Long-Term Real Dynamic Investment Planning

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ABSTRACT

When long-term savers plan for retirement they need to know their investment prospects in terms of real income (Merton, 2014). While inflation has traditionally been considered as a complication in financial analysis and financial practice, we obtain enhanced predictability and model fit if the real returns are targeted in conjunction with earnings-by-price minus inflation as predictor. For this latter case, we propose an investment strategy of updating the simple classical Merton proportion as we go along. This simple strategy is very close to the complicated theoretically optimal solution but has comparably much lower parameter uncertainty.

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1 Introduction

Recent scientific developments provide a long list of innovations for long term pension products, when the underlying econometric investment assumptions are stable, see e.g. Donnelly, Guillen, Nielsen, and Pérez-Marín (2018); Zhu, Hardy, and Saunders (2018); van Bilsen and Bovenberg (2018); Mei (2019); van Bilsen and Linders (2019); Bernhardt and Donnelly (2019). The purpose of this paper is to provide econometric investment dynamics for long term real income pension products to provide the long term saver with an optimal, or close to optimal, investment strategy in a dynamic market. In this paper the prefix *real* denotes net of inflation.

It has long been accepted in the financial academic literature that predictability of the stock returns can be exploited by the short-term as well as the long-term investor. One of the more recent examples is Golez and Koudijs (2018), who find a 2% to 10% out-of-sample R-squared (ROOS) when exploiting the predictive power of dividends. Another strand of research exploits this type of predictive power to provide optimal long-term investment strategies. For example Barberis (2000) uses dividends to drive future investments according to a discrete econometric version of the financial optimization methodology of Kim and Omberg (1996). The latter work provides a dynamic optimal investment structure taking into account the time-varying nature of expected returns. The empirical part in Barberis (2000) is conducted with the following three ad-hoc choices: monthly data, targeting returns in excess of short term interest and dividends as predictor. These three seemingly little choices have a rather big impact with respect to model fit, returns predictability and investment performance. With regards to the first choice, a long-term econometric model with time steps of one year seems more appropriate when the target is for example a pension half a lifetime into the future. One year steps may eliminate correlations arising from small movements noting that model parameters are very sensitive to the time-steps considered; see Harrison and Zhang (1999); Engsted and Tanggaard (2002); Kim and In (2005) among many others. With regards to the second choice, we argue that one should target returns in excess of inflation when considering long-term investments. Targeting returns in the excess of inflation ensures real-income protection by maximizing the purchasing power of our terminal investment value (Merton, 2014). Appropriate benchmarking via inflation additionally aids communication with the pension saver; see Merton (2014); Gerrard, Hiabu, Kyriakou, and Nielsen (2018, 2019). With regards to the third choice, Scholz, Nielsen, and Sperlich (2015); Kyriakou, Mousavi, Nielsen, and Scholz (2019a,b); Mammen, Nielsen, Scholz, and Sperlich (2019) employ a machine learning approach to validating long-term nonparametric smoothing of returns. They consider dividends, earnings and a number of other macro financial key drivers of the stock market and find that the long-term investor interested in real income, that is, returns in excess of inflation, should focus on real earnings, that is, earnings minus inflation. When targeting real returns, nominal earnings-by-price as such has no value as a predictor with a validated-R-squared (RV) (RV is an alternative to ROOS, see (Nielsen and

Sperlich, 2003; Bergmeir, Hyndman, and Koo, 2018)) of only -1.5%. However, real earnings-by-price is a powerful predictor for the long-term investor with a RV value of 12.2% (Nominal and real dividends have an RV value of -0.2% and 10.4%, respectively), see Kyriakou et al. (2019a,b).

When calibrating our model on S&P 500 data from 1873 to 2018 and changing those three choices, that is, when employing real earnings as the predictor for the real returns by utilizing yearly data, then we find that there is virtually no correlation between the market price of risk – measured in the real terms – and the real returns.

Such a lack of correlation simplifies considerably the complex optimal investment structure provided in Kim and Omberg (1996). Zero correlation turns the complicated optimal investment structure into a simple regular update of the well known deterministic optimal investment structure provided by Merton (1969) and Merton (1975). We call this simple regular update dynamic Merton strategy and the main conclusion of our paper is that the long-term investor should use dynamic Merton based on real earnings while planning for their retirement.

In Section 2 we define the optimal dynamic financial investment policy, which is non-myopic. Section 3 and Section 4 find that a financial optimal investment policy adds one extra term – the intertemporal hedging demand – to the dynamic Merton rule. After fitting the model in an econometric setting using S&P 500 data from 1873 to 1988 in Section 5, the intertemporal hedging demand turns out to be of marginal interest when considering long-term real investment strategies. This is fortunate because the extra term comes with a significant extra number of parameters to be estimated. The extra instability from the added parameter uncertainty – the main focus of Barberis (2000) – could alone lead to an advocacy for the dynamic Merton strategy. When the major customer-selected adjustment of the long-term hedging policy as advocated in Merton (2014) and Gerrard et al. (2019) is added to this equation, then there is a strong argument that the extra trouble of implementing the theoretically optimal but complicated nonmyopic strategy might not be worth it for the long-term real investor. In Sections 5 and 6 of this paper we take a historical look at the S&P 500 Index and how the classical Merton approach of Merton (1969) and Merton (1975) compares to our simple suggested dynamic Merton approach and the complicated but theoretically optimal nonmyopic strategy. For the last 30 years, utilizing real earnings as predictor we derive ratios of geometric mean return to average stock exposure of 5.46%, 7.84% 7.86% for classical Merton, dynamic Merton and optimal nonmyopic, respectively. Hence, while a dynamic strategy seems necessary for real long-term investments, the simplicity of the dynamic Merton approach with little parameter uncertainty makes us recommend it as the dynamic investment strategy when optimizing for real long-term investments. In the appendix we derive the necessary financial mathematics to understand the theoretical details behind the myopic and nonmyopic discussion in this paper. The results are a special case of the more general results of Kim and Omberg (1996).

2 From the classical Merton model to a dynamic model

2.1 Starting from the classical Merton world

We start with a financial market based on simple Brownian motions and deterministic parameters μ, σ, $r > 0$:

$$dS(t) = \mu S(t)\, dt + \sigma S(t)\, dW(t),$$

$$dX(t) = rX(t)\, dt + (\theta\, dt + \, dW(t))\, \sigma\pi(t),$$

where $\theta = (\mu - r)/\sigma$ is the market price of risk and $\pi(t)$ is the amount of the current wealth $X(t)$ invested in the risky asset with the remainder $X(t) - \pi(t)$ in the risk-free asset. The customer's wealth is governed by a standard Brownian motion W defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The filtration available to the customer is represented by the filtration $\mathcal{F}(t) = \sigma\{W(s),\ s \in [0,t] \vee \mathcal{N}(\mathbb{P}), t \in [0,T]\}$, where $\mathcal{N}(\mathbb{P})$ denotes the collection of all $\mathbb{P}$-null sets so that the filtration obeys the usual conditions.

This simple deterministic model is not appropriate for the long-term investor. It is not realistic that parameters stay the same over the half-century or more that could represent many long-term investors' time horizon. Therefore we define a more general financial market incorporating a more general class of parameters providing a more realistic model framework for the long-term investor. Instead of the deterministic opportunity set of parameters (μ,

σ) we consider its stochastic counterpart. This is done by replacing the time-independent parameters (μ, σ) with time-dependent stochastic parameters ($\mu(t)$, $\sigma(t)$). We do restrict the space of admissible stochastic processes for ($\mu(t)$, $\sigma(t)$) to predictable functions. Therefore we assume, that while parameters can vary stochastically, the correct value of the parameters are known at any point in time and do not depend on future events. Formally, we therefore assume that ($\mu(t)$, $\sigma(t)$) are predictable with respect to filtration $\mathcal{F}(t)$ according to which all considered Brownian motions of our financial market are measurable.

Below in Subsection 2.2 we firstly define our general dynamic market framework for the real life-cycle investor. In Subsection 2.3 the pension saver is accumulating wealth via a wealth process and a savings function g. This savings ratio g is itself assumed to be a predictable function that cannot assume any knowledge of the future. Notice that the savings function g can be positive as well as negative. When g is positive the pension saver is in the so called accumulation phase saving for the retirement; when g is negative the pension saver has started the pay-out phase. In Subsection 2.4 we state the unconstrained optimization problem and the Hamilton–Jacobi–Bellman equation.

2.2 The long-term saver in the dynamic market

We consider a real long-term investor who manages investments in a continuous-time market model over a target date horizon $[0, T]$, where $T > 0$ refers to the terminal time when retirement occurs. The pension fund is built in a market with a constant risk-free rate asset, called *a bond*, which pays interest at constant rate r, and one risky asset (or "security"), called *a stock*, whose price $S(t)$ is governed by an one-dimensional, standard Brownian motion $W = \{W(t), \mathcal{F}(t); t \in [0,T]\}$ defined on a complete probability space described by a triplet $(\Omega, \mathcal{F}(t), \mathbb{P})$. The price $S(t)$ evolves according to the linear stochastic differential equation

$$dS(t) = \mu(t)S(t)\,dt + \sigma(t)S(t)\,dW(t), \tag{1}$$

where $\mu = \{\mu(t), \mathcal{F}(t); t \in [0,T]\}$ and positive $\sigma = \{\sigma(t), \mathcal{F}(t); t \in [0,T]\}$ are adapted and measurable one-dimensional Itô processes of the mean rate of returns and standard deviation respectively. Both stochastic processes are fulfilling conditions of $\mathbb{P}\Big[\int_0^t \mid \mu(s) \mid ds < \infty \text{ for all } t \geq 0\Big] = 1$, $\mathbb{P}\Big[\int_0^t \sigma(s)^2\,ds < \infty \text{ for all } t \geq 0\Big] = 1$. Taking into account market timing we define a stochastic market price of risk $\theta = \{\theta(t), \mathcal{F}(t); t \in [0,T]\}$ as the Sharpe ratio

$$\theta(t) := \frac{\mu(t) - r}{\sigma(t)}. \tag{2}$$

The information available is represented by the filtration $\mathcal{F}(t) := \sigma\{W(s), s \in [0,t]\} \vee \mathcal{N}(t,\mathbb{P})$, $\forall t \in [0,T]$, where $\mathcal{N}(t,\mathbb{P})$ is an increasing family of $\sigma-$algebras of subsets of Ω and denotes the collection of all $\mathbb{P}-$null events in the probability space $(\Omega, \mathcal{F}(t), \mathbb{P})$. Since $\mu(t)$, $\sigma(t)$ and $\theta(t)$ are all related by the single equation (2), we choose θ and σ as the main quantities that vary over time.

