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BLOCKS WITH NORMAL ABELIAN DEFECT AND ABELIAN p' INERTIAL QUOTIENT

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ABSTRACT. Let k be an algebraically closed field of characteristic p , and let \mathcal{O} be either k or its ring of Witt vectors $W(k)$. Let G be a finite group and B a block of $\mathcal{O}G$ with normal abelian defect group and abelian p' inertial quotient L . We show that B is isomorphic to its second Frobenius twist. This is motivated by the fact that bounding Frobenius numbers is one of the key steps towards Donovan's conjecture. For $\mathcal{O} = k$, we give an explicit description of the basic algebra of B as a quiver with relations. It is a quantised version of the group algebra of the semidirect product $P \rtimes L$.

1. INTRODUCTION

Let p be a prime number. The purpose of this paper is to bound the Frobenius numbers and to give a structure theorem for p blocks of finite groups with normal abelian defect groups and abelian p' inertial quotients. This extends the results of Benson and Green [2], Holloway and Kessar [7], Benson and Kessar [3].

We show that these blocks are isomorphic to their second Frobenius twist. By [8], bounding Frobenius numbers is a key step towards Donovan's conjecture; see for instance [4], [5]. We obtain further a complete description of the basic algebra of such a block over a field by means of quiver with relations.

Our main theorems are as follows. Let k be an algebraically closed field of characteristic p and let $W(k)$ be the ring of Witt vectors over k . Let $\mathcal{O} \in \{k, W(k)\}$. For q a power of p , the Frobenius automorphism $\lambda \mapsto \lambda^q$ of the field k lifts uniquely to an automorphism of the ring $W(k)$, and we denote its inverse in both cases by $\mu \mapsto \mu^{\frac{1}{q}}$ (see [10, Chapter 3, Theorem 3, Proposition 10, Theorem 8]).

Recall from [3] that for an \mathcal{O} -algebra A , the Frobenius twist $A^{(q)}$ is the \mathcal{O} -algebra which equals A as a ring, and where scalar multiplication is twisted via the Frobenius map; that is, for $\lambda \in \mathcal{O}$, and $a \in A$, the action on $A^{(q)}$ is given by $\lambda \cdot a = \lambda^{\frac{1}{q}} a$.

Theorem 1.1. *Let P be a finite abelian p -group, L an abelian p' -subgroup of $\mathbf{Aut}(P)$ and $\alpha \in H^2(L, \mathcal{O}^\times)$. The twisted group algebra $\mathcal{O}_\alpha(P \rtimes L)$ is isomorphic to its second Frobenius twist $\mathcal{O}_\alpha(P \rtimes L)^{(p^2)}$.*

Theorem 1.1 is proved in Section 2.

Theorem 1.2. *Let P be a finite abelian p -group, L an abelian p' -subgroup of $\mathbf{Aut}(P)$ and $\alpha \in H^2(L, k^\times)$ and \mathfrak{A} the basic algebra of the twisted group algebra $k_\alpha(P \rtimes L)$. Then $k_\alpha(P \rtimes L)$ is a matrix algebra over \mathfrak{A} and \mathfrak{A} has an explicit presentation as a quantised version of the group algebra of the semidirect product $P \rtimes L$.*

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The explicit generators and relations for \mathfrak{A} are given in Theorem 4.14.

By a theorem of Külshammer [9], any block of a finite group algebra over \mathcal{O} with a normal defect group is isomorphic to a matrix algebra of a twisted group algebra of the semi-direct product of the defect group of the block with the inertial quotient of the block. Combining [9] with the two results above yields the following.

Corollary 1.3. *Let B be a block of a finite group algebra over \mathcal{O} with a normal defect group P and abelian inertial quotient L . Then B is isomorphic to its second Frobenius twist and if $\mathcal{O} = k$, then B is a matrix algebra over a quantised version of the group algebra of the semidirect product $P \rtimes L$.*

Remark 1.4. It seems unclear whether the same bound holds for strong Frobenius numbers, introduced by Eaton and Livesey in [5]. One issue is that we do not have a sufficiently explicit description of the automorphism φ of $P \rtimes L$ constructed in Lemma 2.3 below.

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2. PROOF OF THEOREM 1.1.

For L a group, $\phi \in \mathbf{Aut}(L)$ and $\alpha : L \times L \rightarrow \mathcal{O}^\times$, denote by ${}^\phi\alpha : L \times L \rightarrow \mathcal{O}^\times$ the map defined by ${}^\phi\alpha(x, y) = \alpha(\phi^{-1}(x), \phi^{-1}(y))$ and by $(\phi, \alpha) \mapsto {}^\phi\alpha$ the induced action of $\mathbf{Aut}(L)$ on $H^2(L, \mathcal{O}^\times)$. For q a power of p , denote by $\alpha^{(q)}$ the map $L \times L \rightarrow \mathcal{O}^\times$ defined by $\alpha^{(q)}(x, y) = \alpha(x, y)^{\frac{1}{q}}$ and by $\alpha^{(q)}$ the image of α under the induced isomorphism $H^2(L, \mathcal{O}^\times) \cong H^2(L, \mathcal{O}^\times)$.

