Mitigating product shortage due to disruptions in multi-stage supply chains through inventory and reserve capacity

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We focus on the optimal use of risk mitigation inventory (RMI) and reserve capacity to manage disruption risk in serial multi-stage supply chains where product transformation occurs at each stage. We find that under reasonable conditions it is better to hold more RMI downstream than upstream even when the upstream holding costs are lower. We also find that it is often optimal to hold more reserve capacity downstream than upstream. While in one-stage supply chains RMI and reserve capacity always behave as substitutes, it turns out that in multi-stage serial supply chains the interplay between RMI and reserve capacity is more nuanced. We find that echelon RMI and reserve capacity at each stage are substitutes. In contrast, RMI at a stage complements reserve capacity at the adjacent downstream stage.

Key words: Disruption risk management, inventory, reserve capacity, serial supply chain

1. Introduction

Consider a multi-national pharmaceutical company that manufactures and markets a life-saving cancer drug. The supply chain typically consists of multiple stages with product transformation occurring at each stage. The first stage is typically the production of the active pharmaceutical ingredient (API). The second stage is typically the production of the finished product in the form of a pill or liquid. The final stage is typically packaging to meet local requirements. Demand for such a drug remains relatively stable, given that neither the price nor the number of patients vary significantly. In such circumstances, the pharmaceutical company's primary concern is the risk of disruption. The complex pharmaceutical production process is exposed to disruption risks such as a biological contamination at a production site, which results in production stops for extended time periods.

Observe that a disruption at a stage results in no product transformation occurring at that stage. As a result, no input can be fed to the next stage unless the disrupted stage either has inventory of its finished product or has extra capacity that can be brought online in the event of a disruption to the primary capacity. This results in two main disruption risk mitigation levers. First, firms may hold *Risk Mitigation Inventory* (RMI) at a stage. RMI at a stage is held in the form of its output that is used to supply the next downstream stage in the event of a supply disruption. Second, firms may hold *reserve capacity* for a stage at another reliable manufacturing site. In the event of a disruption at the primary production site, the reserve capacity can be used to produce output to supply the next downstream site.

In this paper we focus on understanding the optimal use of RMI and reserve capacity in serial multi-stage supply chains to deal with disruption risk at each stage. An important goal in our research is to understand what factors influence the location (e.g. the stages) and quantity of RMI and reserve capacity held in a serial supply chain. Recall that RMI at each stage is held in the form of output of that stage. Given that inputs are transformed into higher value outputs at each stage, we assume that holding RMI at an upstream stage is cheaper than holding inventory at a downstream stage closer to the customer. However, it is important to recognize that RMI at a stage can only be used to cover disruptions at that stage or further upstream. It cannot be used to cover a downstream disruption because the transformation capacity downstream is lost and thus cannot convert the upstream RMI into output.

Unlike RMI that may be used in the event of a disruption at that stage or any upstream stage, reserve capacity at a stage is only used if a disruption occurs at that stage. It serves no purpose if an upstream disruption results in no input arriving at the stage.

We illustrate the use of RMI and reserve capacity in Figure 1, where we present a 3-stage serial supply chain where the middle stage 2 is disrupted. During the disruption we can use RMI at stage

1 to meet customer demand and RMI located at stage 2 to supply stage 1 where it is transformed to meet customer demand. If reserve capacity is available at stage 2, we can also use it to supply stage 1. The RMI located at stage 3 does not help in the event of a disruption at stage 2. Similarly, reserve capacity at stages 1 and 3 is not used in the event of a disruption at stage 2.

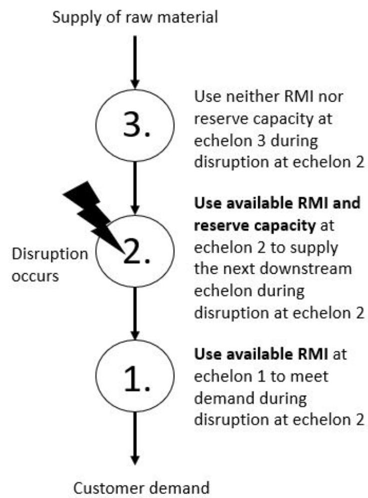

Figure 1 Serial supply chain with disruption at stage 2.

