Distributed LQR-based Suboptimal Control for Coupled Linear Systems ⋆

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Abstract: A well-established distributed LQR method for decoupled systems is extended to the dynamically coupled case where the open-loop systems are dynamically dependent. First, a fully centralized controller is designed which is subsequently substituted by a distributed state-feedback gain with sparse structure. The control scheme is obtained by optimizing an LQR performance index with a tuning parameter utilized to emphasize/de-emphasize relative state difference between interconnected systems. Overall stability is guaranteed via a simple test applied to a convex combination of Hurwitz matrices, the validity of which leads to stable global operation for a class of interconnection schemes. It is also shown that the suboptimality of the method can be assessed by measuring a certain distance between two positive definite matrices which can be obtained by solving two Lyapunov equations.

Keywords: distributed LQR, coupled linear systems, multi-agent control, networked control.

1. INTRODUCTION

Networked systems have attracted attention from the control community in recent years due to their broad range of applications. Such schemes are often referred to as multi-agent networks, with each agent having autonomous actuation capacity. The need for forming networks in many cases arises from the fact that some problems may not admit a solution at the individual system level. Thus agents despite their independent operation have also the ability to cooperate with certain of their counterparts within the network towards a common objective. In other cases, the topology of the network may be imposed by structural links such as in power systems where agents denote power generators and interconnections represent power transmission lines.

In this paper, we focus on multi-agent networks composed of identical dynamically coupled linear time-invariant systems. We consider that these dynamical couplings can be expressed in a state-space form of a certain structure. In our case each system representing an agent can produce actuation signals independently and is dynamically coupled with certain number of its peers referred to as neighboring agents with whom it can exchange state information. Effectively, we assume that the topology of physical couplings and the topology of information exchange among agents coincide and are described by the same graph. Network stabilization is one of the most challenging problems in multi-agent network control. In this work, we propose a stabilizing distributed LQR-based controller for networks formed of identical agents with dynamical couplings.

Over the past few years, there has been a renewal of interest in control of networks composed of a large number of interacting systems. Rigorous methods for cooperative control design for multi-agent systems with distributed or decentralized pattern have been provided in Olfati-Saber (2006); Fax and Murray (2002); Wang et al. (2018). A thorough procedure for designing distributed controllers for a class of coupled systems based on a decomposition approach has been presented in Massioni and Verhaegen (2009). The fundamental works of Borrelli and Keviczky (2008); Deshpande et al. (2012) discuss distributed LQR design for a set of identical decoupled dynamical systems. Unfortunately, there is no documented distributed LQR-based approach to networked systems with dynamical couplings in the context of relevant literature.

In this work, we follow a top-down method to approximate a centralized LQR optimal controller by a distributed control scheme. It is shown that overall network stability is guaranteed via a stability test applied to a convex combination of Hurwitz matrices. The validity of this condition is consistent with the stability of a class of network interconnection structures which is identified. Sufficient condition for stability of convex combination of Hurwitz matrices can be found in Białas (2004). The control scheme is obtained by optimizing an LQR performance index with a tuning parameter which can be used to emphasize/de-emphasize relative state difference between interconnected systems. Our approach was inspired by the powerful results proposed in Borrelli and Keviczky (2008).

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Therein, the subsystems constituting the network are dynamically decoupled, and the stability of the distributed scheme designed relies on the stability margins of LQR control. A complementary distributed LQR method has also been proposed in Deshpande et al. (2012), which consists of a bottom-up approach in which optimal interactions between self-stabilizing agents are defined so as to minimize an upper bound of the global LQR criterion.

Our definition of multi-agent networks with dynamical couplings has been motivated by the structure of a multi-area power system. The proposed method has been successfully applied to multi-area power system control design numerous simulations of which can be found in Vlahakis et al. (2019). A major assumption of our work is that the dynamical models of each system are identical. Although this assumption may be unrealistic in practice, it simplifies the design problem considerably, which is especially hard due to the coupling terms appearing in the model. Future work will attempt to eliminate or relax this assumption. Preliminary results in this direction can be found in Vlahakis and Halikias (2018a,b).

The remaining of the paper is organized in four sections. In Section 2 notation and some preliminaries on graph theory are presented. The main results of our work are presented in Section 3 and 4 where LQR properties of coupled linear systems and the distributed control algorithm are derived, respectively. Section 5 summarizes the main conclusions of the work where a discussion of the main results and suggestions for future work are also included.

2. NOTATION AND GRAPH THEORY PRELIMINARIES

The field of real and complex numbers are denoted by $\mathbb{R}$ and $\mathbb{C}$, respectively. $\mathbb{R}^n$ denotes the n-dimensional vector space over the field $\mathbb{R}$. $\mathbb{R}^{n\times m}$ denotes the set of $n \times m$ real matrices. ξ' and Ξ' denote the transpose of ξ and Ξ, respectively. Matrix $\Xi \in \mathbb{R}^{n\times n}$ is called symmetric if $\Xi' = \Xi$. The identity matrix of dimension $m\times m$ is denoted by $I_m \in \mathbb{R}^{m\times m}$. The $n\times m$ zero matrix is denoted by $0_{n,m}$. $\mathbb{C}_{-\,-} = \{s \in \mathbb{C} : \mathbf{Re}(s) < 0\}$. $\mathbb{C}_{-} = \{s \in \mathbb{C} : \mathbf{Re}(s) \leq 0\}$.

