Nonlocal gauge equivalence: Hirota versus extended continuous Heisenberg and Landau-Lifschitz equation

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Abstract: We exploit the gauge equivalence between the Hirota equation and the extended continuous Heisenberg equation to investigate how nonlocality properties of one system are inherited by the other. We provide closed generic expressions for nonlocal multi-soliton solutions for both systems. By demonstrating that a specific auto-gauge transformation for the extended continuous Heisenberg equation becomes equivalent to a Darboux transformation, we use the latter to construct the nonlocal multi-soliton solutions from which the corresponding nonlocal solutions to the Hirota equation can be computed directly. We discuss properties and solutions of a nonlocal version of the nonlocal extended Landau-Lifschitz equation obtained from the nonlocal extended continuous Heisenberg equation or directly from the nonlocal solutions of the Hirota equation.

1. Introduction

Traditionally the study of solutions to nonlinear wave equations has mainly focused on systems that involve local fields that all depend on a single point in space-time. However, there are many well-known phenomena in nature related to events that appear to be correlated to each other even though they are separated in a spacelike or timelike fashion. In quantum mechanics entanglement, see e.g. [1], is a well studied phenomenon that in some settings seems to exclude the possibility of signalling [2] and can be implemented even for many particle systems [3]. There are, however, also physical phenomena that are of a nonlocal nature that are describable with nonlinear wave equations, such as nonlocal rogue waves [4, 5], weather forecast models in which nonlocality is caused by feedback loops [6, 7], gravitational waves [8], etc.

Recently a simple principle was identified [9, 10] that introduces nonlocality into nonlinear integrable systems in a systematic and mathematically well-defined manner. This is achieved by exploring various versions of $\mathcal{PT}$-symmetry present in the zero curvature condition that relates fields in the theory to each other in a nonlocal fashion. One particular type of these new nonlocal nonlinear Schrödinger equation has attracted a lot of attention [11, 12, 13, 14, 15, 16, 17, 18, 19]. These studies were extended to other types of systems, such as Fordy-Kulish equations [20], Davey-Stewartson equations [21, 22], Sasa-Satsuma equations [23], Kadomtsev-Petviashvili equations [24] and Korteweg de-Vries systems [25, 26, 27]. Here we will build on a particular case of the various nonlocal versions of the Hirota equation [28]. In the local case the extension from the nonlinear Schrödinger equation to the Hirota equation is suggested by experiments in the high-intensity and short pulse subpicosecond regime [29, 30] where the accurate description of the former equation [31] breaks down. The nonlocality is known to be implementable by the applications of various variants of $\mathcal{PT}$-symmetry [32] to the equations resulting from an AKNS zero curvature construction [33]. On the other hand one may also directly decompose the fields in some local systems into field depending of different points in space-time and obtain nonlocal systems in this manner, as described in [25, 26, 27]. Here we explore the possibility to exploit the gauge equivalence of two systems and investigate how the nonlocality properties of one system is inherited by the other. As concrete systems we investigate the gauge equivalent pair of the extended versions of the continuous limit of the Heisenberg equation (ECHE) [34, 35, 36, 37, 38] and the extended Landau-Lifschitz equations (ELLE) [39, 40]. The local version of the original Landau-Lifschitz equation famously describes the precession of the magnetization in a solid when subjected to a torque resulting from an effective external magnetic field. Various extended versions have been proposed, such as for instance the Landau-Lifshitz-Gilbert equation [41] to take damping into account. The nonlocal versions of these equation studied here provide further extensions with complex components. We will see how the nonlocality may be incorporated most naturally into a pair of auxiliary equations occurring this setting.

We will also show how the nonlocality is implemented into two standard types of solution procedures for nonlinear systems, Hirota's direct method and the method of using Darboux transformations. Similarly as in [28] we find new types of solutions in the nonlocal setting which have no counterpart in the local case.

Our manuscript is organized as follows: In section 2 we introduce our three systems the nonlocal Hirota equation, the extended versions of the continuous limit of the Heisenberg equation and the extended Landau Lifschitz equations, and explain how they are related to each other. In section 3 we explain in general how specific choices for the gauge transformations can become equivalent to Darboux transformations that when iterated may be used to construct multi-soliton solutions. We then utilize this scheme to derive explicit expressions for the multi-soliton solutions to the continuous limit of the Heisenberg equation. These solutions may then be used to calculate directly solutions to the Hirota equation as explained in section 4. In section 5 we discuss how to obtain nonlocal multi-soliton solutions to the extended Landau-Lifschitz equation. We provide two alternative ways to achieve this, directly via the nonlocal solutions of the Hirota equation or by implementing the nonlocality on some auxiliary equations that emerge in the solution process of the continuous limit of the Heisenberg equation. Our conclusions are stated in section 6.

2. Nonlocal gauge equivalence

Many integrable systems are related to each other by means of gauge transformations, often in an unexpected way. Such type of correspondences can be exploited to gain insight into either system from its gauge partner, for instance by transforming solutions of one system to solutions of the other. Often this process can only be carried out in one direction. One may also consider auto-gauge transformation from a system to itself, which when iterated can be used to generate new types of solutions, similar to Darboux or auto-Bäcklund transformations. In general, we consider here two systems whose auxiliary functions Ψ_1 and Ψ_2 are related to each other by means of a gauge field operator G as $\Psi_1 = G\Psi_2$. Formally the system can be cast into two gauge equivalent zero curvature conditions for the two sets of two operators, U_1, V_1 and U_2, V_2, together with their equivalent two linear first order differential equations involving the auxiliary function Ψ_1 and Ψ_2

$$\partial_t U_i - \partial_x V_i + [U_i, V_i] = 0 \qquad \Leftrightarrow \qquad \Psi_{i,t} = V_i\Psi_i,\ \Psi_{i,x} = U_i\Psi_i \quad i = 1, 2. \tag{2.1}$$

Given the transformation from Ψ_1 to Ψ_2, the operators U_1, V_1 and U_2, V_2 are related as

$$U_1 = GU_2G^{-1} + G_xG^{-1}, \qquad \text{and} \qquad V_1 = GV_2G^{-1} + G_tG^{-1}. \tag{2.2}$$

The relations (2.1) and (2.2) are entirely generic providing a connection between two types of integrable systems, assuming the invertible gauge transformation map G exists. Specific systems are obtained by concrete choices of the two sets of two operators U_1, V_1 and U_2, V_2. Concrete versions for the gauge equivalence for specific systems can be found in many places of the literature; for nonlocal versions of Hirota type systems see for instance [42].

2.1 The nonlocal Hirota system

Let us first specify the system 1, by taking U_1 and V_1 to be of the form

$$U_1 = A_0 + \lambda A_1, \qquad V_1 = B_0 + \lambda B_1 + \lambda^2 B_2 + \lambda^3 B_3, \tag{2.3}$$

where

$$A_0 = \begin{pmatrix} 0 & q(x,t) \\ r(x,t) & 0 \end{pmatrix}, \quad A_1 = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} = -i\sigma_3, \tag{2.4}$$

$$B_0 = i\alpha\left[\sigma_3 (A_0)_x - \sigma_3 A_0^2\right] + \beta\left[2A_0^3 + (A_0)_x A_0 - A_0 (A_0)_x - (A_0)_{xx}\right], \tag{2.5}$$

$$B_1 = 2\alpha A_0 + 2i\beta\sigma_3\left[(A_0)_x - A_0^2\right], \tag{2.6}$$

$$B_2 = 4\beta A_0 - 2i\alpha\sigma_3, \tag{2.7}$$

$$B_3 = -4i\beta\sigma_3, \tag{2.8}$$

with σ_i, $i = 1, 2, 3$ denoting the Pauli spin matrices, λ the spectral parameter and α, β are real constants. Using the explicit expressions (2.3)-(2.8) the zero curvature condition (2.1) becomes equivalent to the Hirota system [43, 28] for the fields $q(x,t)$ and $r(x,t)$ as

$$q_t - i\alpha q_{xx} + 2i\alpha q^2 r + \beta\left[q_{xxx} - 6qrq_x\right] = 0, \tag{2.9}$$

$$r_t + i\alpha r_{xx} - 2i\alpha qr^2 + \beta\left(r_{xxx} - 6qrr_x\right) = 0. \tag{2.10}$$

These equations may be viewed with $q(x,t)$ and $r(x,t)$ as entirely independent functions, but most commonly one imposes the relation $r(x,t) = \kappa q^*(x,t)$ with $\kappa = \pm 1$, such that the two equations become their mutual conjugates and are therefore essentially reduced to one equation only - the Hirota equation. Recently [28] alternative possibilities that exploit $\mathcal{PT}$-symmetry have been proposed, such as taking $r(x,t) = \kappa q^*(-x,t)$, $r(x,t) = \kappa q^*(x,-t)$, $r(x,t) = \kappa q^*(-x,-t)$, $r(x,t) = \kappa q(-x,t)$, $r(x,t) = \kappa q(x,-t)$ or $r(x,t) = \kappa q(-x,-t)$ with $\kappa \in \mathbb{R}$ and a suitable adaptation of the parameters α and β. As for these type of choices the equations contain fields that depend simultaneously on x and $-x$, and/or t and $-t$, they are referred to as *nonlocal.* These type of novel variants of integrable systems are the main focus of this manuscript. It was shown in [28] that the different versions display quite distinct and varied behaviour and therefore deserve to be investigated in their own right. However, in what follows we will exclusively focus on the complex parity extended version corresponding to the choice $r(x,t) = \kappa q^*(-x,t)$ together with $\beta = i\delta$, $\delta \in \mathbb{R}$, and refer to it as the *nonlocal* case throughout the manuscript. The treatment of the other cases goes along the same lines, but will not be discussed here.

