PARETO-OPTIMAL INSURANCE CONTRACTS WITH PREMIUM BUDGET AND MINIMUM CHARGE CONSTRAINTS

Alexandru V. Asimit

Cass Business School, City University, London EC1Y 8TZ, United Kingdom.
E-mail: asimit@city.ac.uk

Ka Chun Cheung

Department of Statistics and Actuarial Science, The University of Hong Kong, Pokfulam Road, Hong Kong. E-mail: kccg@hku.hk

Wing Fung Chong*

Department of Mathematics and Department of Statistics, University of Illinois at Urbana-Champaign, Illinois, United States. E-mail: wfchong@illinois.edu

Junlei Hu

Department of Mathematical Sciences, University of Essex, Colchester CO4 3SQ, United Kingdom. E-mail: j.hu@essex.ac.uk

August 7, 2020

Abstract

In view of the fact that minimum charge and premium budget constraints are natural economic considerations in any risk-transfer between the insurance buyer and seller, this paper revisits the optimal insurance contract design problem in terms of Pareto optimality with imposing these practical constraints. Pareto optimal insurance contracts, with indemnity schedule and premium payment, are solved in the cases when the risk preferences of the buyer and seller are given by Value-at-Risk or Tail Value-at-Risk. The effect of our constraints and the relative bargaining powers of the buyer and seller on the Pareto optimal insurance contracts are highlighted. Numerical experiments are employed to further examine these effects for some given risk preferences.

Keywords: Bargaining power; Minimum charge; Optimal insurance contract design; Pareto optimality; Premium budget; Proportional Hazard Transformation; Tail Value-at-Risk; Value-at-Risk.

JEL codes: C6; C7; G22.

1 Introduction

The optimal (re)insurance contract design problem studies the rationale underneath any decision, including indemnity schedule and premium payment, made by the parties involved during the

*Corresponding author. Phone: +1 217-300-7129.

insurance risk-transfer. This is one of the most trending research areas in actuarial science over the past decade and was pioneered by Borch (1960; [8]) and Arrow (1963; [1]). They studied unilateral optimal insurance problems via optimizing the buyer's variance and expected utility respectively. A vast literature has been readdressing their proposed unilateral optimal insurance problems via various objective functions in [6, 11, 14, 16, 17, 20, 31, 32, 34], premium principles in [12, 15], practical constraints in [13, 23, 27, 28, 35, 36, 37], and more recently, heterogeneous beliefs in [7, 18, 24], as well as background risks in [19]; see also a recent work [22], and the references therein.

An insurance contract design decision should be bilateral, which is evidently a more realistic approach as it takes into account the objectives of all insurance players. Indeed, Raviv (1979; [30]) proposed to seek the optimal insurance contract in terms of Pareto optimality, and the problem has been revisited ever since. To name a few, Ludkovski and Young (2009; [29]) studied a multiple agent Pareto optimal risk sharing problem with concave distortion risk measures and constraints; Asimit *et al.* (2017; [2]) constructed the robust and Pareto optimal insurance contracts (see also Asimit *et al.* (2019) in [5]); together with premium constraints, Asimit *et al.* (2018; [4]) numerically found the Pareto optimality; Asimit and Boonen (2018; [3]) characterized the Pareto optimal insurance contracts as the optimal solutions of aggregate risk minimization problems; under general model settings, Cai *et al.* (2017; [10]) provided the Pareto optimal insurance contracts characterization and sufficient conditions for their existences; by utilizing the Neyman–Pearson perspective developed by Lo (2017; [27]), under a general setting with distortion risk measures, the Pareto optimal insurance contracts were explicitly solved by Jiang *et al.* (2018; [26]) and Lo and Tang (2019; [28]).

This paper revisits the optimal insurance contract design problem in terms of Pareto optimality. On one hand, we consider a premium budget constraint of the buyer; such a budget constraint has indeed been studied in unilateral optimal insurance design problems, such as Zheng and Cui (2014; [35]), Lo (2017; [27]), and Cheung *et al.* (2019; [13]). On the other hand, this paper considers a minimum charge constraint for the seller; such a minimum charge constraint is well-justified for covering indirect costs of the seller due to the risk-transfer; this minimum charge is a fairly general lower bound of the premium payment that imposes a non-negative risk loading condition on the premium payment. In practice, the buyer should be the one who bears all indirect costs, which include overhead and administrative expenses, in the up front premium, so that the expected profit of the seller, after receiving the up front premium, settling the claim, if any, and paying out these indirect costs, is still non-negative; otherwise, in a long run, the seller will bankrupt. See, for example, Section 4.8.1 in [21] for its theoretical justification via copula; see also [29]. This motivates us to incorporate these indirect costs into optimal insurance models in this paper; specifically, the up front premium paid by the buyer to the seller is bounded below by these total indirect costs; hence, our model is in line with practice that the buyer bears all indirect costs, which do not add to the loss of the seller. Notably, this minimum charge constraint has not been studied in the optimal insurance contract design literature, neither the unilateral nor the bilateral setting, except of [4]. However, only numerical methods are employed by Asimit *et al.* (2018; [4]).

This paper is indeed the first work within the optimal insurance contract design literature to address both minimum charge and premium budget constraints. We should mention that we discuss the Pareto optimal insurance contracts under the general setting, but closed-form solutions are only given for some risk preferences due to some challenging derivations that will be reconsidered in the coming future. That is, explicit Pareto optimal insurance contracts are provided when the buyer and seller risk preferences are given by Value-at-Risk (VaR) or Tail Value-at-Risk (TVaR). For VaR, all Pareto optimal insurance contracts are explicitly solved, with complete solutions being displayed to Table A.1 in Appendix A. The TVaR Pareto optimal insurance contracts are shown to be a single-layer counterpart, which reduces the corresponding family of fully non-linear and infinite dimensional optimization problems to that with finite dimensional optimization problems, which further requires numerical programming. The Pareto optimality for the case of TVaR is solved via modification arguments; see, for example, [12, 17].

Throughout this paper, the effect of the minimum charge, premium budget constraints, and relative bargaining powers of the buyer and seller on the Pareto optimal insurance contracts are specifically highlighted. The VaR closed-form Pareto optimal solutions lead to reasonable interpretations; moreover, the TVaR and Proportional Hazard Transformation (PHT) preferences require numerical optimization.

We finally conclude the introduction by comparing our work with [29], since a version of minimum premium constraint is mentioned and employed therein. On one hand, [29] considered general objective functions in concave distortion risk measures, together with n-agents, where n was allowed to be greater than two. On the other hand, there are four key dimensions that differentiate this paper from [29], which are inclusion of VaR, premium as decision, direct constraints on premium, and feasibility analysis. First, since the distortions in [29] are concave, their models and solutions do not include VaR as an example; but, we explicitly solve below all Pareto optimal insurance contracts for the case of VaR in Proposition 4.2, Theorem 4.1, and Table A.1 in Appendix A. Second, the premium in [29] is given by the expected value premium principle of the indemnity loss; but, our premium is a pure decision variable that it is part of the optimal solutions, which consist of a pair of disentangled premium constant and indemnity function. Most importantly, the version of minimum premium constraint in [29] is induced from rationality constraints, instead of their risk constraints further assumed in later sections; but, we impose both the rationality constraints, and thus the version of minimum premium constraint as in [29], as well as the direct constraints on the pure decision variable premium, by both minimum charge and premium budget, throughout this paper. Finally, the admissible sets in [29] are assumed to be non-empty; but, we carefully study below the feasibility issue due to the minimum charge and premium budget constraints in Section 3.1, Proposition 4.1, and Proposition 4.3.

This paper is organized as follows. Section 2 reviews our general Pareto optimal insurance contract design problem together. Section 3 investigates the feasibility and characterization of the

Pareto optimality with Pareto optimal premium payments, and Section 4 solves the cases in which the risk preferences are either VaR or TVaR. Section 5 relies on numerical optimization to study the effect of the constraints and bargaining powers on the optimal contracts for TVaR and PHT risk preferences. The concluding remarks of the paper appear in Section 6, while Appendix A lists the complete VaR Pareto optimal insurance contracts. Finally, all proofs are relegated into Appendix B.

2 Problem Formulation

Consider the current time $t = 0$. A risk holder has a random loss X, which is payable at a fixed time $T > 0$ in the future and is defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The loss X satisfies the properties that $X \geq 0$, $\mathbb{P}$-a.s., and $0 < \mathbb{E}[X] < \infty$. The risk holder, or insurance buyer, aims to share this loss X at time T with another party, or insurance seller. At time T, the seller indemnifies the buyer a portion $I(X)$ of the loss X, in which I is known as an indemnity function; in return, at time 0, the buyer pays to the seller an up front premium P.

Any admissible indemnity function I lies in the set of comonotonic risk-transfers:

$$\mathcal{I} := \{I : [0, \operatorname{ess\,sup} X] \to [0, \operatorname{ess\,sup} X] : 0 \leq I \leq \mathrm{Id},\ I \text{ and } R \text{ are both non-decreasing}\},$$

where Id is denoted as the identity function and $R := \mathrm{Id} - I$ is known as the retention function corresponding to an indemnity function $I \in \mathcal{I}$. The first condition in $\mathcal{I}$ is motivated by the fact that any indemnity loss $I(X)$ paid by the seller to the buyer at time T is at least non-negative and is at most the loss X. The second condition removes *ex post* moral hazard from both the buyer and the seller, as suggested by [25].

Any admissible up front premium P lies in the interval $\mathcal{P} := \left[\underline{P}, \overline{P}\right]$. The condition that $P \leq \overline{P}$ depicts the premium budget constraint of the buyer. The condition that $\underline{P} \leq P$ represents the minimum charge for the premium by the seller. As we discussed in the introduction, such a minimum charge is justified for covering the seller's indirect costs due to the risk-transfer, such as administrative expenses, being paid by the buyer. Another justification is imposing the non-negative risk loading property for the premium P when $\underline{P} = \mathbb{E}[X]$. To rule out trivial cases, assume further that $0 < \underline{P} \leq \overline{P} < \infty$; indeed, if $\overline{P} < \underline{P}$, there is obviously no feasible risk-transfer between the buyer and the seller.

Let Ψ_B and Ψ_S be two risk measures for the buyer and the seller respectively to order their risk preferences at time 0. For each admissible pair of indemnity function $I \in \mathcal{I}$ and premium $P \in \mathcal{P}$, the time-T post-transfer risk positions of the buyer and the seller are respectively given by $X - I(X) + P$, which equals to $R(X) + P$, and $I(X) - P$. Therefore, the time-0 objective functions of the buyer and the seller are respectively given by their time-0 post-transfer risk positions: for any $I \in \mathcal{I}$ and $P \in \mathcal{P}$,

$$B(I, P) := \Psi_B(X - I(X) + P) \text{ and } S(I, P) := \Psi_S(I(X) - P).$$

Unless otherwise specified, the following assumption holds throughout this paper.

