrm{braid}_{\tilde{s},s}$ *or its dual. In other words, the relations* (3.1)–(3.11) *and the Zamolodchikov relations* [8, (5.8)–(5.12)] *which do not involve an* $\tilde{s}$*-colored braid still hold when the morphisms* $\mathrm{fork}_s^{(F)}$, $\mathrm{fork}_s^{(F)}$, *and* $\mathrm{braid}_{s,t}^{(F)}$ *are substituted for all corresponding dots, forks, and braids respectively.*

(ii) *For each $s \in S_{\mathrm{f}}$ such that $m_{s\tilde{s}} < \infty$, there is a unique bi-valued morphism*

$$\mathrm{braid}^{(F)}_{\tilde{s},s} : \overbrace{B^F_{\tilde{s}} \otimes B_s \otimes B^F_{\tilde{s}} \otimes \cdots}^{m_{s\tilde{s}}} \longrightarrow \overbrace{B_s \otimes B^F_{\tilde{s}} \otimes B_s \otimes \cdots}^{m_{s\tilde{s}}}$$

which satisfies all relations in $\mathcal{D}^F_{\mathrm{BS}}$ which involve $\mathrm{braid}^F_{\tilde{s},s}$ *and its dual, after substituting* $\mathrm{dot}^{(F)}_t$, $\mathrm{fork}^{(F)}_t$, *and* $\mathrm{braid}^{(F)}_{t,u}$ *for all corresponding dots, forks, and braids respectively.*

Proof. The only non-trivial relations to check are those which involve $\tilde{s}$-colored dots, forks, or braids. Instead of checking these relations individually, we will use localization to check them all at once.

Recall that $B_{\tilde{s}}$ is isomorphic to $Q_s \oplus Q$ in $\mathcal{D}_{\mathrm{BS,std}}$. This induces the following isomorphism $B^{(F)}_{\tilde{s}} \to Q_{\tilde{s}_p} \oplus Q$ in $\mathcal{D}_{\mathrm{BS,std}}$:

$$\begin{gathered}
\left(Q_{(\tilde{s}s_{\alpha_{\mathrm{h}}})^{(p-1)/2}} \otimes Q_{\tilde{s}} \otimes Q_{(s_{\alpha_{\mathrm{h}}}\tilde{s})^{(p-1)/2}}\right) \oplus \left(Q_{(\tilde{s}s_{\alpha_{\mathrm{h}}})^{(p-1)/2}} \otimes Q \otimes Q_{(s_{\alpha_{\mathrm{h}}}\tilde{s})^{(p-1)/2}}\right) \\
\Big\downarrow{\scriptstyle \mathrm{id} \oplus \epsilon_{\underline{(\tilde{s}s_{\alpha_{\mathrm{h}}})^{(p-1)/2}}}}\ \wr \\
Q_{\tilde{s}_p} \oplus Q.
\end{gathered}$$

Using this isomorphism, $\mathrm{dot}^{(F)}_{\tilde{s}}$ corresponds to the matrix

$$\begin{aligned}
&\begin{pmatrix} 0 & \epsilon_{\underline{(\tilde{s}s_{\alpha_{\mathrm{h}}})^{(p-1)/2}}} \circ \left(1 \otimes \cdots \otimes 1 \otimes \binom{a_{\tilde{s}}}{1} \otimes 1 \otimes \cdots \otimes 1\right) \circ \overline{\epsilon_{\underline{(\tilde{s}s_{\alpha_{\mathrm{h}}})^{(p-1)/2}}}} \end{pmatrix} \\
&\qquad = \begin{pmatrix} 0 & (-1)^{\ell(\tilde{s}s_{\alpha_{\mathrm{h}}})(p-1)/2} (\tilde{s}s_{\alpha_{\mathrm{h}}})^{(p-1)/2} \left(\binom{a_{\tilde{s}}}{1}\right) \end{pmatrix} \\
&\qquad = \begin{pmatrix} 0 & \binom{a^F_{\tilde{s}}}{1} \end{pmatrix},
\end{aligned}$$

where we have used the fact that $\ell(\tilde{s}s_{\alpha_{\mathrm{h}}})$ is even.

Similarly, there is an isomorphism $B^{(F)}_{\tilde{s}} \otimes B^{(F)}_{\tilde{s}} \to Q \oplus Q_{\tilde{s}_p} \oplus Q_{\tilde{s}_p} \oplus Q$ in $\mathcal{D}_{\mathrm{BS,std}}$:

$$\begin{gathered}
\left(Q_{(\tilde{s}s_{\alpha_{\mathrm{h}}})^{(p-1)/2}} \otimes Q_{\tilde{s}} \otimes Q_{(s_{\alpha_{\mathrm{h}}}\tilde{s})^{(p-1)/2}}\right) \otimes \left(Q_{(\tilde{s}s_{\alpha_{\mathrm{h}}})^{(p-1)/2}} \otimes Q_{\tilde{s}} \otimes Q_{(s_{\alpha_{\mathrm{h}}}\tilde{s})^{(p-1)/2}}\right) \\
\oplus \left(Q_{(\tilde{s}s_{\alpha_{\mathrm{h}}})^{(p-1)/2}} \otimes Q_{\tilde{s}} \otimes Q_{(s_{\alpha_{\mathrm{h}}}\tilde{s})^{(p-1)/2}}\right) \otimes \left(Q_{(\tilde{s}s_{\alpha_{\mathrm{h}}})^{(p-1)/2}} \otimes Q \otimes Q_{(s_{\alpha_{\mathrm{h}}}\tilde{s})^{(p-1)/2}}\right) \\
\oplus \left(Q_{(\tilde{s}s_{\alpha_{\mathrm{h}}})^{(p-1)/2}} \otimes Q \otimes Q_{(s_{\alpha_{\mathrm{h}}}\tilde{s})^{(p-1)/2}}\right) \otimes \left(Q_{(\tilde{s}s_{\alpha_{\mathrm{h}}})^{(p-1)/2}} \otimes Q_{\tilde{s}} \otimes Q_{(s_{\alpha_{\mathrm{h}}}\tilde{s})^{(p-1)/2}}\right) \\
\oplus \left(Q_{(\tilde{s}s_{\alpha_{\mathrm{h}}})^{(p-1)/2}} \otimes Q \otimes Q_{(s_{\alpha_{\mathrm{h}}}\tilde{s})^{(p-1)/2}}\right) \otimes \left(Q_{(\tilde{s}s_{\alpha_{\mathrm{h}}})^{(p-1)/2}} \otimes Q \otimes Q_{(s_{\alpha_{\mathrm{h}}}\tilde{s})^{(p-1)/2}}\right) \\
\Bigg\downarrow{\scriptstyle \begin{array}{r} \mathrm{stdcap}^{(F)}_{\tilde{s}} \\ \oplus \mathrm{id} \otimes \epsilon_{\underline{(\tilde{s}s_{\alpha_{\mathrm{h}}})^{(p-1)/2}}} \\ \oplus \epsilon_{\underline{(\tilde{s}s_{\alpha_{\mathrm{h}}})^{(p-1)/2}}} \otimes \mathrm{id} \\ \oplus \epsilon_{\underline{(\tilde{s}s_{\alpha_{\mathrm{h}}})^{(p-1)/2}}} \otimes \epsilon_{\underline{(\tilde{s}s_{\alpha_{\mathrm{h}}})^{(p-1)/2}}} \end{array}}\ \wr \\
Q \oplus Q_{\tilde{s}_p} \oplus Q_{\tilde{s}_p} \oplus Q.
\end{gathered}$$

Using these isomorphisms, $\mathrm{fork}^{(F)}_{\tilde{s}}$ corresponds to the matrix

$$\begin{pmatrix} 0 & \binom{1}{-(a^F_{\tilde{s}})^{-1}} & \binom{1}{(a^F_{\tilde{s}})^{-1}} & 0 \\ \binom{1}{-(a^F_{\tilde{s}})^{-1}} & 0 & 0 & \binom{1}{(a^F_{\tilde{s}})^{-1}} \end{pmatrix}.$$

Both matrices are identical to the localization matrices for the $\tilde{s}$-colored dot and fork respectively in $\mathcal{D}^F_{\mathrm{BS}}$. Since localization is faithful, the first result follows.

Before we prove the second result, we introduce some notation. For each $s \in S$, write $\mathrm{bivalent}_s : B_s \to R_s$ for the s-colored bivalent morphism in $\mathcal{D}_{\mathrm{BS,std}}$, and set $\mathrm{bivalent}^{(F)}_s : B^{(F)}_s \to \hat{R}^{(F)}_s$ to be the following morphism in $\mathcal{D}_{\mathrm{BS,std}}$:

$$\mathrm{bivalent}^{(F)}_s = \begin{cases} \mathrm{bivalent}_s & \text{if } s \in S_{\mathrm{f}}, \\ \mathrm{id}^{\otimes(p-1)/2}_{\hat{R}_{\underline{\tilde{s}s_{\alpha_{\mathrm{h}}}}}} \otimes \mathrm{bivalent}^{(F)}_{\tilde{s}} \otimes \mathrm{id}^{\otimes(p-1)/2}_{\hat{R}_{\underline{s_{\alpha_{\mathrm{h}}}\tilde{s}}}} & \text{if } s = \tilde{s}. \end{cases}$$

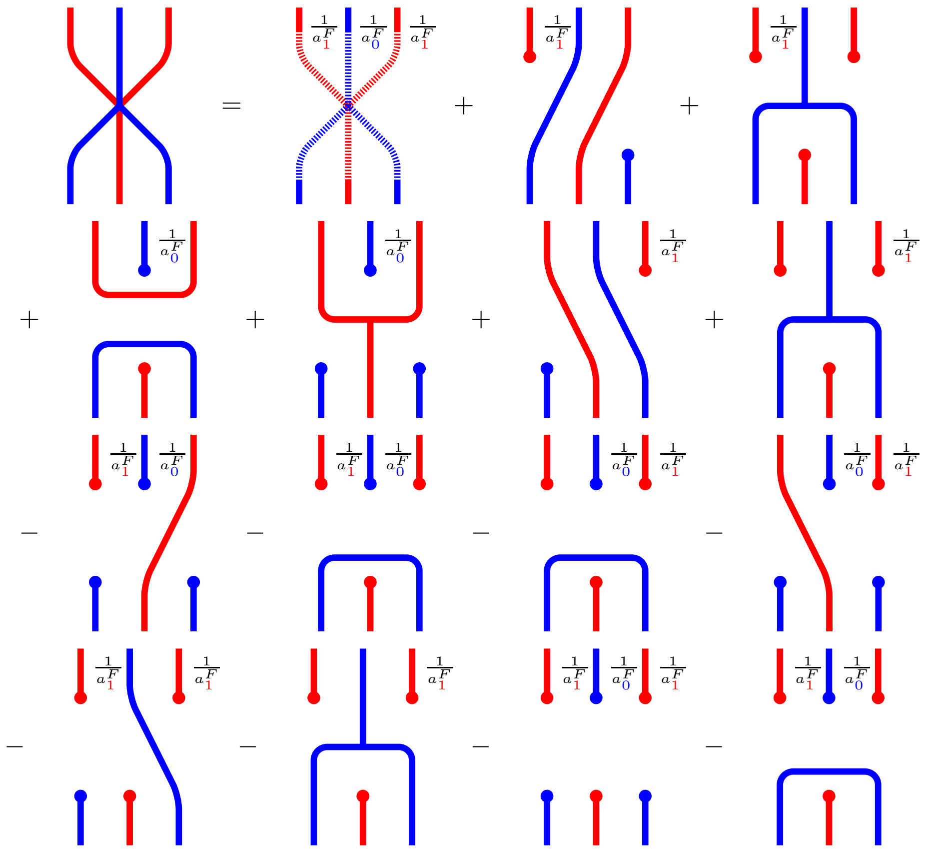


FIGURE 4. A braid decomposition for $\Phi_{\mathrm{f}} = A_2$.

Now suppose $s \in S_{\mathrm{f}}$ such that $m_{\tilde{s}s} < \infty$. We construct the morphism $\mathrm{braid}_{\tilde{s},s}^{(F)}$ as follows. First we decompose $\mathrm{braid}_{\tilde{s},s}^{F}$ into a sum of a mixed diagram with $\mathrm{stdbraid}_{\tilde{s},s}^{F}$ and Bott–Samelson diagrams which only involve forks and dots. For example, one such decomposition is depicted in Figure 4.

One method for finding such a decomposition is to apply (3.16) to the all the strings in the codomain and using the Jones–Wenzl relation. Now for each $t \in \{s, \tilde{s}\}$ we substitute the morphisms $\mathrm{dot}_t^{(F)}$, $\mathrm{fork}_t^{(F)}$, and $\mathrm{bivalent}_t^{(F)}$ for all corresponding dots, forks, and bivalent vertices and the morphism $\mathrm{stdbraid}_{\tilde{s}s}^{(F)}$ for $\mathrm{stdbraid}_{\tilde{s}s}^{F}$ in the first term of the decomposition. The resulting morphism is $\mathrm{braid}_{\tilde{s},s}^{(F)}$.

As with $\mathrm{dot}_{\tilde{s}}^{(F)}$ and $\mathrm{fork}_{\tilde{s}}^{(F)}$ above, we can decompose the domain and codomain of $\mathrm{braid}_{\tilde{s},s}^{(F)}$ into a direct sum of standard bimodules, and write the matrix form of $\mathrm{braid}_{\tilde{s},s}^{(F)}$. But then it is immediately clear that this matrix equals the localization matrix of $\mathrm{braid}_{\tilde{s},s}$ in $\mathcal{D}_{\mathrm{BS}}^{F}$ by construction. Uniqueness follows because localization is faithful.

$\square$

An immediate consequence of this result is the long-awaited construction of the Frobenius functor F.

Theorem 4.15. *There is a well-defined monoidal functor* $F : \mathcal{D}_{\mathrm{BS}}^F \to \hat{R} \otimes_{R\,R} \mathcal{D}_{\mathrm{BS,std}}$ *which for all* $s \in S$ *maps*

$$\begin{aligned} B_s^F &\longmapsto B_s^{(F)} \\ a_s^F \mathrm{id}_{R^F} &\longmapsto a_s^F \mathrm{id}_R \\ v_{\mathrm{fix}} \mathrm{id}_{R^F} &\longmapsto v_{\mathrm{fix}} \mathrm{id}_R \\ \mathrm{dot}_s^F &\longmapsto \mathrm{dot}_s^{(F)} \\ \mathrm{fork}_s^F &\longmapsto \mathrm{fork}_s^{(F)} \end{aligned}$$

and for all distinct $s, t \in S$ *such that* $m_{st} < \infty$ *maps*

$$\mathrm{braid}_{s,t}^F \longmapsto \mathrm{braid}_{s,t}^{(F)}.$$

5. The category $\mathcal{D}_{p|*}$

Using localization, it is clear that the Frobenius functor induces a monoidal equivalence of categories between $\mathcal{D}_{\mathrm{BS}}^F$ and its image. In this section we will define a category $\mathcal{D}_{p|*}$ extending this image and show that it categorifies the bimodule $\mathbb{H}_{p|*}$.

5.1. **Construction.** In this section, let $\mathcal{D}_{\mathrm{BS}}^{\mathrm{deg}}$ denote the *degrading* of $\mathcal{D}_{\mathrm{BS}}$, which is just the category $\mathcal{D}_{\mathrm{BS}}$ but without the grading on the Hom-spaces. Similarly let $\mathcal{D}^{\mathrm{deg}}$ denote the additive Karoubi envelope of $\mathcal{D}_{\mathrm{BS}}^{\mathrm{deg}}$.

Definition 5.1. The diagrammatic $(p|*)$-Bott–Samelson category $\mathcal{D}_{\mathrm{BS},p|*}$ is the following $\hat{R}$-linear subcategory of $\hat{R} \otimes_{R\,R} \mathcal{D}_{\mathrm{BS,std}}$. It is equipped with the structure of a $(\mathcal{D}_{\mathrm{BS}}^F, \mathcal{D}_{\mathrm{BS}}^{\mathrm{deg}})$-bimodule.

