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ON THE LIE ALGEBRA STRUCTURE OF $HH^1(A)$ OF A FINITE-DIMENSIONAL ALGEBRA A

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ABSTRACT. Let A be a split finite-dimensional associative unital algebra over a field. The first main result of this note shows that if the Ext-quiver of A is a simple directed graph, then $HH^1(A)$ is a solvable Lie algebra. The second main result shows that if the Ext-quiver of A has no loops and at most two parallel arrows in any direction, and if $HH^1(A)$ is a simple Lie algebra, then $\text{char}(k) \neq 2$ and $HH^1(A) \cong \mathfrak{sl}_2(k)$. The third result investigates symmetric algebras with a quiver which has a vertex with a single loop.

1. INTRODUCTION

Let k be a field. Our first result is a sufficient criterion for $HH^1(A)$ to be a solvable Lie algebra, where A is a split finite-dimensional k -algebra (where the term ‘algebra’ without any further specifications means an associative and unital algebra).

Theorem 1.1. *Let A be a split finite-dimensional k -algebra. Suppose that the Ext-quiver of A is a simple directed graph. Then the derived Lie subalgebra of $HH^1(A)$ is nilpotent; in particular the Lie algebra $HH^1(A)$ is solvable.*

The recent papers [4] and [8] contain comprehensive results regarding the solvability of $HH^1(A)$ of tame algebras and blocks, and [8] also contains a proof of Theorem 1.1 with different methods. We will prove Theorem 1.1 in Section 3 as part of the more precise Theorem 3.1, bounding the derived length of the Lie algebra $HH^1(A)$ and the nilpotency class of the derived Lie subalgebra of $HH^1(A)$ in terms of the Loewy length $\ell(A)$ of A . The hypothesis on the quiver of A is equivalent to requiring that $\text{Ext}_A^1(S, S) = 0$ for any simple A -module S and $\dim_k(\text{Ext}_A^1(S, T)) \leq 1$ for any two simple A -modules S, T . If in addition A is monomial, then Theorem 1.1 follows from work of Strametz [10]. The hypotheses on A are not necessary for the derived Lie subalgebra of $HH^1(A)$ to be nilpotent or for $HH^1(A)$ to be solvable; see [2, Theorem 1.1] or [8] for examples.

The Lie algebra structure of $HH^1(A)$ is invariant under derived equivalences, and for symmetric algebras, also invariant under stable equivalences of Morita type. Therefore, the conclusions of Theorem 1.1 remain true for any finite-dimensional k -algebra B which is derived equivalent to an algebra A satisfying the hypotheses of this theorem, or for a symmetric k -algebra B which is stably equivalent of Morita type to a symmetric algebra A satisfying the hypotheses of the theorem.

If we allow up to two parallel arrows in the same direction in the quiver of A but no loops, then it is possible for $HH^1(A)$ to be simple as a Lie algebra. The only simple Lie algebra to arise in that case is $\mathfrak{sl}_2(k)$, with $\text{char}(k) \neq 2$.

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Theorem 1.2. *Let A be a split finite-dimensional k -algebra. Suppose that $\text{Ext}_A^1(S, S) = 0$ for any simple A -module S and that $\dim_k(\text{Ext}_A^1(S, T)) \leq 2$ for any two simple A -modules S, T . If $HH^1(A)$ is not solvable, then $\text{char}(k) \neq 2$ and $HH^1(A)/\text{rad}(HH^1(A))$ is a direct product of finitely many copies of $\mathfrak{sl}_2(k)$. In particular, the following hold.*

- (i) *If $HH^1(A)$ is a simple Lie algebra, then $\text{char}(k) \neq 2$, and $HH^1(A) \cong \mathfrak{sl}_2(k)$.*
- (ii) *If $\text{char}(k) = 2$, then $HH^1(A)$ is a solvable Lie algebra.*

This will be proved in Section 3; for monomial algebras this follows as before from Strametz [10]. An example of an algebra A satisfying the hypotheses of this theorem is the Kronecker algebra, a 4-dimensional k -algebra, with $\text{char}(k) \neq 2$, given by the directed quiver with two vertices e_0, e_1 and two parallel arrows α, β from e_0 to e_1 . This example is a special case of more general results on monomial algebras; see in particular [10, Corollary 4.17]. As in the case of the previous Theorem, the conclusions of Theorem 1.2 remain true for an algebra B which is derived equivalent to an algebra A satisfying the hypotheses of this theorem, or for a symmetric algebra B which is stably equivalent of Morita type to a symmetric algebra A satisfying the hypotheses of the theorem.

We have the following partial result for symmetric algebras whose quiver has a single loop at some vertex.

Theorem 1.3. *Suppose that k is algebraically closed. Let A be a finite-dimensional symmetric k -algebra, and let S be a simple A -module. Suppose that $\dim_k(\text{Ext}_A^1(S, S)) = 1$ and that for any primitive idempotent i in A satisfying $iS \neq 0$ we have $J(iAi)^2 = iJ(A)^2i$. If $HH^1(A)$ is a simple Lie algebra, then $\text{char}(k) = p > 2$ and $HH^1(A)$ is isomorphic to either $\mathfrak{sl}_2(k)$ or the Witt Lie algebra $W = \text{Der}(k[x]/(x^p))$.*

This will be proved in Section 4, along with some general observations regarding the compatibility of Schur functors and the Lie algebra structure of $HH^1(A)$. Section 5 contains some examples.