2.3 The contribution plan and the wealth process

The long-term saver starts with a fixed non-random initial wealth $\tilde{x}(0) > 0$. Future contributions in the form of deterministic cash flows are given by $dC(t)$. Let us assume that the saver knows at time 0 how much money they will save towards their retirement. We define *a contribution plan* g as a mapping of the discounted sum of their planned savings from $[0, T]$ to $[0, \infty)$

$$g(t) := \int_t^T e^{-r(s-t)}\, dC(s), \;\; \forall t \in [0, T]. \tag{3}$$

A portfolio process $\pi = \{\pi(t, x, \theta, \sigma); t \in [0, T]\}$ is a real-valued, square-integrable and $\mathcal{F}(t)$-progressively measurable process for which $\int_0^T \mathbb{E}[\pi^2(t)]\, dt < \infty$ a.s. The process describes a time-varying investing strategy, called *a policy*, which is giving us information about the choice to allocate financial resources in the stock at time t. Let $\mathcal{A}$ be the collection of such portfolio processes. The saver allows for a dynamic self-financed strategy and invests at each instant $t \in [0, T]$ an amount $\pi(t)$ in the stock. Use $\tilde{X}^\pi = \{\tilde{X}^\pi(t); t \in [0, T]\}$ to denote an $\mathcal{F}(t)$-adapted wealth process representing the trajectory of $\tilde{X}(t)$ if the policy π and the saving plan g is applied. The wealth equation, with its initial value, reads as

$$d\tilde{X}^\pi(t) = r\tilde{X}^\pi(t)\, dt + (\mu(t) - r)\, \pi(t)\, dt + \sigma(t)\pi(t)\, dW(t) + dC(t), \quad \tilde{X}^\pi(0) = \tilde{x}(0) \quad \text{a.s.}$$

If we write $X^\pi(t) = \tilde{X}^\pi(t) + g(t)$, then we have

$$dX^\pi(t) = rX^\pi(t)\,dt + \sigma(t)\pi(t)(\theta(t)\,dt + \,dW(t)), \quad X^\pi(0) = x(0) \quad \text{a.s.} \tag{4}$$

We define the set of admissible trading portfolios at time t for the saver's initial endowment

$$\mathcal{A}(t) := \{\pi(t) \in \mathcal{A} : \Omega \times [0,T] \to \mathbb{R} : X^\pi(0) = x(0)\}, \tag{5}$$

and we say that a portfolio process π at time t is admissible if $\pi(t) \in \mathcal{A}(t)$. We define the wealth process $Y(t)$

$$Y^\pi(t) = e^{r(T-t)}X^\pi(t), \tag{6}$$

in order to remove dependence on the risk-free rate r. We describe a financial market as in Chapter 1 of Karatzas and Shreve (1998) – see Definition 1.3 and Remark 1.4 – using state processes driven by stochastic differential equations of the form

$$dY^\pi(t) = \sigma(t)\pi(t)e^{r(T-t)}\big(\theta(t)\,dt + dW(t)\big), \tag{7}$$

$$d\theta(t) = \delta^{(\theta)}(t,\theta(t))\,dt + \tau^{(\theta)}(t,\theta(t))\,dW_\theta(t), \tag{8}$$

$$d\sigma(t) = \delta^{(\sigma)}(t,\sigma(t))\,dt + \tau^{(\sigma)}(t,\sigma(t))\,dW_\sigma(t). \tag{9}$$

$\delta^{(\theta)}(t, \theta(t))$ and $\delta^{(\sigma)}(t, \sigma(t))$ represent the mean rates of returns for both state processes, $\tau^{(\theta)}(t, \theta(t))$ and $\tau^{(\sigma)}(t, \sigma(t))$ are diffusion coefficients. Note that for dependent Brownian motions we have

$$\mathbb{E}[W(t)W_\theta(t)] = \rho_{y\theta} t, \quad \mathbb{E}[W(t)W_\sigma(t)] = \rho_{y\sigma} t \quad \text{and} \quad \mathbb{E}[W_\theta(t)W_\sigma(t)] = \rho_{\theta\sigma} t,$$

and the fixed coefficients $\rho_{y\theta}$, $\rho_{y\sigma}$ and $\rho_{\theta\sigma}$ are the correlations between W, W_θ and W_σ respectively. We first investigate the general framework above. In Section 4, we will simplify the model by imposing more structure. We will follow Kim and Omberg (1996) and impose mean reversion of the risk premium and we will follow Mammen et al. (2019) and proceed with a constant return volatility.

2.4 The financial optimization problem of the long-term saver

Within the stochastic control formulation setup in the previous Subsection 2.2 and Subsection 2.3, we focus on the following optimization problem:

Problem 1. ***Unconstrained problem.*** *The long-term saver seeks to maximize the expected utility of their terminal reward $U\big(X^\pi(T)\big)$ or, equivalently, $U\big(Y^\pi(T)\big)$. The saver needs to find the optimal control $\pi^\star \in \mathcal{A}$ to allocate their monetary resources such that for $t \in [0, T]$*

$$\mathbb{E}_t\Big[U\big(Y^{\pi^\star}(T)\big)\Big] = \sup_{\pi \in \mathcal{A}} \Big\{\mathbb{E}_t\left[U\big(Y^\pi(T)\big)\right]\Big\},$$

subject to the initial wealth $Y(0)$, where the dynamics of $Y^{\pi}(t)$ are defined in (7).

The notation U typically stands for the utility function and its desirable properties are formulated in following remark:

Remark 1. ***Utility function.*** *We consider only strictly concave and C^2 utility functions $U : (0,\infty) \to \mathcal{R}$ with limits $U(0) = \lim_{y\to 0} U(y) \geq -\infty$, $U'(\infty) = \lim_{y\to\infty} U'(y) = 0$ and allowing the possibility $U'(0) = \lim_{y\to 0} U'(y) = \infty$.*

To derive the optimal controls of Problem 1, the technique of dynamic programming is used. Alternative approaches are to operate with the general martingale method (Björk, 2009, Chapter 20) or the simplified utility gradient method (Duffie and Skiadas, 1994; Duffie, 2010, Chapter 9H). We denote the supremum over all admissible controls at time t by

$$V^{\pi}(t,y,\theta,\sigma) = \sup_{\pi\in\mathcal{A}(t)} \Big\{ \mathbb{E}_t\big[U\big(Y^{\pi}(T)\big) \mid Y(t) = y, \theta(t) = \theta, \sigma(t) = \sigma\big] \Big\}$$

as the optimal value function. $\mathbb{E}_t$ is the conditional expectation operator. We assume that the value function admits infinite utility, $V^{\pi}(t,y,\theta,\sigma) = -\infty$, if there is no admissible control given to initial state variables y, σ and θ. We solve the nonlinear partial differential equation of second order called the Hamilton–Jacobi–Bellman equation at time t:

$$0 = \sup_{\pi \in \mathcal{A}(t)} \Big\{ V_t + \sigma\pi\theta e^{r(T-t)} V_y + \delta^{(\theta)} V_\theta + \delta^{(\sigma)} V_\sigma + \frac{1}{2}\sigma^2\pi^2 e^{2r(T-t)} V_{yy} + \frac{1}{2}\left(\tau^{(\theta)}\right)^2 V_{\theta\theta} + \frac{1}{2}\left(\tau^{(\sigma)}\right)^2 V_{\sigma\sigma} + \sigma\pi e^{r(T-t)}\rho_{\theta y}\tau^{(\theta)} V_{\theta y} + \sigma\pi e^{r(T-t)}\rho_{\sigma y}\tau^{(\sigma)} V_{\sigma y} + \tau^{(\theta)}\tau^{(\sigma)}\rho_{\theta\sigma} V_{\theta\sigma} \Big\} \tag{10}$$

with respect to the set of admissible trading portfolios defined in (5). We seek a solution in a separable form

$$V(t, y, \theta, \sigma) = U(y) e^{b(t,\theta,\sigma)} \tag{11}$$

with the boundary condition $b(T, \theta, \sigma) = 0$ for all states of θ and σ. One may think of $V(t, \cdot)$ as the life-cycle saver's indirect utility function for their wealth at time t. Obtaining the candidate for the optimal policy $\pi^\star$, and proving that the candidate does indeed verify the PDE is not in itself sufficient: the HJB method relies on the so-called verification theorem to accomplish the proof.

3 General solution to the financial optimization problem of the long-term saver

In the previous Section 2 we have formulated the general financial market model described by (7)-(9). The solution of the Problem 1 is attained by solving (10). In this section, Theorem 1 provides the optimal nonmyopic investment policy for the long-term saver in the case of the general financial market model.

Theorem 1. ***Optimal nonmyopic solution.*** *An optimal investment strategy for the Problem 1 is to invest at every time t the amount*

$$\pi^*(t, y, \theta, \sigma) := -\frac{e^{-r(T-t)}}{\sigma(t)V_{yy}} \Big[\theta(t)V_y + \rho_{\theta y}\tau^{(\theta)}(\theta) V_{\theta y} + \rho_{\sigma y}\tau^{(\sigma)}(\sigma) V_{\sigma y}\Big], \tag{12}$$

in the risky asset. V_y, $V_{\theta y}$ and $V_{\sigma y}$ are the corresponding partial derivatives of the value function. Both intertemporal hedging demands are given by sensitivities of our customer's value function of anticipated portfolio wealth gains due to the market price of risk $\rho_{\theta y}\tau^{(\theta)}(\theta) V_{\theta y}$ and anticipated portfolio gains due to the stock returns $\rho_{\sigma y}\tau^{(\sigma)}(\sigma) V_{\sigma y}$.

Proof. See Appendix A. □

Theorem 1 becomes more comprehensible when we write it as analytical representations of the partial derivatives of the value function. Corollary 1 follows intuitively from Theorem 1.

Corollary 1. ***Simplified optimal nonmyopic solution.*** *Suppose that of the market price of risk θ in (2) is independent of the evolution of the volatility of returns, i.e., $\rho_{\sigma\theta} = 0$. If the partial differential equation*

$$b_t + \delta^{(\theta)} b_\theta + \frac{1}{2}\left(\tau^{(\theta)}\right)^2 \left(b_{\theta\theta} + b_\theta^2\right) = -\frac{1}{2}\eta\left(\theta + \rho_{\theta y}\tau^{(\theta)} b_\theta\right)^2, \qquad b(T, \theta) = 0 \text{ for all } \theta.$$

has a solution b which is bounded for $0 \le t \le T$*, an optimal investment strategy for Problem 1 is to invest at every time t the amount*

$$\pi^*(t, y, \theta, \sigma) := -\frac{U'(y)}{\sigma(t)U''(y)} e^{-r(T-t)} \left[\theta(t) + \rho_{\theta y}\tau^{(\theta)}(\theta) b_\theta(t, \theta)\right] \tag{13}$$

in the risky stock. If we use the optimal policy (13) and the power utility function, i.e.,

$$U(y) := \frac{1}{\gamma} y^\gamma \text{ for } y > 0, \qquad \gamma \in (-\infty, 1) \setminus \{0\},$$

with a constant relative risk aversion, then the trajectory $\{Y(t), 0 \le t \le T\}$ *of the saver's optimal wealth is governed by an Itô process.*

$$Y^*(T) = Y^*(t) \exp\left[\int_t^T \left(\frac{Q(u)}{1-\gamma}\theta(u) - \frac{1}{2(1-\gamma)^2} Q(u)^2\right) du + \int_t^T \frac{Q(u)}{1-\gamma}\, dW(u)\right]. \tag{14}$$

where the function $Q(t)$ *is given by*

$$Q(t) \equiv \theta(t) + \rho_{\theta y}\tau^{(\theta)}(\theta)\, b_\theta\left(t, \theta(t)\right). \tag{15}$$

Proof. See Appendix A. □

We now consider the case when the Brownian motion, W_θ, the driver of the risk premium of the stock, is independent of the Brownian motion, W, which is driving the process of the stock returns. We call the resulting strategy Dynamic Merton (DM).

Corollary 2 follows also intuitively from Theorem 1.

Corollary 2. ***Dynamic Merton proportion.*** *If the evolution of the market price of risk (2), is independent of the evolution of asset price (1), i.e., if $\rho_{y\theta} = 0$, an optimal investment strategy for Problem 1 is to invest at every time t the monetary amount*

$$\pi^*(t, y, \theta, \sigma) := -\frac{U'(y)}{\sigma(t)U''(y)} e^{-r(T-t)} \theta(t), \tag{16}$$

in the risky asset.