Objects: For each $\underline{x}|\underline{w} \in \underline{S}_{p|1}$ there is an object $B_{\underline{x}|\underline{w}}^{(p|*)} = B_{F^{-1}(\underline{x})}^{(F)} \otimes B_{\underline{w}}$ called a $(p|*)$-*Bott–Samelson bimodule.* The $(\mathcal{D}_{\mathrm{BS}}^F, \mathcal{D}_{\mathrm{BS}}^{\mathrm{deg}})$-bimodule structure is defined by

$$B_{\underline{y}}^F \otimes B_{\underline{x}|\underline{w}}^{(p|*)} \otimes B_{\underline{z}} = B_{F(\underline{y})\underline{x}|\underline{wz}}^{(p|*)}.$$

Morphisms: All morphisms in $\mathcal{D}_{\mathrm{BS}}^{\mathrm{deg}}$ and all morphisms in the image of F are morphisms in $\mathcal{D}_{\mathrm{BS},p|*}$. Another morphism in $\mathcal{D}_{\mathrm{BS},p|*}$ is the localization inclusion map

$$B_{\tilde{s}}^{(F)} = (\hat{R}_{\underline{\tilde{s}s_{\alpha_{\mathrm{h}}}}})^{\otimes(p-1)/2} \otimes B_{\tilde{s}} \otimes (\hat{R}_{\underline{s_{\alpha_{\mathrm{h}}}\tilde{s}}})^{\otimes(p-1)/2} \longrightarrow B_{(\underline{\tilde{s}s_{\alpha_{\mathrm{h}}}})^{(p-1)/2}\underline{\tilde{s}}(\underline{s_{\alpha_{\mathrm{h}}}\tilde{s}})^{(p-1)/2}}$$

which we call the *(orthodox) menorah morphism.* These morphisms generate (in the sense of a $(\mathcal{D}_{\mathrm{BS}}^F, \mathcal{D}_{\mathrm{BS}}^{\mathrm{deg}})$-bimodule) all morphisms in $\mathcal{D}_{\mathrm{BS},p|*}$.

For each $\underline{x} \in \underline{S}_p$, we also write $B_{\underline{x}}^{(p)}$ for the object $B_{F^{-1}(\underline{x})}^{(F)}$ in $\mathcal{D}_{\mathrm{BS},p|*}$. For all $s \in S$ and $\underline{w}, \underline{w}' \in \underline{S}$ such that $\ell(\underline{w}) = \ell(\underline{w}')$ and $\underline{wsw'}$ is a rex for $\tilde{s}_p$, we also write $\mathrm{menorah}_{\underline{w},s,\underline{w}'}$ for any morphism of the following form

$$B_{\tilde{s}}^{(F)} \xrightarrow[\text{menorah}]{\text{orthodox}} B_{(\underline{\tilde{s}s_{\alpha_{\mathrm{h}}}})^{(p-1)/2}\underline{\tilde{s}}(\underline{s_{\alpha_{\mathrm{h}}}\tilde{s}})^{(p-1)/2}} \xrightarrow{\text{rex move}} B_{\underline{wsw'}}$$

in $\mathcal{D}_{\mathrm{BS},p|*}$. We will also call such morphisms "menorah morphisms".

Example 5.2. Let $p = 3$ and $\Phi_{\mathrm{f}} = A_2$. Following the same diagrammatic convention as Example 4.3, we depict the orthodox menorah morphism in Figure 5.

Remark 5.3.

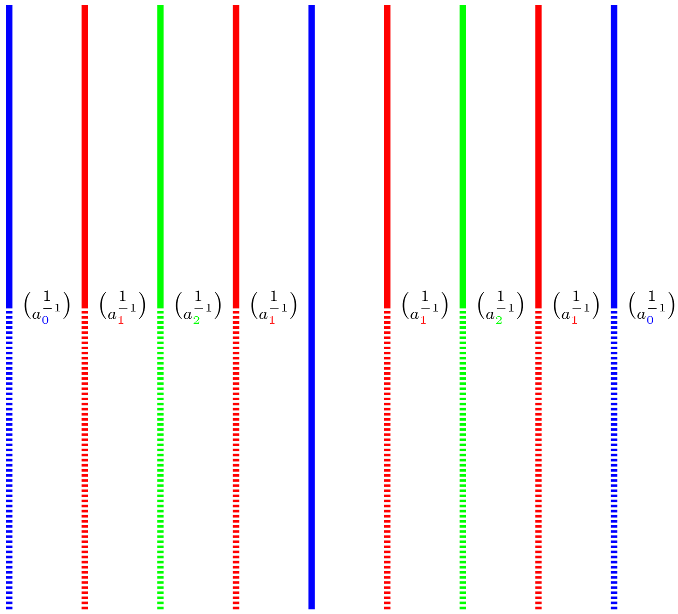
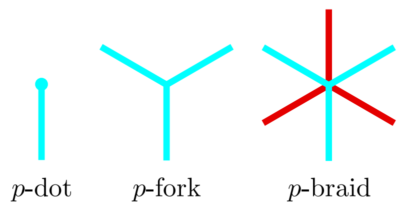


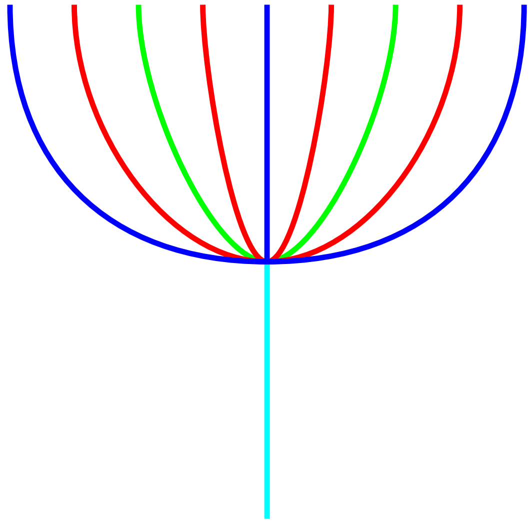
FIGURE 5. A menorah morphism for $p = 3$ and $\Phi_{\mathrm{f}} = A_2$.

(1) The Frobenius functor and the category $\mathcal{D}_{\mathrm{BS},p|*}$ both use the same fixed rex for $\tilde{s}_p$ introduced in Example 1.12; however, we could easily have chosen to use a different rex for $\tilde{s}_p$, as all such choices turn out to equivalent categories. Similarly, all choices of rex move to define $\mathrm{menorah}_{\underline{w},s,\underline{w}'}$ for fixed s and $\underline{w}, \underline{w}'$ give the same morphism.
(2) The diagrams defining morphisms in $\mathcal{D}_{\mathrm{BS},p|*}$ are not quite "graphs up to isotopy" since bivalent vertices can change sign under arbitrary isotopies. However, if we restrict to diagrams that never factor through a non-$(p|*)$-Bott–Samelson bimodule, then isotopy classes of such diagrams do define a unique morphism, not just up to sign.
(3) Menorah morphisms are, strictly speaking, not cyclic, since some rotations of a menorah morphism do not correspond to a morphism in $\mathcal{D}_{\mathrm{BS},p|*}$. However, menorah morphisms of the form $\mathrm{menorah}_{\underline{w},s,\underline{w}^{-1}}$ for some $\underline{w} \in \underline{S}$ and $s \in S$ are what we call *semi-cyclic*. In other words, if we twist such a menorah morphism by 180 degrees clockwise we get its dual, and vice-versa.

Notation 5.4. We assign a lighter version of the color corresponding to $\tilde{s}$ to the W_p-generator $\tilde{s}_p$ (e.g. if $\tilde{s}$ is colored blue then $\tilde{s}_p$ is colored cyan). In the diagrams we use this color to abbreviate morphisms which involve $B_{\tilde{s}}^{(F)}$ or $B_{\tilde{s}}^{F}$, by using solid $\tilde{s}_p$-colored lines. For example, the morphisms corresponding to $\mathrm{dot}_{\tilde{s}}^{(F)}$, $\mathrm{fork}_{\tilde{s}}^{(F)}$, and $\mathrm{braid}_{\tilde{s}s}^{(F)}$ abbreviate to



so that they look exactly the same as their counterparts in $\mathcal{D}^F$. Similarly, the menorah morphism in Figure 5 abbreviates to



menorah

For this reason we will also call these morphisms "vertices".

We also have some special terminology for a menorah vertex. The $\tilde{s}_p$-colored edge is called the *handle*, while the middle edge among the S-colored edges (corresponding to s above) is called the *shamash*. The remaining edges are called *candles*.[1]

The grading on $\mathcal{D}_{\mathrm{BS},p|*}$ inherited from $\mathcal{D}_{\mathrm{BS,std}}$ is not a very useful invariant because $\hat{R}$ is no longer meaningfully graded. However we can define a valuation on morphisms in the sense of Definition 4.8.

Definition 5.5. Let L be a morphism in $\mathcal{D}_{\mathrm{BS},p|*}$. Localize S-colored strings as in §3.3 and the S_p-colored strings as in the proof of Proposition 4.14 to obtain a bi-valued matrix of standard morphisms. We can simplify the entries of this matrix so that they each consist of a single basic standard diagram with a bi-valued coefficient on the left. The *valuation* of L, which we write as $\operatorname{val} L$, is defined to be the minimal valuation of all these coefficients.

Note that this valuation is a special case of the induced valuation from Example 4.9.

Example 5.6.

(1) In Example 3.15 we showed that for each $s \in S$ the localization matrix of dot_s is
$$\begin{pmatrix} 0 & \binom{a_s}{1} \end{pmatrix}.$$
Recall that $\operatorname{val} a_s = 2$ if $s \in S_{\mathrm{f}}$ but $\operatorname{val} a_{\tilde{s}} = 0$. Thus
$$\operatorname{val} \mathrm{dot}_s = \begin{cases} 1 & \text{if } s \in S_{\mathrm{f}}, \\ 0 & \text{if } s = \tilde{s}. \end{cases}$$

(2) Recall from the proof of Proposition 4.14 that $\mathrm{fork}_{\tilde{s}}^{(F)}$ is equivalent via localization to the matrix
$$\begin{pmatrix} 0 & \binom{1}{-(a_{\tilde{s}}^F)^{-1}} & \binom{1}{(a_{\tilde{s}}^F)^{-1}} & 0 \\ \binom{1}{-(a_{\tilde{s}}^F)^{-1}} & 0 & 0 & \binom{1}{(a_{\tilde{s}}^F)^{-1}} \end{pmatrix}.$$

[1]Outside of mathematics, a menorah is a multi-branched candelabrum associated with the ancient Temple in Jerusalem and the Jewish festival of Hanukkah. On a Hanukkah menorah, one of the branches (usually the middle one) is called the shamash.

Since $\operatorname{val} a^F_{\tilde{s}} = 2$, this means that $\operatorname{val}(\mathrm{fork}^{(F)}_{\tilde{s}}) = -1$.

The previous example generalizes as follows.

Lemma 5.7.

(i) *Let* $\mathbb{LL}$ *be a double leaves map in* $\mathcal{D}^F_{\mathrm{BS}}$. *Then* $\operatorname{val} F(\mathbb{LL}) = \deg \mathbb{LL}$.
(ii) *Let* L *be a homogeneous morphism in* $\mathcal{D}^F_{\mathrm{BS}}$ *and* P *a morphism in* $\mathcal{D}_{\mathrm{BS},p|*}$. *Then* $\operatorname{val}(F(L) \otimes P) = \operatorname{val} F(L) + \operatorname{val} P$.

Proof.

(i) Every double leaves map is the composition of some combination of tensor products of dots, forks, and braids. In particular, there are no non-trivial scalars in any region of the Soergel diagram for $\mathbb{LL}$. The localization matrices of all dots, forks, and braids have entries in the subfield $\Bbbk(a^F_s : s \in S) \subseteq Q$. But for all $s \in S$ we see that $\operatorname{val} a^F_s = 2 = \deg a^F_s$, and the localization matrix of $F(\mathbb{LL})$ is identical to that of $\mathbb{LL}$ by the proof of Proposition 4.14. This means that every left coefficient in the localization matrix for $F(\mathbb{LL})$ has valuation equal to the degree of $\mathbb{LL}$.
(ii) The left coefficients in the localization matrix of $F(L) \otimes P$ are all of the form $fw(g)$, where f is the left coefficient of some basic standard diagram with domain Q_w in the localization matrix of $F(L)$, while g is a left coefficient from the localization matrix of P. From (i) the only generating morphism of $\mathcal{D}^F_{\mathrm{BS}}$ whose valuation is *not* equal to its degree is $v_{\mathrm{fix}} \mathrm{id}_{R^F}$, which has degree 2 but valuation 0. Thus all left coefficients in the localization matrix of $F(L)$ have the same valuation. Moreover, each entry in the localization matrix of $F(L)$ has domain isomorphic to $Q_{F(x)}$ for some $x \in W$, i.e. $w = F(x) \in W_p$. For any $s \in S$ and any $x \in W$

$$F(x)(a_s) \in a_s + \bigoplus_{t \in S} \Bbbk a^F_t = a_s + \bigoplus_{t \in S_{\mathrm{f}}} \Bbbk a_t,$$

so $\operatorname{val} a_s = \operatorname{val}(F(x)(a_s))$. This means that $\operatorname{val} w(g) = \operatorname{val} g$, so $\operatorname{val}(fw(g)) = \operatorname{val} f + \operatorname{val} g$ has minimal value $\operatorname{val} F(L) + \operatorname{val} P$.

□

Definition 5.8. Let $\mathcal{D}^{\langle - \rangle}_{\mathrm{BS},p|*}$ be the following extension of $\mathcal{D}_{\mathrm{BS},p|*}$. The objects of $\mathcal{D}^{\langle - \rangle}_{\mathrm{BS},p|*}$ are $B\langle m \rangle$, for each $m \in \mathbb{Z}$ and each object B in $\mathcal{D}_{\mathrm{BS},p|*}$. The morphisms in $\mathcal{D}_{\mathrm{BS},p|*}$, $\mathcal{D}^{\langle - \rangle}_{\mathrm{BS},p|*}$ are given by

$$\mathrm{Hom}^\bullet_{\mathcal{D}^{\langle - \rangle}_{\mathrm{BS},p|*}}(B\langle m \rangle, B'\langle n \rangle) = (\mathrm{Hom}^\bullet_{\mathcal{D}_{\mathrm{BS},p|*}}(B, B'))\langle n - m \rangle$$

for all objects B, B' in $\mathcal{D}_{\mathrm{BS},p|*}$ and all $m, n \in \mathbb{Z}$. We define the *diagrammatic* $(p|*)$*-Hecke category* $\mathcal{D}_{p|*}$ to be the additive Karoubi envelope of $\mathcal{D}^{\langle - \rangle}_{\mathrm{BS},p|*}$. In other words, $\mathcal{D}_{p|*}$ is the closure of $\mathcal{D}_{\mathrm{BS},p|*}$ with respect to all finite direct sums, direct summands, and valuation shifts.

Remark 5.9.

(1) Even with the valuation structure on $\mathcal{D}_{\mathrm{BS},p|*}$, the right $\mathcal{D}^{\deg}_{\mathrm{BS}}$-module structure on $\mathcal{D}_{\mathrm{BS},p|*}$ does *not* extend to a graded $\mathcal{D}_{\mathrm{BS}}$-module structure. The primary reason for this is a mismatch between the degree and valuation of morphisms in $\mathcal{D}_{\mathrm{BS}}$ (e.g. $\deg(\mathrm{dot}_{\tilde{s}}) = 1 \neq 0 = \operatorname{val}(\mathrm{dot}_{\tilde{s}})$). Unlike degree, valuation does not behave well algebraically in $\mathcal{D}_{\mathrm{BS}}$; for example there are cases where $\operatorname{val}(L \otimes L') \neq \operatorname{val} L + \operatorname{val} L'$ for L, L' two morphisms in $\mathcal{D}_{\mathrm{BS}}$. We will return to this problem in §6.3.

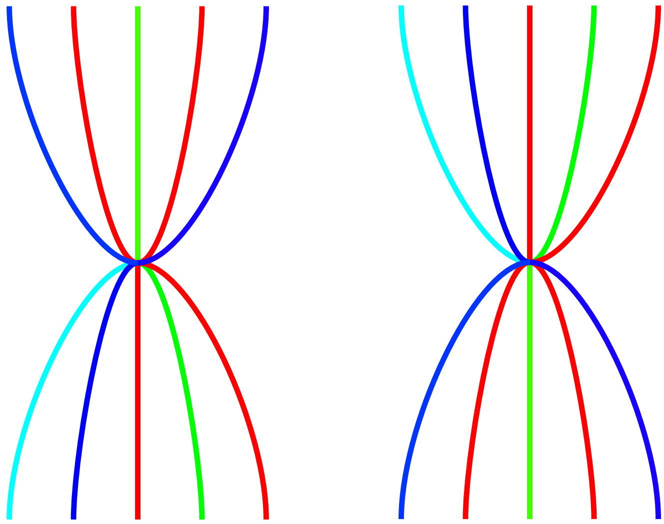


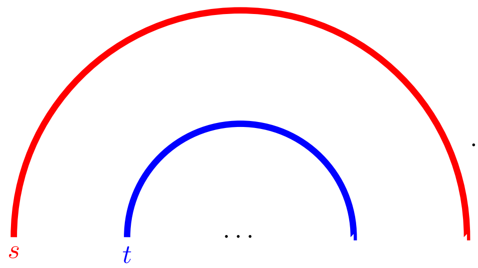
FIGURE 6. Two braid-like menorahs, for $p = 3$ and $\Phi_{\mathrm{f}} = A_2$.

