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Equivalent Choice Functions and Stable Mechanisms

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Abstract

We study conditions for the existence of stable and (group)-strategy-proof mechanisms in a many-to-one matching model with contracts if students' preferences are monotone in contract terms. We show that "equivalence", properly defined, to a choice profile under which contracts are substitutes and the law of aggregate demand holds is a necessary and sufficient condition for the existence of a stable and (group)-strategy-proof mechanism.

Our result can be interpreted as a (weak) embedding result for choice functions under which contracts are observable substitutes and the observable law of aggregate demand holds. *JEL-classification:* C78, D47
Keywords: Matching with contracts; College admission; Substitutes; Observable Substitutes; Strategy-Proofness; Deferred Acceptance

1 Introduction

Centralized clearing houses based on the deferred-acceptance mechanism are at the heart of many successful matching markets (Roth, 1984a; Abdulkadiroglu and Sönmez, 2003; Sönmez and Switzer, 2013; Sönmez, 2013). Deferred-acceptance

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mechanisms are appealing because they produce stable outcomes, meaning that no subgroup of agents can find a mutually beneficial deviation and, thus, would have a reason to contract outside the market.[1] Moreover, it is safe for the applying side of the market to report true preferences to the mechanism. Thus, the mechanism successfully aggregates the information in the market and levels the playing field for both naive and sophisticated participants.

In some applications, the market does not only match agents, but determines also the contractual details of the match. In a labor market, firms and workers may have some discretion on how to set the salary. In a college admission problem, students can be admitted with or without a stipend. In the cadet-to-branch match (Sönmez and Switzer, 2013), cadets can choose between different lengths of service time in exchange for a higher priority in their branch of choice. These markets can be understood as hybrids between matching markets and auctions and have first been analyzed in the seminal paper of Kelso and Crawford (1982), with later important extensions by Roth (1984b); Fleiner (2003); Hatfield and Milgrom (2005); Hatfield and Kojima (2009) among others. The central mechanism design result in this context states that in a model of many-to-one matching with endogenous contracting, a generalized version of the deferred-acceptance mechanism can be defined, and is stable and group-strategy-proof under the assumption that contracts are substitutes for colleges[2] and the law of aggregate demand holds.[3] Moreover, the deferred acceptance algorithm is outcome equivalent to the cumulative offer algorithm which is a natural modification of the deferred acceptance algorithm. In the cumulative offer algorithm not only acceptances, but also rejections are deferred; in each round of the algorithm, colleges are allowed to choose among all applications they have received so far including those that they have previously rejected.

In this paper, we study stable and (group)-strategy-proof mechanisms for matching markets with contracts in which students have monotone preferences in contract-terms,[4] and explore how much the assumptions of substitutability and the law of aggregate demand can be relaxed in this context. Our research question is motivated by the observation that in many applications, there is a natural ordering over contracts terms, and it is reasonable to assume that preferences are monotone with respect to the ordering: In college admission problems with stipends (Hassidim et al., 2017; Abizada and Dur, 2017), it is natural to assume that students prefer being admitted with a stipend to being admitted without a stipend at the same college, or more generally, being admitted with a higher stipend than a lower stipend at the same college.[5] In the cadet-to-branch match-

[1] Roth (1991) provides evidence that clearing houses using unstable mechanisms tend to fail in practice.

[2] In the following we call the applying side of the market "students" and the admitting side of the market "colleges" motivated by the application of college admission. However, the model equally applies to the other applications mentioned in this introduction.

[3] This means that if we expend the choice set of a college, an equal or larger number of contracts is chosen.

[4] Colleges' preferences are not necessarily monotone in our analysis.

[5] There is empirical evidence that monotonicity is violated for reported preferences of some participants in the Israeli psychology match, see Hassidim et al., 2020. But it seems likely

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ing model of Sönmez and Switzer (2013) the contract-term is the service time in the military and it is natural to assume that cadets prefer a shorter to a longer service in the same branch. This assumption is for example made in the analysis of Jagadeesan (2019). For the medical match, mechanisms which allow for flexible salaries have been proposed (Crawford, 2008), and it seems reasonable that doctors in these mechanisms would prefer working for a higher salary rather than a lower salary at the same hospital.

We study conditions on choice functions of colleges under which a stable and (group)-strategy-proof matching mechanism exists for our model. Since we work with monotone preferences for students, sufficient conditions on choice functions of colleges are generally weaker. This is because certain preference manipulations are ruled out by our model. A student must report monotone preferences. Thus, a student cannot, e.g., rank being admitted to a college without a stipend above being admitted to the same college with a stipend.[6] Our main result is a characterization of a sufficient and necessary (in the maximal domain sense) condition for the existence of a stable and (group)-strategy-proof mechanism in terms of "equivalence" of choice functions: Two choice functions are equivalent if the cumulative offer algorithm yields the same outcome under either choice function, for

that these monotonicity violations can be attributed to students failing to play the weakly dominant strategy of revealing preferences truthfully, rather than to them having non-monotone preferences.

[6]Similarly, weaker conditions are sufficient to guarantee the existence of stable allocations than those for markets with non-monotone preferences. To illustrate this point, consider a college admission problem of the following kind: There are two colleges c_1, c_2 and three students s_1, s_2, s_3. Suppose there are two kinds of contracts: A student can be admitted with stipend (represented by the contract term "1") or without stipend (represented by the contract term "0"). The colleges have choice functions induced (in the usual way) by the following preferences

$$\{(s_1,0),(s_2,0),(s_3,0)\} \succ_{c_1} \{(s_2,0)\} \succ_{c_1} \{(s_2,1)\} \succ_{c_1} \emptyset \succ_{c_1} \dots$$
$$\{(s_1,0)\} \succ_{c_2} \{(s_3,0)\} \succ_{c_2} \emptyset \succ_{c_2} \dots$$

Suppose student always prefer to be admitted at a college under a stipend to being admitted at the same college without a stipend. Going through all different cases, one can show that, for any preferences satisfying this monotonicity assumption, a stable allocation (in the matching with contracts sense) exists. This changes if students can report non-monotonic preferences. Consider the following preferences:

$$(c_1,1) \succ_{s_1} (c_2,1) \succ_{s_1} (c_1,0) \succ_{s_1} (c_2,0) \succ_{s_1} \emptyset$$
$$(c_1,0) \succ_{s_2} (c_1,1) \succ_{s_2} \emptyset \succ_{s_2} \dots$$
$$(c_2,1) \succ_{s_3} (c_1,1) \succ_{s_3} (c_2,0) \succ_{s_3} (c_1,0) \succ_{s_3} \emptyset$$

Student s_2 prefers to go to college c_1 without a stipend rather than a stipend. Thus, in a stable allocation it will never be the case that s_2 goes to college c_1 with a stipend, because otherwise c_1 and s_2 could block that allocation. This in turn implies that no stable allocation exists: The allocation that matches all three students to c_1 without stipend is blocked by student s_3 and college c_2. Any allocation that matches s_1 to c_2 without stipend is blocked by students s_1, s_2 and s_3 being admitted to college c_1 without stipend. Any allocation that matches s_3 to c_2 without stipend is blocked by student s_1 being admitted to college c_2 without a stipend. Finally, all other allocations are either not individually rational or blocked by students s_1, s_2 and s_3 being admitted to college c_1 without a stipend.

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any preferences and choice functions for the other agents. If for each college an equivalent choice function exists under which contracts are substitutes and the law of aggregate demand holds, then the mechanism that assigns to each profile of preferences the outcome of the cumulative offer algorithm, or equivalently the outcome of the deferred acceptance algorithm, is group-strategy-proof, both under the equivalent choice profile and, by definition of equivalence, under the original choice profile.

We proceed as follows: First we show (Theorems 1 and 2) that an equivalent choice function under which contracts are substitutes and the law of aggregate demand holds exists, if and only if under the original choice function contracts are observable substitutes (in the sense of Hatfield et al., 2020) and the observable law of aggregate demand holds.[7] The result crucially relies on our monotonicity assumption on preferences: In independent work, Jagadeesan (2019) has considered the corresponding notion of equivalence between choice functions for the general preference domain. He shows that equivalence in that context implies unilateral substitutability which is a much more stringent restriction than observable substitutability. Even if the notion of equivalence is relaxed, by allowing, as in the construction of substitutable completions (Hatfield and Kominers, 2015), for non-feasible equivalent choice functions, the corresponding result does not hold on the general preference domain (Hatfield et al., 2020).