2.2 The nonlocal extended continuous Heisenberg equation

Having committed to a fixed form of the system 1, we elaborate next on a more precise form of the system 2 following from that concrete choice. Employing the expansion (2.3), we obtain from (2.2) the expressions

$$U_2 = -i\lambda G^{-1}\sigma_3 G, \qquad V_2 = \lambda G^{-1}B_1 G + \lambda^2 G^{-1}B_2 G + \lambda^3 G^{-1}B_3 G \tag{2.11}$$

together with

$$G_x = A_0 G, \qquad \text{and} \qquad G_t = B_0 G. \tag{2.12}$$

With given A_0 and B_0, it is the solution for these two equations in (2.12) that determines the precise form of the gauge map G for a particular set of models.

An interesting and universally applied equation emerges when we use the gauge field G to define a new field operator

$$S := G^{-1}\sigma_3 G. \tag{2.13}$$

The following properties follow directly from above

$$S^2 = 1, \quad S_x = 2G^{-1}\sigma_3 A_0 G, \quad SS_x = -S_x S = 2G^{-1}A_0 G, \quad [S, S_{xx}] = 2\left(SS_{xx} + S_x^2\right). \tag{2.14}$$

Next we notice that instead of expressing the operators U_2 and V_2 in terms of the gauge field G, one can express them entirely in terms of the operator S as

$$U_2 = -i\lambda S, \qquad V_2 = \alpha\left(\lambda SS_x - \lambda^2 2iS\right) + \beta\left[\lambda\left(i\frac{3}{2}SS_x^2 + iS_{xx}\right) + \lambda^2 2SS_x - \lambda^3 4iS\right]. \tag{2.15}$$

Using this variant we evaluate the zero curvature condition (2.1) with (2.11) and the identities (2.14), to obtain the equation of motion for the S-operator

$$S_t = i\alpha\left(S_x^2 + SS_{xx}\right) - \beta\left[\frac{3}{2}\left(SS_x^2\right)_x + S_{xxx}\right] \tag{2.16}$$

$$= \frac{i}{2}\alpha\,[S, S_{xx}] - \frac{\beta}{2}\left(3S_x^3 + S\,[S, S_{xxx}]\right). \tag{2.17}$$

For $\beta = 0$ this equation reduces to the well-known continuous limit of the Heisenberg spin chain [34, 35, 36, 37, 38] and for $\beta \neq 0$ to the first member of the corresponding hierarchy [44]. We refer to this equation as the extended continuous Heisenberg (ECH) equation. The equation (2.16) is rather universal as it also emerges for other types of integrable higher order equations of nonlinear Schrödinger type, such as the modified Korteweg-de Vries equation [45, 46] or the Sasa–Satsuma equation [45, 47]. The distinction towards specific models of this general type is obtained by specifying the gauge map G.

Given the above gauge correspondence one may now obtain solutions to the nonlinear equations of a member of the nonlinear Schrödinger hierarchy from the equations of motion of the corresponding member the continuous Heisenberg hierarchy, or vice versa. For instance, given a solution $q(x,t)$ and $r(x,t)$ to the Hirota equations (2.9), (2.10) one may use equation (2.12) to construct the gauge field operator G and subsequently simply compute S, that solves (2.16) by construction, by means of the relation (2.13). In reverse, from a solution S to (2.16) we may construct G by (2.13) and subsequently $q(x,t)$ and $r(x,t)$ from (2.12). We elaborate below on the details of this correspondence.

2.3 The nonlocal extended Landau-Lifschitz equation

Equation (2.16) possesses an interesting and well known vector variant with many physical applications that arises when decomposing S in the standard fashion as $S = \mathbf{s} \cdot \boldsymbol{\sigma}$, where $\boldsymbol{\sigma}$ is a vector whose entries are Pauli matrices $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$. Then the equation of motion (2.16) becomes equivalent to an extended version of the Landau-Lifschitz equation

$$\mathbf{s}_t = -\alpha \mathbf{s} \times \mathbf{s}_{xx} - \frac{3}{2}\beta \left(\mathbf{s}_x \cdot \mathbf{s}_x\right) \mathbf{s}_x + \beta \mathbf{s} \times \left(\mathbf{s} \times \mathbf{s}_{xxx}\right). \tag{2.18}$$

The ELL equations (2.18) is easily derived from (2.16) when using the standard identity $\sigma_i \sigma_j = \delta_{ij}\mathbb{I} + i\varepsilon_{ijk}\sigma_k$ with ε_{ijk} denoting the Levi-Civita tensor. Since $S^2 = 1$, we immediately obtain that $\mathbf{s}$ is a unit vector $\mathbf{s} \cdot \mathbf{s} = 1$. For $\beta \to 0$ this equation reduces to the standard Landau-Lifschitz equation [39, 40].

3. Nonlocal multi-solitons for the ECHE from Darboux transformations

3.1 Darboux transformations as auto-gauge transformation

Let us now explain how to solve the above nonlinear systems and construct their nonlocal multi-soliton solutions by means of repeated gauge transformations. The key idea is that the gauge is chosen in such a way that the transformation becomes equivalent to a Darboux transformation. We start with the extended continuous Heisenberg equation and introduce for convenience $T := -iS$, $\psi := \Psi_2$, $U := U_2$ and $V := V_2$ so that the spectral problem in (2.1) for system 2 with (2.15) reads

$$\psi_x = U\psi = \lambda T \psi, \qquad \psi_t = V\psi = \sum_{k=0}^{2} \lambda^{3-k} V^{(k)} \psi \tag{3.1}$$

where

$$V^{(0)} = 4\beta T, \quad V^{(1)} = 2\alpha T - 2\beta T T_x, \quad V^{(2)} = \frac{3}{2}\beta T T_x^2 - \alpha T T_x - \beta T_{xx}. \tag{3.2}$$

Next we carry out another gauge transformation $\hat{G}$ on the system (3.1) relating the eigenstates ψ to new eigenstates $\hat{\psi}$ as $\hat{\psi} = \hat{G}\psi$, so that similarly to the relations in (2.2) we obtain a new spectral problem with

$$\hat{\psi}_x = \hat{U}\hat{\psi}, \qquad \hat{\psi}_t = \hat{V}\hat{\psi}, \tag{3.3}$$

in which the new operators $\hat{U}$, $\hat{V}$ are related to the original U, V as

$$\hat{U} = \hat{G}U\hat{G}^{-1} + \hat{G}_x\hat{G}^{-1}, \qquad \text{and} \qquad \hat{V} = \hat{G}V\hat{G}^{-1} + \hat{G}_t\hat{G}^{-1}. \tag{3.4}$$

The key ingredient to achieve the equivalence between the gauge transformation and the Darboux transformation [48, 49] lies in the right choice of the gauge transformation $\hat{G}$. Following essentially [44], we take now

$$\hat{G}(\lambda) := -\mathbb{I} + \lambda L, \qquad \text{with} \quad L := H\Lambda^{-1}H^{-1},\ H := [\psi(\lambda_1), \psi(\lambda_2)]\,,\ \Lambda = \operatorname{diag}(\lambda_1, \lambda_2). \tag{3.5}$$

Thus H is taken to be a 2×2-matrix with column vectors $\psi(\lambda_1)$ and $\psi(\lambda_2)$ denoting solution to (3.1) for some specific values of the spectral parameter $\lambda_1 \neq \lambda_2 \neq 0$. We notice that $\det L = \lambda_1^{-1}\lambda_2^{-1} \neq 0$, so that the inverse of L exists. Using (3.1) we then compute the derivatives

$$H_x = TH\Lambda, \tag{3.6}$$

$$H_t = \sum_{k=0}^{2} V^{(k)} H \Lambda^{3-k}, \tag{3.7}$$

$$L_x = T - LTL^{-1}, \tag{3.8}$$

$$L_t = \sum_{k=0}^{2} \left(V^{(k)} L^{k-2} - LV^{(k)} L^{k-3} \right), \tag{3.9}$$

which allows us to evaluate the right hand sides of the equations in (3.4) to

$$\hat{U} = LUL^{-1}, \tag{3.10}$$

$$\hat{V}^{(0)} = LV^{(0)}L^{-1}, \tag{3.11}$$

$$\hat{V}^{(1)} = LV^{(1)}L^{-1} - V^{(0)}L^{-1} + \hat{V}^{(0)}L^{-1}, \tag{3.12}$$

$$\hat{V}^{(2)} = LV^{(2)}L^{-1} - V^{(1)}L^{-1} + \hat{V}^{(1)}L^{-1}. \tag{3.13}$$

The matrix $\hat{V}$ is of the same form as V, that is $\hat{V} = \sum_{k=0}^{2} \lambda^{3-k}\hat{V}^{(k)}$. Equation (3.10) is equivalent to $\hat{S}L = LS$, which is reminiscent of the intertwining relations employed in Darboux transformations. Since the gauge system is of the same form as the original equation, this means that if S is also a solution to (2.16), then $\hat{S}$ is a solution to the same equation.