(2) Unlike with Soergel diagrams, the valuation of a diagram in $\mathcal{D}_{\mathrm{BS},p|*}$ is not necessarily the sum of the valuations of all vertices and scalars in the diagram. This is because for L, L' two composable morphisms in $\mathcal{D}_{\mathrm{BS},p|*}$, we only have an inequality $\mathrm{val}(L \circ L') \geq \mathrm{val}\, L + \mathrm{val}\, L'$ rather than an equality.

5.2. **$(p|*)$-light leaves.** We will construct a basis for the Hom-spaces in $\mathcal{D}_{\mathrm{BS},p|*}$ analogous to the light leaves basis for $\mathcal{D}_{\mathrm{BS}}$. Before we describe the construction it is necessary to generalize the notion of a rex move in $\mathcal{D}_{\mathrm{BS}}$ to $\mathcal{D}_{\mathrm{BS},p|*}$.

Definition 5.10. For $\underline{w} = \underline{st\cdots} \in \underline{S}$ write

$\mathrm{cap}_{\underline{w}} =$



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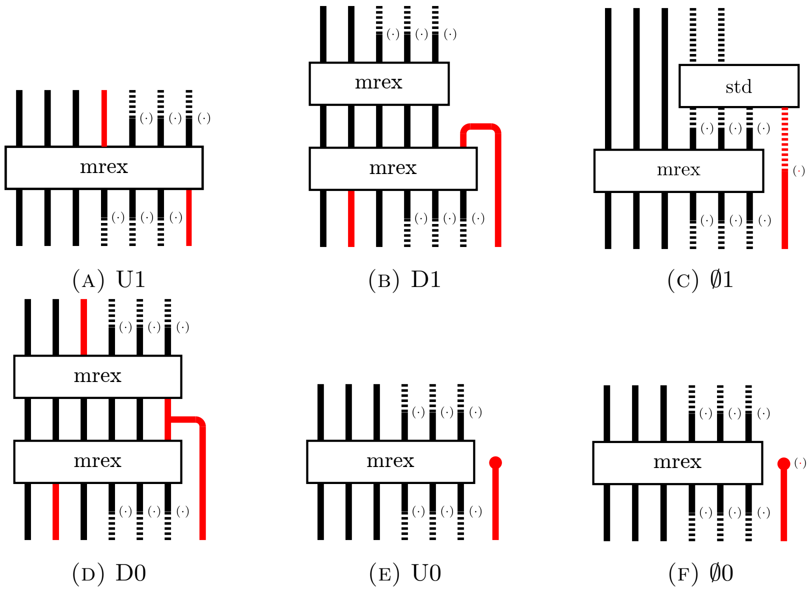
A *braid-like menorah* is a morphism in $\mathcal{D}_{\mathrm{BS},p|*}$ of the form

$$B^{(F)}_{\tilde{s}} \otimes B^{(F)}_{(\underline{w}')^{-1}} \xrightarrow{\mathrm{menorah}_{\underline{w},s,\underline{w}'} \otimes \mathrm{id}} B^{(F)}_{\underline{w}} \otimes B^{(F)}_{s} \otimes B^{(F)}_{\underline{w}'} \otimes B^{(F)}_{(\underline{w}')^{-1}} \xrightarrow{\mathrm{id} \otimes \mathrm{cap}_{\underline{w}'}} B^{(F)}_{\underline{w}} \otimes B^{(F)}_{s}$$

or its dual. We call a morphism in $\mathcal{D}_{\mathrm{BS},p|*}$ an *mrex move* if it can be generated (using composition and the tensor product) from identity morphisms, braid morphisms and braid-like menorahs.

In other words, mrex moves correspond to morphisms in $\mathcal{D}_{\mathrm{BS},p|*}$ which do not factor through $(p|*)$-Bott–Samelson bimodules of shorter length than the domain or codomain.

Example 5.11. Let $p = 3$ and $\Phi_{\mathrm{f}} = A_2$. Following the same diagrammatic convention as Example 4.3, Figure 6 depicts two braid-like menorahs.



(A) U1 (B) D1 (C) ∅1 (D) D0 (E) U0 (F) ∅0

FIGURE 7. Six maps for constructing $(p|*)$-light leaves.

Construction 5.12. Let $\underline{x} \in \underline{S}_{p|1}$, $\underline{r} \in [\underline{x}]_{p|*}$ and $\mathbf{c} \in [\underline{r}]$. Suppose $\underline{w} \in \underline{S}_p$ and $\underline{z} \in \underline{S}$ are rexes for c and $\hat{r}$. The *$(p|*)$-light leaves morphism* ${}^{p|*}\mathrm{LL}_{\mathbf{c},\underline{w},\underline{z}} : B_{\underline{x}}^{(p|*)} \to B_{\underline{w}}^{(p)} \otimes \hat{R}_{\underline{z}}$ is a morphism in $\mathcal{D}_{p|*}$ constructed inductively as follows. For each non-negative integer $i \leq \ell(\underline{x})$ let $\underline{x}_{\leq i}$, $\underline{r}_{\leq i}$, and $\mathbf{c}_{\leq i}$ consist of the first i terms of $\underline{x}$, $\underline{r}$, and $\mathbf{c}$ respectively. Fix rexes $\underline{w}_{\leq i} \in \underline{S}_p$ and $\underline{z}_{\leq i} \in \underline{S}$ for $c_{\leq i}$ and $\hat{r}_{\leq i}$ respectively such that $\underline{w} = \underline{w}_{\leq \ell(\underline{x})}$ and $\underline{z} = \underline{z}_{\leq \ell(\underline{x})}$. We set ${}^{p|*}\mathrm{LL}_i = {}^{p|*}\mathrm{LL}_{\mathbf{c}_{\leq i},\underline{w}_{\leq i},\underline{z}_{\leq i}}$ and define ${}^{p|*}\mathrm{LL}_i = \phi_i \circ ({}^{p|*}\mathrm{LL}_{i-1} \otimes \mathrm{id}_{B_{s_i}^{(p|*)}})$, where ϕ_i depends on the decorated type of $\mathbf{c}_i$. There are six possibilities for ϕ_i, which are illustrated in Figure 7. In these pictures, boxes labeled "mrex" are mrex moves, and boxes labeled "std" are basic standard diagrams with codomain corresponding to a rex. We also write $(\cdot)$ to denote the bi-valued normalizing factor for the nearest bivalent vertex to the left (see (3.21) and (3.20)).

As with the light leaves morphisms, the construction of ${}^{p|*}\mathrm{LL}_{\mathbf{c},\underline{w},\underline{z}}$ depends on several non-canonical choices and as such is not uniquely defined.

Example 5.13. Let $p = 3$ and $\Phi_{\mathrm{f}} = A_2$. Following the notational convention in Example 2.11 set $\underline{x} = \underline{0}_p|\underline{101202122} \in \underline{S}_{p|1}$. We depict a light leaves morphism for the match of type 1111110110 for the pattern $**1111**** \in [\underline{x}]_{p|*}$ in Figure 8, following the same diagrammatic conventions as Example 4.3.

Lemma 5.14. *Let $\underline{x} \in \underline{S}_{p|1}$, $\underline{r} \in [\underline{x}]_{p|*}$, and $\mathbf{c} \in [\underline{r}]$, and suppose $\underline{w} \in \underline{S}_p$ and $\underline{z} \in \underline{S}$ are rexes for c and $\hat{r}$ respectively. Then* $\operatorname{val} {}^{p|*}\mathrm{LL}_{\mathbf{c},\underline{w},\underline{z}} = \deg {}^{p|*}\mathrm{LL}_{\mathbf{c},\underline{w},\underline{z}} = d(\mathbf{c})$.

Proof. The localization matrices of the morphisms in Figure 7 have left coefficients lying in the subfield of Q generated by $\{a_s^F : s \in S\}$, which is also generated by $\{a_s : s \in S_{\mathrm{f}}\}$. This means that the valuation of these morphisms is equal to the degree. It is easy to check that the U0 morphism has degree $+1$, the D0 morphism has degree -1, and all other morphisms have degree 0. The total degree of ${}^{p|*}\mathrm{LL}_{\mathbf{c},\underline{w},\underline{z}}$ is the sum of these degrees and therefore equals $d(\mathbf{c})$. □

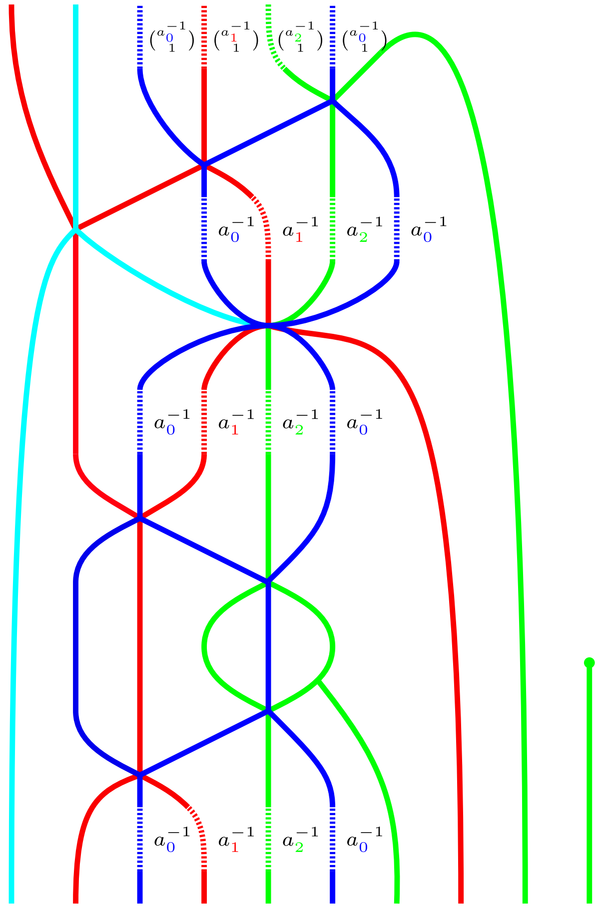


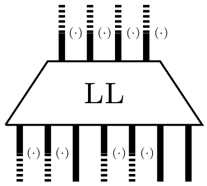
FIGURE 8. A $(p|*)$-light leaves morphism.

Let $\underline{w} \in \underline{S}_p$ and $\underline{z} \in \underline{S}$ be rexes. We write ${}^{p|*}\mathrm{LL}_{[[\underline{x}]],\underline{w},\underline{z}}$ to denote a complete selection of $(p|*)$-light leaves morphisms ${}^{p|*}\mathrm{LL}_{\mathbf{c},\underline{w},\underline{z}}$ over all patterns $\underline{r} \in [\underline{x}]_{p|*}$ and all matches $\mathbf{c} \in [\underline{r}]$ such that $c = w$ and $\hat{r} = z$. As with ordinary light leaves, for expressions $\underline{x}, \underline{y} \in \underline{S}_{p|1}$ and patterns $\underline{q} \in [\underline{x}]_{p|*}$ and $\underline{r} \in [\underline{y}]_{p|*}$, if we have matches $\mathbf{e} \in [\underline{q}]$ and $\mathbf{f} \in [\underline{r}]$ such that $e\hat{q} = f\hat{r}$, then we can construct the $(p|*)$-*double leaves morphism* ${}^{p|*}\mathbb{LL}_{\mathbf{e}}^{\mathbf{f}} = \overline{{}^{p|*}\mathrm{LL}_{\mathbf{f},\underline{w},\underline{z}}} \circ {}^{p|*}\mathrm{LL}_{\mathbf{e},\underline{w},\underline{z}} : B_{\underline{x}}^{(p|*)} \to B_{\underline{y}}^{(p|*)}$ in $\mathcal{D}_{\mathrm{BS},p|*}$, where $\underline{w}$ and $\underline{z}$ are rexes for $e = f$ and $\hat{q} = \hat{r}$ respectively. Now suppose for each $w \in W_p$ and $z \in {}^pW$ we have fixed corresponding rexes $\underline{w} \in \underline{S}_p$ and $\underline{z} \in \underline{S}$. We write ${}^{p|*}\mathbb{LL}_{[[\underline{x}]]}^{[[\underline{y}]]}$ to denote a complete selection of $(p|*)$-double leaves morphisms $B_{\underline{x}}^{(p|*)} \to B_{\underline{y}}^{(p|*)}$.

For $\underline{x} \in \underline{S}_{p|1}$, write $\underline{x}_1 \in \underline{S}$ for the expanded S-generator form of $\underline{x}$, where $(\underline{\tilde{s}s_{\alpha_h}})^{(p-1)/2}\underline{\tilde{s}}(\underline{s_{\alpha_h}\tilde{s}})^{(p-1)/2}$ is substituted for each $\tilde{s}_p$.

Theorem 5.15. *Let $\underline{x} \in \underline{S}_{p|1}$, and let $\underline{w} \in \underline{S}_p$ and $\underline{z} \in \underline{S}$ be rexes. Suppose we have fixed a complete selection ${}^{p|*}\mathrm{LL}_{[[\underline{x}]],\underline{w}|\underline{z}}$ of $(p|*)$-light leaves morphisms. Fix a rex $\underline{y} \in \underline{S}$ for wz. Then there exists a complete set of partially standardized ordinary light leaves morphisms $\mathrm{LL}'_{[\underline{x}_1],\underline{y}}$, each of the form*

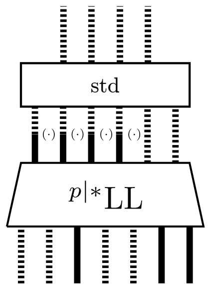
$$\mathrm{LL}'_{\mathbf{e},\underline{w}_1} : B_{\underline{x}}^{(p|*)} \xrightarrow{\textit{inclusion}} B_{\underline{x}_1} \xrightarrow{\mathrm{LL}} B_{\underline{y}} \xrightarrow{\textit{projection}} \hat{R}_{\underline{y}}$$



,

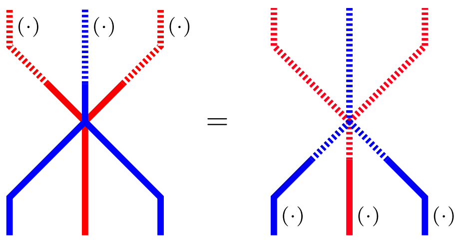
which is spanned by partially standardized $(p|*)$-light leaves morphisms ${}^{p|*}\mathrm{LL}'_{[[\underline{x}]],\underline{w},\underline{z}}$, each of the form

$$
{}^{p|*}\mathrm{LL}'_{\mathbf{c},\underline{w},\underline{z}} : B^{(p|*)}_{\underline{x}} \xrightarrow{{}^{p|*}\mathrm{LL}} B^{(p)}_{\underline{w}} \otimes \hat{R}_{\underline{z}} \xrightarrow{\textit{projection}} \hat{R}_{\underline{w}_1\underline{z}} \xrightarrow{\textit{standard}} \hat{R}_{\underline{y}}
$$



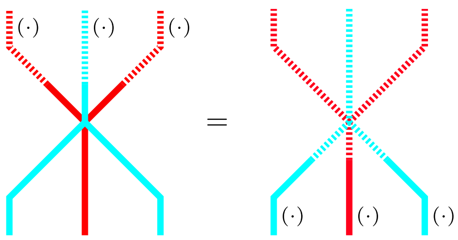
.

Proof. First we determine the effect of partially standardizing an mrex move. We already know that projectors placed on the top of an ordinary solid braid (i.e. one only involving S-generators) "propagate" through the braid:



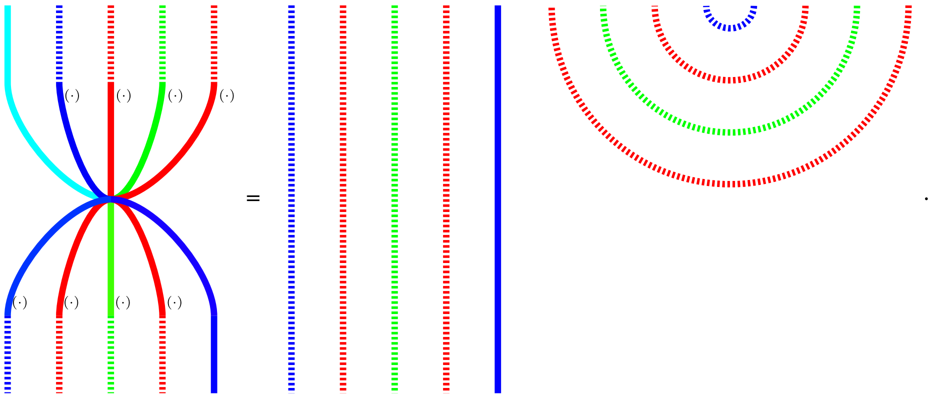
.

Doing the same thing to a p-affine braid results in a standard morphism, plus some inclusions on the bottom:



.

