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Model-Matching type-methods and Stability of Networks consisting of non-Identical Dynamic Agents ^{*}

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Abstract: Many recent approaches of distributed control over networks of dynamical agents rely on the assumption of identical agent dynamics. In this paper we propose a systematic method for removing this assumption, leading to a general approach for distributed-control stabilization of networks of non-identical dynamics. Local agents are assumed to share a minimal set of structural properties, such as input dimension, state dimension and controllability indices, which are generically satisfied for parametric families of systems. Our approach relies on the solution of certain model-matching type problems using local state-feedback and input matrix transformations which map the agent dynamics to a target system, selected to minimize the joint control effort of the local feedback-control schemes. By adapting a well-established distributed LQR control design methodology to our framework, the stabilization problem for a network of non-identical dynamical agents is solved. The applicability of our approach is illustrated via a simple UAV formation control problem.

Keywords: Model-matching, distributed LQR, non-identical systems, networked control.

1. INTRODUCTION

Multi-agent networks have attracted a lot of attention of the control community in recent years. In such schemes, each agent is represented by a dynamical system and has the ability to communicate with certain of its counterparts within the network. The interactions established among agents determine the network topology and define a distributed communication and control pattern, often modelled as a graph, the nodes of which represent agents exchanging information through the links (edges) of the graph. The need for forming networks of systems typically arises from the consideration that some problems are not easily resolved at the individual system level. Military applications, transport networks and supply chains are typical paradigms which indicate that difficult tasks may be accomplished cooperatively (Jacoby and Chang (2008)). In other cases, the topology of the network may be imposed by physical links, such as in power systems where the agents take the role of power generators and the interconnections are represented by power transmission lines (Andreasson et al. (2012)).

Network stabilization is the most challenging problem in multi-agent network control (Olfati-Saber (2006); Fax and Murray (2002)). In typical situations the mere complexity of the system makes centralized control schemes either impossible or undesirable. Hence, distributed cooperative control is typically needed to ensure stable network operation. In cases where networks are composed of sufficiently small number of agents, the interconnections among the systems may not be limited and fully-centralized cooper-

ative controllers can be established. Nevertheless, bandwidth limitations as well as cost factors are the main reasons for imposing restrictions to network's communication capacity, resulting in sparsity of interactions among individual agents.

Two complementary distributed LQR methods have been proposed in Borrelli and Keviczky (2008) and Deshpande et al. (2012). The first is a top-down approach (Borrelli and Keviczky (2008)) in which the centralized optimal LQR controller is approximated by a distributed control scheme whose stability is guaranteed by the stability margins of LQR control. The second (Deshpande et al. (2012)) consists of a bottom-up approach in which optimal interactions between self-stabilizing agents are defined so as to minimize an upper bound of the global LQR criterion. A limitation of both methods is the assumption that networks are formed by identical systems, which is often unrealistic in applications. The motivation for this work is to remove this major assumption, thereby generalizing the approaches in Borrelli and Keviczky (2008) and Deshpande et al. (2012) with certain modifications.

In this paper, rather than assuming identical models for all agents, we consider a general class of models which share the same characteristics in terms of input and state dimensions and other structural properties (e.g. controllability, controllability indices). State-feedback and input transformations are used to solve several model-matching type-problems and compensate for the mismatch among the models of the agents. In the present context the definition of "model-matching" (in contrast to other "exact model matching" problems defined in the literature) gives us considerable flexibility as the output matrices of the

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mapped systems are required to be square and invertible but are otherwise arbitrary. Here the model of each agent matches the input-to-state part of a target system via state-feedback control and input matrix scaling. The selection of the target model is specified such that the perturbations in the agents' models produced by state-feedback controllers are minimal in a sense which is clearly defined. Existence conditions for the proposed model-matching schemes are established. Single-input plants are first investigated and then the multi-input case is analyzed assuming identical controllability indices for the agents' dynamical models. The definition of the target's model is achieved by minimizing a measure of the joint model-matching control effort. This allows closed-loop network performance to depend predominantly on the LQR optimality criterion, defined and optimized in the second stage of our approach. This adapts the state-feedback distributed control scheme presented in Borrelli and Keviczky (2008), leading to the solution of the stabilization problem for networks with non-identical agent dynamics. The formation control for a network of experimental UAV's with widely different parameters is finally used as design study to demonstrate the effectiveness of the method.

The rest of the paper is organized in four sections. In section 2 a few preliminaries on graph theory are presented. The main results of our work are presented in section 3, where model-matching type problems are solved for various classes of systems along with optimization techniques for specifying the target's model. The extension of the distributed scheme presented in Borrelli and Keviczky (2008), followed by a numerical example are included in the fourth section. The fifth section presents the main conclusions of the work where a discussion of the main results and suggestions for future work are given.