We can now iterate this systems like a standard Darboux-Crum transformation [48, 50]. Indexing all quantities, we have at each stage the spectral problem

$$\psi_x^{(n-1)}(\lambda) = U^{(n-1)}(\lambda)\ \psi^{(n-1)}(\lambda), \qquad \psi_t^{(n-1)}(\lambda) = V^{(n-1)}(\lambda)\ \psi^{(n-1)}(\lambda), \qquad n \in \mathbb{N}, \tag{3.14}$$

which when solved for $\psi^{(n-1)}(\lambda)$ allows to define the new quantities

$$H^{(n-1)}(\lambda_{2n-1}, \lambda_{2n}) := \left(\psi^{(n-1)}(\lambda_{2n-1}), \psi^{(n-1)}(\lambda_{2n})\right), \tag{3.15}$$

$$\Lambda_n := \mathrm{diag}(\lambda_{2n-1}, \lambda_{2n}), \tag{3.16}$$

$$L^{(n)}(\lambda_{2n-1}, \lambda_{2n}) := H^{(n-1)}(\lambda_{2n-1}, \lambda_{2n})\Lambda_n^{-1}\left[H^{(n-1)}(\lambda_{2n-1}, \lambda_{2n})\right]^{-1}, \tag{3.17}$$

where $\lambda_i \neq \lambda_j \neq 0$, $i, j \in \mathbb{N}$. By means of the intertwining operator $L^{(n)}$ we can now specify the gauge transformations as

$$\hat{G}^{(n)}(\lambda) := -\mathbb{I} + \lambda L^{(n)}, \tag{3.18}$$

so that we can construct the solution to the spectral problem (3.14) at the next level as

$$\psi^{(n)}(\lambda) = \hat{G}^{(n)}(\lambda)\psi^{(n-1)}(\lambda) = \mathcal{G}^{(n)}(\lambda)\psi^{(0)}(\lambda), \tag{3.19}$$

$$U^{(n)}(\lambda) = L^{(n)}U^{(n-1)}(\lambda)\left[L^{(n)}\right]^{-1} = \mathcal{L}^{(n)}U^{(0)}(\lambda)\left(\mathcal{L}^{(n)}\right)^{-1}, \tag{3.20}$$

with $\mathcal{L}^{(n)} := L^{(n)}L^{(n-1)}\ldots L^{(1)}$ and $\mathcal{G}^{(n)}(\lambda) := \hat{G}^{(n)}(\lambda)\hat{G}^{(n-1)}(\lambda)\ldots\hat{G}^{(1)}(\lambda)$. Noting that

$$\det \mathcal{L}^{(n)} = \prod_{i=1}^{2n} \lambda_i^{-1} = (-1)^n \prod_{i=1}^{n} |\lambda_{2i-1}|^{-2} =: \chi_n, \tag{3.21}$$

with $\lambda_{2i-1} = -\lambda_{2i}^*$, the inverse of $\mathcal{L}^{(n)}$ is guaranteed to always exist with the restrictions on the λ_i as introduced above. Extrapolating from (3.11)-(3.13) there are naturally also generic formulae for $V^{(n)}(\lambda)$, but since we are mainly interested in $U^{(n)}(\lambda)$ we will not report them here. It is clear from (3.14)-(3.20) that once the initial spectral problem involving $U^{(0)}(\lambda)$, $V^{(0)}(\lambda)$ and $\psi^{(0)}(\lambda)$ has been solved all higher levels follow simply by iteration.

We apply this scheme now to construct multi-soliton solutions to equation (3.1) and in particular explain how nonlocality is naturally introduced into these systems.

3.2 Nonlocal multi-soliton solutions

We start by parameterizing a matrix field solution S to the extended continuous Heisenberg equation (2.16) as

$$S = \begin{pmatrix} -\omega & u \\ v & \omega \end{pmatrix}, \qquad \omega^2 + uv = 1, \tag{3.22}$$

where the form of S is dictated by (2.13) with the constraint on the entries resulting from the first property in (2.14). Substituting this expression into equation (2.16), we identify from the off-diagonal components of this matrix equation the two constraining nonlinear differential equations

$$u_t = i\alpha(u\omega_x - \omega u_x)_x - \beta\left[u_{xx} + 3/2u(u_x v_x + \omega_x^2)\right]_x, \tag{3.23}$$

$$v_t = -i\alpha(v\omega_x - \omega v_x)_x - \beta\left[v_{xx} + 3/2v(v_x u_x + \omega_x^2)\right]_x, \tag{3.24}$$

that u, v and ω have two satisfy. The diagonal entries are trivially satisfied when (3.23) and (3.24) hold. Similarly as the equations (2.9) and (2.10), one may treat (3.23) and (3.24) as independent equations for the functions $u(x,t)$ and $v(x,t)$, with $\omega(x,t)$ obtained from the constraint in (3.22). However, just as for the Hirota system one could also make the choice $u(x,t) = \kappa v^*(x,t)$ so that equation (3.24) simply becomes the complex conjugate of equation (3.23). Likewise we can make the nonlocal choice $u(x,t) = \kappa v^*(-x,t)$ with $\beta = i\delta$, $\delta \in \mathbb{R}$, in which case equation (3.24) becomes the complex conjugate parity transformed of equation (3.23). This means for the matrix S that the nonlocality can be imposed as $S(x,t) = \kappa S^\dagger(-x,t)$, which holds with $\omega(x,t) = \kappa\omega^*(-x,t)$.

The multi-soliton solutions to the ECH equation are then computed from the Darboux-Crum transformations as explained above. From (3.20) we obtain therefore

$$S_n = \mathcal{L}^{(n)} S_0 \left(\mathcal{L}^{(n)}\right)^{-1}, \tag{3.25}$$

with factors $L^{(i)}(\lambda_{2i-1}, \lambda_{2i}), i = 1, \ldots, n$, being evaluated with the appropriate two component solutions $(\psi(\lambda_{2i-1}), \psi(\lambda_{2i}))$ to the spectral problems (3.14) for the iterated U and V operators. As the entire procedure relies on the solutions to the initial spectral problem, it is the choice of the so-called seed functions $\psi^{(0)}(\lambda_i) = (\varphi_i, \phi_i)$ and their implementation into the definition $H^{(n-1)}$ that will introduce the nonlocality properties. This mechanism is similar to what we observed in [28]. From the iteration procedure we obtain the closed solutions

$$\mathcal{L}_{11}^{(n)} = \frac{\det \Omega_n}{\det \mathcal{W}_n}, \qquad \mathcal{L}_{12}^{(n)} = \frac{\det \mathcal{U}_n}{\det \mathcal{W}_n}, \qquad \mathcal{L}_{21}^{(n)} = \frac{\det \mathcal{V}_n}{\det \mathcal{W}_n}, \qquad \mathcal{L}_{22}^{(n)} = \frac{\det \Upsilon_n}{\det \mathcal{W}_n}, \tag{3.26}$$

with $(2n \times 2n)$-matrices $\mathcal{W}_n$, Ω_n, Υ_n, $\mathcal{U}_n$ and $\mathcal{V}_n$ defined in terms of the seed function components as

$$(\mathcal{W}_n)_{ij} = \begin{cases} \lambda_i^{n+1-j}\varphi_i \\ \lambda_i^{2n+1-j}\phi_i \end{cases}, (\Omega_n)_{ij} = \begin{cases} \lambda_i^{j-1}\varphi_i \\ \lambda_i^{j-n}\phi_i \end{cases}, (\Upsilon_n)_{ij} = \begin{cases} \lambda_i^{j}\varphi_i & j = 1, \ldots, n, \\ \lambda_i^{j-n-1}\phi_i & j = n+1, \ldots, 2n, \end{cases} \tag{3.27}$$

$$(\mathcal{U}_n)_{ij} = \begin{cases} \lambda_i^{2n-j}\varphi_i & j = n, \ldots, 2n, \\ \lambda_i^{n-j}\phi_i & j = 1, \ldots, n-1 \end{cases}, \quad (\mathcal{V}_n)_{ij} = \begin{cases} \lambda_i^{j-1}\phi_i & j = 1, \ldots, n+1, \\ \lambda_i^{j-n-1}\varphi_i & j = n+2, \ldots, 2n, \end{cases} \tag{3.28}$$

with $i = 1, \ldots, 2n$.

Keeping the matrix S in the same functional form as in the parameterization (3.22) at each step of the iteration procedure

$$S_n = \begin{pmatrix} -\omega_n & u_n \\ v_n & \omega_n \end{pmatrix}, \qquad \omega_n^2 + u_n v_n = 1, \tag{3.29}$$

and abbreviating for convenience the entries of the matrix $\mathcal{L}^{(n)}$ by $A_n := \mathcal{L}_{11}^{(n)}$, $B_n := \mathcal{L}_{12}^{(n)}$, $C_n := \mathcal{L}_{21}^{(n)}$, $D_n := \mathcal{L}_{22}^{(n)}$ we evaluate the entries to the S-matrix as

$$u_n = \left(A_n^2 u_0 - B_n^2 v_0 + 2A_n B_n \omega_0\right)/\chi_n, \tag{3.30}$$

$$v_n = \left(D_n^2 v_0 - C_n^2 u_0 - 2C_n D_n \omega_0\right)/\chi_n, \tag{3.31}$$

$$\omega_n = \left[A_n C_n u_0 - B_n D_n v_0 + (A_n D_n + B_n C_n)\omega_0\right]/\chi_n, \tag{3.32}$$

We also derive the identity

$$(A_n)_x D_n - B_n (C_n)_x = A_n (D_n)_x - (B_n)_x C_n = 0, \tag{3.33}$$

that will be crucial below. As mentioned, in order to obtain the nonlocal solutions we need to impose the constraint $u_n(x,t) = \kappa v_n^*(-x,t)$. Let us now explain how this is achieved by discussing the explicit solutions in more detail.