Proof. See Appendix A. □

Note that the logarithmic, exponential and power utility functions are all consistent with Remark 1. In the setting of Corollary 2, the optimal investment strategy has the same form as the Merton proportion in the original work Merton (1969), with the acknowledgement that θ is now allowed to vary over time t.

Remark 2. ***Intertemporal hedging demand.*** *The key difference between the optimal nonmyopic strategy in (13) and the dynamic Merton proportion in (16) is the time-varying quantity*

$$\rho_{\theta y} \tau^{(\theta)}(\theta) b_\theta(t, \theta). \tag{17}$$

The correction term is also known as the intertemporal hedging demand.

4 Kim and Omberg model

In this section we specify further the general market model given in Section 2. Our objective is to get closer to the observed dynamics of the econometric data considered in Section 5 and also to be able to provide closed-form solutions to the general dynamic Merton hedge and the simplified dynamic Merton hedge derived in Section 3. For that we need to derive a closed form solution for b_θ. Until now we have retained the possibility that $\tau^{(\theta)}$ is an arbitrary function of θ. Let us now instead assume that

$$\tau^{(\theta)}(\theta) = \tau,$$

a constant, and for simplicity we write ρ instead of $\rho_{\theta y}$. The assumption of constant volatility is supported by the recent findings of Mammen et al. (2019) who conclude a non-predictability of the volatility when real returns are targeted by real earnings as predictor. We simplify the financial model (7)-(9) in Section 2 and we model θ as a non-central Ornstein-Uhlenbeck process

$$d\theta(t) = -\kappa(\theta(t) - \mu_\theta) + \tau \, dW_\theta(t). \tag{18}$$

The solution $b(t, \theta, \sigma)$ must satisfy the following partial differential equation resulting from (10)

$$0 = b_t + \kappa(\mu_\theta - \theta)b_\theta + \frac{1}{2}\tau^2(b_{\theta\theta} + b_\theta^2) + \frac{\eta}{2}\left(\theta + \rho_{\theta y}\tau b_\theta\right)^2, \tag{19}$$

where $\eta = -U'(y)^2/\{U''(y)U(y)\}$. There is no canonical method for solving the Hamilton–Jacobi–Bellman equation in general, so we guess the form of the solution and then provide a proof that our solution satisfies (19). By nature (19) is a parabolic nonlinear second order differential equation and we explore a solution of the form quadratic in θ, that is, we assume that the function $b\,(t, \theta)$ appearing in the value function, $V(t, y, \theta, \sigma) = U(y)e^{b(t,\theta)}$, is a quadratic polynomial in θ:

$$b(t, \theta) = b_0(T - t) + b_1(T - t)\theta(t) + b_2(T - t)\theta(t)^2; \tag{20}$$

see also Kim and Omberg (1996). The assumptions in this section, that is, to model the stochastic risk premium as a mean-reverting process with the mean-reversion speed κ and volatility τ in (18), is the specific extension of the Black-Scholes model considered in Kim and Omberg (1996). The authors place particular emphasis on solving the mathematical forms of the portfolio strategies and identifying the badly-behaved or nirvana solution regions. In Proposition 1 below we highlight the only solution that is admissible. The Riccati differential equation for b_2 is

$$\frac{db_2(s)}{ds} = 2\tau^2 B b_2^2(s) - 2\tau A b_2(s) + \frac{\eta}{2}, \quad A = \xi - \eta\rho, \ B = 1 + \eta\rho^2, \ \xi = \frac{\kappa}{\tau}. \tag{21}$$

A central role is played by whether the discriminant

$$\Delta = 4\tau^2 A^2 - 4\eta\tau^2 B = 4\tau^2 \left(\xi^2 - \eta\left(1 + 2\rho\xi\right)\right),$$

takes a positive value, a negative value, or the value 0. The corresponding equation for b_1,

$$\frac{db_1(s)}{ds} = 2\tau^2 B b_1(s) b_2(s) - 2\tau A b_1(s) + 2\kappa\mu_\theta b_2(s)$$

can be solved straightforwardly once b_2 is known.

Proposition 1. ***Admissible optimal nonmyopic solution under mean-reverting assumption and constant volatility.*** *Assume that the discriminant Δ is positive. Let us consider the market in which the stochastic mean-reverting market price of risk in (18) is traded continuously.The functions b_1 and b_2 from the intertemporal hedging term in (20), admit the following hyperbolic representations*

$$\begin{aligned} b_1(s) &= \frac{\xi\mu_\theta\eta}{2\tau^2 R_p(1+\eta\rho^2)} \cdot \frac{e^{R_p\tau s} - 2 + e^{-R_p\tau s}}{\phi_1 e^{R_p\tau s} - \phi_2 e^{-R_p\tau s}}, \\ b_2(s) &= \frac{\eta}{4\tau^2(1+\eta\rho^2)} \cdot \frac{e^{R_p\tau s} - e^{-R_p\tau s}}{\phi_1 e^{R_p\tau s} - \phi_2 e^{-R_p\tau s}}, \end{aligned} \tag{22}$$

where $\phi_1, \phi_2 = \frac{A \pm R_p}{2\tau B}$, $R_p = \sqrt{\xi^2 - \eta(1+2\rho\xi)}$, $s = T - t$.

Proof. See Appendix B. □

Remark 3. ***Hyperbolic tangent form.*** *Proposition 1 allows us to write the derivative of the optimal solution with respect to the market price of risk in (20) using the hyperbolic tangent*

$$b_\theta(t, \theta(t)) = b_2(T-t)\left(\frac{2\xi\mu_\theta}{R_p}\tanh\left(\frac{R_p\tau(T-t)}{2}\right) + 2\theta(t)\right). \tag{23}$$

By the boundary condition on b, the hedging demand vanishes as we approach maturity. See Appendix B for the limiting behaviour of the correction term.

Remark 4. ***A nirvana solution for the long-term saver.*** *Assume the same market setting from Proposition 1. If we restrict* γ *to take only positive values, i.e.,* $\gamma \in (0,1)$ *and if* $\Delta < 0$ *then the functions* b_1, b_2 *admit following trigonometric representations*

$$\begin{aligned} b_1(s) &= \frac{2\kappa\mu_\theta\eta}{\tau^2 R_m} \cdot \frac{\sin^2\left(\frac{1}{2}R_m\tau s\right)}{(\xi - \eta\rho)\sin(R_m\tau s) + R_m\cos(R_m\tau s)}, \\ b_2(s) &= \frac{\eta}{2\tau} \cdot \frac{\sin(R_m\tau s)}{(\xi - \eta\rho)\sin(R_m\tau s) + R_m\cos(R_m\tau s)}, \end{aligned} \tag{24}$$

where $R_m = \sqrt{(1+2\rho\xi)\eta - \xi^2}$, $\xi = \frac{\kappa}{\tau}$, $s = T - t$. *The denominators of* b_1 *and* b_2 *become infinite periodically and lead us to a strategy which is not admissible* $\pi(t) \notin \mathcal{A}$, *see the acceptable set defined by (5).*

Remark 5. ***Optimal solution for mean-reverting Kim and Omberg model with uncorrelated Brownian motions.*** *Assume the same market setting from Proposition 1. If the evolution of the mean-reverting market price of risk is independent of the evolution of the asset price then the intertemporal hedging demand is zero and the dynamic Merton investment strategy applies, see Corollary 2.*

5 Real market data illustration

In this section we construct an econometric setting to derive parameter estimates guiding our investment strategy developed in the previous sections. A similar analysis has been conducted by Campbell and Viceira (1999) and Barberis (2000). Three important points are different in our analysis. We will verify later in the next section that these differences have a significant effect both on the predictability of the mean equity returns and also on the correlation, ρ, between returns and the market price of risk. The latter determines the size of the hedging demand, i.e., how much the nonmyopic strategy differs from the simple dynamic Merton strategy via (13).

Firstly, the correlation between the state variable(s) and the returns tends to be sensitive to the length of the parameter forecasting horizon considered; see Harrison and Zhang (1999); Engsted and Tanggaard (2002); Kim and In (2005) among many others. Our interest lies in investing for retirement, i.e., we are concerned with the long-term view. We will look at yearly data; we expect that this will eliminate correlations arising from small movements with comparably high noise. This is a distinction between our work and that of Campbell and Viceira (1999) and Barberis (2000) who look at a shorter time interval.

Secondly, we will benchmark all returns with respect to inflation. Most financial literature – and also Barberis (2000) and Campbell and Viceira (1999) – aims to predict returns in excess of short-term interest rate. However, we believe that for long-term investing one should instead consider the annual returns in excess of inflation. This ensures that the right thing is hedged, i.e., we maximize the purchasing power of our terminal investment value. Appropriate benchmarking via inflation also aids communication with the pension saver; see Gerrard et al. (2018, 2019). This amounts to setting the short interest r equal to zero and consider all derivations in real terms. From a strict financial mathematical point of view this is only fully correct if one can buy a fund returning exactly zero in real terms. In other words, a fund returning exactly the inflation.

Thirdly and lastly, we utilize a recent and extensive analysis of the predictive power of various state variables conducted by Kyriakou et al. (2019a,b). The analysis in (Kyriakou et al., 2019a,b) is different from the analysis in Welch and Goyal (2007) and Golez and Koudijs (2018). The latter two only look at returns in excess of short-term interest when analyzing the predictive strength of their state variables. The first part of (Kyriakou et al., 2019a,b) confirm the finding of Welch and Goyal (2007): returns in excess of short term interest are difficult to predict. However, when we are predicting real returns via state-variables benchmarked for inflation then higher predictive power can be attained. Hence, conveniently, the object of greater interest for long-term investments – returns in excess of inflation – is actually also simpler to predict.

5.1 Data source and our predictor

The basis of the following illustration is data from the U.S. stock market, the S&P 500 Index. We use the dataset provided by Robert Shiller, which can be found online[2]. We select the annual series of the nominal composite stock price index, P, the nominal dividends paid during the year, D, and inflation as measured by the consumer price index, CPI, all observed between the years 1871 and 2018. In our long-term prediction exercise we investigate the historical performance for the last 30 years, from 1989 to 2018.

Definition 1. ***Real returns process.*** *We define the returns process as the one-year excess stock returns process inclusive of dividends*

$$r(n) = \log\left(\frac{P(n) + D(n)}{P(n-1)}\right) - \log\left(1 + \frac{CPI(n-1)}{CPI(n-2)}\right) \quad \text{for} \quad n = 1873, \ldots, 2018. \tag{25}$$

Kyriakou et al. (2019a,b) look at a variety of lagged variables, i.e., observations at $n-1$, in order to predict the stock returns, $r(n)$, of the coming year. Candidates considered were (i) the dividend-by-price ratio, (ii) the earnings-by-price ratio, (iii) the short-term interest rate, (iv) the long-term interest rate, (v) inflation, (vi) the term spread and (vi) the lagged

[2] `http://www.econ.yale.edu/~shiller/`

excess stock returns; as well as the two-dimensional and the three-dimensional interactions between them. In that comparison the earnings-by-price ratio adjusted by inflation as a single predictor turned out to be a very strong candidate, performing 12.2% better than using the historical mean returns as the predictor and beating most other ones as well as the multi-dimensional candidates as well. Additionally, while the approach was non-parametric, i.e., non-linear relationships were considered, the best relationship between the annual stock returns and the earnings-by-price ratio turned out to be linear. This is is why we choose the earnings-by-price ratio as our predictor.

