Quasi-hereditary twisted category algebras

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Abstract

We give a sufficient criterion for when a twisted finite category algebra over a field is quasi-hereditary, in terms of the partially ordered set of $\mathscr{L}$-classes in the morphism set of the category. We show that this is a common generalisation of a long list of results in the context of EI-categories, regular monoids, Brauer algebras, Temperley-Lieb algebras, partition algebras.

1 Introduction

Let k be a field and $\mathcal{C}$ a small category. Let α be a 2-cocycle in $Z^2(\mathcal{C};k^\times)$; that is, α is a map sending any two morphisms φ, ψ in $\mathrm{Mor}(\mathcal{C})$ for which $\psi\circ\varphi$ is defined to an element $\alpha(\psi,\varphi)$ in $k^\times$ such that for any three morphisms φ, ψ, τ for which the compositions $\psi\circ\varphi$ and $\tau\circ\psi$ are defined, we have the 2-*cocycle identity* $\alpha(\tau,\psi\circ\varphi)\alpha(\psi,\varphi)=\alpha(\tau\circ\psi,\varphi)\alpha(\tau,\psi)$. The *twisted category algebra* $k_\alpha\mathcal{C}$ is the k-vector space having the morphism set $\mathrm{Mor}(\mathcal{C})$ as a k-basis, with a k-bilinear multiplication given by $\psi\varphi=\alpha(\psi,\varphi)(\psi\circ\varphi)$ if $\psi\circ\varphi$ is defined, and $\psi\varphi=0$, otherwise. The 2-cocycle identity is equivalent to the associativity of this multiplication. The isomorphism class of $k_\alpha\mathcal{C}$ depends only on the class of α in $H^2(\mathcal{C};k^\times)$, with $k^\times$ here understood as a constant contravariant functor on $\mathcal{C}$. If α represents the trivial class, then $k_\alpha\mathcal{C}\cong k\mathcal{C}$, the category algebra of $\mathcal{C}$ over k. For any idempotent endomorphism e of an object X in $\mathcal{C}$ we denote by G_e the group of all invertible elements in the monoid $e\circ\mathrm{End}_\mathcal{C}(X)\circ e$. As in [7], we consider the well-known extension to categories of the notion of Green relations in semigroups. In particular, two morphism s, t in $\mathcal{C}$ are called $\mathscr{L}$-equivalent, if $\mathrm{Mor}(\mathcal{C})\circ s=\mathrm{Mor}(\mathcal{C})\circ t$, they are called $\mathscr{R}$-equivalent if $s\circ\mathrm{Mor}(\mathcal{C})=t\circ\mathrm{Mor}(\mathcal{C})$, and they are called $\mathscr{J}$-equivalent if $\mathrm{Mor}(\mathcal{C})\circ s\circ\mathrm{Mor}(\mathcal{C})=\mathrm{Mor}(\mathcal{C})\circ t\circ\mathrm{Mor}(\mathcal{C})$. The relations $\mathscr{L}$, $\mathscr{R}$, $\mathscr{J}$ are equivalence relations on the set $\mathrm{Mor}(\mathcal{C})$. The set of $\mathscr{L}$-classes is partially ordered: if L, L' are two $\mathscr{L}$-classes in $\mathrm{Mor}(\mathcal{C})$ we write $L'\leq L$ if $\mathrm{Mor}(\mathcal{C})\circ s'\subseteq\mathrm{Mor}(\mathcal{C})\circ s$ for some (and hence any) morphisms $s'\in L'$, $s\in L$. In a similar way, the sets of $\mathscr{R}$-classes and of $\mathscr{J}$-classes are partially ordered. A morphism $s:X\to Y$ in $\mathcal{C}$ is called *split* if there is a morphism $t:Y\to X$ such that $s\circ t\circ s=s$. An $\mathscr{L}$-class L in $\mathrm{Mor}(\mathcal{C})$ is called *split* if it contains a split morphism. In that case, all morphisms in L are split, and L contains an idempotent endomorphism of some object in $\mathcal{C}$. Similarly, an $\mathscr{R}$-class or a $\mathscr{J}$-class is called *split* if it contains a split morphism, in which case all morphisms in that class are split. A (possibly empty) subset I of $\mathrm{Mor}(\mathcal{C})$ is called an *ideal* in $\mathrm{Mor}(\mathcal{C})$ if $\mathrm{Mor}(\mathcal{C})\circ I\circ\mathrm{Mor}(\mathcal{C})=I$.

Theorem 1.1. *Let $\mathcal{C}$ be a finite category, k a field and $\alpha\in Z^2(\mathcal{C};k^\times)$. Suppose that the set of nonsplit morphisms in $\mathcal{C}$ is an ideal in* $\mathrm{Mor}(\mathcal{C})$*, and suppose that for any $\mathscr{L}$-class L in* $\mathrm{Mor}(\mathcal{C})$ *there is a unique minimal split $\mathscr{L}$-class L' such that $L\leq L'$. Suppose that either* $\mathrm{char}(k)=$

0 *or that* char(k) > 0 *does not divide the group orders* $|G_e|$*, where* e *runs over the idempotent endomorphisms in* $\mathcal{C}$*. Then the* k*-algebra* $k_\alpha\mathcal{C}$ *is quasi-hereditary.*

The conclusion holds also with the uniqueness hypothesis on minimal split $\mathscr{L}$-classes replaced by the analogous condition on $\mathscr{R}$-classes. This hypothesis is not necessary in the sense that there are examples of categories with quasi-hereditary category algebras but where neither the $\mathscr{L}$-classes nor the $\mathscr{R}$-classes satisfy this uniqueness property - see Example 6.1 below. The hypotheses on $\mathcal{C}$ are trivially satisfied if all morphisms in $\mathcal{C}$ are split:

Corollary 1.2. *Let* $\mathcal{C}$ *be a finite category,* k *a field and* $\alpha \in Z^2(\mathcal{C}; k^\times)$*. Suppose that all morphisms in* $\mathcal{C}$ *are split, and that either* char(k) $= 0$ *or that* char(k) > 0 *does not divide the group orders* $|G_e|$*, where* e *runs over the idempotent endomorphisms in* $\mathcal{C}$*. Then the* k*-algebra* $k_\alpha\mathcal{C}$ *is quasi-hereditary.*

This corollary has been proved independently by Boltje and Danz [2]. Specialising 1.2 to regular monoids (that is, monoids in which all elements are split or, following semigroup terminology, von Neumann regular) yields the following:

Corollary 1.3. *Let* M *be a finite regular monoid,* k *a field and* $\alpha \in Z^2(M; k^\times)$*. Suppose that either* char(k) $= 0$ *or that* char(k) > 0 *does not divide the order of any subgroup of* M*. Then the algebra* $k_\alpha M$ *is quasi-hereditary.*

Further specialised to $\alpha = 1$ and $k = \mathbb{C}$ we obtain a result due to Putcha [10, Theorem 2.1]: if M is a finite regular monoid, then the algebra $\mathbb{C}M$ is quasi-hereditary.