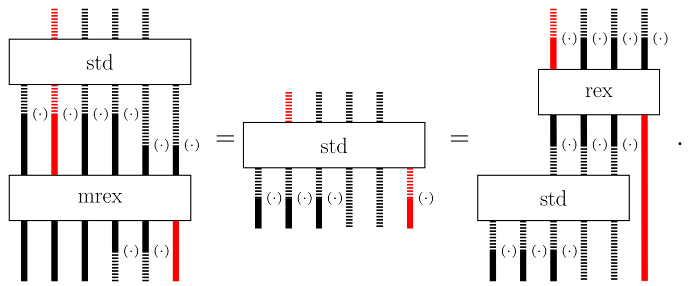
Finally if the candles of a braid-like menorah are standardized then the resulting morphism is just the identity tensored with a standard morphism:



Similarly, using the Jones–Wenzl relations we can "pull" a dot placed on the top of an ordinary braid or a p-affine braid through the braid to get a rex move on a smaller expression, plus a dot on the bottom. The same is true for dots on the shamash or the handle of a braid-like menorah, as long as all the candles are standardized.

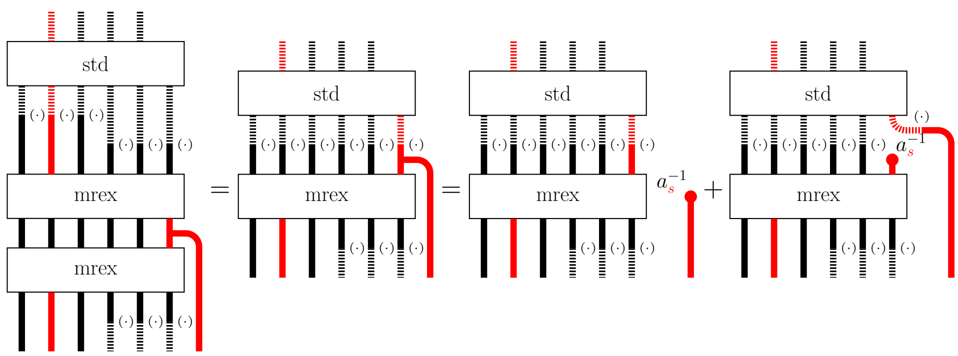
Next we try partially standardizing the maps ϕ_i above. As in Figure 7, boxes labeled "rex" are rex moves between two ordinary reduced expressions, boxes labeled "mrex" are mrex moves between two reduced $(p|1)$-expressions, and boxes labeled "std" are basic standard diagrams to a standard bimodule corresponding to some reduced expression.

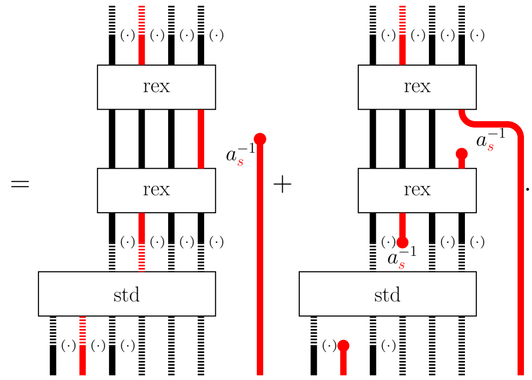
When i is an indeterminate index with decoration U, we can easily show that the partially standardized version of ϕ_i is nearly the same as that in the ordinary case. For example, when i is of decorated type U1 we have



The calculation for U0 is similar.

When i has decoration D we have to split the diagram into a sum. For example, when i is of decorated type D0 we have

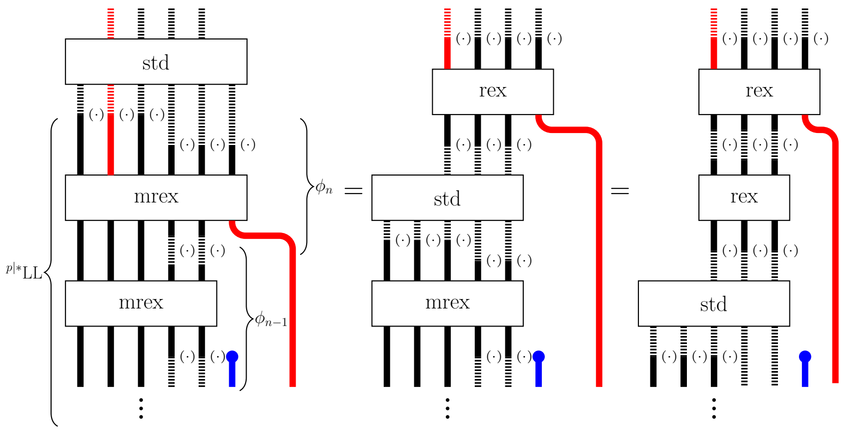




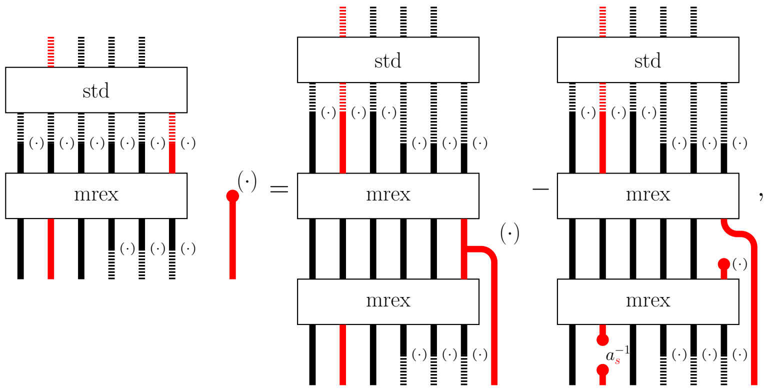
Again, the calculation for D1 is similar. In each of these cases, we get a partially standardized version of one of the four maps used for defining ordinary light leaves [8, Figure 2].

Now let $\mathbf{e} \in [\underline{x}_1]$ be a subsequence expressing w. Suppose we have already calculated $\mathrm{LL}_{\mathbf{f},\underline{y}}$ for all $\mathbf{f} < \mathbf{e}$, where the subsequences are ordered using the path dominance order from [8, §2.4]. If any of the standardized indices in the domain (i.e. any generator except the middle one in the S-expansion $(\underline{\tilde{s}s_{\alpha_{\mathrm{h}}}})^{(p-1)/2}\underline{\tilde{s}}(\underline{s_{\alpha_{\mathrm{h}}}\tilde{s}})^{(p-1)/2}$ of $\tilde{s}_p$) has type 0, then by pulling bivalent vertices and dots through braid moves, the partially standardized morphism $\mathrm{LL}'_{\mathbf{e},\underline{y}}$ vanishes. So without loss of generality all of these indices must have type 1, and there is a unique $\mathbf{c} \in [\underline{r}]$ for some $\underline{r} \in [\underline{x}]_{p|*}$ which as a subsequence equals $\mathbf{e}$.

Consider ${}^{p|*}\mathrm{LL}'_{\mathbf{c},\underline{w},\underline{z}}$. We use the above calculations to obtain $\mathrm{LL}'_{\mathbf{e},\underline{y}}$ by pulling the bivalent inclusions/projections (and any dots introduced by D-decorated indices) through the ϕ_i down to the bottom of the diagram. The goal is to get the resulting map to look like a light leaves morphism. The first step might look like



and continue downwards to the bottom of the diagram. For indeterminate indices i of $\mathbf{c}$ the resulting diagram is (possibly a scalar multiple of) a light leaves morphism. Fixed indices are similar except those corresponding to an index of $\mathbf{e}$ of type D0. In this situation, we use the relation



which is a difference of ordinary light leaves morphisms.

After pulling through all the ϕ_i and getting to the bottom we have shown that ${}^{p|*}\mathrm{LL}'_{\mathbf{c},\underline{w},\underline{z}}$ is equal to a partially standardized light leaves morphism $\mathrm{LL}'_{\mathbf{e},\underline{y}}$, plus some other partially standardized light leaves morphisms that we already know are spanned by $(p|*)$-light leaves morphisms. □

Let $w \in W$, and suppose for all $\underline{x}, \underline{y} \in \underline{S}_{p|1}$ we have fixed a complete selection of double leaves morphisms ${}^{p|*}\mathbb{LL}_{[[\underline{x}]]}^{[[\underline{y}]]}$. Recall that $I_{\not\geq w} = \{z \in W : z \not\geq w\}$ is an ideal in the Bruhat order and let ${}^{p|*}\mathbb{LL}_{\not\geq w}$ denote the span of the ${}^{p|*}\mathbb{LL}$ maps

$$\{{}^{p|*}\mathbb{LL}_{\mathbf{e}}^{\mathbf{f}} : \underline{x}, \underline{y} \in \underline{S}_{p|1},\ \underline{q} \in [\underline{x}]_{p|*},\ \underline{r} \in [\underline{y}]_{p|*},\ \mathbf{e} \in [\underline{q}],\ \mathbf{f} \in [\underline{r}],\ e\hat{q} = f\hat{r} \in I_{\not\geq w}\}.$$

In other words, ${}^{p|*}\mathbb{LL}_{\not\geq w}$ is spanned by double leaves morphisms which factor through $I_{\not\geq w}$. It follows from Theorem 5.15 and [8, Claim 6.19] that ${}^{p|*}\mathbb{LL}_{\not\geq w}$ is a 2-sided ideal of morphisms in $\mathcal{D}_{\mathrm{BS},p|*}$, namely the $\hat{R}$-extension of the ideal $\mathbb{LL}_{\not\geq w}$ in $\mathcal{D}_{\mathrm{BS}}$. Let $\mathcal{D}_{\mathrm{BS},p|*}^{\geq w} = \mathcal{D}_{\mathrm{BS},p|*} / {}^{p|*}\mathbb{LL}_{\not\geq w}$, and let $\mathcal{D}_{p|*}^{\geq w}$ be the corresponding quotient of $\mathcal{D}_{p|*}$. Using Theorem 5.15 and Lemma 5.14 in conjunction with Theorem 3.5 we obtain a light leaves basis result.

Corollary 5.16. *Let $\underline{x} \in \underline{S}_{p|1}$, and let $\underline{w} \in \underline{S}_p$ and $\underline{z} \in \underline{S}$ be rexes. The light leaves morphisms ${}^{p|*}\mathrm{LL}_{[[\underline{x}]],\underline{w},\underline{z}}$ form a valuation basis for $\mathrm{Hom}^{\bullet}_{\mathcal{D}_{p|*}^{\geq w}}(B_{\underline{x}}^{(p|*)}, B_{\underline{w}}^{(p)} \otimes \hat{R}_{\underline{z}})$.*

The analogous double leaves basis result follows by the same reasoning as in [8, §7.3].

Corollary 5.17. *Let $\underline{x}, \underline{y} \in \underline{S}_{p|1}$. The double leaves morphisms ${}^{p|*}\mathbb{LL}_{[[\underline{x}]]}^{[[\underline{y}]]}$ form a valuation basis for $\mathrm{Hom}^{\bullet}_{\mathcal{D}_{\mathrm{BS},p|*}}(B_{\underline{x}}^{(p|*)}, B_{\underline{y}}^{(p|*)})$.*

5.3. **Indecomposable objects.** As with Theorem 3.4, Corollary 5.17 is strong enough to give a complete classification of the indecomposable objects in $\mathcal{D}_{p|*}$. We closely follow the strategy of [8, §6.6].

Lemma 5.18. *The category $\mathcal{D}_{p|*}$ is Krull–Schmidt.*

Proof. It is sufficient to show that $E = \mathrm{End}^{\bullet}_{\mathcal{D}_{p|*}}(B)$ is local for any indecomposable object B in $\mathcal{D}_{p|*}$. By Corollary 5.17 E is a finite $\hat{R}$-algebra. Any finite algebra over a complete local ring is either local or contains an idempotent, but E cannot contain an idempotent as $\mathcal{D}_{p|*}$ is a Karoubi envelope. Thus E is local and $\mathcal{D}_{p|*}$ is Krull–Schmidt. □

The Frobenius functor extends naturally to a functor $F : \mathcal{D}^F \to \mathcal{D}_{p|*}$. We will similarly extend our notation by writing $B_x^{(F)}$ and $B_x^{(p)}$, where $x \in W$ and $y \in W_p$, for the images under F of the indecomposable Soergel bimodules B_x^F and $B_{F^{-1}(y)}^F$ in $\mathcal{D}^F$ respectively.

Lemma 5.19. *Let $\underline{x}, \underline{y} \in \underline{S}$ and $w \in {}^pW$. The map*

$$F \otimes \mathrm{id}_{\hat{R}_w} : \mathrm{Hom}^\bullet_{\hat{R}\otimes\mathcal{D}^F}(B_{\underline{x}}^F, B_{\underline{y}}^F) \longrightarrow \mathrm{Hom}^\bullet_{\mathcal{D}_{p|*}}(B_{\underline{x}}^{(F)} \otimes \hat{R}_w, B_{\underline{y}}^{(F)} \otimes \hat{R}_w)$$
$$f \longmapsto F(f) \otimes \mathrm{id}_{\hat{R}_w}$$

is a valuation isomorphism.

Proof. Let $\underline{x}' = F(\underline{x})$ and $\underline{y}' = F(\underline{y})$. Fix a rex $\underline{w} \in \underline{S}$ for w, and let M and M' denote $\mathrm{Hom}^\bullet_{\mathcal{D}_{p|*}}(B_{\underline{x}}^{(F)} \otimes \hat{R}_{\underline{w}}, B_{\underline{y}}^{(F)} \otimes \hat{R}_{\underline{w}})$ and $\mathrm{Hom}^\bullet_{\mathcal{D}_{p|*}}(B_{\underline{x}'|\underline{w}}^{(p|*)}, B_{\underline{y}'|\underline{w}}^{(p|*)})$ respectively. By Corollary 5.17 the $(p|*)$-double leaves morphisms ${}^{p|*}\mathbb{LL}_{[[\underline{x}'|\underline{w}]]}^{[[\underline{y}'|\underline{w}]]}$ form a basis for M'. Since $B_{\underline{x}}^{(F)} \otimes \hat{R}_{\underline{w}}$ is a summand of $B_{\underline{x}'|\underline{w}}^{(p|*)}$ (and similarly for $\underline{y}$) this means M is spanned by $p_{\underline{y}} \circ {}^{p|*}\mathbb{LL}_{[[\underline{x}'|\underline{w}]]}^{[[\underline{y}'|\underline{w}]]} \circ i_{\underline{x}}$, where $i_{\underline{x}}$ and $p_{\underline{y}}$ are the inclusion and projection maps for these summands.

Now note that $p_{\underline{y}} \circ {}^{p|*}\mathbb{LL}_{\mathbf{e}}^{\mathbf{f}} \circ i_{\underline{x}}$ is non-zero only when the matches $\mathbf{e}$ and $\mathbf{f}$ come from the pattern $* \cdots * |1 \cdots 1 \in [\underline{x}'|\underline{w}]_{p|*}$ and $* \cdots * |1 \cdots 1 \in [\underline{y}'|\underline{w}]_{p|*}$ respectively. This shows that M has basis $F(\mathbb{LL}_{[\underline{x}]}^{[\underline{y}]}) \otimes \mathrm{id}_{\hat{R}_{\underline{w}}}$, which proves the result. ☐

Theorem 5.20. *Let $x \in W_p$ and $w \in {}^pW$. The object $B_x^{(p)} \otimes \hat{R}_w$ is indecomposable in $\mathcal{D}_{p|*}$.*

Proof. Let $\underline{x} \in \underline{S}_p$ be a rex for x, and let $\underline{y} = F^{-1}(\underline{x}) \in \underline{S}$. By Theorem 3.6 B_y^F is an indecomposable summand of $B_{\underline{y}}^F$. This means that $B_x^{(p)} \otimes \hat{R}_w$ is a summand of $B_{\underline{x}}^{(p)} \otimes \hat{R}_w$. Let E and E' denote the endomorphism algebras of $B_x^{(p)} \otimes \hat{R}_w$ and $B_{\underline{x}}^{(p)} \otimes \hat{R}_w$. By Lemma 5.19 E' is isomorphic to $\hat{R} \otimes \mathrm{End}^\bullet_{\mathcal{D}^F}(B_{\underline{y}}^F)$, and this isomorphism restricts to an isomorphism $E \cong \hat{R} \otimes \mathrm{End}^\bullet_{\mathcal{D}^F}(B_y^F)$.