Let $A \in \mathbb{R}^{n\times m}$ and $B \in \mathbb{R}^{q\times p}$. Then, the Kronecker product of A and B is denoted by $A \otimes B$ and defined as

$$A \otimes B = \begin{bmatrix} a_{11}B & \cdots & a_{1m}B \\ \vdots & \ddots & \vdots \\ a_{n1}B & \cdots & a_{nn}B \end{bmatrix} \in \mathbb{R}^{nq\times mp},$$

where a_{ij} is the (i, j)-th entry of A, with $i = 1, \cdots , n$ and $j = 1, \cdots , m$.

Let $\lambda_i(\Xi)$ denote the i-th eigenvalue of $\Xi \in \mathbb{R}^{n\times n}$, $i = 1, \cdots , n$. Then, the spectrum of Ξ is denoted by $S(\Xi) = \{\lambda_1(\Xi), \cdots , \lambda_n(\Xi)\}$.

Definition 1. Matrix $\Xi \in \mathbb{R}^{n\times n}$ is called Hurwitz (or stable) if all its eigenvalues have negative real part, i.e., $\lambda_i(\Xi) \in \mathbb{C}_{-\,-}$, $i = 1, \cdots , n$.

Proposition 2. Let $A_1 = aI_m$ and $A_2 \in \mathbb{R}^{m\times m}$. Then $\lambda_i(A_1 + A_2) = a + \lambda_i(A_2)$, $i = 1, \cdots , m$.

Proof. Let $\lambda_i(A_2)$ be any eigenvalue of A_2 with corresponding eigenvector $v_i \in \mathbb{C}^m$. Then $(A_1 + A_2)v_i = A_1v_i + A_2v_i = av_i + \lambda_i(A_2)v_i = (a + \lambda_i(A_2))v_i$.

Proposition 3. Consider matrices $A_1, A_2 \in \mathbb{R}^{m\times m}$ and $\Xi \in \mathbb{R}^{n\times n}$ and let $\bar{A}_1 = I_n \otimes A_1$ and $\bar{A}_2 = \Xi \otimes A_2$ with $\bar{A}_1, \bar{A}_2 \in \mathbb{R}^{nm\times nm}$. Then $S(\bar{A}_1 + \bar{A}_2) = \cup_{i=1}^{n} S(A_1 + \lambda_i(\Xi)A_2)$ where $\lambda_i(\Xi)$ represents the i-th eigenvalue of Ξ.

Proof. Let $v \in \mathbb{C}^n$ be an eigenvector of Ξ associated with eigenvalue $\lambda(\Xi)$ and $u \in \mathbb{C}^m$ be an eigenvector of $M = A_1+\lambda(\Xi)A_2$ associated with eigenvalue $\lambda(M)$. Define the vector $v \otimes u \in \mathbb{C}^{nm}$ and consider

$$\begin{aligned}(\bar{A}_1 + \bar{A}_2)(v \otimes u) &= v \otimes A_1u + \Xi u \otimes A_2u \\ &= v \otimes A_1u + \lambda(\Xi)v \otimes u \\ &= v \otimes (A_1u + \lambda(\Xi)A_2u).\end{aligned}$$

Since $(A_1 + \lambda(\Xi)A_2)u = \lambda(\Xi)u$, we get $(\bar{A}_1 + \bar{A}_2)(v \otimes u) = \lambda(\Xi)(v \otimes u)$.

A graph $\mathcal{G}$ is defined as the ordered pair $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V}$ is the set of nodes (or vertices) $\mathcal{V} = \{1, \cdots , N\}$ and $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ the set of edges (i, j) with $i \in \mathcal{V}$, $j \in \mathcal{V}$. The degree d_j of a graph vertex j is the number of edges which start from j. Let $d_{max}(\mathcal{G})$ denote the maximum vertex degree of the graph $\mathcal{G}$. We denote by $\mathcal{A}(\mathcal{G})$ the $(0, 1)$ adjacency matrix of the graph $\mathcal{G}$. In particular, the (ij)th element of $\mathcal{A}$, $A_{ij} = 1$ if $(i, j) \in \mathcal{E}\ \forall\ i, j = 1, \cdots , N$, $i \neq j$ and zero otherwise. Let $j \in \mathcal{N}_i$ if $(i, j) \in \mathcal{E}$ and $i \neq j$. We call $\mathcal{N}_i$ the neighborhood of node i. The adjacency matrix $\mathcal{A}(\mathcal{G})$ of undirected graphs is symmetric. We define the Laplacian matrix as $\mathcal{L}(\mathcal{G}) = \mathcal{D}(\mathcal{G}) - \mathcal{A}(\mathcal{G})$, where $\mathcal{D}(\mathcal{G})$ is the diagonal matrix of vertex degrees d_i (also called the valence matrix). Let $S(\mathcal{L}(\mathcal{G})) = \{\lambda_1(\mathcal{L}(\mathcal{G})), \cdots , \lambda_N(\mathcal{L}(\mathcal{G}))\}$ be the spectrum of the Laplacian matrix $\mathcal{L}$ associated with an undirected graph $\mathcal{G}$ arranged in nondecreasing semi-order. The following Proposition is derived straight forward from Proposition 3

Proposition 4. Let A, B be matrices of appropriate dimensions and $\mathcal{L}$ be Laplacian matrix of graph $\mathcal{G}$ with spectrum $S(\mathcal{L}) = \{\lambda_1(\mathcal{L}), \cdots , \lambda_N(\mathcal{L})\}$. Then,

$$S(I_N \otimes A + \mathcal{L} \otimes B) = \cup_{i\in[1:N]} S(A + \lambda_i(\mathcal{L})B),$$

with $\lambda_i(\mathcal{L}) \in S(\mathcal{L})$.