Our proof is constructive, by providing a natural construction of an equivalent choice function that we call a "virtual choice function", and by showing that the virtual choice function satisfies substitutability and the law of aggregate demand if and only if the original choice function satisfies observable substitutes and the observable law of aggregate demand. Importantly, the constructed virtual choice function is monotone in contract terms, even if the original choice function is non-monotone in contract terms.

Theorems 1 and 2 have several important consequences that are subsequently spelled out: First, Theorem 1 implies (Corollory 1) that the deferred acceptance mechanisms is group-strategy-proof if contracts are observable substitutes and the observable law of aggregate demand holds. Second, combining the theorems with a result of Hatfield et al. (2020), we obtain a maximal domain result (Corollary 2): The domain of choice profiles that are equivalent to a choice profile under which contracts are substitutes for colleges and the law of aggregate demand holds, form a maximal Cartesian and unital[8] domain for the existence of a stable and (group)-strategy-proof mechanism. Equivalently this domain is the domain of profiles under which contracts are observable substitutes and the observable law of aggregate demand holds. Our results are stronger than the result obtained by Hatfield et al. (2020) for monotone preferences: For monotone preferences, strategy-proofness of the deferred acceptance mechanism can be strengthened to group-strategy-

[7]Importantly, observability is meant with respect to offer sequences consistent with monotone preferences for students. Thus, the notions of observable substitutability and the observable law of aggregate demand are weaker than the corresponding notions for the general model with possibly non-monotone preferences.

[8]A domain of choice profiles is unital if it contains all profile in which each college has a unit demand choice function, i.e. it chooses at most one contract from each set.

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proofness, and the axiom of "non-manipulability via contractual terms" in the characterization of the maximal domain becomes redundant. Furthermore, the notion of equivalence yields an easier and more interpretable condition on choice functions: In particular, group-strategy-proof of the deferred acceptance mechanism on the maximal domain now follows immediately from well-known previous results due to Hatfield and Milgrom (2005) and Hatfield and Kojima (2009) applied to the equivalent choice profile. Third, since an equivalent choice profile can be constructed and the equivalent choice profile is monotone in contract terms (Theorem 1), we obtain an embedding result (Corollary 3) in the sense of Jagadeesan (2019) for the class of choice functions under which contracts are observable substitutes and the observable law of aggregate demand holds. Thus, if attention is restricted to the case of monotone preferences for students, it is, in some sense without loss of generality to work with the model of matching with salaries of Kelso and Crawford (1982) rather than the full matching with contracts model.

1.1 Related Literature

Stable many-to-one matching mechanisms and their incentive properties have been extensively studied (Hatfield and Kojima, 2010; Hatfield and Kominers, 2015; Chen et al., 2016; Kominers and Sönmez, 2016; Hirata and Kasuya, 2017; Hatfield et al., 2020). Most papers focus on the pure matching model or on the matching with contracts model. Working with monotone preferences makes our model less general than the full matching with contracts model. In particular, sufficient conditions for stability and the existence of a stable and (group)-strategy-proof mechanism from the literature on matching with contracts also apply to our model. However, conditions that are necessary for the general model with contracts are not necessary conditions for the model with monotone preferences. Thus, our results are independent of previous results for the matching with contracts model.

In recent related work, Abizada and Dur (2017) consider a model of college admissions with stipends where complementarities in contract terms are present for colleges: In their model three contract terms $\{t_+, t_0, t_-\}$ are available, interpreted as admission with stipend, admission without stipend but with tuition waiver, and admission without either of the two, and the number of t_--contracts signed by a college constraints the number of t_+-contracts it signs. Importantly, students have monotone preferences in these contract terms. The model is a special case of ours. In particular, the result for "Max-Min Responsive" preferences can be obtained as a special case of ours and for this case, their strategy-proofness result can be strengthened to group-strategy-proofness. However, Abizada and Dur (2017) also analyze pairwise-stable outcomes and this part of their analysis does not have a counter-part in our paper.

Our original working paper (Schlegel, 2016) contained a version of our maximal domain result for a model of matching with contracts where also colleges' choice functions are monotone in contract-terms. Technically the two maximal domain results are independent. However, the adaption to obtain the previous version

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of the theorem from the current one are minimal.[9] The current version of the maximal domain result can also be obtained with a similar proof as the one in the previous version. For the original version of the theorem, we refer the interested reader to the original version of the working paper.

After, our original working paper (Schlegel, 2016), Hatfield et al. (2020) released a new version of their paper, where the authors extend their analysis to the case of restricted preference domains for the applicant side and show that their analysis of observable substitutes and the observable law of aggregate demand extends to restricted preference domains if the notion of "observability" is adjusted to the domain restriction. Their additional work allows us to shorten the proof of our maximal domain result, since we can now make use of their Theorem 5. As remarked previously, our results in the current paper for monotone preferences strengthen the results in Hatfield et al. (2020) by strengthening strategy-proofness to group-strategy-proofness.

2 Model and Known Results

2.1 Model

There are two finite disjoint sets of agents, a set of **colleges** C and a set of **students** S. There is a finite set of possible **contract-terms** T which are totally ordered by $\rhd$. Contract terms could, for example, be different amounts of stipends that a college can pay a student, ordered from high to low stipends; in a labor market, contract terms could be different possible wages ordered from high to low, or working hours from short to long, etc.[10] Colleges can accept students under different bilateral contracts. The set of possible **contracts** is $X \subseteq C \times S \times T$. For a contract $x \in X$, we denote by $x_C \in C$ the college involved in x, by $x_S \in S$ the student involved in X, and by $x_T \in T$ the contract term involved in x. We write $x_T \trianglerighteq x'_T$ whenever $x_T \rhd x'_T$ or $x = x'$. For $Y \subseteq X$ and college $c \in C$ we denote by $Y_c = \{x \in Y : x_C = c\}$ the set of contracts in Y involving c, and similarly for a student $s \in S$ we denote by $Y_s := \{x \in Y : y_S = s\}$ the set of contracts in Y involving s.

Each college c has a choice function $Ch_c : 2^{X_c} \to 2^{X_c}$ that from each set $Y \subseteq X_c$ chooses a subset of contracts. Each college can only sign one contract with any given student, i.e. for each $x, y \in Ch_c(Y)$ with $x \neq y$ we have $x_S \neq y_S$. Throughout this paper, we assume that all considered choice functions satisfy the **irrelevance of rejected contracts (IRC)** (Aygün and Sönmez, 2013), which

[9] One has to make sure that in the "necessity part" of the maximal domain proof, the profile of unit demand choice functions for the other colleges can be chosen to be induced by monotone preferences.

[10] More generally, we can deal with the case where the ordering is college-student pair specific, i.e. for each college $c \in C$ and student $s \in S$ there is a total ordering $\rhd_{c,s}$ over contract terms. If "Pareto separability" (Roth, 1984b) of contracts holds, the set T can be interpreted as the set of "generalized salaries" in the sense of Roth (1984b) and $\rhd_{c,s}$ is derived from a parameterization of the efficient contract frontier. Thus, also in this case, our results apply.

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means that for all $Y \subseteq X_c$, $x \in X_c \setminus Y$,

$$x \notin Ch_c(Y \cup \{x\}) \Rightarrow Ch_c(Y) = Ch_c(Y \cup \{x\}).$$

We also define a rejection function $R_c : 2^{X_c} \to 2^{X_c}$ by $R_c(Y) := Y \setminus Ch_c(Y)$. We denote the set of all choice functions for college $c \in C$ that satisfy IRC, by $\mathcal{C}_c$.

Each student s has preferences $\succeq_s$ over different contracts involving him, and an outside option which we denote by "$\emptyset$". We make the following assumption on students' preferences:

1. Preferences are **strict**, for $x, x' \in X$ with $x'_S = x_S$, we have

   $$x \neq x' \Rightarrow x \succ_{x_S} x' \text{ or } x' \succ_{x_S} x,$$

   and

   $$x \succ_{x_S} \emptyset \text{ or } \emptyset \succ_{x_S} x.$$

2. Preferences are **monotone** in contract terms, for each $x, x' \in X$ with $x_S = x'_S$ and $x_C = x'_C$ we have

   $$x_T \rhd x'_T \Rightarrow x \succ_{x_S} x'.$$

We denote the set of all strict and monotone preferences for student $s \in S$ by $\mathcal{R}_s$. A **problem** is a pair $(Ch, \succeq)$ consisting of a **choice profile** $Ch = (Ch_c)_{c \in C} \in \bigtimes_{c \in C} \mathcal{C}_c$ and a **preference profile** $\succeq = (\succeq_s)_{s \in S} \in \bigtimes_{s \in S} \mathcal{R}_s$.