Many of the combinatorially defined cellular algebras, such as Brauer algebras, Temperley-Lieb algebras, partition algebras, are known to be quasi-hereditary for 'most' choices of parameters. Following Wilcox [15], these algebras can be interpreted as twisted monoid algebras, and hence Corollary 1.3 can be used to show that they are quasi-hereditary in many instances. The Brauer algebra $B_r(\delta)$ on $2r$ vertices and a nonzero parameter δ in the field k can be viewed as a twisted monoid algebra. By [15, §8], the underlying monoid is regular, and its maximal subgroups are symmetric groups on at most r letters, hence their twisted group algebras are semisimple if char(k) is either zero or greater than r. Thus Corollary 1.3 implies the following result, due to Graham and Lehrer:

Corollary 1.4 ([3, 4.16, 4.17])**.** *Let* r *be a positive integer and* δ *a nonzero element in* k*. Suppose that either* char(k) $= 0$ *ar that* char(k) $> r$*. Then the Brauer algebra* $B_r(\delta)$ *is quasi-hereditary.*

The maximal subgroups of the monoid underlying the cyclotomic Brauer algebra $Br_{r,m}$ are wreath products of a cyclic group of order m and symmetric groups on at most r letters. We obtain a part of a result of Rui and Yu (not requiring the hypothesis on k being a splitting field of $x^m - 1$, but requiring that all parameters are nonzero):

Corollary 1.5 ([12, 5.9])**.** *Let* r*,* m *be positive integers. Suppose that* char(k) $= 0$ *or that* char(k) $> r$ *and* char(k) *does not divide* m*. Then the cyclotomic Brauer algebra* $B_{r,m}$ *with all parameters nonzero is quasi-hereditary.*

The Temperley-Lieb algebra $TL_r(\delta)$ is a subalgebra of $B_r(\delta)$. Following [15, §8], it can be identified as a twisted monoid algebra, where the underlying monoid is regular and has trivial maximal subgroups. Thus we obtain a result of Westbury:

Corollary 1.6 ([14, §6]). *Let r be a positive integer and δ a nonzero element in k. Then the Temperley-Lieb algebra $TL_r(\delta)$ is quasi-hereditary.*

The maximal subgroups of the monoid underlying a cyclotomic Temperley-Lieb algebra $TL_{r,m}$ are direct products of cyclic groups of order m. As before, we obtain a part of a result of Rui and Xi:

Corollary 1.7 ([11, 6.2]). *Let r, m be positive integers. Suppose that* $\mathrm{char}(k) = 0$ *or that* $\mathrm{char}(k) > 0$ *does not divide m. Then the Temperley-Lieb algebra $TL_{r,m}$ with all parameters nonzero is quasi-hereditary.*

Similarly, the partition algebra $P_r(\delta)$ admits a description as a twisted monoid algebra, where the underlying monoid is regular and has maximal subgroups isomorphic to symmetric groups on at most r letters. This yields a result of Martin [9] and Xi [16]:

Corollary 1.8 ([9, Prop. 3], [16, 4.12]). *Let r be a positive integer and δ a nonzero element in k. Suppose that either* $\mathrm{char}(k) = 0$ *or that* $\mathrm{char}(k) > r$. *Then the partition algebra $P_r(\delta)$ is quasi-hereditary.*

König and Xi have given in [5] a characterisation of quasi-hereditary cellular algebras; this has been used to determine precisely the possible choices of the field and parameter for which the above diagram algebras are quasi-hereditary.

The fact that Theorem 1.1 is a common generalisation of the results on diagram algebras mentioned above, has been one of the motivations for the present paper. Boltje and Danz showed recently in [1] that certain double Burnside rings give rise to twisted monoid algebras as well. Theorem 1.1 can be generalised further to a statement to also encompass results in [6, 2.4, 4.5] showing that weight algebras of EI-categories (and hence in particular, of fusion systems) are quasi-hereditary. Following [7, 1.2], the isomorphism classes of simple $k_\alpha\mathcal{C}$-modules are parametrised by isomorphism classes of pairs (e, T), where e is an idempotent endomorphism of an object of $\mathcal{C}$ and where T is a simple module over the twisted group algebra $k_\alpha G_e$, with α here understood as the restriction of α to G_e. The following definition is from [7, 1.4]; the special case of finite EI-categories has been defined in [6, 4.4].

Definition 1.9. Let $\mathcal{C}$ be a finite category, k a field, and $\alpha \in Z^2(\mathcal{C}; k^\times)$. A *weight of* $k_\alpha\mathcal{C}$ is a pair (e, T) consisting of an idempotent endomorphism e of an object X in $\mathcal{C}$ and a projective simple $k_\alpha G_e$-module T. A *weight algebra* $W(k_\alpha\mathcal{C})$ of $k_\alpha\mathcal{C}$ is a k-algebra of the form $W(k_\alpha\mathcal{C}) = ck_\alpha\mathcal{C}c$, where c is an idempotent in $k_\alpha\mathcal{C}$ with the property that $cS = S$ for every simple $k_\alpha\mathcal{C}$-module S parametrised by a weight, and $cS' = \{0\}$ for every simple $k_\alpha\mathcal{C}$-module S' which is not parametrised by a weight.

Remarks 1.10. Let $\mathcal{C}$ be a finite category, k a field and $\alpha \in Z^2(\mathcal{C}; k^\times)$.

(1) An idempotent $c \in k_\alpha\mathcal{C}$ as in the Definition 1.9 is a sum of pairwise orthogonal primitive idempotents i in $k_\alpha\mathcal{C}$ such that the unique simple quotient of the projective indecomposable $k_\alpha\mathcal{C}$-module $k_\alpha\mathcal{C}i$ is parametrised by a weight, and such that no primitive idempotent ocurring in a decomposition of $1 - c$ has this property. This defines the idempotent c uniquely up to conjugation in $k_\alpha\mathcal{C}$ and hence the algebra $W(k_\alpha\mathcal{C})$ is determined uniquely up to isomorphism. The number of isomorphism classes of simple $W(k_\alpha\mathcal{C})$-modules is equal to the number of isomorphism classes of weights of $k_\alpha\mathcal{C}$.

(2) If $\mathrm{char}(k) = 0$, or if $\mathrm{char}(k) > 0$ does not divide the order of $|G_e|$ for any idempotent endomorphism e in $\mathcal{C}$, then $W(k_\alpha\mathcal{C}) = k_\alpha\mathcal{C}$. Indeed, in that case the twisted group algebras $k_\alpha G_e$ are semisimple, hence every simple $k_\alpha\mathcal{C}$ module is parametrised by a weight.

(3) Suppose that k is algebraically closed and that $\mathrm{char}(k) = p > 0$. Let $\mathcal{F}$ be a fusion system of a p-block b of a finite group G for which the 2-cocycle gluing problem [6, 4.2] has a solution α. If $\mathcal{C}$ is the full subcategory $\bar{\mathcal{F}}^c$ of the orbit category $\bar{\mathcal{F}}$ of $\mathcal{F}$-centric subgroups of P, then $W(k_\alpha\mathcal{C})$ is isomorphic to the algebra $\bar{\mathcal{F}}(b)$ defined in [6, 4.4].