Lemma 2.1. *Let L be a finite abelian p' -group and let $\phi : L \rightarrow L$ be the group automorphism defined by $\phi(x) = x^p$ for all $x \in L$. Then for all $\alpha \in H^2(L, \mathcal{O}^\times)$, we have ${}^\phi\alpha = \alpha^{(p^2)}$.*

Proof. It is well-known that since L is a finite p' -group, it follows that the canonical map $\mathcal{O} \rightarrow k$ induces an isomorphism $H^2(L, \mathcal{O}^\times) \cong H^2(L, k^\times)$. Thus we may assume that $\mathcal{O} = k$.

Consider the universal coefficient sequence

$$0 \rightarrow \mathrm{Ext}^1(H_1(L, \mathbb{Z}), k^\times) \rightarrow H^2(L, k^\times) \rightarrow \mathrm{Hom}(H_2(L, \mathbb{Z}), k^\times) \rightarrow 0.$$

Since k is algebraically closed, k^\times is divisible, and therefore injective as an abelian group. So the first term in this sequence is zero. For the third term, we have $H_2(L, \mathbb{Z}) \cong \Lambda^2(L)$, the exterior square in the category of abelian groups. Therefore we obtain an isomorphism

$$H^2(L, k^\times) \cong \mathrm{Hom}(\Lambda^2(L), k^\times)$$

which by naturality is $\mathbf{Aut}(L)$ equivariant and which commutes with the Frobenius morphism of k . More precisely, if $\alpha \in H^2(L, k^\times)$ corresponds to $\tau \in \mathrm{Hom}(\Lambda^2(L), k^\times)$ under the above isomorphism, then for any $\psi \in \mathbf{Aut}(L)$, and any power q of p , ${}^\psi\alpha$ corresponds to the homomorphism ${}^\psi\tau$ defined by ${}^\psi\tau(x \wedge y) = \tau(\psi^{-1}(x) \wedge \psi^{-1}(y))$ and $\alpha^{(q)}$ corresponds to the homomorphism $\tau^{(q)}$ defined by $\tau^{(q)}(x \wedge y) = \tau(x \wedge y)^{\frac{1}{q}}$. The result follows since for any $\tau \in \mathrm{Hom}(\Lambda^2(L), k^\times)$ we have $\tau(x^q \wedge y) = \tau(x \wedge y)^q = \tau(x \wedge y^q)$. \square

The isomorphism $H^2(L, k^\times) \cong \mathbf{Hom}(\Lambda^2(L), k^\times)$ in the above proof can be explicitly described as follows. If $\alpha \in Z^2(L, k^\times)$, then the image of the class of α in $\mathbf{Hom}(\Lambda^2(L), k^\times)$ is the group homomorphism $x \wedge y \mapsto \alpha(x, y)\alpha(y, x)^{-1}$, where $x, y \in L$. One can either verify directly, using the 2-cocycle identity, that this assignment is a group homomorphism in each component, or one can observe that $\alpha(x, y)\alpha(y, x)^{-1}$ is equal to the commutator of lifts of x, y in a central extension determined by α . More precisely, let

$$1 \rightarrow k^\times \rightarrow \tilde{L} \rightarrow L \rightarrow 1$$

be a central extension defined by α . For each $x \in L$, choose an element $\tilde{x} \in \tilde{L}$ lifting x . An easy calculation shows that $\alpha(x, y)\alpha(y, x)^{-1} = [\tilde{x}, \tilde{y}]$. This commutator does not depend on the choices of the lifts \tilde{x} and since \tilde{L} is a central extension of the abelian group L , this commutator is a group homomorphism in each component. In particular, $[\tilde{x}^p, \tilde{y}^p] = [\tilde{x}, \tilde{y}]^{p^2}$, which explains the statement of the above Lemma.

Lemma 2.2. *Let P be a finite p -group and let $\Phi(P)$ be the Frattini subgroup of P . The kernel of the natural group homomorphism $\mathbf{Aut}(P) \rightarrow \mathbf{Aut}(P/\Phi(P))$ is a p -group. If P is homocyclic, then the map $\mathbf{Aut}(P) \rightarrow \mathbf{Aut}(P/\Phi(P))$ is surjective.*

Proof. For the first assertion see [6, Chapter 5, Theorem 1.4]. Assume that P is homocyclic and let $\{x_1, x_2, \dots, x_r\}$ be a minimal generating set of P . Let $\psi \in \mathbf{Aut}(P/\Phi(P))$. For each i , $1 \leq i \leq r$, pick an element $u_i \in P$ such that $\psi(x_i\Phi(P)) = u_i\Phi(P)$. Since P is homocyclic, there exists a homomorphism $\tilde{\psi} : P \rightarrow P$ such that $\tilde{\psi}(x_i) = u_i$, $1 \leq i \leq r$. Clearly, $\tilde{\psi}$ lifts ψ . Since ψ is an automorphism, $\text{Im}(\tilde{\psi})\Phi(P) = P$ whence $\text{Im}(\tilde{\psi}) = P$ and $\tilde{\psi} \in \mathbf{Aut}(P)$. \square

Lemma 2.3. *Let P be a finite abelian p -group and let L be an abelian p' -group acting on P . There exists an automorphism ϕ of $P \rtimes L$ such that $\phi(L) = L$ and $\phi(x) = x^p$ for all $x \in L$.*