Our collaborating pharmaceutical company used to hold large quantities of RMI at the upstream (API) stage of the supply chain due to the lower costs of holding RMI there. However, the company also recognized the importance of holding RMI at the drug product / finished goods stages that are further downstream, but committed only to small quantities of RMI downstream. The company justified holding most of its RMI upstream based on the lower holding costs upstream.

In contrast to RMI, reserve capacity was held up- and downstream as there was no major cost difference of holding reserve capacity across the different stages. Our goal in this research is to better understand how RMI and reserve capacity should be best positioned in a multi-stage serial supply chain and how RMI and reserve capacity influence each other.

One of our main findings is that in a serial supply chain it often pays to push more RMI downstream even though RMI is cheaper to hold upstream. We refer to this finding as *downstream commitment to RMI*. This finding is driven primarily by the observation that in a serial supply chain with product transformation at each stage, downstream RMI can help buffer against disruptions at all upstream stages. These results are counter to what the pharmaceutical company was doing where it was holding most of its RMI at the upstream stage.

Counter to the practice at the pharmaceutical firm, we also find that it is often optimal to hold more reserve capacity at a downstream stage when compared to the adjacent upstream stage.

The existing literature suggests that RMI and reserve capacity are substitutes in one-stage supply chains (Tomlin 2006). We extend this result to multi-stage supply chains where the interactions is more nuanced. We find that in multi-stage supply chains, echelon RMI and reserve capacity at any stage are substitutes. In contrast, RMI at any stage and reserve capacity at the adjacent downstream stage are complements.

Multi-stage supply chains in the presence of disruption risk have been studied in the literature. However, most literature differs from our work by either not explicitly considering product transformation occurring at each stage (for example Ang et al. (2017)), or by providing simulation based insights (for example Schmitt and Singh (2012)).

The remainder of this paper is structured as follows. In Section 2 we review the relevant literature, focusing mainly on multi-stage and disruption risk literature. In Sections 3 to 5 we present our mathematical models with managerial insights. Finally, we provide concluding remarks in Section 6.

2. Literature Review

Several authors have studied serial multi-stage supply chains with regards to safety inventory in the absence of disruption probabilities (Clark and Scarf 1960, Federgruen and Zipkin 1984, Chen and Zheng 1994, Graves 1985, Rosling 1989, Schmidt and Nahmias 1985). DeCroix (2013) showed that these results do not extend to the presence of supply disruptions in general.

Supply disruptions have been studied extensively in one-stage supply chains in the context of both RMI and reserve capacity (or contingent rerouting / volume flexibility). Tomlin (2006) studies a model where a firm can source from a reliable supplier and a cheaper but less reliable supplier.

The reliable supplier allows for contingent rerouting due to volume flexibility. The author identifies conditions when different mitigation strategies are optimal. He shows that contingent rerouting is preferred to RMI as a risk mitigation strategy if disruptions are rare but long, whereas RMI is preferred if disruptions are frequent but short. Chopra et al. (2007) study reserve capacity and RMI decisions in the presence of disruption risk and recurrent risk (demand uncertainty). The authors find that a firm should order more from a reliable source if disruption risk dominates against recurrent risk but hold inventory and order more from the cheaper but less reliable source if recurrent risk dominates disruption risk. Qi (2013) extends this work to incorporate the concept of a waiting time after a disruption. The waiting time allows the decision maker to distinguish between operational fluctuations and disruptions. The author studies optimal sourcing decisions and provides structural insights on optimal decision variables. Qi and Lee (2015) study the role of expedited shipping on the optimal risk mitigation strategy. Their research demonstrates that expedited shipping is a valuable alternative when the cost of maintaining reserve capacity is high.

There are a few papers that study reserve capacity and RMI decisions in multi-stage supply chains under supply disruption risks. Ang et al. (2017) study a non-centralized three-stage supply chain where different sourcing decisions are analyzed besides the use of inventory. The authors show that penalty contracts alleviate the coordination problem in the supply chain. Schorpp et al. (2018) study a three-tier supply chain consisting of a buyer, a supplier, and a sub-supplier. The authors provide insights on the optimal use of reserve capacity at the suppliers in a decentralized setting. Schmitt and Singh (2012) perform a simulation study and elaborate on the importance of analyzing supply chain networks as a whole as opposed to the single-stage approach. They find that firms should focus on minimizing the disruption time rather than the disruption frequency. Closely related to our research is the paper by Hopp and Yin (2010) who study reserve capacity and RMI decisions in two-stage supply chains under disruption risk. Through numerical experiments they find that long upstream disruptions tend to push the reserve capacity upstream. Their model, however, does not consider product transformation at each stage.