Assumption 2.1. *The risk measures Ψ_B and Ψ_S are translational invariant, monotonic, positively homogeneous, comonotonic additive, and normalized to* 0*, with $\Psi_B(X) < \infty$.*

It is well-known that distortion risk measures satisfy the conditions in Assumption 2.1. The distortion risk measures, which will be recalled in Section 5, include the VaR, TVaR, and PHT. These three risk measures will be recalled and discussed in detail in Sections 4 and 5.

To ensure the risk-transfer being feasible, both the buyer and the seller expect that it does not create any extra risk at time 0. In other words, the following individual rationality constraints have to hold: for any $I \in \mathcal{I} \backslash \{0\}$ and $P \in \mathcal{P}$,

$$B(I, P) \leq B(0,0) = \Psi_B(X) \text{ and } S(I, P) \leq S(0,0) = \Psi_S(0) = 0.$$

Together with translation invariance and comonotonic additivity, these can be rewritten as additional premium constraints:

$$\Psi_S(I(X)) \leq P \leq \Psi_B(I(X)). \tag{2.1}$$

Moreover, to rule out the status quo with no-insurance and no-premium strategy of the buyer and seller after any risk-transfer, i.e. $(I, P) = (0,0)$, $I = 0$ cannot be an admissible strategy; otherwise, the rationality constraints also enforce $P = 0$, and thus the no-insurance and no-premium strategy. Therefore, the joint admissible set $\mathcal{A}$ of indemnity functions and premiums contains any $I \in \mathcal{I}$ and $P \in \mathcal{P}$ such that (2.1) holds:

$$\mathcal{A} := \{(I, P) \in \mathcal{I} \backslash \{0\} \times \mathcal{P} : \Psi_S(I(X)) \leq P \leq \Psi_B(I(X))\}.$$

At time 0, both the buyer and the seller agree on achieving an optimality, in terms of their time-0 objective functions. They compromise the indemnity function and premium in the admissible set $\mathcal{A}$, such that they cannot find another admissible contract for the indemnity function and premium that reduces the time-0 post-transfer risk position of either one of them, without increasing the risk position of the counterparty. Such an optimality concept is coined *Pareto optimal*.

Definition 2.1. *An indemnity function and premium pair $(I^*, P^*) \in \mathcal{I} \times \mathcal{P}$ is called Pareto optimal in $\mathcal{A}$, if $(I^*, P^*) \in \mathcal{A}$, and there is no admissible pair $(I, P) \in \mathcal{A}$ such that $B(I, P) \leq B(I^*, P^*)$ and $S(I, P) \leq S(I^*, P^*)$, with at least one of the inequalities being strict.*

The aim of this paper is to solve Pareto optimal indemnity function and premium payment pairs $(I^*, P^*) \in \mathcal{A}$. On one hand, by definition, the non-emptiness of the admissible set $\mathcal{A}$ is crucial; on the other hand, the definition is not yet mathematically convenient for being studied. Therefore, the next section delves into discussing the feasibility and deriving a useful characterization of Pareto optimality.

3 Feasibility and Characterization

3.1 Feasibility

Due to the existence of minimum charge and budget constraints for premium, there could be no feasible risk-transfer between the buyer and the seller. This subsection devotes to discussing the feasibility of the Pareto optimality problem formulated in Section 2. Such a feasibility analysis relates to whether the buyer and the seller would keep the status quo with no-insurance and no-premium strategy, when $\mathcal{A}$ is an empty set. We first state the following equivalence, where the proof follows by simple arguments, and hence is omitted.

Proposition 3.1. *Define a subset of admissible indemnity functions*

$$\mathcal{I}_0 := \left\{ I \in \mathcal{I} \backslash \{0\} : \Psi_S \left(I \left(X \right) \right) \leq \Psi_B \left(I \left(X \right) \right) \right\}.$$

Then $\mathcal{A}$ is an empty set if and only if, either one of the following conditions holds:

(i) $\mathcal{I}_0$ is an empty set;

(ii) $\mathcal{I}_0$ is a non-empty set but, for any $I \in \mathcal{I}_0$, either $\overline{P} < \Psi_S \left(I \left(X \right) \right) \leq \Psi_B \left(I \left(X \right) \right)$, or $\Psi_S \left(I \left(X \right) \right) \leq \Psi_B \left(I \left(X \right) \right) < \underline{P}$.

Indeed, if $\mathcal{I}_0$ is an empty set, then all non-status quo strategy do not satisfy the rationality constraints. Even when there exists an admissible risk-transfer which satisfies the rationality constraints, if the budget of the buyer is too low, or if the minimum charge to the seller is too high, then all non-status quo strategy are not feasible. However, this equivalence result demands to first identify the subset $\mathcal{I}_0$ of admissible non-trivial indemnity functions, which satisfy the rationality constraints; this depends on the risk measures Ψ_B and Ψ_S, as well as the distribution of ground up loss X.

Intuitively, if the minimum charge for premium is greater than the buyer's time-0 risk position of retaining the ground up loss X, with $I \equiv 0$, the rational buyer would rather retain the ground up loss X herself at time T, than pay a tremendous amount of premium to the seller at time 0 with being indemnified at time T. That is indeed a sufficient condition for the non-existence of feasible risk-transfer between the buyer and the seller. Unlike Proposition 3.1, the following proposition is valid regardless of the choices of risk measures Ψ_B and Ψ_S, as well as the distribution of ground up loss X. The proof follows directly from Proposition 3.1.

Proposition 3.2. *If $\Psi_B \left(X \right) < \underline{P}$, then $\mathcal{A}$ is an empty set.*

Hence, unless otherwise specified, assume that $\underline{P} \leq \Psi_B \left(X \right)$ throughout this paper. Notice, however, that such a condition is not sufficient for the existence of feasible risk-transfer between the buyer and the seller. Indeed, even if $\underline{P} \leq \Psi_B \left(X \right)$ holds, but when, for any $I \in \mathcal{I} \backslash \{0\}$, $\Psi_B \left(I \left(X \right) \right) < \Psi_S \left(I \left(X \right) \right)$, by Proposition 3.1, the buyer and seller will keep the status quo with no-insurance and no-premium strategy, i.e. $\mathcal{A}$ is an empty set. This suggests that the feasibility indeed depends on the time-0 objectives of the buyer and seller. Thus, for the moment, assume that the joint admissible set $\mathcal{A}$ is non-empty. In Section 4, the existence of feasible risk-transfer between the buyer and the seller will be discussed in details when both objectives of the buyer and the seller are modeled by the VaR and the TVaR.

3.2 Characterization of Pareto Optimality

This subsection devotes to deriving a handy characterization of being Pareto optimal for an indemnity function and premium pair $(I, P) \in \mathcal{A}$. To this end, consider the following weighted aggregate risk minimization problem for the buyer and the seller: for any $\lambda \in (0, 1)$,

$$\min_{(I,P)\in\mathcal{A}} \lambda B\left(I, P\right) + (1-\lambda)\, S\left(I, P\right). \tag{3.1}$$

Furthermore, define $\mathcal{S} := \cup_{\lambda\in(0,1)}\mathcal{S}_\lambda$, where $\mathcal{S}_\lambda := \operatorname{argmin}_{(I,P)\in\mathcal{A}} \lambda B\left(I, P\right) + (1-\lambda)\, S\left(I, P\right)$ is the set of all minimizer(s) for Problem (3.1), for each $\lambda \in (0, 1)$.

Theorem 3.1. *$(I^*, P^*) \in \mathcal{A}$ is Pareto optimal if and only if $(I^*, P^*) \in \mathcal{S}$.*

By Theorem 3.1, together with the translation invariance and comonotonic additivity of Ψ_B and Ψ_S, *all* Pareto optimal indemnity function and premium pairs in $\mathcal{A}$ are given by the optimizers of the following family of weighted aggregate risk minimization problem for the buyer and seller: for any $\lambda \in (0, 1)$,

$$\min_{(I,P)\in\mathcal{A}} \lambda\Psi_B\left(X\right) - \lambda\Psi_B\left(I\left(X\right)\right) + (1-\lambda)\,\Psi_S\left(I\left(X\right)\right) + (2\lambda-1)\,P, \tag{3.2}$$

where $\lambda \in (0, 1)$ represents the bargaining power of the buyer relative to the seller, while $(1 - \lambda) \in (0, 1)$ represents the bargaining power of the seller relative to the buyer.

We compare our problem formulation and Pareto optimality characterization in Theorem 3.1 with those in the literature. In particular, [3] showed that the weight $\lambda \in (0, 1)$ in the family of aggregate risk minimization problems (3.2) must be given by 0.5, as long as the risk measures Ψ_B and Ψ_S are translational invariant and monotonic. However, due to the potential existence of premium budget and minimum charge constraints, the argument in [3] of modifying the premium does not hold in our problem setting. On the other hand, our Pareto optimality characterization in Theorem 3.1 is in line with [10, 26, 28]. Yet, the up front premium P in our problem setting does not necessarily satisfy a certain premium principle as in [10, 26, 28]; instead, as in [3], the up front premium P in our problem setting is purely a decision variable of both the buyer and the seller, which is disentangled from the indemnity function via the premium principle, though they are still tangled by the rationality constraints in the joint admissible set $\mathcal{A}$. Moreover, [10, 26, 28] do not impose any premium budget or minimum charge constraint on the up front premium P. Finally, as we discussed in the introduction, our problem formulation is substantially different from that in [29], that (i) our risk measures Ψ_B and Ψ_S are allowed to include VaR as an example, (ii) our up front premium P is a pure decision variable disentangled from the indemnity function via the premium principle, and most importantly, (iii) our version of minimum charge constraint acts directly on the pure decision variable premium.

3.3 Characterization of Pareto Optimal Premium Payment

Although the family of aggregate risk minimization problems (3.2), to characterize the Pareto optimality problem formulated in Section 2, is fully non-linear and infinite dimensional in terms of indemnity function $I \in \mathcal{I}$, it is in fact linear and finite dimensional with respect to premium $P \in \mathcal{P}$, such that $(I, P) \in \mathcal{A}$. The proof follows by simple arguments, and hence is omitted.