Proof. Denote by L' the image of L in $\mathbf{Aut}(P)$. Proving the existence of ϕ is equivalent to showing that there exists $\tau \in \mathbf{Aut}(P)$ such that $\tau y \tau^{-1} = y^p$ for all $y \in L'$. Suppose first that $P = P_1 \times P_2$ for L' -invariant subgroups P_1 and P_2 of P . If for each $i = 1, 2$, there exists $\tau_i \in \mathbf{Aut}(P_i)$ with $\tau_i(y \downarrow_{P_i}) \tau_i^{-1} = (y \downarrow_{P_i})^p$ for all $y \in L'$, then the map $\tau : P \rightarrow P$ sending $x_1 x_2$ to $\tau_1(x_1) \tau_2(x_2)$ for $x_1 \in P_1, x_2 \in P_2$ has the required properties. Hence we may assume that P is indecomposable for the action of L' and consequently that P is homocyclic (see [6, Chapter 5, Theorem 2.2]).

We claim that it suffices to prove the result for the case that P is elementary abelian. Indeed, let U be the kernel of the map $\mathbf{Aut}(P) \rightarrow \mathbf{Aut}(P/\Phi(P))$. By Lemma 2.2, U is a p -group. Let \bar{L}' be the image of L' in $\mathbf{Aut}(P/\Phi(P))$ and suppose that there exists $\bar{\tau} \in \mathbf{Aut}(P/\Phi(P))$ such that $\bar{\tau} \eta \bar{\tau}^{-1} = \eta^p$ for all $\eta \in \bar{L}'$. By Lemma 2.2, there exists $\tau \in \mathbf{Aut}(P)$ lifting $\bar{\tau}$. Since $L'U$ is the full inverse image of \bar{L}' in $\mathbf{Aut}(P)$, and \bar{L}' is $\bar{\tau}$ -invariant, $\tau L'U \tau^{-1} = L'U$. Hence L' and $\tau L' \tau^{-1}$ are both complements to the normal Sylow p -subgroup U of $L'U$. By the Schur–Zassenhaus theorem, there exists $u \in U$ such that $u L' u^{-1} = \tau L' \tau^{-1}$. Replacing τ by $u^{-1} \tau$ we may assume that $\tau L' \tau^{-1} = L'$. Then for any $y \in L'$, $\tau y \tau^{-1}$ and y^p are elements of the p' -group L' lifting the same element of \bar{L}' . The claim follows by Lemma 2.2.

By the discussion above we may assume that P is elementary abelian and that P is an indecomposable, faithful $\mathbb{F}_p L'$ -module. Since L' is an abelian p' -group, P is in fact an irreducible $\mathbb{F}_p L'$ -module and L' is cyclic. Let $L' = \langle y \rangle$ and let $f(X) \in \mathbb{F}_p[X]$ be the characteristic polynomial of y as an element of $\text{End}_{\mathbb{F}_p}(P)$. Since $f(y^p) = f(y)^p = 0$, $f(X)$

is also the characteristic polynomial of y^p . Thus, y and y^p are conjugate in $\mathrm{GL}(P) \cong \mathrm{Aut}(P)$. \square

Proof of Theorem 1.1. Let ϕ be as in Lemma 2.3. Then ϕ induces an \mathcal{O} -algebra isomorphism $\mathcal{O}_\alpha(P \rtimes L) \cong \mathcal{O}_{\phi\alpha}(P \rtimes L)$. The result follows by Lemma 2.1 since for any power q of p , $\mathcal{O}_\alpha(P \rtimes L)^{(q)} \cong \mathcal{O}_{\alpha^{(q)}}(P \rtimes L)$ as \mathcal{O} -algebras. \square

3. ON CHARACTERS OF GROUPS OF CLASS TWO

For a finite group H denote by $\mathrm{Irr}(H)$ the set of ordinary irreducible characters of H . If N is a normal subgroup of H and $\chi \in \mathrm{Irr}(N)$, denote by $\mathrm{Irr}(H | \chi)$ the subset of $\mathrm{Irr}(H)$ covering χ . Recall that if H/N is abelian, then the group of irreducible (i.e. linear) characters of H/N acts on $\mathrm{Irr}(H | \chi)$ via multiplication and this action is transitive.

Proposition 3.1. *Let H be a finite group which is nilpotent of class 2. Let χ be a faithful irreducible character of $Z := [H, H]$. Set $m = \sqrt{|H : Z(H)|}$.*

- (i) *For any $\phi \in \mathrm{Irr}(Z(H) | \chi)$, $\phi \uparrow^H = m\tau_\phi$ for some $\tau_\phi \in \mathrm{Irr}(H)$. In particular, $m = \tau_\phi(1)$ is an integer.*
- (ii) *The map $\phi \mapsto \tau_\phi$, $\phi \in \mathrm{Irr}(Z(H) | \chi)$, is a bijection between $\mathrm{Irr}(Z(H) | \chi)$ and $\mathrm{Irr}(H | \chi)$.*
- (iii) *The actions of $\mathrm{Irr}(H/Z)$ on $\mathrm{Irr}(H | \chi)$ and of $\mathrm{Irr}(Z(H)/Z)$ on $\mathrm{Irr}(Z(H) | \chi)$ are compatible with the bijection in (ii). More precisely, let $\eta \in \mathrm{Irr}(H/Z)$, and let $\phi \in \mathrm{Irr}(Z(H) | \chi)$. Then $\tau_{\eta \downarrow_{Z(H)} \phi} = \eta \tau_\phi$. Consequently, $\eta \tau_\phi = \tau_\phi$ if and only if η restricts to the trivial character of $Z(H)$.*