2. PRELIMINARIES

A graph \mathcal{G} is defined as $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where \mathcal{V} is the set of nodes (or vertices) $\mathcal{V} = \{1, \dots, N\}$ and $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ the set of edges (i, j) with $i \in \mathcal{V}$, $j \in \mathcal{V}$. The degree d_j of a graph vertex j is the number of edges which start from j . Let $d_{max}(\mathcal{G})$ denote the maximum vertex degree of the graph \mathcal{G} . We denote by $\mathbf{A}(\mathcal{G})$ the $(0, 1)$ adjacency matrix of the graph \mathcal{G} . Let $\mathbf{A}_{i,j} \in \mathbb{R}$ be its i, j element, then $\mathbf{A}_{i,j} = 1$ if $(i, j) \in \mathcal{E}$, $\forall i, j = 1, \dots, N$, $i \neq j$. Let \mathcal{N}_i represent the neighbourhood of the i^{th} node if $(i, j) \in \mathcal{E}$ and $i \neq j$. The adjacency matrix $\mathbf{A}(\mathcal{G})$ of undirected graphs is symmetric. We define the Laplacian matrix as $L(\mathcal{G}) = \mathbf{D}(\mathcal{G}) - \mathbf{A}(\mathcal{G})$ where $\mathbf{D}(\mathcal{G})$ is the diagonal matrix of vertex degrees d_i (also called the valence matrix). Let $S(\mathbf{L}(\mathcal{G})) = \{\lambda_1(\mathbf{L}(\mathcal{G})), \dots, \lambda_N(\mathbf{L}(\mathcal{G}))\}$ be the spectrum of the Laplacian matrix \mathbf{L} associated with an undirected graph \mathcal{G} arranged in nondecreasing semi-order.

3. MODEL-MATCHING PROBLEMS WITH OPTIMAL SELECTION OF TARGET SYSTEM

In this section, a specific type of model-matching problem is solved for various general classes of systems via state-feedback and input matrix transformations, as the first stage of the solution to the problem of stabilizing a network of non-identical agents via distributed LQR control, which is implemented in the second stage of the design. The plants representing the dynamical agents of the network are assumed to belong to a family of systems sharing

certain structural properties, such as system size, input dimension, controllability indices, etc; these are defined precisely later in the section. It is shown that under these transformations the open-loop agent dynamics can be mapped to a pre-specified target model. This effectively gives all agents identical input-to-state dynamics (and, in general, a different invertible output matrix for each agent). We also show that the target system can be selected so that the joint control effort of this scheme is minimized in a specific sense also made precise later. We impose this objective because we wish to use the "minimum amount of feedback" in the first stage of the design, so that the overall performance and stability properties of the closed-loop system are effectively defined by the weighting matrices of the quadratic performance index of the LQR problem solved in the second stage. Although local stabilization by state feedback is not necessary, it is possible to impose it as an additional constraint to the problem of joint control-effort minimization. This offers the design integrity and autonomous stable operation of individual agents when communication with neighbouring agents is disrupted in the final distributed control scheme (although of course this is at the expense of an increase of the index of joint local control effort). The main assumptions used are summarized in Table 1.

Table 1. Assumptions, structural properties.

	Single-input case	Multi-input case
Agent models	$\begin{aligned} \dot{x}_i &= A_i x_i + b_i u_i \\ y_i &= x_i, i = 1, \dots, N \end{aligned} \quad (1)$	$\begin{aligned} \dot{x}_i &= A_i x_i + B_i u_i \\ y_i &= x_i, i = 1, \dots, N \end{aligned} \quad (2)$
Target model	$\begin{aligned} \dot{x} &= A_d x + b_d u \\ y &= x \end{aligned} \quad (3)$	$\begin{aligned} \dot{x} &= A_{N+1} x + B_{N+1} u \\ y &= x \end{aligned} \quad (4)$
Assumptions	$\{(A_i, b_i)\} \cup \{(A_d, b_d)\}$ have same dimensions and are controllable with $A_i \in \mathbb{R}^{n \times n}$, $b_i \in \mathbb{R}^n$.	$\{(A_i, B_i)\}$, $i = 1, \dots, N+1$ have identical controllability indices $\{\mu_j\}_{j=1}^m$, $\sum_{j=1}^m \mu_j = n$.

3.1 Single-input case

The first class of agent models is defined as the set of single-input controllable plants with fixed state dimension and is summarized in the first column of Table 1. Consider a network formed of N single-input linear systems with state size equal to n and state-space represented by (1). Let (A_d, b_d) be the target system, chosen from the same system class. It can be easily shown that there are always uniquely defined state-feedback gain-vectors f_i , $i = 1, \dots, N$ such that

$$T_i(sI - A_i - b_i f_i^T)^{-1} b_i = T_d(sI - A_d)^{-1} b_d, \quad i = 1, \dots, N \quad (5)$$

Here the T_i 's and T_d are appropriate similarity transformations that bring the state-space form of the i^{th} plant and the target system, respectively, into controllable canonical forms. Note that there is always a similarity transformation that matches the state-space forms of two controllable single-input linear systems as long as they have the same state-dimension and common poles. The same argument is also true for N such systems. Therefore, the model-matching of N linear systems with a target model will affect only their characteristic polynomial. Thus, the state coordinate system to be employed for control purposes depends on either the designer's choice or even the physical meaning of the system variables themselves. In this respect, our goal here is to find the target characteristic polynomial of A_d , $p(s, d) = s^n + d_{n-1} s^{n-1} + \dots + d_1 s + d_0$

that minimizes a cost function $J(\cdot)$ which penalizes joint control effort. This cost function is defined next.