An **allocation** is a set $Y \subseteq X$ that contains at most one contract per student. We denote the set of allocations by $\mathcal{A}$. For $Y \in \mathcal{A}$, we write $Y_S := \{y_S : y \in Y\}$. An allocation Y is

individually rational in $(Ch, \succeq)$ if for each $c \in C$, we have $Y_c = Ch_c(Y)$, and for each $y \in Y$ we have $y \succ_{y_S} \emptyset$,

blocked in $(Ch, \succeq)$ if there are $c \in C$ and an allocation Z with $Z_c \neq Y_c$, such that $Z_c = Ch_c(Y \cup Z)$ and for each $z \in Z_c$ we have $z \succeq_{z_S} Y_{z_S}$,

stable in $(Ch, \succeq)$ if it is individually rational and not blocked.

In the following, it will be useful to define for each set of contracts $Y \subseteq X_c$ with a college c the allocation

$$Y^{\min} := \{y \in Y : \nexists y' \in Y, y'_S = y_S, y_T \rhd y'_T\}$$

of contracts that gives each student the worst contract among the contract in Y, and the set

$$\mathcal{U}(Y) := \{x \in X_c : x_T \trianglerighteq y_T \text{ for some } y \in Y \text{ with } x_S = y_S\}$$

of contracts, not necessarily in Y, which are as least as good for the involved student as his worst contract in Y.[11] We call choice function Ch_c for college $c \in C$ **monotone in contract-terms** if for each $Y \subseteq X_c$ we have

$$Ch_c(Y) \subseteq Y^{\min}.$$

[11] Note that the definitions of $Y^{\min}$ and $\mathcal{U}(Y)$ only depend on $\rhd$.

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2.2 Mechanisms

A **mechanism** (for the students) is a mapping from preference profiles to allocations $\mathcal{M} : \bigtimes_{s\in S} \mathcal{R}_s \to \mathcal{A}$. Mechanism $\mathcal{M}$ is **strategy-proof** if it is a weakly dominant strategy for students to report their true preferences to the mechanism, i.e. for each $s \in S$, $\succeq_{-s} \in \bigtimes_{s'\in S\setminus\{s\}} \mathcal{R}_{s'}$ and $\succeq_s, \succeq'_s \in \mathcal{R}_s$ we have

$$\mathcal{M}(\succeq_s, \succeq_{-s}) \succeq_s \mathcal{M}(\succeq'_s, \succeq_{-s}).$$

Mechanism $\mathcal{M}$ is **group-strategy-proof** if for each $S' \subseteq S$, $\succeq_{-S'} \in \bigtimes_{s\in S\setminus S'} \mathcal{R}_s$ and $\succeq_{S'}, \succeq'_{S'} \in \bigtimes_{s\in S'} \mathcal{R}_s$, there is a $s' \in S'$ with

$$\mathcal{M}(\succeq_{S'}, \succeq_{-S'}) \succeq_{s'} \mathcal{M}(\succeq'_{S'}, \succeq_{-S'}).$$

Let Ch be a choice profile. Mechanism $\mathcal{M}$ is Ch**-stable** if for each $\succeq \in \bigtimes_{s\in S} \mathcal{R}_s$ allocation $\mathcal{M}(\succeq)$ is stable in $(Ch, \succeq)$.

2.3 Examples

Several examples from applied marked design fit into our model, such as the model of cadet-to-branch matching of Sönmez (2013) with the preference restriction of Jagadeesan (2019), the discrete version of the job matching model of Kelso and Crawford (1982), the college admission models of Hassidim et al. (2017) and of Abizada and Dur (2017). The job matching model of Kelso and Crawford (1982) will play an important role in Section 3.3.

2.3.1 Job matching with salaries

A finite **Kelso-Crawford economy** consists of a finite set of firms F, a finite set of workers W, a finite set of salaries $\Sigma \subseteq \mathbb{R}_+$ and a profile $u_{i\in F\cup W}$ of utility functions, where for each $f \in F$, utility is a functions $u_f : \{(W', p) : W' \subseteq W, p \in \mathbb{R}_+^{W'}\} \to \mathbb{R}$ that is strictly decreasing in $\mathbb{R}_+^{W'}$ for each W', and for each $w \in W$, utility is a function $u_w : F \times \mathbb{R}_+ \cup \{\emptyset\} \to \mathbb{R}$ that is strictly increasing in $\mathbb{R}_+$ for each f.

The model fits in our framework with $C = F$, $S = W$, $T = \Sigma$, $x_T \rhd x'_T \Leftrightarrow x_T > x'_T$, $X = F \times W \times \Sigma$, choice functions are defined by

$$Ch_c(Y) = \max_{Y'\subseteq Y^{\min}} u_c(Y'_S, (y_T)_{y\in Y'}),$$

and preferences $(\succeq_s)_{s\in S}$ are induced by utility functions,

$$(c, s, t) \succ_s (c', s, t') \Leftrightarrow u_s(c, t) > u_s(c', t'), \quad (c, s, t) \succ_s \emptyset \Leftrightarrow u_s(c, t) > u_s(\emptyset).$$

Note that the constructed market with contracts $(Ch, \succeq)$ does not only satisfy monotonicity of students' preferences, but also monotonicity of colleges' choice function. We say that workers are **gross substitutes** for firm f if for each $p, p' \in \Sigma^W$ such that $p \leq p'$, if $p'_w = p_w$ and $w \in \operatorname{argmax}_{W'\subseteq W} u_f(W', p)$, then $w \in \operatorname{argmax}_{W'\subseteq W} u_f(W', p')$.

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2.4 Stable Allocations

In general, a stable allocation does not need to exist for our model. A sufficient condition for stability is that contracts are substitutes for colleges, i.e. if a contract is rejected from some set of contracts, then this contract is also rejected from each superset of that set of contracts.

Substitutability (Roth, 1984b; Hatfield and Milgrom, 2005): For each $Z \subseteq Y \subseteq X_c$,

$$R_c(Z) \subseteq R_c(Y).$$

Not only is substitutability sufficient for the existence of a stable allocation but it also guarantees that the set of stable allocations has a lattice structure. If contracts are substitutes for colleges, then the set of stable allocation forms a lattice with respect to the preferences of students (Blair, 1988). In particular, there is a unique stable allocation that is most preferred by all students among all stable allocations. We call this allocation the **student-optimal stable** allocation. It can be found by the **cumulative offer (CO) algorithm** that is defined as follows.

1. Each student applies under his favorite acceptable and unrejected contract or stays alone if he finds no unrejected contract acceptable.

2. Each college tentatively accepts the contracts it chooses among the applications it has received so far and rejects all other contracts.

3. If no application has been rejected in the current round, the algorithm stops. If some applications are rejected we repeat.

We denote the set of contracts chosen by the colleges in the final round of the CO algorithm for problem $(Ch, \succeq)$ by $\mathcal{CO}(Ch, \succeq)$. Alternatively, we can use the deferred acceptance algorithm to find the student optimal stable allocation. In the **deferred acceptance (DA) algorithm** rejections are permanent: Thus, in 2., colleges only choose among contracts that are proposed in the current round of the algorithm, but not contracts that have been proposed and rejected in a previous round. If contracts are substitutes for colleges, then the deferred acceptance and the cumulative offer algorithm yield the same outcome. The **deferred acceptance mechanism** for Ch assigns to each $\succeq \in \bigtimes_{s \in S} \mathcal{R}_s$ the outcome of the deferred acceptance algorithm in $(Ch, \succeq)$.[12]

Student-optimality is related to group-strategy-proofness. Under substitutability and the following additional condition on the colleges' choice functions the deferred acceptance mechanism is group-strategy-proof.