3.2.1 Nonlocal one-soliton solutions

We start with a simple constant solution to the ECH equation (2.16) of the general form (3.29) describing the free case

$$S_0 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \omega_0 = 1,\ u_0 = v_0 = 0. \tag{3.34}$$

In order to define the matrix operator H as in (3.5), we need to construct the seed solution $\psi(\lambda)$ to the spectral problem (3.1) and evaluate it for two different and nonzero spectral parameters $\psi(\lambda_1) = (\varphi_1, \phi_1)$ and $\psi(\lambda_2) = (\varphi_2, \phi_2)$. The first intertwining operator can be computed directly and acquires the form

$$\mathcal{L}^{(1)} = \frac{1}{\lambda_1\lambda_2 \det H} \begin{pmatrix} \lambda_2\varphi_1\phi_2 - \lambda_1\varphi_2\phi_1 & (\lambda_1 - \lambda_2)\varphi_1\varphi_2 \\ (\lambda_2 - \lambda_1)\phi_1\phi_2 & \lambda_1\varphi_1\phi_2 - \lambda_2\varphi_2\phi_1 \end{pmatrix}. \tag{3.35}$$

We confirm that this expression can be cast into the form of the generic expression (3.26) with matrices

$$\mathcal{W}_1 = \begin{pmatrix} \lambda_1\varphi_1 & \lambda_1\phi_1 \\ \lambda_2\varphi_2 & \lambda_2\phi_2 \end{pmatrix}, \qquad \Omega_1 = \begin{pmatrix} \varphi_1 & \lambda_1\phi_1 \\ \varphi_2 & \lambda_2\phi_2 \end{pmatrix}, \qquad \Upsilon_1 = \begin{pmatrix} \lambda_1\varphi_1 & \phi_1 \\ \lambda_2\varphi_2 & \phi_2 \end{pmatrix}, \tag{3.36}$$

$$\mathcal{U}_1 = \begin{pmatrix} \lambda_1\varphi_1 & \varphi_1 \\ \lambda_2\varphi_2 & \varphi_2 \end{pmatrix}, \qquad \mathcal{V}_1 = \begin{pmatrix} \phi_1 & \lambda_1\phi_1 \\ \phi_2 & \lambda_2\phi_2 \end{pmatrix}. \tag{3.37}$$

Given the intertwining operator $\mathcal{L}^{(1)}$, we can now calculate the one-soliton solution S_1 directly from (3.30)-(3.32), obtaining

$$u_1 = \frac{2\varphi_1\varphi_2(\lambda_2\varphi_1\phi_2 - \lambda_1\varphi_2\phi_1)(\lambda_1 - \lambda_2)}{\lambda_1\lambda_2(\varphi_2\phi_1 - \varphi_1\phi_2)^2}, \tag{3.38}$$

$$v_1 = \frac{2\phi_1\phi_2(\lambda_1\varphi_1\phi_2 - \lambda_2\varphi_2\phi_1)(\lambda_1 - \lambda_2)}{\lambda_1\lambda_2(\varphi_2\phi_1 - \varphi_1\phi_2)^2}, \tag{3.39}$$

$$\omega_1 = 1 - \frac{2\varphi_1\varphi_2\phi_1\phi_2(\lambda_1 - \lambda_2)^2}{\lambda_1\lambda_2(\varphi_2\phi_1 - \varphi_1\phi_2)^2}. \tag{3.40}$$

So far, these expressions hold for any solution to the spectral problem. Imposing next the nonlocality condition $u_1(x,t) = \kappa v_1^*(-x,t)$ leads for instance to the constraints

$$\varphi_2(x,t) = -\kappa\phi_1^*(-x,t), \quad \phi_2(x,t) = \varphi_1^*(-x,t), \quad \text{with } \lambda_1 = -\lambda_2^* =: \lambda. \tag{3.41}$$

We can now solve the spectral problem (3.1) with S_0 for $\psi(\lambda_1 = \lambda)$ to

$$\psi_1(\lambda) = \begin{pmatrix} e^{\xi_\lambda(x,t)+\gamma_1} \\ e^{-\xi_\lambda(x,t)+\gamma_2} \end{pmatrix}, \tag{3.42}$$

where we introduced the function

$$\xi_\lambda(x,t) := i\lambda x + 2\lambda^2(i\alpha - 2\delta\lambda)t. \tag{3.43}$$

and the additional constants $\gamma_1, \gamma_2 \in \mathbb{C}$ to account for boundary conditions. The second solution is then simply obtained from the constraint (3.41) to

$$\psi_2(\lambda^*) = \begin{pmatrix} -\kappa e^{-\xi^*_\lambda(-x,t)+\gamma^*_2} \\ e^{\xi^*_\lambda(-x,t)+\gamma^*_1} \end{pmatrix}. \tag{3.44}$$

Notice that $\psi_2(\lambda^*)$ is the solution to the parity transformed and conjugated spectral problem (3.1). Given these solutions we are in a position to compute the functions in (3.38)-(3.40)

$$u_1(x,t) = \frac{4\kappa \operatorname{Re}\lambda \left(\kappa\lambda e^{-2\xi^*_\lambda(-x,t)-\gamma^*_1+\gamma^*_2} - \lambda^* e^{2\xi_\lambda(x,t)+\gamma_1-\gamma_2}\right) e^{2\operatorname{Re}\gamma_1+2\operatorname{Re}\gamma_2}}{|\lambda|^2 \left(e^{\xi_\lambda(x,t)+\xi^*_\lambda(-x,t)+2\operatorname{Re}\gamma_1} + \kappa e^{-\xi_\lambda(x,t)-\xi^*_\lambda(-x,t)+2\operatorname{Re}\gamma_2}\right)^2}, \tag{3.45}$$

$$v_1(x,t) = \frac{4\operatorname{Re}\lambda \left(\kappa\lambda^* e^{-2\xi_\lambda(x,t)-\gamma_1+\gamma_2} - \lambda e^{2\xi^*_\lambda(-x,t)+\gamma^*_1-\gamma^*_2}\right) e^{2\operatorname{Re}\gamma_1+2\operatorname{Re}\gamma_2}}{|\lambda|^2 \left(e^{\xi_\lambda(x,t)+\xi^*_\lambda(-x,t)+2\operatorname{Re}\gamma_1} + \kappa e^{-\xi_\lambda(x,t)-\xi^*_\lambda(-x,t)+2\operatorname{Re}\gamma_2}\right)^2}, \tag{3.46}$$

$$\omega_1(x,t) = 1 - \frac{8\kappa(\operatorname{Re}\lambda)^2 e^{2\operatorname{Re}\gamma_1+2\operatorname{Re}\gamma_2}}{|\lambda|^2 \left(e^{\xi_\lambda(x,t)+\xi^*_\lambda(-x,t)+2\operatorname{Re}\gamma_1} + \kappa e^{-\xi_\lambda(x,t)-\xi^*_\lambda(-x,t)+2\operatorname{Re}\gamma_2}\right)^2}. \tag{3.47}$$

We verify that these expressions do indeed satisfy the nonlinear differential equations (3.23) and (3.24) for the component functions of S, together with the locality constraint $u_1(x,t) = \kappa v_1^*(-x,t)$ that converts the two equations (3.23) and (3.24) into each other via a parity transformation and a complex conjugation.

3.2.2 Nonlocal two-soliton solutions

The two-soliton solution is obtained in the next iterative step. With $L^{(1)}$ already computed in (3.35), we evaluate the gauge transformation $\hat{G}^{(1)}$ and the matrix $H^{(1)}$

$$\hat{G}^{(1)}(\lambda) = -\mathbb{I} + \lambda L^{(1)}, \quad H^{(1)}(\lambda_3,\lambda_4) := \left(\hat{G}^{(1)}(\lambda_3)\psi^{(0)}(\lambda_3), \hat{G}^{(1)}(\lambda_4)\psi^{(0)}(\lambda_4)\right), \tag{3.48}$$

from which we compute $L^{(2)}(\lambda_3,\lambda_4)$ as defined in (3.17). Subsequently we compute the complete intertwining operator $\mathcal{L}^{(2)} = L^{(2)}(\lambda_3,\lambda_4)L^{(1)}(\lambda_1,\lambda_2)$ with entries given by the general formula (3.26) with explicit matrices

$$\mathcal{W}_2 = \begin{pmatrix} \lambda_1^2\varphi_1 & \lambda_1\varphi_1 & \lambda_1^2\phi_1 & \lambda_1\phi_1 \\ \lambda_2^2\varphi_2 & \lambda_2\varphi_2 & \lambda_2^2\phi_2 & \lambda_2\phi_2 \\ \lambda_3^2\varphi_3 & \lambda_3\varphi_3 & \lambda_3^2\phi_3 & \lambda_3\phi_3 \\ \lambda_4^2\varphi_4 & \lambda_4\varphi_4 & \lambda_4^2\phi_4 & \lambda_4\phi_4 \end{pmatrix}, \tag{3.49}$$

$$\Omega_2 = \begin{pmatrix} \varphi_1 & \lambda_1\varphi_1 & \lambda_1\phi_1 & \lambda_1^2\phi_1 \\ \varphi_2 & \lambda_2\varphi_2 & \lambda_2\phi_2 & \lambda_2^2\phi_2 \\ \varphi_3 & \lambda_3\varphi_3 & \lambda_3\phi_3 & \lambda_3^2\phi_3 \\ \varphi_4 & \lambda_4\varphi_4 & \lambda_4\phi_4 & \lambda_4^2\phi_4 \end{pmatrix}, \qquad \Upsilon_2 = \begin{pmatrix} \lambda_1\varphi_1 & \lambda_1^2\varphi_1 & \phi_1 & \lambda_1\phi_1 \\ \lambda_2\varphi_2 & \lambda_2^2\varphi_2 & \phi_2 & \lambda_2\phi_2 \\ \lambda_3\varphi_3 & \lambda_3^2\varphi_3 & \phi_3 & \lambda_3\phi_3 \\ \lambda_4\varphi_4 & \lambda_4^2\varphi_4 & \phi_4 & \lambda_4\phi_4 \end{pmatrix}, \tag{3.50}$$

$$\mathcal{U}_2 = \begin{pmatrix} \lambda_1\phi_1 & \lambda_1^2\varphi_1 & \lambda_1\varphi_1 & \varphi_1 \\ \lambda_2\phi_2 & \lambda_2^2\varphi_2 & \lambda_2\varphi_2 & \varphi_2 \\ \lambda_3\phi_3 & \lambda_3^2\varphi_3 & \lambda_3\varphi_3 & \varphi_3 \\ \lambda_4\phi_4 & \lambda_4^2\varphi_4 & \lambda_4\varphi_4 & \varphi_4 \end{pmatrix}, \qquad \mathcal{V}_2 = \begin{pmatrix} \phi_1 & \lambda_1\phi_1 & \lambda_1^2\phi_1 & \lambda_1\varphi_1 \\ \phi_2 & \lambda_2\phi_2 & \lambda_2^2\phi_2 & \lambda_2\varphi_2 \\ \phi_3 & \lambda_3\phi_3 & \lambda_3^2\phi_3 & \lambda_3\varphi_3 \\ \phi_4 & \lambda_4\phi_4 & \lambda_4^2\phi_4 & \lambda_4\varphi_4 \end{pmatrix}. \tag{3.51}$$

