ON THE LIE ALGEBRA STRUCTURE OF $HH^1(A)$ OF A FINITE-DIMENSIONAL ALGEBRA A

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Abstract. Let A be a split finite-dimensional associative unital algebra over a field. The first main result of this note shows that if the Ext-quiver of A is a simple directed graph, then $HH^1(A)$ is a solvable Lie algebra. The second main result shows that if the Ext-quiver of A has no loops and at most two parallel arrows in any direction, and if $HH^1(A)$ is a simple Lie algebra, then $\mathrm{char}(k) \neq 2$ and $HH^1(A) \cong \mathfrak{sl}_2(k)$. The third result investigates symmetric algebras with a quiver which has a vertex with a single loop.

1. Introduction

Let k be a field. Our first result is a sufficient criterion for $HH^1(A)$ to be a solvable Lie algebra, where A is a split finite-dimensional k-algebra (where the term 'algebra' without any further specifications means an associative and unital algebra).

Theorem 1.1. *Let A be a split finite-dimensional k-algebra. Suppose that the* Ext*-quiver of A is a simple directed graph. Then the derived Lie subalgebra of $HH^1(A)$ is nilpotent; in particular the Lie algebra $HH^1(A)$ is solvable.*

The recent papers [4] and [8] contain comprehensive results regarding the solvability of $HH^1(A)$ of tame algebras and blocks, and [8] also contains a proof of Theorem 1.1 with different methods. We will prove Theorem 1.1 in Section 3 as part of the more precise Theorem 3.1, bounding the derived length of the Lie algebra $HH^1(A)$ and the nilpotency class of the derived Lie subalgbra of $HH^1(A)$ in terms of the Loewy length $\ell\ell(A)$ of A. The hypothesis on the quiver of A is equivalent to requiring that $\mathrm{Ext}^1_A(S,S) = 0$ for any simple A-module S and $\dim_k(\mathrm{Ext}^1_A(S,T)) \leq 1$ for any two simple A-modules S, T. If in addition A is monomial, then Theorem 1.1 follows from work of Strametz [10]. The hypotheses on A are not necessary for the derived Lie subalgebra of $HH^1(A)$ to be nilpotent or for $HH^1(A)$ to be solvable; see [2, Theorem 1.1] or [8] for examples.

The Lie algebra structure of $HH^1(A)$ is invariant under derived equivalences, and for symmetric algebras, also invariant under stable equivalences of Morita type. Therefore, the conclusions of Theorem 1.1 remain true for any finite-dimensional k-algebra B which is derived equivalent to an algebra A satisfying the hypotheses of this theorem, or for a symmetric k-algebra B which is stably equivalent of Morita type to a symmetric algebra A satisfying the hypotheses of the theorem.

If we allow up to two parallel arrows in the same direction in the quiver of A but no loops, then it is possible for $HH^1(A)$ to be simple as a Lie algebra. The only simple Lie algebra to arise in that case is $\mathfrak{sl}_2(k)$, with $\mathrm{char}(k) \neq 2$.

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Theorem 1.2. *Let A be a split finite-dimensional k-algebra. Suppose that $\mathrm{Ext}^1_A(S,S)=0$ for any simple A-module S and that $\dim_k(\mathrm{Ext}^1_A(S,T))\leq 2$ for any two simple A-modules S, T. If $HH^1(A)$ is not solvable, then $\mathrm{char}(k)\neq 2$ and $HH^1(A)/\mathrm{rad}(HH^1(A))$ is a direct product of finitely many copies of $\mathfrak{sl}_2(k)$. In particular, the following hold.*

(i) *If $HH^1(A)$ is a simple Lie algebra, then $\mathrm{char}(k)\neq 2$, and $HH^1(A)\cong\mathfrak{sl}_2(k)$.*
(ii) *If $\mathrm{char}(k)=2$, then $HH^1(A)$ is a solvable Lie algebra.*

This will be proved in Section 3; for monomial algebras this follows as before from Strametz [10]. An example of an algebra A satisfying the hypotheses of this theorem is the Kronecker algebra, a 4-dimensional k-algebra, with $\mathrm{char}(k)\neq 2$, given by the directed quiver with two vertices e_0, e_1 and two parallel arrows α, β from e_0 to e_1. This example is a special case of more general results on monomial algebras; see in particular [10, Corollary 4.17]. As in the case of the previous Theorem, the conclusions of Theorem 1.2 remain true for an algebra B which is derived equivalent to an algebra A satisfying the hypotheses of this theorem, or for a symmetric algebra B which is stably equivalent of Morita type to a symmetric algebra A satisfying the hypotheses of the theorem.

We have the following partial result for symmetric algebras whose quiver has a single loop at some vertex.

Theorem 1.3. *Suppose that k is algebraically closed. Let A be a finite-dimensional symmetric k-algebra, and let S be a simple A-module. Suppose that $\dim_k(\mathrm{Ext}^1_A(S,S))=1$ and that for any primitive idempotent i in A satisfying $iS\neq 0$ we have $J(iAi)^2=iJ(A)^2i$. If $HH^1(A)$ is a simple Lie algebra, then $\mathrm{char}(k)=p>2$ and $HH^1(A)$ is isomorphic to either $\mathfrak{sl}_2(k)$ or the Witt Lie algebra $W=\mathrm{Der}(k[x]/(x^p))$.*

This will be proved in Section 4, along with some general observations regarding the compatibility of Schur functors and the Lie algebra structure of $HH^1(A)$. Section 5 contains some examples.