[12] We use the deferred acceptance algorithm and not the cumulative offer algorithm to define a mechanism, because it yields a well-defined mechanism even in situations where contracts are not substitutes for colleges. The deferred acceptance algorithm always ends in a (potentially unstable) allocation where each student is accepted at at most one college, whereas the cumulative offer algorithm can yield an outcome where multiple colleges accept a student.

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Law of Aggregate Demand (Hatfield and Milgrom, 2005): For each $Z \subseteq Y \subseteq X_c$:

$$Z \subseteq Y \Rightarrow |Ch_c(Y)| \geq |Ch_c(Z)|.$$

The following proposition summarizes known results about side-optimal stable allocations, the invariance of the set of matched students in stable allocations (the rural hospitals theorem), and group-strategy-proofness of the deferred acceptance mechanism.

Proposition 1 (Kelso and Crawford, 1982; Blair, 1988; Fleiner, 2003; Hatfield and Milgrom, 2005; Hatfield and Kojima, 2009)**.**

1. *If contracts are substitutes for colleges, then the deferred acceptance and the cumulative offer algorithm are equivalent and converge to a stable allocation that is most preferred by all students among all stable allocations.*

2. *If choice functions satisfy, moreover, the law of aggregate demand, then*

   (a) *the set of accepted students is the same in all stable allocations and each college accepts the same number of students in each stable allocation,*

   (b) *the deferred acceptance mechanism is group-strategy-proof.*

3 Results

3.1 Equivalent Choice Functions

It is a natural question whether the conditions of Section 2.4 for the stability and group-strategy-proofness of the deferred acceptance mechanism are also necessary. Next we provide a counterexample that shows that substitutability and the law of aggregate demand are not necessary for the deferred-acceptance mechanism to be stable and group-strategy-proof. The example will have the following structure: There is one college c for which contracts are not substitutes. For each college except for c, contracts are substitutes and the law of aggregate demand holds. However, c's choice function can be replaced by another choice function, such that

1. under the replacing choice function, contracts are substitutes for c and the law of aggregate demand holds,

2. the outcome of the cumulative offer algorithm (which here coincides with the outcome of the deferred acceptance algorithm) is, for all monotone preference profiles for students, the same under the original choice profile and the profile where c's choice function is replaced.

As the deferred acceptance mechanism is group-strategy-proof for the profile where we have replaced c's choice function by the other choice function, it is by outcome-equivalence also group-strategy-proof for the original choice profile.

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Example 1. The original choice function is from Example 2 in Hassidim et al. (2017). Consider a college admission problem with stipends where $T = \{0, 1\}$, with the interpretation that contract term "0" corresponds to admission without stipend, and contract term "1" corresponds to admission with a stipend. Thus, $1 \rhd 0$. For simplicity, we assume that there are two students $S = \{s_1, s_2\}$. The example can be generalized to more students. Consider a college c_1 which can admit up to two students but only has one stipend available. Students are admitted based on merit, and student s_1 is the higher merit student than s_2. If both students are available, the college prefers to allocate the stipend to student s_2 rather than s_1, e.g. because s_2 comes from a lower income background. The choice function of the college Ch_{c_1} is induced by preferences

$$\{(c_1, s_1, 0), (c_1, s_2, 0)\} \succ_{c_1} \{(c_1, s_1, 0), (c_1, s_2, 1)\} \succ_{c_1} \{(c_1, s_1, 1), (c_1, s_2, 0)\}$$
$$\succ_{c_1} \{(c_1, s_1, 0)\} \succ_{c_1} \{(c_1, s_1, 1)\} \succ_{c_1} \{(c, s_2, 0)\} \succ_{c_1} \{(c_1, s_2, 1)\}$$

in the usual way. Now consider the alternative choice function Ch'_{c_1} induced by preferences:

$$\{(c_1, s_1, 0), (c_1, s_2, 0)\} \succ'_{c_1} \{(c_1, s_1, 1), (c_1, s_2, 0)\} \succ'_{c_1} \{(c_1, s_1, 0)\}$$
$$\succ'_{c_1} \{(c_1, s_1, 1)\} \succ'_{c_1} \{(c_1, s_2, 0)\} \succ'_{c_1} \{(c_1, s_2, 1)\}$$

Thus, the preferences differ only in so far as allocation $\{(c_1, s_1, 0), (c_1, s_2, 1)\}$ where s_1 is admitted without stipend and s_2 is admitted with stipend is unacceptable under $\succ'_{c_1}$. The ranking of all other allocations is the same under both preferences.

Note that under Ch_{c_1} contracts are not substitutes as

$$(c_1, s_2, 1) \in Ch_{c_1}(\{(c_1, s_1, 0), (c_1, s_1, 1), (c_1, s_2, 1)\}) = \{(c_1, s_1, 0), (c_1, s_2, 1)\}$$

but

$$(c_1, s_2, 1) \notin Ch_{c_1}(\{(c_1, s_1, 1), (c_1, s_2, 1)\}) = \{(c_1, s_1, 1)\},$$

and that under Ch'_{c_1} contracts are substitutes and the law of aggregate demand holds.

Suppose colleges $C \setminus \{c_1\}$ have choice functions $Ch_{-c_1} = (Ch_c)_{c \neq c_1}$ under which contracts are substitutes and the law of aggregate demand holds. Define $Ch := (Ch_{c_1}, Ch_{-c_1})$ and $Ch' := (Ch'_{c_1}, Ch_{-c_1})$. Let $\succeq \in \bigtimes_{s \in S} \mathcal{R}_s$. We show that the cumulative offer algorithm converges to the same allocation for the problems $(Ch, \succeq)$ and $(Ch', \succeq)$. Observe that for all sets $Y \subseteq X$ with $Ch_{c_1}(Y) \neq Ch'_{c_1}(Y)$ we have $\{(c_1, s_1, 0), (c_1, s_2, 1)\} \subseteq Y$. In particular, for the cumulative offer algorithm to differ for the two problems, student s_1 must apply to c_1 for admission without stipend during the cumulative offer algorithm in $(Ch, \succeq)$. Note however that before s_1 applies to c_1 for admission without stipend, he applies to c_1 for admission with stipend, as $1 \rhd 0$. But once college c_1 tentatively accepts s_1 with a stipend, the college will not subsequently drop student s_1, as s_1 has higher merit than s_2. Thus, s_1 will never apply to c_1 for admission without a stipend during the cumulative offer algorithm. Hence, the cumulative offer algorithm for the two problems converges to the same allocation, which is the student-optimal stable allocation in $(Ch', \succeq)$.

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Note, moreover, that the equivalent choice function Ch'_c prioritizes merit over need in the following sense: From each set of contracts, an allocation is chosen, where the stipend is either un-allocated or allocated to the higher merit student s_1; whereas under choice function, Ch_c the stipend would be allocated to student s_2, if both students are available and it is possible to admit student s_1 without a stipend. □

The example motivates the following definition.[13]

CO-equivalence: Two choice profiles Ch and Ch' are CO-equivalent if for each $\succeq \in \bigtimes_{s\in S} \mathcal{R}_s$ we have $\mathcal{CO}(Ch, \succeq) = \mathcal{CO}(Ch', \succeq)$. Two choice functions Ch_c and Ch'_c are CO-equivalent if for each $Ch_{-c} \in \bigtimes_{c'\in C\setminus\{c\}} \mathcal{C}_{c'}$ profiles Ch and $Ch' := (Ch'_c, Ch_{-c})$ are CO-equivalent.