The following result is standard (Mohar (1991)) and is stated without proof.

Proposition 5. Let $\mathcal{G}$ be a complete graph (with all possible edges) with N_L vertices and $\mathcal{L}(\mathcal{G})$ be the corresponding Laplacian matrix. Then the following hold: $\mathcal{L}(\mathcal{G})^p = N_L^{p-1}\mathcal{L}(\mathcal{G})$ and $S(\mathcal{L}(\mathcal{G})) = \{0, N_L, \cdots , N_L\}$.

3. LARGE-SCALE LQR FOR DYNAMICALLY COUPLED SYSTEMS

Consider a network of N_L dynamically coupled LTI systems referred to as agents. At local level the dynamics of the i-th agent is represented in state-space form as:

$$\dot{x}_i = A_1x_i + A_2 \sum_{j=1, j\neq i}^{N_L} (x_i - x_j) + Bu_i,\ x_{0,i} = x_i(0) \quad (1)$$

where $x_i \in \mathbb{R}^n$, $u_i \in \mathbb{R}^m$ denote states and inputs of the i-th system, respectively. A complete graph (with all possible edges) $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ with Laplacian matrix $\mathcal{L}_c$ is utilized to model the topology of the physical links among the agents. Node $i \in \mathcal{V}$ of $\mathcal{G}$ corresponds to local state x_i while edge $(i, j) \in \mathcal{E}$ corresponds to the $x_i - x_j$ term in (1). Now construct the aggregate state $\tilde{x} \in \mathbb{R}^{nN_L}$ and input vector $\tilde{u} \in \mathbb{R}^{mN_L}$ by stacking all state and input vectors, respectively, of all N_L systems taken in ascending order depending on their label in graph $\mathcal{G}$. The aggregate state-space of the network becomes:

$$\dot{\tilde{x}} = \tilde{A}\tilde{x} + \tilde{B}\tilde{u},\ \tilde{x}_0 = \tilde{x}(0) \tag{2}$$

with

$$\tilde{A} = I_{N_L} \otimes A_1 + \mathcal{L}_c \otimes A_2,\ \tilde{B} = I_{N_L} \otimes B \tag{3}$$

Consider now LQR control problem for the network of N_L coupled systems:

$$\min_{\tilde{u}}\ J(\tilde{u}, \tilde{x}_0) \text{ s.t. } \dot{\tilde{x}} = \tilde{A}\tilde{x} + \tilde{B}\tilde{u},\ \tilde{x}_0 = \tilde{x}(0) \tag{4}$$

where the cost function

$$J(\tilde{u}, \tilde{x}_0) = \int_0^\infty \tilde{x}'\tilde{Q}\tilde{x} + \tilde{u}'\tilde{R}\tilde{u}\ dt \tag{5}$$

with

$$\tilde{Q} = I_{N_L} \otimes Q_1 + \mathcal{L}_c \otimes Q_2 \text{ and } \tilde{R} = I_{N_L} \otimes R. \tag{6}$$

Here $Q_1 = Q_1' \geq 0$ and $Q_2 = Q_2' \geq 0$ penalize local state and relative state difference $x_i - x_j$ between the nodes $i, j \in \mathcal{V}$, respectively. The matrix $R = R' > 0$ weighs inputs of each subsystem. The following stabilizability and observability assumptions guarantee a unique stabilizing solution to LQR problem (4).

Assumption 6. Let $C_1'C_1 = Q_1$. The pair (A_1, B) is stabilizable and (A_1, C_1) is observable.

Assumption 7. Let $C_{12}'C_{12} = Q_1 + N_LQ_2$. The pair $(A_1 + N_LA_2, B)$ is stabilizable and $(A_1 + N_LA_2, C_{12})$ is observable.

Under Assumption 6,7, problem (4) has a unique stabilizing solution $\tilde{u} = \tilde{K}\tilde{x}$, which gives finite performance index (5) equal to $\tilde{x}_0'\tilde{P}\tilde{x}_0$. The optimal state-feedback gain $\tilde{K} = -\tilde{R}^{-1}\tilde{B}'\tilde{P}$, where $\tilde{P}$ is the symmetric positive definite (s.p.d.) solution to the (large-scale) Algebraic Riccati Equation (ARE):

$$\tilde{A}'\tilde{P} + \tilde{P}\tilde{A} - \tilde{P}\tilde{B}\tilde{R}^{-1}\tilde{B}'\tilde{P} + \tilde{Q} = 0. \tag{7}$$

Due to special formulation of (4), $\tilde{K}$ and $\tilde{P}$ retain certain structure, which will prove essential for designing stabilizing distributed controllers in the next section. The specific structure of these matrices is proved in Theorem 8. In the following, we set $X = BR^{-1}B'$.