2. On derivations and the radical

We start with a brief review of some basic terminology. The *nilpotency class* of a nilpotent Lie algebra $\mathcal{L}$ is the smallest positive integer m such that $\mathcal{L}^m=0$, where $\mathcal{L}^1=\mathcal{L}'$ and $\mathcal{L}^{m+1}=[\mathcal{L},\mathcal{L}^m]$ for $m\geq 1$. In addition, the *derived length* of a solvable Lie algebra is the smallest positive integer n such that $\mathcal{L}^{(n)}=0$, where $\mathcal{L}^{(1)}=\mathcal{L}'$ and $\mathcal{L}^{(n+1)}=[\mathcal{L}^{(n)},\mathcal{L}^{(n)}]$ for $n\geq 1$. A Lie algebra $\mathcal{L}$ is called *strongly solvable* if its derived subalgebra is nilpotent. A Lie algebra $\mathcal{L}$ of finite dimension n is called *completely solvable* (also called *supersolvable*) if there exists a sequence of ideals $\mathcal{L}_1=\mathcal{L}\supset\mathcal{L}_2\supset\cdots\supset\mathcal{L}_n\supset 0$ such that $\dim_k(\mathcal{L}_i)=n+1-i$ for $1\leq i\leq n$.

Remark 2.1. If k is algebraically closed of characteristic zero, then, as a consequence of Lie's theorem, the classes of strongly and completely solvable Lie algebras coincide with the class of solvable Lie algebras. Lie's theorem does not hold in positive characteristic. If k is algebraically closed of prime characteristic p, then by [3, Theorem 3], a finite-dimensional Lie algebra $\mathcal{L}$ over k is strongly solvable if and only if $\mathcal{L}$ is completely solvable.

Let A be a finite-dimensional k-algebra. We denote by $\ell(A)$ the number of isomorphism classes of simple A-modules. The *Loewy length* $\ell\ell(A)$ of A is the smallest positive integer m such that $J(A)^m=0$, where $J(A)$ denotes the Jacobson radical of A. We denote by $[A,A]$ the k-subspace of A generated by the set of additive commutators $ab-ba$, where a, $b\in A$. A *derivation on A* is a k-linear map $f:A\to A$ satisfying $f(ab)=f(a)b+af(b)$ for all a, $b\in A$. If f, g are derivations on A, then so is $[f,g] = f \circ g - g \circ f$, and the space $\mathrm{Der}(A)$ of derivations on A becomes a Lie algebra in this way. If $c \in A$, then the map $[c,-]$ defined by $[c,a] = ca - ac$ is a derivation; any derivation of this form is called an *inner derivation*. The space $\mathrm{IDer}(A)$ of inner derivations is a Lie ideal in $\mathrm{Der}(A)$, and we have a canonical isomorphism $HH^1(A) \cong \mathrm{Der}(A)/\mathrm{IDer}(A)$; see [12, Lemma 9.2.1]. It is easy to see that any derivation on A preserves the subspace $[A,A]$, and that any inner derivation of A preserves any ideal in A. A finite-dimensional k-algebra A is called *split* if $\mathrm{End}_A(S) \cong k$ for every simple A-module S. If A is split, then by the Wedderburn-Malcev Theorem, A has a separable subalgebra E such that $A = E \oplus J(A)$. Moreover, E is unique up to conjugation by elements in the group $A^\times$ of invertible elements in A. A primitive decomposition I of 1 in E remains a primitive decomposition of 1 in A.

For convenience, we mention the following well-known descriptions of certain Ext^1-spaces.

Lemma 2.2. *Let A be a split finite-dimensional k-algebra, let i be a primitive idempotent in A. Set $S = Ai/J(A)i$ and $S^\vee = iA/iJ(A)$. We have k-linear isomorphisms*

$$HH^1(A; S \otimes_k S^\vee) \cong \mathrm{Ext}^1_A(S,S) \cong \mathrm{Hom}_A(J(A)i/J(A)^2 i, S) \cong \mathrm{Hom}_{A\otimes_k A^{\mathrm{op}}}(J(A)/J(A)^2, S \otimes_k S^\vee)\ .$$

Lemma 2.3. *Let A be a split finite-dimensional k-algebra. Let i be a primitive idempotent in A, and set $S = Ai/J(A)i$. We have $\mathrm{Ext}^1_A(S,S) = 0$ if and only if $iJ(A)i \subseteq J(A)^2$.*

Proof. By Lemma 2.2, we have $\mathrm{Ext}^1_A(S,S) = 0$ if and only if $J(A)/J(A)^2$ has no simple bimodule summand isomorphic to $S \otimes_k S^\vee$. This is equivalent to $i \cdot (J(A)/J(A)^2) \cdot i = 0$, hence to $iJ(A)i \subseteq J(A)^2$ as stated. □

Lemma 2.4. *Let A be a split finite-dimensional k-algebra, and let E be a separable subalgebra of A such that $A = E \oplus J(A)$. Every class in $HH^1(A)$ has a representative $f \in \mathrm{Der}(A)$ satisfying $E \subseteq \ker(f)$.*

Proof. Let $f : A \to A$ be a derivation. Since E is separable, it follows that for any E-E-bimodule M we have $HH^1(E;M) = 0$. In particular, the derivation $f|_E : E \to A$ is inner; that is, there is an element $c \in A$ such that $f(x) = [c,x]$ for all $x \in E$. Thus the derivation $f - [c,-]$ on A vanishes on E and represents the same class as f in $HH^1(A)$. □