In the following, we use the term "equivalent" instead of "CO-equivalent", since we only consider one notion of equivalence throughout the paper. From the definition and Proposition 1, we obtain the following result:

Proposition 2. *If Ch is equivalent to a profile under which contracts are substitutes and the law of aggregate demand holds, then the deferred acceptance mechanism for Ch is Ch-stable and group-strategy-proof.*

Proof. Let Ch' be the equivalent profile. Let $\succeq \in \bigtimes_{s\in S} \mathcal{R}_s$. Let $Y_1 \subseteq Y_2 \subseteq \ldots \subseteq Y_\tau$ be the set of accumulated offers during the cumulative offer algorithm for $(Ch', \succeq)$ and $Z_1, Z_2, \ldots, Z_\tau$ be the set of proposed contracts during the deferred acceptance algorithm for $(Ch', \succeq)$. Since contracts under Ch' are substitutes, Part (1) of Proposition 1 implies $Ch'_c(Y_t) = Ch'_c(Z_t)$ for $t = 1, \ldots, \tau$ and each $c \in C$. Since Ch and Ch' are equivalent we have $Ch_c(Y_t) = Ch'_c(Y_t)$ for $t = 1, \ldots, \tau$ and each $c \in C$. Thus, $Ch_c(Y_t) = Ch'_c(Y_t) = Ch'_c(Z_t) \subseteq Z_t$ for $t = 1, \ldots, \tau$ and each $c \in C$. By IRC for Ch this implies $Ch_c(Z_t) = Ch_c(Y_t) = Ch'_c(Z_t)$ for $t = 1, \ldots, \tau$ and each $c \in C$, and the deferred acceptance algorithm in $(Ch, \succeq)$ and in $(Ch', \succeq)$ yield the same outcome. In particular, this implies the stability in $(Ch, \succeq)$ of the outcome $\bigcup_{c\in C} Ch_c(Z_\tau)$ of the deferred acceptance algorithm, since otherwise, if college $c \in C$ blocks, we would have $Ch_c(Y_\tau) \not\subseteq Z_\tau$. Moreover, if under Ch' contracts are substitutes and the law of aggregate demand holds, this implies by Part 2(b) of Proposition 1 the group-strategy-proofness of the deferred acceptance mechanism for Ch. □

As our first main result, we show that for each choice function under which contracts are observable substitutes and the observable law of aggregate demand holds we can construct an equivalent choice function under which contracts are substitutes and the law of aggregate demand holds. As an immediate corollary, we will obtain the group-strategy-proofness of the deferred acceptance mechanism under observable substitutes and the observable law of aggregate demand.

[13] A similar notion has been introduced independently by Jagadeesan (2019). However, since Jagadeesan (2019) does not assume monotonicity on students' preferences, his notion of equivalence is much stronger. See our Remark 1.

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Importantly, we define observable substitutability and the observable law of aggregate demand for monotone preferences by only requiring that the properties hold for observable sequences that can occur under monotone preferences: In the following, a sequence of contracts $x^1, x^2, \ldots, x^\tau$ is **generated from monotone preferences** if for $1 \leq t \leq \tau$ and each $x \in X$ with $x_S = x_S^t$ and $x_T \rhd x_T^t$, we have $x \in \{x^1, \ldots, x^{t-1}\}$. A sequence $x^1, x^2, \ldots, x^\tau$ is **observable** under Ch_c if for $1 \leq t \leq \tau - 1$ we have $x_S^{t+1} \notin [Ch_c\{x^1, \ldots, x^t\}]_S$. We call a set of contracts $Y \subseteq X_c$ **observable** under Ch_c, if there is a sequence of contracts $x^1, \ldots, x^{|Y|}$ that is observable under Ch_c and that is generated from monotone preferences such that $Y = \{x^1, \ldots, x^{|Y|}\}$. We denote the set of all observable sets of contracts under Ch_c by $\mathcal{O}(Ch_c)$. Observable substitutability for monotone preferences is defined as follows:[14]

Observable Substitutability (for Monotone Preferences) (Hatfield et al., 2020): Contracts are observably substitutes for monotone preferences under choice function Ch_c if for each observable sequence $x^1, x^2, \ldots, x^\tau$ under Ch_c that is generated from monotone preferences, we have

$$R_c\{x^1, \ldots x^t\} \subseteq R_c\{x^1, \ldots, x^{t+1}\} \text{ for each } 1 \leq t < \tau.$$

Similarly, we can define an observable version of the law of aggregate demand.

Observable Law of Aggregate Demand (for Monotone Preferences) (Hatfield et al., 2020): A choice function Ch_c satisfies the observable law of aggregate demand for monotone preferences if for each observable sequence under Ch_c that is generated from monotone preferences, we have

$$|Ch_c\{x^1, \ldots x^t\}| \leq |Ch_c\{x^1, \ldots, x^{t+1}\}| \text{ for each } 1 \leq t < \tau.$$

Since we exclusively deal with monotone preferences, from now one we drop the term "for monotone preferences." However, the reader should be aware that the conditions are weaker than the corresponding conditions for general preferences.[15] With the definition, we have the following result.

Theorem 1. *For each choice function Ch_c under which contracts are observable substitutes and the observable law of aggregate demand holds, there exists an equivalent choice function Ch'_c that is monotone in contract terms such that contracts are substitutes and the law of aggregate demand holds.*

To prove the result, we define for each choice function Ch_c a related choice function Ch'_c that we call the **virtual choice function** for Ch_c by

$$Ch'_c(Y) := Y^{\min} \setminus \bigcup_{\tilde{Y} \subseteq \mathcal{U}(Y), \tilde{Y} \in \mathcal{O}(Ch_c)} R_c(\tilde{Y}).$$

[14] Alternatively, we could also define observable substitutabilty for monotone preferences, by requiring that the substitutes condition holds for observable sets, i.e. by requiring that for $Y, Z \in \mathcal{O}(Ch_c)$ and $Z \subseteq Y$ we have $R_c(Z) \subseteq R_c(Y)$. See the first part of Lemma 2 in the appendix. Note that this definition is only equivalent for the case of monotone preferences, but is stronger on the general preference domain.

[15] In the example in Footnote 6, for example, college c_1's choice function satisfies the monotone versions of the two conditions but not the general preference versions.

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The construction has the following intuition: The set of accumulated offers during an instance of the cumulative offer algorithm is observable. Hence a sufficient condition for equivalence between two choice functions Ch_c and Ch'_c is that $Ch_c(Y) = Ch'_c(Y)$ and therefore $R_c(Y) = R'_c(Y)$ for each $Y \in \mathcal{O}(Ch_c)$. If contracts are substitutes under Ch'_c, then, in particular, contracts that are rejected from an observable (under Ch_c) set are also rejected from each, potentially unobservable, superset of that set. Hence assuming that choices agree on observable sets and contracts are substitutes under Ch'_c for each $Y \subseteq X_c$ we have $\bigcup_{\tilde{Y}\subseteq Y, \tilde{Y}\in\mathcal{O}(Ch_c)} R_c(\tilde{Y}) = \bigcup_{\tilde{Y}\subseteq Y, \tilde{Y}\in\mathcal{O}(Ch_c)} R'_c(\tilde{Y}) \subseteq R'_c(Y)$. Thus,

$$Ch'_c(Y) \subseteq Y \setminus \bigcup_{\tilde{Y}\subseteq Y, \tilde{Y}\in\mathcal{O}(Ch_c)} R_c(\tilde{Y}).$$

To guarantee feasibility, $Ch'_c(Y)$ generally has to be a strict subset in the previous expression. Choosing only contracts in $Y^{\min}$ guarantees feasibility and monotonicity of the choice function Ch'_c. To guarantee the IRC condition for Ch'_c on sets with $Y \neq \mathcal{U}(Y)$ we need in general to consider observable subsets of $\mathcal{U}(Y)$ rather than Y. The full proof of the theorem is in the appendix.

Remark 1. A corresponding result to Theorem 1 does not hold for the case of non-monotonic preferences. Without monotonicity, equivalence requires that choices coincide on all sets of contracts which contain at most one contract per student (see Theorem 1 of Jagadeesan, 2019), and equivalence to a choice function under which contracts are substitutes and the law of aggregate demand holds implies unilateral substitutability and the law of aggregate demand. These properties are much stronger than observable substitutability and the observable law of aggregate demand. □

Remark 2. One can show that, for the job matching model with salaries, if the choice function in Theorem 1 is generated from maximizing preferences that are quasi-linear in salaries, then the choice function and the equivalent virtual choice function are the same. Combining this observation with the maximal domain results of Hatfield et al. (2020), one can show that for quasi-linear preferences, choice profiles under which contracts are substitutes form a maximal domain of choice profiles generated from quasi-linear preferences for the existence of a stable (and strategy-proof) mechanism. This yields an alternative way of proving a result due to Hatfield and Milgrom, 2005; Hatfield and Kojima, 2008. □

Immediately from Theorem 1 and Proposition 2 it follows that the deferred acceptance mechanism is stable and group-strategy-proof if contracts are observable substitutes and the observable law of aggregate demand holds.