For the nonlocal case we define ψ_1 and ψ_2 as in (3.42) and (3.44). In addition, we use

$$\psi_3(\lambda_3 = \rho) = \begin{pmatrix} \varphi_3 \\ \phi_3 \end{pmatrix} = \begin{pmatrix} e^{\xi_\rho(x,t)+\gamma_3} \\ e^{-\xi_\rho(x,t)+\gamma_4} \end{pmatrix}, \quad \psi_4(\lambda_4 = \rho^*) = \begin{pmatrix} \varphi_4 \\ \phi_4 \end{pmatrix} = \begin{pmatrix} -\kappa e^{-\xi_\rho^*(-x,t)+\gamma_4^*} \\ e^{\xi_\rho^*(-x,t)+\gamma_3^*} \end{pmatrix}, \tag{3.52}$$

so that with (3.30)-(3.32) we determine the nonlocal two-soliton solutions as

$$\omega_2 = \sqrt{1 - u_2(x,t)v_2(x,t)}, \tag{3.53}$$

$$u_2 = \frac{2\left(L_{1234} - L_{2134} + L_{3124} - L_{4123}\right)\left(R_{1234} + R_{1342} + R_{1423} + R_{2314} + R_{2431} + R_{3412}\right)}{\lambda_1\lambda_2\lambda_3\lambda_4\left(\Gamma_{1234} + \Gamma_{1342} + \Gamma_{1423} + \Gamma_{2314} + \Gamma_{3412} + \Gamma_{4213}\right)^2},$$

$$v_2 = \frac{2\left(K_{2134} - K_{1234} + K_{4123} - K_{3124}\right)\left(T_{1234} + T_{1342} + T_{1423} + T_{2314} + T_{2431} + T_{3412}\right)}{\lambda_1\lambda_2\lambda_3\lambda_4\left(\Gamma_{1234} + \Gamma_{1342} + \Gamma_{1423} + \Gamma_{2314} + \Gamma_{3412} + \Gamma_{4213}\right)^2},$$

where we defined the shorthand symbols

$$\Gamma_{ijkl} := (\lambda_i - \lambda_j)(\lambda_k - \lambda_l)\varphi_i\varphi_j\phi_k\phi_l, \quad R_{ijkl} = \lambda_k\lambda_l\Gamma_{ijkl}, \quad T_{ijkl} = \lambda_i\lambda_j\Gamma_{ijkl}, \tag{3.54}$$

$$L_{ijkl} := \lambda_i(\lambda_j - \lambda_k)(\lambda_j - \lambda_l)(\lambda_k - \lambda_l)\phi_i\varphi_j\varphi_k\varphi_l, \tag{3.55}$$

$$K_{ijkl} := \lambda_i(\lambda_j - \lambda_k)(\lambda_j - \lambda_l)(\lambda_k - \lambda_l)\varphi_i\phi_j\phi_k\phi_l, \tag{3.56}$$

Once more we verify that these expressions satisfy the nonlinear differential equations (3.23) and (3.24) for the component functions of S and in addition are nonlocal, i.e. satisfying $u_2(x,t) = \kappa v_2^*(-x,t)$, which is required to convert the two equations into each other via a parity transformation and a complex conjugation.

3.3 Nonlocal n-soliton solutions

We proceed further in the same way for the nonlocal multi-soliton solutions for $n > 2$. In general, for a nonlocal n-soliton solution we choose the spectral parameters as

$$\lambda_{2i} = -\lambda_{2i-1}^* \neq 0, \qquad \lambda_{2i} \neq \lambda_{2j} \quad i,j = 1,2,\ldots,n, \tag{3.57}$$

and the seed functions computed at these values as

$$\psi_{2i-1}(\lambda_{2i-1}) = \begin{pmatrix} \varphi_{2i-1} \\ \phi_{2i-1} \end{pmatrix} = \begin{pmatrix} e^{\xi_{\lambda_{2i-1}}(x,t)+\gamma_{2i-1}} \\ e^{-\xi_{\lambda_{2i-1}}(x,t)+\gamma_{2i}} \end{pmatrix}, \tag{3.58}$$

$$\psi_{2i}(\lambda_{2i}) = \begin{pmatrix} \varphi_{2i} \\ \phi_{2i} \end{pmatrix} = \begin{pmatrix} -\kappa e^{-\xi^*_{\lambda_{2i-1}}(-x,t)+\gamma^*_{2i}} \\ e^{\xi^*_{\lambda_{2i-1}}(-x,t)+\gamma^*_{2i-1}} \end{pmatrix}. \tag{3.59}$$

We may then apply directly the formulae (3.30)-(3.32) and evaluate u_n, v_n, and ω_n. We find the nonlocality property $u_n(x,t) = \kappa v_n^*(-x,t)$ for all solutions.

4. Nonlocal solutions to Hirota's equation from the ECH equation

Let us now demonstrate how to obtain nonlocal solutions for the Hirota equation from those of the extended continuous Heisenberg equation. With S being parameterized as in (3.22) we solve for this purpose equation (2.13) for G

$$G = \begin{pmatrix} a & a\frac{\omega+1}{v} \\ c & c\frac{\omega-1}{v} \end{pmatrix}, \tag{4.1}$$

where the functions $a(x,t)$ and $c(x,t)$ remain unknown at this point. They can be determined when substituting G into the equations (2.12). Solving the first equation for $q(x,t)$ and $r(x,t)$ we find

$$q(x,t) = \frac{\mu(t)}{2}\left(\frac{v_x}{v} + \frac{\omega v_x - \omega_x v}{v}\right)\exp\left[\int^x \frac{\omega(s,t)v_s(s,t) - \omega_s(s,t)v(s,t)}{v(s,t)}ds\right], \tag{4.2}$$

$$r(x,t) = \frac{1}{2\mu(t)}\left(\frac{v_x}{v} - \frac{\omega v_x - \omega_x v}{v}\right)\exp\left[-\int^x \frac{\omega(s,t)v_s(s,t) - \omega_s(s,t)v(s,t)}{v(s,t)}ds\right], \tag{4.3}$$

where $\mu(t)$ is an arbitrary function of t at this stage. Notice that the integral representations (4.2) and (4.3) are valid for *any* solution to the ECH equation (2.16). Next we demonstrate how to solve these integrals. Using the expression in (3.30)-(3.32), with suppressed subscripts n and S_0 chosen as in (3.22), we can re-express the terms in (4.2) and (4.3) via the components of the intertwining operator $\mathcal{L}$ as

$$\frac{\omega v_x - \omega_x v}{v} = -2\frac{A_x D - BC_x}{AD - BC} + \partial_x \ln\left[\frac{C}{D}(AD - BC)\right] = \partial_x \ln\left(\frac{C}{D}\right), \tag{4.4}$$

$$\frac{v_x}{v} = \frac{D}{C}\frac{AC_x - A_x C}{AD - BC} - \frac{C}{D}\frac{BD_x - B_x D}{AD - BC} = \frac{C_x}{C} + \frac{D_x}{D}, \tag{4.5}$$

where we used the property (3.33). With these relations the integral representations (4.2), (4.3) simplify to

$$q_n(x,t) = c\mu_n \frac{(C_n)_x}{D_n} = c\mu_n\left(\frac{(\det \mathcal{V}_n)_x}{\det \Upsilon_n} - \frac{(\det \mathcal{W}_n)_x \det \mathcal{V}_n}{\det \mathcal{W}_n \det \Upsilon_n}\right), \tag{4.6}$$

$$r_n(x,t) = \frac{1}{c\mu_n}\frac{(D_n)_x}{C_n} = \frac{1}{c\mu_n}\left(\frac{(\det \Upsilon_n)_x}{\det \mathcal{V}_n} - \frac{(\det \mathcal{W}_n)_x \det \Upsilon_n}{\det \mathcal{W}_n \det \mathcal{V}_n}\right), \tag{4.7}$$

where c is an integration constant. Thus, we have now obtained a simple relation between the spectral problem of the extended continuous Heisenberg equation and the solutions to the Hirota equation. It appears that this is a novel relation even for the local scenario. The nonlocality property of the solutions to the ECH equation is then naturally inherited by the solutions to the Hirota equation. Using the nonlocal choices for the seed functions as specified in (3.58) and (3.59) we may compute directly the right hand sides in (4.6) and (4.7). Crucially these solutions satisfy the nonlocality property

$$r_n(x,t) = \frac{\kappa}{c^2\mu_n^2} q_n^*(-x,t). \tag{4.8}$$

We discuss this in more detail for the one-soliton solution for which the more explicit expressions are less lengthy.