Lemma 2.5. *Let A be a split finite-dimensional k-algebra, and let E be a separable subalgebra of A such that $A = E \oplus J(A)$. Let $f : A \to A$ be a derivation such that $E \subseteq \ker(f)$. For any two idempotents i, j in E we have $f(iAj) \subseteq iAj$ and $f(AiAj) \subseteq AiAj$.*

Proof. Let i, j be idempotents in E, and let a, $b \in A$. We have $f(iaj) = f(i^2aj) = if(iaj) + f(i)iaj = if(iaj)$, since $i \in E \subseteq \ker(f)$. Thus $f(iaj) \in iA$. A similar argument shows that $f(iaj) \in Aj$, and hence $f(iaj) \in iAj$. This shows the first statement. The second statement follows from this and the equality $f(biaj) = f(b)iaj + bf(iaj)$. □

Lemma 2.6. *Let A be a split finite-dimensional k-algebra such that $\mathrm{Ext}^1_A(S,S) = 0$ for all simple A-modules S. Then for any derivation $f : A \to A$ we have $f(J(A)) \subseteq J(A)$.*

Proof. Let E be a separable subalgebra of A such that $A = E \oplus J(A)$. Let I be a primitive decomposition of 1 in E (hence also in A). Note that if i, $j \in I$ are not conjugate in $A^\times$, then $iAj \subseteq J(A)$. The hypotheses on A imply that $J(A)i/J(A)^2 i$ has no summand isomorphic to $Ai/J(A)i$, and hence that $iJ(A)i \subseteq J(A)^2$ for any $i \in I$. Then $iJ(A)j \subseteq J(A)^2$ for any two i, $j \in I$ which are conjugate in $A^\times$. Let now $f : A \to A$ be a derivation. As noted above, any inner derivation preserves $J(A)$. Thus, by Lemma 2.4, we may assume that $f|_E = 0$. Since $J(A) = \oplus_{i\in I} J(A)i$, it suffices to show that $f(J(A)i) \subseteq J(A)i$, where $i \in I$. If j is conjugate to i, then $AjJ(A)i \subseteq J(A)^2 i$. Since $J(A)i = \sum_{j\in I} AjJ(A)i$, it follows from Nakayama's Lemma that $J(A)i = \sum_j AjAi$, where j runs over the subset I' of all j in I which are not conjugate to i. Now f preserves the submodules $AjAi$ in this sum, thanks to Lemma 2.5. The result follows. □

Definition 2.7. Let A be a split finite-dimensional k-algebra, and let E be a separable subalgebra of A such that $A = E \oplus J(A)$. For $m \geq 1$, denote by D_m the subspace of $\mathrm{Der}(A)$ consisting of all derivations $f : A \to A$ such that $E \subseteq \ker(f)$ and such that $f(J(A)) \subseteq J(A)^m$.

The following observations are variations of the statements in [6, Proposition 3.5].

Proposition 2.8. *Let A be a split finite-dimensional k-algebra, and let E be a separable subalgebra of A such that $A = E \oplus J(A)$. The following hold.*

(i) *For any positive integers m, n we have $[D_m, D_n] \subseteq D_{m+n-1}$.*
(ii) *The space D_1 is a Lie subalgebra of* $\mathrm{Der}(A)$*, and for any positive integer m, the space D_m is a Lie ideal in D_1.*
(iii) *The space D_2 is a nilpotent ideal in D_1. More precisely, if $\ell\ell(A) \leq 2$, then $D_2 = 0$, and if $\ell\ell(A) > 2$, then the nilpotency class of D_2 is at most $\ell\ell(A) - 2$.*

Proof. The space of derivations on A which vanish on E is easily seen to be closed under the Lie bracket on $\mathrm{Der}(A)$. Thus statement (i) follows from [6, Lemma 3.4]. Statement (ii) is an immediate consequence of (i). If $m \geq \ell\ell(A)$, then $J(A)^m = 0$, and hence $D_m = 0$. Together with (i), this implies (iii). □

Proposition 2.9. *Let A be a split finite-dimensional k-algebra, and let E be a separable subalgebra of A such that $A = E \oplus J(A)$. Suppose that every derivation f on A satisfies $f(J(A)) \subseteq J(A)$. Then the canonical algebra homomorphism $A \to A/J(A)^2$ induces a Lie algebra homomorphism $\Phi : HH^1(A) \to HH^1(A/J(A)^2)$. The following hold.*

(i) *The canonical surjection* $\mathrm{Der}(A) \to HH^1(A)$ *maps D_1 onto $HH^1(A)$.*
(ii) *The canonical surjection* $\mathrm{Der}(A) \to HH^1(A)$ *maps D_2 onto* $\ker(\Phi)$*; in particular,* $\ker(\Phi)$ *is a nilpotent ideal in the Lie algebra $HH^1(A)$.*
(iii) *The Lie algebra $HH^1(A)$ is solvable if and only if $HH^1(A)/\ker(\Phi)$ is solvable.*
(iv) *If the derived Lie algebra of $HH^1(A)$ is contained in* $\ker(\Phi)$*, then $HH^1(A)$ is nilpotent.*
(v) *If the Lie algebra $HH^1(A)$ is simple, then Φ is injective.*