Proof. Let τ be an irreducible character of H covering χ . We claim that $\tau(x) = 0$ for all $x \in H \setminus Z(H)$. Indeed, since $Z(H)$ is the intersection of all maximal abelian subgroups of H , it suffices to prove that if A is a maximal subgroup of H , then $\tau(x) = 0$ if $x \notin A$. So, let A be a maximal abelian subgroup of H . Then $Z(H) \leq A$ and since H is of class 2, A is normal in H . Let ψ be an irreducible constituent of the restriction of τ to A and suppose that $g \in H$ is such that ${}^g\psi = \psi$. Then for all $a \in A$, $\psi(gag^{-1}) = \psi(a)$ and so $\psi(gag^{-1}a^{-1}) = 1$. Since the restriction of ψ to Z equals χ , we have that $\chi(gag^{-1}a^{-1}) = 1$ for all $a \in A$. The faithfulness of χ and the maximality of A now imply that $g \in C_G(A) = A$. Consequently, $\tau = \psi \uparrow^H$ and $\tau(x) = 0$ for all $x \notin A$, proving the claim.

Let ϕ be the unique linear character of $Z(H)$ covering χ and which is covered by τ . Since ϕ is linear, the restriction of τ to $Z(H)$ consists of $\tau(1)$ copies of ϕ . By the claim above,

$$1 = \langle \tau, \tau \rangle = \frac{1}{|H|} \sum_{x \in Z(H)} \tau(x)\tau(x^{-1}) = \frac{\tau(1)^2}{|H|} \sum_{x \in Z(H)} \phi(x)\phi(x^{-1}) = \frac{\tau(1)^2}{m^2}.$$

Thus $\tau(1) = m$ and

$$\tau(1)m = |H : Z(H)| = \phi \uparrow^H(1).$$

On the other hand, by Frobenius reciprocity the multiplicity of τ as a constituent of $\phi \uparrow^H$ equals $\tau(1)$. So $\phi \uparrow^H = m\tau$. Setting $\tau_\phi = \tau$ proves part (i) of the proposition. Part (ii) is immediate from (i) and the fact that every element of $\mathrm{Irr}(H | \chi)$ covers a unique element of $\mathrm{Irr}(Z(H) | \chi)$. By the induction formula, $\eta(\phi \uparrow^H) = (\eta \downarrow_{Z(H)} \phi) \uparrow^H$, hence (i) gives that $\tau_{\eta \downarrow_{Z(H)} \phi} = \eta \tau_\phi$. Now (ii) yields that $\eta \tau_\phi = \tau_\phi$ if and only if $\eta \downarrow_{Z(H)} \phi = \phi$ if and only if $\eta \downarrow_{Z(H)}$ is trivial. \square

4. THE BASIC ALGEBRA.

Lemma 4.1. *Let $1 \rightarrow A \rightarrow B \rightarrow C \rightarrow 1$ be a short exact sequence of abelian groups and let $\pi : D \rightarrow A$ be a surjective homomorphism of abelian groups. For each $\alpha \in C$, choose a pre-image u_α in B . Then there exists a 2-cocycle $(\alpha, \beta) \mapsto f_{\alpha, \beta}$ from $C \times C$ to D such that $\pi(f_{\alpha, \beta}) = u_\alpha^{-1} u_\beta^{-1} u_{\alpha\beta}$ and $f_{\alpha, \beta} = f_{\beta, \alpha}$ for all $\alpha, \beta \in C$.*

Proof. It is well-known that $\text{Ext}_{\mathbb{Z}}^n(C, A) = \{0\}$ for $n \geq 2$, and hence the connecting homomorphism $\text{Ext}_{\mathbb{Z}}^1(C, A) \rightarrow \text{Ext}_{\mathbb{Z}}^2(C, \ker(\pi)) = \{0\}$ is zero. Thus the map $\text{Ext}_{\mathbb{Z}}^1(C, D) \rightarrow \text{Ext}_{\mathbb{Z}}^1(C, A)$ induced by π is surjective. In particular, the element in $\text{Ext}_{\mathbb{Z}}^1(C, A)$ represented by the given short exact sequence lifts to an element in $\text{Ext}_{\mathbb{Z}}^1(C, D)$. Rephrased in terms of extensions this means that there is a commutative diagram of abelian groups with exact rows of the form

$$\begin{array}{ccccccc} 1 & \longrightarrow & D & \longrightarrow & \hat{B} & \longrightarrow & C \longrightarrow 1 \\ & & \downarrow \pi & & \downarrow \tau & & \parallel \\ 1 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C \longrightarrow 1 \end{array}$$

Note that τ is surjective and restricts to the map π on D . For $\alpha \in C$, choose a preimage v_α of u_α in \hat{B} and set $f_{\alpha, \beta} = v_\beta^{-1} v_\alpha^{-1} v_{\alpha\beta}$ for all $\alpha, \beta \in C$. Clearly, $f_{\alpha, \beta} \in D$. Since \hat{B} is abelian, $f_{\beta, \alpha} = f_{\alpha, \beta}$. Thus $(\alpha, \beta) \mapsto f_{\alpha, \beta}$ is a 2-cocycle with the properties as stated. \square

We recall the following result on the structure of twisted group algebras.