We now give the solution to the model-matching problem for the single-input case of finding appropriate f_i for $i = 1, \dots, N$ that satisfy (5) without setting any constraint on the polynomial $p(s, d)$. Let $a^{iT} = [a_{n-1}^i \ a_{n-2}^i \ \dots \ a_0^i]$ and $d^T = [d_{n-1} \ d_{n-2} \ \dots \ d_0]$ be row vectors representing the coefficients of the characteristic polynomial of the i^{th} agent model and the target system, respectively. Let also, $T_i = (\Gamma_{ci} H_{ai})^{-1}$ for $i = 1, \dots, N$ where $\Gamma_{ci} = [b_i \ A_i b_i \ \dots \ A_i^{n-1} b_i]$ and H_{ai} represent the controllability matrix and the Hankel matrix formed by the coefficients of the characteristic polynomial, respectively, of the i^{th} plant (Antsaklis and Michel (2005)). The state-feedback gain-vector which matches the characteristic polynomial of the i^{th} system with $p(s, d)$ is given uniquely by

$$f_i = (\Gamma_{ci} H_{ai})^{-T} (a^i - d) = T_i^T (a^i - d) \quad (6)$$

Since T_i and a^i are fixed, the freedom to select state-feedback gains relies only on d . Consider the following cost function which penalizes joint state-feedback control effort: $J_1(\cdot) = \sum_{i=1}^N \|f_i\|^2$. This can be written as a function of d : $J_1(d) = d^T (\sum_{i=1}^N T_i T_i^T) d - 2 (\sum_{i=1}^N T_i T_i^T a^i)^T d + \sum_{i=1}^N \|T_i^T a^i\|^2$. Note that $J_1(d)$ is strictly convex and thus the unique minimum for d^* is attained as:

$$\frac{\partial J_1}{\partial d} = 0 \Rightarrow d^* = \left(\sum_{i=1}^N T_i T_i^T \right)^{-1} \left(\sum_{i=1}^N T_i T_i^T a^i \right) \quad (7)$$

Optimal state-feedback gains in the above least-squares sense are obtained by substituting d^* to (6). An alternative cost function penalizing worst-case control effort among the N agents can be defined as the following discrete minimax problem to derive optimal d :

$$\min_{d \in \mathbb{R}^{1 \times n}} \phi(d), \text{ with } \phi(d) = \max_{i=1:N} M_i, \text{ where } M_i = \|f_i\|^2 \quad (8)$$

and f_i is given by (6). Since $\phi(d)$ is continuous and convex by the continuity and convexity of the M_i , $i = 1, \dots, N$ and its sub-level sets are bounded the minimizing solution exists and is unique. Efficient ϵ -approximation algorithms described in Dem'yanov and Malozemov (2014) can be employed to calculate the optimal solution.

As mentioned earlier in many cases it makes sense to add an additional constraint to the problem, namely the stability of the target polynomial $d(s)$. Of course, the distributed LQR controller which is synthesized at the second stage of the design will guarantee closed-loop network stability, however local stabilization in many cases is still desirable as it ensures that stable operation is maintained if communication with neighbouring agents is disrupted (possibly at poor levels of performance). Since the optimization in the single-input case is carried out over the coefficients of the target polynomial, the extra constraints are derived from the Routh-Hurwitz stability criterion. Unfortunately these are highly nonlinear so they can be used effectively only for low-dimensional problems. An alternative approach is to enforce "local" stability conditions by specifying a nominal target polynomial and calculating the maximum region in coefficient space around the nominal coefficients in which the perturbed polynomial remains stable. This can be achieved by specifying the maximum stability radius $r(d^o)$ of a nominal Hurwitz polynomial $p_o(s, d^o) = s^n + d_{n-1}^o s^{n-1} + \dots + d_1^o s + d_0^o$, e.g. defined via the Euclidian or infinity norms of the coefficient vector. First

we illustrate the application of Routh-Hurwitz conditions via a small numerical example.