Let $E_0 \cong \hat{R} \otimes_R \mathrm{End}^0_{\mathcal{D}^F}(B_y^F)$ be the $\hat{R}$-subalgebra of E generated by the (F-images of) degree 0 morphisms in the ordinary diagrammatic category $\mathcal{D}^F$. Note that $a_{\tilde{s}}^{-1} \in \hat{R}$, so if f is a morphism of negative degree $-n$ and $r \in \hat{R}$, then

$$rf = ra_{\tilde{s}}^{-n}(a_{\tilde{s}}^n f) \in \hat{R}\,\mathrm{End}^0_{\mathcal{D}^F}(B).$$

This shows that $\hat{R} \otimes_R \mathrm{End}^{<0}_{\mathcal{D}^F}(B_{F^{-1}(x)}^F) \subseteq E_0$. In addition, for any morphism L in $\mathcal{D}^F$ we have $\deg L \leq \mathrm{val}\, F(L)$ by Lemma 5.7, which implies that

$$\mathrm{End}^{>0}_{\mathcal{D}_{p|*}} B_x^{(p)} \geq \hat{R} \otimes_R \mathrm{End}^{>0}_{\mathcal{D}^F}(B_{F^{-1}(x)}^F),$$

where the first term is the ideal of all morphisms in E of positive valuation. Since $E \cong \sum_i \hat{R} \otimes_R \mathrm{End}^i_{\mathcal{D}^F}(B)$, combining these facts gives

$$E = E_0 + \mathrm{End}^{>0}_{\mathcal{D}_{p|*}} B_x^{(p)}. \tag{5.1}$$

The category $\mathcal{D}^F$ is Krull–Schmidt [8, Lemma 6.25], so the ring $\mathrm{End}^0_{\mathcal{D}^F}(B_y^F)$ is local. Let $\mathfrak{m}$ be the maximal ideal of this ring, and let

$$I = \hat{R}\mathfrak{m} + (a_s^F)_{s\in S} F\left(\mathrm{End}^0_{\mathcal{D}^F}(B_y^F)\right) + \mathrm{End}^{>0}_{\mathcal{D}_{p|*}} B_x^{(p)} \subseteq E,$$

where $(a_s^F)_{s\in S}$ is the maximal ideal of $\hat{R}$. The first two terms are ideals in E_0, so the decomposition (5.1) shows that I is an ideal in E. Clearly $E = E_0 + I$ follows from (5.1) as well.

We will show that all morphisms in $E \setminus I$ are invertible, and thus that E is local with maximal ideal I and that $B_x^{(p)}$ is indecomposable in $\mathcal{D}_{p|*}$. Suppose $f \in E \setminus I$. We write

$$f = r_0 f_0 + r_{\mathfrak{m}} f_{\mathfrak{m}} + r^F f^F + r_{>0} f_{>0}$$

where $r_0, r_{\mathfrak{m}}, r_{>0} \in \hat{R}$, $r^F \in (a_s^F)_{s \in S}$, $f_0, f^F \in F(\mathrm{End}^0_{\mathcal{D}^F}(B_y^F))$, $f_{\mathfrak{m}} \in \mathfrak{m}$, and $f_{>0} \in \mathrm{End}^{>0}_{\mathcal{D}_{p|*}}(B^{(p)_x})$.

Clearly $r_0 \notin (a_s^F)_{s \in S_{\mathrm{f}}}$ and $f_0 \notin \mathfrak{m}$ as $f \notin I$. Thus we can write

$$r_0 f_0 + r_{\mathfrak{m}} f_{\mathfrak{m}} = r_0 f_0 \left(1 + \frac{r_{\mathfrak{m}}}{r_0} f_0^{-1} f_{\mathfrak{m}}\right).$$

Since $\mathrm{End}^0_{\mathcal{D}^F}(B_y^F)$ is finite-dimensional the maximal ideal $\mathfrak{m}$ is nilpotent. But $f_0^{-1} f_{\mathfrak{m}}$ is contained in $\mathfrak{m}$, so the above morphism is invertible using the formula $(1+x)^{-1} = 1 + x + x^2 + \cdots$.

The remaining two terms in the sum for f are contained in an ideal $J = (a_s^F)_{s \in S} \mathrm{End}^0_{\mathcal{D}^F}(B_y^F) + \mathrm{End}^{>0}_{\mathcal{D}_{p|*}} B_x^{(p)}$. From Theorem 3.4 and Corollary 5.17 J is generated as an $\hat{R}$-module by morphisms $a_s^F \mathbb{LL}_{\mathbf{e}}^{\mathbf{f}}$ (for $d(\mathbf{e}) + d(\mathbf{f}) = 0$ and any $s \in S$) and ${}^{p|*}\mathbb{LL}_{\mathbf{e}}^{\mathbf{f}}$ (for $d(\mathbf{e}) + d(\mathbf{f}) > 0$). This basis is finite, so for sufficiently large n we have $J^n \leq (a_s^F) J$. Yet $\hat{R}$ is complete with respect to its maximal ideal $(a_s^F)_{s \in S}$ so $f \in (r_0 f_0 + r_{\mathfrak{m}} f_{\mathfrak{m}}) + J$ is invertible using the same trick as above. □

For $x \in W_p$ and $w \in {}^pW$, we will now write $B_{x|w}^{(p|*)}$ for the indecomposable object $B_x^{(p)} \otimes \hat{R}_w$ from the previous theorem.

Theorem 5.21. *Every indecomposable object in $\mathcal{D}_{p|*}$ is valuation isomorphic to an object of the form $B_{y|w}^{(p|*)}\langle m \rangle$ for some $y \in W_p$, $w \in {}^pW$, and $m \in \mathbb{Z}$.*

Proof. Suppose B is a direct summand of $B_{\underline{x}}^{(p|*)}$ for some $\underline{x} \in \underline{S}_{p|1}$, and that $\eta \in \mathrm{End}^\bullet_{\mathcal{D}_{\mathrm{BS},p|*}}(B_{\underline{x}}^{(p|*)})$ is an idempotent corresponding to this summand. We can write

$$\eta = \sum \lambda_{\mathbf{e},z,\mathbf{f}} \, {}^{p|*}\mathbb{LL}_{\mathbf{e}}^{\mathbf{f}},$$

where the sum is over matches $\mathbf{e}, \mathbf{f}$ for patterns in $[\underline{x}]_{p|*}$ corresponding to the same group element $z \in W$ and $\lambda_{\mathbf{e},z,\mathbf{f}} \in \hat{R}$. Pick $z' \in W$ maximal in the Bruhat order such that there is a non-zero coefficient $\lambda_{\mathbf{e},z',\mathbf{f}} \neq 0$ for some matches $\mathbf{e}, \mathbf{f}$. In $\mathcal{D}_{p|*}^{\geq z'}$ we get

$$\eta = \sum \gamma_{\mathbf{e},\mathbf{f}} (\overline{{}^{p|*}\mathrm{LL}_{\mathbf{f},\underline{y},\underline{w}}} \circ {}^{p|*}\mathrm{LL}_{\mathbf{e},\underline{y},\underline{w}}),$$

where the sum is over matches $\mathbf{e}, \mathbf{f}$ which both correspond to z', for some rexes $\underline{y} \in \underline{S}_p$ and $\underline{w} \in \underline{S}$ with $w \in {}^pW$, and $\gamma_{\mathbf{e},\mathbf{f}} \in \hat{R}$. Now assume that for all matches in the sum we have

$${}^{p|*}\mathrm{LL}_{\mathbf{e},\underline{y},\underline{w}} \circ \eta \circ \overline{{}^{p|*}\mathrm{LL}_{\mathbf{f},\underline{y},\underline{w}}} \in (a_s^F)_{s \in S} \leq \hat{R} = \mathrm{End}^\bullet_{\mathcal{D}_{p|*}^{\geq z'}}(B_{\underline{y}}^{(p)} \otimes \hat{R}_{\underline{w}})$$

Then by expanding out $\eta^3 = \eta$ we get $\gamma_{\mathbf{e},\mathbf{f}} \in (a_s^F)_{s \in S}$. But this implies that

$$\eta \in (a_s^F)_{s \in S} \mathrm{End}^\bullet_{\mathcal{D}_{\mathrm{BS},p|*}}(B_{\underline{x}}^{(p|*)}) \leq J\left(\mathrm{End}^\bullet_{\mathcal{D}_{\mathrm{BS},p|*}}(B_{\underline{x}}^{(p|*)})\right)$$

where $J(\cdot)$ denotes the Jacobson radical. Since η is idempotent, we obtain a contradiction. Hence there must be some matches $\mathbf{e}, \mathbf{f}$ for which the following composition

$$B_{\underline{y}}^{(p)} \otimes \hat{R}_w \xrightarrow{\overline{{}^{p|*}\mathrm{LL}_{\mathbf{f},\underline{y},\underline{w}}}} B \xrightarrow{{}^{p|*}\mathrm{LL}_{\mathbf{e},\underline{y},\underline{w}}} B_{\underline{y}}^{(p)} \otimes \hat{R}_w$$

is invertible in $\mathcal{D}_{p|*}^{\geq z'}$. This induces an invertible morphism

$$B_y^{(p)} \otimes \hat{R}_w \xrightarrow{i} B \xrightarrow{p} B_y^{(p)} \otimes \hat{R}_w$$

which proves the result. □

Let $M \cong \bigoplus_i \hat{R}\langle d_i \rangle$ be a free $\hat{R}$-valuation module. We write

$$\operatorname{valrk} M = \sum_i v^{d_i} \in \mathbb{Z}[v^{\pm 1}],$$

which we call the *valuation rank* of M.

Definition 5.22. The $(p|*)$-*diagrammatic character* is the $\mathbb{Z}[v^{\pm 1}]$-linear map

$$\begin{aligned} \operatorname{ch}_{p|*} : [\mathcal{D}_{p|*}] &\longrightarrow \mathbb{H}_{p|*} \\ [B] &\longmapsto \sum_{x \in W} (\operatorname{valrk} \operatorname{Hom}^\bullet_{\mathcal{D}_{p|*}^{\geq x}}(B, B_x^{(p|*)})) H_x^{(p|*)}. \end{aligned}$$

Since $\mathcal{D}_{p|*}$ has a left $\mathcal{D}^F$-action, the Grothendieck group $[\mathcal{D}_{p|*}]$ has the structure of a left $[\mathcal{D}^F]$-module. By Corollary 3.8 we have $[\mathcal{D}^F] \cong \mathbb{H} \overset{F}{\cong} \mathbb{H}_p$, so $[\mathcal{D}_{p|*}]$ has the structure of a left $\mathbb{H}_p$-module via F.

Suppose $\underline{y} \in \underline{S}_p$ and $\underline{w} \in \underline{S}$ are rexes with $w \in {}^pW$. In the category $\mathcal{D}_{p|*}^{\geq yw}$, the objects $B_{y|w}^{(p|*)}$, $B_y^{(p)} \otimes \hat{R}_w$, and $B_{\underline{y}|\underline{w}}^{(p|*)}$ are all isomorphic. Thus using Corollary 5.16 we observe that

$$\operatorname{ch}_{p|*} [B_{\underline{x}}^{(p|*)}] = H_{\underline{x}}^{(p|*)} \tag{5.2}$$

for all $\underline{x} \in \underline{S}_{p|1}$.

Theorem 5.23. *The map* $\operatorname{ch}_{p|*} : [\mathcal{D}_{p|*}] \to \mathbb{H}_{p|*}$ *is an isomorphism of left* $\mathbb{H}_p$-*modules.*

Proof. The category $\mathcal{D}_{p|*}$ is the closure of $\mathcal{D}_{\mathrm{BS},p|*}$ with respect to direct sums, direct summands, and valuation shifts. This means that the $\mathbb{Z}[v^{\pm 1}]$-module $[\mathcal{D}_{p|*}]$ is spanned by the isomorphism classes $\{[B_{\underline{x}}^{(p|*)}] : \underline{x} \in \underline{S}_{p|1}\}$. The calculation (5.2) along with Lemma 2.13 then immediately implies that $\operatorname{ch}_{p|*}$ is a left $\mathbb{H}_p$-module homomorphism. Moreover, the isomorphism classes $\{[B_x^{(p|*)}] : x \in W\}$ form a $\mathbb{Z}[v^{\pm 1}]$-basis for $[\mathcal{D}_{p|*}]$ by Theorem 5.21. For any $x \in W$ we have

$$\operatorname{ch}_{p|*} [B_x^{(p|*)}] \in H_x^{(p|*)} + \sum_{y < x} \mathbb{Z}[v^{\pm 1}] H_y^{(p|*)}$$

using the construction of $B_x^{(p|*)}$ and the definition of $\operatorname{ch}_{p|*}$. This means that $\{\operatorname{ch}_{p|*} [B^{(p|*)_x}] : x \in W\}$ is a basis for $\mathbb{H}_{p|*}$ as well, which shows that $\operatorname{ch}_{p|*}$ is an isomorphism. □

6. The functor Ψ

The category $\mathcal{D}_{p|*}$ has a $(\mathcal{D}^F, \mathcal{D}^{\deg})$-bimodule structure. So far we have focused our attention on the left $\mathcal{D}^F$-module structure of this category. For example, we can reinterpret Theorem 5.21 as the following categorical decomposition

$$\mathcal{D}_{p|*} \cong \bigoplus_{w \in {}^pW} \mathcal{D}^F \otimes \hat{R}_w \tag{6.1}$$

of left $\mathcal{D}^F$-modules. This decomposition is also a *block decomposition*, because

$$\operatorname{Hom}_{\mathcal{D}_{p|*}}(\mathcal{D}^F \otimes \hat{R}_w, \mathcal{D}^F \otimes \hat{R}_{w'}) = 0$$

for any distinct $w, w' \in {}^pW$. We will construct a functor Ψ on $\mathcal{D}^{\text{deg}}$ by writing the right $\mathcal{D}^{\text{deg}}$-action in terms of this decomposition.

6.1. **Construction.** In this section we will mostly refer to standard diagrams by their isomorphism class, i.e. we will write $\hat{R}_w$ instead of $\hat{R}_{\underline{w}}$. To deal with potential ambiguity we introduce several notions of equivalence between morphisms.

Definition 6.1. We say that two morphisms f, g in $\mathcal{D}_{\text{std}}$ are *basic standard equivalent* if there exist basic standard diagrams σ, σ' such that $f = \sigma \circ g \circ \sigma'$. More generally, we say that two morphisms h, k in $\mathcal{D}_{\text{BS,std}}$ are basic standard equivalent if there exist matrices Θ, Θ' whose entries are basic standard diagrams such that for H, K the localization matrices of h, k respectively, we have $H = \Theta \circ K \circ \Theta'$.

So far we have only used patterns in the context of matches. By themselves, patterns describe how to localize a subset of the boundary strings of a Soergel diagram.

Definition 6.2. Let $\underline{x} \in \underline{S}$. Suppose $\underline{r} = (\underline{r}_1, \underline{r}_2, \dots, \underline{r}_m)$ is a pattern for $\underline{x}$. We write

$$B_{\underline{x}} = \bigotimes_{i=1}^{m} \begin{cases} B_{s_i} & \text{if } \underline{r}_i = (s_i, *), \\ \hat{R}_{s_i} & \text{if } \underline{r}_i = (s_i, 1), \\ \hat{R} & \text{if } \underline{r}_i = (s_i, 0) \end{cases}$$

in $\mathcal{D}_{p|*}$. Similarly we write

$$p_{\underline{r}} = \bigotimes_{i=1}^{m} \begin{cases} \mathrm{id}_{B_{s_i}} & \text{if } \underline{r}_i = (s_i, *), \\ \mathrm{bivalent}_{s_i} \binom{a_{s_i}^{-1}}{1} & \text{if } \underline{r}_i = (s_i, 1), \\ \mathrm{dot}_{s_i} \binom{a_{s_i}^{-1}}{1} & \text{if } \underline{r}_i = (s_i, 0), \end{cases}$$

a morphism $B_{\underline{x}} \longrightarrow B_{\underline{r}}$ in $\mathcal{D}_{p|*}$.

Lemma 6.3. *Let $\underline{x} \in \underline{S}$ and $w \in {}^pW$. Then*

$$\hat{R}_w \otimes B_{\underline{x}} \xrightarrow{\oplus_{\underline{r}} p_{\underline{r}}} \bigoplus_{\underline{r} \in [\underline{x}]_*(w)} B_{\underline{r}}$$

is an isomorphism in $\mathcal{D}_{p|*}$.

Proof. Let $\underline{r} = (\underline{r}_1, \underline{r}_2, \dots, \underline{r}_m) \in [\underline{x}]_*(w)$. By the construction of $[\underline{x}]_*$, for each fixed index i with generator s_i we have $w\hat{r}_{\leq i-1} s_i (w\hat{r}_{\leq i-1})^{-1} \notin S_p$. This means that $(w\hat{r}_{\leq i-1})a_{s_i}$ is invertible in $\hat{R}$ by Lemma 1.11. This means we have a splitting

$$\hat{R}_w \otimes B_{\underline{x}} = \hat{R}_w \otimes B_{\underline{x}_{\leq i-1}} \otimes B_{s_i} \otimes B_{\underline{x}_{\geq i+1}} \cong \hat{R}_w \otimes B_{\underline{x}_{\leq i-1}} \otimes \left(\hat{R} \oplus \hat{R}_{s_i}\right) \otimes B_{\underline{x}_{\geq i+1}}$$

in $\mathcal{D}_{p|*}$. Applying this splitting to each fixed index of a pattern, over all patterns in $[\underline{x}]_*(w)$, gives the decomposition. $\square$

For $w \in {}^pW$ and $\underline{x} \in \underline{S}$ let $\underline{r} = (\underline{r}_1, \underline{r}_2, \dots, \underline{r}_m) \in [\underline{x}]_*(w)$. As in the Bruhat stroll let $\underline{r}_{\leq i}$ denote the truncated pattern for an integer $0 \leq i \leq m$. For each indeterminate term $\underline{r}_i = (s_i, *)$ we have $w\hat{r}_{\leq i-1} s_i (w\hat{r}_{\leq i-1})^{-1} \in S_p$ by the construction of $[\underline{x}]_*$. Let $w\underline{r}(w\hat{r})^{-1}$ denote the expression in $\underline{S}_p$ formed by these generators. By inserting standard caps where necessary we obtain an isomorphism $B_{\underline{r}} \to B^{(p)}_{w\underline{r}(w\hat{r})^{-1}}$ which is basic standard equivalent to the identity. This gives a reinterpretation of Lemma 6.3.

