BLOCKS WITH NORMAL ABELIAN DEFECT AND ABELIAN p' INERTIAL QUOTIENT

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Abstract. Let k be an algebraically closed field of characteristic p, and let $\mathcal{O}$ be either k or its ring of Witt vectors $W(k)$. Let G be a finite group and B a block of $\mathcal{O}G$ with normal abelian defect group and abelian p' inertial quotient L. We show that B is isomorphic to its second Frobenius twist. This is motivated by the fact that bounding Frobenius numbers is one of the key steps towards Donovan's conjecture. For $\mathcal{O} = k$, we give an explicit description of the basic algebra of B as a quiver with relations. It is a quantised version of the group algebra of the semidirect product $P \rtimes L$.

1. Introduction

Let p be a prime number. The purpose of this paper is to bound the Frobenius numbers and to give a structure theorem for p blocks of finite groups with normal abelian defect groups and abelian p' inertial quotients. This extends the results of Benson and Green [2], Holloway and Kessar [7], Benson and Kessar [3].

We show that these blocks are isomorphic to their second Frobenius twist. By [8], bounding Frobenius numbers is a key step towards Donovan's conjecture; see for instance [4], [5]. We obtain further a complete description of the basic algebra of such a block over a field by means of quiver with relations.

Our main theorems are as follows. Let k be an algebraically closed field of characteristic p and let $W(k)$ be the ring of Witt vectors over k. Let $\mathcal{O} \in \{k, W(k)\}$. For q a power of p, the Frobenius automorphism $\lambda \mapsto \lambda^q$ of the field k lifts uniquely to an automorphism of the ring $W(k)$, and we denote its inverse in both cases by $\mu \mapsto \mu^{\frac{1}{q}}$ (see [10, Chapter 3, Theorem 3, Proposition 10, Theorem 8]).

Recall from [3] that for an $\mathcal{O}$-algebra A, the Frobenius twist $A^{(q)}$ is the $\mathcal{O}$-algebra which equals A as a ring, and where scalar multiplication is twisted via the Frobenius map; that is, for $\lambda \in \mathcal{O}$, and $a \in A$, the action on $A^{(q)}$ is given by $\lambda \cdot a = \lambda^{\frac{1}{q}} a$.

Theorem 1.1. *Let P be a finite abelian p-group, L an abelian p'-subgroup of $\mathsf{Aut}(P)$ and $\alpha \in H^2(L, \mathcal{O}^\times)$. The twisted group algebra $\mathcal{O}_\alpha(P \rtimes L)$ is isomorphic to its second Frobenius twist $\mathcal{O}_\alpha(P \rtimes L)^{(p^2)}$.*

Theorem 1.1 is proved in Section 2.

Theorem 1.2. *Let P be a finite abelian p-group, L an abelian p'-subgroup of $\mathsf{Aut}(P)$ and $\alpha \in H^2(L, k^\times)$ and $\mathfrak{A}$ the basic algebra of the twisted group algebra $k_\alpha(P \rtimes L)$. Then $k_\alpha(P \rtimes L)$ is a matrix algebra over $\mathfrak{A}$ and $\mathfrak{A}$ has an explicit presentation as a quantised version of the group algebra of the semidirect product $P \rtimes L$.*

2010 *Mathematics Subject Classification.* 20C20, 20J06.

Key words and phrases. Finite groups, block theory, abelian defect, Frobenius twist.

The explicit generators and relations for $\mathfrak{A}$ are given in Theorem 4.14.

By a theorem of Külshammer [9], any block of a finite group algebra over $\mathcal{O}$ with a normal defect group is isomorphic to a matrix algebra of a twisted group algebra of the semi-direct product of the defect group of the block with the inertial quotient of the block. Combining [9] with the two results above yields the following.

Corollary 1.3. *Let B be a block of a finite group algebra over $\mathcal{O}$ with a normal defect group P and abelian inertial quotient L. Then B is isomorphic to its second Frobenius twist and if $\mathcal{O} = k$, then B is a matrix algebra over a quantised version of the group algebra of the semidirect product $P \rtimes L$.*

Remark 1.4. It seems unclear whether the same bound holds for strong Frobenius numbers, introduced by Eaton and Livesey in [5]. One issue is that we do not have a sufficiently explicit description of the automorphism φ of $P \rtimes L$ constructed in Lemma 2.3 below.

Acknowledgement. This material is based on work supported by the National Science Foundation under Grant No. DMS-1440140 while the authors were in residence at the Mathematical Sciences Research Institute in Berkeley, California, during the Spring 2018 semester. The first author thanks City, University of London for its hospitality during part of the preparation of this paper. The third author acknowledges support from EPSRC grant EP/M02525X/1. The authors thank Gunter Malle for a careful reading of the paper.

2. Proof of Theorem 1.1.

For L a group, $\phi \in \mathsf{Aut}(L)$ and $\alpha : L \times L \to \mathcal{O}^\times$, denote by ${}^\phi\alpha : L \times L \to \mathcal{O}^\times$ the map defined by ${}^\phi\alpha(x, y) = \alpha(\phi^{-1}(x), \phi^{-1}(y))$ and by $(\phi, \alpha) \mapsto {}^\phi\alpha$ the induced action of $\mathsf{Aut}(L)$ on $H^2(L, \mathcal{O}^\times)$. For q a power of p, denote by $\alpha^{(q)}$ the map $L \times L \to \mathcal{O}^\times$ defined by $\alpha^{(q)}(x, y) = \alpha(x, y)^{\frac{1}{q}}$ and by $\alpha^{(q)}$ the image of α under the induced isomorphism $H^2(L, \mathcal{O}^\times) \cong H^2(L, \mathcal{O}^\times)$.