Proof. The hypotheses on $\mathrm{Der}(A)$ together with Lemma 2.4 imply that $HH^1(A)$ is equal to the image of the space D_1 in $HH^1(A)$, whence (i). The canonical surjection $\mathrm{Der}(A) \to HH^1(A)$ clearly maps D_2 to $\ker(\Phi)$; we need to show the surjectivity of the induced map $D_2 \to \ker(\Phi)$. Note first that any inner derivation in D_1 is of the form $[c, -]$ for some c which centralises E. Note further that the centraliser $C_A(E)$ of E in A is canonically isomorphic to $\mathrm{Hom}_{E\otimes_k E^{\mathrm{op}}}(E, A)$ (via the map sending an E-E-bimodule homomorphism $\alpha : E \to A$ to $\alpha(1)$). Since E is separable, hence projective as an E-E-bimodule, it follows that the functor $\mathrm{Hom}_{E\otimes_k E^{\mathrm{op}}}(E, -)$ is exact. In particular, the surjection $A \to A/J(A)^2$ induces a surjection $C_A(E) \to C_{A/J(A)^2}(E)$, where we identify E with its image in $A/J(A)^2$. Let $f \in D_1$ such that the class of f is in $\ker(\Phi)$, or equivalently, such that the induced derivation, denoted $\bar{f}$, on $A/J(A)^2$ is inner. Then there is $c \in A$ such that $\bar{f} = [\bar{c}, -]$, where $\bar{c} = c + J(A)^2$ centralises the image of E in $A/J(A)^2$. By the above, we may choose c such that c centralises E in A. Then the derivation $f - [c, -]$ represents the same class as f, still belongs to D_1, and induces the zero map on $A/J(A)^2$. Thus $f - [c, -]$ belongs in fact to D_2, proving (ii). The remaining statements are immediate consequences of (ii). ☐

The next result includes the special case of Theorem 1.1 where $\ell\ell(A) \leq 2$.

Proposition 2.10. *Let A be a split finite-dimensional k-algebra such that $J(A)^2 = 0$. Suppose that for every simple A-module S we have $\mathrm{Ext}^1_A(S, S) = 0$ and that for any two simple A-modules S, T we have $\dim_k(\mathrm{Ext}^1_A(S, T)) \leq 1$. Let E be a separable subalgebra of A such that $A = E \oplus J(A)$. The following hold.*

(i) *If A is basic and if f, g are derivations on A which vanish on E, then $[f, g] = 0$.*
(ii) *The Lie algebra $HH^1(A)$ is abelian.*
(iii) *Suppose that A is indecomposable as an algebra, and let $e(A)$ be the number of edges in the quiver of A. We have*

$$\dim_k(HH^1(A)) = e(A) - \ell(A) + 1 \leq (\ell(A) - 1)^2 \ .$$

Proof. In order to prove (i), suppose that A is basic. Let I be a primitive decomposition of 1 in A such that $E = \prod_{i \in I} ki$. Let f and g be derivations on A which vanish on E. Then f, g are determined by their restrictions to $J(A)$. By Lemma 2.6, the derivations f, g preserve $J(A)$. By the assumptions, each summand iAj in the vector space decomposition $A = \oplus_{i,j \in I} iAj$ has dimension at most one. By Lemma 2.5, any derivation on A which vanishes on E preserves this decomposition. Therefore, if X is a basis of $J(A)$ consisting of elements of the subspaces iAj, i, $j \in I$, which are nonzero, then $f|_{J(A)} : J(A) \to J(A)$ is represented by a diagonal matrix. Similary for g. But then the restrictions of f and g to $J(A)$ commute. Since both f, g vanish on E , this implies that $[f, g] = 0$, whence (i). If A is basic, then clearly (i) and Lemma 2.4 together imply (ii). Since the hypotheses of the Lemma as well as the Lie algebra $HH^1(A)$ are invariant under Morita equivalences, statement (ii) follows for general A. In order to prove (iii), assume again that A is basic. By the assumptions, $e(A) = \dim_k(J(A)) = |X|$. One verifies that the extension to A by zero on I of any linear map on $J(A)$ which preserves the summands iAj (with $i \neq j$), or equivalently, which preserves the one-dimensional spaces kx, where $x \in X$, is in fact a derivation. By Lemma 2.4, any class in $HH^1(A)$ is represented by such a derivation. Thus the space of derivations on A which vanish on I is equal to $\dim_k(J(A)) = e(A)$. Each $i \in I$ contributes an inner derivation. Since A is indecomposable, it follows that the only k-linear combination of elements in I which belongs to $Z(A)$ are the scalar multiples of $1 = \sum_{i \in I} i$. Thus the space of inner derivations which annihilate I has dimension $\ell(A) - 1$, whence the first equality. Since there are at most $\ell(A) - 1$ arrows starting at any given vertex, it follows that $e(A) \leq (\ell(A) - 1)\ell(A)$, whence the inequality as stated. ☐

The above Proposition can also be proved as a consequence of more general work of Strametz [10], calculating the Lie algebra $HH^1(A)$ for A a split finite-dimensional monomial algebra.

3. Proofs of Theorems 1.1 and 1.2

Theorem 1.1 is a part of the following slightly more precise result.

Theorem 3.1. *Let A be a split finite-dimensional k-algebra. Suppose that for every simple A-module S we have $\mathrm{Ext}^1_A(S, S) = 0$ and that for any two simple A-modules S, T we have $\dim_k(\mathrm{Ext}^1_A(S, T)) \leq 1$. Set $\mathcal{L} = HH^1(A)$, regarded as a Lie algebra.*

(i) *If $\ell\ell(A) \leq 2$ then $\mathcal{L}$ is abelian.*
(ii) *If $\ell\ell(A) > 2$, then the derived Lie algebra $\mathcal{L}' = [\mathcal{L}, \mathcal{L}]$ is nilpotent of nilpotency class at most $\ell\ell(A) - 2$. The derived length of $\mathcal{L}$ is at most $\log_2(\ell\ell(A) - 1) + 1$.*

In particular, $\mathcal{L}$ is solvable, and if k is algebraically closed, then $\mathcal{L}$ is completely solvable.