Definition 2. ***Real earnings-by-price.*** *The real earnings-by-price time series is defined as*

$$e(n) = \left(1 + \frac{E(n)}{P(n)}\right) \frac{CPI(n-2)}{CPI(n-1)} - 1 \quad \text{for} \quad n = 1873, \ldots, 2018, \tag{26}$$

where E are 12-month moving sums of the earnings on the S&P 500 *Index.*

By combining the definition of the real returns in (25) with our financial model in (1) we obtain the following representation

$$r(n) = \log\left(\frac{S_n}{S_{n-1}}\right) = \sigma \int_{n-1}^{n} \theta(t)\, dt - \frac{1}{2}\sigma^2 + \sigma\big(W(n) - W(n-1)\big), \tag{27}$$

noting that we have set $r = 0$ aiding the communication in real terms. We denote the conditional expectation given θ of this quantity by

$$\varepsilon(n) = \sigma \int_{n}^{n+1} \theta(t)\, dt - \frac{1}{2}\sigma^2. \tag{28}$$

Motivated by Kyriakou et al. (2019b) who observe a linear relationship between real earnings and real returns we run an ordinary least squares linear regression

$$\varepsilon(n) = \mathbb{E}\left[r(n+1)|\ e(n)\right] = \beta_0 + \beta_1 e(n) \quad \text{for} \quad n = 1873, \ldots, 2018,$$

and derive

$$\hat{\varepsilon}(n) = 0.004874652 + 1.119331917 e(n), \quad \text{for} \quad n = 1873, \ldots, 2018, \tag{29}$$

We call ε the transformed earnings process. Note that $\varepsilon(n-1)$ is observed at the end of year $n-1$ while $r(n)$ are the returns in the following year n. From Fig. 1 we see that the historical transformed earnings process $\varepsilon(n-1)$ looks like a shrunk version of the return process $r(n)$ – hinting at the predictive power of $\varepsilon(n-1)$. In the last 30 years we observe the minimum value around 0.49%, the maximum value 6.24% and the average value is around 2.92%. The mean over the full range of 147 years is 6.36% with a range from -12% to 34%. On the other hand the historical range of the real returns process is from a minimum value of -45% to a maximal value of 42% with total average value 6.52% and average real returns in the last 30 years is 7.6%.

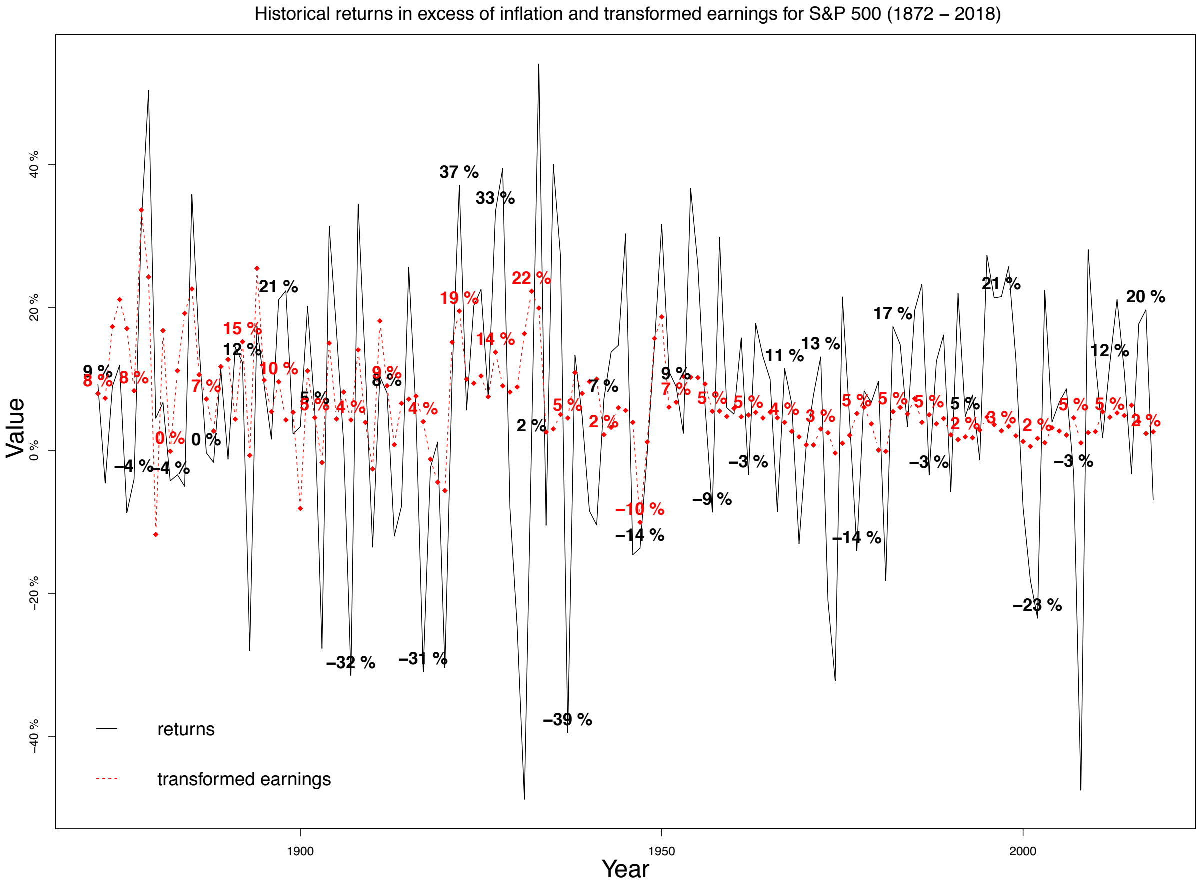


Figure 1. The black solid line represents the real returns $r(n)$ (33) with historical mean $\bar{r} = 6.51\%$ and standard deviation 16.98%. The red dashed line represents the transformed earnings $\hat{\varepsilon}(n-1)$ with historical mean 6.36% and standard deviation 6.66%.

5.2 Parameter estimation

Combining (28) with the solution of (18) implies the following expansion of the transformed earnings

$$\begin{aligned}\varepsilon(n) = \sigma\mu_\theta &- \frac{\sigma}{\kappa}\left[\mu_\theta + (\theta_0 - \mu_\theta)e^{-\kappa(n+1)} + \tau e^{-\kappa(n+1)}\int_0^{n+1} e^{\kappa s}\,dW_\theta(s)\right] \\ &+ \frac{\sigma}{\kappa}\left[\mu_\theta + (\theta_0 - \mu_\theta)e^{-\kappa n} + \tau e^{-\kappa n}\int_0^{n} e^{\kappa s}\,dW_\theta(s)\right] + \frac{\tau\sigma}{\kappa}\big(W_\theta(n+1) - W_\theta(n)\big) \\ &- \frac{1}{2}\sigma^2. \end{aligned} \tag{30}$$

We use this expansion for an estimation procedure described in Appendix C. We arrive at the parameter estimates summarized in Table 1. Interestingly, the point estimate of the correlation coefficient, $\hat{\rho}$, is considerably smaller than the results seen in the work of Barberis (2000) or Campbell and Viceira (1999). Note, again, that the difference in our research is that (a) we use yearly data instead of half-yearly data, (b) we predict the inflation-adjusted returns instead of the short-term interest rate adjusted returns, and lastly (c) we use the

Table 1 Estimated parameters for the Kim and Omberg model with transformed earnings as the state variable, based on annual data from the S&P 500 Index 1871–1988. σ is the volatility parameter of the price process; μ_θ is the mean of the Ornstein—Uhlenbeck process describing the market price of risk; κ is the mean-reversion speed of the market price of the risk process; τ is the volatility parameter of the market price of risk process; ρ is the correlation between market price of risk process and price process.

$\hat{\sigma}$	$\hat{\mu}_\theta$	$\hat{\kappa}$	$\hat{\tau}$	$\hat{\rho}$
0.17	0.44	2.89	1.51	0.03

inflation-adjusted earnings-by-price ratio instead of the dividends as the predictor.

5.3 Historical performance analysis

In our historical performance analysis starting with $\hat{\varepsilon}$, we use estimates of the parameters, $\{\hat{\sigma}, \hat{\mu}_\theta, \hat{\kappa}, \hat{\tau}, \hat{\rho}_{\theta y}\}$, derived in the previous two sections by using annual data only until 1988. We use these estimates to compare three strategies for investing over the period $1989-2018$.

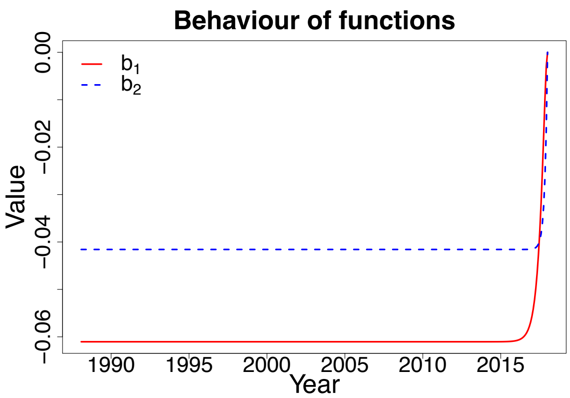
Classical Merton: We invest according to the classical Merton proportion. In particular $\theta(t)$ is considered to be constant in time, yielding the optimal nonmyopic strategy

$$\pi^*(n) = -\frac{U'(y)}{\sigma U''(y)} \frac{\bar{r}_{hist}}{\hat{sd}(r_{hist})}, \quad \text{for} \quad n = 1988, \ldots, 2018,$$

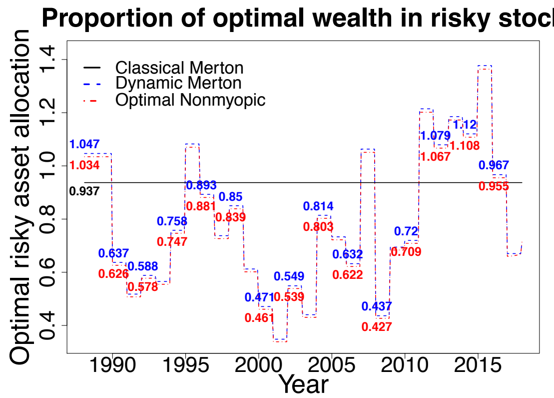
where $\bar{r}_{hist}$, $\hat{sd}(r_{hist})$ are the sample mean and sample standard deviation of $r_{hist} = r(1873), \ldots, r(1988)$.