Proposition 3.3. *All Pareto optimal indemnity function and premium pairs* $(I^*, P^*) \in \mathcal{A}$ *satisfy:*

(i) if $\lambda \in (0, 0.5)$*,* $P^* = \min\left\{\Psi_B\left(I^*\left(X\right)\right), \overline{P}\right\}$*;*

(ii) if $\lambda = 0.5$*,* P^* *is arbitrarily taken in* $\left[\max\left\{\Psi_S\left(I^*\left(X\right)\right), \underline{P}\right\}, \min\left\{\Psi_B\left(I^*\left(X\right)\right), \overline{P}\right\}\right]$*;*

(iii) if $\lambda \in (0.5, 1)$*,* $P^* = \max\left\{\Psi_S\left(I^*\left(X\right)\right), \underline{P}\right\}$*.*

This result entails that the Pareto optimal up front premium depends on the relative bargaining powers of the buyer and the seller. Indeed, when the bargaining power of the seller is higher, i.e. $\lambda \in (0, 0.5)$, the Pareto optimal premium P^*, is charged at the highest rate, which leans towards the objective of the seller, and is capped at the premium budget $\overline{P}$ of the buyer; when the bargaining power of the buyer is higher, i.e. $\lambda \in (0.5, 1)$, the Pareto optimal premium P^*, is charged at the lowest rate, which leans towards the objective of the buyer, and is floored at the minimum charge $\underline{P}$ of the seller. However, when the buyer and seller have equal bargaining powers, i.e. $\lambda = 0.5$, the Pareto optimal premium P^* is arbitrary as long as it is feasible; in this case, the buyer and seller have to impose additional criteria to seek the best premium arrangement among those non-unique Pareto optimal premiums P^* (see, for example, [3]). Moreover, due to the feasibility, all Pareto optimal indemnity functions I^* must satisfy that $\max\left\{\Psi_S\left(I^*\left(X\right)\right), \underline{P}\right\} \leq \min\left\{\Psi_B\left(I^*\left(X\right)\right), \overline{P}\right\}$, and hence, the Pareto optimal premium P^* reduces, when the bargaining power of the buyer is higher than that of the seller, comparing to the case, when the bargaining power of the buyer is lower than that of the seller. Finally, in the case of $\lambda \in (0, 0.5)$ or $\lambda \in (0.5, 1)$, although the Pareto optimal premium P^* depends on, either the minimum charge $\underline{P}$, or the premium budget $\overline{P}$, but not both, the Pareto optimality problem still depends on both constraints; indeed, by the feasibility, all Pareto optimal indemnity functions I^* have to satisfy, not only the constraint $\Psi_S\left(I^*\left(X\right)\right) \leq \Psi_B\left(I^*\left(X\right)\right)$ due to rationality, but also the constraints $\underline{P} \leq \Psi_B\left(I^*\left(X\right)\right)$ and $\Psi_S\left(I^*\left(X\right)\right) \leq \overline{P}$ arising from the minimum charge and premium budget. These two additional constraints pose technical difficulties to solve the family of aggregate risk minimization problems (3.2) with general distortion risk measures Ψ_B and Ψ_S.

4 Analytical Solutions

In this section, Pareto optimal indemnity function and premium pairs $(I^*, P^*) \in \mathcal{A}$ are analytically derived via solving the family of minimization problems (3.2), in which the time-0 risk preferences Ψ_B and Ψ_S of the buyer and the seller are both characterized by the VaR or the TVaR. Let $\alpha \in [0,1]$ and $\beta \in [0,1]$ be the respective confidence levels of the seller and the buyer. For notational simplicity, denote $a := \mathrm{VaR}_\alpha(X)$ and $b := \mathrm{VaR}_\beta(X)$.

4.1 Value-at-Risk

We first recall the definition of VaR as follows. For any confidence level $\gamma \in [0,1]$ and random variable Y defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$,

$$\mathrm{VaR}_\gamma(Y) := \inf\left\{y \in [\operatorname{ess\,inf} Y, \operatorname{ess\,sup} Y] : F_Y(y) \geq \gamma\right\},$$

which is the left-continuous generalized inverse of the distribution function F_Y. Note that the VaR satisfies all properties listed in Assumption 2.1. Also, recall the following important property of VaR that, for any non-decreasing and left-continuous function f with appropriate domain and range, $\mathrm{VaR}_\gamma(f(Y)) = f(\mathrm{VaR}_\gamma(Y))$.

In this subsection, we assume that the time-0 risk preferences Ψ_B and Ψ_S of the buyer and the seller are both characterized by the VaR:

$$\Psi_B(\cdot) = \mathrm{VaR}_\beta(\cdot) \text{ and } \Psi_S(\cdot) = \mathrm{VaR}_\alpha(\cdot).$$

Since any arbitrary $I \in \mathcal{I}$ is non-decreasing and continuous, the joint admissible set $\mathcal{A}$ is given by

$$\mathcal{A} = \left\{(I, P) \in \mathcal{I}\backslash\{0\} \times \mathcal{P} : I(a) \leq P \leq I(b)\right\};$$

moreover, the family of minimization problems (3.2) is given by: for any $\lambda \in (0,1)$,

$$\min_{(I,P)\in\mathcal{A}} \lambda b - \lambda I(b) + (1-\lambda) I(a) + (2\lambda - 1) P. \tag{4.1}$$

Prior to explicitly solving the family of minimization problems (4.1), we first discuss the existence of feasible risk-transfer between the buyer and the seller. Recall that, by Proposition 3.2, if $b < \underline{P}$, then $\mathcal{A}$ must be empty. It turns out that, when Ψ_B and Ψ_S are both given by the VaR, the condition is also necessary.

Proposition 4.1. *Suppose that $\Psi_B(\cdot) = VaR_\beta(\cdot)$ and $\Psi_S(\cdot) = VaR_\alpha(\cdot)$, for some $\alpha, \beta \in [0,1]$. Then, $\underline{P} \leq b$ if and only if $\mathcal{A}$ is a non-empty set.*

Due to Proposition 4.1, together with the assumption that $\underline{P} \leq \Psi_B(X) (= b)$ throughout this paper, the joint admissible set $\mathcal{A}$ is non-empty when both Ψ_B and Ψ_S are characterized by the VaR.

The following proposition explicitly solves the family of minimization problems (4.1), to characterize the Pareto optimal indemnity function and premium pairs $(I^*, P^*) \in \mathcal{A}$, when the VaR of the buyer is no larger than that of the seller for the ground up loss X. The proof follows by simple arguments, and hence is omitted.

Proposition 4.2. *Suppose that* $\Psi_B(\cdot) = VaR_\beta(\cdot)$ *and* $\Psi_S(\cdot) = VaR_\alpha(\cdot)$*, for some* $\alpha, \beta \in [0,1]$*, and* $b \leq a$*. Then, the Pareto optimal indemnity function and premium pairs* $(I^*, P^*) \in \mathcal{A}$ *are given by any* $I^* \in \mathcal{I}\backslash\{0\}$ *with* $I^*(a) = P^* = I^*(b) \in \left[\underline{P}, \min\left\{\overline{P}, b\right\}\right]$*.*

Proposition 4.2 implies three interesting results. Firstly, as long as the buyer and the seller share risk in a rational way, i.e., satisfying their individual rationality constraints, all feasible contracts are Pareto optimal, in which all Pareto optimal indemnity functions have gradient zero on the interval $[b, a]$; moreover, the buyer's and the seller's time-0 risk positions stay the same before and after the risk-transfer:

$$B(I^*, P^*) = B(0,0) = b \text{ and } S(I^*, P^*) = S(0,0) = 0.$$

Secondly, all Pareto optimal contract pairs $(I^*, P^*) \in \mathcal{A}$ are independent of the relative bargaining powers of the buyer and the seller. Finally, the flexibility of negotiating different Pareto optimal indemnity functions and premiums between the buyer and the seller depends on the sizes of the premium budget and the minimum charge for the premium; indeed, on one hand, when the difference $\left(\overline{P} - \underline{P}\right)$ between the premium budget and the minimum charge for the premium decreases, the size of the interval $\left[\underline{P}, \min\left\{\overline{P}, b\right\}\right]$ for $I^*(a) = P^* = I^*(b)$ reduces; on the other hand, when both the premium budget and the minimum charge for the premium vanish, i.e. $\underline{P} \to 0$ and $\overline{P} \to \infty$, all Pareto optimal indemnity function and premium pairs satisfy $I^*(a) = P^* = I^*(b) \in [0, b]$.

In practice, the seller usually has a lower confidence level than that of the buyer. The following theorem explicitly solves the family of minimization problems (4.1), to characterize the Pareto optimal indemnity function and premium pairs $(I^*, P^*) \in \mathcal{A}$, when the VaR of the seller is strictly less than that of the buyer for the ground up loss X.

Theorem 4.1. *Suppose that* $\Psi_B(\cdot) = VaR_\beta(\cdot)$ *and* $\Psi_S(\cdot) = VaR_\alpha(\cdot)$*, for some* $\alpha, \beta \in [0,1]$*, and* $a < b$*. Then, the Pareto optimal indemnity function and premium pairs* $(I^*, P^*) \in \mathcal{A}$ *are summarized in Table A.1 of Appendix A; in particular, all Pareto optimal indemnity functions* $I^* \in \mathcal{I}\backslash\{0\}$ *satisfy* $I^*(b) - I^*(a) = b - a$*, and thus Table A.1 of Appendix A only shows either the values of* $I^*(a)$ *or* $I^*(b)$*.*

Theorem 4.1 has several interesting implications. Firstly, all Pareto optimal indemnity functions have gradient one on the interval $[a, b]$. This conclusion can also be derived by modification arguments on the objective functions in the family of minimization problems (4.1); see, for example, [12, 17] and the discussion in the TVaR case below. Although the gradient is independent of the premium budget and the minimum charge, it should be noted that the optimal indemnity values $I^*(a)$ and $I^*(b)$, as well as the optimal up front premium P^*, depend on both $\underline{P}$ and $\overline{P}$; see Table A.1 in Appendix A.1 for details.

Secondly, unlike Proposition 4.2, the Pareto optimal contract pairs $(I^*, P^*) \in \mathcal{A}$, in Theorem 4.1, obviously depend on the relative bargaining powers of the buyer and the seller. However, regardless of the dependence, their bargaining powers are not a priori chosen; instead, the buyer and the seller first compromise on a Pareto optimal contract pair, and then the value of λ is implied by their agreement. Moreover, regardless of the potential existences of the premium budget and the minimum charge for the premium, if the implied $\lambda < 0.5$, then the seller shows more bargaining power during the negotiation, with more premium, which echoes Proposition 3.3, but less indemnity coverage agreed; if the implied $\lambda > 0.5$, then the buyer shows more bargaining power during the negotiation, with less premium, which echoes Proposition 3.3, but more indemnity coverage agreed.

Finally, those sophisticated (sub-)case conditions in Theorem 4.1 are due to the potential existences of the premium budget and the minimum charge for the premium; indeed, when both the premium budget and the minimum charge for the premium vanish, i.e. $\underline{P} \to 0$ and $\overline{P} \to \infty$, the Pareto optimal contract pairs are completely characterized by the case and the sub-case, when $\underline{P} \leq a < b \leq \overline{P}$ and $\underline{P} < b - a$, in Table A.1, with $\underline{P} \to 0$ and $\overline{P} \to \infty$. Moreover, regardless of the relative bargaining powers of the buyer and the seller, when the budget $\overline{P}$ of the buyer gradually decreases, the buyer and the seller agree that the (maximum) acceptable premium and the indemnity coverage decrease; when the minimum charge $\underline{P}$ of the seller gradually increases, the buyer and the seller agree that the (minimum) acceptable premium and the indemnity coverage increase.