Theorem 1.11. *Let $\mathcal{C}$ be a finite category, k a field, and $\alpha \in Z^2(\mathcal{C}; k^\times)$. Let c be an idempotent in $k_\alpha\mathcal{C}$ such that for any simple A-module S we have $cS = S$ if S is parametrised by a weight, and $cS = \{0\}$, otherwise. Suppose that the set N of nonsplit morphism in $\mathcal{C}$ is an ideal in $\mathrm{Mor}(\mathcal{C})$, and suppose that for any $\mathscr{L}$-class L in $\mathrm{Mor}(\mathcal{C})$ there is a unique minimal split $\mathscr{L}$-class L' such that $L \leq L'$. Suppose that c commutes with the images in $k_\alpha\mathcal{C}$ of all idempotent endomorphisms in $\mathcal{C}$. Then the weight algebra $W(k_\alpha\mathcal{C}) = ck_\alpha\mathcal{C}c$ is quasi-hereditary.*

As mentioned above, if the group orders $|G_e|$ are invertible in k, then $c = 1$, and the additional hypotheses on c holds trivially; thus Theorem 1.1 is indeed a special case of Theorem 1.11. The extra hypothesis on c ensures that the chain of ideals constructed in the proof of Theorem 1.1 becomes a hereditary chain upon multiplying the ideals by c on both sides. This hypothesis can be replaced by an a priori weaker but more technical hypothesis - see Remark 6.3 below. The extra hypothesis on c holds if $\mathcal{C}$ is a finite EI-category (that is, a category in which all endomorphisms are isomorphisms, a concept introduced by Lück [8]), and hence we obtain the following result:

Corollary 1.12 ([6, 2.4])**.** *Let $\mathcal{C}$ be a finite EI-category, k a field, and $\alpha \in Z^2(\mathcal{C}; k^\times)$. Then the k-algebra $W(k_\alpha\mathcal{C})$ is quasi-hereditary.*

Proof. The only idempotent endomorphisms in $\mathcal{C}$ are the identity morphisms Id_X, where X runs over the objects of $\mathcal{C}$. The split morphisms in $\mathcal{C}$ are isomorphisms, and the nonisomorphisms in $\mathcal{C}$ form an ideal. For $s : X \to Y$ a morphism in $\mathcal{C}$, the $\mathscr{L}$-class $L = L_{\mathrm{Id}_X}$ of Id_X is the unique minimal split $\mathscr{L}$-class satisfying $L_s \leq L$. Any idempotent in a twisted finite category algebra $k_\alpha\mathcal{C}$ has a conjugate which commutes with all identity morphisms, and hence we may choose c to commute with Id_X for all objects X in $\mathcal{C}$. Thus the hypotheses of Theorem 1.11 are satisfied; the result follows. □

Applied to fusion systems of blocks, this yields also [6, 4.5].

Remark 1.13. It is possible for a nonunitary semigroup S to have a unitary semigroup algebra kS over some field k (for instance, this is always the case if S is a finite inverse semigroup). In the same vein, the results of this paper can be adapted for finite *semicategories* (that is, categories without the requirement for the existence of identity morphisms at each object) so long as their twisted category algebras are unitary.

2 On idempotent ideals in algebras

Let A be a finite-dimensional algebra over a field k. We will use without further comment the following standard facts on idempotents in finite-dimensional algebras. The isomorphism class of a simple A-module S is parametrised by a unique conjugacy class of primitive idempotents i in A satisfying $iS \neq \{0\}$, and then Ai is a projective cover of S. The image of a primitive idempotent i in any quotient of A is either zero or a primitive idempotent. A primitive decomposition of an idempotent e in A is a set E of pairwise orthogonal primitive idempotents such that $\sum_{i \in E} i = e$. Any two primitive decompositions of e are conjugate (this is the algebra theoretic version of the Krull-Schmidt Theorem). This can be used to show that for any two idempotents e, f in A, there is a conjugate f' of f which commutes with e, and then $ef' = f'e$ is either zero or an idempotent. It is well-known that an ideal I in A satisfies $I^2 = I$ if and only if $I = AeA$ for some idempotent e in A. If in addition $I = AeA$ is projective as a left A-module, then every indecomposable direct summand of I as a left A-module is isomorphic to an indecomposable direct summand of Ae; for the convenience of the reader, we include the argument to show this in the proof of the next result. If every indecomposable direct summand of a finitely generated projective A-module U is isomorphic to a direct summand of Ae, then eU is a projective eAe-module.

Proposition 2.1. *Let A be a finite-dimensional algebra over a field k, e an idempotent in A and f an idempotent in $Z(eAe)$. Suppose that the ideal AeA is projective as a left A-module. Then the ideal AfA is a direct summand of AeA as a left A-module; in particular, AfA is projective as a left A-module.*

Proof. Let X be a k-basis of eA, and set $n = |X| = \dim_k(eA)$. The inclusions $Aex \subseteq AeA$, where $x \in X$, induce a surjective homomorphism of left A-modules $\oplus_{x \in X} Aex \to AeA$. Each summand Aex is isomorphic to a quotient of Ae. Thus, as a left A-module, AeA is a quotient of a direct sum $(Ae)^n$ of n copies of Ae. If AeA is projective as a left A-module, then AeA is isomorphic to a direct summand of $(Ae)^n$ as a left A-module. Thus there is an isomorphism of left A-modules $\alpha : AeA \cong \oplus_{i=1}^n Ae_i$, where each of the e_i is an idempotent in eAe or zero. Since α is an A-homomorphism, we have $\alpha(fA) \subseteq \oplus_{i=1}^n fAe_i = \oplus_{i=1}^n fAfe_i$, where the second equality uses the fact that $f \in Z(eAe)$. Again, since α is an A-homomorphism, we have $\alpha(AfA) \subseteq \oplus_{i=1}^n Afe_i$. Since α is an isomorphism, there exist unique elements $y_i \in AeA$ such that $\alpha(y_i) = fe_i$, for $1 \leq i \leq n$. Then also $\alpha(fy_i) = fe_i$, and hence $\alpha(AfA) = \oplus_{i=1}^n Afe_i$, which is a direct summand of $\alpha(AeA)$ as a left A-module. The result follows. □

Proposition 2.2. *Let A be a finite-dimensional algebra over a field k and e, c idempotents in A. Suppose that c is conjugate to an idempotent c' in A such that $ec' = c'e \in Z(eAe)$. Then $cAec'Ac$ is an ideal in cAc. Moreover, if AeA is projective as a left A-module, then $cAc'eAc$ is projective as a left cAc-module.*

Proof. Two conjugate elements in an algebra generate the same ideal, and hence we may assume that $c = c'$ commutes with e and that $ec \in Z(eAe)$. Then ec is either zero or an idempotent in $Z(eAe)$. Thus 2.1 applied to $f = ec$ implies that $AecA$ is projective as a left A-module. Moreover, every indecomposable direct summand of $AecA$ as a left A-module is isomorphic to a direct summand of Aec, hence of Ac. It follows that $AecAc$ is a projective A-module with this property, and therefore $cAecAc$ is projective as a cAc-module. □

Lemma 2.3. *Let $\mathcal{C}$ be a finite category, k a field, and $\alpha \in Z^2(\mathcal{C}, k^\times)$. Let e, f be idempotent endomorphisms of some objects in $\mathcal{C}$.*

(i) The elements $\hat{e} = \alpha(e,e)^{-1}e$ and $\hat{f} = \alpha(f,f)^{-1}f$ are idempotents in the algebra $k_\alpha\mathcal{C}$.

(ii) If e and f belong to the same $\mathscr{J}$-class, then the idempotents $\hat{e}$ and $\hat{f}$ are conjugate in $k_\alpha\mathcal{C}$.

Proof. Set $A = k_\alpha\mathcal{C}$. The fact that $\hat{e}$ and $\hat{f}$ are idempotents in the twisted category algebra A follows immediately from the definition of the multiplication of morphisms in A. Suppose that $J_e = J_f$. Set $M = \mathrm{Mor}(\mathcal{C})$. Then $M \circ e \circ M = M \circ f \circ M$. Thus $e = u \circ f \circ v$ for some morphisms $u \in e \circ M \circ f$ and $v \in f \circ M \circ e$. Right multiplication by u on $Ae = A\hat{e}$ induces an A-homomorphism $A\hat{e} \to A\hat{f}$, and right multiplication by v on $Af = A\hat{f}$ induces an A-homomorphism $A\hat{f} \to A\hat{e}$. The composition of these yields an automorphism of Ae, hence Af has a direct summand isomorphic to Ae. Exchanging the roles of e and f shows that Ae has a direct summand isomorphic to Af, hence both are isomorphic. This implies that $\hat{e}$ and $\hat{f}$ are conjugate. □

Remark 2.4. The results in this section remain true with k replaced by a complete discrete valuation ring $\mathcal{O}$. See for instance [13, §3, §4] for background material on idempotents in $\mathcal{O}$-algebras.