Theorem 8. Assume $\tilde{P}$ is the s.p.d solution to (7) associated with the optimal solution to (4). Let $\tilde{P} \in \mathbb{R}^{nN_L \times nN_L}$ be partitioned into N_L^2 blocks of dimension $n \times n$, each denoted by $\tilde{P}_{ij}$ and referred to as the (i, j)-block of $\tilde{P}$. Then, the following hold:

I. For $i = 1, \cdots, N_L$, $\sum_{j=1}^{N_L} \tilde{P}_{ij} = P$ where

$$A_1'P + PA_1 - PXP + Q_1 = 0. \tag{8}$$

II. $\tilde{P}_{ij} = \tilde{P}_{kl} = \tilde{P}_2$ for all $i, j, k, l = 1, \cdots, N_L$ with $j \neq i$ and $l \neq k$, and $\tilde{P}_2$ symmetric.

Proof. First, we prove part I of the theorem. The equations corresponding to diagonal blocks $\tilde{P}_{ii}$ of $\tilde{P}$ in (7) are:

$$\begin{aligned}&(A_1 + (N_L-1)A_2)'\tilde{P}_{ii} - A_2'\sum_{\substack{j=1\\j\neq i}}^{N_L}\tilde{P}_{ij}\\&+ \tilde{P}_{ii}(A_1 + (N_L-1)A_2) - \sum_{\substack{j=1\\j\neq i}}^{N_L}\tilde{P}_{ij}A_2\\&- \sum_{k=1}^{N_L}\tilde{P}_{ik}X\tilde{P}_{ik} + Q_1 + (N_L-1)Q_2 = 0\end{aligned} \tag{9}$$

for $i = 1, \cdots, N$. Note that $\tilde{P}_{ij} = \tilde{P}_{ji}$ due to symmetry of (7). Now let

$$F_{ii} = \tilde{P}_{ii} + \sum_{\substack{i=1\\j\neq i}}^{N_L}\tilde{P}_{ij}. \tag{10}$$

Substituting (10) to (9) gives:

$$(N_L-1)(A_2'F_{ii} + F_{ii}A_2) - N_L(A_2'\sum_{\substack{j=1\\j\neq i}}^{N_L}\tilde{P}_{ij} - \sum_{\substack{j=1\\j\neq i}}^{N_L}\tilde{P}_{ij}A_2) \tag{11a}$$

$$+ A_1'(F_{ii} - \sum_{\substack{j=1\\j\neq i}}^{N_L}\tilde{P}_{ij}) + (F_{ii} - \sum_{\substack{j=1\\j\neq i}}^{N_L}\tilde{P}_{ij})A_1$$

$$- \sum_{k=1}^{N_L}\tilde{P}_{ik}X\tilde{P}_{ik} + Q_1 + (N_L-1)Q_2 = 0. \tag{11b}$$

Using (10) the equations corresponding to off-diagonal blocks $\tilde{P}_{ij}$, $i \neq j$ of $\tilde{P}$ are

$$\begin{aligned}&(N_L-1)(A_2'\tilde{P}_{ij} + \tilde{P}_{ij}A_2) - A_2'(F_{ii} - \sum_{\substack{k=1\\k\neq i}}^{N_L}\tilde{P}_{ik})\\&- (F_{ii} - \sum_{\substack{k=1\\k\neq i}}^{N_L}\tilde{P}_{ik})A_2 - A_2'\sum_{\substack{l=1\\l\neq i\\l\neq j}}^{N_L}\tilde{P}_{il} - \sum_{\substack{l=1\\l\neq i\\l\neq j}}^{N_L}\tilde{P}_{il}A_2\end{aligned} \tag{12a}$$

$$+ A_1'\tilde{P}_{ij} + \tilde{P}_{ij}A_1 - \sum_{k=1}^{N_L}\tilde{P}_{ik}X\tilde{P}_{kj} - Q_2 = 0 \tag{12b}$$

Summing up (12a) for all $j \neq i$ block-wise and adding this summation to (11a) gives

$$\begin{aligned}&(N_L-1)(A_2'F_{ii} + F_{ii}A_2) - (N_L-1)A_2'F_{ii}\\&- F_{ii}(N_L-1)A_2 - N_LA_2'\sum_{\substack{j=1\\j\neq i}}^{N_L}\tilde{P}_{ij} - \sum_{\substack{j=1\\j\neq i}}^{N_L}\tilde{P}_{ij}N_LA_2\\&+ (N_L-1)(A_2'\sum_{\substack{j=1\\j\neq i}}^{N_L}\tilde{P}_{ij} + \sum_{\substack{j=1\\j\neq i}}^{N_L}\tilde{P}_{ij}A_2)\\&+ (N_L-1)(A_2'\sum_{\substack{k=1\\k\neq i}}^{N_L}\tilde{P}_{ik} + \sum_{\substack{k=1\\k\neq i}}^{N_L}\tilde{P}_{ik}A_2)\\&- (N_L-1)(A_2'\sum_{\substack{l=1\\l\neq i\\l\neq j}}^{N_L}\tilde{P}_{il} - \sum_{\substack{l=1\\l\neq i\\l\neq j}}^{N_L}\tilde{P}_{il}A_2) = 0\end{aligned} \tag{13}$$

where all terms associated with A_2 cancel out. By summing up now (12) for all $j \neq i$ block-wise and adding this summation to (11) gives

$$A_1' F_{ii} + F_{ii} A_1 - F_{ii} X F_{ii} + \sum_{\substack{k=1\\k\neq i}}^{N_L} \big(\tilde{P}_{ik} X \big(F_{ii} - F_{kk}\big)\big) + Q_1 = 0. \quad (14)$$