Before proceeding to the next subsection which models the objectives of the buyer and the seller by the TVaR, we make a final remark that both Proposition 4.2 and Theorem 4.1 solve *all* possible Pareto optimal contract pairs $(I^*, P^*) \in \mathcal{A}$. Indeed, both results are obtained by directly solving the family of minimization problems (4.1), which exhaust all minimizers in $\mathcal{S}$.

4.2 Tail Value-at-Risk

We first recall the definition of TVaR as follows. For any confidence level $\gamma \in [0, 1]$ and random variable Y defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$,

$$\mathrm{TVaR}_\gamma(Y) := \begin{cases} \frac{1}{1-\gamma}\int_\gamma^1 \mathrm{VaR}_\eta(Y)\, d\eta & \text{if } \gamma \in [0,1) \\ \operatorname{ess\,sup} Y & \text{if } \gamma = 1 \end{cases}.$$

Note that the TVaR satisfies all properties listed in Assumption 2.1. Also, recall the following dual representation of TVaR that

$$\mathrm{TVaR}_\gamma(Y) = \inf_{y \in [\operatorname{ess\,inf} Y, \operatorname{ess\,sup} Y]} \left(y + \frac{1}{1-\gamma}\mathbb{E}\left[(Y - y)_+\right] \right),$$

where the minimizer is given by the VaR: $y^* = \mathrm{VaR}_\gamma(Y)$.

In this subsection, we assume that the time-0 risk preferences Ψ_B and Ψ_S of the buyer and the seller are both characterized by the TVaR:

$$\Psi_B(\cdot) = \mathrm{TVaR}_\beta(\cdot) \text{ and } \Psi_S(\cdot) = \mathrm{TVaR}_\alpha(\cdot),$$

where $\alpha \in [0,1]$ and $\beta \in [0,1]$ are recalled as the respective confidence levels of the seller and the buyer. In turn, the joint admissible set $\mathcal{A}$ is given by

$$\mathcal{A} = \{(I,P) \in \mathcal{I} \setminus \{0\} \times \mathcal{P} : \mathrm{TVaR}_\alpha(I(X)) \leq P \leq \mathrm{TVaR}_\beta(I(X))\},$$

while the family of minimization problems (3.2) is given by: for any $\lambda \in (0,1)$,

$$\min_{(I,P)\in\mathcal{A}} \lambda \mathrm{TVaR}_\beta(X) - \lambda \mathrm{TVaR}_\beta(I(X)) + (1-\lambda)\mathrm{TVaR}_\alpha(I(X)) + (2\lambda - 1)P. \tag{4.2}$$

Again, prior to explicitly solving the family of minimization problems (4.2), we first discuss the existence of feasible risk-transfer between the buyer and the seller. Unlike Proposition 4.1 in the case of the VaR, the condition, $\underline{P} \leq \Psi_B(X)\,(=\mathrm{TVaR}_\beta(X))$, alone is not sufficient for the non-emptiness of the joint admissible set $\mathcal{A}$. It turns out, when Ψ_B and Ψ_S are both given by the TVaR, together with the condition that the seller has a lower confidence level than that of the buyer, which is common in practice, the condition $\underline{P} \leq \mathrm{TVaR}_\beta(X)$ is then sufficient.

Proposition 4.3. *Suppose that $\Psi_B(\cdot) = TVaR_\beta(\cdot)$ and $\Psi_S(\cdot) = TVaR_\alpha(\cdot)$, for some $\alpha, \beta \in [0,1]$ with $\alpha \leq \beta$. Then, $\underline{P} \leq TVaR_\beta(X)$ if and only if $\mathcal{A}$ is a non-empty set.*

Due to Proposition 4.3, together with the assumption that $\underline{P} \leq \Psi_B(X)\,(=\mathrm{TVaR}_\beta(X))$ throughout this paper, assume further that $\alpha \leq \beta$ in the remains of this subsection, so that the joint admissible set $\mathcal{A}$ is non-empty when both Ψ_B and Ψ_S are characterized by the TVaR.

Unlike Theorem 4.1 in the case of the VaR, the family of minimization problems (4.2), to characterize the Pareto optimal indemnity function and premium pairs $(I^*, P^*) \in \mathcal{A}$ when both Ψ_B and Ψ_S are characterized by the TVaR, cannot be solved explicitly by the linear programming method as in the proof of Theorem 4.1. Instead, in this subsection, a modification argument (see, for example, [12], [17], [15], and [9]) is applied to identify a sub-class of Pareto optimal solutions, which has the *least* finite number of parameters to be determined, via exhausting some minimizers in $\mathcal{S}$. Due to the existence of a feasible risk-transfer between the buyer and the seller, the modification argument is valid.

To this end, denote, for any $\lambda \in (0,1)$, the objective function in the minimization problem (4.2) as F_λ. Define a subset of the admissible indemnity functions

$$\tilde{\mathcal{I}} := \{I \in \mathcal{I} : \text{there exist } d_1 \in [0, \operatorname{ess\,sup} X] \text{ and } d_2 \in [d_1, \operatorname{ess\,sup} X] \text{ such that } I(x) = (x-d_1)_+ - (x-d_2)_+\}.$$

Theorem 4.2. *For any $\lambda \in (0,1)$ and $(I,P) \in \mathcal{A}$, there exists an $\tilde{I} \in \tilde{\mathcal{I}}$ such that $(\tilde{I}, P) \in \mathcal{A}$ and $F_\lambda(\tilde{I}, P) \leq F_\lambda(I,P)$.*

Theorem 4.2 states that any admissible indemnity function $I \in \mathcal{I}$ is inferior to a single-layer indemnity function $\tilde{I} \in \tilde{\mathcal{I}}$, with the same premium P. The family of infinite dimensional minimization problems (4.2) can be reduced to a family of finite dimensional minimization problems, to obtain Pareto optimal indemnity function and premium pairs with the least finite number of parameters: for any $\lambda \in (0,1)$,

$$\min_{(I,P)\in\tilde{\mathcal{A}}} \lambda \mathrm{TVaR}_{\beta}\left(X\right) - \lambda \mathrm{TVaR}_{\beta}\left(I\left(X\right)\right) + \left(1-\lambda\right)\mathrm{TVaR}_{\alpha}\left(I\left(X\right)\right) + \left(2\lambda - 1\right)P,$$

where $\tilde{\mathcal{A}} = \{(I,P) \in \mathcal{A} : I \in \tilde{\mathcal{I}}\}$. In other words, there exists at least one Pareto optimal indemnity function and premium pair $(I^*, P^*) \in \mathcal{A}$, when both Ψ_B and Ψ_S are characterized by the TVaR, such that the optimal indemnity function $I^* \in \mathcal{I}$ is given by a single-layer.

5 Numerical Solutions

We first recall the definition of distortion risk measures as follows. For any non-decreasing distortion function $g : [0,1] \to [0,1]$ such that $g\left(0\right) = 0$ and $g\left(1\right) = 1$, a distortion risk measure ρ_g is defined by, for any non-negative random variable Y defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$,

$$\rho_g\left(Y\right) := \int_0^{\infty} g\left(1 - F_Y\left(y\right)\right) dy.$$

Note that distortion risk measures satisfy all properties listed in Assumption 2.1. The VaR, TVaR, and PHT are distortion risk measures, with their respective distortion functions, $\mathbb{1}_{[1-\gamma,1]}\left(x\right)$, $\min\left\{\frac{x}{1-\gamma}, 1\right\}$, and x^c, for some $\gamma \in [0,1]$ and $c \in (0,1]$.

In Section 4, all Pareto optimal indemnity function and premium pairs $(I^*, P^*) \in \mathcal{A}$ are explicitly solved, when both risk preferences Ψ_B and Ψ_S are characterized by the VaR; yet, although the existence of Pareto optimal single layer indemnity function is established, no Pareto optimal contract pair $(I^*, P^*) \in \mathcal{A}$ is explicitly solved, when both risk preferences Ψ_B and Ψ_S are characterized by the TVaR. This section devotes to adopting a well-established numerical approach to solve Pareto optimal contract pairs $(I^*, P^*) \in \mathcal{A}$; see, for example, [4] and [33]. In particular, this section illustrates the approach on the TVaR and the PHT; however, we emphasize that this numerical approach can be applied, in an equal manner, to the case when both risk preferences are characterized by the TVaR.

To this end, the loss X is represented by its discretized samples $\mathbf{x} := (x_1, x_2, \ldots, x_n)^{\mathrm{tr}}$, which could be either empirically observed or simulated from a pre-specified parametric model for the loss X. This section only illustrates the latter case, but the approach applies indifferently for the former case. Without loss of generality, assume that the samples are sorted in the ascending order, with $x_1 \leq x_2 \leq \cdots \leq x_n$. For any admissible indemnity function $I \in \mathcal{I}$, define $\mathbf{y} := (y_1, y_2, \ldots, y_n)^{\mathrm{tr}}$, in which, for any $i = 1, 2, \ldots, n$, $y_i = I\left(x_i\right)$, and thus, the admissibility conditions in $\mathcal{I}$ are translated into $\mathbf{0} \le \mathbf{A}\mathbf{y} \le \mathbf{A}\mathbf{x}$, where

$$\mathbf{A} := \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ -1 & 1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & -1 & 1 & 0 \\ 0 & \cdots & 0 & 0 & -1 & 1 \end{pmatrix}.$$

Following similar arguments as in [4], for any $I \in \mathcal{I}$, the risk preference of the buyer $\Psi_B(I(X)) = \phi_B^{\mathrm{tr}}\mathbf{y}$, where $\phi_B = (\phi_{B1}, \phi_{B2}, \dots, \phi_{Bn})^{\mathrm{tr}}$, with $\phi_{Bi} = g_B\left(\frac{n-i+1}{n}\right) - g_B\left(\frac{n-i}{n}\right)$, for any $i = 1, 2, \dots, n$, and g_B is the distortion function of the buyer. Similarly, for any $I \in \mathcal{I}$, the risk preference of the seller $\Psi_S(I(X)) = \phi_S^{\mathrm{tr}}\mathbf{y}$, with self-evident notations. Therefore, the family of weighted aggregate risk minimization problem (3.2) for the buyer and seller, which is fully non-linear and infinite dimensional, becomes a finite dimensional linear programming problem with constraints: for any $\lambda \in (0, 1)$,

$$\begin{aligned} \min_{(\mathbf{y},P)\in\mathbb{R}^n\times\mathbb{R}} \quad & \lambda\phi_B^{\mathrm{tr}}\mathbf{x} - \lambda\phi_B^{\mathrm{tr}}\mathbf{y} + (1-\lambda)\,\phi_S^{\mathrm{tr}}\mathbf{y} + (2\lambda - 1)\,P \\ \text{such that} \quad & \mathbf{0} \le \mathbf{A}\mathbf{y} \le \mathbf{A}\mathbf{x} \\ & \underline{P} \le P \le \overline{P} \\ & \phi_S^{\mathrm{tr}}\mathbf{y} \le P \le \phi_B^{\mathrm{tr}}\mathbf{y}. \end{aligned} \tag{5.1}$$

Putting this numerical solution approach into concrete settings, assume, in the sequel and unless otherwise specified, that $n = 300$, $\alpha = 0.75$, $\beta = 0.8$, $\underline{P} = 10\% \times \mathrm{TVaR}_{0.75}(X)$, and $\overline{P} = 30\% \times \mathrm{TVaR}_{0.8}(X)$, where the loss X follows a Pareto distribution with $F_X(x) = 1 - \left(\frac{10{,}000}{10{,}000+x}\right)^3$, for any $x \ge 0$.