Lemma 2.1. *Let L be a finite abelian p'-group and let $\phi : L \to L$ be the group automorphism defined by $\phi(x) = x^p$ for all $x \in L$. Then for all $\alpha \in H^2(L, \mathcal{O}^\times)$, we have ${}^\phi\alpha = \alpha^{(p^2)}$.*

Proof. It is well-known that since L is a finite p'-group, it follows that the canonical map $\mathcal{O} \to k$ induces an isomorphism $H^2(L, \mathcal{O}^\times) \cong H^2(L, k^\times)$. Thus we may assume that $\mathcal{O} = k$.

Consider the universal coefficient sequence

$$0 \to \mathsf{Ext}^1(H_1(L, \mathbb{Z}), k^\times) \to H^2(L, k^\times) \to \mathsf{Hom}(H_2(L, \mathbb{Z}), k^\times) \to 0.$$

Since k is algebraically closed, $k^\times$ is divisible, and therefore injective as an abelian group. So the first term in this sequence is zero. For the third term, we have $H_2(L, \mathbb{Z}) \cong \Lambda^2(L)$, the exterior square in the category of abelian groups. Therefore we obtain an isomorphism

$$H^2(L, k^\times) \cong \mathsf{Hom}(\Lambda^2(L), k^\times)$$

which by naturality is $\mathsf{Aut}(L)$ equivariant and which commutes with the Frobenius morphism of k. More precisely, if $\alpha \in H^2(L, k^\times)$ corresponds to $\tau \in \mathsf{Hom}(\Lambda^2(L), k^\times)$ under the above isomorphism, then for any $\psi \in \mathsf{Aut}(L)$, and any power q of p, ${}^\psi\alpha$ corresponds to the homomorphism ${}^\psi\tau$ defined by ${}^\psi\tau(x \wedge y) = \tau(\psi^{-1}(x) \wedge \psi^{-1}(y))$ and $\alpha^{(q)}$ corresponds to the homomorphism $\tau^{(q)}$ defined by $\tau^{(q)}(x \wedge y) = \tau(x \wedge y)^{\frac{1}{q}}$. The result follows since for any $\tau \in \mathsf{Hom}(\Lambda^2(L), k^\times)$ we have $\tau(x^q \wedge y) = \tau(x \wedge y)^q = \tau(x \wedge y^q)$. □

The isomorphism $H^2(L, k^\times) \cong \mathsf{Hom}(\Lambda^2(L), k^\times)$ in the above proof can be explicitly described as follows. If $\alpha \in Z^2(L, k^\times)$, then the image of the class of α in $\mathsf{Hom}(\Lambda^2(L), k^\times)$ is the group homomorphism $x \wedge y \mapsto \alpha(x, y)\alpha(y, x)^{-1}$, where x, $y \in L$. One can either verify directly, using the 2-cocycle identity, that this assignment is a group homomorphism in each component, or one can observe that $\alpha(x, y)\alpha(y, x)^{-1}$ is equal to the commutator of lifts of x, y in a central extension determined by α. More precisely, let

$$1 \to k^\times \to \tilde{L} \to L \to 1$$

be a central extension defined by α. For each $x \in L$, choose an element $\tilde{x} \in \tilde{L}$ lifting x. An easy calculation shows that $\alpha(x, y)\alpha(y, x)^{-1} = [\tilde{x}, \tilde{y}]$. This commutator does not depend on the choices of the lifts $\tilde{x}$ and since $\tilde{L}$ is a central extension of the abelian group L, this commutator is a group homomorphism in each component. In particular, $[\bar{x^p}, \bar{y^p}] = [\bar{x}, \bar{y}]^{p^2}$, which explains the statement of the above Lemma.

Lemma 2.2. *Let P be a finite p-group and let $\Phi(P)$ be the Frattini subgroup of P. The kernel of the natural group homomorphism $\mathsf{Aut}(P) \to \mathsf{Aut}(P/\Phi(P))$ is a p-group. If P is homocyclic, then the map $\mathsf{Aut}(P) \to \mathsf{Aut}(P/\Phi(P))$ is surjective.*

Proof. For the first assertion see [6, Chapter 5, Theorem 1.4]. Assume that P is homocyclic and let $\{x_1, x_2, \ldots, x_r\}$ be a minimal generating set of P. Let $\psi \in \mathsf{Aut}(P/\Phi(P))$. For each i, $1 \le i \le r$, pick an element $u_i \in P$ such that $\psi(x_i\Phi(P)) = u_i\Phi(P)$. Since P is homocyclic, there exists a homomorphism $\tilde{\psi} : P \to P$ such that $\tilde{\psi}(x_i) = u_i$, $1 \le i \le r$. Clearly, $\tilde{\psi}$ lifts ψ. Since ψ is an automorphism, $\mathrm{Im}(\tilde{\psi})\Phi(P) = P$ whence $\mathrm{Im}(\tilde{\psi}) = P$ and $\tilde{\psi} \in \mathsf{Aut}(P)$. □

Lemma 2.3. *Let P be a finite abelian p-group and let L be an abelian p'-group acting on P. There exists an automorphism ϕ of $P \rtimes L$ such that $\phi(L) = L$ and $\phi(x) = x^p$ for all $x \in L$.*