Corollary 1. *For each choice profile under which contracts are observable substitutes and the observable law of aggregate demand holds, the deferred acceptance mechanism is stable and group-strategy-proof.*

Remark 3. Hatfield et al. (2020) prove that observable substitutability, the observable law of aggregate demand and "non-manipulability via contractual terms"

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are sufficient conditions for the stability and strategy-proofness of the deferred acceptance mechanism. Non-manipulability via contractual terms requires that for each profile in the domain and each college $c \in C$ the deferred-acceptance mechanism is strategy-proof on the domain of preference profiles where only contracts with c are acceptable for students. For the case of monotone preferences, Corollary 1 strengthens the result in two directions: First, strategy-proofness can be replaced by the stronger property of group-strategy-proofness. Second, whereas non-manipulability via contractual terms is in general independent of the other two properties, Corollary 1 implies that this property is implied by observable substitutability and the observable law of aggregate demand for monotone preferences. □

The assumption of observable substitutability turns out to be necessary for Theorem 1, as we show in the following theorem.

Theorem 2. *If Ch is equivalent to a profile Ch' under which contracts are substitutes, then contracts are observable substitutes under Ch. If, moreover, the law of aggregate demand holds under Ch', then the observable law of aggregate demand holds under Ch.*

3.2 A Maximal Domain Result

Next, we derive several implications of Theorems 1 and 2. Combining the results with Theorem 5 of Hatfield et al. (2020) we first characterize a maximal domain of choice profiles for the existence of a stable and strategy-proof mechanism using our notion of equivalence. In the following a **choice domain** is a set of choice profiles $\mathcal{D} \subseteq \bigtimes_{c \in C} \mathcal{C}_c$. A choice domain $\mathcal{D}$ is **Cartesian** if $\mathcal{D} = \bigtimes_{c \in C} \mathcal{D}_c$ where $\mathcal{D}_c \subseteq \mathcal{C}_c$ for each college $c \in C$. A choice function Ch_c for college c is **unit demand** if $|Ch_c(Y)| \leq 1$ for all $Y \subseteq X_c$. A choice domain is **unital** if it includes all profiles of unit demand choice functions. We obtain the following corollary of Theorems 1 and 2.

Corollary 2. *For a domain $\mathcal{D}$ of choice profiles the following statements are equivalent:*

1. *$\mathcal{D}$ is a maximal Cartesian, unital domain of choice profiles such that a stable and strategy-proof mechanism is guaranteed to exist,*

2. *$\mathcal{D}$ is the domain of choice profiles that are equivalent to a choice profile under which contracts are substitutes and the law of aggregate demand holds.*

Proof. Let $\mathcal{D}_1$ be a maximal Cartesian, unital domain of choice profiles such that a stable and strategy-proof mechanism is guaranteed to exist. Let $\mathcal{D}_2$ he domain of choice profiles that are equivalent to a choice profile under which contracts are substitutes and the law of aggregate demand holds. By Theorems 1 and 2, $\mathcal{D}_2$ is the domain of choice profiles under which contracts are observable substitutes and the observable law of aggregate demand holds. By Theorem 5 of Hatfield et al. (2020), $\mathcal{D}_1$ is a subdomain of the domain of profiles under which contracts

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are observable substitutes and the observable law of aggregate demand holds, thus $\mathcal{D}_1 \subseteq \mathcal{D}_2$. By Proposition 2, for each $Ch \in \mathcal{D}_2$, the deferred acceptance mechanism is stable and strategy-proof. Thus, by maximality $\mathcal{D}_1 = \mathcal{D}_2$.

□

Remark 4. Combining the result with Proposition 2, the maximal domain result also holds if strategy-proofness is replaced by group-strategy-proofness. □

3.3 An embedding result

As a third corollary of Theorem 1, we obtain an embedding result. In recent work Jagadeesan (2019) shows that for each BfYC choice profile as introduced by Sönmez (2013) there is an equivalent choice profile such that the equivalent problem can be embedded into a Kelso-Crawford economy. Formally, an **isomorphism** between a matching market with contracts $(Ch, \succeq)$ and a Kelso-Crawford economy (Σ, u) is a one-to-one mapping $(f, w, \sigma) : X \to F \times W \times \Sigma$ such that

1. for each $x, x' \in X$, $f(x) = f(x')$ if and only if $x'_C = x_C$, and $w(x) = w(x')$ if and only if $x_S = x'_S$,
2. for each $x, x' \in X$ we have

$$x \succ_{x_S} x' \Leftrightarrow u_{w(x)}(f(x), \sigma(x)) > u_{w(x)}(f(x), \sigma(x))$$

and

$$x \succ_{x_S} \emptyset \Leftrightarrow u_{w(x)}(f(x), \sigma(x)) > u_{w(x)}(\emptyset),$$

3. for each $c \in C$, $Y \subseteq X_c$ and $f \in F$ such that $f(x) = f$ for each $x \in Y$ we have

$$Ch_c(Y) = \text{argmax}_{Y' \subseteq Y^{\min}} u_f(\{(w(x), \sigma(x)) : x \in Y'\}),$$

4. for each $w \in W, f \in F$ and $\sigma \in \Sigma \setminus \{\sigma(x) : x \in X, f(x) = f, w(x) = w\}$, we have $u_w(f, \sigma) < u_w(\emptyset)$.[16]

Remark 5. The above notion of an isomorphism is due to Jagadeesan (2019). It is different from the notion introduced by Echenique (2012), and used by Kominers (2012), and Schlegel (2015): Utility function in the Kelso-Crawford economy satisfy stronger regularity conditions in Jagadeesan (2019). In Echenique (2012), monotonicity of utility functions is only required for salaries corresponding to "undominated" contracts (see the discussion in Schlegel, 2015), whereas in Jagadeesan (2019) monotonicity can be achieved for all salaries.

To guarantee that utility functions satisfy the stronger regularity condition, Jagadeesan (2019) first constructs an equivalent choice profile. The embedding is then performed for a problem where the original choice profile is replaced by the equivalent choice profile. In contrast to this, the embedding result of Echenique (2012) establishes an isomorphism (with his weaker notion of isomorphism) for

[16]This item is not required in Jagadeesan (2019), as he only considers situations in which there is the same number of contracts between any college and student.

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the original market with contracts, without the need to first construct an equivalent choice profile. Put differently, the embedding result of Jagadeesan (2019) establishes an isomorphism between the cumulative offer algorithm in the original problem and the salary adjustment process in the Kelso-Crawford economy, whereas the embedding result of Echenique (2012) establishes an isomorphism between the sets of stable allocations in the original problem and the Kelso-Crawford economy, but under a weaker notion of an isomorphism.

In the following, we will establish an embedding result in the sense of Jagadeesan (2019). An embedding result in the sense of Echenique (2012) does not, in general, hold in our setting. □

Corollary 3. *For each choice profile Ch such that contracts are observable substitutes and the observable law of aggregate demand holds, there exists an equivalent choice profile Ch', such that for each preference profile $\succeq$, the market $(Ch', \succeq)$ is isomorphic to a Kelso-Crawford economy (Σ, u) such that workers are gross substitutes for firms.*

Proof. By Theorem 1, there exist an equivalent profile Ch' under which contracts are substitutes and the law of aggregate demand holds. Moreover, Ch' can be chosen to be monotone in contract-terms for each college. For each $c \in C$, Ch'_c can be rationalized (see Theorem 1 in Alva, 2018) by a strict preference relation $\succeq'_c$ over $\mathcal{A}_c$. The preference relation $\succeq'_c$ can be chosen to be monotonic in contract terms. Let $(\succeq_s)_{s\in S}$ be a preference profile for students. For the preference profile $((\succeq'_c)_{c\in C}, (\succeq_s)_{s\in S})$, we can construct as in the proof of Theorem 1 in Echenique (2012) a corresponding Kelso-Crawford economy (Σ, u) such that Gross Substitutability for firms hold. Since $\succeq'_c$ is monotonic in contract terms for each $c \in C$ there are no undominated contracts in $(Ch', \succeq)$. Thus, the embedding is one-to-one and satisfies our definition of an isomorphism, in particular monotonicity of the utility functions in the Kelso-Crawford economy is satisfied.