4.1 Nonlocal one-soliton solutions

Reading off the entries C_1 and D_1 from the $\mathcal{L}^{(1)}$-operator in (3.35), the one-soliton solution in (4.6) and (4.7) acquires the form

$$q_1(x,t) = c\mu_1(\lambda_1-\lambda_2)\frac{\phi_1^2\left[\varphi_2(\phi_2)_x-\phi_2\left(\varphi_2\right)_x\right]+\phi_2^2\left[\phi_1\left(\varphi_1\right)_x-\varphi_1(\phi_1)_x\right]}{(\varphi_2\phi_1-\varphi_1\phi_2)(\lambda_1\varphi_1\phi_2-\lambda_2\varphi_2\phi_1)}, \tag{4.9}$$

$$r_1(x,t) = \frac{1}{c\mu_1}\frac{\phi_1\phi_2\left[\varphi_2\left(\varphi_1\right)_x-\varphi_1\left(\varphi_2\right)_x\right]-\varphi_1\varphi_2\left[\phi_2\left(\phi_1\right)_x+\phi_1\left(\phi_2\right)_x\right]}{\phi_1\phi_2\left(\varphi_2\phi_1-\varphi_1\phi_2\right)}. \tag{4.10}$$

Specifying the solutions to the spectral problem as in (3.42) and (3.44) with $\lambda_1=\lambda$, $\lambda_2=-\lambda^*$ and $c=-1$ we obtain the nonlocal one-soliton solution

$$q_1(x,t) = 4i\mu_1\operatorname{Re}\lambda\frac{e^{-\xi(x,t)+\xi^*(-x,t)+\gamma_2+\gamma_1^*}}{e^{\xi(x,t)+\xi^*(-x,t)+\gamma_1+\gamma_1^*}+\kappa e^{-\xi(x,t)-\xi^*(-x,t)+\gamma_2+\gamma_2^*}}, \tag{4.11}$$

$$r_1(x,t) = -\frac{4i\kappa\operatorname{Re}\lambda}{\mu_1}\frac{e^{\gamma_1+\gamma_2^*}}{e^{2\xi^*(-x,t)+\gamma_1+\gamma_1^*}+\kappa e^{-2\xi(x,t)+\gamma_2+\gamma_2^*}}. \tag{4.12}$$

We verify that (4.11) and (4.12) are solution to the nonlocal Hirota equation (2.9) and (2.10) for any constant value of μ_1. The nonlocality property inherited from the extended continuous Heisenberg equation is

$$r_1(x,t) = \frac{\kappa}{\mu_1^2}q_1^*(-x,t). \tag{4.13}$$

Thus for $\mu_1^2=1$ the nonlocality property between q_1 and r_1 becomes the same as the one between the functions v_1 and u_1. Clearly, we may proceed in the same manner and use (4.6) and (4.7) to calculate directly the nonlocal n-soliton solutions to the to the Hirota equation from the spectral problem of the nonlocal ECH equation.

5. Nonlocal solutions to the extended Landau-Lifschitz equation

5.1 Nonlocal solutions to the ELLE from the ECHE or Hirota equation

Given the nonlocal solutions to the ECH equation (2.16), it is now also straightforward to construct nonlocal solutions to the ELL equation (2.18) from them simply by using the representation $S_n=\mathbf{s}_n\cdot\boldsymbol{\sigma}$ with S_n taken to be in the parameterization (3.29). Suppressing the index n, a direct expansion then yields

$$s_1=\frac{1}{2}(u+v), \qquad s_2=\frac{i}{2}(u-v), \qquad s_3=-\omega=\pm\sqrt{1-uv}. \tag{5.1}$$

For the local choice $u(x,t)=v^*(x,t)$ these function are evidently real

$$s_1(x,t)=\operatorname{Re}u, \qquad s_2=-\operatorname{Im}u, \qquad s_3=\pm\sqrt{1-|u|^2}. \tag{5.2}$$

Thus, since $\mathbf{s}$ is a real unit vector function and $\mathbf{s}\cdot\mathbf{s}=1$, its endpoint traces out a curve on the unit sphere, as demonstrated with an example of a one-soliton solution in figure 1 below. However, for the nonlocal choice $u(x,t)=\kappa v^*(-x,t)$, the vector function $\mathbf{s}$ is no longer real so that we may decompose it into $\mathbf{s} = \mathbf{m} + i\mathbf{l}$, where now $\mathbf{m}$ and $\mathbf{l}$ are real valued vector functions. From the relation $\mathbf{s} \cdot \mathbf{s} = 1$ it follows directly that $\mathbf{m}^2 - \mathbf{l}^2 = 1$ and that these vector functions are orthogonal to each other $\mathbf{m} \cdot \mathbf{l} = 0$. The nonlocal extended Landau Lifschitz equation (2.18) then becomes a set of coupled equations for the real valued vector functions $\mathbf{m}$ and $\mathbf{l}$

$$\begin{aligned} \mathbf{m}_t &= \alpha\left(\mathbf{l} \times \mathbf{l}_{xx} - \mathbf{m} \times \mathbf{m}_{xx}\right) + \frac{3}{2}\delta\left[\left(\mathbf{m}_x \cdot \mathbf{m}_x\right)\mathbf{m}_x + 2\left(\mathbf{l}_x \cdot \mathbf{m}_x\right)\mathbf{m}_x - \left(\mathbf{l}_x \cdot \mathbf{l}_x\right)\mathbf{l}_x\right] \qquad (5.3)\\ &\quad +\delta\left[\mathbf{l}\times\left(\mathbf{l} \times \mathbf{l}_{xxx}\right) - \mathbf{m}\times\left(\mathbf{l} \times \mathbf{m}_{xxx}\right) - \mathbf{m}\times\left(\mathbf{m} \times \mathbf{l}_{xxx}\right) - \mathbf{l}\times\left(\mathbf{m} \times \mathbf{m}_{xxx}\right)\right],\\ \mathbf{l}_t &= -\alpha\left(\mathbf{l} \times \mathbf{m}_{xx} + \mathbf{m} \times \mathbf{l}_{xx}\right) + \frac{3}{2}\delta\left[\left(\mathbf{l}_x \cdot \mathbf{l}_x\right)\mathbf{m}_x + 2\left(\mathbf{l}_x \cdot \mathbf{m}_x\right)\mathbf{l}_x - \left(\mathbf{m}_x \cdot \mathbf{m}_x\right)\mathbf{m}_x\right] \qquad (5.4)\\ &\quad +\delta\left[\mathbf{m}\times\left(\mathbf{m} \times \mathbf{m}_{xxx}\right) - \mathbf{l}\times\left(\mathbf{m} \times \mathbf{l}_{xxx}\right) - \mathbf{l}\times\left(\mathbf{l} \times \mathbf{m}_{xxx}\right) - \mathbf{m}\times\left(\mathbf{l} \times \mathbf{l}_{xxx}\right)\right], \end{aligned}$$

Given $\mathbf{s}$, the real component entries of the new vectors are trivially obtained from (5.1) to $m_i = \left(s_i + s_i^*\right)/2$ and $l_i = i\left(s_i^* - s_i\right)/2$ that is

$$m_1(x,t) = \frac{1}{4}\left[u(x,t) + v(x,t) + \kappa v(-x,t) + \kappa^{-1}u(-x,t)\right], \qquad (5.5)$$

$$m_2(x,t) = \frac{i}{4}\left[u(x,t) - v(x,t) - \kappa v(-x,t) + \kappa^{-1}u(-x,t)\right], \qquad (5.6)$$

$$m_3(x,t) = -\frac{1}{2}\left[\omega(x,t) + \omega(-x,t)\right] \qquad (5.7)$$

$$l_1(x,t) = \frac{i}{4}\left[-u(x,t) - v(x,t) + \kappa v(-x,t) + \kappa^{-1}u(-x,t)\right], \qquad (5.8)$$

$$l_2(x,t) = \frac{1}{4}\left[u(x,t) - v(x,t) + \kappa v(-x,t) - \kappa^{-1}u(-x,t)\right], \qquad (5.9)$$

$$l_3(x,t) = \frac{i}{2}\left[\omega(x,t) - \omega(-x,t)\right]. \qquad (5.10)$$

Clearly despite the fact that $\mathbf{s} \cdot \mathbf{s} = 1$, the real and imaginary components no longer trace out a curve on the unit sphere.

When solving the ECH equation directly we have implemented to nonlocality through the compatibility relations between the auxiliary equations (3.23) and (3.24), which was then inherited by $\mathbf{s}$. We may also attempt to implement the nonlocality from the Hirota system directly into S and therefore $\mathbf{s}$. For this purpose we make use of the fact that so far the gauge operator G, that relates the spectral problem of the Hirota system to the spectral problem of the ECH equation has been left completely generic and the entries of the matrix A_0 are constrained by the equations of motion (2.9) and (2.10).

As commented above, when specifying the $(A_0)_{21}$-entry to $r(x,t) = \kappa q^*(x,t)$ with $\kappa = \pm 1$, the two equations (2.9) and (2.10) reduce to standard local Hirota equation [43, 28]. The first equation in (2.12) then implies that $G_{11} = G_{22}^*$ and $G_{21} = \kappa G_{12}^*$, reducing the four equations resulting from each matrix entry to the two equations

$$a_x = \kappa b^* u, \qquad b_x = b^* u, \qquad (5.11)$$

where we used the more compact notation $G_{11} =: a$, $G_{12} =: b$. Having specified the gauge transformation G, we may compute the matrix S directly from its defining relation (2.13) so that the components of the vector $\mathbf{s}$ become in this case

$$s_1 = \frac{a^* b - \kappa a b^*}{|a|^2 - \kappa |b|^2}, \quad s_2 = i\frac{a^* b + \kappa a b^*}{|a|^2 - \kappa |b|^2}, \quad s_3 = \frac{|a|^2 + \kappa |b|^2}{|a|^2 - \kappa |b|^2}. \tag{5.12}$$

Hence for the choice $\kappa = -1$ the vector $\mathbf{s}$ is real valued.