Example 1. Let the state-space form of two unstable systems $\dot{x}_i = A_i x_i + e_3 u_i$, $i = 1, 2$ is given in controllable canonical form, with $A_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -3 & 1 \end{bmatrix}$, $A_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -2 & 1 \end{bmatrix}$, $e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. We

seek the best target Hurwitz characteristic polynomial in the sense that minimizes the joint state-feedback control effort (f_1, f_2) . The constrained optimization problem is:

$$\min_{d \in \mathbb{R}^{1 \times 3}} \sum_{i=1}^2 \|a^i - d\|^2 \text{ s.t. } d_1 > 0, \ d_3 > 0 \text{ and } d_3 d_2 > d_1 \quad (9)$$

where a^i represents the coefficients of the characteristic polynomial of A_i , $i = 1, 2$. The infimizing solution was obtained in MatLab using the *fmincon* function and the coefficient vector d^* of the best target polynomial is: $d = [0 \ 2.5 \ 0]$. Note that two poles of the optimal solution lie on the imaginary axis which defines the boundary of the constrained set. This may be rectified, if desired, by redefining the stable region.

In order to formulate the constrained optimization problem for the second case, we give an explicit formula for the distance ρ of a Hurwitz polynomial from the set of non-Hurwitz polynomials in the coefficient space via Proposition 1 which can originally be found in Hinrichsen and Pritchard (1988). For a Hurwitz polynomial $p(s, a)$ this distance determines the largest ρ for which the open cube in \mathbb{R}^n with center a and radius $\rho > 0$ the set $\{p(s, b) : |b_i - a_i| < \rho \text{ for } i = 1, \dots, n\}$ only consists of Hurwitz polynomials.

Proposition 1. If $p(s, a)$, $a \in \mathbb{R}^n$ is a Hurwitz polynomial, $p(s, a) = p_1(-s^2) + s p_2(-s^2)$ where $p_j(-s^2)$, $j = 1, 2$ are real polynomials in $-s^2$, then $\rho(a) = \min\{a_0, [\max_{\omega^2 \in \mathbb{R}^+} f(\omega^2)]^{-1/2}\}$ where $f(\omega^2)$ equals

$$\frac{1 + \omega^4 + \dots + \omega^{2n-4}}{p_1^2(\omega^2) + p_2^2(\omega^2)} \text{ if } n \text{ is even. If } n \text{ is odd, } f(\omega^2) = \frac{(1 + \omega^4 + \dots + \omega^{2n-6})(1 + \omega^4 + \dots + \omega^{2n-2})}{p_1^2(\omega^2)(1 + \omega^4 + \dots + \omega^{2n-6}) + p_2^2(\omega^2)(1 + \omega^4 + \dots + \omega^{2n-2})}$$

The optimization problem is now outlined. Let a Hurwitz polynomial $p_o(s, d^o)$ be specified a priori and have maximum stability radius $r^o = \rho$ given by Proposition 1. Let also $S \subseteq \mathbb{R}^n$ be such that $\delta \in S$ if and only if $|d_i^o - \delta_i| < \rho$ for all i . Consider N non-identical single-input systems with state-dimension equal to n and given as in (1). Then the state-feedback gain-vectors f_i which are defined in (6) and match all the characteristic polynomials of the N systems with a Hurwitz polynomial with coefficient vector $d^* \in S$ with minimum joint effort in the least-squares sense can be obtained by solving

$$\inf_{d \in S} \sum_{i=1}^N \|f_i\|^2 = \inf_{d \in S} \sum_{i=1}^N \|T_i^T (a^i - d)\|^2 \quad (10)$$

where T_i , $i = 1, \dots, N$ is the similarity transformation which brings the i^{th} state-space agent model in controllable canonical form and a^i represents the coefficient vector of the characteristic polynomial of the i^{th} system. Note that for nontrivial problems S is bounded, so constraining the problem on the closure of S will result in a unique solution via standard Quadratic Programming algorithms. Similarly, defining a worst-case control effort problem of the form:

$$\inf_{d \in \mathcal{S}} \max_{i \in \{1, 2, \dots, N\}} \|T_i^T(a^i - d)\| \quad (11)$$

involves the minimization of a continuous convex function over a compact set and therefore a unique minimum exists which can be calculated efficiently with the algorithms described in Dem'yanov and Malozemov (2014).

3.2 Multi-input case

The class of systems to be considered in this paragraph consists of multi-input linear systems with fixed controllability indices $\{\mu_j\}$ and is summarized in the second column of Table 1. Recall that $\sum_{j=1}^m \mu_j = n$ where m stands for the number of inputs and n the state dimension. Note that the controllability indices define completely the class without the need for specifying input and state sizes. The following Lemma is standard and is included without proof.

Lemma 1. Given (A, B) controllable, then $(P(A+BF)P^{-1}, PBG)$ has the same controllability indices, up to reordering, for any P, F and G ($\det(P) \neq 0, \det(G) \neq 0$) of appropriate dimensions.