Eq. (14) has been established in Theorem 1 of Borrelli and Keviczky (2008). It is true also here due to (13). Summing up (14) for all $i = 1, \cdots, N_L$ we get

$$\sum_{i=1}^{N_L} \Big(A_1' F_{ii} + F_{ii} A_1 - F_{ii} X F_{ii} + Q_1 \Big) = 0 \quad (15)$$

which is sum of N_L identical ARE's, i.e.,

$$A_1' F_{ii} + F_{ii} A_1 - F_{ii} X F_{ii} + Q_1 = 0. \quad (16)$$

Equation (10) implies $F_{ii} = \sum_{i=1}^{N_L} \tilde{P}_{ij}$ which, along with (16), proves part I. The ARE (7) has a repetitive structure and essentially can be decomposed into N_L identical equations since $\tilde{B}$, $\tilde{R}$ are block diagonal and $\tilde{A}$, $\tilde{Q}$ have repetitive pattern. This implies that matrices $\tilde{P}_{ij}$ with $j \neq i$ are all equal and symmetric. This proves part II of the theorem. $\square$

The following corollary of Theorem 8 is stated without proof due to lack of space.

Corollary 9. Let $\tilde{P}_{ij} \in \mathbb{R}^{n\times n}$, $i, j = 1, \cdots, N_L$, denote the (i, j)-block of $\tilde{P}$ in (7) associated with the optimal solution to (4). Then, the following hold:

I. $\tilde{P}_{ii} = P - (N_L - 1)\tilde{P}_2$, for all $i = 1, \cdots, N_L$ where P is the s.p.d solution to ARE (8).
II. $\tilde{P}_{ij} = \tilde{P}_2$, for $i, j = 1, \cdots, N_L$ and $i \neq j$, where $\tilde{P}_2$ is symmetric matrix associated with node-level ARE:

$$(A_1 + N_L A_2)'(P - N_L\tilde{P}_2) + (P - N_L\tilde{P}_2)(A_1 + N_L A_2) - (P - N_L\tilde{P}_2)X(P - N_L\tilde{P}_2) + Q_1 + N_L Q_2 = 0. \quad (17)$$

By assumption, the matrices $\tilde{R}$ and $\tilde{B}$ are selected block diagonal. Consequently, the state-feedback gain $\tilde{K} = -\tilde{R}^{-1}\tilde{B}'\tilde{P}$ associated with the optimal solution to (4) retains the same structure with $\tilde{P}$. This leads to the following Corollary.

Corollary 10. Assume $\tilde{K} = -\tilde{R}^{-1}\tilde{B}'\tilde{P}$ is the optimal state-feedback gain obtained from the solution to (4) which gives minimum performance index $\tilde{x}_0'\tilde{P}\tilde{x}_0$ with $\tilde{P}$ being the s.p.d solution to (7). Let $\tilde{K} \in \mathbb{R}^{mN_L\times nN_L}$ and $\tilde{P} \in \mathbb{R}^{nN_L\times nN_L}$ be partitioned into N_L^2 blocks of dimension $m \times n$ and $n \times n$, denoted by $\tilde{K}_{ij}$ and $\tilde{P}_{ij}$, respectively each referred to as (i, j)-block of the respective matrix. Then, the following are true;

I. $\tilde{P} = I_{N_L} \otimes P - \mathcal{L}_c \otimes \tilde{P}_2$.
II. $\sum_{j=1}^{N_L} \tilde{K}_{ij} = -R^{-1}B'P$ for $i = 1, \cdots, N_L$.
III. $\tilde{K}_{ii} = -R^{-1}B'P + (N_L - 1)R^{-1}B'\tilde{P}_2$ for $i = 1, \cdots, N_L$.
IV. $\tilde{K}_{ij} = -R^{-1}B'\tilde{P}_2$ for $i, j = 1, \cdots, N_L$ and $j \neq i$.
V. $\tilde{K} = -I_{N_L} \otimes R^{-1}B'P + \mathcal{L}_c \otimes R^{-1}B'\tilde{P}_2$.

Theorem 8 along with the results stated in Corollary 9 suggest that due to special formulation of the cost function (5) and the structure of the aggregate state-space form (2), the large-scale LQR problem (4) under Assumption 6,7 can effectively be reduced to finding the solution of two node-level ARE's. This feature may be highly beneficial for problems involving networks, the topology of which is modeled by graph with an excessively large number of vertices (N_L).