3 On ideals in categories

We refer to [7] for notation and well-known background material on Green relations for categories. For t a morphism in a small category $\mathcal{C}$, we denote by L_t, R_t, J_t its $\mathscr{L}$-class, $\mathscr{R}$-class, $\mathscr{J}$-class, respectively. That is, the set L_t consists of all morphisms t' satisfying $\mathrm{Mor}(\mathcal{C}) \circ t = \mathrm{Mor}(\mathcal{C}) \circ t'$, the set R_t consists of all morphisms t' satisfying $t \circ \mathrm{Mor}(\mathcal{C}) = t' \circ \mathrm{Mor}(\mathcal{C})$, and the set J_t consists of all morphisms t' satisfying $\mathrm{Mor}(\mathcal{C}) \circ t \circ \mathrm{Mor}(\mathcal{C}) = \mathrm{Mor}(\mathcal{C}) \circ t' \circ \mathrm{Mor}(\mathcal{C})$. For any two $\mathscr{L}$-classes L, L' we write $L \leq L'$ if $\mathrm{Mor}(\mathcal{C}) \circ L \subseteq \mathrm{Mor}(\mathcal{C}) \circ L'$; this defines a partial order on the set of $\mathscr{L}$-classes. Similarly, we have partial orders on the sets of $\mathscr{R}$-classes and $\mathscr{J}$-calsses given for any two $\mathscr{R}$-classes R, R' by by $R \leq R'$ if $R \circ \mathrm{Mor}(\mathcal{C}) \subseteq R' \circ \mathrm{Mor}(\mathcal{C})$, and for any two $\mathscr{J}$-classes J, J' by $J \leq J'$ if $\mathrm{Mor}(\mathcal{C}) \circ J \circ \mathrm{Mor}(\mathcal{C}) \subseteq \mathrm{Mor}(\mathcal{C}) \circ J' \circ \mathrm{Mor}(\mathcal{C})$. Any ideal I in $\mathrm{Mor}(\mathcal{C})$ is a disjoint union of $\mathscr{J}$-classes, and the k-span $k_\alpha I$ is an ideal in $k_\alpha\mathcal{C}$, where $\alpha \in Z^2(\mathcal{C}; k^\times)$. We adopt the convention that the empty ideal in $\mathrm{Mor}(\mathcal{C})$ corresponds to the zero ideal in $k_\alpha\mathcal{C}$.

Proposition 3.1. *Let $\mathcal{C}$ be a finite category and let t, $t' \in \mathrm{Mor}(\mathcal{C})$. If $L_t < L_{t'}$ or $R_t < R_{t'}$, then $J_t < J_{t'}$.*

Proof. Set $M = \mathrm{Mor}(\mathcal{C})$. Arguing by contradiction, suppose that $L_t < L_{t'}$ and $J_t = J_{t'}$. That is, $M \circ t$ is a proper subset of $M \circ t'$, and $M \circ t \circ M = M \circ t' \circ M$. In particular, $t' = u \circ t \circ v$ for some morphisms u, v. Thus there is a surjective map $M \circ u \circ t \to M \circ t'$ sending a morphism w to $w \circ v$. In particular, $|M \circ u \circ t| \geq |M \circ t'|$. However, $M \circ u \circ t$ is a subset of $M \circ t$, hence $|M \circ t| \geq |M \circ t'|$. This contradicts the inequality $|M \circ t| < |M \circ t'|$, and shows that if $L_t < L_{t'}$, then $J_t < J_{t'}$. A similar argument yields the statement for $\mathscr{R}$-classes instead of $\mathscr{L}$-classes. □

If a $\mathscr{J}$-class J of a morphism in a category $\mathcal{C}$ contains a split morphism, then all morphisms in J are split.

Corollary 3.2. *Let $\mathcal{C}$ be a finite category, I a proper ideal in $\mathrm{Mor}(\mathcal{C})$, and t a split morphism in $\mathrm{Mor}(\mathcal{C}) \smallsetminus I$. The following are equivalent:*

(i) The $\mathscr{J}$-class J_t is minimal in the set of split $\mathscr{J}$-classes not contained in I.

(ii) The $\mathscr{L}$-class L_t is minimal in the set of split $\mathscr{L}$-classes not contained in I.

(iii) The $\mathscr{R}$-class R_t is minimal in the set of split $\mathscr{R}$-classes not contained in I.

The next proposition describes some basic properties of finite categories in which the nonsplit morphisms form an ideal.

Proposition 3.3. *Let $\mathcal{C}$ be a finite category such that the set N of nonsplit morphisms is an ideal in $\mathrm{Mor}(\mathcal{C})$. Then the following hold.*

(i) Every endomorphism of an object in $\mathcal{C}$ is split.

(ii) The set of split morphisms in $\mathrm{Mor}(\mathcal{C})$ is closed under composition of morphisms.

(iii) If X, Y are objects in $\mathcal{C}$ such that $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ contains a nonsplit morphism, then every morphism in $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ is nonsplit, and $\mathrm{Hom}_{\mathcal{C}}(Y, X) = \emptyset$.

(iv) There is a positive integer i such that $N^i = \emptyset$.

(v) If t is a split morphism in $\mathrm{Mor}(\mathcal{C})$ then there are idempotents e, f in $\mathrm{Mor}(\mathcal{C})$ such that $L_t = L_e$ and $R_t = R_f$.

Proof. Let s be an endomorphism of an object in $\mathcal{C}$. Since $\mathcal{C}$ is finite, there is a positive integer n such that s^n is an idempotent endomorphism; in particular, s^n is split. Since the set N of nonsplit morphisms is an ideal it follows that s is split, whence (i). Let now s, t be split morphisms in $\mathcal{C}$ such that $t \circ s$ is defined. Since s, t are split, there are morphisms s', t' satisfying $s = s \circ s' \circ s$ and $t = t \circ t' \circ t$. Then $s' \circ t' \circ t \circ s$ is an endomorphism of an object in $\mathcal{C}$, hence split by (i). Since N is an ideal it follows that $t \circ s$ is split. This proves (ii). Let $u : X \to Y$ be a morphism in $\mathcal{C}$. If there is a morphism $v : Y \to X$, then $v \circ u$ is an endomorphism of X, hence split by (i), and thus u is split because N is an ideal. This implies (iii). It follows from (iii) that if

$$X_1 \xrightarrow{u_1} X_2 \xrightarrow{u_2} \cdots \xrightarrow{u_{r-1}} X_r$$

is a sequence of nonsplit morphisms u_j, then the objects X_j in this sequence are pairwise different, and hence the integer r is bounded by the number of objects in $\mathcal{C}$, which proves (iv). To prove (v), one can take $e = t' \circ t$ and $f = t \circ t'$, where t' satisfies $t \circ t' \circ t = t$. This completes the proof. □

Lemma 3.4. *Let $\mathcal{C}$ be a finite category such that the set N of nonsplit morphisms is an ideal in $M = \mathrm{Mor}(\mathcal{C})$. Let e be an idempotent endomorphism in M. We have*

$$M \circ e \circ M = (\cup_{e'} \; M \circ e') \; \cup \; (M \circ e \circ N) \; ,$$

where the first union runs over the set of idempotent endomorphisms e' contained in $M \circ e \circ M$.