□

Remark 6. While we use the same notion of isomorphism as Jagadeesan (2019), we do not assume quasi-linearity of firm utility functions in the Kelso-Crawford economy. In this sense, our result is weaker. However, our result applies to a larger domain of choice functions. □

4 Conclusion

We have studied a model of matching with contracts with a natural monotonicity condition on applicants' preferences that is satisfied in many practically important matching markets, e.g., for the matching of students to colleges with different stipends or fees, the medical match with flexible salaries, or the cadet-to-branch match if cadets rank branch-of-choice contracts monotonically. Our domain restriction allowed us to obtain a characterization of a maximal domain of choice profiles for which a stable and strategy-proof mechanism exists, which - arguably - is simpler than the corresponding characterization for the general model with

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non-monotone preferences (Hatfield et al., 2020). This indicates that the domain restriction to monotone preferences removes some complexity from the matching model with contracts and, hence, simplifies its analysis while practical relevance is maintained since the monotonicity assumption is satisfied in many real-world matching markets. We hope that these features will make the model with monotone preferences also a useful tool for future research on matching markets with contracts.

Our results also raise interesting questions for the general matching model with contracts: For our model and choice domain, strategy-proofness of the deferred acceptance mechanism implies its group-strategy-proofness. This is established through the construction of an equivalent choice profile under which contracts are substitutes. Substitutability implies a lattice structure for the set of stable allocations under the equivalent choice profile, and with the lattice structure it is easy to show that strategy-proofness implies group-strategy-proofness (see e.g. the discussion in Barberà et al., 2016). For the general matching model with contracts similar constructions, such as the substitutable completion procedure of Hatfield and Kominers (2015), are known. For the same reason as before - the set of stable allocations forms a lattice under the substitutable completion of the choice profile - strategy-proofness of the deferred acceptance mechanism implies its group-strategy-proofness. However, in contrast to our model, for the general model with non-monotone preferences, a substitutable completion does not exist for each profile in the maximal domain of Hatfield et al. (2020). It, thus, remains an interesting open question under which conditions the strategy-proofness of a stable matching mechanism for the general model of matching with contracts implies its group-strategy-proofness.

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A Proof of Theorem 1

The proof relies on the following two lemmata that we prove first.

Lemma 1. *If contracts are observable substitutes under Ch_c, then for each $Y \subseteq X_c$ and for any two maximal sequences $x^1, \ldots, x^\tau$ and $y^1, \ldots, y^{\tau'}$ in Y that are observable under Ch_c and generated from monotone preferences we have $\{x^1, \ldots, x^\tau\} = \{y^1, \ldots, y^{\tau'}\}$.*

Proof. The proof strategy is due to Hirata and Kasuya (2014). We use induction on the size of the set Y.
Induction Basis: If $|Y| = 0$, then $Y = \emptyset$ and trivially the empty sequence is the only observable sequence under Ch_c generated from monotone preferences in Y.
Induction Assumption: For each $Y \subseteq X_s$ with $|Y| \leq n$ for any two maximal sequences $x^1, \ldots, x^\tau$ and $y^1, \ldots, y^{\tau'}$ in Y that are observable under Ch_c and generated from monotone preferences we have $\{x^1, \ldots, x^\tau\} = \{y^1, \ldots, y^{\tau'}\}$.
Induction Step: Let $|Y| = n + 1$. Consider two maximal sequences $x^1, \ldots, x^\tau$ and $y^1, \ldots, y^{\tau'}$ in Y that are observable under Ch_c and generated from monotone preferences. Suppose $\{x^1, \ldots, x^\tau\} \neq \{y^1, \ldots, y^{\tau'}\}$. Then w.l.o.g. there is a

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$1 \leq t \leq \tau'$ with $y^t \notin \{x^1, \ldots, x^\tau\}$. Choose the smallest such t and consider the set $Y' := Y \setminus \{y^t\}$. The sequence $y^1, \ldots, y^{t-1}$ is contained in Y', generated from monotone preferences and is observable under Ch_c. Extend $y^1, \ldots y^{t-1}$ to a maximal sequence $y^1, \ldots, y^{t-1}, \tilde{y}^t, \ldots, \tilde{y}^{\tilde{\tau}}$ in Y' that is observable under Ch_c and generated from monotone preferences. Since $y^t \notin \{x^1, \ldots, x^\tau\}$, sequence $x^1, \ldots, x^\tau$ is a maximal sequence in Y' that is observable under Ch_c and generated from monotone preferences. By the induction assumption, we have $\{y^1 \ldots, y^{t-1}, \tilde{y}^t, \ldots, \tilde{y}^{\tilde{\tau}}\} = \{x^1, \ldots, x^\tau\}$. Moreover, $y^t_S \in Ch_c\{x^1, \ldots, x^\tau\}_S$, as otherwise $x^1, \ldots, x^\tau, y^t$ would be observable under Ch_c and generated from monotone preferences, contradicting the maximality of $x^1, \ldots, x^\tau$ in Y. Since $y^1, \ldots, y^t$ is observable and generated from monotone preferences, we have $y \in R_c\{y^1, \ldots, y^{t-1}\}$ for each $y \in X_c$ with $y_S = y^t_S$ and $y_T \rhd y^t_T$. By observable substitutability, we have $y \in R_c\{y^1 \ldots, y^{t-1}, \tilde{y}^t, \ldots, \tilde{y}^{\tilde{\tau}}\}$ for each $y \in X_c$ with $y_S = y^t_S$ and $y_T \rhd y^t_T$. As $y^t \notin \{y^1, \ldots, y^{t-1}, \tilde{y}^t, \ldots, \tilde{y}^{\tilde{\tau}}\}$ this implies $y^t_S \notin Ch_c\{y^1 \ldots, y^{t-1}, \tilde{y}^t, \ldots, \tilde{y}^{\tilde{\tau}}\}_S$. Since $y^t_S \in Ch_c\{x^1, \ldots, x^\tau\}_S$, as previously observed, this contradicts

$$\{y^1 \ldots, y^{t-1}, \tilde{y}^t, \ldots, \tilde{y}^{\tilde{\tau}}\} = \{x^1, \ldots, x^\tau\}.$$

□

Lemma 2. *If contracts are observable substitutes under Ch_c, then*

1. *for $Y, Z \in \mathcal{O}(Ch_c)$ with $Z \subseteq Y$ we have*

$$R_c(Z) \subseteq R_c(Y).$$

If moreover, the observable law of aggregate demand holds for Ch_c, then

$$|Ch_c(Z)| \leq |Ch_c(Y)|,$$

2. *for $Ch_{-c} \in \bigtimes_{c' \in C \setminus \{c\}} \mathcal{C}_c$ and $\succeq \in \bigtimes_{s \in S} \mathcal{R}_s$ the sets of contracts $Y_1 \subseteq Y_2 \subseteq \ldots \subseteq Y_\tau \subseteq X_c$ proposed to c during the cumulative offer algorithm in $(Ch, \succeq)$ are observable under Ch_c.*

Proof. For the first part let $Y, Z \in \mathcal{O}(Ch_c)$ with $Z \subseteq Y$. Since Z is observable, we can find a sequence $x^1, \ldots, x^{|Z|}$ generated from monotone preferences with $Z = \{x^1, \ldots, x^{|Z|}\}$ that is observable under Ch_c. Maximally extend the sequence in Y to obtain a sequence $x^1, \ldots, x^{|Z|}, x^{|Z|+1}, \ldots, x^\tau$ generated from monotone preferences that is observable under Ch_c. By Lemma 1 and observability of Y we have $Y = \{x^1, \ldots, x^\tau\}$ and $\tau = |Y|$. The result follows by applying observable substitutability (resp. the observable law of aggregate demand) to the sequence $x^1, \ldots, x^{|Y|}$.