Example 5.1. *Assume that $\Psi_B(\cdot) = TVaR_{0.8}(\cdot)$ and $\Psi_S(\cdot) = PHT(\cdot; c_S)$, where the seller's PHT parameter c_S is calibrated such that $PHT(X; c_S) = TVaR_{0.75}(X)$. In other words, the distortion functions g_B and g_S of the buyer and the seller in problem (5.1) are respectively given by $\min\left\{\frac{x}{1-0.8}, 1\right\}$ and x^{c_S}.*

Figure 5.1 illustrates the Pareto optimal premium P^* against various relative bargaining power of the buyer $\lambda \in (0, 1)$, in black solid line; it also represents the buyer's and seller's optimal risk preferences $\Psi_B(I^*(X))$, in blue inverted triangles, and $\Psi_S(I^*(X))$, in red upright triangles, with the premium minimum charge $\underline{P}$ and the premium budget $\overline{P}$ indicated on the right-vertical axis.

The left-hand panel of Figure 5.1 demonstrates the status-quo assumption that the premium budget $\overline{P} = 30\% \times TVaR_{0.8}(X)$. In this case, the Pareto optimal premium P^* is independent of the relative bargaining power of the buyer and the seller. This phenomenon can be explained by Proposition 3.3; the Pareto optimal premium P^* is indeed given by $\min\left\{\overline{P}, \Psi_B(I^*(X))\right\}$, which is a smaller value of $\overline{P}$ or blue inverted triangle, when $\lambda \in (0, 0.5)$, and $\max\{\underline{P}, \Psi_S(I^*(X))\}$, which is a larger value of $\underline{P}$ or red upright triangle, when $\lambda \in (0.5, 1)$. Most importantly,

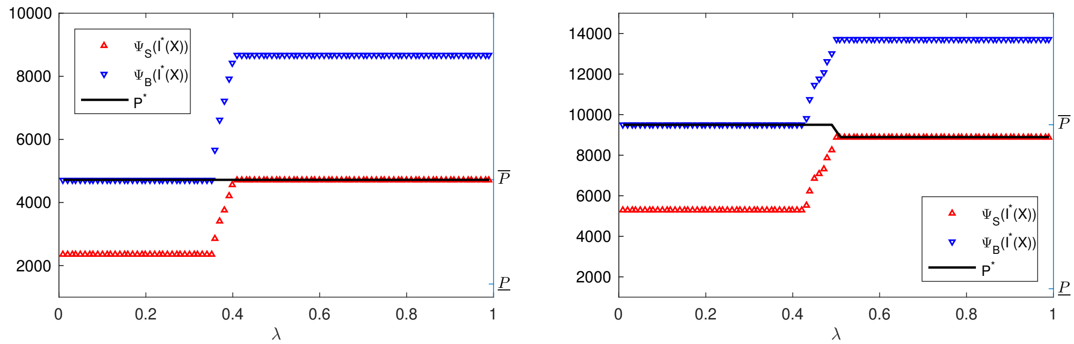

Figure 5.1: Pareto optimal premium P^* – left-hand panel with $\overline{P} = 30\% \times \mathrm{TVaR}_{0.8}(X)$; right-hand panel with $\overline{P} = 60\% \times \mathrm{TVaR}_{0.8}(X)$.

since $\max\left\{\underline{P}, \Psi_S\left(I^*(X)\right)\right\} \leq \min\left\{\overline{P}, \Psi_B\left(I^*(X)\right)\right\}$, *when the relative bargaining power of the buyer* $\lambda \in (0,1)$ *increases, or equivalently, when the relative bargaining power of the seller* $(1-\lambda) \in (0,1)$ *decreases, the Pareto optimal premium* P^* *reduces, and vice versa (which also echoes Proposition 3.3 and the theoretical findings in the case of VaR in Section 4). To better illustrate this numerically, the right-hand panel of Figure 5.1 depicts the case when the premium budget* $\overline{P} = 60\% \times TVaR_{0.8}(X)$, *in which the Pareto optimal premium* P^* *can be accounted by similar arguments. Observe that, by comparing the left-hand and right-hand panels of Figure 5.1, when the premium budget* $\overline{P}$ *is increased, not only the Pareto optimal premium* P^*, *but also the buyer's and seller's optimal risk preferences,* $\Psi_B\left(I^*(X)\right)$ *and* $\Psi_S\left(I^*(X)\right)$, *are varied.*

The left-hand panel of Figure 5.2 illustrates the Pareto optimal indemnity function I^*, *or equivalently* $\boldsymbol{y}^*$, *in blue circles, when the relative bargaining powers of the buyer and the seller are equal. The right-hand panel of Figure 5.2 depicts the heat map, which illustrates the gradient of the Pareto optimal indemnity function* I^*, *or equivalently the sample-wise increment* $\boldsymbol{y}^*$ *over the sample-wise increment* $\boldsymbol{x}$, *against various relative bargaining power of the buyer* $\lambda \in (0,1)$, *in which the x-axis contains indexes of the samples* $\boldsymbol{x}$.

The top panel of Figure 5.2 demonstrates the status-quo assumption that the premium budget $\overline{P} = 30\% \times TVaR_{0.8}(X)$. *Observe that, for each relative bargaining power of the buyer* $\lambda \in (0,1)$, *the gradient of the Pareto optimal indemnity function* I^*, *first changes from* 0 *to* 1, *and then changes back to* 0, *which deduces the Pareto optimal single layer indemnity function* I^*. *Moreover, when the relative bargaining power of the buyer* $\lambda \in (0,1)$ *increases, or equivalently, when the relative bargaining power of the seller* $(1-\lambda) \in (0,1)$ *decreases, the Pareto optimal single layer indemnity function* I^* *fully indemnifies more and more moderate losses, and vice versa (which again echoes the theoretical findings in the case of VaR in Section 4); yet, recall that, from the left-hand panel of Figure 5.1, the Pareto optimal premium* P^* *remains unchanged. The bottom panel of Figure 5.2 shows the case when the premium budget* $\overline{P} = 60\% \times TVaR_{0.8}(X)$. *Observe that, by comparing the top and bottom panels of Figure 5.2, when the premium budget* $\overline{P}$ *is increased,*

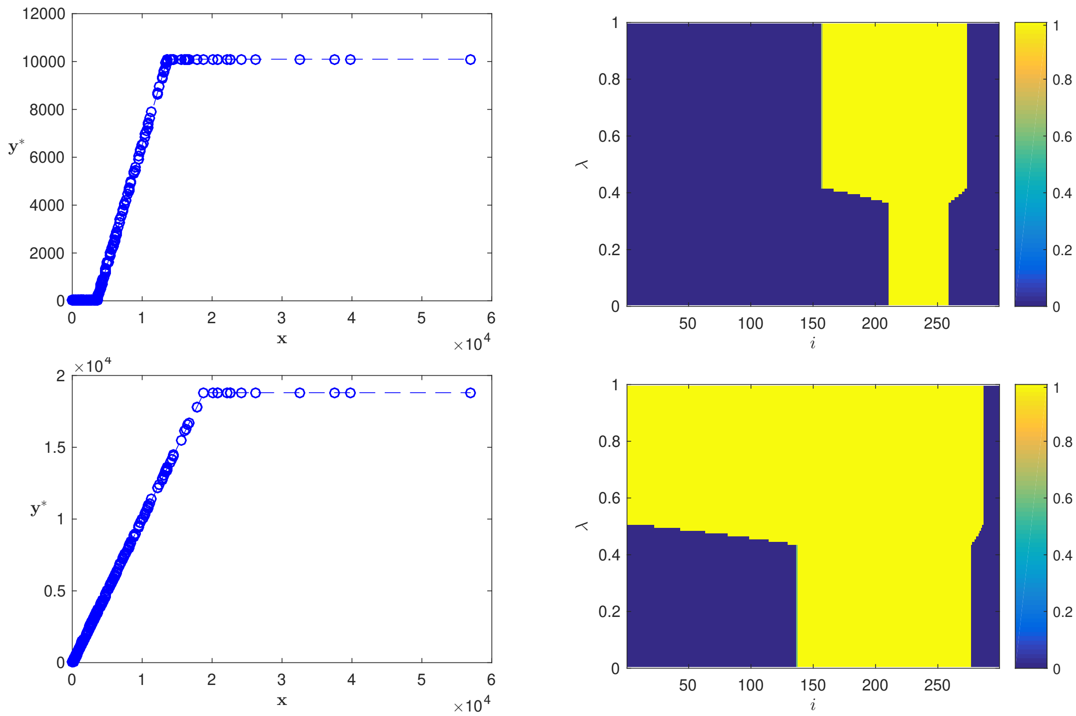

Figure 5.2: Pareto optimal indemnity function I^* – left-hand panel with $\lambda = 0.5$; top panel with $\overline{P} = 30\% \times \text{TVaR}_{0.8}(X)$; bottom panel with $\overline{P} = 60\% \times \text{TVaR}_{0.8}(X)$.

both small and moderate losses are fully indemnified by the Pareto optimal indemnity function I^*, when the buyer has a larger bargaining power than the seller; yet, in return, the buyer has to pay a higher Pareto optimal premium P^*, by comparing the left-hand and right-hand panels of Figure 5.1.

It is also interesting to explore the effects on the optimal solutions if we lift the minimum charge and budget constraints by setting $\underline{P} \to 0$ and $\overline{P} \to \infty$. Therefore, we solve the optimization model (5.1) once again with the constraints $\underline{P} \leq P \leq \overline{P}$ being replaced by $P \geq 0$, and the results are illustrated in Figures 5.3 and 5.4. It is worth reminding that although the constraints $\underline{P} \leq P \leq \overline{P}$ have been lifted, the optimal premium P^* is still bounded by the rationality constraints $\phi_S^{tr}\boldsymbol{y} \leq P \leq \phi_B^{tr}$. Therefore, as shown in Figure 5.3, the optimal premium P^* is charged at its upper boundary when the bargaining power of the insurance buyer, λ, is small, while it is charged at its lower boundary when λ becomes large, even though the constraints have been lifted.