Lemma 4.2. *Let G be a finite group and let $\alpha \in H^2(G, k^\times)$. Then there exists a central extension*

$$1 \rightarrow Z \rightarrow \tilde{G} \rightarrow G \rightarrow 1$$

with Z a finite cyclic p' -group and a linear character $\chi : Z \rightarrow k^\times$ such that $k_\alpha G \cong k\tilde{G}e$ where $e = \frac{1}{|Z|} \sum_{z \in Z} \chi(z^{-1})z$ is the idempotent of kZ corresponding to χ . Moreover, Z may be chosen to be contained in $[\tilde{G}, \tilde{G}]$.

Proof. This is well known, but for completeness we provide a proof. Let m be the order of the cohomology class α . Since k is algebraically closed, k^\times is a divisible group. So we have a short exact sequence

$$0 \rightarrow \mu_m \rightarrow k^\times \rightarrow k^\times \rightarrow 0,$$

where μ_m denotes the subgroup of m -th roots of unity in k^\times . Note that μ_m has order m as m is relatively prime to p . Consider the corresponding maps of universal coefficient sequences

$$\begin{array}{ccccccc} 0 & \longrightarrow & \text{Ext}^1(H_1(G, \mathbb{Z}), \mu_m) & \longrightarrow & H^2(G, \mu_m) & \longrightarrow & \text{Hom}(H_2(G, \mathbb{Z}), \mu_m) \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \text{Ext}^1(H_1(G, \mathbb{Z}), k^\times) & \longrightarrow & H^2(G, k^\times) & \xrightarrow{\cong} & \text{Hom}(H_2(G, \mathbb{Z}), k^\times) \longrightarrow 0 \\ & & \downarrow & & \downarrow m & & \downarrow m \\ 0 & \longrightarrow & \text{Ext}^1(H_1(G, \mathbb{Z}), k^\times) & \longrightarrow & H^2(G, k^\times) & \xrightarrow{\cong} & \text{Hom}(H_2(G, \mathbb{Z}), k^\times) \longrightarrow 0 \end{array}$$

We have $\text{Ext}^1(H_1(G, \mathbb{Z}), k^\times) = 0$ since k^\times is divisible, so $H^2(G, k^\times) \rightarrow \text{Hom}(H_2(G, \mathbb{Z}), k^\times)$ is an isomorphism. Since α has order m , its image in $\text{Hom}(H_2(G, \mathbb{Z}), k^\times)$ lifts to a *surjective*

element of order m in $\mathbf{Hom}(H_2(G, \mathbb{Z}), \mu_m)$. An inverse image $\tilde{\alpha} \in H^2(G, \mu_m)$ again has order m . Let

$$1 \rightarrow Z \rightarrow \tilde{G} \rightarrow G \rightarrow 1$$

be a central extension corresponding to $\tilde{\alpha}$, with $Z \cong \mu_m$.

Now choose a presentation of G by generators and relations

$$1 \rightarrow R \rightarrow F \rightarrow G \rightarrow 1.$$

By freeness, the identity map on G lifts to a map $F \rightarrow \tilde{G}$. This map sends R into Z , and $[F, F]$ into $[\tilde{G}, \tilde{G}]$. It has $[F, R]$ in its kernel since μ_m is central. This gives us a map

$$H_2(G, \mathbb{Z}) = (R \cap [F, F]) / [F, R] \rightarrow Z$$

which is the lift of α in $\mathbf{Hom}(H_2(G, \mathbb{Z}), \mu_m)$. This map is surjective, but it lands in $Z \cap [\tilde{G}, \tilde{G}]$ and hence $Z \subseteq [\tilde{G}, \tilde{G}]$.

The formula for the idempotent e can be found in the statement and proof of Thévenaz [11, Chapter 2, Proposition 10.5]. The linear character χ sends each element $z \in Z$ to its image under $Z \cong \mu_m \hookrightarrow k^\times$. \square

Let P be an abelian p -group, L an abelian p' -subgroup of $\mathbf{Aut}(P)$ and $\alpha \in H^2(L, k^\times)$. Let Z , \tilde{G} and χ be as in the conclusion of Lemma 4.2 applied to $G = P \rtimes L$ and with α regarded as an element of $H^2(G, k^\times)$ via the pull back along $G \rightarrow G/P$. Let H be the full inverse image of L in \tilde{G} . Then $\tilde{G} = P \rtimes H$ and e is a central idempotent of H .

We have a natural homomorphism

$$\rho: H \rightarrow \mathbf{Hom}(H, k^\times)$$

sending g to $\rho(g): h \mapsto \chi[g, h]$. The kernel of this map is $Z(H)$ and the image is $\mathbf{Hom}(H/Z(H), k^\times)$. We denote by

$$\bar{\rho}: H/Z(H) \rightarrow \mathbf{Hom}(H/Z, k^\times)$$

the induced isomorphism.