2. ON DERIVATIONS AND THE RADICAL

We start with a brief review of some basic terminology. The *nilpotency class* of a nilpotent Lie algebra \mathcal{L} is the smallest positive integer m such that $\mathcal{L}^m = 0$, where $\mathcal{L}^1 = \mathcal{L}$ and $\mathcal{L}^{m+1} = [\mathcal{L}, \mathcal{L}^m]$ for $m \geq 1$. In addition, the *derived length* of a solvable Lie algebra is the smallest positive integer n such that $\mathcal{L}^{(n)} = 0$, where $\mathcal{L}^{(1)} = \mathcal{L}$ and $\mathcal{L}^{(n+1)} = [\mathcal{L}^{(n)}, \mathcal{L}^{(n)}]$ for $n \geq 1$. A Lie algebra \mathcal{L} is called *strongly solvable* if its derived subalgebra is nilpotent. A Lie algebra \mathcal{L} of finite dimension n is called *completely solvable* (also called *supersolvable*) if there exists a sequence of ideals $\mathcal{L}_1 = \mathcal{L} \supset \mathcal{L}_2 \supset \cdots \supset \mathcal{L}_n \supset 0$ such that $\dim_k(\mathcal{L}_i) = n + 1 - i$ for $1 \leq i \leq n$.

Remark 2.1. If k is algebraically closed of characteristic zero, then, as a consequence of Lie's theorem, the classes of strongly and completely solvable Lie algebras coincide with the class of solvable Lie algebras. Lie's theorem does not hold in positive characteristic. If k is algebraically closed of prime characteristic p , then by [3, Theorem 3], a finite-dimensional Lie algebra \mathcal{L} over k is strongly solvable if and only if \mathcal{L} is completely solvable.

Let A be a finite-dimensional k -algebra. We denote by $\ell(A)$ the number of isomorphism classes of simple A -modules. The *Loewy length* $\ell(A)$ of A is the smallest positive integer m such that $J(A)^m = 0$, where $J(A)$ denotes the Jacobson radical of A . We denote by $[A, A]$ the k -subspace of A generated by the set of additive commutators $ab - ba$, where $a, b \in A$. A *derivation on A* is a k -linear map $f : A \rightarrow A$ satisfying $f(ab) = f(a)b + af(b)$ for all $a, b \in A$. If f, g are derivations

on A , then so is $[f, g] = f \circ g - g \circ f$, and the space $\text{Der}(A)$ of derivations on A becomes a Lie algebra in this way. If $c \in A$, then the map $[c, -]$ defined by $[c, a] = ca - ac$ is a derivation; any derivation of this form is called an *inner derivation*. The space $\text{IDer}(A)$ of inner derivations is a Lie ideal in $\text{Der}(A)$, and we have a canonical isomorphism $HH^1(A) \cong \text{Der}(A)/\text{IDer}(A)$; see [12, Lemma 9.2.1]. It is easy to see that any derivation on A preserves the subspace $[A, A]$, and that any inner derivation of A preserves any ideal in A . A finite-dimensional k -algebra A is called *split* if $\text{End}_A(S) \cong k$ for every simple A -module S . If A is split, then by the Wedderburn-Malcev Theorem, A has a separable subalgebra E such that $A = E \oplus J(A)$. Moreover, E is unique up to conjugation by elements in the group A^\times of invertible elements in A . A primitive decomposition I of 1 in E remains a primitive decomposition of 1 in A .

For convenience, we mention the following well-known descriptions of certain Ext^1 -spaces.

Lemma 2.2. *Let A be a split finite-dimensional k -algebra, let i be a primitive idempotent in A . Set $S = Ai/J(A)i$ and $S^\vee = iA/iJ(A)$. We have k -linear isomorphisms*

$$HH^1(A; S \otimes_k S^\vee) \cong \text{Ext}_A^1(S, S) \cong \text{Hom}_A(J(A)i/J(A)^2i, S) \cong \text{Hom}_{A \otimes_k A^{\text{op}}}(J(A)/J(A)^2, S \otimes_k S^\vee).$$

Lemma 2.3. *Let A be a split finite-dimensional k -algebra. Let i be a primitive idempotent in A , and set $S = Ai/J(A)i$. We have $\text{Ext}_A^1(S, S) = 0$ if and only if $iJ(A)i \subseteq J(A)^2$.*

Proof. By Lemma 2.2, we have $\text{Ext}_A^1(S, S) = 0$ if and only if $J(A)/J(A)^2$ has no simple bimodule summand isomorphic to $S \otimes_k S^\vee$. This is equivalent to $i \cdot (J(A)/J(A)^2) \cdot i = 0$, hence to $iJ(A)i \subseteq J(A)^2$ as stated. \square

Lemma 2.4. *Let A be a split finite-dimensional k -algebra, and let E be a separable subalgebra of A such that $A = E \oplus J(A)$. Every class in $HH^1(A)$ has a representative $f \in \text{Der}(A)$ satisfying $E \subseteq \ker(f)$.*

Proof. Let $f : A \rightarrow A$ be a derivation. Since E is separable, it follows that for any E - E -bimodule M we have $HH^1(E; M) = 0$. In particular, the derivation $f|_E : E \rightarrow A$ is inner; that is, there is an element $c \in A$ such that $f(x) = [c, x]$ for all $x \in E$. Thus the derivation $f - [c, -]$ on A vanishes on E and represents the same class as f in $HH^1(A)$. \square