Proof. Denote by L' the image of L in $\mathsf{Aut}(P)$. Proving the existence of ϕ is equivalent to showing that there exists $\tau \in \mathsf{Aut}(P)$ such that $\tau y \tau^{-1} = y^p$ for all $y \in L'$. Suppose first that $P = P_1 \times P_2$ for L'-invariant subgroups P_1 and P_2 of P. If for each $i = 1, 2$, there exists $\tau_i \in \mathsf{Aut}(P_i)$ with $\tau_i(y{\downarrow_{P_i}})\tau_i^{-1} = (y{\downarrow_{P_i}})^p$ for all $y \in L'$, then the map $\tau \colon P \to P$ sending x_1x_2 to $\tau_1(x_1)\tau_2(x_2)$ for $x_1 \in P_1$, $x_2 \in P_2$ has the required properties. Hence we may assume that P is indecomposable for the action of L' and consequently that P is homocyclic (see [6, Chapter 5, Theorem 2.2]).

We claim that it suffices to prove the result for the case that P is elementary abelian. Indeed, let U be the kernel of the map $\mathsf{Aut}(P) \to \mathsf{Aut}(P/\Phi(P))$. By Lemma 2.2, U is a p-group. Let $\bar{L}'$ be the image of L' in $\mathsf{Aut}(P/\Phi(P))$ and suppose that there exists $\bar{\tau} \in \mathsf{Aut}(P/\Phi(P))$ such that $\bar{\tau}\eta\bar{\tau}^{-1} = \eta^p$ for all $\eta \in \bar{L}$. By Lemma 2.2, there exists $\tau \in \mathsf{Aut}(P)$ lifting $\bar{\tau}$. Since $L'U$ is the full inverse image of $\bar{L}'$ in $\mathsf{Aut}(P)$, and $\bar{L}'$ is $\bar{\tau}$-invariant, $\tau L'U\tau^{-1} = L'U$. Hence L' and $\tau L'\tau^{-1}$ are both complements to the normal Sylow p-subgroup U of $L'U$. By the Schur–Zassenhaus theorem, there exists $u \in U$ such that $uL'u^{-1} = \tau L'\tau^{-1}$. Replacing τ by $u^{-1}\tau$ we may assume that $\tau L'\tau^{-1} = L'$. Then for any $y \in L'$, $\tau y'\tau^{-1}$ and y'^p are elements of the p'-group L' lifting the same element of $\bar{L}'$. The claim follows by Lemma 2.2.

By the discussion above we may assume that P is elementary abelian and that P is an indecomposable, faithful $\mathbb{F}_pL'$-module. Since L' is an abelian p'-group, P is in fact an irreducible $\mathbb{F}_pL'$-module and L' is cyclic. Let $L' = \langle y \rangle$ and let $f(X) \in \mathbb{F}_p[X]$ be the characteristic polynomial of y as an element of $\mathrm{End}_{\mathbb{F}_p}(P)$. Since $f(y^p) = f(y)^p = 0$, $f(X)$

is also the characteristic polynomial of y^p. Thus, y and y^p are conjugate in $\mathrm{GL}(P) \cong \mathsf{Aut}(P)$. □

Proof of Theorem 1.1. Let ϕ be as in Lemma 2.3. Then ϕ induces an $\mathcal{O}$-algebra isomorphism $\mathcal{O}_\alpha(P \rtimes L) \cong \mathcal{O}_{\phi\alpha}(P \rtimes L)$. The result follows by Lemma 2.1 since for any power q of p, $\mathcal{O}_\alpha(P \rtimes L)^{(q)} \cong \mathcal{O}_{\alpha^{(q)}}(P \rtimes L)$ as $\mathcal{O}$-algebras. □

3. On characters of groups of class two

For a finite group H denote by $\mathrm{Irr}(H)$ the set of ordinary irreducible characters of H. If N is a normal subgroup of H and $\chi \in \mathrm{Irr}(N)$, denote by $\mathrm{Irr}(H \,|\, \chi)$ the subset of $\mathrm{Irr}(H)$ covering χ. Recall that if H/N is abelian, then the group of irreducible (i.e. linear) characters of H/N acts on $\mathrm{Irr}(H \,|\, \chi)$ via multiplication and this action is transitive.

Proposition 3.1. *Let H be a finite group which is nilpotent of class 2. Let χ be a faithful irreducible character of $Z := [H, H]$. Set $m = \sqrt{|H : Z(H)|}$.*

(i) *For any $\phi \in \mathrm{Irr}(Z(H) \,|\, \chi)$, $\phi{\uparrow}^H = m\tau_\phi$ for some $\tau_\phi \in \mathrm{Irr}(H)$. In particular, $m = \tau_\phi(1)$ is an integer.*

(ii) *The map $\phi \mapsto \tau_\phi$, $\phi \in \mathrm{Irr}(Z(H) \,|\, \chi)$, is a bijection between $\mathrm{Irr}(Z(H) \,|\, \chi)$ and $\mathrm{Irr}(H \,|\, \chi)$.*