Corollary 6.4. *Let $\underline{x} \in \underline{S}$ and $w \in {}^pW$. Any morphism in $\mathcal{D}_{p|*}$ of the form*

$$\hat{R}_w \otimes B_{\underline{x}} \longrightarrow \bigoplus_{z \in {}^pW} \left(\bigoplus_{\substack{\underline{r} \in [\underline{x}]_*(w) \\ w\hat{r}=z}} B^{(p)}_{w\underline{r}z^{-1}} \right) \otimes \hat{R}_z,$$

with $(z, \underline{r})$-components basic standard equivalent to $p_{\underline{r}}$ over all $\underline{r} \in [\underline{x}]_*(w)$ such that $w\hat{r} = z$, is an isomorphism.

Note that by Theorem 3.12 the components of the decomposition above are uniquely determined by the expressions $\underline{w}$ and $\underline{z}$ used to define the standard bimodules in the domain and codomain. The two decompositions in Lemma 6.3 and Corollary 6.4 give rise to two different ways to localize a Soergel diagram with respect to patterns.

Definition 6.5. Let $f : B_{\underline{x}} \to B_{\underline{y}}$ be a morphism in $\mathcal{D}^{\text{deg}}_{\text{BS}}$. Suppose $w, z \in {}^pW$ with $\underline{q} \in [\underline{x}]_*(w)$ and $\underline{r} \in [\underline{y}]_*(w)$ such that $w\hat{q} = w\hat{r} = z$.

(1) The *partial localization* of $\text{id}_{\hat{R}_w} \otimes f$ with respect to $\underline{q}$ and $\underline{r}$ is the composition

$$B_{\underline{q}} \xrightarrow{\overline{p_{\underline{q}}}} \hat{R}_w \otimes B_{\underline{x}} \xrightarrow{\text{id}_{\hat{R}_w} \otimes f} \hat{R}_w \otimes B_{\underline{y}} \xrightarrow{p_{\underline{r}}} B_{\underline{r}}.$$

(2) The *basic standard equivalent partial localization* of $\text{id}_{\hat{R}_w} \otimes f \otimes \text{id}_{\hat{R}_{z^{-1}}}$ with respect to $\underline{q}$ and $\underline{r}$ is the composition

$$B^{(p)}_{w\underline{q}(w\hat{q})^{-1}} \xrightarrow{\sim} B_{\underline{q}} \xrightarrow{\overline{p_{\underline{q}}}} \hat{R}_w \otimes B_{\underline{x}} \xrightarrow{\text{id}_{\hat{R}_w} \otimes f} \hat{R}_w \otimes B_{\underline{y}} \xrightarrow{p_{\underline{r}}} B_{\underline{r}} \xrightarrow{\sim} B^{(p)}_{w\underline{r}(w\hat{r})^{-1}},$$

where the isomorphisms are basic standard equivalent to the identity.

From these two localization methods, we obtain two related analogues of the localization matrix. Let $\hat{R} \otimes \mathcal{D}^{F,\text{dev}}_{\text{BS}}$ be the *devaluation* of $\hat{R} \otimes \mathcal{D}^{F}_{\text{BS}}$, analogous to the degrading of $\mathcal{D}_{\text{BS}}$. For convenience we write $\mathcal{D}^{\text{deg},\oplus}_{\text{BS}}$ and $\hat{R} \otimes \mathcal{D}^{F,\text{dev},\oplus}_{\text{BS}}$ for the additive envelopes of $\mathcal{D}^{\text{deg}}_{\text{BS}}$ and $\hat{R} \otimes \mathcal{D}^{F,\text{dev}}_{\text{BS}}$ respectively.

Definition 6.6.

(1) Let $\underline{x} \in \underline{S}$. We define two bi-valued ${}^pW \times {}^pW$ matrices $\Psi'(B_{\underline{x}})$ and $\Psi(B_{\underline{x}})$ of objects in $\mathcal{D}_{p|*}$ and $\hat{R} \otimes \mathcal{D}^{F,\text{dev},\oplus}_{\text{BS}}$ respectively as follows.

(i) The (w, z)-entry of $\Psi'(B_{\underline{x}})$ is

$$\bigoplus_{\substack{\underline{q} \in [\underline{x}]_*(w) \\ w\hat{q}=z}} B_{\underline{q}}.$$

(ii) The (w, z)-entry of $\Psi(B_{\underline{x}})$ is

$$\bigoplus_{\substack{\underline{q} \in [\underline{x}]_*(w) \\ w\hat{q}=z}} B^F_{F^{-1}(w\underline{q}z^{-1})}$$

(2) Let $f : B_{\underline{x}} \to B_{\underline{y}}$ be a morphism in $\mathcal{D}^{\text{deg}}_{\text{BS}}$. We define two bi-valued ${}^pW \times {}^pW$ matrices $\Psi'(f)$ and $\Psi(f)$ of morphisms in $\mathcal{D}_{p|*}$ and $\hat{R} \otimes \mathcal{D}^{F,\text{dev},\oplus}_{\text{BS}}$ as follows.

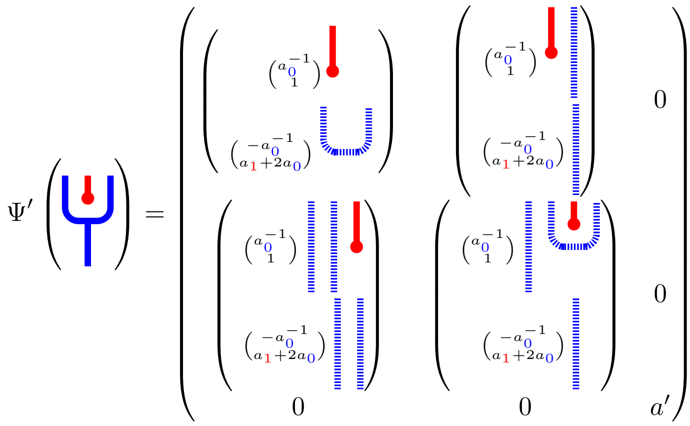
(i) The (w, z)-entry of $\Psi'(f)$ is the direct sum of all partial localizations of $\text{id}_{\hat{R}_w} \otimes f$ with respect to patterns $\underline{q} \in [\underline{x}]_*(w)$ and $\underline{r} \in [\underline{y}]_*(w)$ for which $w\hat{q} = w\hat{r} = z$.

(ii) The (w, z)-entry of $\Psi(f)$ is the direct sum of all basic standard equivalent partial localizations of $\text{id}_{\hat{R}_w} \otimes f \otimes \text{id}_{\hat{R}_{z^{-1}}}$ with respect to patterns $\underline{q} \in [\underline{x}]_*(w)$ and $\underline{r} \in [\underline{y}]_*(w)$ for which $w\hat{q} = w\hat{r} = z$.

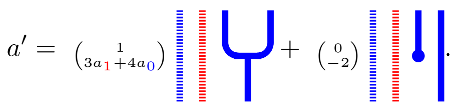
These constructions extend to both $\mathcal{D}_{\mathrm{BS}}^{\mathrm{deg},\oplus}$ and to $\mathcal{D}^{\mathrm{deg}}$ in the natural way.

For $f : B_{\underline{x}} \to B_{\underline{y}}$ note that the matrix of the domains (resp. codomains) of the entries of $\Psi(f)$ is given by $\Psi(B_{\underline{x}})$ (resp. $\Psi'(B_{\underline{y}})$) and similarly for Ψ'. We will show in the next section that Ψ is a functor and that it describes a connection between Soergel bimodules in $\mathcal{D}^{\mathrm{deg}}$ and "smaller" Soergel bimodules in $\mathcal{D}^{F}$ as claimed in the Introduction. Strictly speaking we do not need to define Ψ' in order to define Ψ, but we have included this construction because it is usually much easier to calculate $\Psi(f)$ from $\Psi'(f)$, as the following example shows.

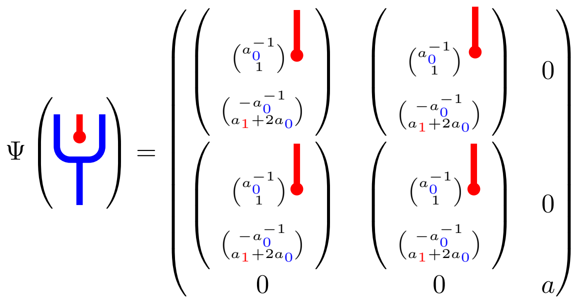
Example 6.7. Let $p = 3$ and $\Phi_{\mathrm{f}} = A_1$. Label the unique finite generator 1 (colored red) and the affine generator 0 (colored blue). Here is an example of $\Psi'(f)$ for a morphism $f : B_{\underline{0}} \to B_{\underline{010}}$ in $\mathcal{D}_{\mathrm{BS}}^{\mathrm{deg}}$:



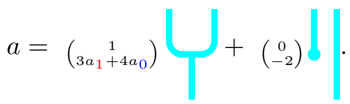
where



Here is $\Psi(f)$ for the same morphism:



where



6.2. Categorical results.

Definition 6.8. We define the *matrix category* $\mathcal{M}_{{}^pW}(\hat{R} \otimes \mathcal{D}_{\mathrm{BS}}^{F,\mathrm{dev},\oplus})$ to be the following $\hat{R}$-linear monoidal category.

Objects: The objects are $W_p \times W_p$ matrices of objects in $\hat{R} \otimes \mathcal{D}_{\mathrm{BS}}^{F,\mathrm{dev},\oplus}$. The tensor product of two matrices of objects A and A' is the categorified matrix product, with entries

$$(AA')_{z,w} = \bigoplus_{y \in {}^pW} A_{z,y} \otimes A'_{y,w}. \tag{6.2}$$

Morphisms: The morphisms $A \to A'$ are $W_p \times W_p$ matrices L, with each entry $L_{z,w} : A_{z,w} \to A'_{z,w}$ a morphism in $\hat{R} \otimes \mathcal{D}_{\mathrm{BS}}^{F,\mathrm{dev},\oplus}$. The composition of two composable matrices of morphisms L and L' is the categorified Hadamard product or entrywise product, with entries

$$(L \circ_{{}^pW} L')_{z,w} = L_{z,w} \circ L'_{z,w}. \tag{6.3}$$

The tensor product of two arbitrary matrices of morphisms $(L_{z,w})$ and $(L'_{z,w})$ is the categorified product matrix

$$(LL')_{z,w} = \bigoplus_{y \in {}^pW} L_{z,y} \otimes L'_{y,w}. \tag{6.4}$$

Remark 6.9. The matrix category can be motivated as follows. Let

$$(\hat{R} \otimes \mathcal{D}_{\mathrm{BS}}^{F,\mathrm{dev}\oplus})^{{}^pW} = \bigoplus_{w \in {}^pW} (\hat{R} \otimes \mathcal{D}_{\mathrm{BS}}^{F,\mathrm{dev},\oplus})_w$$

denote the (external) direct sum category of $|{}^pW|$ copies of $(\hat{R} \otimes \mathcal{D}_{\mathrm{BS}}^{F,\mathrm{dev}\oplus})$. The monoidal structure on $\mathcal{D}_{\mathrm{BS}}^{F,\mathrm{dev},\oplus}$ induces a left $(\hat{R} \otimes \mathcal{D}_{\mathrm{BS}}^{F,\mathrm{dev},\oplus})$-module structure on $(\hat{R} \otimes \mathcal{D}_{\mathrm{BS}}^{F,\mathrm{dev},\oplus})^{{}^pW}$. The category of endofunctors of $(\hat{R} \otimes \mathcal{D}_{\mathrm{BS}}^{F,\mathrm{dev},\oplus})^{{}^pW}$ has the structure of a monoidal category, with endofunctor composition as the tensor product. The matrix category $\mathcal{M}_{{}^pW}(\hat{R} \otimes \mathcal{D}_{\mathrm{BS}}^{F,\mathrm{dev},\oplus})$ naturally embeds into this endofunctor category via the categorified "row vector times matrix" action

$$\begin{aligned} A : \bigoplus_{w \in {}^pW} (\hat{R} \otimes \mathcal{D}_{\mathrm{BS}}^{F,\mathrm{dev}\oplus})_w &\longrightarrow \bigoplus_{w \in {}^pW} (\hat{R} \otimes \mathcal{D}_{\mathrm{BS}}^{F,\mathrm{dev},\oplus})_w \\ (C_w)_{w \in {}^pW} &\longmapsto \left(\bigoplus_{z \in {}^pW} C_z \otimes A_{z,w} \right)_{w \in {}^pW}. \end{aligned}$$

From this embedding we may recover the formulas (6.2)–(6.4).

As Ψ maps objects and morphisms to $\mathcal{M}_{{}^pW}(\hat{R} \otimes \mathcal{D}_{\mathrm{BS}}^{F,\mathrm{dev},\oplus})$, we are now prepared to state our main theorem.

Theorem 6.10. *The mapping Ψ is a monoidal functor*

$$\Psi : \mathcal{D}_{\mathrm{BS}}^{\mathrm{deg}} \longrightarrow \mathcal{M}_{{}^pW}(\hat{R} \otimes \mathcal{D}_{\mathrm{BS}}^{F,\mathrm{dev},\oplus}),$$

which extends on the additive Karoubi envelopes to a monoidal functor

$$\Psi : \mathcal{D}^{\mathrm{deg}} \longrightarrow \mathcal{M}_{{}^pW}(\hat{R} \otimes \mathcal{D}^{F,\mathrm{dev}}).$$

Proof. Let $\underline{x}, \underline{y} \in \underline{S}$ and $w \in {}^pW$. From the construction of $[\underline{xy}]_*$ there is a bijection

$$\begin{aligned} \{(\underline{q}', \underline{r}') : \underline{q}' \in [\underline{x}]_*(w),\ \underline{r}' \in [\underline{y}]_*(w\hat{q}')\} &\longrightarrow [\underline{xy}]_*(w) \\ (\underline{q}', \underline{r}') &\longmapsto \underline{q'r'}. \end{aligned}$$

For any $z \in {}^pW$ this bijection restricts to

$$\{(\underline{q}', \underline{r}') : \underline{q}' \in [\underline{x}]_*(w),\ \underline{r}' \in [\underline{y}]_*(w\hat{q}'),\ w\hat{q}'\hat{r}' = z\} \longrightarrow \{\underline{q} \in [\underline{xy}]_*(w) : w\hat{q} = z\}.$$

The set $P_{w,z}$ on the left-hand side is partitioned by the sets

$$P_{w,z}^{z'} = \{(\underline{q}', \underline{r}') : \underline{q}' \in [\underline{x}]_*(w),\ \underline{r}' \in [\underline{y}]_*(w\hat{q}'),\ w\hat{q}' = z',\ z'\hat{r}' = z\}$$

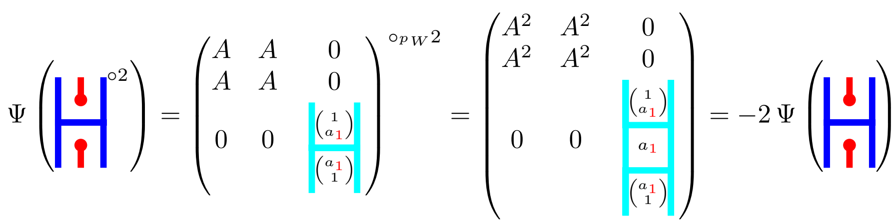
indexed over $z' \in {}^pW$. Now we observe that the (w, z)-entry of $\Psi(B_{\underline{xy}})$ is

$$\begin{aligned}
\bigoplus_{\substack{\underline{q} \in [\underline{x}]_*(w) \\ w\hat{q} = z}} B^F_{F^{-1}(w\underline{q}z^{-1})} &= \bigoplus_{(\underline{q}', \underline{r}') \in P_{w,z}} B^F_{F^{-1}(w\underline{q'r'}z^{-1})} \\
&= \bigoplus_{(\underline{q}', \underline{r}') \in P_{w,z}} B^F_{F^{-1}(w\underline{q}'(w\hat{q}')^{-1})} \otimes B^F_{F^{-1}((w\hat{q}')\underline{r}'z^{-1})} \\
&= \bigoplus_{z' \in {}^pW} \left(\bigoplus_{(\underline{q}', \underline{r}') \in P^{z'}_{w,z}} B^F_{F^{-1}(w\underline{q}'(z')^{-1})} \otimes B^F_{F^{-1}(z'\underline{r}'z^{-1})} \right) \\
&= \bigoplus_{z' \in {}^pW} (\Psi(B_{\underline{x}}))_{w,z'} \otimes (\Psi(B_{\underline{y}}))_{z',z}.
\end{aligned}$$

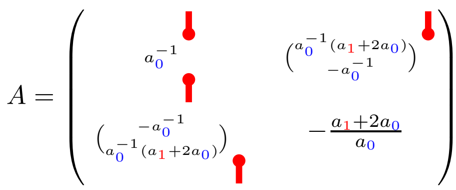
This shows that Ψ is monoidal on objects, and a similar argument shows that it is monoidal on morphisms as well.