Lemma 2.5. *Let A be a split finite-dimensional k -algebra, and let E be a separable subalgebra of A such that $A = E \oplus J(A)$. Let $f : A \rightarrow A$ be a derivation such that $E \subseteq \ker(f)$. For any two idempotents i, j in E we have $f(iAj) \subseteq iAj$ and $f(AiAj) \subseteq AiAj$.*

Proof. Let i, j be idempotents in E , and let $a, b \in A$. We have $f(iaj) = f(i^2aj) = if(iaj) + f(i)iaj = if(iaj)$, since $i \in E \subseteq \ker(f)$. Thus $f(iaj) \in iA$. A similar argument shows that $f(iaj) \in Aj$, and hence $f(iaj) \in iAj$. This shows the first statement. The second statement follows from this and the equality $f(biaj) = f(b)iaj + bf(iaj)$. \square

Lemma 2.6. *Let A be a split finite-dimensional k -algebra such that $\text{Ext}_A^1(S, S) = 0$ for all simple A -modules S . Then for any derivation $f : A \rightarrow A$ we have $f(J(A)) \subseteq J(A)$.*

Proof. Let E be a separable subalgebra of A such that $A = E \oplus J(A)$. Let I be a primitive decomposition of 1 in E (hence also in A). Note that if $i, j \in I$ are not conjugate in A^\times , then $iAj \subseteq J(A)$. The hypotheses on A imply that $J(A)i/J(A)^2i$ has no summand isomorphic to $Ai/J(A)i$, and hence that $iJ(A)i \subseteq J(A)^2$ for any $i \in I$. Then $iJ(A)j \subseteq J(A)^2$ for any two $i, j \in I$ which are conjugate in A^\times . Let now $f : A \rightarrow A$ be a derivation. As noted above,

any inner derivation preserves $J(A)$. Thus, by Lemma 2.4, we may assume that $f|_E = 0$. Since $J(A) = \bigoplus_{i \in I} J(A)i$, it suffices to show that $f(J(A)i) \subseteq J(A)i$, where $i \in I$. If j is conjugate to i , then $AjJ(A)i \subseteq J(A)^2i$. Since $J(A)i = \sum_{j \in I} AjJ(A)i$, it follows from Nakayama's Lemma that $J(A)i = \sum_j AjAi$, where j runs over the subset I' of all j in I which are not conjugate to i . Now f preserves the submodules $AjAi$ in this sum, thanks to Lemma 2.5. The result follows. \square

Definition 2.7. Let A be a split finite-dimensional k -algebra, and let E be a separable subalgebra of A such that $A = E \oplus J(A)$. For $m \geq 1$, denote by D_m the subspace of $\text{Der}(A)$ consisting of all derivations $f : A \rightarrow A$ such that $E \subseteq \ker(f)$ and such that $f(J(A)) \subseteq J(A)^m$.

The following observations are variations of the statements in [6, Proposition 3.5].

Proposition 2.8. *Let A be a split finite-dimensional k -algebra, and let E be a separable subalgebra of A such that $A = E \oplus J(A)$. The following hold.*

- (i) *For any positive integers m, n we have $[D_m, D_n] \subseteq D_{m+n-1}$.*
- (ii) *The space D_1 is a Lie subalgebra of $\text{Der}(A)$, and for any positive integer m , the space D_m is a Lie ideal in D_1 .*
- (iii) *The space D_2 is a nilpotent ideal in D_1 . More precisely, if $\ell(A) \leq 2$, then $D_2 = 0$, and if $\ell(A) > 2$, then the nilpotency class of D_2 is at most $\ell(A) - 2$.*

Proof. The space of derivations on A which vanish on E is easily seen to be closed under the Lie bracket on $\text{Der}(A)$. Thus statement (i) follows from [6, Lemma 3.4]. Statement (ii) is an immediate consequence of (i). If $m \geq \ell(A)$, then $J(A)^m = 0$, and hence $D_m = 0$. Together with (i), this implies (iii). \square

Proposition 2.9. *Let A be a split finite-dimensional k -algebra, and let E be a separable subalgebra of A such that $A = E \oplus J(A)$. Suppose that every derivation f on A satisfies $f(J(A)) \subseteq J(A)$. Then the canonical algebra homomorphism $A \rightarrow A/J(A)^2$ induces a Lie algebra homomorphism $\Phi : HH^1(A) \rightarrow HH^1(A/J(A)^2)$. The following hold.*

- (i) *The canonical surjection $\text{Der}(A) \rightarrow HH^1(A)$ maps D_1 onto $HH^1(A)$.*
- (ii) *The canonical surjection $\text{Der}(A) \rightarrow HH^1(A)$ maps D_2 onto $\ker(\Phi)$; in particular, $\ker(\Phi)$ is a nilpotent ideal in the Lie algebra $HH^1(A)$.*
- (iii) *The Lie algebra $HH^1(A)$ is solvable if and only if $HH^1(A)/\ker(\Phi)$ is solvable.*
- (iv) *If the derived Lie algebra of $HH^1(A)$ is contained in $\ker(\Phi)$, then $HH^1(A)$ is nilpotent.*
- (v) *If the Lie algebra $HH^1(A)$ is simple, then Φ is injective.*