(iii) *The actions of $\mathrm{Irr}(H/Z)$ on $\mathrm{Irr}(H \,|\, \chi)$ and of $\mathrm{Irr}(Z(H)/Z)$ on $\mathrm{Irr}(Z(H) \,|\, \chi)$ are compatible with the bijection in (ii). More precisely, let $\eta \in \mathrm{Irr}(H/Z)$, and let $\phi \in \mathrm{Irr}(Z(H) \,|\, \chi)$. Then $\tau_{\eta\downarrow_{Z(H)}\phi} = \eta\tau_\phi$. Consequently, $\eta\tau_\phi = \tau_\phi$ if and only if η restricts to the trivial character of $Z(H)$.*

Proof. Let τ be an irreducible character of H covering χ. We claim that $\tau(x) = 0$ for all $x \in H \setminus Z(H)$. Indeed, since $Z(H)$ is the intersection of all maximal abelian subgroups of H, it suffices to prove that if A is a maximal subgroup of H, then $\tau(x) = 0$ if $x \notin A$. So, let A be a maximal abelian subgroup of H. Then $Z(H) \le A$ and since H is of class 2, A is normal in H. Let ψ be an irreducible constituent of the restriction of τ to A and suppose that $g \in H$ is such that ${}^g\psi = \psi$. Then for all $a \in A$, $\psi(gag^{-1}) = \psi(a)$ and so $\psi(gag^{-1}a^{-1}) = 1$. Since the restriction of ψ to Z equals χ, we have that $\chi(gag^{-1}a^{-1}) = 1$ for all $a \in A$. The faithfulness of χ and the maximality of A now imply that $g \in C_G(A) = A$. Consequently, $\tau = \psi{\uparrow}^H$ and $\tau(x) = 0$ for all $x \notin A$, proving the claim.

Let ϕ be the unique linear character of $Z(H)$ covering χ and which is covered by τ. Since ϕ is linear, the restriction of τ to $Z(H)$ consists of $\tau(1)$ copies of ϕ. By the claim above,

$$1 = \langle \tau, \tau \rangle = \frac{1}{|H|} \sum_{x \in Z(H)} \tau(x)\tau(x^{-1}) = \frac{\tau(1)^2}{|H|} \sum_{x \in Z(H)} \phi(x)\phi(x^{-1}) = \frac{\tau(1)^2}{m^2}.$$

Thus $\tau(1) = m$ and

$$\tau(1)m = |H : Z(H)| = \phi{\uparrow}^H(1).$$

On the other hand, by Frobenius reciprocity the multiplicity of τ as a constituent of $\phi{\uparrow}^H$ equals $\tau(1)$. So $\phi{\uparrow}^H = m\tau$. Setting $\tau_\phi = \tau$ proves part (i) of the proposition. Part (ii) is immediate from (i) and the fact that every element of $\mathrm{Irr}(H \,|\, \chi)$ covers a unique element of $\mathrm{Irr}(Z(H) \,|\, \chi)$. By the induction formula, $\eta(\phi{\uparrow}^H) = (\eta{\downarrow}_{Z(H)}\phi){\uparrow}^H$, hence (i) gives that $\tau_{\eta\downarrow_{Z(H)}\phi} = \eta\tau_\phi$. Now (ii) yields that $\eta\tau_\phi = \tau_\phi$ if and only if $\eta{\downarrow}_{Z(H)}\phi = \phi$ if and only if $\eta{\downarrow}_{Z(H)}$ is trivial. □

4. The basic algebra.

Lemma 4.1. *Let $1 \to A \to B \to C \to 1$ be a short exact sequence of abelian groups and let $\pi : D \to A$ be a surjective homomorphism of abelian groups. For each $\alpha \in C$, choose a pre-image u_α in B. Then there exists a 2-cocycle $(\alpha, \beta) \to f_{\alpha,\beta}$ from $C \times C$ to D such that $\pi(f_{\alpha,\beta}) = u_\alpha^{-1} u_\beta^{-1} u_{\alpha\beta}$ and $f_{\alpha,\beta} = f_{\beta,\alpha}$ for all $\alpha, \beta \in C$.*

Proof. It is well-known that $\mathsf{Ext}^n_{\mathbb{Z}}(C, A) = \{0\}$ for $n \geq 2$, and hence the connecting homomorphism $\mathsf{Ext}^1_{\mathbb{Z}}(C, A) \to \mathsf{Ext}^2_{\mathbb{Z}}(C, \ker(\pi)) = \{0\}$ is zero. Thus the map $\mathsf{Ext}^1_{\mathbb{Z}}(C, D) \to \mathsf{Ext}^1_{\mathbb{Z}}(C, A)$ induced by π is surjective. In particular, the element in $\mathsf{Ext}^1_{\mathbb{Z}}(C, A)$ represented by the given short exact sequence lifts to an element in $\mathsf{Ext}^1_{\mathbb{Z}}(C, D)$. Rephrased in terms of extensions this means that there is a commutative diagram of abelian groups with exact rows of the form

$$\begin{array}{ccccccccc} 1 & \longrightarrow & D & \longrightarrow & \hat{B} & \longrightarrow & C & \longrightarrow & 1 \\ & & {\scriptstyle\pi}\big\downarrow & & \big\downarrow{\scriptstyle\tau} & & \big\| & & \\ 1 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 1 \end{array}$$

Note that τ is surjective and restricts to the map π on D. For $\alpha \in C$, choose a preimage v_α of u_α in $\hat{B}$ and set $f_{\alpha,\beta} = v_\beta^{-1} v_\alpha^{-1} v_{\alpha\beta}$ for all α, $\beta \in C$. Clearly, $f_{\alpha,\beta} \in D$. Since $\hat{B}$ is abelian, $f_{\beta,\alpha} = f_{\alpha,\beta}$. Thus $(\alpha, \beta) \mapsto f_{\alpha,\beta}$ is a 2-cocycle with the properties as stated. □

We recall the following result on the structure of twisted group algebras.