Dynamic Merton: The nonmyopic strategy without the intertemporal hedging term (16) is executed on a yearly basis, i.e.,

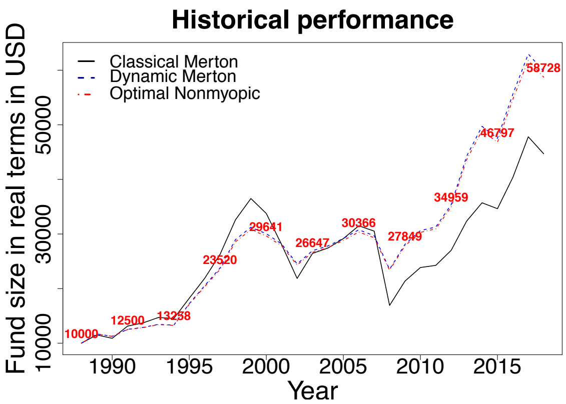
$$\pi^*(n) = -\frac{U'(y)}{\sigma U''(y)} \left(\frac{\hat{\varepsilon}(n) + \frac{1}{2}\hat{\sigma}^2}{\hat{\sigma}} \right), \quad \text{for} \quad n = 1988, \ldots, 2018.$$



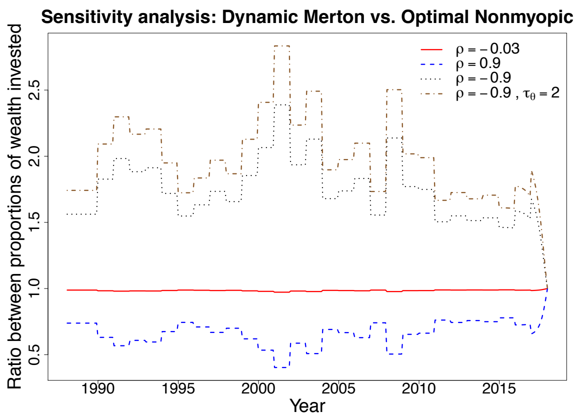
(a) Real earnings targeting real returns



(b) Real earnings targeting real returns



(c) Real earnings targeting real returns



(d) Real earnings targeting real returns

Figure 2. Real earnings targeting real returns. Fig. 2(a) illustrates the behaviour of the functions b_1 (red solid line) and b_2 (blue dashed line) in the last 3 years of the long-term saver's contribution planning. Fig. 2(b) compares optimal allocations in the risky stock following different strategies: classical Merton (black solid line), dynamic Merton (blue dashed line) and optimal nonmyopic (red dotdash line). Fig. 2(c) compares the wealth size performances: classical Merton (black solid line), dynamic Merton (blue dashed line) and optimal nonmyopic (red dotdash line). The initial wealth is $Y(0)$ =USD 10,000. The saver is assumed to obey a power utility with risk appetite parameter $\gamma = -1$ and the investment horizon is $T = 30$ years from 1988 to 2018. Fig. 2(d) is giving insight into the multiplicative sensitivity of the investment ratio of the optimal nonmyopic strategy to the dynamic Merton when the correlation is close to 1, when the correlation is close to -1, and when in addition the volatility of market price of risk is high.

Optimal Nonmyopic Strategy: The optimal nonmyopic strategy includes also the intertemporal hedging demand, see (13), so that

$$\pi^*(n) = -\frac{U'(y)}{\sigma U''(y)} \left(\hat{\theta}(n) + \hat{\tau}\hat{\rho}_{\theta y}\hat{b}_\theta(n, \hat{\theta}(n))\right), \quad \text{for} \quad n = 1988, \ldots, 2018,$$
$$\hat{\theta}(n) = \left(\frac{\hat{\varepsilon}(n) + \frac{1}{2}\hat{\sigma}^2}{\hat{\sigma}}\right),$$

where $\hat{b}_\theta$ is derived from b_θ in (23) by replacing the unknown quantities by their estimates from Subsection 5.2. For all three strategies we only report the results for the power utility with risk appetite parameter $\gamma = -1$. We note that changing the parameter value did not alter any conclusion. Figure 2 shows the result of our investigation. Fig. 2(a) visualizes how the intertemporal hedging demand vanishes when we are moving closer to the end of the investment horizon. Fig. 2(b) and Fig. 2(c) deliver the main message of our paper: in contrast to Barberis (2000) and Campbell and Viceira (1999), the correlation ρ turns out to be very small ($\hat{\rho} = 0.03$). For that reason the dynamic Merton and the optimal nonmyopic strategy are practically indistinguishable. Running a simulation with 1,000 repetitions suggests that the estimator of ρ has a standard deviation of 0.14. This means that the null hypothesis of the dynamic Merton being optimal (which is the case for $\rho = 0$) cannot be rejected. Finally, Fig. 2(d) shows that if the correlation coefficient, ρ, takes higher values, close to one, the investment strategies, the optimal nonmyopic and the dynamic Merton, behave quite differently. In such a market setting one cannot hedge with the dynamic Merton instead of the optimal nonmyopic investment strategy. The small correlation encountered in our investigation is favourable. As pointed out in Barberis (2000), the parameters involved in the intertemporal hedging demand can only be estimated with considerable parameter uncertainty, making it hard to obtain the actual optimal nonmyopic strategy with enough certainty. In our long-term perspective, however, the dynamic Merton is very close to the optimal. The dynamic Merton has a much easier structure by not including the intertemporal hedging demand and can hence be estimated with comparably much higher accuracy and can easily be implemented and communicated to the pensioner.

6 Comparison to different predictors and different target

In this section we perform two comparisons. Firstly, we look at the dividends as an alternative for the earnings as the long-term predictor. Secondly, we change the object of interest: returns in excess of short-term interest rate instead of the returns in excess of inflation.

6.1 Dividends as an alternative for earnings?

We define the dividends-by-price and the dividends-by-price adjusted by inflation.

Definition 3. ***Nominal dividends.*** *We define the nominal dividends-by-price ratio as*

$$d^N(n) = \frac{D(n)}{P(n)} \quad \text{for} \quad n = 1873, \ldots, 2018. \tag{31}$$

Definition 4. ***Real dividends.*** *We define the real dividends-by-price ratio as*

$$d^R(n) = \frac{D(n)}{P(n)} \frac{CPI(n-2)}{CPI(n-1)} \quad \text{for} \quad n = 1873, \ldots, 2018. \tag{32}$$

We follow closely the estimation procedure described in Section 5.1 and 5.2, replacing in each step $e(n)$ with $d^N(n)$ and then with $d^R(n)$. New parameter estimates are given in Table 2. Quite interestingly, the parameter estimates differ substantially when using

Table 2 Estimated parameters for the Kim and Omberg model with transformed earnings, transformed dividends (real and nominal) as state variables, based on yearly data from the S&P 500 Index 1871–1988. σ is the volatility parameter of the price process; μ_θ is the mean of the Ornstein—Uhlenbeck process describing the market price of risk; κ is the mean-reverting speed of the market price of the risk process; τ is the volatility parameter of the market price of risk process; ρ is the correlation between market price of risk process and price process.

Predictor	$\hat{\sigma}$	$\hat{\mu}_\theta$	$\hat{\kappa}$	$\hat{\tau}$	$\hat{\rho}$
real earnings (i.e. inflation-adjusted)	0.17	0.44	2.89	1.51	0.03
nominal dividends	0.18	0.36	0.67	0.11	0.09
real dividends (i.e. inflation-adjusted)	0.18	0.377	0.70	0.18	0.15

different predictors. The greater value for κ (2.89 compared to 0.67 and 0.70) is a hint that earnings have a better fit to the mean-reverting assumption. In Figure 3 we have plotted the historical performance of dividends as predictor. In both cases, the dynamic Merton and the optimal nonmyopic strategy are practically identical. This is because the factor $\hat{\tau}\hat{\rho}$ is small in all cases. The optimal nonmyopic strategy, while theoretical optimal, is hard to follow because the model complexity is making the parameter estimates very noisy. The Dynamic Merton on the other hand is much simpler to calibrate. Finally, the comparably smaller predictive power of divididens (real or nominal) compared to real earnings, as hinted at in Kyriakou et al. (2019a), and Mammen et al. (2019) manifests in more volatile returns and lower performance.

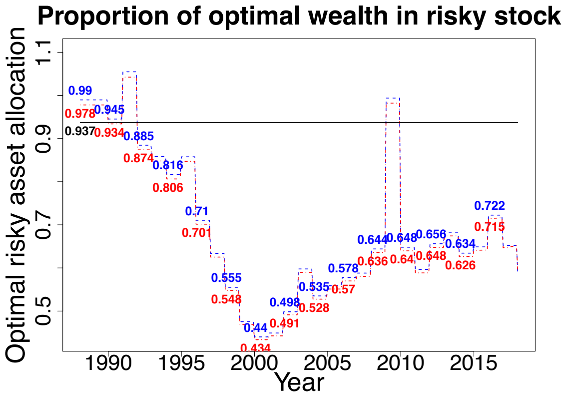
6.2 Predicting returns in excess of short-term interest rate

It is returns in excess of inflation that should be targeted when considering long-term investments. However, most of the financial literature aims to predict returns in excess of short-term interest rate. We now investigate what happens if one performs an investment strategy optimised for returns adjusted with the short-term interest rate. For this we will use, as is done in most financial literature, nominal state variables: earnings and dividends. We introduce the following definitions:

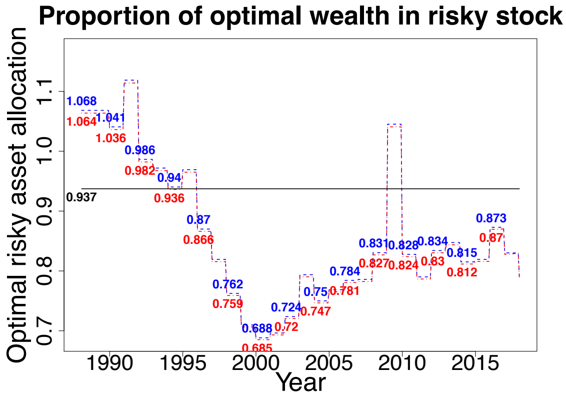
Definition 5. ***Returns in excess of the short-term interest rate.*** *We define the returns process in excess of the short-term interest rate as*

$$r^R(n) = \log\left(\frac{P(n) + D(n)}{P(n-1)}\right) - \log\left(\frac{R(n-1)}{100} + 1\right) \quad \text{for} \quad n = 1873, \dots, 2018, \tag{33}$$

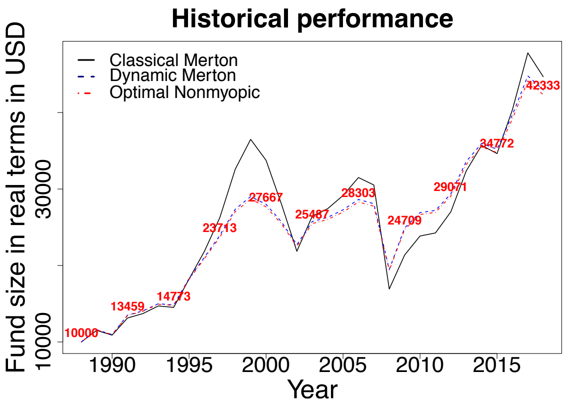
where R is the short-term interest rate.



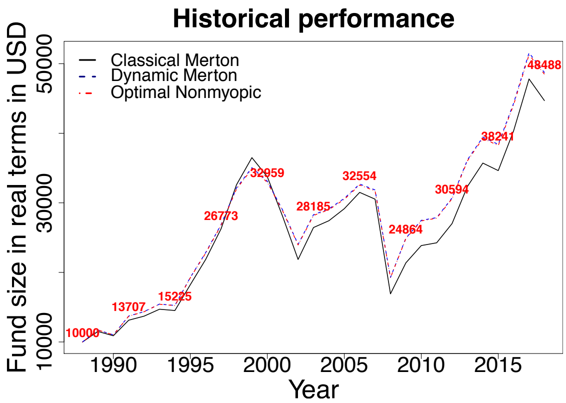
(a) Real dividends targeting real returns



(b) Nominal dividends targeting real returns



(c) Real dividends targeting real returns



(d) Nominal dividends targeting real returns

Figure 3. Fig. 3(a) and 3(c) use real dividends and Fig. 3(b) and 3(d) nominal dividends; both cases target real returns. Fig. 3(a) and 3(b) compare optimal allocations using different strategies: classical Merton (black solid line), dynamic Merton (blue dashed line) and optimal nonmyopic (red dotdash line). Fig. 3(c) and 3(d) compare the wealth size performances with classical Merton (black solid line), dynamic Merton (blue dashed line) and optimal nonmyopic (red dotdash line). The initial wealth is $Y(0)$ =USD 10,000. The saver is assumed to obey a power utility with risk appetite parameter $\gamma = -1$ and the investment horizon is of $T = 30$ years from 1988 to 2018.