Let the pair (A, B) be controllable with controllability indices $\{\mu_j\}$ where $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times m}$. There is always similarity transformation P (see Antsaklis and Michel (2005) for how to construct matrix P) such that the pair can be reduced to controller canonical form, namely, (A_c, B_c) where

$$A_c = \bar{A}_c + \bar{B}_c A_m, \quad B_c = \bar{B}_c B_m \quad (12)$$

with $A_m \in \mathbb{R}^{m \times n}$ and $B_m \in \mathbb{R}^{m \times m}$ being free. The pair (\bar{A}_c, \bar{B}_c) is called the Brunovsky canonical form (Antsaklis and Michel (2005)) and is unique (up to reordering) for the class of pairs (A_i, B_i) with common controllability indices. The matrices $(\bar{A}_c, \bar{B}_c) = (\text{diag}(\bar{A}_{11}, \dots, \bar{A}_{mm}), \text{diag}(\bar{B}_{11}, \dots, \bar{B}_{mm}))$ where

$$\bar{A}_{jj} = \begin{bmatrix} 0 & & & \\ & \ddots & & \\ & & I_{\mu_j-1} & \\ 0 & & & 0 \end{bmatrix} \in \mathbb{R}^{\mu_j \times \mu_j}, \quad \bar{B}_{jj} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \in \mathbb{R}^{\mu_j} \text{ for } j = 1, \dots, m.$$

Consider now a set of $N+1$ systems arbitrarily chosen from the class defined by the controllability indices $\{\mu_j\}$. Their state-space equations are:

$$\dot{x}_i = A_i x_i + B_i u_i, \quad y_i = x_i \text{ for } i = 1, \dots, N+1 \quad (13)$$

where $A_i \in \mathbb{R}^{n \times n}$, $B_i \in \mathbb{R}^{n \times m}$. The $(N+1)^{\text{th}}$ index corresponds to a target system the state-space form of which is assumed to be in controllable canonical form. There are similarity transformations P_i such that the state-space representation of the i^{th} plant in the set can be reduced to canonical form with dynamics and input matrices being given as in (12). The state-space equations of the N plants in the new coordinates ($x_{ci} = P_i x_i$) are:

$$\dot{x}_{ci} = A_{ci} x_{ci} + B_{ci} u_i, \quad y_i = P_i^{-1} x_{ci} \text{ for } i = 1, \dots, N \quad (14)$$

with the pairs (A_{ci}, B_{ci}) and (A_{N+1}, B_{N+1}) having identical Brunovsky forms (\bar{A}_c, \bar{B}_c) . The N plants can match the input-to-state part of the target by applying state-feedback and input transformations $u_i = F_i x_i + G_i v_i$ with the corresponding matrices (F, G) being given as

$$F_{ci} = B_{mi}^{-1}(A_{mN+1} - A_{mi}), \quad G_i = B_{mi}^{-1} B_{mN+1} \quad (15)$$

where the state-feedback gain in the original coordinates can be recovered by $F_i = F_{ci} P_i$. The state-space equations of the closed-loop systems take the following form

$$\dot{x}_{ci} = (A_{ci} + B_{ci} F_{ci}) x_{ci} + B_{ci} G_i v_i, \quad y_i = P_i^{-1} x_{ci} \quad (16)$$

with $A_{ci} + B_{ci} F_{ci} = A_{N+1}$ and $B_{ci} G_i = B_{N+1}$ which are identical to the corresponding matrices of the target system. Since all the N closed-loop systems have the same dynamics and input matrices with the target in the transformed coordinates, the state-space equations can be rewritten in the form

$$\dot{\xi} = A_{N+1} \xi + B_{N+1} v, \quad x_i = P_i^{-1} \xi \text{ for } i = 1, \dots, N. \quad (17)$$

Possible selection of the canonical form of the target $(N+1)^{\text{th}}$ system is given next. Let B_{mN+1} be the identity matrix I_m and $J_3 = \sum_{i=1}^N \|F_i\|_F^2 = \sum_{i=1}^N \|B_{mi}^{-1}(\bar{A}_m - A_{mi})P_i\|_F^2$ be cost function that measures the joint effort of the model-matching control, where all the matrices are as given earlier and \bar{A}_m represent the free term in (12). Then, the unique optimal solution \bar{A}_m of $\min(J_3)$ can be calculated using Kronecker products and vectorization arguments. Due to lack of space details are omitted. The target's state-space form becomes: $(A_{N+1}, B_{N+1}) = (\bar{A}_c + \bar{B}_c \bar{A}_m, \bar{B}_c I_m)$ and the optimal state-feedback gains and the input transformations are respectively given by

$$\bar{F}_i = B_{mi}^{-1}(\bar{A}_m - A_{mi})P_i, \quad \bar{G}_i = B_{mi}^{-1} I_m, \quad i = 1, \dots, N. \quad (18)$$

4. DISTRIBUTED CONTROL DESIGN

In this section a generalization of the distributed LQR method proposed in Borrelli and Keviczky (2008) is analyzed and applied to networks formed of systems which belong to the classes of linear systems examined earlier.