Lemma 4.2. *Let G be a finite group and let $\alpha \in H^2(G, k^\times)$. Then there exists a central extension*

$$1 \to Z \to \tilde{G} \to G \to 1$$

with Z a finite cyclic p'-group and a linear character $\chi \colon Z \to k^\times$ such that $k_\alpha G \cong k\tilde{G}e$ where $e = \frac{1}{|Z|}\sum_{z \in Z} \chi(z^{-1}) z$ is the idempotent of kZ corresponding to χ. Moreover, Z may be chosen to be contained in $[\tilde{G}, \tilde{G}]$.

Proof. This is well known, but for completeness we provide a proof. Let m be the order of the cohomology class α. Since k is algebraically closed, $k^\times$ is a divisible group. So we have a short exact sequence

$$0 \to \mu_m \to k^\times \to k^\times \to 0,$$

where μ_m denotes the subgroup of m-th roots of unity in $k^\times$. Note that μ_m has order m as m is relatively prime to p. Consider the corresponding maps of universal coefficient sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathsf{Ext}^1(H_1(G,\mathbb{Z}), \mu_m) & \longrightarrow & H^2(G, \mu_m) & \longrightarrow & \mathsf{Hom}(H_2(G,\mathbb{Z}), \mu_m) & \longrightarrow & 0 \\ & & \big\downarrow & & \big\downarrow & & \big\downarrow & & \\ 0 & \longrightarrow & \mathsf{Ext}^1(H_1(G,\mathbb{Z}), k^\times) & \longrightarrow & H^2(G, k^\times) & \xrightarrow{\cong} & \mathsf{Hom}(H_2(G,\mathbb{Z}), k^\times) & \longrightarrow & 0 \\ & & \big\downarrow & & \big\downarrow{\scriptstyle m} & & \big\downarrow{\scriptstyle m} & & \\ 0 & \longrightarrow & \mathsf{Ext}^1(H_1(G,\mathbb{Z}), k^\times) & \longrightarrow & H^2(G, k^\times) & \xrightarrow{\cong} & \mathsf{Hom}(H_2(G,\mathbb{Z}), k^\times) & \longrightarrow & 0 \end{array}$$

We have $\mathsf{Ext}^1(H_1(G,\mathbb{Z}), k^\times) = 0$ since $k^\times$ is divisible, so $H^2(G, k^\times) \to \mathsf{Hom}(H_2(G,\mathbb{Z}), k^\times)$ is an isomorphism. Since α has order m, its image in $\mathsf{Hom}(H_2(G,\mathbb{Z}), k^\times)$ lifts to a *surjective* element of order m in $\mathsf{Hom}(H_2(G,\mathbb{Z}),\mu_m)$. An inverse image $\tilde{\alpha} \in H^2(G,\mu_m)$ again has order m. Let

$$1 \to Z \to \tilde{G} \to G \to 1$$

be a central extension corresponding to $\tilde{\alpha}$, with $Z \cong \mu_m$.

Now choose a presentation of G by generators and relations

$$1 \to R \to F \to G \to 1.$$

By freeness, the identity map on G lifts to a map $F \to \tilde{G}$. This map sends R into Z, and $[F,F]$ into $[\tilde{G},\tilde{G}]$. It has $[F,R]$ in its kernel since μ_m is central. This gives us a map

$$H_2(G,\mathbb{Z}) = (R \cap [F,F])/[F,R] \to Z$$

which is the lift of α in $\mathsf{Hom}(H_2(G,\mathbb{Z}),\mu_m)$. This map is surjective, but it lands in $Z \cap [\tilde{G},\tilde{G}]$ and hence $Z \subseteq [\tilde{G},\tilde{G}]$.

The formula for the idempotent e can be found in the statement and proof of Thévenaz [11, Chapter 2, Proposition 10.5]. The linear character χ sends each element $z \in Z$ to its image under $Z \cong \mu_m \hookrightarrow k^\times$. $\square$

Let P be an abelian p-group, L an abelian p'-subgroup of $\mathsf{Aut}(P)$ and $\alpha \in H^2(L,k^\times)$. Let Z, $\tilde{G}$ and χ be as in the conclusion of Lemma 4.2 applied to $G = P \rtimes L$ and with α regarded as an element of $H^2(G,k^\times)$ via the pull back along $G \to G/P$. Let H be the full inverse image of L in $\tilde{G}$. Then $\tilde{G} = P \rtimes H$ and e is a central idempotent of H.

We have a natural homomorphism

$$\rho \colon H \to \mathsf{Hom}(H,k^\times)$$

sending g to $\rho(g)\colon h \mapsto \chi[g,h]$. The kernel of this map is $Z(H)$ and the image is $\mathsf{Hom}(H/Z(H),k^\times)$. We denote by

$$\bar{\rho} \colon H/Z(H) \to \mathsf{Hom}(H/Z,k^\times)$$

the induced isomorphism.