Proof. The hypotheses on $\text{Der}(A)$ together with Lemma 2.4 imply that $HH^1(A)$ is equal to the image of the space D_1 in $HH^1(A)$, whence (i). The canonical surjection $\text{Der}(A) \rightarrow HH^1(A)$ clearly maps D_2 to $\ker(\Phi)$; we need to show the surjectivity of the induced map $D_2 \rightarrow \ker(\Phi)$. Note first that any inner derivation in D_1 is of the form $[c, -]$ for some c which centralises E . Note further that the centraliser $C_A(E)$ of E in A is canonically isomorphic to $\text{Hom}_{E \otimes_k E^{\text{op}}}(E, A)$ (via the map sending an E - E -bimodule homomorphism $\alpha : E \rightarrow A$ to $\alpha(1)$). Since E is separable, hence projective as an E - E -bimodule, it follows that the functor $\text{Hom}_{E \otimes_k E^{\text{op}}}(E, -)$ is exact. In particular, the surjection $A \rightarrow A/J(A)^2$ induces a surjection $C_A(E) \rightarrow C_{A/J(A)^2}(E)$, where we identify E with its image in $A/J(A)^2$. Let $f \in D_1$ such that the class of f is in $\ker(\Phi)$, or equivalently, such that the induced derivation, denoted \bar{f} , on $A/J(A)^2$ is inner. Then there is $c \in A$ such that $\bar{f} = [\bar{c}, -]$, where $\bar{c} = c + J(A)^2$ centralises the image of E in $A/J(A)^2$. By the above,

we may choose c such that c centralises E in A . Then the derivation $f - [c, -]$ represents the same class as f , still belongs to D_1 , and induces the zero map on $A/J(A)^2$. Thus $f - [c, -]$ belongs in fact to D_2 , proving (ii). The remaining statements are immediate consequences of (ii). \square

The next result includes the special case of Theorem 1.1 where $\ell(A) \leq 2$.

Proposition 2.10. *Let A be a split finite-dimensional k -algebra such that $J(A)^2 = 0$. Suppose that for every simple A -module S we have $\text{Ext}_A^1(S, S) = 0$ and that for any two simple A -modules S, T we have $\dim_k(\text{Ext}_A^1(S, T)) \leq 1$. Let E be a separable subalgebra of A such that $A = E \oplus J(A)$. The following hold.*

- (i) *If A is basic and if f, g are derivations on A which vanish on E , then $[f, g] = 0$.*
- (ii) *The Lie algebra $HH^1(A)$ is abelian.*
- (iii) *Suppose that A is indecomposable as an algebra, and let $e(A)$ be the number of edges in the quiver of A . We have*

$$\dim_k(HH^1(A)) = e(A) - \ell(A) + 1 \leq (\ell(A) - 1)^2 .$$

Proof. In order to prove (i), suppose that A is basic. Let I be a primitive decomposition of 1 in A such that $E = \prod_{i \in I} ki$. Let f and g be derivations on A which vanish on E . Then f, g are determined by their restrictions to $J(A)$. By Lemma 2.6, the derivations f, g preserve $J(A)$. By the assumptions, each summand iAj in the vector space decomposition $A = \bigoplus_{i,j \in I} iAj$ has dimension at most one. By Lemma 2.5, any derivation on A which vanishes on E preserves this decomposition. Therefore, if X is a basis of $J(A)$ consisting of elements of the subspaces iAj , $i, j \in I$, which are nonzero, then $f|_{J(A)} : J(A) \rightarrow J(A)$ is represented by a diagonal matrix. Similarly for g . But then the restrictions of f and g to $J(A)$ commute. Since both f, g vanish on E , this implies that $[f, g] = 0$, whence (i). If A is basic, then clearly (i) and Lemma 2.4 together imply (ii). Since the hypotheses of the Lemma as well as the Lie algebra $HH^1(A)$ are invariant under Morita equivalences, statement (ii) follows for general A . In order to prove (iii), assume again that A is basic. By the assumptions, $e(A) = \dim_k(J(A)) = |X|$. One verifies that the extension to A by zero on I of any linear map on $J(A)$ which preserves the summands iAj (with $i \neq j$), or equivalently, which preserves the one-dimensional spaces kx , where $x \in X$, is in fact a derivation. By Lemma 2.4, any class in $HH^1(A)$ is represented by such a derivation. Thus the space of derivations on A which vanish on I is equal to $\dim_k(J(A)) = e(A)$. Each $i \in I$ contributes an inner derivation. Since A is indecomposable, it follows that the only k -linear combination of elements in I which belongs to $Z(A)$ are the scalar multiples of $1 = \sum_{i \in I} i$. Thus the space of inner derivations which annihilate I has dimension $\ell(A) - 1$, whence the first equality. Since there are at most $\ell(A) - 1$ arrows starting at any given vertex, it follows that $e(A) \leq (\ell(A) - 1)\ell(A)$, whence the inequality as stated. \square

The above Proposition can also be proved as a consequence of more general work of Strametz [10], calculating the Lie algebra $HH^1(A)$ for A a split finite-dimensional monomial algebra.

3. PROOFS OF THEOREMS 1.1 AND 1.2

Theorem 1.1 is a part of the following slightly more precise result.

Theorem 3.1. *Let A be a split finite-dimensional k -algebra. Suppose that for every simple A -module S we have $\text{Ext}_A^1(S, S) = 0$ and that for any two simple A -modules S, T we have $\dim_k(\text{Ext}_A^1(S, T)) \leq 1$. Set $\mathcal{L} = HH^1(A)$, regarded as a Lie algebra.*

- (i) If $\ell(A) \leq 2$ then \mathcal{L} is abelian.
- (ii) If $\ell(A) > 2$, then the derived Lie algebra $\mathcal{L}' = [\mathcal{L}, \mathcal{L}]$ is nilpotent of nilpotency class at most $\ell(A) - 2$. The derived length of \mathcal{L} is at most $\log_2(\ell(A) - 1) + 1$.