Proof. If $\ell\ell(A) \leq 2$, then $J(A)^2 = 0$, and hence (i) follows from Proposition 2.10. Suppose that $\ell\ell(A) > 2$. We may assume that A is basic. Note that A and $A/J(A)^2$ have the same Ext-quiver, and hence we may apply Proposition 2.10 to the algebra $A/J(A)^2$; in particular, $HH^1(A/J(A)^2)$ is abelian. Thus the kernel of the canonical Lie algebra homomorphism $\mathcal{L} = HH^1(A) \to HH^1(A/J(A)^2)$ contains $\mathcal{L}'$. Proposition 2.9 implies that $\mathcal{L}'$ is contained in the image of D_2, hence nilpotent of nilpotency class at most $\ell\ell(A) - 2$ by Proposition 2.8. From the same proposition we have that if $f \in \mathcal{L}^{(n)}$, then $f(J(A)) \subseteq J(A)^{2^{n-1}+1}$ for $n \geq 1$. Therefore the derived length is at most $\log_2(\ell\ell(A) - 1) + 1$. Since $\mathcal{L}'$ is nilpotent, it follows that if k is algebraically closed, then $\mathcal{L}$ is completely solvable. □

Proof of Theorem 1.2. By Lemma 2.6, every derivation $f : A \to A$ preserves $J(A)$, and hence sends $J(A)^2$ to $J(A)^2$. Thus the canonical map $A \to A/J(A)^2$ induces a Lie algebra homomorphism $\varphi : \mathrm{Der}(A) \to \mathrm{Der}(A/J(A)^2)$ which in turn induces a Lie algebra homomorphism $\Phi : HH^1(A) \to HH^1(A/J(A)^2)$. By Proposition 2.9, $\ker(\Phi)$ is a nilpotent ideal. If $\mathrm{char}(k) = 2$, then $HH^1(A/J(A)^2)$ is solvable by [10, Corollary 4.12], and hence $HH^1(A)$ is solvable. Suppose now that $HH^1(A)$ is not solvable. Then, by the above, we have $\mathrm{char}(k) \neq 2$. Then, by [10, Corollary 4.11, Remark 4.16], the Lie algebra $HH^1(A/J(A)^2)$ is a finite direct product of copies of $\mathfrak{sl}_2(k)$. Thus $HH^1(A)/\ker(\Phi)$ is a subalgebra of a finite direct product of copies of $\mathfrak{sl}_2(k)$, and hence $HH^1(A)/\mathrm{rad}(HH^1(A))$ is a subquotient of a finite direct product of copies of $\mathfrak{sl}_2(k)$. Since any proper Lie subalgebra of $\mathfrak{sl}_2(k)$ is solvable, it follows easily that the semisimple Lie algebra $HH^1(A)/\mathrm{rad}(HH^1(A))$ is a finite direct product of copies of $\mathfrak{sl}_2(k)$. □

4. Schur functors and proof of Theorem 1.3

The hypothesis $J(iAi)^2 = iJ(A)^2i$ in the statement of Theorem 1.3 means that for any primitive idempotent j not conjugate to i in A we have $iAjAi \subseteq J(iAi)^2$; that is, the image in iAi of any path parallel to the loop at i which is different from that loop is contained in $J(iAi)^2$. We start by collecting some elementary observations which will be used in the proof of Theorem 1.3.

Lemma 4.1. *Let A be a k-algebra and e an idempotent in A. Let $f : A \to A$ be a derivation. The following hold.*

(i) *We have $f(AeA) \subseteq AeA$.*
(ii) *We have $ef(e)e = 0$.*
(iii) *We have $(1 - e)f(e)(1 - e) = 0$.*
(iv) *We have $f(e) \in eA(1 - e) \oplus (1 - e)Ae$.*
(v) *We have $f(e) = [[f(e), e]], e]$; equivalently, the derivation $f - [[f(e), e], -]$ vanishes at e.*
(vi) *If $f(e) = 0$, then for any $a \in A$ we have $f(eae) = ef(a)e$; in particular, $f(eAe) \subseteq eAe$ and f induces a derivation on eAe.*
(vii) *If $f(e) = 0$ and if f is an inner derivation on A, then f restricts to an inner derivation on eAe.*

Proof. Let a, $b \in A$. Then $aeb = aeeb$, hence $f(aeb) = aef(eb) + f(ae)eb \in AeA$, implying the first statement. We have $f(e) = f(e^2) = f(e)e + ef(e)$. Right multiplication of this equation by e yields $f(e)e = f(e)e + ef(e)e$, whence the second statement. Right and left multiplication of the same equation by $1-e$ yields the third statement. Statement (iv) follows from combining the statements (ii) and (iii). We have $[[f(e), e], e] = [f(e)e - ef(e), e]$. Using that $ef(e)e = 0$ this is equal to $f(e)e + ef(e) = f(e)$, since f is a derivation. This shows (v). Suppose that $f(e) = 0$. Let $a \in A$. Then $f(eae) = f(e)ae + ef(a)e + eaf(e) = ef(a)e$, whence (vi). If in addition $f = [c, -]$ for some $c \in A$, then the hypothesis $f(e) = 0$ implies that $ec = ce$, and hence (vi) implies that the restriction of f to eAe is equal to the inner derivation $[ce, -]$. This completes the proof of the Lemma. □