Now $P/\Phi(P)$ is naturally a faithful $\mathbb{F}_pL$-module. The extension of scalars $k \otimes_{\mathbb{F}_p} P/\Phi(P)$ gives a kL-module isomorphic to $J(kP)/J^2(kP)$. Let ψ be the character of kL on this module, and write

$$\psi = \bigoplus_{i=1}^{r} \psi_i$$

where r is the rank of $P/\Phi(P)$ and the ψ_i are one dimensional kL-modules (there may be repetitions). We choose an H-invariant complement W for $J^2(kP)$ in $J(kP)$, and a basis w_i of W so that for $g \in H$ we have

$$g w_i g^{-1} = \psi_i(g) w_i. \tag{4.3}$$

Since a p'-group of automorphisms of an abelian p-group preserves some decomposition into homocyclic summands (see for example Chapter 5, Theorem 2.2 in Gorenstein [6]), we may assume that

$$kP = k[w_1,\dots,w_r]/(w_1^{p^{n_1}},\dots,w_r^{p^{n_r}})$$

with $n_1 \geqslant \dots \geqslant n_r$ and $|P| = p^{n_1+\dots+n_r}$. Thus we have relations

$$w_i^{p^{n_i}} = 0. \tag{4.4}$$

Applying the results of Section 3, the irreducible characters τ_ϕ of H lying over χ are in one to one correspondence with the one dimensional characters ϕ of $Z(H)$ lying over χ. The corresponding central idempotents are

$$e_\phi = \frac{1}{|Z(H)|} \sum_{h \in Z(H)} \phi(h^{-1})h. \tag{4.5}$$

Choose one of these, say $\tau = \tau_{\phi_0}$, and choose a matrix representation $T_{\phi_0}\colon H \to \mathsf{Mat}_m(k)$ affording τ_{ϕ_0}. Then for each ϕ choose a one dimensional representation ξ_ϕ of H whose restriction to $Z(H)$ is $\phi\phi_0^{-1}$ (and hence whose restriction to Z is trivial) and chosen so that $\xi_{\phi_0} = 1$. We assume that these ξ_ϕ have been chosen, one for each ϕ, and we define $T_\phi\colon H \to \mathsf{Mat}_m(k)$ via $T_\phi(h) = \xi_\phi(h).T_{\phi_0}(h)$. Then T_ϕ is a matrix representation affording τ_ϕ. So the map

$$\begin{aligned} kHe &\to \mathsf{Mat}_m(k) \times \cdots \times \mathsf{Mat}_m(k) \qquad\qquad (|Z(H):Z| \text{ copies}) \\ he &\mapsto (T_{\phi_0}(h), \dots, T_\phi(h), \dots) \end{aligned}$$

is an isomorphism. Elements of kHe of the form $\sum_\phi \xi_\phi^{-1}(h)e_\phi.h$ are sent to diagonal elements $(T_{\phi_0}(h), \dots, T_{\phi_0}(h))$, and therefore span a copy of $\mathsf{Mat}_m(k)$ in kHe containing e as its identity element. Let us write $\mathfrak{M}$ for this subalgebra of kHe.

Now for each ψ_i and each ϕ, the character $\phi.(\psi_i|_{Z(H)})$ is some ϕ', which we denote $\phi\psi_i$ for convenience. So $\xi_{\phi\psi_i}\xi_\phi^{-1}\psi_i^{-1}$ is trivial on $Z(H)$. Thus there exists an element $g_{i,\phi} \in H$ such that

$$\rho(g_{i,\phi}) = \xi_{\phi\psi_i}\xi_\phi^{-1}\psi_i^{-1}, \tag{4.6}$$

where $\rho(g_{i,\phi})(h) = \chi([g_{i,\phi}, h])$. We choose such elements $g_{i,\phi}$, one for each ψ_i and ϕ.

On the other hand, using (4.3) and (4.5), we have

$$\sum_{h \in Z(H)} \phi(h^{-1})w_i h = \sum_{h \in Z(H)} \phi(h^{-1})\psi_i(h^{-1})hw_i$$

and so

$$w_i e_\phi = e_{\phi\psi_i} w_i. \tag{4.7}$$

Lemma 4.8. *For $h \in H$ we have*

$$(g_{i,\phi}w_i)(\xi_\phi(h)^{-1}e_\phi.h) = (\xi_{\phi\psi_i}(h)^{-1}e_{\phi\psi_i}.h)(g_{i,\phi}w_i).$$

Thus $g_{i,\phi}w_ie_\phi = e_{\phi\psi_i}g_{i,\phi}w_i$ commutes with $\mathfrak{M}$.

Proof. Scalars commute with everything, and the e_ϕ, being in $Z(kH)$, commute with all $h \in H$ and all $g_{i,\psi}$. By (4.3) we have $hw_i = \psi_i(h)w_ix$, and by (4.7) we have $w_ie_\phi = e_{\phi\psi_i}w_i$. We are in kGe, and $eg_{i,\phi}h = e\chi([g_{i,\phi}, h])^{-1}hg_{i,\phi}$. Putting these together gives

$$\xi_\phi(x)^{-1}(g_{i,\phi}w_i)(e_\phi.h) = \xi_\phi(h)^{-1}\psi_i(h)^{-1}\chi([g_{i,\phi}, h])^{-1}(e_{\phi\psi_i}h)(g_{i,\phi}w_i).$$

Finally, applying (4.6), the scalar on the right hand side is equal to $\xi_{\phi\psi_i}(h)^{-1}$.