Now $P/\Phi(P)$ is naturally a faithful $\mathbb{F}_p L$ -module. The extension of scalars $k \otimes_{\mathbb{F}_p} P/\Phi(P)$ gives a kL -module isomorphic to $J(kP)/J^2(kP)$. Let ψ be the character of kL on this module, and write

$$\psi = \bigoplus_{i=1}^r \psi_i$$

where r is the rank of $P/\Phi(P)$ and the ψ_i are one dimensional kL -modules (there may be repetitions). We choose an H -invariant complement W for $J^2(kP)$ in $J(kP)$, and a basis w_i of W so that for $g \in H$ we have

$$(4.3) \quad gw_i g^{-1} = \psi_i(g) w_i.$$

Since a p' -group of automorphisms of an abelian p -group preserves some decomposition into homocyclic summands (see for example Chapter 5, Theorem 2.2 in Gorenstein [6]), we may assume that

$$kP = k[w_1, \dots, w_r] / (w_1^{p^{n_1}}, \dots, w_r^{p^{n_r}})$$

with $n_1 \geq \dots \geq n_r$ and $|P| = p^{n_1 + \dots + n_r}$. Thus we have relations

$$(4.4) \quad w_i^{p^{n_i}} = 0.$$

Applying the results of Section 3, the irreducible characters τ_ϕ of H lying over χ are in one to one correspondence with the one dimensional characters ϕ of $Z(H)$ lying over χ . The corresponding central idempotents are

$$(4.5) \quad e_\phi = \frac{1}{|Z(H)|} \sum_{h \in Z(H)} \phi(h^{-1})h.$$

Choose one of these, say $\tau = \tau_{\phi_0}$, and choose a matrix representation $T_{\phi_0}: H \rightarrow \mathbf{Mat}_m(k)$ affording τ_{ϕ_0} . Then for each ϕ choose a one dimensional representation ξ_ϕ of H whose restriction to $Z(H)$ is $\phi\phi_0^{-1}$ (and hence whose restriction to Z is trivial) and chosen so that $\xi_{\phi_0} = 1$. We assume that these ξ_ϕ have been chosen, one for each ϕ , and we define $T_\phi: H \rightarrow \mathbf{Mat}_m(k)$ via $T_\phi(h) = \xi_\phi(h).T_{\phi_0}(h)$. Then T_ϕ is a matrix representation affording τ_ϕ . So the map

$$\begin{aligned} kHe &\rightarrow \mathbf{Mat}_m(k) \times \cdots \times \mathbf{Mat}_m(k) && (|Z(H) : Z| \text{ copies}) \\ he &\mapsto (T_{\phi_0}(h), \dots, T_\phi(h), \dots) \end{aligned}$$

is an isomorphism. Elements of kHe of the form $\sum_\phi \xi_\phi^{-1}(h)e_\phi.h$ are sent to diagonal elements $(T_{\phi_0}(h), \dots, T_\phi(h))$, and therefore span a copy of $\mathbf{Mat}_m(k)$ in kHe containing e as its identity element. Let us write \mathfrak{M} for this subalgebra of kHe .

Now for each ψ_i and each ϕ , the character $\phi.(\psi_i|_{Z(H)})$ is some ϕ' , which we denote $\phi\psi_i$ for convenience. So $\xi_{\phi\psi_i}\xi_\phi^{-1}\psi_i^{-1}$ is trivial on $Z(H)$. Thus there exists an element $g_{i,\phi} \in H$ such that

$$(4.6) \quad \rho(g_{i,\phi}) = \xi_{\phi\psi_i}\xi_\phi^{-1}\psi_i^{-1},$$

where $\rho(g_{i,\phi})(h) = \chi([g_{i,\phi}, h])$. We choose such elements $g_{i,\phi}$, one for each ψ_i and ϕ .

On the other hand, using (4.3) and (4.5), we have

$$\sum_{h \in Z(H)} \phi(h^{-1})w_i h = \sum_{h \in Z(H)} \phi(h^{-1})\psi_i(h^{-1})hw_i$$

and so

$$(4.7) \quad w_i e_\phi = e_{\phi\psi_i} w_i.$$

Lemma 4.8. *For $h \in H$ we have*

$$(g_{i,\phi} w_i)(\xi_\phi(h)^{-1} e_\phi . h) = (\xi_{\phi\psi_i}(h)^{-1} e_{\phi\psi_i} . h)(g_{i,\phi} w_i).$$

Thus $g_{i,\phi} w_i e_\phi = e_{\phi\psi_i} g_{i,\phi} w_i$ commutes with \mathfrak{M} .

Proof. Scalars commute with everything, and the e_ϕ , being in $Z(kH)$, commute with all $h \in H$ and all $g_{i,\psi}$. By (4.3) we have $hw_i = \psi_i(h)w_i x$, and by (4.7) we have $w_i e_\phi = e_{\phi\psi_i} w_i$. We are in kGe , and $eg_{i,\phi} h = e\chi([g_{i,\phi}, h])^{-1} h g_{i,\phi}$. Putting these together gives

$$\xi_\phi(x)^{-1} (g_{i,\phi} w_i)(e_\phi . h) = \xi_\phi(h)^{-1} \psi_i(h)^{-1} \chi([g_{i,\phi}, h])^{-1} (e_{\phi\psi_i} h)(g_{i,\phi} w_i).$$

Finally, applying (4.6), the scalar on the right hand side is equal to $\xi_{\phi\psi_i}(h)^{-1}$.