Applying the stabilizing optimal state-feedback control $\tilde{u} = \tilde{K}\tilde{x}$ to (2) results in a closed-loop matrix, which is Hurwitz and is written as:

$$A_{cl} = I_{N_L} \otimes (A_1 - XP) + \mathcal{L}_c \otimes (A_2 + X\tilde{P}_2). \quad (18)$$

Due to Proposition 3 the spectrum of A_{cl} can be decomposed into:

$$S(A_{cl}) = \bigcup_{i=0}^{N_L} S\big(A_1 + BK + \lambda_{c,i}(A_2 - BK_2)\big) \quad (19)$$

where $\lambda_{c,i} \in S(\mathcal{L}_c)$.

Remark 11. From Proposition 5, the matrix $A_1 + BK + \alpha N_L(A_2 - BK_2)$ is Hurwitz for $\alpha = 0$ and $\alpha = 1$.

In the sequel we require that:

Condition 12. The matrix $A_1 + BK + \alpha N_L(A_2 - BK_2)$ is Hurwitz for all $\alpha \in [0, 1]$.

Condition 12 states that all convex combinations of two Hurwitz matrices

$$\mu\bar{A}_1 + (1 - \mu)\bar{A}_2 \text{ with } \mu \in [0, 1] \quad (20)$$

are Hurwitz, where $\bar{A}_1 = A_1 + BK + N_L(A_2 - BK_2)$ and $\bar{A}_2 = A_1 + BK$. Sufficient conditions for Hurwitz stability of convex combination of Hurwitz matrices can be found in Theorem 2.2 in Białas (2004). In essence, Condition 12 characterizes a class of LQR problems (4) which admit of solutions for which the Condition 12 holds. This will be used later for the design of distributed stabilizing controllers. For a given selection of weighting matrices (Q_1, Q_2, R) of the LQR problem (4), the validity of Condition 12 can be verified by searching for a symmetric positive definite matrix $\bar{P}$ for which the following Linear Matrix Inequality (LMI),

$$\begin{bmatrix} -(\bar{A}_1'\bar{P} + \bar{P}\bar{A}_1) & 0_{n\times n} & 0_{n\times n} \\ 0_{n\times n} & -(\bar{A}_2'\bar{P} + \bar{P}\bar{A}_2) & 0_{n\times n} \\ 0_{n\times n} & 0_{n\times n} & \bar{P} \end{bmatrix} > 0, \quad (21)$$

is feasible. Obviously, if matrix $\bar{P}$ exists then premultiplying and postmultiplying (21) by $[\sqrt{\mu}I_n \ \sqrt{1-\mu}I_n \ 0_{n\times n}]'$ and $[\sqrt{\mu}I_n \ \sqrt{1-\mu}I_n \ 0_{n\times n}]$, respectively, for $\mu \in [0, 1]$ leads to Lyapunov inequality:

$$(\mu\bar{A}_1 + (1 - \mu)\bar{A}_2)'\bar{P} + \bar{P}(\mu\bar{A}_1 + (1 - \mu)\bar{A}_2) < 0, \quad (22)$$

which admits of a solution $\bar{P} = \bar{P}' > 0$. This demonstrates that $\mu\bar{A}_1 + (1 - \mu)\bar{A}_2$ is a Hurwitz matrix for all $\mu \in [0, 1]$. Alternatively, the stability of $\mu\bar{A}_1 + (1 - \mu)\bar{A}_2$ can be examined via a simple graphical test by plotting the eigenvalue with the maximum real part of the matrix $\mu\bar{A}_1 + (1 - \mu)\bar{A}_2$ for $\mu \in [0, 1]$.

The following arguments are given without proof due to lack of space. Consider now the aggregate input matrix

$$\tilde{B} = I_{N_L} \otimes B_1 + \mathcal{L}_c \otimes B_2 \quad (23)$$

where $B_1 \in \mathbb{R}^{n\times m}$, $B_2 \in \mathbb{R}^{n\times m}$ represent input channels corresponding to local inputs and relative input difference $u_i - u_j$ between the node $i, j \in \mathcal{V}$ with $i \neq j$, respectively. In this case the state-space form at node level is written as:

$$\dot{x}_i = A_1 x_i + A_2 \sum_{j=1, j\neq i}^{N_L} (x_i - x_j) + B_1 u_i + B_2 \sum_{j=1, j\neq i}^{N_L} (u_i - u_j),\ x_{0,i} = x_i(0). \quad (24)$$

Then solving LQR problem (4) with $\tilde{A}$ as given in (3), $\tilde{B}$ as given in (23) and weighting matrices $(\tilde{Q}, \tilde{R})$ as selected in (6), results in optimal solution $\tilde{P}$ in (7). This has structure as defined in Theorem 8 and Corollary 9, i.e., $\tilde{P} = I_{N_L} \otimes P - \mathcal{L}_c \otimes \tilde{P}_2$ where P is the solution to ARE (8) and $\tilde{P}_2$ is symmetric matrix associated with ARE (17). The optimal state-feedback controller $\tilde{K}$ can then be derived by $\tilde{K} = -\tilde{R}^{-1}\tilde{B}'\tilde{P} = I_{N_L} \otimes K - \mathcal{L}_c \otimes K_2$ with

$$K = -R^{-1}B_1'P, \quad (25)$$

$$K_2 = -R^{-1}(B_1'\tilde{P}_2 + B_2'P - N_L B_2'\tilde{P}_2) \quad (26)$$

where the property $\mathcal{L}_c^2 = N_L \mathcal{L}_c$ (see Proposition 5) was used. Proof of closed-loop stability of this control scheme along with derivation of all equations will be included in an extended work of this paper. In the sequel, we consider multi-agent systems with dynamical couplings in their states only.