For the nonlocal choice $r(x,t) = \kappa q^*(-x,t)$ the first equation in (2.12) implies that $G_{11} = \tilde{G}_{22}^*$ and $G_{21} = -\kappa \tilde{G}_{12}^*$. We adopt here the notation from [28] and suppress the explicit dependence on (x,t), indicating the functional dependence on $(-x,t)$ by a tilde, i.e. $\tilde{q} := q(-x,t)$, $\tilde{G}_{12}^* := G_{12}^*(-x,t)$, etc. The first equation in (2.12) then reduces to the two equations

$$a_x = -\kappa \tilde{b}^* u, \qquad b_x = \tilde{a}^* u, \tag{5.13}$$

so that in this case the components of the vector $\mathbf{s}$ become

$$s_1 = \frac{\tilde{a}^* b - \kappa a \tilde{b}^*}{a\tilde{a}^* - \kappa b \tilde{b}^*}, \quad s_2 = i\frac{\tilde{a} b + \kappa a \tilde{b}^*}{a\tilde{a}^* - \kappa b \tilde{b}^*}, \quad s_3 = i\frac{a\tilde{a}^* + \kappa b \tilde{b}^*}{a\tilde{a}^* - \kappa b \tilde{b}^*}, \tag{5.14}$$

which are solutions to the nonlocal extended Landau-Lifschitz equations (5.3), (5.4).

5.2 Nonlocal one-soliton solutions to the ELL equation

We will now discuss some concrete soliton solutions obtained as outlined in the previous section. We start by solving the constraint (5.11) for a local solution first that determines the gauge transformation G. Making the additional assumption $b = cq(x,t)$ for some constant c, equation (5.11) becomes

$$a_x = \kappa c |q|^2, \qquad a = c (\ln q^*)_x . \tag{5.15}$$

The compatibility between the two equations in (5.15) implies that

$$|q|^2 = \kappa (\ln q^*)_{xx} \quad \Leftrightarrow \quad D_x^2 f \cdot f = -2\kappa |g|^2, \quad (\ln g)_{xx} = 0, \tag{5.16}$$

with $q = g/f$, $g \in \mathbb{C}$, $f \in \mathbb{R}$. We notice that the first relation in (5.16), following directly from (5.15), corresponds to one of the bilinear equations into which the Hirota equation can be converted with an additional constraint. Here D_x denotes a Hirota derivative, see [28] for more details. Evidently the additional constraint is not satisfied by all solutions to the Hirota equation. The second equation in (2.12) is then satisfied trivially for solutions of (5.16). We have therefore obtained a solution for the gauge field operator in the form

$$G = c \begin{pmatrix} (\ln q^*)_x & q \\ \kappa q^* & (\ln q)_x \end{pmatrix}. \tag{5.17}$$

From the definition of S and its decomposition the components of $\mathbf{s}$ are computed to

$$s_1(x,t) = 1 + \frac{2\kappa |q|^4}{q_x q_x^* - \kappa |q|^4}, \quad s_2(x,t) = \frac{|q|^2 (q_x - \kappa q_x^*)}{q_x q_x^* - \kappa |q|^4}, \quad s_3(x,t) = i\frac{|q|^2 (q_x + \kappa q_x^*)}{q_x q_x^* - \kappa |q|^4}. \tag{5.18}$$

It is trivial to verify that $\mathbf{s} \cdot \mathbf{s} = 1$. Thus for any solution to the Hirota equation, with the additional constraint as specified in (5.16), the vector $\mathbf{s}$ constitutes a solution to the ELL equation as given in (2.18).

One such solution we may employ is for instance the local one-soliton solution obtained in [28]

$$q(x,t) = \frac{(\mu+\mu^*)^2 \exp[\gamma + \mu x + \mu^2 t(i\alpha - \beta\mu)]}{(\mu+\mu^*)^2 + \exp[\gamma + \gamma^* + i\alpha t\,(\mu^2 - \mu^{*2}) - \beta t\,(\mu^3 + \mu^{*3}) + x(\mu+\mu^*)]}. \tag{5.19}$$

We briefly discuss some of the key characteristic behaviours of $\mathbf{s}$ for various choices of the parameters. When $\beta = 0$, the solutions correspond to the one-soliton solutions of the nonlinear Schrödinger equation. For real parameters μ we obtain the well known periodic solutions to the ELL equation as seen in the left panel of figure 1. However, when the shift parameters μ is taken to be complex we obtain decaying solutions tending towards a fixed point.

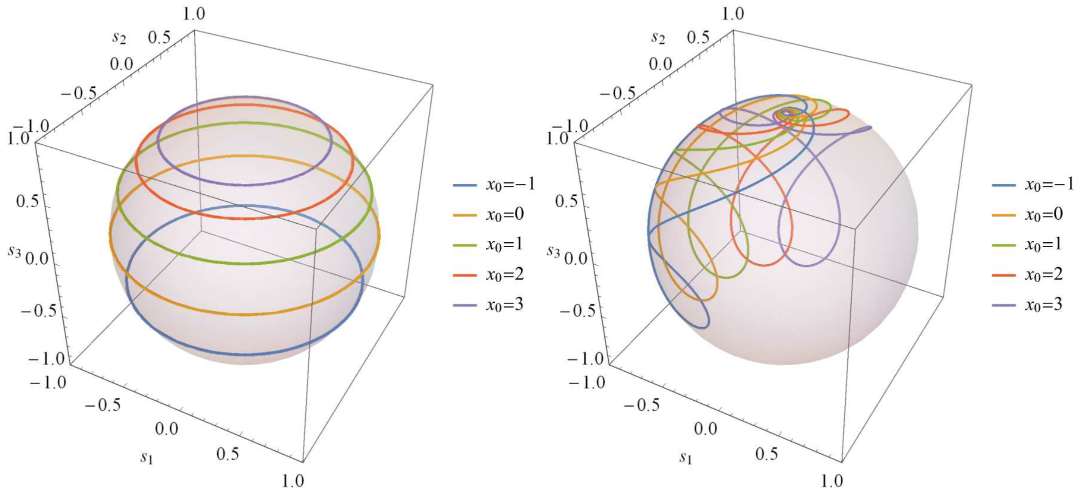

Figure 1: Local solutions to the extended Landau Lifschitz equation (2.18) from a gauge equivalent one-soliton solution (5.19) of the nonlinear Schrödinger equation for different initial values x_0, complex shift $\gamma = 0.4 + i0.2$, $\alpha = 0.3$ and $\beta = 0$. In the left panel the spectral parameter is real $\mu = 0.3$ and in the right panel it is complex $\mu = 0.3 + i0.1$.

When taking $\beta \neq 0$, that is the solutions to the Hirota equation even for real values μ the bahaviour of the trajectories is drastically changed even for small values of β, as they become more knotty and convoluted as seen in the left panel of figure 2. Complex values of μ are once more decaying solutions tending towards a fixed point.

Next we discuss the nonlocal solutions obtained by solving (5.11), by making the same additional assumption as for the construction of the local solutions $b = cq(x,t)$ for some real constant c. In this case equation (5.13) becomes

$$a_x = -\kappa c q \tilde{q}^*, \qquad a = c\,(\ln \tilde{q}^*)_x\,. \tag{5.20}$$

Now the compatibility between the two equations in (5.20) implies that

$$\kappa q \tilde{q}^* = -\,(\ln \tilde{q}^*)_{xx} \quad \Leftrightarrow \quad D_x^2 f \cdot f = \kappa g h, \quad (\ln h)_{xx} = 0. \tag{5.21}$$

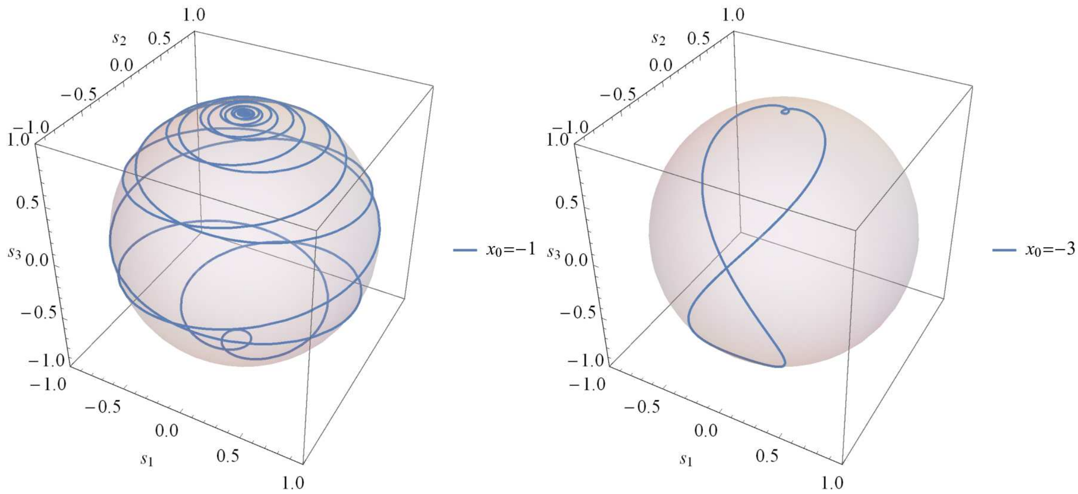

Figure 2: Local solutions to the extended Landau Lifschitz equation (2.18) from a gauge equivalent one-soliton solution (5.19) of the Hirota for a fixed value of x_0, complex shift $\gamma = 0.4 + i0.2$, $\alpha = 0.3$ and $\beta = 0.1$. In the left panel the spectral parameter is real $\mu = 0.3$ and in the right panel it is complex $\mu = 0.3 + i0.1$.