In particular, \mathcal{L} is solvable, and if k is algebraically closed, then \mathcal{L} is completely solvable.

Proof. If $\ell(A) \leq 2$, then $J(A)^2 = 0$, and hence (i) follows from Proposition 2.10. Suppose that $\ell(A) > 2$. We may assume that A is basic. Note that A and $A/J(A)^2$ have the same Ext-quiver, and hence we may apply Proposition 2.10 to the algebra $A/J(A)^2$; in particular, $HH^1(A/J(A)^2)$ is abelian. Thus the kernel of the canonical Lie algebra homomorphism $\mathcal{L} = HH^1(A) \rightarrow HH^1(A/J(A)^2)$ contains \mathcal{L}' . Proposition 2.9 implies that \mathcal{L}' is contained in the image of D_2 , hence nilpotent of nilpotency class at most $\ell(A) - 2$ by Proposition 2.8. From the same proposition we have that if $f \in \mathcal{L}^{(n)}$, then $f(J(A)) \subseteq J(A)^{2^{n-1}+1}$ for $n \geq 1$. Therefore the derived length is at most $\log_2(\ell(A) - 1) + 1$. Since \mathcal{L}' is nilpotent, it follows that if k is algebraically closed, then \mathcal{L} is completely solvable. \square

Proof of Theorem 1.2. By Lemma 2.6, every derivation $f : A \rightarrow A$ preserves $J(A)$, and hence sends $J(A)^2$ to $J(A)^2$. Thus the canonical map $A \rightarrow A/J(A)^2$ induces a Lie algebra homomorphism $\varphi : \text{Der}(A) \rightarrow \text{Der}(A/J(A)^2)$ which in turn induces a Lie algebra homomorphism $\Phi : HH^1(A) \rightarrow HH^1(A/J(A)^2)$. By Proposition 2.9, $\ker(\Phi)$ is a nilpotent ideal. If $\text{char}(k) = 2$, then $HH^1(A/J(A)^2)$ is solvable by [10, Corollary 4.12], and hence $HH^1(A)$ is solvable. Suppose now that $HH^1(A)$ is not solvable. Then, by the above, we have $\text{char}(k) \neq 2$. Then, by [10, Corollary 4.11, Remark 4.16], the Lie algebra $HH^1(A/J(A)^2)$ is a finite direct product of copies of $\mathfrak{sl}_2(k)$. Thus $HH^1(A)/\ker(\Phi)$ is a subalgebra of a finite direct product of copies of $\mathfrak{sl}_2(k)$, and hence $HH^1(A)/\text{rad}(HH^1(A))$ is a subquotient of a finite direct product of copies of $\mathfrak{sl}_2(k)$. Since any proper Lie subalgebra of $\mathfrak{sl}_2(k)$ is solvable, it follows easily that the semisimple Lie algebra $HH^1(A)/\text{rad}(HH^1(A))$ is a finite direct product of copies of $\mathfrak{sl}_2(k)$. \square

4. SCHUR FUNCTORS AND PROOF OF THEOREM 1.3

The hypothesis $J(iAi)^2 = iJ(A)^2i$ in the statement of Theorem 1.3 means that for any primitive idempotent j not conjugate to i in A we have $iAjAi \subseteq J(iAi)^2$; that is, the image in iAi of any path parallel to the loop at i which is different from that loop is contained in $J(iAi)^2$. We start by collecting some elementary observations which will be used in the proof of Theorem 1.3.

Lemma 4.1. *Let A be a k -algebra and e an idempotent in A . Let $f : A \rightarrow A$ be a derivation. The following hold.*

- (i) We have $f(AeA) \subseteq AeA$.
- (ii) We have $ef(e)e = 0$.
- (iii) We have $(1 - e)f(e)(1 - e) = 0$.
- (iv) We have $f(e) \in eA(1 - e) \oplus (1 - e)Ae$.
- (v) We have $f(e) = [[f(e), e], e]$; equivalently, the derivation $f - [[f(e), e], -]$ vanishes at e .
- (vi) If $f(e) = 0$, then for any $a \in A$ we have $f(eae) = ef(a)e$; in particular, $f(Ae) \subseteq Ae$ and f induces a derivation on eAe .
- (vii) If $f(e) = 0$ and if f is an inner derivation on A , then f restricts to an inner derivation on eAe .

Proof. Let $a, b \in A$. Then $aeb = aebe$, hence $f(aeb) = aef(eb) + f(ae)eb \in AeA$, implying the first statement. We have $f(e) = f(e^2) = f(e)e + ef(e)$. Right multiplication of this equation by e yields $f(e)e = f(e)e + ef(e)e$, whence the second statement. Right and left multiplication of the same equation by $1 - e$ yields the third statement. Statement (iv) follows from combining the statements (ii) and (iii). We have $[[f(e), e], e] = [f(e)e - ef(e), e]$. Using that $ef(e)e = 0$ this is equal to $f(e)e + ef(e) = f(e)$, since f is a derivation. This shows (v). Suppose that $f(e) = 0$. Let $a \in A$. Then $f(eae) = f(e)ae + ef(a)e + eaf(e) = ef(a)e$, whence (vi). If in addition $f = [c, -]$ for some $c \in A$, then the hypothesis $f(e) = 0$ implies that $ec = ce$, and hence (vi) implies that the restriction of f to eAe is equal to the inner derivation $[ce, -]$. This completes the proof of the Lemma. \square