We have seen in Figure 5.2 that when the minimum charge and budget constraints are in place, small losses are usually fully retained by the insurance buyer, especially when their bargaining power is relatively small. However, as shown in Figure 5.4, when the constraints are lifted, small losses are always fully indemnified, regardless of the relative bargaining power of the buyer.

Example 5.2. *In this example, assume, the other way around, that $\Psi_B(\cdot) = PHT(\cdot; c_B)$ and*

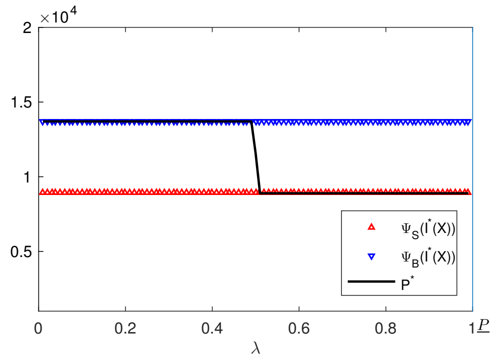

Figure 5.3: Pareto optimal premium P^* when minimum charge and budget constraints are lifted.

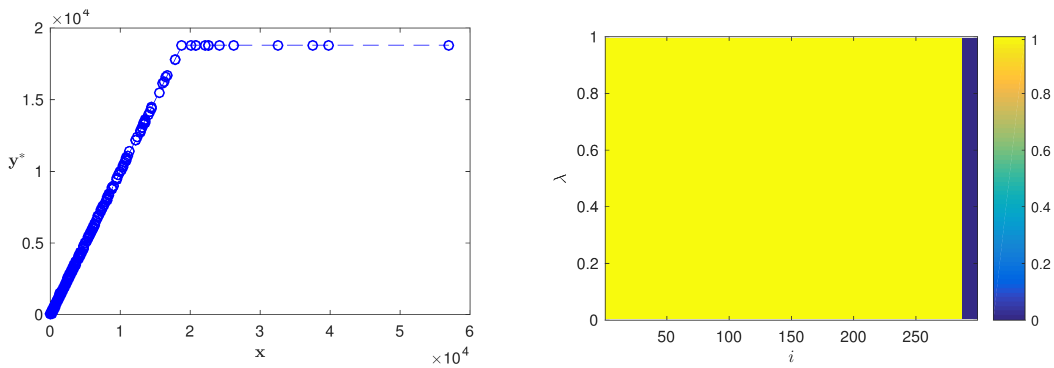

Figure 5.4: Pareto optimal indemnity function I^* when minimum charge and budget constraints are lifted – left-hand panel with $\lambda = 0.5$.

$\Psi_S(\cdot) = TVaR_{0.75}(\cdot)$, *where the buyer's PHT parameter* c_B *is calibrated such that* $PHT(X; c_B) = TVaR_{0.8}(X)$. *The left-hand panel in Figure 5.5 and the top panel in Figure 5.6 depict the Pareto optimal premium* P^* *and Pareto optimal indemnity function* I^* *respectively, which can be accounted in a similar manner as in Example 5.1. This example, however, highlights that even a minor twist in the model setting could result in a dramatic change for the Pareto optimal contract; in Example 5.1, moderate, or even small, but not large, losses are fully indemnified, while, in this example, large, but not small nor moderate losses, are fully covered.*

When the minimum charge and budget constraints are lifted, the optimal premium P^* *is charged at its upper boundary of the rationality constraints when the insurance buyer's relative bargaining power,* λ*, is small, while it is reduced to its lower boundary of the rationality constraints when the* λ *becomes large. This is illustrated by the right-hand panel in Figure 5.5. On the other hand, the optimal indemnity function* I^* *does not change its shape much when the constraints are lifted as illustrated by the bottom panel in Figure 5.6. For small* λ*, we do observe that the yellow area in the heat map has enlarged a bit when* $\underline{P} \to 0$ *and* $\overline{P} \to \infty$*. This indicates that, when the buyer is able to afford insurance at any price and the seller does not charge any indirect cost, a* *larger amount of losses will be indemnified even when the buyer has a relatively lower bargaining power.*

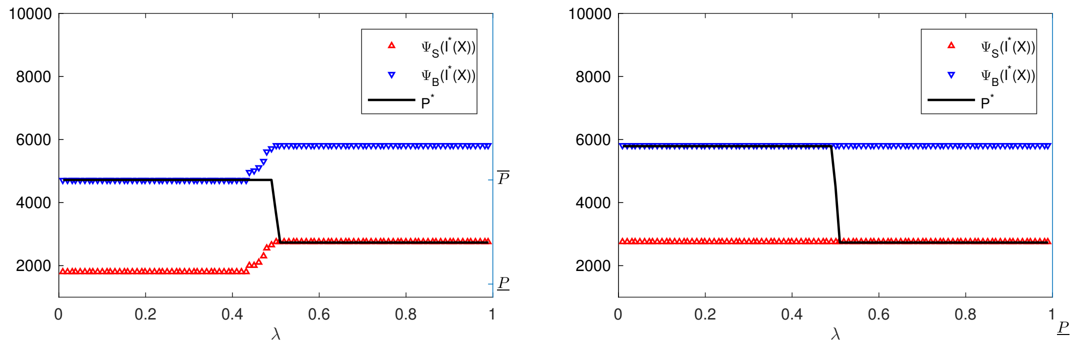

Figure 5.5: Pareto optimal premium P^* – left-hand panel with $\overline{P} = 30\% \times \text{TVaR}_{0.8}(X)$; right-hand panel with $\underline{P} \to 0$ and $\overline{P} \to \infty$.

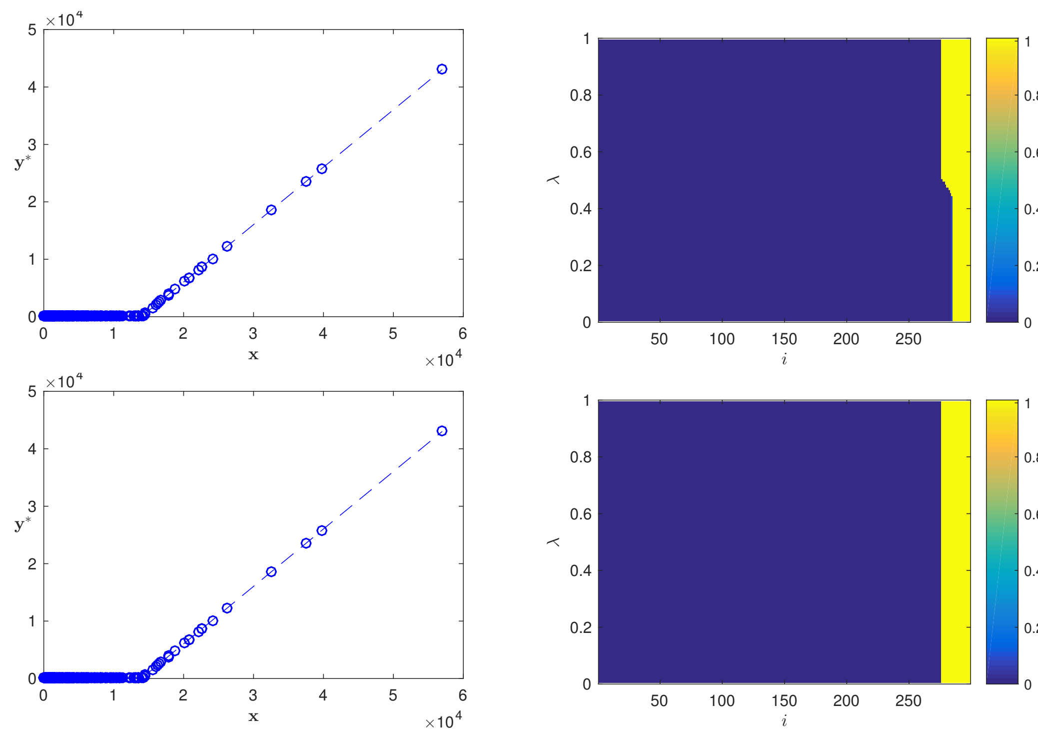

Figure 5.6: Pareto optimal indemnity function I^* – left-hand panel with $\lambda = 0.5$; top panel with $\overline{P} = 30\% \times \text{TVaR}_{0.8}(X)$; bottom panel with $\underline{P} \to 0$ and $\overline{P} \to \infty$.

6 Concluding Remarks and Future Direction

This paper revisited the Pareto optimal insurance contract design problem with both minimum charge and premium budget constraints. In addition to rationality constraints, technical difficulties arise from the two extra constraints on the indemnity schedule. This paper closely investigated the effects on the feasibility, Pareto optimal indemnity schedule, and Pareto optimal premium payment, by the premium constraints and bargaining powers of the buyer and seller. All Pareto optimal solutions were explicitly solved for the case of Value-at-Risk, while Pareto optimal single-layer indemnity schedule was identified for the case of Tail Value-at-Risk. A numerical solution approach was implemented to further study the case of Tail Value-at-Risk, as well as the Proportional Hazard Transformation; reasonable interpretations were obtained. As the first work incorporating both practical constraints in the Pareto optimal insurance contract design problem, we did not attempt to explicitly solve the Pareto optimal insurance contracts with general distortion risk measures. However, this is indeed an important topic to explore further, and shall be left as a future research direction.

7 Acknowledgment

The authors are grateful to anonymous reviewers for their careful reading and insightful comments.