Proof. Clearly the union on the right side is contained in the left side. For the converse inclusion, let s, $t \in M$ such that $s \circ e \circ t$ is defined. If $t \in N$, then $s \circ e \circ t \in M \circ e \circ N$, which is a subset of the right side. If $t \notin N$, then t is split and by 3.3 (ii) and (v), $e \circ t$ is split and $L_{e \circ t} = L_{e'}$ for some idempotent endomorphism e'. Then clearly $e' \in M \circ e \circ M$ and $s \circ e \circ t \in M \circ e'$ as required. □

Proposition 3.5. *Let $\mathcal{C}$ be a finite category such that the set N of nonsplit morphisms is an ideal in $M = \mathrm{Mor}(\mathcal{C})$. Let I be a proper ideal in M, and let J be a split $\mathscr{J}$-class which is minimal in the set of split $\mathscr{J}$-classes not contained in I. Suppose that $J \circ N \subseteq I$. We have*

$$M \circ J \circ M \subseteq M \circ J \cup I .$$

Proof. By Lemma 3.4 we have $M \circ e \circ M = \cup_{e'} M \circ e' \cup M \circ e \circ N$, with e' running over the idempotent endomorphisms contained in $M \circ e \circ M$. The term $M \circ e \circ N$ as well as the terms $M \circ e'$ with $J_{e'} < J_e$ are contained in I by the assumptions. The remaining terms are contained in $M \circ J$, whence the result. □

Proposition 3.6. *Let $\mathcal{C}$ be a finite category such that the set N of nonsplit morphisms is an ideal in $\mathrm{Mor}(\mathcal{C})$. Let I be a proper ideal in $\mathrm{Mor}(\mathcal{C})$. Then there exists a $\mathscr{J}$-class J in $\mathrm{Mor}(\mathcal{C})$ with the following properties.*

(i) J is minimal in the set of split $\mathscr{J}$-classes not contained in I.

(ii) $J \circ N \subseteq I$.

Moreover, if J is a $\mathscr{J}$-class satisfying (i) and (ii), then every $\mathscr{L}$-class L contained in J is minimal in the set of split $\mathscr{L}$-classes not contained in I, and satisfies $L \circ N \subseteq I$.

Proof. Since I is a proper ideal in $\mathrm{Mor}(\mathcal{C})$, it follows that $\mathrm{Mor}(\mathcal{C}) \smallsetminus I$ contains a split morphism. Indeed, otherwise I would contain the identity morphisms Id_X for each object X in $\mathcal{C}$ hence I would be equal to $\mathrm{Mor}(\mathcal{C})$. Choose a minimal split $\mathscr{J}$-class J not contained in I such that $N^i \circ J \neq \emptyset$, with $i \geq 0$ maximal possible with the convention $N^0 = \mathrm{Mor}(\mathcal{C})$. The integer i is well-defined by 3.3 (iv). Suppose that $s \circ N$ is not contained in I for some morphism $s : Y \to Z$ in J. That is, there is a morphism $u : X \to Y$ in N such that $s \circ u \notin I$. Then u, and hence Id_X, are not contained in the ideal I. Let J' be a minimal split $\mathscr{J}$-class not contained in I such that $J' \leq J_{\mathrm{Id}_X}$. Note that every morphism in J' factors through Id_X. Since J' is split, J' contains an endomorphism s' of X. Thus the composition $s \circ u \circ s'$ is defined. Since $N^i \circ s$ is nonempty and $u \in N$, hence $s \circ u \in N$, it follows that $N^{i+1} \circ s'$ is nonempty. This contradicts the initial choice of J, and shows that $J \circ N \subseteq I$. If J satisfies (i) and (ii), it follows from 3.2 that every $\mathscr{L}$-class L contained in J is minimal in the set of split $\mathscr{L}$-classes not contained in I, whence the last statement. □

Proposition 3.7. *Let $\mathcal{C}$ be a small category. Set $M = \mathrm{Mor}(\mathcal{C})$. Suppose that for any $\mathscr{L}$-class L in M there is a unique minimal split $\mathscr{L}$-class L' such that $L \leq L'$. Let I be a proper ideal in M, and let L_1, L_2 be two different minimal split $\mathscr{L}$-classes not contained in I. Then*

$$M \circ L_1 \cap M \circ L_2 \subseteq I .$$

Proof. Let L be an $\mathscr{L}$-class contained in $M \circ L_1 \cap M \circ L_2$. Then $L \leq L_1$ and $L \leq L_2$. By the assumptions, there is a unique minimal split $\mathscr{L}$-class L' such that $L \leq L'$. The uniquenss of L' implies that $L' \leq L_1$ and $L' \leq L_2$. Since $L_1 \neq L_2$, this implies $L' < L_1$ and $L' < L_2$. The minimality of L_1, L_2 forces $L' \subseteq I$. Since I is an ideal and since $L \leq L'$, this implies that $L \subseteq I$. □

Proposition 3.8. *Let $\mathcal{C}$ be a finite category. Suppose that the set of nonsplit morphisms N in $\mathcal{C}$ is an ideal in $M = \mathrm{Mor}(\mathcal{C})$. Suppose that for any $\mathscr{L}$-class L in M there is a unique minimal split $\mathscr{L}$-class L' such that $L \leq L'$. Let I be a proper ideal in M, and let J, J' be two different minimal split $\mathscr{J}$-classes not contained in I. Suppose that $J \circ N \subseteq I$ and $J' \circ N \subseteq I$. Then $M \circ J \circ M \cap M \circ J' \circ M \subseteq I$.*

Proof. By 3.5 we have $M \circ J \circ M \cap M \circ J' \circ M \subseteq (M \circ J \cup I) \cap (M \circ J' \cup I) = (M \circ J \cap M \circ J') \cup I$. Since the $\mathscr{J}$-classes J, J' are different, they contain no common $\mathscr{L}$-class. Moreover, by 3.6, any $\mathscr{L}$-class L contained in J or in J' is a minimal split $\mathscr{L}$-class not contained in I and satisfies $L \circ N \subseteq I$. It follows from 3.7 that the intersection $M \circ J \cap M \circ J'$ is contained in I, whence the result. □

Proposition 3.9. *Let $\mathcal{C}$ be a finite category, k a field, and $\alpha \in Z^2(\mathcal{C}; k^\times)$. Suppose that the set of nonsplit morphisms N in $\mathcal{C}$ is an ideal in $M = \mathrm{Mor}(\mathcal{C})$. Suppose that for any $\mathscr{L}$-class L in M there is a unique minimal split $\mathscr{L}$-class L' such that $L \leq L'$. Let I be a proper ideal in M, and let J be a minimal split $\mathscr{J}$-class not contained in I and satisfying $J \circ N \subseteq I$. Then $k_\alpha[M \circ J \circ M]/k_\alpha[M \circ J \circ M \cap I]$ is projective as a left $k_\alpha\mathcal{C}/k_\alpha I$-module.*

Proof. By 3.5, we have $M \circ J \circ M \subseteq M \circ J \cup I$. Thus

$$k_\alpha[M \circ J \circ M]/k_\alpha[M \circ J \circ M \cap I] = k_\alpha[M \circ J]/k_\alpha[M \circ J \cap I]$$