For the final statement, we have

$$(g_{i,\phi} w_i e_\phi) \left(\sum_{\phi'} \xi_{\phi'}^{-1}(h) e_{\phi'} . h \right) = (g_{i,\phi} w_i)(\xi_\phi^{-1}(h) e_\phi . h)$$

$$\begin{aligned}
&= (\xi_{\phi\psi_i}^{-1}(h)e_{\phi\psi_i}.h)(g_{i,\phi}w_i) \\
&= \left(\sum_{\phi'} \xi_{\phi'}^{-1}(h)e_{\phi'}.h \right) (e_{\phi\psi_i}g_{i,\phi}w_i). \quad \square
\end{aligned}$$

Definition 4.9. Let \mathfrak{A} be the subalgebra of kHe generated by the elements e_ϕ and $g_{i,\phi}w_ie_\phi$. Thus by the lemma, \mathfrak{A} and \mathfrak{M} commute.

We claim that \mathfrak{A} is a basic algebra of dimension $|P| \cdot |H : Z(H)|$, and that multiplication in kHe induces an isomorphism

$$\mathfrak{A} \otimes_k \mathfrak{M} \rightarrow kHe,$$

so that $kHe \cong \text{Mat}_m(\mathfrak{A})$. For this purpose, we shall use the following.

Lemma 4.10. *Let $A \leq B$ be k -algebras with A an Azumaya algebra (i.e., a finite dimensional central separable k -algebra). Then the map $A \otimes_k C_B(A) \rightarrow B$ is an isomorphism.*

Proof. See for example Chapter 3, Corollary 4.3, in Bass [1]. □

Remark 4.11. Note that the hypotheses of the lemma include the assumption that the identity element of A is equal to the identity element of B .

We display \mathfrak{A} as kQ/I where Q is a quiver and $I \leq J^2(kQ)$ is an ideal of relations. The quiver Q has $|Z(H) : Z|$ vertices labelled $[\phi]$ corresponding to the idempotents $e_\phi \in kZ(H)$ lying over χ , and directed edges labelled with the w_i corresponding to

$$g_{i,\phi}w_ie_\phi = e_{\phi\psi_i}g_{i,\phi}w_i = e_{\phi\psi_i}g_{i,\phi}w_ie_\phi.$$

going from $[\phi]$ to $[\phi\psi_i]$. For brevity, we can illustrate these vertices and directed edges as

$$[\phi] \xrightarrow{i} [\phi\psi_i].$$

Lemma 4.12. (i) *For suitable elements $z_{i,j,\phi} \in Z(H)$, we have*

$$g_{j,\phi\psi_i}g_{i,\phi} = g_{i,\phi\psi_j}g_{j,\phi}z_{i,j,\phi}.$$

(ii) *The following relations hold in \mathfrak{A} :*

$$(g_{j,\phi\psi_i}w_je_{\phi\psi_i})(g_{i,\phi}w_ie_\phi) = q_{i,j,\phi}(g_{i,\phi\psi_j}w_je_{\phi\psi_j})(g_{j,\phi}w_je_\phi)$$

where $q_{i,j,\phi} = \phi(z_{i,j,\phi}) \in k^\times$.

(iii) *By changing the choices of $g_{i,\phi}$ by elements of $Z(H)$, we may ensure that $z_{i,j,\phi} \in Z$ and $q_{i,j,\phi} = \chi(z_{i,j,\phi})$.*

Proof. (i) By (4.6), we have

$$\begin{aligned}
\rho(g_{j,\phi\psi_i}g_{i,\phi}) &= \rho(g_{j,\phi\psi_i})\rho(g_{i,\phi}) \\
&= (\xi_{\phi\psi_i\psi_j}\xi_{\phi\psi_i}^{-1}\psi_j^{-1})(\xi_{\phi\psi_i}\xi_\phi^{-1}\psi_i^{-1}) \\
&= \xi_{\phi\psi_i\psi_j}\xi_\phi^{-1}\psi_j^{-1}\psi_i^{-1}.
\end{aligned}$$

This is symmetric in i and j , and so

$$\rho(g_{j,\phi\psi_i}g_{i,\phi}) = \rho(g_{i,\phi\psi_j}g_{j,\phi}).$$

Since the kernel of ρ is $Z(H)$ it follows that for some element $z_{i,j,\phi} \in Z(H)$ we have

$$g_{j,\phi\psi_i}g_{i,\phi} = g_{i,\phi\psi_j}g_{j,\phi}z_{i,j,\phi}.$$

(ii) This follows from the fact that we have $z_{i,j,\phi} e_\phi = \phi(z_{i,j,\phi})e_\phi$.

(iii) We apply Lemma 4.1 with $A = \text{Irr}(H/Z(H))$, $B = \text{Irr}(H/Z)$, $C = \text{Irr}(Z(H)/Z)$, the map from B to C the restriction map, $D = H/Z$, π the composition of the natural surjection $H/Z \rightarrow H/Z(H)$ with $\bar{\rho}$ and $u_\alpha = \xi_{\alpha\phi_0}$, $\alpha \in \text{Irr}(Z(H)/Z)$. Let $f_{\alpha,\beta} \in H/Z$ be as in the conclusion of the lemma, and let $\tilde{f}_{\alpha,\beta} \in H$ be any lift of $f_{\alpha,\beta}$ to H . Denote also by ψ_i the restriction of ψ to $Z(H)$. So $\psi_i^{-1}u_{\psi_i}$ is an element of $\text{Irr}(H/Z(H))$. Choose an element $g_i \in H$ such that $\rho(g_i) = \psi_i^{-1}u_{\psi_i}$ and set