4.1 Network-based LQR problem

First the analysis of a network-based LQR problem defined in Borrelli and Keviczky (2008) is briefly presented here. Let N_L identical linear agents (A, B) constitute a network described by a complete graph (i.e., the graph with all possible interconnections) which has the ability to exchange state information between any two nodes. Since the systems in the original method are considered identical the collective state-space form of the network shown next is obtained from augmenting the individual systems using Kronecker products:

$$\dot{\tilde{x}} = (I \otimes A) \tilde{x} + (I \otimes B) \tilde{u}, \quad \tilde{x}_0 = [x_1^T(0), \dots, x_{N_L}^T(0)]^T \quad (19)$$

where $\tilde{x} = [x_1^T, \dots, x_{N_L}^T]^T$ and $\tilde{u} = [u_1^T, \dots, u_{N_L}^T]^T$. Consider now a performance index that couples the dynamical behavior of the individual systems:

$$J(\tilde{u}, \tilde{x}_0) = \int_0^\infty \sum_{i=1}^{N_L} \left(x_i^T Q_{ii} x_i + u_i^T R u_i + \sum_{j \neq i}^{N_L} (x_i - x_j)^T Q_{ij} (x_i - x_j) \right) d\tau \quad (20)$$

with $Q_{ii} \geq 0$, $Q_{ij} \geq 0$ and $R > 0$ or, written in a more compact form: $J(\tilde{u}, \tilde{x}_0) = \int_0^\infty \sum_{i=1}^{N_L} (\tilde{x}^T \tilde{Q} \tilde{x} + \tilde{u}^T \tilde{R} \tilde{u}) d\tau$ where

$$\tilde{Q} = \begin{bmatrix} \tilde{Q}_{11} & \dots & \tilde{Q}_{1N_L} \\ \vdots & \ddots & \vdots \\ \tilde{Q}_{N_L 1} & \dots & \tilde{Q}_{N_L N_L} \end{bmatrix}, \quad \tilde{R} = I_{N_L} \otimes R \quad (21)$$

with $\tilde{Q}_{ii} = \sum_{k=1}^{N_L} Q_{ik}$ and $\tilde{Q}_{ij} = -Q_{ij}$. Let $Q_{ii} = C_{ii}^T C_{ii}$ and $Q_{ij} = C_{ij}^T C_{ij}$. Under the assumption that (A, B) is controllable and all pairs (A, C_{ii}) and (A, C_{ij}) are observable, the following network-based LQR problem

$$\min_{\tilde{u}} J(\tilde{u}, \tilde{x}_0) \quad \text{s.t.} \quad \dot{\tilde{x}} = (I \otimes A) \tilde{x} + (I \otimes B) \tilde{u}, \quad \tilde{x}_0 \quad (22)$$

leads to the networked state-feedback gain $\tilde{K} = -\tilde{R}^{-1}\tilde{B}^T\tilde{P}$ where the Lyapunov function \tilde{P} is the unique symmetric positive definite solution to the (large scale) ARE: $\tilde{A}^T\tilde{P} + \tilde{P}\tilde{A} - \tilde{P}\tilde{B}\tilde{R}^{-1}\tilde{B}^T\tilde{P} + \tilde{Q} = 0$ where $\tilde{A} = I_{N_L} \otimes A$ and $\tilde{B} = I_{N_L} \otimes B$. Let now $Q_{ii} = Q_1$ and $Q_{ij} = Q_2$ with $Q_1 \geq 0$ and $Q_2 \geq 0$ then

$$\tilde{P} = \begin{bmatrix} P - (N_L - 1)\tilde{P}_2 & \tilde{P}_2 & \cdots & \tilde{P}_2 \\ \vdots & \ddots & \ddots & \vdots \\ \tilde{P}_2 & \cdots & \cdots & P - (N_L - 1)\tilde{P}_2 \end{bmatrix} \quad (23)$$

where P is the unique symmetric positive definite solution to the single-node ARE: $A^T P + P A - P B R^{-1} B^T P + Q_1 = 0$. Since $\tilde{K} = -\tilde{R}^{-1}\tilde{B}^T\tilde{P}$ the structure of \tilde{K} is also preserved in the networked state-feedback gain \tilde{K} . For more details see Borrelli and Keviczky (2008).

4.2 Distributed LQR for networks of non-identical systems

Consider N interconnected, non-identical and dynamically decoupled linear systems (A_i, B_i) , $i = 1, \dots, N$ characterized by the same controllability indices $\{\mu_j\}$. The collective state-space form of the network is given by

$$\dot{\tilde{x}} = \text{diag}(A_1, \dots, A_N)\tilde{x} + \text{diag}(B_1, \dots, B_N)\tilde{u}, \quad \tilde{x}(0) = \tilde{x}_0 \quad (24)$$

where $\tilde{x} = [x_1^T, \dots, x_N^T]^T$ and $\tilde{u} = [u_1^T, \dots, u_N^T]^T$. A stabilizing distributed control scheme is described next.