Now let $g : B_{\underline{x}} \to B_{\underline{y}}$ be a morphism in $\mathcal{D}^{\mathrm{deg}}_{\mathrm{BS}}$. For $w \in {}^pW$, the localization matrix of $\mathrm{id}_{R_w} \otimes g$ can be viewed as a block diagonal matrix, with a separate block $G_{w,w'}$ for each $w' \in {}^pW$ corresponding to entries which are standard morphisms on $Q_{w'}$. By construction the (w, w')-entry of $\Psi'(g)$ has localization matrix equal to $G_{w,w'}$. If $h : B_{\underline{y}} \to B_{\underline{z}}$ is another morphism in $\mathcal{D}^{\mathrm{deg}}_{\mathrm{BS}}$ with blocks $H_{w,w'}$, then the localization matrix of $h \circ g$ has blocks $H_{w,w'} \circ G_{w,w'}$. This implies that the entries of $\Psi'(h \circ g)$ and $\Psi'(h) \circ_{{}^pW} \Psi'(g)$ have identical localization matrices. As localization is faithful, it follows that Ψ' is a functor. We conclude that Ψ is also a functor, because $F(\Psi(g)_{w,w'}) \otimes \mathrm{id}_{\hat{R}_{w'}}$ is basic standard equivalent to $\Psi'(g)_{w,w'}$ by definition. Finally the matrix category is equivalent as an additive category to a direct sum of copies of $\hat{R} \otimes \mathcal{D}^{F,\mathrm{dev},\oplus}_{\mathrm{BS}}$, so its additive Karoubi envelope is just $\mathcal{M}(\hat{R} \otimes \mathcal{D}^{F,\mathrm{dev}})$ and the functor extends to $\mathcal{D}^{\mathrm{deg}}$ as claimed. $\square$

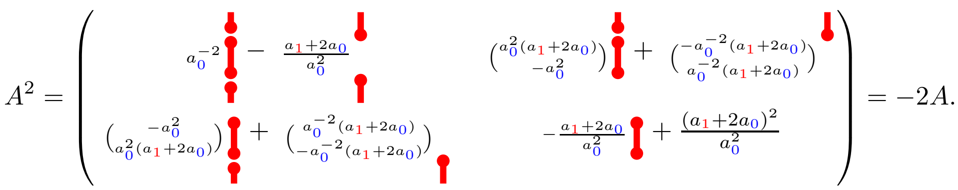
Example 6.11. As before suppose $p = 3$ and $\Phi_{\mathrm{f}} = A_1$ with labeling as in Example 6.7. Here is an example of the Hadamard product on an idempotent (up to scaling):



where



and



The next result will show that the monoidal action of $\mathcal{D}^{\deg}$ on $(\hat{R} \otimes \mathcal{D}^{F,\mathrm{dev}})^{{}^pW}$ defined by Ψ (see Remark 6.9) is essentially the same as the right monoidal action on $\mathcal{D}_{p|*}$.

Proposition 6.12. *The left $(\hat{R} \otimes \mathcal{D}^{F,\mathrm{dev}})$-module equivalence*

$$(\hat{R} \otimes \mathcal{D}^{F,\mathrm{dev}})^{{}^pW} \longrightarrow \bigoplus_{w \in {}^pW} (\mathcal{D}^{F,\mathrm{dev}} \otimes \hat{R}_w) \cong \mathcal{D}_{p|*}$$

$$(C_w)_{w \in {}^pW} \longmapsto (C_w \otimes \hat{R}_w)_{w \in {}^pW}$$

is $\mathcal{D}^{\deg}$-equivariant up to natural isomorphism.

Proof. For equivariance up to (possibly unnatural) isomorphism, it's enough to check a similar statement on all Bott–Samelson bimodules. Namely, for all $\underline{x}, \underline{y} \in \underline{S}$ and $w \in {}^pW$, we must show that

$$B^F_{\underline{x}} \otimes \hat{R}_w \otimes B_{\underline{y}} \cong B_{\underline{x}} \otimes \bigoplus_{z \in {}^pW} \Psi(B_{\underline{y}})_{w,z}.$$

But by the definition of Ψ, this isomorphism follows by tensoring $B^F_{\underline{x}}$ with the decomposition of $\hat{R}_w \otimes B_{\underline{y}}$ from Corollary 6.4. This decomposition is clearly natural on Bott–Samelson bimodules, so the equivariance is natural as well. $\square$

6.3. **Decategorified results.** In this section we aim to decategorify the functor Ψ. Although Ψ is inherently ungraded, we will present these results using certain categories which do have valuation structure. We hope that future work will explain precisely how the grading on $\mathcal{D}$ interacts with this valuation structure.

Taking the valuation on $\hat{R} \otimes \mathcal{D}^F$ into account, consider the category $\mathcal{M}_{{}^pW}(\hat{R} \otimes \mathcal{D}^F)$. This matrix category has a valuation on morphisms in the sense of Definition 4.8; namely, for L a matrix of morphisms we set $\operatorname{val} L$ be the minimal valuation of all the entries. For each $\underline{x} \in \underline{S}$ we write $B^{(*)}_{\underline{x}}$ for the object in $\mathcal{M}_{{}^pW}(\hat{R} \otimes \mathcal{D}^F)$ corresponding to $\Psi(B_{\underline{x}})$ with each entry having valuation shift 0. In this matrix category it is also useful to define a selective version of a valuation shift.

Definition 6.13. For each $A \in {}^p\mathcal{U}$ and $m \in \mathbb{Z}$ we define $\langle m \rangle_A \in \mathcal{M}_{{}^pW}(\hat{R} \otimes \mathcal{D}^{F,\oplus}_{\mathrm{BS}})$ by

$$(\langle m \rangle_A)_{z,w} = \begin{cases} \hat{R}\langle m \rangle & \text{if } z = w \in A, \\ \hat{R} & \text{if } z = w \notin A, \\ 0 & \text{otherwise.} \end{cases}$$

In the special case that $A = {}^pW(s,*) = \{w \in {}^pW : W_p ws = W_p w\}$ we write $\langle m \rangle_s$ for $\langle m \rangle_A$.

Remark 6.14. We can rewrite the valuation shift functor on $\mathcal{M}_{{}^pW}(\hat{R} \otimes \mathcal{D}^F)$ as $\langle 1 \rangle = \langle 1 \rangle_{{}^pW}$ using the language of partial valuation shifts. In general partial valuation shifts do *not* commute with tensor products, e.g. for any $s \in S$ and $A \in {}^p\mathcal{U}$

$$B^{(*)}_s \langle 1 \rangle_A \cong \langle 1 \rangle_{As} B^{(*)}_s.$$

This particular isomorphism is a categorification of the relation $\underline{H}^{(*)}_s u_A = u_{As} \underline{H}^{(*)}_s$ (or equivalently (2.15)) inside $\mathbb{H}_*$.

Definition 6.15. The category $\mathcal{D}^{(*),\oplus}_{\mathrm{BS}}$ is the full additive $\hat{R}$-linear monoidal subcategory of $\mathcal{M}_{{}^pW}(\hat{R} \otimes \mathcal{D}^F)$ generated by $B^{(*)}_{\underline{x}}$ and $\langle m \rangle_A$ for all $\underline{x} \in \underline{S}$, $m \in \mathbb{Z}$, and $A \in {}^p\mathcal{U}$.

The category $\mathcal{D}_{\mathrm{BS}}^{(*),\oplus}$ inherits a monoidal action on $\bigoplus_{w\in {}^pW}(\mathcal{D}^F \otimes \hat{R}_w)$, defined by

$$\begin{aligned} D : \bigoplus_{w\in {}^pW}(\mathcal{D}^F \otimes \hat{R}_w) &\longrightarrow \bigoplus_{w\in {}^pW}(\mathcal{D}^F \otimes \hat{R}_w) \\ (C_w \otimes \hat{R}_w)_{w\in {}^pW} &\longmapsto \left(\left(\bigoplus_{z\in {}^pW} C_z \otimes D_{z,w}\right) \otimes \hat{R}_w\right)_{w\in {}^pW}, \end{aligned}$$

which comes from a valuation module version of the equivalence in Proposition 6.12. This defines a monoidal action on $\mathcal{D}_{p|*}$.

Theorem 6.16. *There is an isomorphism of $\mathbb{Z}[v^{\pm 1}]$-algebras mapping*

$$\begin{aligned} \mathbb{H}_* &\longrightarrow [\mathcal{D}_{\mathrm{BS}}^{(*),\oplus}] \\ \underline{H}_s^{(*)} &\longmapsto [B_s^{(*)}] \\ u_A &\longmapsto [\langle 1\rangle_A] \end{aligned}$$

for all $s \in S$ and $A \in {}^p\mathcal{U}$. Moreover, $\mathrm{ch}_{p|*} : [\mathcal{D}_{p|*}] \to \mathbb{H}_{p|*}$ is an isomorphism of right $[\mathcal{D}_{\mathrm{BS}}^{(*),\oplus}] \cong \mathbb{H}_*$-modules.

Proof. For each $\underline{x} \in \underline{S}$ and $w \in W$, we calculate

$$\begin{aligned} \mathrm{ch}_{p|*}[\hat{R}_w \cdot B_{\underline{x}}^{(*)}] &= \mathrm{ch}_{p|*}\left[\left(\bigoplus_{\substack{\underline{q}\in[\underline{x}]_*(w) \\ w\hat{q}=z}} B^F_{F^{-1}(w\underline{q}z^{-1})}\right) \otimes \hat{R}_z\right] \\ &= \sum_{\substack{\underline{q}\in[\underline{x}]_*(w) \\ \mathbf{e}\in[\underline{q}]}} v^{d_w(\mathbf{e})} H^{(p|*)}_{we\hat{q}} \\ &= H_w^{(p|*)} \cdot \underline{H}_{\underline{x}}^{(*)} \end{aligned}$$

by Lemma 2.13, using the fact that the defect of a subsequence of $w\underline{q}z^{-1}$ (where $w\hat{q} = z$) is equal to the w-twisted defect of the corresponding match. Similarly $[\langle 1\rangle_A]$ acts on $[\mathcal{D}_{p|*}]$ as u_A. As the monoidal action commutes with the left $(\hat{R} \otimes \mathcal{D}^F)$-module action and the objects $\hat{R}_w$ span $\mathcal{D}_{p|*}$ with respect to this action, the isomorphism classes $[B_{\underline{x}}^{(*)}]$ and $[\langle 1\rangle]_A$ have the correct action on all of $[\mathcal{D}_{p|*}]$. Thus some quotient of $[\mathcal{D}_{\mathrm{BS}}^{(*),\oplus}]$ is isomorphic to $\mathbb{H}_*$ by Lemma 2.7. But the monoidal action of $\mathcal{D}_{\mathrm{BS}}^{(*),\oplus}$ on $\mathcal{D}_{p|*}$ is faithful because it is the restriction of the faithful action of $\mathcal{M}_{{}^pW}(\hat{R} \otimes \mathcal{D}^F)$ on $\mathcal{D}_{p|*}$, so we are done. $\square$

Let

$$\psi : \mathbb{H}_* \longrightarrow M_{|{}^pW|}(\mathbb{H}_*)$$

be the homomorphism defined by

$$\psi(H_s^{(*)})_{w,z} = \begin{cases} H_s^{(*)} & \text{if } W_p ws = W_p z \text{ and } w = z, \\ 1 & \text{if } W_p ws = W_p z \text{ and } w \neq z, \\ 0 & \text{otherwise} \end{cases}$$

for all $w, z \in {}^pW$ and $s \in S$. It is easy to check that ψ is just another form of the right $\mathbb{H}_*$-action on $\mathbb{H}_{p|*}$, in the sense that

$$H^{(p|*)}_{x|w} H = \sum_{z\in {}^pW} H_x^{(p)} F(\psi(H)_{w,z}) H_z^{(p|*)}$$

for all $H \in \mathbb{H}_*$, $x \in W_p$, and $w \in {}^pW$. We call ψ the *matrix recursion representation* on $\mathbb{H}_*$.

Now write ${}_{v=1}\mathbb{H}_*$ and ${}_{v=1}\mathbb{H}_{p|*}$ for the quotients $\mathbb{H}_{p|*}/(v-1)$ and $\mathbb{H}_*/(v-1)$, and for H in $\mathbb{H}_*$ or $\mathbb{H}_{p|*}$ write ${}_{v=1}H$ for the image of H in the quotient. We have a ring isomorphism and a $(\mathbb{Z}W_p, \mathbb{Z}W)$-bimodule isomorphism mapping

$$\begin{aligned} {}_{v=1}\mathbb{H}_* &\longrightarrow \mathbb{Z}W \qquad & {}_{v=1}\mathbb{H}_{p|*} &\longrightarrow \mathbb{Z}W \\ {}_{v=1}H_s^{(*)} &\longmapsto s & {}_{v=1}H_{x|w} &\longmapsto xw \end{aligned}$$

for all $s \in S$ and $x|w \in W$. It is easy to check that the ungraded matrix recursion representation ${}_{v=1}\psi$ is the matrix recursion representation ξ on $\mathbb{Z}W$ from the Introduction. The ungraded embedding of $\mathcal{D}_{\mathrm{BS}}^{\mathrm{deg},\oplus}$ into $\mathcal{D}_{\mathrm{BS}}^{(*),\oplus}$, along with Theorem 6.16 immediately implies the following.

Corollary 6.17. *The functor $\Psi : \mathcal{D}^{\mathrm{deg}} \to \mathcal{M}_{{}^pW}(\hat{R} \otimes \mathcal{D}^{F,\mathrm{dev}})$ is a categorification of the ungraded matrix recursion representation ${}_{v=1}\psi$.*

We can now prove our lower bound on the p-canonical basis as discussed in the Introduction. Recall that the *p-canonical basis* is defined by ${}^p\underline{H}_x = \mathrm{ch}[B_x]$ for all $x \in W$.

Corollary 6.18. *Let $x \in W$. Then*

$${}_{v=1}^{\ p}\underline{H}_x \in \sum_{\substack{y \in W \\ w \in {}^pW \\ F(y)w \leq x}} \mathbb{Z}_{\geq 0} F({}_{v=1}^{\ p}\underline{H}_y)\, {}_{v=1}H_w.$$

Proof. Consider $1 \cdot {}_{v=1}^{\ p}\underline{H}_x$ in ${}_{v=1}\mathbb{H}_{p|*}$. By the previous corollary this corresponds to sum of the ungraded characters in the first row of $\Psi([B_x])$. The result follows immediately. $\square$

Example 6.19. Let $\Phi_{\mathrm{f}} = A_1$ and $p = 3$. We have ${}^3\underline{H}_{010}\underline{H}_1 = \underline{H}_{0101} + \underline{H}_{01}$. By [14, Proposition 4.2(6)] this is a sum of p-canonical basis elements, so we can apply Corollary 6.18. Setting $v = 1$ we get

$${}_{v=1}^{\ 3}\underline{H}_{010}\underline{H}_1 = 2({}_{v=1}^{\ 3}\underline{H}_1)H_{\mathrm{id}} + ({}_{v=1}^{\ 3}\underline{H}_1 + {}_{v=1}^{\ 3}\underline{H}_{0_3})H_0 + ({}_{v=1}^{\ 3}\underline{H}_1 + {}_{v=1}^{\ 3}\underline{H}_{0_3})H_{01}.$$

We depict this using *weight diagrams* in Figure 9, where the alcove corresponding to $y \in W$ is marked with a number of dots equal to the coefficient of H_y. One can visualize the two decompositions above by coloring the dots (i.e. standard subquotients) according to which underlined terms (i.e. indecomposable summands) they lie in. Since the decompositions lead to different colorings, we draw a complete colored weight diagram for each decomposition. Corollary 6.18 implies that the p-canonical summands partition the colors in the W_p-weight diagram. In particular, it is easy to see that ${}^3\underline{H}_{0101} \neq \underline{H}_{0101}$. Otherwise the green dots and the black dots in the W-weight diagram correspond to different p-canonical basis elements, but this cannot be the case because it is impossible to partition the colors in the W_p-weight diagram below in the same manner. Weight diagrams are very similar to the diagrams in [13] depicting tilting characters, and the processes of applying Corollary 6.18 or Andersen's tilting character lower bounds to a potential diagram are essentially identical.