Definition 6. ***Nominal earnings-by-price.*** *The nominal earnings-by-price time series is defined as*

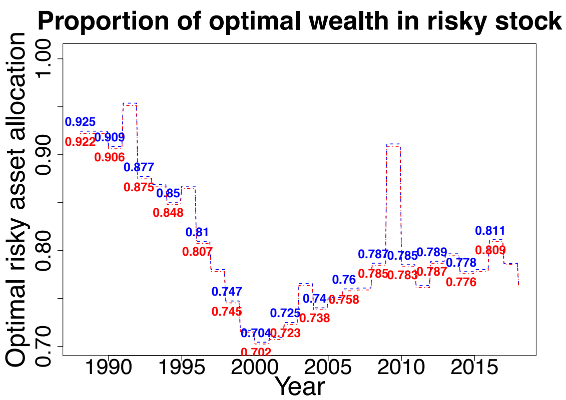
$$e^N(n) = \frac{E(n)}{P(n)} \quad \text{for} \quad n = 1873, \dots, 2018, \tag{34}$$

where E are 12-month moving sums of earnings on the S&P 500 *Index.*

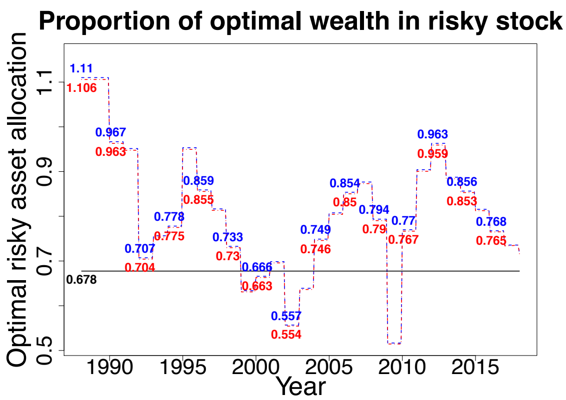
Again, we follow closely the estimation procedure described in Section 5.1 and 5.2 replacing in each step $r(n)$ by $r^R(n)$ as well as $e(n)$ first with $e^N(n)$ and then with $d^N(n)$. New parameter estimates are given in Table 3. Compared to Table 1, i.e. real earnings predicting real returns, we again derive much smaller κ values (0.58 and 0.67 compared to 2.89) – hinting at an inferior model fit. In Figure 4 we visualize the historical investment strategy and performance resulting from our parameter estimates. The overall conclusion is clear. The nominal earnings and dividends predicting stock returns in excess of short-term interest give inferior model suitability, poorer performance as measured by both, the volatility of the stock returns and also terminal performance.

Table 3 Estimated parameters for the Kim and Omberg model with transformed earnings (nominal), transformed dividends (nominal) as state variables predicting returns in excess of short-term interest rate, based on yearly data from the S&P 500 Index from 1871 to 1988. σ is the volatility parameter of the price process; μ_θ is the mean of the Ornstein—Uhlenbeck process describing the market price of risk; κ is the mean-reverting speed of the market price of the risk process; τ is the volatility parameter of the market price of risk process; ρ is the correlation between market price of risk process and price process.

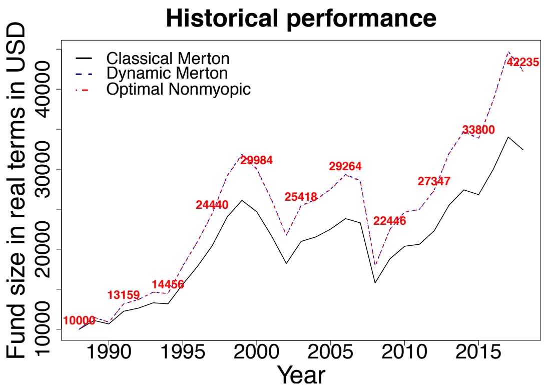
Predictor	**Target**	$\hat{\sigma}$	$\hat{\mu}_\theta$	$\hat{\kappa}$	$\hat{\tau}$	$\hat{\rho}$
nominal earnings	returns in excess of short-term interest	0.17	0.28	0.58	0.10	0.09
nominal dividends	returns in excess of short-term interest	0.17	0.29	0.67	0.06	0.11



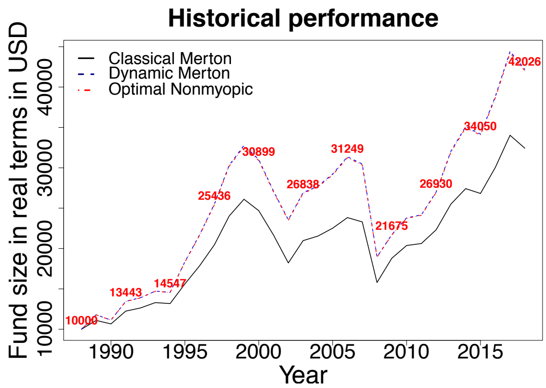
(a) Nominal dividends targeting returns in excess of single short-term interest



(b) Nominal earnings targeting returns in excess of single short-term interest



(c) Nominal dividends targeting returns in excess of single short-term interest. Fund size is inflation corrected.



(d) Nominal earnings targeting returns in excess of single short-term interest. Fund size is inflation corrected.

Figure 4. Optimal investment strategies for stock returns in excess of the short-term interest rate. Fig. 4(a) and 4(c) use nominal dividends and Fig. 4(b) and 4(d) nominal earnings as predictor. Fig. 4(a) and 4(b) compare optimal allocations in the risky stock following different strategies: classical Merton (black solid line), dynamic Merton (blue dashed line) and optimal nonmyopic (red dotdash line). Fig. 4(c) and 4(d) compare wealth size performances between the classical Merton (black solid line), dynamic Merton (blue dashed line) and optimal nonmyopic (red dotdash line). The initial wealth is $Y(0)$=USD 10,000. The saver is assumed to obey a power utility with risk appetite parameter $\gamma = -1$ and the investment horizon is of $T = 30$ years from 1988 to 2018.

6.3 Comparison

Table 4 presents the results of applying the three strategies under investigation to historical data of the S&P 500. Parameters were estimated on the basis of observations up to 1988. Different sections of the table consider different predictors and different targets while within each section the three different strategies are considered. In all cases real returns are considered, i.e., in excess of inflation. On this data set, we observe two key points. Firstly, the classical Merton strategy, which does not depend on the choice of a predictor, is outperformed by the other two dynamic strategies in all cases, when considering the ratio of mean return and average exposure or the ratio of mean return and standard deviation of returns. Secondly, in the real earnings targeting real return case the optimal nonmyopic strategy is practically indistinguishable from the dynamic Merton strategy and these two strategies also outperform all other strategies. when considering the ratio of mean return and average exposure or the ratio of mean return and standard deviation of returns. We note that changing the risk appetite parameter does not change any conclusion. For example a value of $\gamma = -2$ lead to around 33% more exposure in the stock market and around 1% higher ratio of average return per exposure, uniformly for all strategies.

Strategy	Avg exposure	Sd of returns	Mean return	Mean return/ exposure	Mean return/ sd
Real earnings targeting real returns					
Classical Merton	93.72 %	15.75 %	5.12%	5.46 %	32.50%
Dynamic Merton	78.40%	11.53 %	6.14%	7.84 %	53.28%
Optimal Nonmyopic	77.31%	11.36 %	6.08%	7.86 %	53.50%
Real dividends targeting real returns					
Classical Merton	93.72 %	15.75 %	5.12 %	5.46 %	32.50%
Dynamic Merton	68.32%	11.77%	4.98 %	7.29 %	42.31%
Optimal Nonmyopic	67.48%	11.62 %	4.93%	7.30 %	42.40 %
Nominal dividends targeting real returns					
Classical Merton	93.72%	15.75%	5.12%	5.46%	32.50%
Dynamic Merton	85.15 %	14.41%	5.42%	6.37%	37.62 %
Optimal Nonmyopic	84.79%	14.35%	5.40%	6.37 %	37.66%
Nominal earnings targeting returns in excess of short term interest					
Classical Merton	67.75 %	11.38%	4.00%	5.90%	35.13%
Dynamic Merton	79.46 %	13.07 %	4.92 %	6.19%	37.62%
Optimal Nonmyopic	79.15%	13.02 %	4.90 %	6.19%	37.65%
Nominal dividends targeting returns in excess of short term interest					
Classical Merton	67.75%	11.38 %	4.00%	5.90%	35.13%
Dynamic Merton	79.90%	13.45%	4.93 %	6.17 %	36.64 %
Optimal Nonmyopic	79.69 %	13.42 %	4.92 %	6.17 %	36.66%

Table 4 Numerical values of the historical performance of the pensioner's funds with respect to different predictors, different targets and different investment strategies. The results of applying the classical Merton strategy (based on a single estimated value of θ), the dynamic Merton strategy (not involving the intertemporal hedging terms) and the optimal non-myopic strategy when using the power utility function with risk appetite $\gamma = -1$. Parameters were estimated on the basis of yearly data 1858–1988 and the strategies were applied over the period from 1988 to 2018. The "exposure" column presents the average proportion invested into the risky asset over the 30-year period; The "Sd of returns" column shows the empirical standard deviation of the historical returns, the "mean returns" column shows the geometric mean $(X_{2018}/X_{1988})^{1/30} - 1$. The last two columns measure the overall performance as ratios of former colums.

7 Concluding Remarks

This paper has taken advantage of very recent insights regarding long-term predictions of stock returns – see Scholz et al. (2015); Kyriakou et al. (2019a,b) and it concludes that real earnings is an excellent driver for optimizing future investments for the long-term income. Using S&P 500 data from 1873 to 2018, we draw the conclusion that the optimal investment strategy is a simple one that we call the dynamic Merton, involving a simple regular update, according to the market conditions, of the classical investment strategy as given in Merton (1969) and Merton (1975), which is optimal when parameters are deterministic. We also note that recent research into the design of long-term products tailored to circumvent the pension crisis as defined in Merton (2014), Gerrard et al. (2018) and Gerrard et al. (2019) show that, if an optimal investment strategy is combined with easy-to-explain upper and lower bounds designed in Merton (2014), then the exact details of the optimal nonmyopic strategy are not very important, as long as the optimal nonmyopic strategy has sufficient risk appetite. The reason is that, when the long-term saver hedges in order to stay within the upper and lower financial bounds, he or she ends up with the correct risk appetite because the risky starting point is stabilized according to the financial hedging.

Future research should concentrate on incorporating this kind of dynamic strategies into modern approaches to pension products, see for example Gerrard et al. (2018, 2019). One challenge will be to find ways to incorporate non-gaussian properties into to asset-allocation strategies. Here a recent study by van Bilsen and Linders (2019) could serve as inspiration. The authors provide a similar idea as in the Time Pension product of Guillén, Jørgensen, and Nielsen (2006), where investment shocks are smoothly adjusted to future pension payment output in such a way that the pension income becomes remarkable stable, see also Jørgensen and Linnemann (2012) and Linnemann, Bruhn, and Steffensen (2015). The approach of van Bilsen and Linders (2019) is, however, more general allowing for more general distributions of stock returns including skewness and other higher order moments and postponing mechanism of financial shocks. We believe that future approaches to pensions should incorporate advanced econometric models as in van Bilsen and Linders (2019), that include communicative simplicity of pension outcomes as advocated for in Gerrard et al. (2018, 2019) without losing the financial advantages of financial dynamics as advocated for in this paper.