$$g_{i,\phi} = g_i \tilde{f}_{\psi_i, \phi\phi_0^{-1}}.$$

Then

$$\begin{aligned} \rho(g_{i,\phi}) &= \psi_i^{-1}u_{\psi_i} \rho(\tilde{f}_{\psi_i, \phi\phi_0^{-1}}) \\ &= \psi_i^{-1}u_{\psi_i} u_{\psi_i}^{-1} u_{\phi\phi_0^{-1}}^{-1} u_{\psi_i \phi\phi_0^{-1}} \\ &= \psi_i^{-1} \xi_\phi^{-1} \xi_{\psi_i \phi} \end{aligned}$$

and

$$\begin{aligned} g_{j,\phi\psi_i} g_{i,\phi} Z &= g_j \tilde{f}_{\psi_j, \phi\psi_i\phi_0^{-1}} g_i \tilde{f}_{\psi_i, \phi\phi_0^{-1}} Z \\ &= g_j g_i \tilde{f}_{\psi_j, \phi\psi_i\phi_0^{-1}} \tilde{f}_{\psi_i, \phi\phi_0^{-1}} Z \\ &= g_j g_i \tilde{f}_{\psi_j, \phi\psi_i\phi_0^{-1}} \tilde{f}_{\phi\phi_0^{-1}, \psi_i} Z \\ &= g_j g_i \tilde{f}_{\psi_j, \phi\phi_0^{-1}} \tilde{f}_{\psi_j\phi\phi_0^{-1}, \psi_i} \\ &= g_j g_i \tilde{f}_{\psi_j, \phi\phi_0^{-1}} \tilde{f}_{\psi_i, \psi_j\phi\phi_0^{-1}} Z \\ &= g_{i,\phi\psi_j} g_{j,\phi} Z. \end{aligned} \quad \square$$

Lemma 4.13. *For each i and each ϕ we have*

$$(g_{i,\phi\psi_i^{p^{n_i}-1}} w_i e_{\phi\psi_i^{p^{n_i}-1}}) \cdots (g_{i,\phi\psi_i^2} w_i e_{\phi\psi_i^2}) (g_{i,\phi\psi_i} w_i e_{\phi\psi_i}) (g_{i,\phi} w_i e_\phi) = 0.$$

Here, we have written down the only composable sequence of arrows in Q beginning with e_ϕ with each arrow involving w_i , and there are p^{n_i} terms in the product:

$$[\phi] \xrightarrow{i} [\phi\psi_i] \xrightarrow{i} [\phi\psi_i^2] \xrightarrow{i} \cdots \xrightarrow{i} [\phi\psi_i^{p^{n_i}}].$$

Proof. Relations (4.3) and (4.7) allow us to push the w_i terms past the other terms so that they are directly multiplied together. Then we can use relation (4.4) to conclude that we get zero. \square

Theorem 4.14. *The relations on the quiver algebra kQ given by*

$$(g_{j,\phi\psi_i} w_j e_{\phi\psi_i}) (g_{i,\phi} w_i e_\phi) = q_{i,j,\phi} (g_{i,\phi\psi_j} w_i e_{\phi\psi_j}) (g_{j,\phi} w_j e_\phi)$$

and

$$(g_{i,\phi\psi_i^{p^{n_i}-1}} w_i e_{\phi\psi_i^{p^{n_i}-1}}) \cdots (g_{i,\phi\psi_i^2} w_i e_{\phi\psi_i^2}) (g_{i,\phi\psi_i} w_i e_{\phi\psi_i}) (g_{i,\phi} w_i e_\phi) = 0$$

as in Lemmas 4.12 and 4.13 are a complete set of relations among the arrows $g_{i,\phi} w_i e_\phi$ of Q to define the quotient \mathfrak{A} . Thus $\mathfrak{A} \cong kQ/I$ where $I \leq J^2(kQ)$ is the two-sided ideal generated by these relations.

Proof. Using Lemmas 4.12 and 4.13, we have an obvious homomorphism from the algebra kQ/I given by these generators and relations to \mathfrak{A} . Now kQ/I is a finite dimensional algebra whose socle elements are products which involve $p^{n_i} - 1$ arrows of type i for each i . Such an element maps to something of the form (element of H)($w_1^{p^{n_1}-1} \dots w_r^{p^{n_r}-1} e_\phi$) in \mathfrak{A} , and such an element is non-zero in kG . \square

Theorem 4.15. *The multiplication in kGe induces an isomorphism $\mathfrak{A} \otimes_k \mathfrak{M} \rightarrow kGe$.*

Proof. Applying Lemma 4.10 with $A = \mathfrak{M}$ and B the subalgebra generated by \mathfrak{A} and \mathfrak{M} , we see that the given map is injective. The dimensions are given by $\dim(\mathfrak{A}) = |Z(H) : Z| \cdot |P|$, $\dim(\mathfrak{M}) = |H : Z(H)|$ and $\dim(kGe) = |G : Z|$, so $\dim(kGe) = \dim(\mathfrak{A}) \cdot \dim(\mathfrak{M})$ and the map is an isomorphism. \square

Corollary 4.16. *The algebra kGe is isomorphic to $\text{Mat}_m(\mathfrak{A})$. In particular, \mathfrak{A} is the basic algebra of kGe , and is Morita equivalent to it.* \square

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