Proposition 4.2. *Let A be a k -algebra, and let e be an idempotent in A . For any derivation f on A satisfying $f(e) = 0$ denote by $\varphi(f)$ the derivation on eAe sending eae to $ef(a)e$, for all $a \in A$. The correspondence $f \mapsto \varphi(f)$ induces a Lie algebra homomorphism $HH^1(A) \rightarrow HH^1(eAe)$. If A is an algebra over a field of prime characteristic p , then this map is a homomorphism of p -restricted Lie algebras.*

Proof. Let f be an arbitrary derivation on A . By Lemma 4.1 (v), the derivation $f - [[f(e), e], -]$ vanishes at e . Thus every class in $HH^1(A)$ has a representative in $\text{Der}(A)$ which vanishes at e . By Lemma 4.1 (vi), any derivation on A which vanishes at e restricts to a derivation on eAe , and by Lemma 4.1 (vii), this restriction sends inner derivations on A to inner derivations on eAe , hence induces a map $HH^1(A) \rightarrow HH^1(eAe)$. A trivial verification shows that if f, g are two derivations on A which vanish at e , then so does $[f, g]$, and an easy calculation shows that therefore the above map $HH^1(A) \rightarrow HH^1(eAe)$ is a Lie algebra homomorphism. If A is an algebra over a field of characteristic $p > 0$, and if f is a derivation on A which vanishes at e , then the derivation f^p vanishes on e and the restriction to eAe commutes with taking p -th powers by Lemma 4.1 (vi). This shows the last statement. \square

We call the Lie algebra homomorphism $HH^1(A) \rightarrow HH^1(eAe)$ in Proposition 4.2 the *canonical Lie algebra homomorphism* induced by the Schur functor given by multiplication with the idempotent e .

For A a finite-dimensional k -algebra and m a positive integer, denote by $HH^1_{(m)}(A)$ the subspace of $HH^1(A)$ of classes which have a representative $f \in \text{Der}(A)$ satisfying $f(J(A)) \subseteq J(A)^m$.

Proposition 4.3. *Let A be a split finite-dimensional k -algebra. Let i be a primitive idempotent in A . Set $S = Ai/J(A)i$. Suppose that $\text{Ext}_A^1(S, S) = 0$. Then the image of the canonical map $HH^1(A) \rightarrow HH^1(iAi)$ is contained in $HH^1_{(1)}(iAi)$.*

Proof. By Lemma 2.3 we have $iJ(A)i = iJ(A)^2i$. By Lemma 4.1 (v), any class in $HH^1(A)$ is represented by a derivation f satisfying $f(i) = 0$. Thus if $a \in J(A)$, then $iai = ibci$ for some $b, c \in J(A)$, and hence $f(iai) = if(b)ci + ibf(c)i \in iJ(A)i$. \square

Proposition 4.4. *Let A be a split symmetric k -algebra. Let i be a primitive idempotent in A . Set $S = Ai/J(A)i$. Suppose that $\text{Ext}_A^1(S, S) \neq 0$. Then the canonical Lie algebra homomorphism $HH^1(A) \rightarrow HH^1(iAi)$ is nonzero.*

Proof. Set $S^\vee = iA/iJ(A)$. Choose a maximal semisimple subalgebra E of A . Since $\text{Ext}_A^1(S, S)$ is nonzero, it follows from Lemma 2.2 that $J(A)/J(A)^2$ has a direct summand isomorphic to

$S \otimes_k S^\vee$ as an A - A -bimodule. Since A is symmetric, we have $\text{soc}(A) \cong A/J(A)$, and hence $\text{soc}(A)$ has a bimodule summand isomorphic to $S \otimes_k S^\vee$. Thus there is a bimodule homomorphism $J(A)/J(A)^2 \rightarrow \text{soc}(A)$ with image isomorphic to $S \otimes_k S^\vee$. Composing with the canonical map $J(A) \rightarrow J(A)/J(A)^2$ yields a bimodule homomorphism $f : J(A) \rightarrow \text{soc}(A)$ with kernel containing $J(A)^2$ and with image isomorphic to $S \otimes_k S^\vee$. Extending f by zero on E yields a derivation \hat{f} on A , by Lemma 2.4. Restricting \hat{f} to $iJ(A)i$ sends $iJ(A)i$ to a nonzero subspace of $\text{soc}(A)$ isomorphic to $iS \otimes_k S^\vee i$, hence onto $\text{soc}(iAi)$. Thus the image of \hat{f} under the canonical map $\text{Der}(A) \rightarrow \text{Der}(iAi)$ from Proposition 4.2 is a nonzero derivation with kernel containing $ki + J(iAi)^2$ and image in $\text{soc}(iAi)$. By [2, Corollary 3.2], the class in $HH^1(iAi)$ of this derivation is nonzero, whence the result. \square

Proposition 4.5. *Let p be an odd prime and suppose that k is algebraically closed of characteristic p . Set $W = \text{Der}(k[x]/(x^p))$. For $-1 \leq i \leq p-2$ let f_i be the derivation of $k[x]/(x^p)$ sending x to x^{i+1} , where we identify x with its image in $k[x]/(x^p)$. Let L be a simple Lie subalgebra of W . Then either $L = W$, or $L \cong \mathfrak{sl}_2(k)$.*