There is a vast literature on disruption risk management that goes beyond multi-stage inventory models. Dong et al. (2018) analyze the role of insurance in a two-stage supply chain. They find that insurance and operational measures can sometimes be complements rather than substitutes. Lim et al. (2013) study the impact of misestimating disruption probabilities. They show that underestimation in disruption probability results in higher expected total cost than overestimation. A thorough literature review on disruption risk management is presented by Snyder et al. (2016).

Our main contribution to the literature is to derive structural insights on how to best use RMI and reserve capacity to mitigate disruption risks in multi-stage serial supply chains where product transformation occurs at each stage. We find that it is often better to hold more RMI downstream than upstream even when the upstream holding costs are lower. We also find that it is often optimal to hold more reserve capacity downstream than upstream. While in one-stage supply chains RMI and reserve capacity always behave as substitutes, it turns out that in multi-stage serial supply chains the interplay between RMI and reserve capacity is more nuanced. We find that echelon RMI and reserve capacity at each stage are substitutes (a natural extension of the single stage result). In contrast, we find that RMI at a stage complements reserve capacity at the adjacent downstream stage.

3. Modeling framework

We consider an n-stage linear supply chain where stage 1 is closest to the customer and stage n is the farthest from the customer. Stage $i-1$ is considered to be downstream from stage i and stage $i+1$ is considered to be upstream from stage i. At each stage there is unlimited primary production capacity that transforms inputs to outputs. The production capacity at stage i uses the output of stage $i+1$ as input and transforms it into output that in turn becomes input for stage $i-1$. We assume that stage n obtains input from a perfectly reliable external source.

The primary production capacity at each stage is subject to disruption risk. When a disruption occurs at stage i, the primary production capacity at stage i is no longer available. RMI at stage i, I_i, is assumed to be held in the form of output of the production at stage i. However, RMI I_i is held at the adjacent downstream stage $i-1$ where it can be used as input for production at stage $i-1$ in the event of the disruption. Thus, RMI of the output of any stage i is not destroyed in the event of a disruption at stage i. Recall that whereas the RMI at stage $i+1$ (I_{i+1}) can be used to supply stage i if a disruption occurs at any of the stages $i+1$, $i+2$, ... (any of the stages $i+1$ or further upstream), the reserve capacity at stage i (a_i) is only used if the disruption occurs at stage i. The reserve capacity at stage i is not used if stage i is not disrupted because the regular capacity can handle all the required production in that case.

The objective of our model is to determine the optimal RMI and reserve capacity at each stage by minimizing long-run expected cost per unit of time. The decision variables (see Table 1) at each stage i are the amount of RMI to carry (I_i) and the amount of reserve capacity to carry (a_i). Any leftover RMI incurs a holding cost per unit time. The amount of reserve capacity, a_i, is the production rate available at stage i in the event of a disruption at stage i. Reserve capacity a_i incurs an upfront reservation cost of $\hat{c}_i a_i$ per unit time. An additional cost of c_i is incurred for each unit actually produced using the reserve capacity. If a disruption occurs at stage i, the total available input for stage $i-1$ during the entire disruption time (say time d) is constrained by $I_i + da_i$. Any unmet customer demand during a disruption is assumed to be backlogged at a penalty cost of p per unit. Recall that holding RMI upstream is cheaper than holding RMI closer to the customer, i.e., $h_{i+1} < h_i$ for all i (because product is transformed at each stage).

Our model is based on the following key assumptions:

A.1 RMI is not perishable.

A.2 The duration of a disruption is known once a disruption occurs.

A.3 Recovery rates are identical across all stages.

A.4 Disruption durations are exponentially distributed.

A.5 Regular production capacity at each stage is infinite, but the reserve capacity is finite at a_i.

A.6 Customer demand is deterministic and supply lead time is 0.

A.7 Disruption rates are low compared to recovery rates.