For the second part, note that for $t = 1$ the result holds trivially: by the definition of the cumulative offer algorithm each contract in Y_1 is the unique most preferred contract for the involved agent among all contracts. Thus, ordering the contracts in Y_1 arbitrarily yields an observable sequence under Ch_c generated from monotone preferences. Next let $1 < t \leq \tau$ and suppose that $Y_1, \ldots, Y_{t-1} \in \mathcal{O}(Ch_c)$. Take an observable sequence $x^1, \ldots, x^{|Y_{t-1}|}$ under Ch_c generated from

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monotone preferences such that $Y_{t-1} = \{x^1, \ldots, x^{|Y_{t-1}|}\}$ and enumerate contracts in $Y_t \setminus Y_{t-1} = \{x^{|Y_{t-1}|}, \ldots x^{|Y_t|}\}$ arbitrarily. By the definition of the CO algorithm, for each $x \in Y_t \setminus Y_{t-1}$ each $x' \in X_c$ with $x'_S = x_s$ and $x' \rhd x$ has been rejected in some previous round of the algorithm, i.e. there is a $t' < t$ with $x' \in R_c(Y_{t'})$. By observable sustitutability and the second part applied to $Z = Y_{t'}$ and $Y = Y_{t-1}$, we have $x' \in R_c(Y_{t-1}) = R_c\{x^1, \ldots, x^{|Y_{t-1}|}\}$. Thus, for each $x \in Y_t \setminus Y_{t-1}$ we have $x_S \notin Ch_c(Y_{t-1})_S$. Observe furthermore that $Y_t \setminus Y_{t-1}$ contains at most one contract per student. Thus, repeated application of observable substitutability shows that the sequence $x^1, \ldots x^{|Y_t|}$ is observable.

□

With the two lemmata we can prove the theorem.

Proof. For each $Y \subseteq X_c$, let $Y^\vee \in \mathcal{O}(Ch_c)$ be defined by $Y^\vee := \{x^1, \ldots, x^\tau\}$ where $x^1, \ldots, x^\tau$ is a maximal observable subsequence under Ch_c of $\mathcal{U}(Y)$ generated by monotone preferences. By Lemma 1, $Y^\vee$ is well-defined and $\tilde{Y} \subseteq Y^\vee$ for each $\tilde{Y} \in \mathcal{O}(Ch_c)$ with $\tilde{Y} \subseteq \mathcal{U}(Y)$. Thus, by the first part of Lemma 2,

$$\bigcup_{\tilde{Y} \subseteq \mathcal{U}(Y), \tilde{Y} \in \mathcal{O}(Ch_c)} R_c(\tilde{Y}) = R_c(Y^\vee),$$

and therefore the virtual choice function is given by

$$Ch'_c(Y) = Y^{\min} \setminus R_c(Y^\vee).$$

First note that Ch'_c satisfies our assumptions on choice functions: By definition $Ch'_c(Y) \subseteq Y^{\min} \subseteq Y$, and since $Y^{\min}$ contains at most one contract per student, also $Ch'_c(Y)$ contains at most one contract per student. The IRC condition for Ch'_c will follow from substitutability and the law of aggregate demand (see Aygün and Sönmez, 2013) for Ch'_c which we will establish next.

Let $Z \subseteq Y \subseteq X_c$. First we show substitutability, i.e. $R'_c(Z) \subseteq R'_c(Y)$. Let $x \in R'_c(Z)$. If $x \in Y \setminus Y^{\min}$, then $x \in R'_c(Y)$, as $Ch'_c(Y) \subseteq Y^{\min}$. If $x \in Y^{\min}$, then $x \in Z^{\min}$ and therefore $x \in R_c(\tilde{Z})$ for a $\tilde{Z} \in \mathcal{O}(Ch_c)$ with $\tilde{Z} \subseteq \mathcal{U}(Z) \subseteq \mathcal{U}(Y)$. Thus, also $x \in R'_c(Y)$.

By definition of Ch'_c and $Y^\vee$ (resp. $Z^\vee$) we have $Ch'_c(Y)_S = Ch_c(Y^\vee)_S$ and $Ch'_c(Z)_S = Ch_c(Z^\vee)_S$. Lemma 1 implies that $Z^\vee \subseteq Y^\vee$. Thus, by the first part of Lemma 2 we obtain the law of aggregate demand for Ch'_c, as

$$|Ch'_c(Z)| = |Ch'_c(Z)_S| = |Ch_c(Z^\vee)| \leq |Ch_c(Y^\vee)| = |Ch'_c(Y)_S| = |Ch'_c(Y)|.$$

To show that Ch'_c is equivalent to Ch_c, let $Ch_{-c} \in \bigtimes_{c' \neq c} \mathcal{C}_c$ and $\succeq \in \bigtimes_{s \in S} \mathcal{R}_c$. By the second part of Lemma 2, the sets of proposed contracts $Y_1 \subseteq Y_2 \subseteq \ldots \subseteq Y_\tau$ to c during the CO-algorithm in $(Ch, \succeq)$ are observable. Now note that for each $Y \in \mathcal{O}(Ch_c)$ we have $Y = Y^\vee$ and, moreover, by observable substitutability $Ch_c(Y) \subseteq Y^{\min}$. Thus, $Ch'_c(Y) = Y^{\min} \setminus R_c(Y) = Y \setminus R_c(Y) = Ch_c(Y)$. As Ch'_c and Ch_c agree on observable sets, we have $Ch'_c(Y_t) = Ch_c(Y_t)$ for $t = 1, \ldots, \tau$. Hence Ch'_c and Ch_c are equivalent. □

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B Proof of Theorem 2

Proof. Let $c \in C$ and consider a sequence $x^1, \ldots, x^\tau$ that is observable under Ch_c and generated from monotone preferences. We show that $Ch'_c\{x^1, \ldots, x^\tau\} = Ch_c\{x^1, \ldots, x^\tau\}$. Since contracts are substitutes under Ch'_c this will imply that contracts are observable substitutes under Ch_c. Moreover, if the law of aggregate demand holds under Ch'_c, then this will imply the observable law of aggregate demand for Ch_c. In the following, we denote by $\succeq^0 \in \bigtimes_{s\in S} \mathcal{R}_s$ a profile such that no contract is acceptable, and for $1 \leq t \leq \tau$ we denote by $\succeq^t \in \bigtimes_{s\in S} \mathcal{R}_s$ a profile such that $x \succ^t_{x_S} \emptyset$ for $x \in \{x^1, \ldots, x^t\}$ and $\emptyset \succ^t_{x_S} x$ for $x \notin \{x^1, \ldots, x^t\}$. First note that

$$\emptyset = Ch'_c(\emptyset) = \mathcal{CO}(Ch', \succeq^0) = \mathcal{CO}(Ch', \succeq^0) = Ch_s(\emptyset).$$

Next we show that if for $0 \leq t < t' \leq \tau$ we have

$$Ch'_c\{x^1, \ldots, x^t\} = \mathcal{CO}(Ch', \succeq^t) = \mathcal{CO}(Ch, \succeq^t) = Ch_c\{x^1, \ldots, x^t\}$$

then

$$Ch'_c\{x^1, \ldots, x^{t'}\} = \mathcal{CO}(Ch', \succeq^{t'}) = \mathcal{CO}(Ch, \succeq^{t'}) = Ch_c\{x^1, \ldots, x^{t'}\}.$$

Since $Ch_c\{x^1, \ldots, x^t\} = Ch'_c\{x^1, \ldots, x^t\}$ for $0 \leq t < t'$, observability of $x^1, \ldots, x^{t'}$ under Ch_c implies observability of $x^1, \ldots, x^{t'}$ under Ch'_c.

Let Y be the set of proposed contract during the CO-algorithm in $(Ch', \succeq^{t'})$. By (observable) substitutability of Ch'_c, the second part of Lemma 2 applied to $(Ch', \succeq^{t'})$ implies that Y is observable under Ch'_c. Thus, there is a sequence $y^1, \ldots, y^{t''}$ that is observable under Ch_c and generated from monotone preferences such that $Y = \{y^1, \ldots, y^{t''}\} \subseteq \{x^1, \ldots, x^{t'}\}$. Note furthermore that $y^1, \ldots, y^{t''}$ is a maximal such sequence in the set $\{x^1, \ldots, x^{t'}\}$, since otherwise not all students who are unmatched in the outcome of the CO algorithm in $(Ch', \succeq^{t'})$ have proposed under all acceptable contracts. Since $x^1, \ldots, x^{t'}$ is observable under Ch'_c this implies $Y = \{y^1, \ldots, y^{t''}\} = \{x^1, \ldots, x^{t'}\}$. By equivalence of Ch and Ch', we have

$$Ch'_c\{x^1, \ldots, x^{t'}\} = \mathcal{CO}(Ch', \succeq^{t'}) = \mathcal{CO}(Ch, \succeq^{t'}) = Ch_c\{x^1, \ldots, x^{t'}\}.$$

□

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