Proof. Note that the subalgebra S of W spanned by the f_i with $0 \leq i \leq p-2$ is solvable. Thus L is not contained in S . Note further that $\dim_k(L) \geq 3$. Therefore there exist derivations

$$f = \sum_{i=-1}^{p-2} \lambda_i f_i$$

$$g = \sum_{i=t}^{p-2} \mu_i f_i$$

belonging to L with $\lambda_{-1} = 1$, and $\mu_t = 1$, where t is an integer such that $0 \leq t \leq p-2$. Choose g such that t is minimal with this property. But then $[f, g]$ belongs to L . Since $[f_{-1}, f_t] = (t+1)f_{t-1}$, the minimality of $t \geq 0$ forces $t = 0$; that is we have

$$g = \sum_{i=0}^{p-2} \mu_i f_i$$

and $\mu_0 = 1$. Since $\dim_k(L) \geq 3$, it follows that there is a third element h in L not in the span of f, g , and hence, after modifying h by a linear combination of f and g , we can choose h such that

$$h = \sum_{i=s}^{p-2} \nu_i f_i$$

for some s such that $1 \leq s \leq p-2$ and $\nu_s = 1$. Choose h such that s is minimal with this property. Again by considering $[f, h]$, one sees that the minimality of s forces $s = 1$. If L is 3-dimensional, then $L \cong \mathfrak{sl}_2(k)$, where we use that k is algebraically closed. If $\dim_k(L) \geq 4$, then L contains an element of the form

$$u = \sum_{i=r}^{p-2} \tau_i f_i$$

with $2 \leq r \leq p-2$ and $\tau_r = 1$. But then applying $[f, -]$ and $[h, -]$ repeatedly to u shows that L contains a basis of W , hence $L = W$. \square

Remark 4.6. Note that if $\text{char}(k) = p > 2$, then the Witt Lie algebra W contains indeed a subalgebra isomorphic to $\mathfrak{sl}_2(k)$. Let $\mathfrak{f}, \mathfrak{e}, \mathfrak{h}$ be elements of the basis of $\mathfrak{sl}_2(k)$ such that $[\mathfrak{e}, \mathfrak{f}] = \mathfrak{h}$, $[\mathfrak{h}, \mathfrak{f}] = -2\mathfrak{f}$, and $[\mathfrak{h}, \mathfrak{e}] = 2\mathfrak{e}$. Then we have a Lie algebra isomorphism $\mathfrak{sl}_2(k) \cong \langle f_{-1}, f_0, f_1 \rangle$ sending \mathfrak{f} to f_{-1} , \mathfrak{h} to $2f_0$, and \mathfrak{e} to $-f_1$.

Proof of Theorem 1.3. We use the notation and hypotheses of the notation in Theorem 1.3, and we assume that the Lie algebra $HH^1(A)$ is simple. We show that this forces $HH^1(A)$ to be a Lie subalgebra of the Witt Lie algebra W with $\text{char}(k) = p > 2$, and then the result follows from Proposition 4.5.

Since $HH^1(A)$ is simple and since $\text{Ext}_A^1(S, S)$ is nonzero, it follows from Proposition 4.4 that the canonical Lie algebra homomorphism $\Phi : HH^1(A) \rightarrow HH^1(iAi)$ from Proposition 4.2 is injective. By the assumptions, iAi is a local algebra whose quiver has only one loop. Therefore $A \cong k[x]/(v)$ for some polynomial $v \in k[x]$ of degree at least 1. Since k is algebraically closed, v is a product of powers of linear polynomials, say $\prod_i (x - \beta_i)^{n_i}$, with pairwise distinct β_i and positive integers n_i . Therefore $HH^1(iAi)$ is a direct product of the Lie algebras corresponding to these factors. It follows that $HH^1(A)$ is isomorphic to a Lie subalgebra of $HH^1(k[x]/((x - \beta)^n))$ for some positive integer n . After applying the automorphism $x \mapsto x + \beta$ of $k[x]$ we have that $HH^1(A)$ is isomorphic to a Lie subalgebra of $HH^1(k[x]/(x^n))$ for some positive integer n . If $\text{char}(k) = p$ does not divide n or if $\text{char}(k) = 0$, then the linear map sending x to 1 is not a derivation on $k[x]/(x^n)$, and therefore $HH^1(k[x]/(x^n))$ is solvable in that case. Since Lie subalgebras of solvable Lie algebras are solvable, this contradicts the fact that $HH^1(A)$ is simple. Thus we have $\text{char}(k) = p > 0$ and $n = pm$ for some positive integer m . Since $\text{char}(k) = p$, it follows that the canonical surjection $k[x]/(x^n) \rightarrow k[x]/(x^p)$ induces a Lie algebra homomorphism $HH^1(k[x]/(x^n)) \rightarrow W = HH^1(k[x]/(x^p))$ with a nilpotent kernel. Thus $HH^1(A)$ is not contained in that kernel, and hence $HH^1(A)$ is isomorphic to a Lie subalgebra of W . The result follows. \square

To conclude this section we note that although it is not clear which simple Lie algebras might occur as $HH^1(A)$ when $\text{Ext}_A^1(S, S) = 0$ for all simple A -modules S , it is easy to show that $HH^*(A)$ is not a simple graded Lie algebra (with respect to the Gerstenhaber bracket).