with $q = g/f$, $f, g \in \mathbb{C}$ and $h = 2f\ \tilde{g}^*/\tilde{f}^*$. Once again the first relation on the right hand side in (5.21) occurs in the bilinearisation of the nonlocal Hirota equation, see section 4.1 [28]. However, as for the local case in (5.16) the second relation is an additional constraint that is not automatically satisfied by all solutions. We have therefore obtained a solution for the nonlocal gauge field operator in the form

$$G = c \begin{pmatrix} (\ln \tilde{q}^*)_x & q \\ \tilde{q}^* & (\ln q)_x \end{pmatrix}, \tag{5.22}$$

so that the matrix S can be computed directly from its defining relation (2.13). Using the expansion for S in terms of the components of $\mathbf{s}$ we compute

$$s_1(x,t) = \frac{q^2\tilde{q}_x^* - \tilde{q}^{*2}q_x}{\tilde{q}^{*2}q^2 - \tilde{q}_x^* q_x}, \quad s_2(x,t) = i\frac{q^2\tilde{q}_x^* + \tilde{q}^{*2}q_x}{\tilde{q}^{*2}q^2 - \tilde{q}_x^* q_x}, \quad s_3(x,t) = -\frac{q^2\tilde{q}^{*2} + q_x\tilde{q}_x^*}{\tilde{q}^{*2}q^2 - \tilde{q}_x^* q_x}. \tag{5.23}$$

It is trivial to verify that $\mathbf{s} \cdot \mathbf{s} = 1$. Thus for any solution to the nonlocal Hirota equation, with the additional constraint as specified in (5.21), the complex valued vector $\mathbf{s}$ constitutes a solution to the ELLE as given in (2.18).

One such solution one may employ is the nonlocal one-soliton solution obtained in [28]

$$q(x,t) = \frac{(\mu - \mu^*)^2 \exp\left[\gamma + \mu(x + i\mu t(\alpha - \delta\mu))\right]}{(\mu - \mu^*)^2 + \exp\left[\gamma + \gamma^* + it\left(\alpha(\mu^2 - \mu^{*2}) + \delta\left(\mu^{*3} - \mu^3\right)\right) + x(\mu - \mu^*)\right]}. \tag{5.24}$$

Let us analyze how $\mathbf{m}$ and $\mathbf{l}$ behave in this case. As expected, the trajectories will no stay on the unit sphere. However, for certain choices of the parameters it is possible to obtain well localized closed three dimensional trajectories that trace out curves with fixed points at $t = \pm\infty$ as seen for an example in figure 3. Thus the nonlocal nature of the solutions to the Hirota equation has apparently disappeared in the setting of the extended Landau

Lifschitz equation. However, not all solutions are of this type as some of them are now unbounded. A similar approach to construct one-soliton solutions for the local version of the NLSE was pursued in [54].

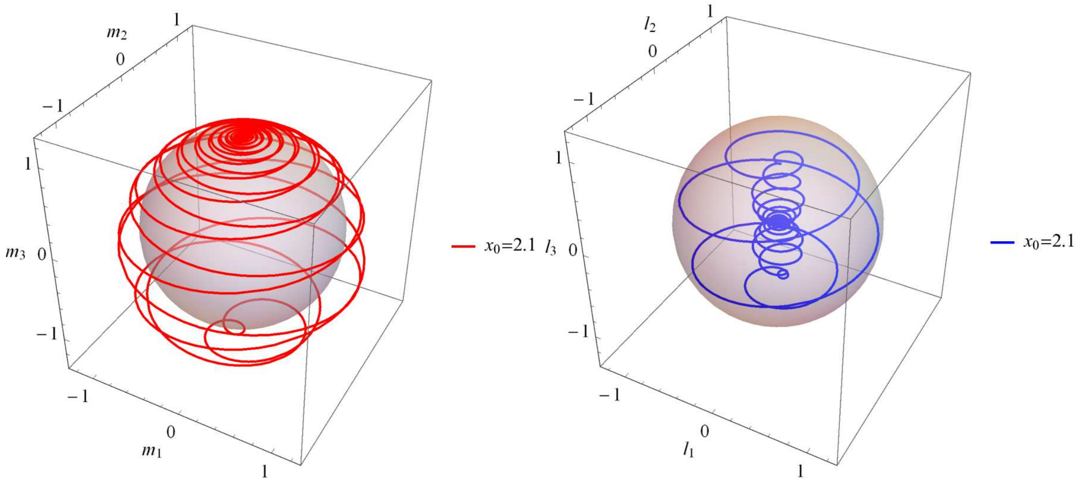

Figure 3: Nonlocal solutions to the extended Landau Lifschitz equation (5.3) and (5.4) from a gauge equivalent one-soliton solution (5.24) of the Hirota equation for a fixed value of x_0, vanishing complex shift $\gamma = 0$, $\mu = i0.55$, $\alpha = 1.5$ and $\delta = 0.15$.

5.3 Nonlocal two-soliton solutions to the ELLE

While the computation of the solutions to the ELL equation is straightforward when computing G directly with some additional constraints, not all Hirota solutions obey them. Let us therefore use the two-soliton solution (3.53) in the representation (5.5)-(5.10) to study the nonlocal two-soliton solutions to the ELL equation. The two-soliton structure is best revealed when plotting it for fixed time over space. In figure 4 we show each component of **m** and **l** separately, displaying clearly two distinct one-soliton structures.

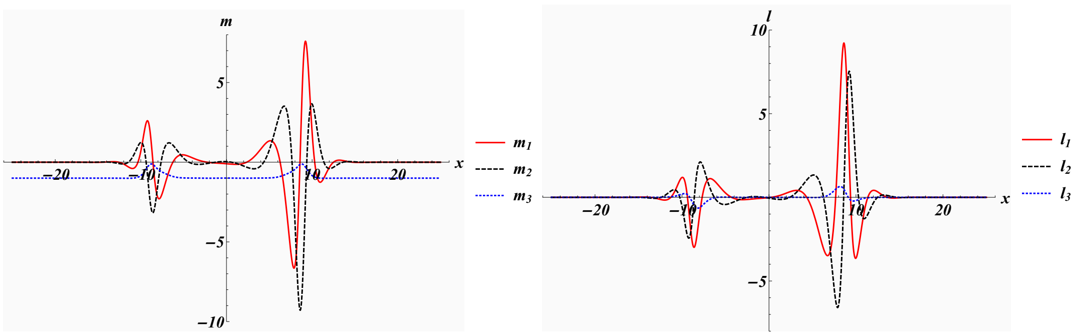

Figure 4: Nonlocal two-soliton solutions to the extended Landau Lifschitz equation for fixed time $t = 3$ as a function of with parameters $\alpha = 1.2$, $\delta = 0.2$, $\kappa = 3$, $\lambda = 0.4 - i0.3$, $\rho = 0.7 + i0.5$, $\gamma_1 = i5.1$, $\gamma_2 = i0.1$, $\gamma_3 = -i1.1$ and $\gamma_4 = i0.2$.

6. Conclusions

We discussed two different types of local/nonlocal gauge transformations: The first of them, G, relates the auxiliary functions in the two spectral problems of the local/nonlocal extended continuous limit of the Heisenberg equation to the local/nonlocal Hirota equation. The explicit form of the gauge functions can be used to establish a concrete relation between solutions of one system to the other. This concrete map when applied to solutions works most efficiently in one direction from the spectral problem the local/nonlocal extended continuous Heisenberg equation to solutions of the (nonlocal) Hirota equation, as stated explicitly in (4.6) and (4.7). This map is not easily invertible and instead we used (5.16) and (5.21) to provide an alternative. While we demonstrated that the (5.16) and (5.21) are equivalent to equations that emerge in the bilinearization process, they also require an additional constraint that is not satisfied by all solutions, so that not all solutions are obtainable in this manner. The second type of gauge transformation, $\hat{G}$, is an auto-gauge transformation that relates the auxiliary functions in the spectral problem of the (nonlocal) extended continuous Heisenberg equation to itself. This gauge transformation can be interpreted as a Darboux transformation and allows to construct a new solution from a known one. In an analogous fashion to Darboux-Crum transformations, it can be iterated to produce multi-soliton solutions.

The nonlocality can be implemented separately in the two systems by applying a parity complex conjugation map to different sets of equations. For the Hirota system it is most naturally applied to the pair of equations (2.9) and (2.10), resulting from the zero curvature formulation as discussed in [28]. For the extended version of continuous limit of the Heisenberg equation it is most obviously applied to its component version (3.23) and (3.24). The two versions of nonlocality in the two systems were shown to be related to each by means of the gauge transformation G as demonstrated by (4.13). As demonstrated, one may, however, also map components of the gauge transformation matrix to each other in a consistent manner. At the level of the spectral problem the nonlocality is implemented via mapping components of the seed functions consistently to each other.

Various issues are worthy of further exploration. It is well known [51, 52, 35, 53] that the Landau Lifschitz equation, i.e. (2.18) for $\beta = 0$, admits a geometric interpretation that directly relates the curvature and torsion of a vector field to any solution of the nonlinear Schrödinger equation when expressed in form of the Hasimoto map [51]. In [55] we demonstrate that these relations and interpretations can be extended to the extended nonlocal versions of this equation. Naturally it would also be interesting to explore the behaviour of the systems arising from the other types of $\mathcal{PT}$-conjugation as discussed in [28].

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