The set $M \circ J$ is the union of the sets $M \circ e$, where e runs over a set of representatives X of the $\mathscr{L}$-classes contained in J. By Proposition 3.3(v) we may chose X which contains only idempotent endomorphisms. This yields the following sum of left $k_\alpha\mathcal{C}$-modules

$$k[M \circ J] = \sum_{e \in X} k[M \circ e] = \sum_{e \in X} k_\alpha \mathcal{C}\hat{e}\ .$$

For different e, e' in X, the intersection $M \circ e \cap M \circ e'$ is contained in I, thanks to 3.7. Thus this sum becomes a direct sum upon taking the quotient by the ideal $k_\alpha I$, and each summand $k_\alpha[M \circ e]/k_\alpha[M \circ e \cap I]$ is isomorphic to $(k_\alpha\mathcal{C}/k_\alpha I)\bar{e}$, where $\bar{e}$ is the image of the idempotent $\hat{e} = \alpha(e,e)^{-1}e$ in the quotient $k_\alpha\mathcal{C}/k_\alpha I$. The statement follows. □

4 Proof of Theorem 1.1

Let $\mathcal{C}$ be a finite category satisfying the assumptions of Theorem 1.1. We set $M = \mathrm{Mor}(\mathcal{C})$ and denote by N the ideal of nonsplit morphisms in M. Let k be a field and $\alpha \in Z^2(\mathcal{C}; k^\times)$. Set $A = k_\alpha\mathcal{C}$. We denote by $\mathrm{rad}(A)$ the Jacobson radical of A.

If e is an idempotent in A and S a simple A-module, then either eS is zero, or eS is a simple eAe-module. By Green [4, 6.2], the correspondence sending S to eS induces a bijection between isomorphism classes of simple A-modules not annihilated by e and isomorphism classes of simple eAe-modules. Moreover, the inverse of this correspondence can be described as follows: if T is a simple eAe-module, then the A-module $Ae \otimes_{eAe} T$ has a unique maximal submodule, hence a unique simple quotient S, and then $eS \cong T$. In conjunction with the description of a projective cover of a simple A-module S, it follows that if e is an idempotent satisfying $eS \neq \{0\}$, then there is a primitive idempotent i such that $i = ie = ei$ and such that Ai is a projective cover of S. It follows further that if $\mathcal{R}$ is a set of isomorphism classes of simple A-modules, then there is an idempotent e such that $eS = S$ for every simple A-module whose isomorphism class belongs to $\mathcal{R}$, and $eS = \{0\}$ for any simple A-module whose isomorphism class does not belong to $\mathcal{R}$. The idempotent e is uniquely determined up to conjugation by the set $\mathcal{R}$, adopting the convention $e = 0$ if $\mathcal{R}$ is empty.

We will further use the parametrisation of isomorphism classes of simple A-modules from [7]. More precisely, the isomorphism class of a simple A-module S is parametrised by the isomorphism class of a pair (e, T), where e is a minimal idempotent endomorphism of an object in $\mathcal{C}$ such that $eS \neq \{0\}$, and where $T \cong eS$ is a simple eAe-module. As mentioned in 2.3, the image in A of an idempotent endomorphism in $\mathcal{C}$ need not be an idempotent, but the scalar multiple $\hat{e} = \alpha(e,e)^{-1}e$ is an idempotent in A, and hence $eAe = \hat{e}A\hat{e}$ is a unitary algebra with unit element $\hat{e}$. If a simple A-module S is parametrised by a pair (e, T) as before, then T is annihilated by the ideal of noninvertible morphisms in $e \circ M \circ e$, hence T can be viewed as a simple $k_\alpha G_e$-module, where G_e is the maximal subgroup of the monoid $e \circ M \circ e$.

Let J be a split $\mathscr{J}$-class J. Choose an idempotent endomorphism e in J; thus G_e is the maximal subgroup of the monoid $e \circ M \circ e$. By [7, 2.6], this monoid is a disjoint union

$$e \circ M \circ e = G_e \cup M_e \ ,$$

where M_e consists of all morphisms in $e \circ M \circ e$ whose $\mathscr{J}$-class is strictly smaller than J. Moreover, M_e is an ideal in $e \circ M \circ e$. Since endomorphisms of objects in $\mathcal{C}$ are all split, it follows that M_e consists of split morphisms. We denote by

$$\sigma_e : k_\alpha(e \circ M \circ e) \to k_\alpha G_e$$

the canonical surjective k-algebra homomorphism with kernel $k_\alpha M_e$.

We construct a chain of ideals in A which will be shown to be a hereditary chain. We start by defining a chain of ideals in M as follows. Set $I_0 = \emptyset$. For $n \geq 0$, if I_n is already defined, define I_{n+1} by

$$I_{n+1} = I_n \cup \ (\cup_J \ M \circ J \circ M) \ ,$$

where J runs over the split $\mathscr{J}$-classes which are minimal in the set of split $\mathscr{J}$-classes not contained in I_n, and which satisfy $J \circ N \subseteq I_n$. Since the ideals I_n in M are generated by split $\mathscr{J}$-classes, hence by idempotent endomorphisms, we have $I_n^2 = I_n$ for $n \geq 0$. It follows from 3.6 that if I_n is a proper ideal, then I_{n+1} is strictly bigger than I_n, and hence for n sufficiently large, we have $I_n = M$. For $n \geq 0$ we set

$$H_n = k_\alpha I_n \ ;$$

that is, H_n is the k-subspace of A spanned by the image of the set I_n in A. Since I_n is an ideal in the morphism set M satisfying $I_n \circ I_n = I_n$, it follows that H_n is an ideal in the algebra A satisfying

$$H_n^2 = H_n \ ,$$

for $n \geq 0$. Let $n \geq 0$ such that I_n is a proper ideal in M, or equivalently, such that H_n is a proper ideal in A. If J_1, J_2 are two different split $\mathscr{J}$-classes contained in $I_{n+1} \setminus I_n$, then

$M \circ J_1 \circ M \cap M \circ J_2 \circ M$ is contained in I_n by 3.8. It follows that H_{n+1}/H_n is the direct sum of the A/H_n-modules

$$\oplus_J \; k_\alpha[M \circ J \circ M]/k_\alpha[M \circ J \circ M \cap I_n] \;,$$

with J running over the split $\mathscr{J}$-classes in $I_{n+1} \smallsetminus I_n$. By 3.9, each of these summands is projective as an A/H_n-module. It follows that H_{n+1}/H_n is projective as an A/H_n-module. In order to show that A is quasi-hereditary, it remains to show that $H_{n+1}\mathrm{rad}(A)H_{n+1} \subseteq H_n$. Since H_{n+1} is generated, as an ideal, by H_n and by idempotent endomorphisms in $I_{n+1} \smallsetminus I_n$, it suffices to show that $e\mathrm{rad}(A)e' \subseteq H_n$, where e, e' are idempotents in $I_{n+1} \smallsetminus I_n$. If e, e' belong to different $\mathscr{J}$-classes, then eAe' is spanned by $e \circ M \circ e'$, and it follows again from 3.8, that this set is contained in I_n, whence $eAe' \subseteq H_n$. If e, e' belong to the same $\mathscr{J}$-class, then the idempotents $\hat{e}$, $\hat{e}'$ in A are conjugate by 2.3. Thus we may assume that $e' = e$. Then

$$eAe = k_\alpha G_e \oplus k_\alpha M_e \;,$$

where $M_e = e \circ M \circ e \smallsetminus G_e$. In particular, since all morphisms in M_e are split and belong to $\mathscr{J}$-classes strictly smaller than the $\mathscr{J}$-class containing e, we have $M_e \subseteq I_n$. The hypothesis on the characteristic of k implies that $\mathrm{rad}(k_\alpha G_e) = \{0\}$, and hence $e\mathrm{rad}(A)e = \mathrm{rad}(eAe) \subseteq k_\alpha M_e \subseteq H_n$ as required. This completes the proof of Theorem 1.1.