Proposition 4.7. *Let A be a finite dimensional k -algebra, and assume that for every simple A -module S we have $\text{Ext}_A^1(S, S) = 0$. Then $HH^*(A)$ is not a perfect graded Lie algebra. In particular, HH^* is not simple.*

Proof. If $f \in C^1(A, A) := \text{Hom}_k(A, A)$ and if $g \in C^0(A, A) := \text{Hom}_k(k, A)$, then the Gerstenhaber bracket is given by $[f, g] = f(g)$, i.e. simply evaluating f in g . Note that $1 \in Z(A) = HH^0(A)$. By Lemma 2.4 and Lemma 2.6, f preserves $J(A)$ and we may assume $E \subseteq \ker(f)$. Therefore the derived Lie subalgebra of $HH^*(A)$ does not contain 1_A . \square

Remark 4.8. Lemma 4.1 and Proposition 4.2 hold for algebras over an arbitrary commutative ring instead of k .

5. EXAMPLES

Theorem 1.1 applies to certain blocks of symmetric groups.

Proposition 5.1. *Suppose that k is a field of prime characteristic p . Let A be a defect 2 block of a symmetric group algebra kS_n or the principal block of kS_{3p} . Then $HH^1(A)$ is a solvable Lie algebra.*

Proof. From [9, Theorem 1] and from [7, Theorem 5.1] we have that the simple modules do not self-extend and the Ext^1 -space between two simple modules is at most one-dimensional. The statement follows from Theorem 1.1. \square

Remark 5.2. A conjecture by Kleshchev and Martin predicts that simple kS_n -modules in odd characteristic do not admit self-extensions.

Proposition 5.3. *Let A be a tame symmetric k -algebra with 3 isomorphism classes of simple modules of type $3\mathcal{A}$ or $3\mathcal{K}$. Then $\text{HH}^1(A)$ is a solvable Lie algebra.*

Proof. From the list at the end of Erdmann's book [5] we have that the simple modules in these cases do not self-extend and that the Ext^1 -space between two simple modules is at most one-dimensional. The statement follows from Theorem 1.1. \square

As mentioned in the introduction, the above Proposition is part of more general results on tame algebras in [4] and [8]. We note some other examples of algebras whose simple modules do not have nontrivial self-extensions.

Theorem 5.4 ([1, Theorem 3.4]). *Let G be a connected semisimple algebraic group defined and split over the field \mathbb{F}_p with p elements, and k be an algebraic closure of \mathbb{F}_p . Assume G is almost simple and simply connected and let $G(\mathbb{F}_q)$ be the finite Chevalley group consisting of \mathbb{F}_q -rational points of G where $q = p^r$ for a non-negative integer r . Let h be the Coxeter number of G . For $r \geq 2$ and $p \geq 3(h-1)$, we have $\text{Ext}_{kG(\mathbb{F}_q)}^1(S, S) = 0$ for every simple $kG(\mathbb{F}_q)$ -module S .*

Remark 5.5. Let G be a simple algebraic group over a field of characteristic $p > 3$, not of type A_1, G_2 and F_4 . Proposition 1.4 in [11] implies that not having self-extensions does not allow to lift to characteristic zero certain simple modular representations. Therefore, for these cases the Lie structure of HH^1 plays a central role.

In the context of blocks with abelian defect groups one expects (by Broué's abelian defect conjecture) every block of a finite group algebra with an abelian defect group P to be derived equivalent to a twisted group algebra of the form $k_\alpha(P \rtimes E)$, where E is the inertial quotient of the block and where α is a class in $H^2(E; k^\times)$, inflated to $P \rtimes E$ via the canonical surjection $P \rtimes E \rightarrow E$. Thus the following observation is relevant in cases where Broué's abelian defect conjecture is known to hold (this includes blocks with cyclic and Klein four defect).

Proposition 5.6. *Suppose that k be a field of prime characteristic p . Let P be a finite p -group and E an abelian p' -subgroup of $\text{Aut}(P)$ such that $[P, E] = P$. Set $A = k(P \rtimes E)$. Suppose that k is large enough for E , or equivalently, that A is split. For any simple A -module S we have $\text{Ext}_A^1(S, S) = 0$.*

Proof. Since E is abelian, it follows that $\dim_k(S) = 1$, and hence that $S \otimes_k -$ is a Morita equivalence. This Morita equivalence sends the trivial A -module k to S , hence induces an isomorphism $\text{Ext}_A^1(k, k) \cong \text{Ext}_A^1(S, S)$. It suffices therefore to show the statement for k instead of S . That is, we need to show that $H^1(P \rtimes E; k) = 0$, or equivalently, that there is no nonzero group homomorphism from $P \rtimes E$ to the additive group k . Since $[P, E] = P$, it follows that every abelian quotient of $P \rtimes E$ is isomorphic to a quotient of E , hence has order prime to p . The result follows. \square

Example 5.7. If B is a block of a finite group algebra over an algebraically closed field k of characteristic $p > 0$ with a nontrivial cyclic defect group P and nontrivial inertial quotient E ,

then $HH^1(B)$ is a solvable Lie algebra, isomorphic to $HH^1(kP)^E$, where E acts on $HH^1(kP)$ via the group action of E on P . Indeed, B is derived equivalent to the Nakayama algebra $k(P \rtimes E)$, which satisfies the hypotheses of Theorem 1.1 (thanks to the assumption $E \neq 1$, which implies $[P, E] = P$). Note that kP is isomorphic to the truncated polynomial algebra $k[x]/(x^{p^d})$, where $p^d = |P|$.

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