Appendix A Table in Theorem 4.1

Case	Sub-case	$\lambda \in (0, 0.5)$	$\lambda = 0.5$	$\lambda \in (0.5, 1)$
$\underline{P} \leq a < b \leq \overline{P}$	$\underline{P} < b - a$	$P^* \in [b-a, b]$ $I^*(b) = P^*$	$P^* \in [\underline{P}, b]$ $I^*(b) \in [\max\{P^*, b-a\}, \min\{P^*, a\} + (b-a)]$	$P^* \in [\underline{P}, a]$ $I^*(a) = P^*$
	$b - a \leq \underline{P}$	$P^* \in [\underline{P}, b]$ $I^*(b) = P^*$	$P^* \in [\underline{P}, b]$ $I^*(b) \in [P^*, \min\{P^*, a\} + (b-a)]$	$P^* \in [\underline{P}, a]$ $I^*(a) = P^*$
$\underline{P} \leq a \leq \overline{P} < b$	$\underline{P} < b - a < \overline{P}$	$P^* \in [b-a, \overline{P}]$ $I^*(b) = P^*$	$P^* \in [\underline{P}, \overline{P}]$ $I^*(b) \in [\max\{P^*, b-a\}, \min\{P^*, a\} + (b-a)]$	$P^* \in [\underline{P}, a]$ $I^*(a) = P^*$
	$\overline{P} \leq b - a$	$P^* = \overline{P}$ $I^*(b) = b - a$	$P^* \in [\underline{P}, \overline{P}]$ $I^*(b) \in [b-a, \min\{P^*, a\} + (b-a)]$	$P^* \in [\underline{P}, a]$ $I^*(a) = P^*$
	$b - a \leq \underline{P}$	$P^* \in [\underline{P}, \overline{P}]$ $I^*(b) = P^*$	$P^* \in [\underline{P}, \overline{P}]$ $I^*(b) \in [P^*, \min\{P^*, a\} + (b-a)]$	$P^* \in [\underline{P}, a]$ $I^*(a) = P^*$
$\underline{P} \leq \overline{P} < a < b$	$\underline{P} < b - a < \overline{P}$	$P^* \in [b-a, \overline{P}]$ $I^*(b) = P^*$	$P^* \in [\underline{P}, \overline{P}]$ $I^*(b) \in [\max\{P^*, b-a\}, P^* + (b-a)]$	$P^* \in [\underline{P}, \overline{P}]$ $I^*(a) = P^*$
	$\overline{P} \leq b - a$	$P^* = \overline{P}$ $I^*(b) = b - a$	$P^* \in [\underline{P}, \overline{P}]$ $I^*(b) \in [b-a, P^* + (b-a)]$	$P^* \in [\underline{P}, \overline{P}]$ $I^*(a) = P^*$
	$b - a \leq \underline{P}$	$P^* \in [\underline{P}, \overline{P}]$ $I^*(b) = P^*$	$P^* \in [\underline{P}, \overline{P}]$ $I^*(b) \in [P^*, P^* + (b-a)]$	$P^* \in [\underline{P}, \overline{P}]$ $I^*(a) = P^*$
$a < \underline{P} \leq b \leq \overline{P}$	$\underline{P} < b - a$	$P^* \in [b-a, b]$ $I^*(b) = P^*$	$P^* \in [\underline{P}, b]$ $I^*(b) \in [\max\{P^*, b-a\}, b]$	$P^* = \underline{P}$ $I^*(b) = b$
	$b - a \leq \underline{P}$	$P^* \in [\underline{P}, b]$ $I^*(b) = P^*$	$P^* \in [\underline{P}, b]$ $I^*(b) \in [P^*, b]$	$P^* = \underline{P}$ $I^*(b) = b$
$a < \underline{P} \leq \overline{P} < b$	$\underline{P} < b - a < \overline{P}$	$P^* \in [b-a, \overline{P}]$ $I^*(b) = P^*$	$P^* \in [\underline{P}, \overline{P}]$ $I^*(b) \in [\max\{P^*, b-a\}, b]$	$P^* = \underline{P}$ $I^*(b) = b$
	$\overline{P} \leq b - a$	$P^* = \overline{P}$ $I^*(b) = b - a$	$P^* \in [\underline{P}, \overline{P}]$ $I^*(b) \in [b-a, b]$	$P^* = \underline{P}$ $I^*(b) = b$
	$b - a \leq \underline{P}$	$P^* \in [\underline{P}, \overline{P}]$ $I^*(b) = P^*$	$P^* \in [\underline{P}, \overline{P}]$ $I^*(b) \in [P^*, b]$	$P^* = \underline{P}$ $I^*(b) = b$

Table A.1: Pareto optimal insurance contracts in Theorem 4.1

Appendix B Proofs of Results

Proof of Theorem 3.1. Suppose that there exists an $(I^*, P^*) \in \mathcal{S}$ such that (I^*, P^*) is not Pareto optimal in $\mathcal{A}$. Since $(I^*, P^*) \in \mathcal{S}$, there exists an $\lambda \in (0,1)$ such that, for any $(I, P) \in \mathcal{A}$,

$$\lambda B\left(I^*, P^*\right) + (1-\lambda) S\left(I^*, P^*\right) \leq \lambda B\left(I, P\right) + (1-\lambda) S\left(I, P\right).$$

On the other hand, since (I^*, P^*) is not Pareto optimal in $\mathcal{A}$, there exists an $(I', P') \in \mathcal{A}$ such that $B\left(I', P'\right) \leq B\left(I^*, P^*\right)$ and $S\left(I', P'\right) \leq S\left(I^*, P^*\right)$, with at least one of the inequalities being strict, and in turn, since $\lambda \in (0,1)$,

$$\lambda B\left(I', P'\right) + (1-\lambda) S\left(I', P'\right) < \lambda B\left(I^*, P^*\right) + (1-\lambda) S\left(I^*, P^*\right)$$

which implies a contradiction. Therefore, any $(I^*, P^*) \in \mathcal{S}$ is Pareto optimal in $\mathcal{A}$.

For the other implication, it is sufficient to show that the set

$$\mathcal{C} := \left\{\left(B\left(I, P\right), S\left(I, P\right)\right) : (I, P) \in \mathcal{A}\right\}$$

is convex, since the remaining arguments are provided by a standard application of the Hahn-Banach separation theorem. Let $(I_1, P_1), (I_2, P_2) \in \mathcal{A}$ and $\theta \in [0,1]$. By the positive homogeneity, comonotonic additivity, and translation invariance of Ψ_B and Ψ_S,

$$\theta B\left(I_1, P_1\right) + (1-\theta) B\left(I_2, P_2\right) = B\left(\theta I_1 + (1-\theta) I_2, \theta P_1 + (1-\theta) P_2\right),$$

and

$$\theta S\left(I_1, P_1\right) + (1-\theta) S\left(I_2, P_2\right) = S\left(\theta I_1 + (1-\theta) I_2, \theta P_1 + (1-\theta) P_2\right).$$

Obviously, $\left(\theta I_1 + (1-\theta) I_2, \theta P_1 + (1-\theta) P_2\right) \in \mathcal{I}\backslash\{0\} \times \mathcal{P}$; furthermore, $\left(\theta I_1 + (1-\theta) I_2, \theta P_1 + (1-\theta) P_2\right) \in \mathcal{A}$; indeed,

$$B\left(\theta I_1 + (1-\theta) I_2, \theta P_1 + (1-\theta) P_2\right) \leq \theta B\left(0,0\right) + (1-\theta) B\left(0,0\right) = B\left(0,0\right),$$

with similar arguments for $S\left(\theta I_1 + (1-\theta) I_2, \theta P_1 + (1-\theta) P_2\right) \leq S\left(0,0\right)$. These justify that the set $\mathcal{C}$ is indeed a convex set. □

Proof of Proposition 4.1. One of the implication is Proposition 3.2. For the other implication, assume that $\underline{P} \leq b$, and let $I(x) := x \wedge \underline{P}$, for any $x \in [0, \operatorname{ess\,sup} X]$. Then $(I, \underline{P}) \in \mathcal{A}$; indeed, $I \in \mathcal{I}\backslash\{0\}$ and $\underline{P} \in \left[\underline{P}, \overline{P}\right]$ are obvious, and consider the following cases.

Case 1: Suppose that $\underline{P} \leq b \leq a$ or $\underline{P} \leq a < b$. Then $I(a) = \underline{P} = I(b)$.

Case 2: Suppose that $a < \underline{P} \leq b$. Then $I(a) = a < \underline{P} = I(b)$. □

Proof of Theorem 4.1. We only prove the case when $\underline{P} \leq a \leq \overline{P} < b$, and when $\lambda \in (0.5, 1)$, since other cases can be shown by similar arguments.

Firstly, notice that, for each fixed $P \in [\underline{P}, \overline{P}]$, the minimization problem (4.1) becomes a linear programming problem with constraints:

$$\begin{aligned} \min_{I(a), I(b) \in \mathbb{R}} \quad & \lambda b - \lambda I(b) + (1-\lambda) I(a) + (2\lambda - 1) P \\ \text{such that} \quad & 0 \le I(a) \le a \\ & 0 \le I(b) \le b \\ & I(b) - I(a) \le b - a \\ & I(a) \le P \le I(b). \end{aligned}$$

This linear programming problem with constraints can be solved explicitly. If $\underline{P} \le P \le a \le \overline{P} < b$ and $P \le b - a$, then $I^*(a) = P$ and $I^*(b) = P + (b - a)$; if $\underline{P} \le P \le a \le \overline{P} < b$ and $b - a \le P$, then $I^*(a) = P$ and $I^*(b) = P + (b - a)$; if $\underline{P} \le a < P \le \overline{P} < b$ and $P \le b - a$, then $I^*(a) = a$ and $I^*(b) = b$; if $\underline{P} \le a < P \le \overline{P} < b$ and $b - a \le P$, then $I^*(a) = a$ and $I^*(b) = b$.

Therefore, if $b - a \le \underline{P}$, then the value function is given by, for any $P \in [\underline{P}, \overline{P}]$,

$$(1-\lambda) I^*(a) - \lambda I^*(b) + (2\lambda - 1) P = \begin{cases} \lambda (a - b) & \text{if } P \in [\underline{P}, a] \\ (1-\lambda) a - \lambda b + (2\lambda - 1) P & \text{if } P \in [a, \overline{P}] \end{cases}.$$

This implies that $P^* \in [\underline{P}, a]$, $I^*(a) = P^*$, and $I^*(b) = P^* + (b - a)$. Other sub-cases can be analyzed in a similar manner. □

Proof of Proposition 4.3. Notice that, since $\alpha \le \beta$, for any $I \in \mathcal{I} \backslash \{0\}$, $\mathrm{TVaR}_\alpha(I(X)) \le \mathrm{TVaR}_\beta(I(X))$. One of the implication is Proposition 3.2. For the other implication, assume that $\underline{P} \le \mathrm{TVaR}_\beta(X)$ and consider the following two cases.

Case 1: Suppose that $\mathrm{TVaR}_\alpha(X) \le \underline{P} \le \mathrm{TVaR}_\beta(X)$. Then, obviously $(X, \underline{P}) \in \mathcal{A}$.

Case 2: Suppose that $\underline{P} < \mathrm{TVaR}_\alpha(X) \le \mathrm{TVaR}_\beta(X)$. The function $g : [0, \operatorname{ess\,sup} X] \to [0, \mathrm{TVaR}_\alpha(X)]$, defined by $g(d) := \mathrm{TVaR}_\alpha\left((X - d)_+\right)$, can be easily shown to be continuous and non-increasing. Therefore, by Intermediate Value Theorem, there exists an $\tilde{d} \in (0, \operatorname{ess\,sup} X]$ such that $\mathrm{TVaR}_\alpha((X - \tilde{d})_+) = \underline{P}$. Since $\tilde{I} \in \mathcal{I}$, where $\tilde{I}(x) := (x - \tilde{d})_+$, for any $x \in [0, \operatorname{ess\,sup} X]$, $\mathrm{TVaR}_\alpha((X - \tilde{d})_+) = \underline{P} \le \mathrm{TVaR}_\beta((X - \tilde{d})_+)$. Therefore, $(\tilde{I}, \underline{P}) \in \mathcal{A}$. □

Proof of Theorem 4.2. Let $\lambda \in (0,1)$ and $(I, P) \in \mathcal{A}$. Define $\tilde{R}(x) := x - (x - R(b))_+ + (x - \tilde{d}_2)_+$, and hence $\tilde{I}(x) = (x - R(b))_+ - (x - \tilde{d}_2)_+$, for any $x \in [0, \operatorname{ess\,sup} X]$, where $\tilde{d}_2 \in [b, \operatorname{ess\,sup} X]$ such that $\mathbb{E}[(\tilde{R}(X) - R(b))_+] = \mathbb{E}[(R(X) - R(b))_+]$. Necessarily, $\mathbb{E}[(\tilde{I}(X) - I(b))_+] = \mathbb{E}[(I(X) - I(b))_+]$, and, for any $x \in [0, b]$, $\tilde{I}(x) \le I(x)$; moreover, by the assumption that $\alpha \le \beta$, $a \le b$; therefore, $\mathbb{E}[(\tilde{I}(X) - I(a))_+] \le \mathbb{E}\left[(I(X) - I(a))_+\right]$.