5 Proof of Theorem 1.11

We use the notation and hypotheses from Theorem 1.11. In particular, we have $A = k_\alpha\mathcal{C}$, and c is an idempotent in A with the property that $cS = S$ for any simple A-module S which is parametrised by a weight, and $cS = \{0\}$ for any simple A-module S which is not parametrised by a weight. This determines c up to conjugation by an element in $A^\times$, and we have $cAc \cong W(k_\alpha\mathcal{C})$. As in the previous section, for $n \geq 0$, we denote by I_n the ideal in $M = \mathrm{Mor}(\mathcal{C})$ constructed inductively by $I_0 = \emptyset$ and $I_{n+1} = I_n \cup (\cup_J M \circ J \circ M)$, with J running over the minimal split $\mathscr{J}$-classes not contained in I_n which satisfy $J \circ N \subseteq I_n$. For any idempotent endomorphism e in M we denote by M_e the complement of the group G_e in $e \circ M \circ e$ and by $\sigma_e : eAe \to k_\alpha G_e$ the split surjective algebra homomorphism with kernel $k_\alpha M_e$. By the assumptions, c commutes with e, hence the product ec is an idempotent in eAe. By the construction of c, the element $z_e = \sigma_e(ec)$ is either zero or it is the central idempotent in $k_\alpha G_e$ which is the sum of all block idempotents of simple block algebras of $k_\alpha G_e$. Equivalently, $k_\alpha G_e z_e$ is the largest semisimple direct factor of the twisted group algebra $k_\alpha G_e$. Combining the above parametrisations of simple A-modules yields the following result.

Lemma 5.1. *Let S be a simple module, parametrised by the pair (e, T) for some idempotent endomorphism e and a simple $k_\alpha G_e$-module T. Then there is a primitive idempotent i in eAe with the following properties.*

(i) The projective indecomposable A-module Ai is a projective cover of S.

(ii) We have $\sigma_e(i) \neq 0$, and $\sigma_e(i)$ is a primitive idempotent in $k_\alpha G_e$.

(iii) The projective indecomposable $k_\alpha G_e$-module $k_\alpha G_e \sigma_e(i)$ is a projective cover of T.

(iv) If (e, T) is a weight, then $T \cong k_\alpha G_e \sigma_e(i)$.

Proof. Since e does not annihilate S, there is a primitive idempotent i in eAe which does not annihilate S, whence (i). Since split morphisms in $\mathscr{J}$-classes strictly smaller than J_e annihilate S, it follows that $\ker(\sigma_e)$ annihilates S, and hence $\sigma_e(i) \neq 0$. Since σ_e is surjective, it follows that $\sigma_e(i)$ is a primitive idempotent in $k_\alpha G_e$ which does not annihilate T. This implies (ii) and (iii). If (e, T) is a weight, then T is a projective $k_\alpha G_e$-module, hence isomorphic to its projective covers. This completes the proof. □

We define ideals H_n in cAc by setting

$$H_n = ck_\alpha I_n c = k_\alpha I_n \cap cAc$$

for any $n \geq 0$, with the convention $H_0 = \{0\}$. We have $H_n = cAc$ for n sufficiently large. In order to show that the ideals H_n form a hereditary chain of ideals in cAc, we need to show that $H_n^2 = H_n$, that H_{n+1}/H_n is projective as a left cAc/H_n-module, and that $H_{n+1}\mathrm{rad}(cAc)H_{n+1} \subseteq H_n$, for $n \geq 0$.

Lemma 5.2. *Let n be a nonnegative integer, and e an idempotent endomorphism in $I_{n+1} \setminus I_n$. Then $\ker(\sigma_e) \subseteq k_\alpha I_n$.*

Proof. The kernel of σ_e is spanned, as a k-vector space, by the image of the complement M_e of G_e in $e \circ M \circ e$. Since the morphisms in M_e have split $\mathscr{J}$-classes strictly smaller than J_e, this space is contained in $k_\alpha I_n$. □

Lemma 5.3. *Let n be a nonnegative integer, and e an idempotent endomorphism in $I_{n+1} \setminus I_n$. We have*

$$cAeAc \subseteq cAecAc + ck_\alpha I_n c\ .$$

Proof. By the construction of I_n and I_{n+1} we have $e \circ N \subseteq I_n$, and we have $e' \in I_n$ for any idempotent e' in $M \circ e \circ M$ such that $J_{e'} < J_e$. It follows from Lemma 3.4 that $cAeAc \subseteq \sum_{e'} cAe'c + ck_\alpha I_n c$, where e' runs over the idempotents in J_e. For any such e', the idempotents $\hat{e}$ and $\hat{e}'$ are conjugate in A, hence so are the idempotents $\hat{e}c$ and $\hat{e}'c$. This implies $AecA = Ae'cA$ for any idempotent $e' \in J_e$, hence $cAe'c \subseteq cAe'cAc = cAecAc$. The result follows. □

Lemma 5.4. *Let n be a nonnegative integer and let E be a set of representatives of the split $\mathscr{J}$-classes contained in $I_{n+1} \setminus I_n$ such that E consists of idempotent endomorphisms. We have $H_{n+1} = H_n + \sum_{e \in E}\ cAecAc$; in particular, we have $H_n^2 = H_n$. Moreover, we have a direct sum decomposition of cAc-cAc-bimodules*

$$H_{n+1}/H_n = \oplus_{e\in E}\ cAecAc/(H_n \cap cAecAc)\ .$$

Proof. By 5.3 we have $cAeAc \subseteq cAecAc + ck_\alpha I_n c = cAecAc + H_n$, where $e \in E$. This shows the first statement. In particular, H_n is generated, as an ideal, by idempotents in cAc, and hence $H_n^2 = H_n$. The fact that upon dividing by H_n the first sum becomes a direct sum decomposition follows from 3.8. Indeed, for different e, $e' \in E$ we have $M \circ e \circ M \cap M \circ e' \circ M \subseteq I_n$, whence the result. □

Lemma 5.5. *Let n be a nonnegative integer. Then H_{n+1}/H_n is projective as a left cAc/H_n-module.*

Proof. It follows from Lemma 5.4 that it suffices to show that the left cAc/H_n-module $U = cAecAc/(H_n \cap cAecAc)$ is projective, where e is an idempotent endomorphism in $I_{n+1} \setminus I_n$. Set $B = cAc/H_n$. By construction, we have $U \cong B\bar{e}\bar{c}B$, where $\bar{e}\bar{c}$ is the image of ec in B. By 5.2 we have $\bar{e}B\bar{e} \cong k_\alpha G_e z_e$. In particular, $\bar{e}\bar{c}$ is a nonzero scalar multiple of a central idempotent in $\bar{e}B\bar{e}$. By 3.9, the quotient $AeA/(AeA \cap k_\alpha I_n)$ is projective as an $A/k_\alpha I_n$-module. Equivalently, $\bar{A}\bar{e}\bar{A}$ is projective as a left $\bar{A}$-module, where $\bar{A} = A/k_\alpha I_n$, and $\bar{e}$ is identified to its image in $\bar{A}$. It follows from 2.2 that U is projective as a left B-module. □

Lemma 5.6. *Let n be a nonnegative integer. We have $H_{n+1}\,\mathrm{rad}(cAc)\,H_{n+1} \subseteq H_n$.*

Proof. Let e, e' be idempotent endomorphisms in $I_{n+1} \setminus I_n$. It suffices to show that we have $ecA\mathrm{rad}(cAc)Ae'c \subseteq k_\alpha I_n$. If $J_e \neq J_{e'}$, this follows from 3.8. If $J_e = J_{e'}$, then $\hat{e}c$ and $\hat{e}c'$ are conjugate idempotents. Thus we may assume that $e = e'$. Since $\sigma_e(ecAec) = k_\alpha G_e z_e$ is semisimple, it follows that $ecA\mathrm{rad}(cAc)Aec \subseteq \mathrm{rad}(ecAec) \subseteq \ker(\sigma_e)$. By Lemma 5.2, we have $\ker(\sigma_e) \subseteq k_\alpha I_n$. The result follows. □

Combining the Lemmas 5.4, 5.5, and 5.6 completes the proof of Theorem 1.11.