Appendices

A Proof of Theorem 1 and Corollaries 1 and 2

The format of the HJB equation (10) arises from expanding the differential of the value function $dV(t)$ using Itô's lemma

$$0 = \sup_{\hat{\pi}(t)\in\mathcal{A}(t)} \Big\{ \mathbb{E}_t\big[\, dV(t) \mid \mathcal{F}(t); \pi(t) = \hat{\pi}(t)\big] \Big\}. \tag{35}$$

If V^π is a function of t, y, θ and σ then

$$
\begin{aligned}
dV^\pi = {} & V_t^\pi dt + V_y^\pi dY(t) + V_\theta^\pi d\theta(t) + V_\sigma^\pi d\sigma(t) \\
& + \frac{1}{2}\left(V_{yy}^\pi d\langle Y, Y\rangle_t + V_{\theta\theta}^\pi d\langle \theta, \theta\rangle_t + V_{\sigma\sigma}^\pi d\langle \sigma, \sigma\rangle_t\right) \\
& + V_{y\theta}^\pi d\langle Y, \theta\rangle_t + V_{y\sigma}^\pi d\langle Y, \sigma\rangle_t + V_{\theta\sigma}^\pi d\langle \theta, \sigma\rangle_t\big) \\
= {} & V_t^\pi dt + \sigma(t)\pi(t)e^{r(T-t)}(\theta(t)dt + dW(t))V_y^\pi \\
& + \left(\delta^{(\theta)}dt + \tau^{(\theta)}dW_\theta(t)\right)V_\theta^\pi + \left(\delta^{(\sigma)}dt + \tau^{(\sigma)}dW_\sigma(t)\right)V_{\sigma\sigma}^\pi dt\big) \\
& + \frac{1}{2}\left(\sigma(t)^2\pi(t)^2 e^{2r(T-t)}V_{yy}^\pi dt + \left(\tau^{(\theta)}\right)^2 V_{\theta\theta}^\pi dt + \left(\tau^{(\sigma)}\right)^2 V_{\sigma\sigma}^\pi dt\right) \\
& + \sigma(t)\pi(t)e^{r(T-t)}\left[\tau^{(\theta)}\rho_{y\theta}V_{y\theta}^\pi + \tau^{(\sigma)}\rho_{y\sigma}V_{y\sigma}^\pi\right]dt + \tau^{(\theta)}\tau^{(\sigma)}\rho_{\theta\sigma}V_{\theta\sigma}^\pi dt.
\end{aligned}
$$

The dW terms all have expectation 0 and the remainder, when divided by dt, gives (10).

We now maximize over π and assume that $V_{yy} < 0$. Note that if V_{yy} were positive we would find that the optimal asset allocation was to put either $+\infty$ or $-\infty$ in the risky asset, making the process inadmissible. We get

$$
\sigma(t)\theta(t)e^{r(T-t)}V_y + \sigma^2(t)\pi(t)e^{2r(T-t)}V_{yy} + \sigma(t)e^{r(T-t)}\rho_{\theta y}\tau^{(\theta)}V_{\theta y} + \sigma(t)e^{r(T-t)}\rho_{\sigma y}\tau^{(\sigma)}V_{\sigma y} = 0
$$

implying that

$$
\pi^* = -\frac{e^{-r(T-t)}}{\sigma(t)V_{yy}}\left(\theta(t)V_y + \rho_{\theta y}\tau^{(\theta)}V_{\theta y} + \rho_{\sigma y}\tau^{(\sigma)}V_{\sigma y}\right). \tag{36}
$$

This completes the proof of Theorem 1.

Plugging this back into the HJB equation (10) gives

$$0 = V_t + \delta^{(\theta)} V_\theta + \delta^{(\sigma)} V_\sigma + \frac{1}{2}\left(\tau^{(\theta)}\right)^2 V_{\theta\theta} + \frac{1}{2}\left(\tau^{(\sigma)}\right)^2 V_{\sigma\sigma} + \tau^{(\theta)}\tau^{(\sigma)}\rho_{\theta\sigma} V_{\theta\sigma}$$
$$- \frac{1}{2V_{yy}}\left(\theta V_y + \rho_{\theta y}\tau^{(\theta)} V_{\theta y} + \rho_{\sigma y}\tau^{(\sigma)} V_{\sigma y}\right)^2.$$

We need the solution to this equation which satisfies the terminal reward condition $V(T, y, \theta, \sigma) = U(y)$. We seek solutions of the form

$$V(t, y, \theta, \sigma) = U(y)e^{b(t,\theta,\sigma)},$$

where $b(T, \theta, \sigma) = 0$ for all θ, σ. Taking partial derivatives of the solution form (11) we have the HJB equation

$$0 = U(y)\Big[b_t + \delta^{(\theta)} b_\theta + \delta^{(\sigma)} b_\sigma + \frac{1}{2}\left(\tau^{(\theta)}\right)^2\left(b_{\theta\theta} + b_\theta^2\right) + \frac{1}{2}\left(\tau^{(\sigma)}\right)^2\left(b_{\sigma\sigma} + b_\sigma^2\right) + \tau^{(\theta)}\tau^{(\sigma)}\rho_{\theta\sigma}\big(b_{\theta\sigma} + b_\theta b_\sigma\big)\Big]$$
$$- \frac{U'(y)^2}{2U''(y)}\left(\theta + \rho_{\theta y}\tau^{(\theta)} b_\theta + \rho_{\sigma y}\tau^{(\sigma)} b_\sigma\right)^2, \tag{37}$$

with the customer's terminal reward

$$b(T, \theta, \sigma) = 0.$$

For the power utility function *the negative inverse Arrow Pratt measure* is

$$\frac{U_p'(y)^2}{U_p''(y)U_p(y)} = -\frac{\gamma}{1-\gamma},$$

whereas for the exponential utility this is simply equal to 1. Therefore we define $\eta = \gamma/(1-\gamma)$ for the power law utility, or $\eta = -1$ in the case of the exponential utility, so that (37) reduces to

$$0 = b_t + \delta^{(\theta)} b_\theta + \delta^{(\sigma)} b_\sigma + \frac{1}{2}\left(\tau^{(\theta)}\right)^2 (b_{\theta\theta} + b_\theta^2) + \frac{1}{2}\left(\tau^{(\sigma)}\right)^2 (b_{\sigma\sigma} + b_\sigma^2) + \tau^{(\theta)}\tau^{(\sigma)}\rho_{\theta\sigma}(b_{\theta\sigma} + b_\theta b_\sigma)$$
$$+ \frac{1}{2}\eta\left(\theta + \rho_{\theta y}\tau^{(\theta)} b_\theta + \rho_{\sigma y}\tau^{(\sigma)} b_\sigma\right)^2.$$

We observe that there is no direct dependence on σ in the HJB equation, nor in the boundary condition. It is our hypothesis, therefore, that b has no dependence on σ. This reduces the PDE still further to

$$b_t + \delta^{(\theta)} b_\theta + \frac{1}{2}\left(\tau^{(\theta)}\right)^2 \left(b_{\theta\theta} + b_\theta^2\right) = -\frac{1}{2}\eta\left(\theta + \rho_{\theta y}\tau^{(\theta)} b_\theta\right)^2. \tag{38}$$

If (38) has a solution b which is bounded on the range $0 \le t \le T$ for each value of θ, then the policy π^* given by (13) is admissible and the function $V(t, y, \theta) = U(y)e^{b(t,\theta)}$ solves the HJB equation. The Verification Theorem allows us to conclude that π^* is the optimal investment strategy and that V is the optimal value function.

As a consequence of this, (7) implies that the optimal trajectory satisfies

$$dY^*(t) = -\frac{U'\big(Y^*(t)\big)}{U''\big(Y^*(t)\big)} \left[\theta(t) + \rho_{\theta y}\tau^{(\theta)}(\theta(t))b_\theta\big(t,\theta(t)\big)\right] (\theta(t)\, dt + dW_\theta(t))\,.$$

In the case of the power-law utility, $-U'\big(Y^*(t)\big)/U''\big(Y^*(t)\big) = (1-\gamma)^{-1}Y^*(t)$, and the equation becomes

$$\frac{dY^*(t)}{Y^*(t)} = \frac{1}{1-\gamma} Q\big(t,\theta(t)\big)\,(\theta(t)\, dt + dW_\theta(t))\,,$$

where

$$Q(t) = \theta(t) + \rho_{\theta y}\tau^{(\theta)}(\theta(t))b_\theta\big(t,\theta(t)\big).$$

Equation (14) follows from this, and the proof of Corollary 1 is complete.

If it happens that the evolution of the market price of risk occurs independently of the evolution of asset prices, in other words, if $\rho_{\theta y} = 0$, then (13) becomes (16), which is all that is required to prove Corollary 2. This leaves us with the problem of finding whether a suitable solution $b(t,\theta,\sigma)$ exists.

B Proof of Proposition 1

On substituting the quadratic expression (20) into the differential equation (19) we obtain the parabolic equation

$$0 = -b_0' - b_1'\theta - b_2'\theta^2 + \frac{\eta\theta^2}{2} + \frac{1}{2}\tau^2(1+\eta\rho^2)(b_1+2b_2\theta)^2 + (\kappa\mu_\theta + (-\kappa+\eta\rho\tau)\theta)(b_1+2b_2\theta) + \tau^2 b_2,$$

where $b_0, b_1, b_2 : [0,T] \to \mathbb{R}$ satisfy the boundary conditions $b_0(0) = b_1(0) = b_2(0) = 0$ and where $'$ represents differentiation with respect to s. We solve parabolic equation by isolating powers of θ, and making the substitution $\xi = \kappa/\tau$. This leads to three simultaneous equations

$$\begin{aligned} b_0' &= \frac{1}{2}\tau^2(1+\eta\rho^2)b_1^2 + \kappa\mu_\theta b_1 + \tau^2 b_2, \\ b_1' &= 2\tau^2(1+\eta\rho^2)b_1 b_2 + 2\kappa\mu_\theta b_2 + \tau(\eta\rho - \xi)b_1, \\ b_2' &= \frac{\eta}{2} + 2\tau^2(1+\eta\rho^2)b_2^2 + 2\tau(\eta\rho - \xi)b_2. \end{aligned}$$

The third of these equations is the place to start. It is a Riccati equation with constant coefficients; as such the form of the solution depends on the sign of the discriminant $\Delta = 4\tau^2\left(\xi^2 - \eta(1+2\rho\xi)\right)$. The solutions of this set of equations are going to depend on the values of the parameters. We distinguish between two cases

- **Case I:** $\xi^2 - \eta(1+2\rho\xi) < 0$.

- **Case II:** $\xi^2 - \eta(1 + 2\rho\xi) > 0$.

In **Case I**, the solutions $b_1(s)$ and $b_2(s)$ are ratios of trigonometric functions for which the denominator takes the value 0 periodically. Both $b_1(s)$ and $b_2(s)$ diverge to $-\infty$ as s approaches such a point from above or to $+\infty$ as s approaches from above.

$$b_1(T-t) = 0 \text{ for } t = T - \frac{2\pi n}{R_m\tau} \text{ and } t = T - \frac{\pi(2n+1)}{R_m\tau},$$
$$b_2(T-t) = 0 \text{ for } t = T - \frac{\pi n}{R_m\tau} \text{ and } t = T - \frac{\pi(2n+1)}{R_m\tau}$$

with $R_m\tau \neq 0$ and $n \in \mathbb{Z}$. Moreover the value function $V^\pi(t, y, \theta, \sigma) = \frac{1}{\gamma} y^\gamma e^{b(t,\theta,\sigma)}$ should be a $C^{12}([0,T]\text{x}\mathbb{R}^3)$ and bounded. The issue arises with the periodic denominator which obtains zero values at time periods

$$t = T - \frac{\pi n - \tan^{-1}\left(\frac{R_m}{\xi - \eta\rho}\right)}{R_m\tau},$$

where $\left(\xi - \eta\rho\right)^2 + R_m^2 \neq 0$, and correspondingly $R_m\tau \neq 0$ and $n \in \mathbb{Z}$. Later b_1 and b_2 change from $-\infty$ to $+\infty$ and consequently V changes from 0 to $+\infty$. Thus V is not a bounded value function and it has no partial derivative with respect to time. Hence this solution is not applicable to investigate further for the optimal investment strategy and the fund management itself.