Remark 6.20. We broadly conjecture that there is a Kazhdan–Lusztig-type construction of a self-dual ${}^p\mathbb{U}$-basis $\{\underline{H}_x^{(*)} : x \in W\}$ of $\mathbb{H}_*$, and that there is a new category related to $\mathcal{D}_{\mathrm{BS}}^{(*),\oplus}$ with indecomposable objects $\{B_x^{(*)} : x \in W\}$. Unfortunately, precisely characterizing such constructions is tricky. On the algebraic side, it is unclear what should take the place of the degree condition on coefficients of

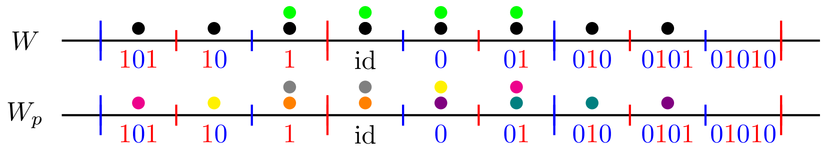


Figure 9. Weight diagrams for ${}^3\underline{H}_{010}\underline{H}_1$.

the standard basis elements $H_x^{(*)}$. On the categorical side, it is unclear which morphisms in $\mathcal{D}_{\mathrm{BS}}^{(*),\oplus}$ should be considered, as extra morphisms will produce too many indecomposable objects. Once the correct definition is found, the next natural step would be to prove a Soergel's conjecture-like result (see [8, Conjecture 3.18]), equating the isomorphism class $[B_x^{(*)}]$ with $\underline{H}_x^{(*)}$ under the isomorphism in Theorem 6.16 for fixed $x \in W$ when p is sufficiently large. Their images in $\mathbb{Z}W$ should then match the images of a "second generation" Kazhdan–Lusztig basis in $\mathbb{H}$ analogous to second generation tilting characters [17, 16].

List of symbols

$(\cdot)$: a bi-valued normalizing factor. 41
a_s, $a_s^\vee$: the vector/covector corresponding to $s \in S$ in the realization V. 5
a_s^F, $(a_s^F)^\vee$: the vector/covector corresponding to $s \in S$ in the realization V^F. 10
α_{h}: the highest short root of Φ_{f}. 5

$\mathrm{bivalent}_s$: the s-colored bivalent morphism in $\mathcal{D}_{\mathrm{BS,std}}$. 23
$\mathrm{braid}_{s,t}$, $\mathrm{braid}_{s,t}^F$: the s-and-t-colored braid morphism in $\mathcal{D}_{\mathrm{BS}}$, $\mathcal{D}_{\mathrm{BS}}^F$. 20
$\mathrm{braid}_{s,t}^{(F)}$: the "$p$-dilated" form of $\mathrm{braid}_{s,t}^F$. 34
$[B]$: the isomorphism class of B. 21
$B_{\underline{r}}$: a pattern summand in $\mathcal{D}_{p|*}$. 51
B_x, B_x^F: an indecomposable Soergel bimodule in $\mathcal{D}_{\mathrm{BS}}$, $\mathcal{D}_{\mathrm{BS}}^F$. 21
$B_{\underline{x}}$, $B_{\underline{x}}^F$: a Bott–Samelson bimodule in $\mathcal{D}_{\mathrm{BS}}$, $\mathcal{D}_{\mathrm{BS}}^F$. 17
$B_{\underline{x}}^{(*)}$: the image under Ψ of $B_{\underline{x}}$ with no valuation shift. 56
$B_x^{(F)}$: the image under F of B_x^F. 47
$B_{\underline{x}}^{(F)}$: a "p-dilated" Bott–Samelson bimodule. 33
$B_x^{(p)}$: the image under F of $B_{F^{-1}(x)}^F$. 47
$B_{x|w}^{(p|*)}$: an indecomposable object in $\mathcal{D}_{p|*}$. 49
$B_{\underline{x}|\underline{w}}^{(p|*)}$, $B_{\underline{x}}^{(p)}$: a $(p|*)$-Bott–Samelson bimodule. 37

ch: the diagrammatic character. 21
$\mathrm{ch}_{p|*}$: the $(p|*)$-diagrammatic character. 50

$d(\mathbf{e})$; $d_w(\mathbf{c})$: the defect of a subsequence $\mathbf{e}$; the w-twisted defect of a match $\mathbf{c}$. 11, 15
$\deg f$: degree of a scalar in R or a morphism in $\mathcal{D}$, $\mathcal{D}^F$. 16
dot_s, dot_s^F: the s-colored dot morphism in $\mathcal{D}_{\mathrm{BS}}$, $\mathcal{D}_{\mathrm{BS}}^F$. 20
$\mathrm{dot}_s^{(F)}$: the "p-dilated" form of dot_s^F. 33
$\mathcal{D}$: the diagrammatic Hecke category over the realization V. 20
$[\mathcal{D}]$, $[\mathcal{D}_{p|*}]$: the Grothendieck ring, module of $\mathcal{D}$, $\mathcal{D}_{p|*}$. 21
$\mathcal{D}_{\mathrm{BS}}$: the diagrammatic Bott–Samelson category over the realization V. 17
$\mathcal{D}_{\mathrm{BS}}^{(*),\oplus}$: the full additive subcategory of $\mathcal{M}_{{}^pW}(\hat{R}\otimes\mathcal{D}^F)$ generated by $B_{\underline{x}}^{(*)}$ and $\langle m\rangle_A$. 56

$\mathcal{D}_{\mathrm{BS}}^{\deg}$, $\mathcal{D}^{\deg}$: the degrading of $\mathcal{D}_{\mathrm{BS}}$, $\mathcal{D}$. 37
$\mathcal{D}_{\mathrm{BS}}^{\deg,\oplus}$: the additive envelope of $\mathcal{D}_{\mathrm{BS}}^{\deg}$. 52
$\mathcal{D}_{\mathrm{BS}}^{F}$, $\mathcal{D}^{F}$, $\mathcal{D}_{\mathrm{std}}^{F}$: the corresponding diagrammatic categories defined over the realization V^F. 26
$\mathcal{D}_{\mathrm{BS},p|*}$: the diagrammatic $(p|*)$-Bott–Samelson category. 37
$\mathcal{D}_{\mathrm{BS,std}}$: the mixed category. 22
${}_R\mathcal{D}_{\mathrm{BS,std}}$: the mixed category. 23
$\mathcal{D}_{p|*}$: the diagrammatic $(p|*)$-Hecke category. 40
$\mathcal{D}_{\mathrm{std}}$: the diagrammatic category of standard bimodules over the realization V. 24

$\mathbf{e}$: a subsequence for an expression; a match for a pattern. 11, 14
$\epsilon_{\underline{x}}$: a bi-valued composition of standard caps. 26

F: the Frobenius map on W or $\mathbb{H}$; the Frobenius functor on $\mathcal{D}_{\mathrm{std}}^{F}$ or $\mathcal{D}^{F}$. 8, 11, 29, 36
$\binom{f}{g}$: a bi-valued scalar. 25
fork_s, fork_s^F: the s-colored fork morphism in $\mathcal{D}_{\mathrm{BS}}$, $\mathcal{D}_{\mathrm{BS}}^{F}$. 20
$\mathrm{fork}_s^{(F)}$: the "p-dilated" form of fork_s^F. 33

$\mathrm{Hom}^{\bullet}_{\mathcal{D}_{\mathrm{BS}}}$, $\mathrm{Hom}^{\bullet}_{\mathcal{D}_{\mathrm{BS},p|*}}$: the Hom-space interpreted as a graded vector space or a valuation module. 20
$\underline{H}_s^{(*)}$, $H_x^{(*)}$, $\underline{H}_{\underline{x}}^{(*)}$: self-dual generators, standard basis elements, and Bott–Samelson elements in $\mathbb{H}_*$. 12
$\underline{H}_s$, $\underline{H}_s^{(p)}$: self-dual generators in $\mathbb{H}$ and $\mathbb{H}_p$. 11
H_x, $H_x^{(p)}$: standard basis elements in $\mathbb{H}$ and $\mathbb{H}_p$. 11
$\underline{H}_{\underline{x}}$, $\underline{H}_{\underline{x}}^{(p)}$: Bott–Samelson elements in $\mathbb{H}$ and $\mathbb{H}_p$. 11
${}^p\underline{H}_x$: the p-canonical basis element $\mathrm{ch}[B_x]$. 58
$H_{x|w}^{(p|*)}$, $\underline{H}_{\underline{x}|\underline{w}}^{(p|*)}$: standard basis element and Bott–Samelson element of $\mathbb{H}_{p|*}$. 13, 15
${}_{v=1}H_x^{(*)}$, ${}_{v=1}H_{x|w}^{(p|*)}$: images of standard basis elements in ${}_{v=1}\mathbb{H}_*$, ${}_{v=1}\mathbb{H}_{p|*}$. 58
$\mathbb{H}$, $\mathbb{H}_p$: the Hecke algebras of W and W_p. 11
$\mathbb{H}_*$: the $*$-Hecke algebra. 12
$\mathbb{H}_{p|*}$: the $(p|*)$-Hecke bimodule. 13
${}_{v=1}\mathbb{H}_*$, ${}_{v=1}\mathbb{H}_{p|*}$: quotient of $\mathbb{H}_*$, $\mathbb{H}_{p|*}$ modulo the ideal $(v-1)$. 58

$\mathrm{LL}_{\mathbf{e},\underline{w}}$; $\mathrm{LL}_{[\underline{x}],\underline{w}}$: a light leaves map; a complete selection of light leaves maps. 20
$\mathbb{LL}_{\mathbf{e}}^{\mathbf{f}}$; $\mathbb{LL}_{[\underline{x}]}^{[\underline{y}]}$: a double leaves map; a complete selection of double leaves maps. 21
${}^{p|*}\mathrm{LL}_{\mathbf{c},\underline{w},\underline{z}}$; ${}^{p|*}\mathrm{LL}_{[[\underline{x}]],\underline{w},\underline{z}}$: a $(p|*)$-light leaves map; a complete selection of $(p|*)$-light leaves maps. 41
${}^{p|*}\mathbb{LL}_{\mathbf{e}}^{\mathbf{f}}$; ${}^{p|*}\mathbb{LL}_{[\underline{x}]}^{\underline{y}}$: a $(p|*)$-double leaves map; a complete selection of $(p|*)$-double leaves maps. 42

(m); $\langle m\rangle$: the degree m grade shift; the degree m valuation shift. 20, 32
$\langle m\rangle_A$: a degree m partial valuation shift. 56
$\mathrm{menorah}_{\underline{w},s,\underline{w}'}$: a menorah morphism. 37
m_{st}: the order of the product st in W. 5
$\mathcal{M}_{{}^pW}(\hat{R}\otimes\mathcal{D}_{\mathrm{BS}}^{F,\mathrm{dev},\oplus})$, $\mathcal{M}_{{}^pW}(\hat{R}\otimes\mathcal{D}^{F})$: the matrix category with entries in $\hat{R}\otimes\mathcal{D}_{\mathrm{BS}}^{F,\mathrm{dev},\oplus}$, $\hat{R}\otimes\mathcal{D}^{F}$. 53

$p_{\underline{r}}$: the projection to the pattern summand $B_{\underline{r}}$. 51
ψ: the matrix recursion representation on $\mathbb{H}_*$. 57
Φ: the twisted affine root system $\widetilde{\Phi_{\mathrm{f}}^{\vee}}^{\vee}$. 4

Φ_{f}: an irreducible finite root system in a Euclidean space E. 4
Ψ'; Ψ: the matrix of partial localizations (basic standard equivalent partial localizations). 52

Q: the fraction field of R; the standard bimodule corresponding to the empty expression. 22
Q_x, Q_x^F: the isomorphism class of a standard bimodule in $\mathcal{D}_{\mathrm{std}}$, $\mathcal{D}_{\mathrm{std}}^F$. 24
$Q_{\underline{x}}$, $Q_{\underline{x}}^F$: a standard bimodule in $\mathcal{D}_{\mathrm{std}}$, $\mathcal{D}_{\mathrm{std}}^F$. 22
$Q_{\underline{x}}^{(F)}$: a "p-dilated" standard bimodule. 26

$\underline{r}$: a pattern for an expression. 14
$[\underline{r}]$: the set of matches for a pattern $\underline{r}$. 14
R: the symmetric algebra of V; the Bott–Samelson or standard bimodule corresponding to the empty expression. 16
$\hat{R}$: a complete discretely valued extension of R. 30
$\hat{R} \otimes \mathcal{D}_{\mathrm{BS}}^{F,\mathrm{dev}}$: the devaluation of $\hat{R} \otimes \mathcal{D}_{\mathrm{BS}}^F$. 52
$\hat{R} \otimes \mathcal{D}_{\mathrm{BS}}^{F,\mathrm{dev},\oplus}$: the additive envelope of of $\hat{R} \otimes \mathcal{D}_{\mathrm{BS}}^{F,\mathrm{dev},\oplus}$. 52
$R_{\underline{x}}$, $\hat{R}_{\underline{x}}$: standard bimodules in ${}_R\mathcal{D}_{\mathrm{BS,std}}$ and $\hat{R} \otimes {}_R\mathcal{D}_{\mathrm{BS,std}}$. 23

$s_{\alpha_{\mathrm{h}}}$: the reflection in W_{f} in the root α_{h}. 10
$\underline{s_{\alpha_{\mathrm{h}}}}$: a fixed reduced expression in $\underline{S}_{\mathrm{f}}$ for $s_{\alpha_{\mathrm{h}}}$. 26
$\tilde{s}$: the unique simple affine reflection in S. 5
$\tilde{s}_p$: the unique simple affine reflection in S_p. 9
$\mathrm{stdbraid}_{s,t}$, $\mathrm{stdcup}_{s,t}^F$: the s-and-t-colored standard braid morphism in $\mathcal{D}_{\mathrm{std}}$, $\mathcal{D}_{\mathrm{std}}^F$. 23
stdcap_s, stdcap_s^F: the s-colored standard cap morphism in $\mathcal{D}_{\mathrm{std}}$, $\mathcal{D}_{\mathrm{std}}^F$. 23
$\mathrm{stdcap}_s^{(F)}$, $\mathrm{stdcup}_s^{(F)}$, $\mathrm{stdbraid}_{s,t}^{(F)}$: the "$p$-dilated" forms of stdcap_s^F, stdcup_s^F, $\mathrm{stdbraid}_{s,t}^F$. 26
stdcup_s, stdcup_s^F: the s-colored standard cup morphism in $\mathcal{D}_{\mathrm{std}}$, $\mathcal{D}_{\mathrm{std}}^F$. 23
$\underline{S}$, $\underline{S}_p$: the sets of expressions in S and S_p. 11
S_{f}, S: the simple reflections of W_{f}, W. 4
S_p: the simple reflections of W_p. 9
$\underline{S}_{p|1}$: the set of $(p|1)$-expressions. 14
Σ_{f}, Σ: the set of simple roots of Φ_{f}, Φ in E. 5

u_s, u_A: monomials in the toral coset algebra ${}^p\mathbb{U}$. 12
${}^p\mathbb{U}$: the toral coset algebra. 12
${}^p\mathcal{U}$: the boolean algebra generated by the sets ${}^pW(s,*)w$. 12

$\operatorname{val} f$: valuation of a scalar in $\hat{R}$ or a morphism in $\mathcal{D}_{p|*}$. 31, 39
V: any $\Bbbk$-realization of W; the universal realization of W with respect to Σ over $\Bbbk$. 16, 26
V^F: the F-twist of the universal realization V over $\Bbbk$. 26

W_{f}, W: the (affine, finite) Weyl group arising from Φ_{f}, Φ. 4
W_p: the p-affine Weyl group $W_{\mathrm{f}} \ltimes p\mathbb{Z}\Phi_{\mathrm{f}}$. 9
${}^pW$: the set of minimal length representatives of the right cosets $W_p \backslash W$. 9
$\circ_{{}^pW}$: the entrywise or Hadamard product on $\mathcal{M}_{{}^pW}(-)$. 54
${}^pW(s,*)$: the set $\{w \in {}^pW : W_p ws = W_p w\}$. 12

$x|w$: a factorization with $x \in W_p$ and $w \in W$. 13
$\underline{x}$: an expression. 11
$\underline{x}|\underline{w}$: an expression in $\underline{S}_{p|1}$. 14

$[\underline{x}]$: the set of subsequences of an expression $\underline{x}$. 11

$[\underline{x}]_*$; $[\underline{x}|\underline{w}]_{p|*}$: a function from ${}^pW$ to sets of patterns; the pattern set $\underline{x}[\underline{w}]_*(1)$. 15

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