6 Examples and further remarks

Example 6.1. Let k be a field. Let $\mathcal{C}$ be a category with two objects X and Y and seven morphisms $\{i, e, e', j, f, f', t\}$ such that i and j are the identity morphisms of X and Y, respectively, and where the remaining morphisms are as follows. The morphism t is the unique morphism from X to Y, and this is the unique nonsplit morphism in $\mathcal{C}$. The morphisms e, e' are idempotent endomorphisms of X satisfying $e \circ e' = e'$ and $e' \circ e = e$; the morphisms f, f' are idempotent endomorphisms satisfying $f \circ f' = f$ and $f' \circ f = f'$. Note that the monoids $\mathrm{End}_\mathcal{C}(X)$ and $\mathrm{End}_\mathcal{C}(Y)$ are opposite to each other. The $\mathscr{L}$-classes of e, e' are different, and they satisfy $L_t \leq L_e$ and $L_t \leq L_{e'}$. Thus the minimality condition on $\mathscr{L}$-classes in Theorem 1.1 does not hold. Similarly, the $\mathscr{R}$-classes of f, f' are different and satisfy $R_t \leq R_f$ and $R_t \leq R_{f'}$. Thus $\mathcal{C}$ does not satisfy the corresponding minimality condition for $\mathscr{R}$-classes either. The algebra $A = k\mathcal{C}$ is, however, quasi-hereditary. This can be seen as follows. The $\mathscr{J}$-classes of e and e' coincide, hence e and e' are conjugate in A. Similarly, the idempotents f and f' are conjugate in A. The idempotents e, $i - e$, f, $j - f$ are easily seen to be primitive, pairwise orthogonal and pairwise nonconjugate in A. Thus A is basic and has four isomorphism classes of simple modules. Denote by S, T, U, V the simple quotients of the projective indecomposable A-modules Ae, $A(i - e)$, Af, $A(j - f)$, respectively. Then $V \cong A(j - f)$. The remaining projective indecomposable modules Ae, $A(i - e)$, Af have dimension 2, and their composition series are $\{S, U\}$, $\{T, S\}$, $\{U, V\}$, respectively. The ideals generated by the idempotents $j - f$, j, $e + j$, $i + j = 1$ are easily seen to form a hereditary chain in A. By contrast, the ideals I_n from the proof of Theorem 1.1 do not yield a hereditary chain: we have $I_1 = \{e, e', t\}$, but kI_1 is not projective as a left A-module. This suggests where to look for improvements of Theorem 1.1: one needs to consider ideals generated not just by idempotents in $\mathcal{C}$ but by idempotents in $k_\alpha\mathcal{C}$.

Example 6.2. It is less clear whether one can avoid the hypothesis in Theorem 1.1 on the set of nonsplit morphisms to be an ideal - the 'smallest' cases which do not satisfy this hypothesis do not yield quasi-hereditary algebras. Let k be a field, and let $M = \{1, a, e\}$ be a monoid consisting of three elements, with identity element 1, such that the product of any two non-identity elements is e. Then M is abelian, e is an idempotent in M, and a is the unique nonsplit element in M. In particular, the set of nonsplit elements in M is not an ideal. The algebra kM is not quasi-hereditary: an easy calculation shows that $kM \cong kMe \times kM(1-e)$, and $kMe \cong k$, while $kM(1-e)$ is a local 2-dimensional algebra (with basis $\{1-e, a-e\}$). In particular, kM is a symmetric non-semisimple k-algebra, hence any non-projective kM-module has infinite projective dimension, and in particular, kM is not quasi-hereditary.

Remark 6.3. With the notation of Theorem 1.11, one can replace the hypothesis that c commutes with idempotent endomorphism by the following hypothesis: for any idempotent endomorphism e in $\mathcal{C}$ there is a conjugate c_e of c in A which commutes with the image of e in A and which satisfies $cAeAc = cAec_eAc + cAek_\alpha Nc$. This is easily seen to hold with $c_e = c$, if c commutes in A with all idempotent endomorphisms e in $\mathcal{C}$. This a priori weaker hypothesis ensures that the statements and proofs of the Lemmas 5.3, 5.4, 5.5, and 5.6 remain correct, with ec replaced in the statements and proofs (as appropriate) by ec_e. The verdict on what would be the weakest hypotheses for the weight algebra of a twisted finite category algebra over a field to be quasi-hereditary is still out.

References

[1] R. Boltje and S. Danz, *Ghost algebras of double Burnside algebras via Schur functors.* Preprint (2012).

[2] R. Boltje and S. Danz, *Twisted split category algebras as quasi-hereditary algebras.* Preprint (2012).

[3] J. J. Graham and G. I. Lehrer, *Cellular algebras*, Invent. Math. **123** (1996), 1–34.

[4] J. A. Green, *Polynomial representations of* GL_n, Lecture Notes in Math. **830** (1980), Springer Verlag, Berlin.

[5] S. König and Ch. Xi, *When is a cellular algebra quasi-hereditary?*, Math. Ann. **315** (1999), 281–293.

[6] M. Linckelmann, Fusion category algebras. *J. Algebra* **277** (2004), 222–235.

[7] M. Linckelmann and M. Stolorz, *On simple modules over twisted finite category algebras*, Proc. Amer. Math. Soc. **140** (2012), 3725–3737.

[8] W. Lück, *Transformation Groups and Algebraic K-Theory*, Springer Lecture Notes Math. **1408** (1989).

[9] P. Martin, *The structure of the partition algebras*, J. Algebra **183** (1996), 319–358.

[10] M. S. Putcha, *Complex representations of finite monoids II. Highest weight categories and quivers*, J. Algebra **205** (1998), 53–76.

[11] H. Rui and Ch. Xi, *The representation theory of cyclotomic Temperley-Lieb algebras*, Comment. Math. Helvet. **79** (2004), 427–450.

[12] H. Rui and W. Yu, *On the semi-simplicity of the cyclotomic Brauer algebras*, J. Algebra **277** (2004), 187–221.

[13] J. Thévenaz, *G-Algebras and Modular Representation Theory*, Oxford Science Publications, Clarendon Press, Oxford (1995).

[14] B. W. Westbury, *The representation theory of the Temperley-Lieb algebras*. Math. Z. **219** (1995), 539–565.

[15] S. Wilcox, *Cellularity of diagram algebras as twisted semigroup algebras*, J. Algebra **309** (2007), 10–31.

[16] Ch. Xi, *Partition algebras are cellular*, Compositio Mathematica **119** (1999), 99–109.