4. DISTRIBUTED CONTROL DESIGN FOR DYNAMICALLY COUPLED SYSTEMS

Let a network be formed of N identical and dynamically coupled LTI systems. The couplings among the systems are modelled by undirected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ with Laplacian matrix $\mathcal{L}$. The neighborhood of the i-th system is denoted by $\mathcal{N}_i \subset \mathcal{V}$. Let the dynamics at local level of the i-th system be

$$\dot{x}_i = A_1 x_i + A_2 \sum_{j\in\mathcal{N}_i} (x_i - x_j) + B u_i,\ x_{0,i} = x_i(0) \quad (27)$$

where $x_i \in \mathbb{R}^n$ and $u_i \in \mathbb{R}^m$. The aggregate state-space of the network becomes

$$\dot{\hat{x}} = \hat{A}\hat{x} + \hat{B}\hat{u},\ \hat{x}_0 = \hat{x}(0) \quad (28)$$

where $\hat{x} \in \mathbb{R}^{nN}$, $\hat{u} \in \mathbb{R}^{mN}$ and

$$\hat{A} = I_N \otimes A_1 + \mathcal{L} \otimes A_2,\ \hat{B} = I_N \otimes B. \quad (29)$$

Note that the Laplacian matrix $\mathcal{L}$ in (29) does not necessarily correspond to a complete graph in contrast to (3) and generically the matrix $\hat{A}$ in (29) is sparse. We note here that the aggregate state-space forms (28) and (2) differ in number of subsystems and the structure of the Laplacian matrix $\mathcal{L}$. We denote aggregate state-space as in (2) when referring to centralized control problems with N_L subsystems and we use aggregate state-space (28) when referring to distributed control problems with N subsystems. Similarly, tilded matrices are referred to centralized problems while hatted matrices to distributed problems.

A stabilizing distributed controller for (28) is constructed in the following Theorem.

Theorem 13. Consider a network of N coupled systems with dynamics described in (27) modelled by graph $\mathcal{G}_N$ with Laplacian matrix $\mathcal{L}_N$. Let λ_N be the maximum eigenvalue of $\mathcal{L}_N$ and denote by d_{max} the smallest integer which is greater than or equal to λ_N. Consider LQR problem (4) for $N_L = d_{max}$, define P and $\tilde{P}_2$ via (8) and (17), respectively, and assume Condition 12 is valid. Define also: $K = -R^{-1}B'P$, $K_2 = -R^{-1}B'\tilde{P}_2$ and

$$\hat{K} = I_N \otimes K - \mathcal{L}_N \otimes K_2. \quad (30)$$

Then, the closed-loop matrix

$$A_{cl} = I_N \otimes (A_1 + BK) + \mathcal{L}_N \otimes (A_2 - BK_2) \quad (31)$$

is Hurwitz.

Proof. Consider the spectrum $S(A_{cl}) = S(I_N \otimes (A_1 + BK) + \mathcal{L}_N \otimes (A_2 - BK_2))$. Let $V_N \otimes I_n$ be state-space transformation where $V_N \in \mathbb{R}^{N\times N}$ is an orthogonal matrix whose columns consist of the eigenvectors of $\mathcal{L}_N$. In the transformed coordinates, $\bar{A}_{cl} = I_N \otimes (A_1 + BK) + \Lambda_N \otimes (A_2 - BK_2)$ where $\Lambda_N = \mathrm{diag}(0, \lambda_2, \cdots, \lambda_N)$ with $\lambda_N \leq d_{max}$. The spectrum of $\bar{A}_{cl}$ is

$$S(\bar{A}_{cl}) = \bigcup_{i=1}^{N} S(A_1 + BK + \lambda_i(A_2 - BK_2)) \quad (32)$$

where λ_i for $i = 1, \cdots, N$ are the eigenvalues of $\mathcal{L}_N$. Condition 12 holds, hence $(A_1+BK)+\alpha d_{max}(A_2-BK_2)$ is Hurwitz for all $\alpha \in [0, 1]$. Consequently $\bar{A}_{cl}$ is also Hurwitz since $\lambda_i \in [0, d_{max}]$ for all $i = 1, \cdots, N$. This proves the Theorem. □

Remark 14. For a time-varying graph $\mathcal{G}(t) = (\mathcal{V}, \mathcal{E}(t))$ with fixed number of vertices (N) and time-varying edges the maximum eigenvalue of the time-varying Laplacian matrix $\mathcal{L}(t)$ is bounded by $2N$. Consequently, solving (4) for $N_L = 2N$ and assuming Condition 12 holds leads to a distributed controller $\hat{K}$ which stabilizes the network for all possible couplings among the N systems. Naturally, this does not imply stability of switching between stable network topologies.