In **Case II**, the solution expansions for large s are

$$b_1(s) = \frac{2\xi\mu_\theta}{R_p}\phi_2\Big(1 - 2e^{-R_p\tau s} + \mathcal{O}(e^{-2R_p\tau s})\Big), \tag{39}$$

$$b_2(s) = \phi_2\Big(1 - \frac{\phi_1 - \phi_2}{\phi_1}e^{-2R_p\tau s} + \mathcal{O}(e^{-4R_p\tau s})\Big) \tag{40}$$

with its constant limits $\lim_{s\to\infty} b_1(s) = \frac{2\xi\mu_\theta}{R_p}\phi_2$ and $\lim_{s\to\infty} b_2(s) = \phi_2$. The Verification Theorem assists us to prove that this solution is indeed the correct solution which we should investigate further. Both hyperbolic functions b_1 and b_2 have terminal cost zero and are monotone decreasing. Because of the existence of finite limits, both functions are bounded. The analytical expression of the solution for b_0 is not particularly required to compute the optimal nonmyopic strategy. Therefore the overall solution b and its exponential e^b are bounded. Similarly as in the previous case, we need to show that optimal value function is sufficiently integrable.

C Parameter estimation

We have

$$\varepsilon(n) = \sigma\int_n^{n+1}\theta(t)\mathrm{d}t - \frac{1}{2}\sigma^2, \tag{41}$$

and we may derive the moments of $\varepsilon(n)$, expanded in (30), as

$$\mathbb{E}\left[\varepsilon(n)\right] = \sigma\mu_\theta + \frac{\sigma}{\kappa}e^{-n\kappa}(1-e^{-\kappa})(\theta(0)-\mu_\theta) - \frac{1}{2}\sigma^2.$$

For $m > 0$

$$\begin{aligned}
\mathrm{Cov}(\varepsilon_n, \varepsilon_{n+m}) =& \frac{\sigma^2\tau^2}{\kappa^2}\mathrm{Cov}\left(e^{-\kappa n}(1-e^{-\kappa})\int_0^n e^{\kappa s}\, dW_\theta(s) + \int_n^{n+1}(1-e^{-\kappa(n+1-s)})\, dW_\theta(s),\right. \\
& \left. e^{-\kappa(n+m)}(1-e^{-\kappa})\int_0^{n+m} e^{\kappa s}\, dW_\theta(s) + \int_{n+m}^{n+m+1}(1-e^{-\kappa(n+m+1-s)})\, dW_\theta(s)\right) \\
=& \frac{\sigma^2\tau^2}{\kappa^2}\left[e^{-\kappa(2n+m)}(1-e^{-\kappa})^2\int_0^n e^{2\kappa s}\, ds + e^{-\kappa(n+m)}(1-e^{-\kappa})\int_n^{n+1}(e^{\kappa s}-e^{\kappa(2s-n-1)})\, ds\right] \\
=& \frac{\sigma^2\tau^2}{\kappa^2}\left[\frac{e^{-\kappa m}}{2\kappa}(1-e^{-\kappa})^2(1-e^{-2\kappa n}) + \frac{e^{-\kappa m}}{\kappa}(1-e^{-\kappa})(e^{\kappa}-1) - \frac{e^{-\kappa m}}{2\kappa}(1-e^{-\kappa})(e^{\kappa}-e^{-\kappa})\right] \\
=& \frac{\sigma^2\tau^2}{2\kappa^3}e^{-\kappa m}(1-e^{-\kappa})^2\left[1-e^{-2\kappa n}+2e^{\kappa}-(e^{\kappa}+1)\right] \\
=& \frac{\sigma^2\tau^2}{2\kappa^3}e^{-\kappa m}(1-e^{-\kappa})^2(e^{\kappa}-e^{-2\kappa n}),
\end{aligned} \tag{42}$$

and for $m = 0$

$$\begin{aligned}
\mathrm{Var}(\varepsilon_n) &= \frac{\sigma^2\tau^2}{\kappa^2}\mathrm{Var}\left(e^{-\kappa n}(1-e^{-\kappa})\int_0^n e^{\kappa s}\, dW_\theta(s) + \int_n^{n+1}(1-e^{-\kappa(n+1-s)})\, dW_\theta(s)\right) \\
&= \frac{\sigma^2\tau^2}{\kappa^2}\left[e^{-2\kappa n}(1-e^{-\kappa})^2\int_0^n e^{2\kappa s}\, ds + \int_n^{n+1}(1-e^{-\kappa(n+1-s)})^2\, ds\right] \\
&= \frac{\sigma^2\tau^2}{\kappa^2}\left[\frac{1}{2\kappa}(1-e^{-\kappa})^2(1-e^{-2\kappa n}) + 1 - \frac{2}{\kappa}(1-e^{-\kappa}) + \frac{1}{2\kappa}(1-e^{-2\kappa})\right] \\
&= \frac{\sigma^2\tau^2}{\kappa^3}\left[e^{-\kappa} - 1 + \kappa - \frac{1}{2}e^{-2\kappa n}(1-e^{-\kappa})^2\right].
\end{aligned} \tag{43}$$

Since $\sigma^2 = \mathrm{Var}\left(r(n) - \varepsilon(n-1)\right)$, we estimate the parameter σ, denoted by $\hat{\sigma}$, as the empirical standard deviation of $r(n) - \varepsilon(n-1)$. Next, we estimate *kappa* from the relationship

$$\frac{\mathrm{Cov}(\varepsilon(n), e^{-\kappa}\varepsilon(n-1))}{\mathrm{Var}(\varepsilon(n))} = \frac{e^{-\kappa}(1-e^{-\kappa})^2(e^{\kappa} - e^{-2\kappa n})}{2\left[e^{-\kappa} - 1 + \kappa - \frac{1}{2}e^{-2\kappa n}(1-e^{-\kappa})^2\right]}$$

by replacing the values on the right hand side by its empirical values and solving the resulting equation numerically. Similarly we estimate μ_θ from

$$\mathbb{E}\left[\varepsilon(n) - e^{-\kappa}\varepsilon(n-1)\right] = \left(1 - e^{-\kappa}\right)\left(\sigma\mu_\theta - \frac{1}{2}\sigma^2\right),$$

and solving for μ_θ after replacing the mean on the left hand side by its empirical value and replacing κ on the ride hand side by $\hat{\kappa}$ derived earlier. In order to remove dependence on $\theta(0)$, we can again look at $\varepsilon(n) - e^{-\kappa}\varepsilon(n-1)$

$$\begin{aligned}
\varepsilon(n) - e^{-\kappa}\varepsilon(n-1) =& \text{const} + \frac{\tau\sigma}{\kappa}e^{-\kappa n}\left(1 - e^{-\kappa}\right)\int_0^n e^{\kappa s}\,dW_\theta(s) + \frac{\tau\sigma}{\kappa}\int_n^{n+1}\left(1 - e^{-\kappa(n+1-s)}\right)dW_\theta(s) \\
& - e^{-\kappa}\left[\frac{\tau\sigma}{\kappa}e^{-\kappa(n-1)}\left(1 - e^{-\kappa}\right)\int_0^{n-1} e^{\kappa s}\,dW_\theta(s) + \frac{\tau\sigma}{\kappa}\int_{n-1}^{n}\left(1 - e^{-\kappa(n-s)}\right)dW_\theta(s)\right] \\
& = \text{const} + \frac{\tau\sigma}{\kappa}\int_{n-1}^{n} dW_\theta(s)\left[e^{-\kappa n}\left(1 - e^{-\kappa}\right)e^{\kappa s} - e^{-\kappa} + e^{-\kappa(n+1-s)}\right] \\
& + \frac{\tau\sigma}{\kappa}\int_n^{n+1}\left(1 - e^{-\kappa(n+1-s)}\right)dW_\theta(s) \\
& = \text{const} + \frac{\tau\sigma}{\kappa}\int_{n-1}^{n} dW_\theta(s)\left[e^{-\kappa(n-s)} - e^{-\kappa}\right] + \frac{\tau\sigma}{\kappa}\int_n^{n+1}\left(1 - e^{-\kappa(n+1-s)}\right)dW_\theta(s).
\end{aligned}$$

Therefore

$$\begin{aligned}
\mathrm{Var}\left[\varepsilon(n) - e^{-\kappa}\varepsilon(n-1)\right] &= \frac{\tau^2\sigma^2}{\kappa^2}\Bigg[\int_{n-1}^{n}\left(e^{-2\kappa(n-s)} - 2e^{-\kappa(n+1-s)} + e^{-2\kappa}\right)\, ds \\
&\quad + \int_{n}^{n+1}\left(1 - 2e^{-\kappa(n+1-s)} + e^{-2\kappa(n+1-s)}\right)\, ds\Bigg] \\
&= \frac{\tau^2\sigma^2}{\kappa^2}\Bigg[\frac{1}{2\kappa}\left(1 - e^{-2\kappa}\right) - \frac{2e^{-\kappa}}{\kappa}\left(1 - e^{-\kappa}\right) + e^{-2\kappa} + 1 \\
&\quad - \frac{2}{\kappa}\left(1 - e^{-\kappa}\right) + \frac{1}{2\kappa}\left(1 - e^{-2\kappa}\right)\Bigg] \\
&= \frac{\tau^2\sigma^2}{\kappa^2}\left[1 + e^{-2\kappa} - \frac{1}{\kappa}\left(1 - e^{-2\kappa}\right)\right],
\end{aligned}$$

and by comparison of the theoretical variance with the variance of the residuals of the linear regression we derive $\hat{\tau}$ as an estimate of τ. In a similar fashion, we estimate the last parameter ρ by calculating the following covariance

$$\begin{aligned}
\mathrm{Cov}\left(r(n) - \varepsilon(n-1), \varepsilon(n-1)\right) &= \mathrm{Cov}\left(\sigma\left(W_1(n) - W_1(n-1)\right), -\frac{\sigma\tau}{\kappa}e^{-\kappa n}\int_0^n e^{\kappa s}\, dW_\theta(s)\right) \\
&\quad + \mathrm{Cov}\left(\sigma\left(W_1(n) - W_1(n-1)\right), \frac{\sigma\tau}{\kappa}e^{-\kappa(n-1)}\int_0^{n-1} e^{\kappa s}\, dW_\theta(s)\right) \\
&\quad + \mathrm{Cov}\left(\sigma\left(W_1(n) - W_1(n-1)\right), \frac{\sigma\tau}{\kappa}\left(W_\theta(n) - W_\theta(n-1)\right)\right) \\
&= \rho_{\theta y}\frac{\sigma^2\tau}{\kappa^2}\left(\kappa - \left(1 - e^{-\kappa}\right)\right).
\end{aligned}$$

Then by comparing the sample covariance with the theoretical covariance in conjunction with plug-in estimates for $\hat{\sigma}, \hat{\kappa}, \hat{\tau}$, we get an estimate, $\hat{\rho}_{\theta y}$, of $\rho_{\theta y}$.

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