Proposition 4.2. *Let A be a k-algebra, and let e be an idempotent in A. For any derivation f on A satisfying $f(e) = 0$ denote by $\varphi(f)$ the derivation on eAe sending eae to $ef(a)e$, for all $a \in A$. The correspondence $f \mapsto \varphi(f)$ induces a Lie algebra homormophism $HH^1(A) \to HH^1(eAe)$. If A is an algebra over a field of prime characteristic p, then this map is a homomorphism of p-restricted Lie algebras.*

Proof. Let f be an arbitrary derivation on A. By Lemma 4.1 (v), the derivation $f - [[f(e), e], -]$ vanishes at e. Thus every class in $HH^1(A)$ has a representative in $\mathrm{Der}(A)$ which vanishes at e. By Lemma 4.1 (vi), any derivation on A which vanishes at e restricts to a derivation on eAe, and by Lemma 4.1 (vii), this restriction sends inner derivations on A to inner derivations on eAe, hence induces a map $HH^1(A) \to HH^1(eAe)$. A trivial verification shows that if f, g are two derivations on A which vanish at e, then so does $[f, g]$, and an easy calculation shows that therefore the above map $HH^1(A) \to HH^1(eAe)$ is a Lie algebra homomorphism. If A is an algebra over a field of characteristic $p > 0$, and if f is a derivation on A which vanishes at e, then the derivation f^p vanishes on e and the restriction to eAe commutes with taking p-th powers by Lemma 4.1 (vi). This shows the last statement. □

We call the Lie algebra homomorphism $HH^1(A) \to HH^1(eAe)$ in Proposition 4.2 the *canonical Lie algebra homomorphism* induced by the Schur functor given by multiplication with the idempotent e.

For A a finite-dimensional k-algebra and m a positive integer, denote by $HH^1_{(m)}(A)$ the subspace of $HH^1(A)$ of classes which have a representative $f \in \mathrm{Der}(A)$ satisfying $f(J(A)) \subseteq J(A)^m$.

Proposition 4.3. *Let A be a split finite-dimensional k-algebra. Let i be a primitive idempotent in A. Set $S = Ai/J(A)i$. Suppose that $\mathrm{Ext}^1_A(S, S) = 0$. Then the image of the canonical map $HH^1(A) \to HH^1(iAi)$ is contained in $HH^1_{(1)}(iAi)$.*

Proof. By Lemma 2.3 we have $iJ(A)i = iJ(A)^2i$. By Lemma 4.1 (v), any class in $HH^1(A)$ is represented by a derivation f satisfying $f(i) = 0$. Thus if $a \in J(A)$, then $iai = ibci$ for some b, $c \in J(A)$, and hence $f(iai) = if(b)ci + ibf(c)i \in iJ(A)i$. □

Proposition 4.4. *Let A be a split symmetric k-algebra. Let i be a primitive idempotent in A. Set $S = Ai/J(A)i$. Suppose that $\mathrm{Ext}^1_A(S, S) \neq 0$. Then the canonical Lie algebra homomorphism $HH^1(A) \to HH^1(iAi)$ is nonzero.*

Proof. Set $S^\vee = iA/iJ(A)$. Choose a maximal semisimple subalgebra E of A. Since $\mathrm{Ext}^1_A(S, S)$ is nonzero, it follows from Lemma 2.2 that $J(A)/J(A)^2$ has a direct summand isomorphic to

$S \otimes_k S^\vee$ as an A-A-bimodule. Since A is symmetric, we have $\mathrm{soc}(A) \cong A/J(A)$, and hence $\mathrm{soc}(A)$ has a bimodule summand isomorphic to $S \otimes_k S^\vee$. Thus there is a bimodule homomorphism $J(A)/J(A)^2 \to \mathrm{soc}(A)$ with image isomorphic to $S \otimes_k S^\vee$. Composing with the canonical map $J(A) \to J(A)/J(A)^2$ yields a bimodule homomorphism $f : J(A) \to \mathrm{soc}(A)$ with kernel containing $J(A)^2$ and with image isomorphic to $S \otimes_k S^\vee$. Extending f by zero on E yields a derivation $\hat{f}$ on A, by Lemma 2.4. Restricting $\hat{f}$ to $iJ(A)i$ sends $iJ(A)i$ to a nonzero subspace of $\mathrm{soc}(A)$ isomorphic to $iS \otimes_k S^\vee i$, hence onto $\mathrm{soc}(iAi)$. Thus the image of $\hat{f}$ under the canonical map $\mathrm{Der}(A) \to \mathrm{Der}(iAi)$ from Proposition 4.2 is a nonzero derivation with kernel containing $ki + J(iAi)^2$ and image in $\mathrm{soc}(iAi)$. By [2, Corollary 3.2], the class in $HH^1(iAi)$ of this derivation is nonzero, whence the result. □

Proposition 4.5. *Let p be an odd prime and suppose that k is algebraically closed of characteristic p. Set $W = \mathrm{Der}(k[x]/(x^p))$. For $-1 \leq i \leq p-2$ let f_i be the derivation of $k[x]/(x^p)$ sending x to x^{i+1}, where we identify x with its image in $k[x]/(x^p)$. Let L be a simple Lie subalgebra of W. Then either $L = W$, or $L \cong \mathfrak{sl}_2(k)$.*