For the final statement, we have

$$(g_{i,\phi}w_ie_\phi)\left(\sum_{\phi'} \xi_{\phi'}^{-1}(h)e_{\phi'}.h\right) = (g_{i,\phi}w_i)(\xi_\phi^{-1}(h)e_\phi.h)$$

$$= (\xi_{\phi\psi_i}^{-1}(h)e_{\phi\psi_i}.h)(g_{i,\phi}w_i)$$
$$= \left(\sum_{\phi'} \xi_{\phi'}^{-1}(h)e_{\phi'}.h\right)(e_{\phi\psi_i}g_{i,\phi}w_i). \qquad \square$$

Definition 4.9. Let $\mathfrak{A}$ be the subalgebra of kHe generated by the elements e_ϕ and $g_{i,\phi}w_ie_\phi$. Thus by the lemma, $\mathfrak{A}$ and $\mathfrak{M}$ commute.

We claim that $\mathfrak{A}$ is a basic algebra of dimension $|P|\cdot|H : Z(H)|$, and that multiplication in kHe induces an isomorphism

$$\mathfrak{A} \otimes_k \mathfrak{M} \to kHe,$$

so that $kHe \cong \mathsf{Mat}_m(\mathfrak{A})$. For this purpose, we shall use the following.

Lemma 4.10. *Let $A \leqslant B$ be k-algebras with A an Azumaya algebra (i.e., a finite dimensional central separable k-algebra). Then the map $A \otimes_k C_B(A) \to B$ is an isomorphism.*

Proof. See for example Chapter 3, Corollary 4.3, in Bass [1]. $\square$

Remark 4.11. Note that the hypotheses of the lemma include the assumption that the identity element of A is equal to the identity element of B.

We display $\mathfrak{A}$ as kQ/I where Q is a quiver and $I \leqslant J^2(kQ)$ is an ideal of relations. The quiver Q has $|Z(H) : Z|$ vertices labelled $[\phi]$ corresponding to the idempotents $e_\phi \in kZ(H)$ lying over χ, and directed edges labelled with the w_i corresponding to

$$g_{i,\phi}w_ie_\phi = e_{\phi\psi_i}g_{i,\phi}w_i = e_{\phi\psi_i}g_{i,\phi}w_ie_\phi.$$

going from $[\phi]$ to $[\phi\psi_i]$. For brevity, we can illustrate these vertices and directed edges as

$$[\phi] \xrightarrow{\ i\ } [\phi\psi_i].$$

Lemma 4.12. (i) *For suitable elements $z_{i,j,\phi} \in Z(H)$, we have*

$$g_{j,\phi\psi_i}\, g_{i,\phi} = g_{i,\phi\psi_j}\, g_{j,\phi}\, z_{i,j,\phi}.$$

(ii) *The following relations hold in $\mathfrak{A}$:*

$$(g_{j,\phi\psi_i}w_je_{\phi\psi_i})(g_{i,\phi}w_ie_\phi) = q_{i,j,\phi}(g_{i,\phi\psi_j}w_ie_{\phi\psi_j})(g_{j,\phi}w_ie_\phi)$$

where $q_{i,j,\phi} = \phi(z_{i,j,\phi}) \in k^\times$.

(iii) *By changing the choices of $g_{i,\phi}$ by elements of $Z(H)$, we may ensure that $z_{i,j,\phi} \in Z$ and $q_{i,j,\phi} = \chi(z_{i,j,\phi})$.*

Proof. (i) By (4.6), we have

$$\begin{aligned}\rho(g_{j,\phi\psi_i}g_{i,\phi}) &= \rho(g_{j,\phi\phi_i})\rho(g_{i,\phi})\\ &= (\xi_{\phi\psi_i\psi_j}\xi_{\phi\psi_i}^{-1}\psi_j^{-1})(\xi_{\phi\psi_i}\xi_\phi^{-1}\psi_i^{-1})\\ &= \xi_{\phi\psi_i\psi_j}\xi_\phi^{-1}\psi_j^{-1}\psi_i^{-1}.\end{aligned}$$

This is symmetric in i and j, and so

$$\rho(g_{j,\phi\psi_i}g_{i,\phi}) = \rho(g_{i,\phi\psi_j}g_{j,\phi}).$$

Since the kernel of ρ is $Z(H)$ it follows that for some element $z_{i,j,\phi} \in Z(H)$ we have

$$g_{j,\phi\psi_i}\, g_{i,\phi} = g_{i,\phi\psi_j}\, g_{j,\phi}\, z_{i,j,\phi}.$$

(ii) This follows from the fact that we have $z_{i,j,\phi}\, e_\phi = \phi(z_{i,j,\phi})e_\phi$.

(iii) We apply Lemma 4.1 with $A = \mathrm{Irr}(H/Z(H))$, $B = \mathrm{Irr}(H/Z)$, $C = \mathrm{Irr}(Z(H)/Z)$, the map from B to C the restriction map, $D = H/Z$, π the composition of the natural surjection $H/Z \to H/Z(H)$ with $\bar\rho$ and $u_\alpha = \xi_{\alpha\phi_0}$, $\alpha \in \mathrm{Irr}(Z(H)/Z)$. Let $f_{\alpha,\beta} \in H/Z$ be as in the conclusion of the lemma, and let $\tilde f_{\alpha,\beta} \in H$ be any lift of $f_{\alpha,\beta}$ to H. Denote also by ψ_i the restriction of ψ_i to $Z(H)$. So $\psi_i^{-1}u_{\psi_i}$ is an element of $\mathrm{Irr}(H/Z(H))$. Choose an element $g_i \in H$ such that $\rho(g_i) = \psi_i^{-1}u_{\psi_i}$ and set