Theorem 1. Consider N non-identical linear agents in a network with state-space form given by (24) and topology specified by a graph with Laplacian matrix L and maximum vertex degree d_{max} . Assume that the agents share the same controllability indices and therefore according to (18) F_i and G_i , $i = 1, \dots, N$ can be found such that the controllable canonical form of the closed-loop systems is identical and given by (17). Consider reduced-order networked LQR problem (22) for $N_L = d_{max} + 1$ identical plants defined by $(A_{N+1}, B_{N+1}) = (P_i(A_i + B_i F_i)P_i^{-1}, P_i B_i G_i)$ where P_i is similarity transformation that bring the i^{th} system into controllable canonical form. Specify P and \tilde{P}_2 according to (23) and let $M = aL$ where $a > \frac{N_L}{\lambda_2(L)}$. Construct the (large-scale) state-feedback gain

$$\tilde{K} = I_N \otimes K_1 + M \otimes K_2 \quad (25)$$

where $K_1 = -R^{-1}B_{N+1}^T P$ and $K_2 = R^{-1}B_{N+1}^T \tilde{P}_2$. Let \mathcal{N}_i represent the neighbourhood of the i^{th} agent. Then the state-space equation

$$\dot{x}_i = [A_i + B_i(F_i + G_i K_1 P_i^{-1})]x_i + a B_i G_i \sum_{j \in \mathcal{N}_i} K_2 (P_i^{-1} x_i - P_j^{-1} x_j) \quad (26)$$

is asymptotically stable for all $i = 1, \dots, N$.

The proof is omitted due to lack of space. Fig. 1 shows a schematic representation of the distributed scheme presented in Theorem 1 at local level i .

4.3 Numerical Example

A formation control problem of four non-identical UAV's is now solved to demonstrate the applicability of our method. The linearized model of a low-speed experimental UAV known as X-RAE1 is utilized here and detailed description of the aircraft can be found in Tomic et al. (2016) and



Fig. 1. Closed-loop configuration of the i^{th} agent.

Elgayar (2013). For a straight, steady, symmetric and horizontal flight at constant velocity $V_{T_0} = 30 \text{ms}^{-1}$ the state-space form of the linearized model for the longitudinal motion of the i^{th} aircraft takes the form

$$\dot{x}_i = A_i x_i + B_i u_i, \quad x_i(0) = x_{i0} \quad (27)$$

where $x_i = [u_i \ w_i \ q_i \ \theta_i]^T$ and $u_i = [n_i \ \delta_{T_i}]^T$. The state vector represents the deviation of the forward velocity, the downward velocity, the pitch angular velocity and the pitch angle while the input vector the deviation of the elevator and the thrust from the operating point. Let four non-identical X-RAE1's move at a nominal height and exchange information about their states according to the interconnection topology shown in Fig. 2. The corresponding interconnected graph has maximum vertex degree $d_{max} = 2$.

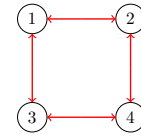


Fig. 2. Interconnection topology of the four X-RAE1's.

Table 2. Dynamics and input matrices (A, B) .

X-RAE1	A_i				B_i	
agent-1	-0.142	-0.227	2.493	-9.771	-1.136	1.444
	-1.033	-4.476	28.639	0.837	-13.060	0
	-0.042	-2.744	15.351	-0.134	-137.157	-2.036
	0	0	1	0	0	0
agent-2	-0.125	-0.155	2.264	-9.773	-0.928	1.3032
	-0.963	-4.053	28.780	0.758	-11.811	0
	-0.022	-2.836	15.383	-0.122	-137.399	-2.036
	0	0	1	0	0	0
agent-3	-0.163	-0.317	2.717	-9.764	-1.392	1.619
	-1.120	-5.002	28.468	0.915	-14.605	0
	-0.059	-2.636	15.315	-0.147	-136.867	-2.036
	0	0	1	0	0	0
agent-4	-0.175	-0.370	2.826	-9.760	-1.545	1.724
	-1.171	-5.314	28.367	0.954	-15.525	0
	-0.065	-2.573	15.294	-0.153	-136.670	-2.036
	0	0	1	0	0	0

The four UAV's of the formation differ from each other with respect to their mass and symmetry about the xz plane of their body axes. The mismatch of the four X-RAE1's in the corresponding matrices of the linear models is shown in Table 2. The formation is represented by:

$$\dot{\tilde{x}} = \text{diag}(A_1, A_2, A_3, A_4)\tilde{x} + \text{diag}(B_1, B_2, B_3, B_4)\tilde{u} \quad (28)$$

where $\tilde{x} = [x_1^T, x_2^T, x_3^T, x_4^T]^T$, $\tilde{u} = [u_1^T, u_2^T, u_3^T, u_4^T]^T$ and $\tilde{x}(0) = \tilde{x}_0$. The main control objective is to stabilize the formation in the presence of impulsive disturbances.