Proof. Note that the subalgebra S of W spanned by the f_i with $0 \leq i \leq p-2$ is solvable. Thus L is not contained in S. Note further that $\dim_k(L) \geq 3$. Therefore there exist derivations

$$f = \sum_{i=-1}^{p-2} \lambda_i f_i$$

$$g = \sum_{i=t}^{p-2} \mu_i f_i$$

belonging to L with $\lambda_{-1} = 1$, and $\mu_t = 1$, where t is an integer such that $0 \leq t \leq p-2$. Choose g such that t is minimal with this property. But then $[f, g]$ belongs to L. Since $[f_{-1}, f_t] = (t+1)f_{t-1}$, the minimality of $t \geq 0$ forces $t = 0$; that is we have

$$g = \sum_{i=0}^{p-2} \mu_i f_i$$

and $\mu_0 = 1$. Since $\dim_k(L) \geq 3$, it follows that there is a third element h in L not in the span of f, g, and hence, after modifying h by a linear combination of f and g, we can choose h such that

$$h = \sum_{i=s}^{p-2} \nu_i f_i$$

for some s such that $1 \leq s \leq p-2$ and $\nu_s = 1$. Choose h such that s is minimal with this property. Again by considering $[f, h]$, one sees that the minimality of s forces $s = 1$. If L is 3-dimensional, then $L \cong \mathfrak{sl}_2(k)$, where we use that k is algebraically closed. If $\dim_k(L) \geq 4$, then L contains an element of the form

$$u = \sum_{i=r}^{p-2} \tau_i f_i$$

with $2 \leq r \leq p-2$ and $\tau_r = 1$. But then applying $[f, -]$ and $[h, -]$ repeatedly to u shows that L contains a basis of W, hence $L = W$. □

Remark 4.6. Note that if $\mathrm{char}(k) = p > 2$, then the Witt Lie algebra W contains indeed a subalgebra isomorphic to $\mathfrak{sl}_2(k)$. Let $\mathfrak{f}, \mathfrak{e}, \mathfrak{h}$ be elements of the basis of $\mathfrak{sl}_2(k)$ such that $[\mathfrak{e}, \mathfrak{f}] = \mathfrak{h}$, $[\mathfrak{h}, \mathfrak{f}] = -2\mathfrak{f}$, and $[\mathfrak{h}, \mathfrak{e}] = 2\mathfrak{e}$. Then we have a Lie algebra isomorphism $\mathfrak{sl}_2(k) \cong \langle f_{-1}, f_0, f_1\rangle$ sending $\mathfrak{f}$ to f_{-1}, $\mathfrak{h}$ to $2f_0$, and $\mathfrak{e}$ to $-f_1$.

Proof of Theorem 1.3. We use the notation and hypotheses of the notation in Theorem 1.3, and we assume that the Lie algebra $HH^1(A)$ is simple. We show that this forces $HH^1(A)$ to be a Lie subalgebra of the Witt Lie algebra W with $\mathrm{char}(k) = p > 2$, and then the result follows from Proposition 4.5.

Since $HH^1(A)$ is simple and since $\mathrm{Ext}^1_A(S, S)$ is nonzero, it follows from Proposition 4.4 that the canonical Lie algebra homomorphism $\Phi : HH^1(A) \to HH^1(iAi)$ from Proposition 4.2 is injective. By the assumptions, iAi is a local algebra whose quiver has only one loop. Therefore $A \cong k[x]/(v)$ for some polynomial $v \in k[x]$ of degree at least 1. Since k is algebraically closed, v is a product of powers of linear polynomials, say $\prod_i (x - \beta_i)^{n_i}$, with pairwise distinct β_i and positive integers n_i. Therefore $HH^1(iAi)$ is a direct product of the Lie algebras corresponding to these factors. It follows that $HH^1(A)$ is isomorphic to a Lie subalgebra of $HH^1(k[x]/((x - \beta)^n))$ for some positive integer n. After applying the automorphism $x \mapsto x + \beta$ of $k[x]$ we have that $HH^1(A)$ is isomorphic to a Lie subalgebra of $HH^1(k[x]/(x^n))$ for some positive integer n. If $\mathrm{char}(k) = p$ does not divide n or if $\mathrm{char}(k) = 0$, then the linear map sending x to 1 is not a derivation on $k[x]/(x^n)$, and therefore $HH^1(k[x]/(x^n))$ is solvable in that case. Since Lie subalgebras of solvable Lie algebras are solvable, this contradicts the fact that $HH^1(A)$ is simple. Thus we have $\mathrm{char}(k) = p > 0$ and $n = pm$ for some positive integer m. Since $\mathrm{char}(k) = p$, it follows that the canonical surjection $k[x]/(x^n) \to k[x]/(x^p)$ induces a Lie algebra homomorphism $HH^1(k[x]/(x^n)) \to W = HH^1(k[x]/(x^p))$ with a nilpotent kernel. Thus $HH^1(A)$ is not containd in that kernel, and hence $HH^1(A)$ is isomorphic to a Lie subalgebra of W. The result follows. □

To conclude this section we note that although it is not clear which simple Lie algebras might occur as $HH^1(A)$ when $\mathrm{Ext}^1_A(S, S) = 0$ for all simple A-modules S, it easy to show that $HH^*(A)$ is not a simple graded Lie algebra (with respect to the Gerstenhaber bracket).