4.1 Measure of suboptimality

At this point, we wish to employ a suboptimality measure which can be cast as a performance loss index of the distributed scheme proposed above. It is necessary so, to define a reference performance index with which the suboptimal scheme is compared. Such an index has been introduced in Borrelli and Keviczky (2008) and is also outlined here.

Let matrix $\Xi \in \mathbb{R}^{mN\times nN}$ be partitioned into N^2 blocks of dimension $m \times n$, each referred to as (i, j)-block of Ξ and denoted by $\Xi_{ij} \in \mathbb{R}^{m\times n}$, with $i, j = 1, \cdots, N$. In particular, the (i, j)-block can be written as: $\Xi_{ij} = \Xi[(i-1)m + 1 : im, (j-1)n + 1 : jn]$. The class of structured matrices $\mathcal{K}_{n,m}^N(\mathcal{G})$ is now defined as follows:

$\mathcal{K}_{m,n}^N(\mathcal{G}) = \{\Xi \in \mathbb{R}^{mN\times nN} \mid \Xi_{ij} = 0_{m,n}$ if $(i, j) \notin \mathcal{E}$, $\Xi_{ij} = \Xi[(i-1)m + 1 : im, (j-1)n + 1 : jn],\ i, j = 1, \cdots, N\}$.

Consider now the following optimal control problem:

$$\min_{\hat{u}} \; J(\hat{u}, \hat{x}_0) = \int_0^\infty (\hat{x}'\hat{Q}\hat{x} + \hat{u}'\hat{R}\hat{u}) \; dt \tag{33a}$$

$$\text{subject to: } \dot{\hat{x}} = \hat{A}\hat{x} + \hat{B}\hat{u}, \; \hat{x}_0 = \hat{x}(0) \tag{33b}$$

$$\hat{u} = \hat{K}\hat{x} \tag{33c}$$

$$\hat{K} \in \mathcal{K}^N_{m,n}(\mathcal{G}) \tag{33d}$$

$$\hat{Q} \in \mathcal{K}^N_{m,n}(\mathcal{G}), \; \hat{R} = I_N \otimes R \tag{33e}$$

where $\hat{Q} = \hat{Q}' \geq 0$ and $\hat{R} = \hat{R}' > 0$. We note that in the absence of constraint (33d), the optimal control problem (33) if feasible, yields a centralized optimal control $u^* = K^*\hat{x}$ where $K^* = -\hat{R}^{-1}\hat{B}'P^*$ and P^* is the symmetric positive definite solution to:

$$\hat{A}'P^* + P^*\hat{A} - P^*\hat{B}\hat{R}^{-1}\hat{B}'P^* + \hat{Q} = 0. \tag{34}$$

Note that, since constraint (33d) is not included in the optimization, K^* and P^* are centralized solutions, and thus, the value $\hat{x}_0'P^*\hat{x}_0$ is the minimum achievable performance index for a given $\hat{x}_0$.

Assume now that constraint (33d) is in force. Then, the distributed state-feedback controller $\hat{K}$ as constructed in (30) can be considered as a suboptimal solution to (33) with $(\hat{A}, \hat{B})$ as given in (29) and $\hat{Q} = I_N \otimes Q_1 + \mathcal{L}_N \otimes Q_2$, with $\mathcal{L}_N$ as defined in Theorem 13. Then, a performance index for this suboptimal distributed scheme can be computed as

$$J(\hat{K}\hat{x}, \hat{x}_0) = \hat{x}_0'\hat{P}\hat{x}_0, \tag{35}$$

where $\hat{P}$ is the positive definite solution to the following Lyapunov equation:

$$(\hat{A} + \hat{B}\hat{K})'\hat{P} + \hat{P}(\hat{A} + \hat{B}\hat{K}) + (\hat{Q} + \hat{K}'\hat{R}\hat{K}) = 0. \tag{36}$$

Since P^* is optimal, $J(K^*\hat{x}, \hat{x}_0) \leq J(\hat{K}\hat{x}, \hat{x}_0)$ for all $\hat{x}_0$ and thus $\Delta P = \hat{P} - P^*$ is a positive semidefinite matrix which is equal to zero if $\hat{K} = K^*$. Any norm of ΔP can be cast as a measure of suboptimality of the distributed controller $\hat{K}$.

5. CONCLUSIONS

Stabilizing distributed state-feedback controller for networks of coupled identical systems was proposed based on a large-scale LQR optimal problem. This method has originally been proposed in Borrelli and Keviczky (2008) for the decoupled case and was extended here to include couplings between the subsystems representing a multi-agent network. An effective condition for stability is also provided which admits of network stable operation for a class of interconnections. The control scheme was obtained by optimizing an LQR performance index with a tuning parameter utilized to emphasize/de-emphasize relative state difference between interconnected agents. The assumption of identical dynamics is clearly restrictive but simplifies the design problem considerably and leads to the derivation of a stability condition which can be easily tested. Attempts to eliminate or relax this assumption will be the topic of future work. Preliminary results in this direction can be found in Vlahakis and Halikias (2018a,b). The control scheme has also been successfully applied to a multi-area power grid numerous simulations for which can be found in Vlahakis et al. (2019).

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