Since (A_i, B_i) for $i = 1, 2, 3, 4$ have the same controllability indices, the model-matching approach proposed earlier

can be applied. Let P_i be similarity transformation which brings the i^{th} linearized system to controllable canonical form given in (12). Minimize the cost function $\sum_{i=1}^N \|F_i\|_F^2 = \sum_{i=1}^N \|B_{m_i}^{-1}(\bar{A}_m - A_{m_i})P_i\|_F^2$ and construct F_i and G_i as in (18). Let $(\bar{A}, \bar{B}) = (P_i^{-1}(A_i + B_i F_i)P_i, P_i^{-1}B_i G_i)$ be the target system. Consider the following LQR problem presented in 4.1 with performance index \tilde{J} having the same structure as in (20) for $N_L = d_{max} + 1 = 3$:

$$\min_{\tilde{K}} \tilde{J}(v, \xi_0) \text{ s.t. } \dot{\xi} = (I_3 \otimes \bar{A})\xi + (I_3 \otimes \bar{B})v, \xi(0) = \xi_0 \quad (29)$$

with weighting matrices being given as $Q_{ii} = 10I_4$, $Q_{ij} = 100I_4$ and $R = \text{diag}(1, 100)$ for $i = 1, 2, 3, 4$ and $j = 1, 2, 3, 4$. The solution of the above LQR problem leads to the structured Lyapunov function \tilde{P} and the distributed state-feedback control $\tilde{u} = \tilde{K}\tilde{x}$ which stabilizes (28) is:

$$\begin{aligned} \tilde{K} = & \text{diag}(F_1, F_2, F_3, F_4) + \\ & \text{diag}(G_1, G_2, G_3, G_4)(-I_4 \otimes R^{-1}\bar{B}^T P + \\ & M \otimes R^{-1}\bar{B}^T \tilde{P}_2) \text{diag}(P_1^{-1}, P_2^{-1}, P_3^{-1}, P_4^{-1}) \quad (30) \end{aligned}$$

where $P = \tilde{P}_1 + 2\tilde{P}_2$, $M = aL$ with $a = 1.7$ reflects the structure of the graph with Laplacian matrix

$$L = \begin{bmatrix} 2 & -1 & -1 & 0 \\ -1 & 2 & 0 & -1 \\ -1 & 0 & 2 & -1 \\ 0 & -1 & -1 & 2 \end{bmatrix} \text{ and } \tilde{P} = \begin{bmatrix} \tilde{P}_1 & \tilde{P}_2 & \tilde{P}_2 \\ \tilde{P}_2 & \tilde{P}_1 & \tilde{P}_2 \\ \tilde{P}_2 & \tilde{P}_2 & \tilde{P}_1 \end{bmatrix}.$$

The simulations depict forward and downward velocity response in the presence of non-uniform wind field which is approximated by arbitrary impulse acceleration along the vertical axis of each UAV. Fig. 3 and Fig. 4 show the recovery of the nominal values. Note that only the

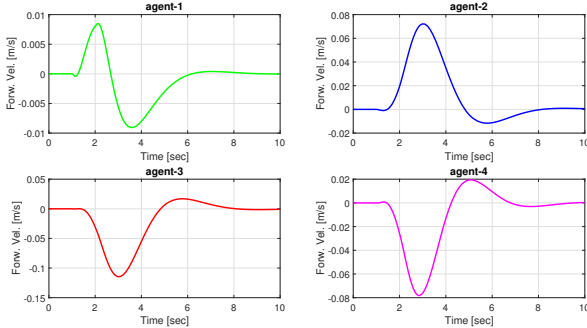


Fig. 3. Forward velocity deviation in the presence of impulse acceleration along the vertical axis.

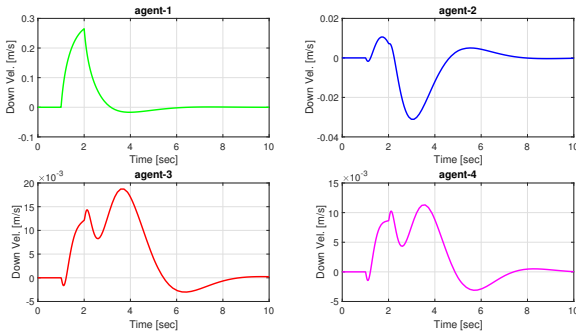


Fig. 4. Downward velocity deviation in the presence of impulse acceleration along the vertical axis.

state-deviation from the nominal values is stabilized but otherwise the initial formation is not recovered along the horizontal and the vertical axis after the occurrence of disturbances. The longitudinal deviation along the x and z axes can be regulated, if required, by additional integral action. The robustness of the method have been tested by numerous simulations which are omitted due to limitation of space.

5. CONCLUSION

We have extended an established technique for solving stability problems for networks formed of non-identical agents which belong to certain general classes of linear systems. The first stage of the method solves model-matching problems and defines the synthesis of local state-feedback controllers which match all the systems in the network with a target which is selected such that the joint control effort is minimized in a pre-specified sense. It has been shown how existing distributed schemes proposed for networks of identical agents can be appropriately adjusted and applied to solve stabilization problems on networks of non-identical dynamics. Further work is needed, however, to extend the method to more generic classes of systems which can be implemented successfully in practical applications.

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