By the dual representation of TVaR,

$$\begin{aligned}\mathrm{TVaR}_\beta(\tilde{R}(X)) &\le \mathrm{VaR}_\beta(R(X)) + \frac{1}{1-\beta}\mathbb{E}\left[\left(\tilde{R}(X) - \mathrm{VaR}_\beta(R(X))\right)_+\right]\\ &= R(b) + \frac{1}{1-\beta}\mathbb{E}\left[\left(\tilde{R}(X) - R(b)\right)_+\right]\\ &= R(b) + \frac{1}{1-\beta}\mathbb{E}\left[(R(X) - R(b))_+\right]\\ &= \mathrm{TVaR}_\beta(R(X)),\end{aligned}$$

and thus, by the comonotonic additivity, $\mathrm{TVaR}_\beta(X) - \mathrm{TVaR}_\beta(\tilde{I}(X)) \le \mathrm{TVaR}_\beta(X) - \mathrm{TVaR}_\beta(I(X))$, which implies that $\mathrm{TVaR}_\beta(\tilde{I}(X)) \ge \mathrm{TVaR}_\beta(I(X))$. Similarly, by the dual representation of TVaR,

$$\begin{aligned}\mathrm{TVaR}_\alpha(\tilde{I}(X)) &\le \mathrm{VaR}_\alpha(I(X)) + \frac{1}{1-\alpha}\mathbb{E}\left[\left(\tilde{I}(X) - \mathrm{VaR}_\alpha(I(X))\right)_+\right]\\ &= I(a) + \frac{1}{1-\alpha}\mathbb{E}\left[\left(\tilde{I}(X) - I(a)\right)_+\right]\\ &\le I(a) + \frac{1}{1-\alpha}\mathbb{E}\left[(I(X) - I(a))_+\right]\\ &= \mathrm{TVaR}_\alpha(I(X)).\end{aligned}$$

Since $(I,P) \in \mathcal{A}$, $\mathrm{TVaR}_\alpha(\tilde{I}(X)) \le \mathrm{TVaR}_\alpha(I(X)) \le P \le \mathrm{TVaR}_\beta(I(X)) \le \mathrm{TVaR}_\beta(\tilde{I}(X))$, which implies $(\tilde{I}, P) \in \mathcal{A}$. Moreover,

$$\begin{aligned}F_\lambda(I,P) &= \lambda\mathrm{TVaR}_\beta(X) - \lambda\mathrm{TVaR}_\beta(I(X)) + (1-\lambda)\mathrm{TVaR}_\alpha(I(X)) + (2\lambda - 1)P\\ &\ge \lambda\mathrm{TVaR}_\beta(X) - \lambda\mathrm{TVaR}_\beta(\tilde{I}(X)) + (1-\lambda)\mathrm{TVaR}_\alpha(\tilde{I}(X)) + (2\lambda - 1)P\\ &= F_\lambda(\tilde{I}, P).\end{aligned}$$

□

References

[1] Arrow, K. J. (1963). Uncertainty and the welfare economics of medical care. *The American Economic Review* 53(5), 941–973.

[2] Asimit, V., Bignozzi, V., Cheung, K. C., Hu, J., and Kim, E.-S. (2017). Robust and Pareto optimality of insurance contracts. *European Journal of Operational Research* 262(2), 720–732.

[3] Asimit, V. and Boonen, T. J. (2018). Insurance with multiple insurers: a game-theoretic approach. *European Journal of Operational Research* 267(2), 778–790.

[4] Asimit, V., Gao, T., Hu, J., and Kim, E.-S. (2018). Optimal risk transfer: a numerical optimization approach. *North American Actuarial Journal* 22(3), 341–364.

[5] Asimit, V., Hu, J., and Xie, Y. (2019). Optimal robust insurance with a finite uncertainty set. *Insurance: Mathematics and Economics* 87, 67–81.

[6] Bernard, C., He, X., Yan, J.-A., and Zhou, X. Y. (2015). Optimal insurance design under rank-dependent expected utility. *Mathematical Finance* 25(1), 154–186.

[7] Boonen, T. J. and Ghossoub, M. (2019). On the existence of a representative reinsurer under heterogeneous beliefs. *Insurance: Mathematics and Economics* 88, 209–225.

[8] Borch, K. (1960). An attempt to determine the optimum amount of stop loss reinsurance. *Proceedings of the Transactions of the 16th International Congress of Actuaries* 1, 597–610.

[9] Cai, J., Lemieux, C., and Liu, F. (2016). Optimal reinsurance from the perspectives of both an insurer and a reinsurer. *ASTIN Bulletin: The Journal of the International Actuarial Association* 46(3), 815–849.

[10] Cai, J., Liu, H., and Wang, R. (2017). Pareto-optimal reinsurance arrangements under general model settings. *Insurance: Mathematics and Economics* 77, 24–37.

[11] Cai, J., Tan, K. S., Weng, C., and Zhang, Y. (2008). Optimal reinsurance under VaR and CTE risk measures. *Insurance: Mathematics and Economics* 43(1), 185–196.

[12] Cheung, K. C. (2010). Optimal reinsurance revisited - a geometric approach. *ASTIN Bulletin: The Journal of the International Actuarial Association* 40(1), 221–239.

[13] Cheung, K. C., Chong, W. F., and Lo, A. (2019). Budget-constrained optimal reinsurance design under coherent risk measures. *Scandinavian Actuarial Journal* 2019(9), 729–751.

[14] Cheung, K. C., Chong, W. F., and Yam, S. C. P. (2015). The optimal insurance under disappointment theories. *Insurance: Mathematics and Economics* 64, 77–90.

[15] Cheung, K. C., Chong, W. F., and Yam, S. C. P. (2015). Convex ordering for insurance preferences. *Insurance: Mathematics and Economics* 64, 409–416.

[16] Cheung, K. C., Sung, K. C. J., Yam, S. C. P., and Yung, S. P. (2014). Optimal reinsurance under general law-invariant risk measures. *Scandinavian Actuarial Journal* 2014(1), 72–91.

[17] Chi, Y. (2012). Reinsurance arrangements minimizing the risk-adjusted value of an insurer's liability. *ASTIN Bulletin: The Journal of the International Actuarial Association* 42(2), 529–557.

[18] Chi, Y. (2019). On the optimality of a straight deductible under belief heterogeneity. *ASTIN Bulletin: The Journal of the International Actuarial Association* 49(1), 243–262.

[19] Chi, Y. and Liu, F. (2017). Optimal insurance design in the presence of exclusion clauses. *Insurance: Mathematics and Economics* 76, 185–195.

[20] Cui, W., Yang, J., and Wu, L. (2013). Optimal reinsurance minimizing the distortion risk measure under general reinsurance premium principles. *Insurance: Mathematics and Economics* 53(1), 74–85.

[21] Denuit, M., Dhaene, J., Goovaerts, M., and Kaas, R. (2005). *Actuarial Theory for Dependent Risks: Measures, Orders and Models*. John Wiley & Sons, Ltd.

[22] Gerber, H. U., Shiu, E. S. W., and Yang, H. (2019). A constraint-free approach to optimal reinsurance. *Scandinavian Actuarial Journal* 2019(1), 62–79.

[23] Ghossoub, M. (2019). Optimal insurance under rank-dependent expected utility. *Insurance: Mathematics and Economics* 87, 51–66.

[24] Ghossoub, M. (2019). Budget-constrained optimal insurance with belief heterogeneity. *Insurance: Mathematics and Economics* 89, 79–91.

[25] Huberman, G., Mayers, D., and Smith, C. W., Jr. (1983). Optimal insurance policy indemnity schedules. *The Bell Journal of Economics* 14(2), 415–426.

[26] Jiang, W., Hong, H., and Ren, J. (2018). On Pareto-optimal reinsurance with constraints under distortion risk measures. *European Actuarial Journal* 8(1), 215–243.

[27] Lo, A. (2017). A Neyman–Pearson perspective on optimal reinsurance with constraints. *ASTIN Bulletin: The Journal of the International Actuarial Association* 47(2), 467–499.

[28] Lo, A. and Tang, Z. (2019). Pareto-optimal reinsurance policies in the presence of individual risk constraints. *Annals of Operations Research* 274(1–2), 395–423.

[29] Ludkovski, M. and Young, V. R. (2009). Optimal risk sharing under distorted probabilities. *Mathematics and Financial Economics* 2, 87–105.

[30] Raviv, A. (1979). The design of an optimal insurance policy. *The American Economic Review* 69(1), 84–96.

[31] Sung, K. C. J., Yam, S. C. P., Yung, S. P., and Zhou, J. H. (2011). Behavioral optimal insurance. *Insurance: Mathematics and Economics* 49(3), 418–428.

[32] Tan, K. S., Weng, C., and Zhang, Y. (2011). Optimality of general reinsurance contracts under CTE risk measure. *Insurance: Mathematics and Economics* 49(2), 175–187.

[33] Tan, K. S. and Weng, C. (2014). Empirical approach for optimal reinsurance design. *North American Actuarial Journal* 18(2), 315–342.

[34] Xu, Z. Q., Zhou, X. Y., and Zhuang, S. C. (2019). Optimal insurance under rank-dependent utility and incentive compatibility. *Mathematical Finance* 29(2), 659–692.

[35] Zheng, Y. and Cui, W. (2014). Optimal reinsurance with premium constraint under distortion risk measures. *Insurance: Mathematics and Economics* 59, 109–120.

[36] Zhuang, S. C., Boonen, T. J., Tan, K. S., and Xu, Z. Q. (2017). Optimal insurance in the presence of reinsurance. *Scandinavian Actuarial Journal* 2017(6), 535–554.

[37] Zhuang, S. C., Weng, C., Tan, K. S., and Assa, H. (2016). Marginal indemnification function formulation for optimal reinsurance. *Insurance: Mathematics and Economics* 67, 65–76.