Proposition 4.7. *Let A be a finite dimensional k-algebra, and assume that for every simple A-module S we have $\mathrm{Ext}^1_A(S, S) = 0$. Then $HH^*(A)$ is not a perfect graded Lie algebra. In particular, HH^* is not simple.*

Proof. If $f \in C^1(A, A) := \mathrm{Hom}_k(A, A)$ and if $g \in C^0(A, A) := \mathrm{Hom}_k(k, A)$, then the Gerstenhaber bracket is given by $[f, g] = f(g)$, i.e. simply evaluating f in g. Note that $1 \in Z(A) = HH^0(A)$. By Lemma 2.4 and Lemma 2.6, f preserves $J(A)$ and we may assume $E \subseteq \ker(f)$. Therefore the derived Lie subalgebra of $HH^*(A)$ does not contain 1_A. □

Remark 4.8. Lemma 4.1 and Proposition 4.2 hold for algebras over an arbitrary commutative ring instead of k.

5. Examples

Theorem 1.1 applies to certain blocks of symmetric groups.

Proposition 5.1. *Suppose that k is a field of prime characteristic p. Let A be a defect 2 block of a symmetric group algebra kS_n or the principal block of kS_{3p}. Then $HH^1(A)$ is a solvable Lie algebra.*

Proof. From [9, Theorem 1] and from [7, Theorem 5.1] we have that the simple modules do not self-extend and the Ext^1-space between two simple modules is at most one-dimensional. The statement follows from Theorem 1.1. □

Remark 5.2. A conjecture by Kleshchev and Martin predicts that simple kS_n-modules in odd characteristic do not admit self-extensions.

Proposition 5.3. *Let A be a tame symmetric k-algebra with 3 isomorphism classes of simple modules of type $3\mathcal{A}$ or $3\mathcal{K}$. Then $\mathrm{HH}^1(A)$ is a solvable Lie algebra.*

Proof. From the list at the end of Erdmann's book [5] we have that the simple modules in these cases do not self-extend and that the Ext^1-space between two simple modules is at most one-dimensional. The statement follows from Theorem 1.1. □

As mentioned in the introduction, the above Proposition is part of more general results on tame algebras in [4] and [8]. We note some other examples of algebras whose simple modules do not have nontrivial self-extensions.

Theorem 5.4 ([1, Theorem 3.4])**.** *Let G be a connected semisimple algebraic group defined and split over the field $\mathbb{F}_p$ with p elements, and k be an algebraic closure of $\mathbb{F}_p$. Assume G is almost simple and simply connected and let $G(\mathbb{F}_q)$ be the finite Chevalley group consisting of $\mathbb{F}_q$-rational points of G where $q = p^r$ for a non-negative integer r. Let h be the Coxeter number of G. For $r \geq 2$ and $p \geq 3(h-1)$, we have $\mathrm{Ext}^1_{kG(\mathbb{F}_q)}(S, S) = 0$ for every simple $kG(\mathbb{F}_q)$-module S.*

Remark 5.5. Let G be a simple algebraic group over a field of characteristic $p > 3$, not of type A_1, G_2 and F_4. Proposition 1.4 in [11] implies that not having self-extensions does not allow to lift to characteristic zero certain simple modular representations. Therefore, for these cases the Lie structure of HH^1 plays a central role.

In the context of blocks with abelian defect groups one expects (by Broué's abelian defect conjecture) every block of a finite group algebra with an abelian defect group P to be derived equivalent to a twisted group algebra of the form $k_\alpha(P \rtimes E)$, where E is the inertial quotient of the block and where α is a class in $H^2(E; k^\times)$, inflated to $P \rtimes E$ via the canonical surjection $P \rtimes E \to E$. Thus the following observation is relevant in cases where Broué's abelian defect conjecture is known to hold (this includes blocks with cyclic and Klein four defect).

Proposition 5.6. *Suppose that k be a field of prime characteristic p. Let P be a finite p-group and E an abelian p'-subgroup of $\mathrm{Aut}(P)$ such that $[P, E] = P$. Set $A = k(P \rtimes E)$. Suppose that k is large enough for E, or equivalently, that A is split. For any simple A-module S we have $\mathrm{Ext}^1_A(S, S) = 0$.*

Proof. Since E is abelian, it follows that $\dim_k(S) = 1$, and hence that $S \otimes_k -$ is a Morita equivalence. This Morita equivalence sends the trivial A-module k to S, hence induces an isomorphism $\mathrm{Ext}^1_A(k, k) \cong \mathrm{Ext}^1_A(S, S)$. It suffices therefore to show the statement for k instead of S. That is, we need to show that $H^1(P \rtimes E; k) = 0$, or equivalently, that there is no nonzero group homomorphism from $P \rtimes E$ to the additive group k. Since $[P, E] = P$, it follows that every abelian quotient of $P \rtimes E$ is isomorphic to a quotient of E, hence has order prime to p. The result follows. □

Example 5.7. If B is a block of a finite group algebra over an algebraically closed field k of characteristic $p > 0$ with a nontrivial cyclic defect group P and nontrivial inertial quotient E, then $HH^1(B)$ is a solvable Lie algebra, isomorphic to $HH^1(kP)^E$, where E acts on $HH^1(kP)$ via the group action of E on P. Indeed, B is derived equivalent to the Nakayama algebra $k(P \rtimes E)$, which satisfies the hypotheses of Theorem 1.1 (thanks to the assumption $E \neq 1$, which implies $[P, E] = P$). Note that kP is isomorphic to the truncated polynomial algebra $k[x]/(x^{p^d})$, where $p^d = |P|$.

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