$$g_{i,\phi} = g_i \tilde f_{\psi_i,\phi\phi_0^{-1}}.$$

Then

$$\begin{aligned}\rho(g_{i,\phi}) &= \psi_i^{-1}u_{\psi_i}\rho(\tilde f_{\psi_i,\phi\phi_0^{-1}})\\ &= \psi_i^{-1}u_{\psi_i}u_{\psi_i}^{-1}u_{\phi\phi_0^{-1}}^{-1}u_{\psi_i\phi\phi_0^{-1}}\\ &= \psi^{-1}\xi_\phi^{-1}\xi_{\psi_i\phi}\end{aligned}$$

and

$$\begin{aligned}g_{j,\phi\psi_i}\, g_{i,\phi}Z &= g_j\tilde f_{\psi_j,\phi\psi_i\phi_0^{-1}}\, g_i\tilde f_{\psi_i,\phi\phi_0^{-1}}Z\\ &= g_jg_i\tilde f_{\psi_j,\phi\psi_i\phi_0^{-1}}\tilde f_{\psi_i,\phi\phi_0^{-1}}Z\\ &= g_jg_i\tilde f_{\psi_j,\phi\psi_i\phi_0^{-1}}\tilde f_{\phi\phi_0^{-1},\psi_i}Z\\ &= g_jg_i\tilde f_{\psi_j,\phi\phi_0^{-1}}\tilde f_{\psi_j\phi\phi_0^{-1},\psi_i}\\ &= g_jg_ji\tilde f_{\psi_j,\phi\phi_0^{-1}}\tilde f_{\psi_i,\psi_j\phi\phi_0^{-1}}Z\\ &= g_{i,\phi\psi_j}\, g_{j,\phi}Z.\end{aligned}$$

□

Lemma 4.13. *For each i and each ϕ we have*

$$(g_{i,\phi\psi_i^{p^{n_i}-1}}w_ie_{\phi\psi_i^{p^{n_i}-1}})\dots(g_{i,\phi\psi_i^2}\,w_ie_{\phi\psi_i^2})(g_{i,\phi\psi_i}\,w_ie_{\phi\psi_i})(g_{i,\phi}\,w_ie_\phi) = 0.$$

Here, we have written down the only composable sequence of arrows in Q beginning with e_ϕ with each arrow involving w_i, and there are p^{n_i} terms in the product:

$$[\phi] \xrightarrow{\;i\;} [\phi\psi_i] \xrightarrow{\;i\;} [\phi\psi_i^2] \xrightarrow{\;i\;} \cdots \xrightarrow{\;i\;} [\phi\psi_i^{p^{n_i}}].$$

Proof. Relations (4.3) and (4.7) allow us to push the w_i terms past the other terms so that they are directly multiplied together. Then we can use relation (4.4) to conclude that we get zero. □

Theorem 4.14. *The relations on the quiver algebra kQ given by*

$$(g_{j,\phi\psi_i}w_je_{\phi\psi_i})(g_{i,\phi}w_ie_\phi) = q_{i,j,\phi}(g_{i,\phi\psi_j}w_ie_{\phi\psi_j})(g_{j,\phi}w_ie_\phi)$$

and

$$(g_{i,\phi\psi_i^{p^{n_i}-1}}w_ie_{\phi\psi_i^{p^{n_i}-1}})\dots(g_{i,\phi\psi_i^2}\,w_ie_{\phi\psi_i^2})(g_{i,\phi\psi_i}\,w_ie_{\phi\psi_i})(g_{i,\phi}\,w_ie_\phi) = 0$$

as in Lemmas 4.12 and 4.13 are a complete set of relations among the arrows $g_{i,\phi}w_ie_\phi$ of Q to define the quotient $\mathfrak{A}$. Thus $\mathfrak{A} \cong kQ/I$ where $I \leqslant J^2(kQ)$ is the two-sided ideal generated by these relations.

Proof. Using Lemmas 4.12 and 4.13, we have an obvious homomorphism from the algebra kQ/I given by these generators and relations to $\mathfrak{A}$. Now kQ/I is a finite dimensional algebra whose socle elements are products which involve $p^{n_i} - 1$ arrows of type i for each i. Such an element maps to something of the form (element of H)$(w_1^{p^{n_1}-1} \dots w_r^{p^{n_r}-1} e_\phi)$ in $\mathfrak{A}$, and such an element is non-zero in kG. □

Theorem 4.15. *The multiplication in kGe induces an isomorphism $\mathfrak{A} \otimes_k \mathfrak{M} \to kGe$.*

Proof. Applying Lemma 4.10 with $A = \mathfrak{M}$ and B the subalgebra generated by $\mathfrak{A}$ and $\mathfrak{M}$, we see that the given map is injective. The dimensions are given by $\dim(\mathfrak{A}) = |Z(H) : Z| \cdot |P|$, $\dim(\mathfrak{M}) = |H : Z(H)|$ and $\dim(kGe) = |G : Z|$, so $\dim(kGe) = \dim(\mathfrak{A}) \cdot \dim(\mathfrak{M})$ and the map is an isomorphism. □

Corollary 4.16. *The algebra kGe is isomorphic to $\mathsf{Mat}_m(\mathfrak{A})$. In particular, $\mathfrak{A}$ is the basic algebra of kGe, and is Morita equivalent to it.* □

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