Indices	
$i \in \{1, 2, .., n\}$	Lower case index labels stage number with 1 being the most downstream stage and n the most upstream stage.
Decision variable	
$I_i \geq 0$	RMI level at stage i
$a_i \geq 0$	Maximum reserve capacity production rate at stage i
Parameters	
k	Disruption time
$\pi(k)$	Probability density function that a stage is disrupted for time $k > 0$
$\Pi(\cdot)$	Cumulative distribution function of $\pi(k)$
$\alpha_i > 0$	Disruption rate at stage i
$\beta > 0$	Recovery rate at any stage
$p > 0$	Penalty cost per unit
$h_i > 0$	Inventory holding cost per unit and per time at stage i
$\bar{h}_i > 0$	Effective inventory holding cost at stage i, $\bar{h}_i = h_i(1 + \sum_{j=1}^{i-1} \frac{\alpha_j}{\beta})$
$\hat{c}_i > 0$	Reservation cost for the reserve capacity per unit and per time at stage i
$c_i > 0$	Production cost through the reserve capacity per unit at stage i
$\mu = 1$	Normalized, deterministic demand per time at downstream stage 1

Table 1 Indices, decision variables and parameters of the model

A.8 At most one stage is disrupted at a time.

Let us comment on these assumptions:

Assumption 1 (A.1). Our assumption that RMI is not perishable is reasonable when the shelf life is long relative to the disruption time. In the pharmaceutical industry the shelf life can be considered as long due to the practice of retesting the API once the expiry date is reached (ICH 2000). Effectively, this practice allows a pharmaceutical company to extend the shelf life of the API beyond the original expiry date. Further, note that in order to ensure that RMI does not get too old, RMI is constantly refreshed through new production while the oldest RMI is used to satisfy the adjacent downstream stage or customer demand. The constant refreshment using a first-in, first-out policy helps ensure that shelf life constraints are satisfied by the RMI.

Assumption 2 (A.2). We assume that the duration of a disruption is known once a disruption occurs. A consequence of this assumption is that reserve capacity is only used if the available RMI is not sufficient to cover demand during a disruption. RMI is used before reserve capacity because holding cost for RMI is assumed to be less than production cost using reserve capacity. We further assume that downstream RMI is used before upstream RMI because holding cost for downstream RMI is larger than upstream RMI.

Assumption 3 (A.3). We assume that once a stage is disrupted, the disrupted stage recovers with a recovery rate β. We assume that β is identical across all stages. This assumption is used to simplify the model. However, our model is general enough to accommodate different recovery rates across all stages. For ease of exposition we use the same recovery rate at each stage.

Assumption 4 (A.4). We assume that disruption durations are exponentially distributed (see Parlar and Perry (1996)). This assumption is equivalent to assuming that disruption and recovery rates are independent of time (memoryless property). Mathematically, we describe disruptions at any stage as a random process governed by the pdf $\pi(k)$, where $k > 0$ is the disruption time (see Table 1 for a complete list of parameters, decision variables and indices). We model the distribution function $\pi(k)$ as an exponential function with recovery rate β: $\pi(k) = \beta \exp(-\beta k)$. The cumulative distribution function gives the probability that the disruption time is less than or equal to the variable K: $\Pi(K) = \int_0^K \pi(k)dk$. We also define a constrained expected disruption time as $E^K = \int_0^K \pi(k)kdk$.

The exponential disruption duration distribution is consistent with the findings of our industrial partner. The company gathered data on the likelihood and impact (in terms of disruption times at a given site) of disruptions at the various production sites (stages). Disruptions can have several causes including a biological contamination at a production site, a fire at a production site, a strike, an earthquake, or a terrorist attack. These disruptions were broadly categorized as external and internal. To assess the external disruptions, our industrial partner collaborated with an insurance company to obtain data on the likelihood and impact of the external disruptions. To assess the internal disruptions the company surveyed its site managers who provided estimates (based on their experience) on likelihood and impact of disruptions. All the data was finally fed into a Monte-Carlo simulation that led to the distribution function of the disruption time (so called site risk profile). The distribution function has the form of an exponential function.

Assumption 5 (A.5). We assume that the regular production capacity is infinite, but that the reserve capacity produces at the contracted finite production rate. Fundamentally, the assumption that the regular production capacity is unlimited states that there is enough regular production capacity available to meet demand and build up RMI quickly after a disruption (that is, all RMI is built up before the next disruption occurs). To simplify our model (and to focus on first-degree effects) we effectively approximate the regular production capacity rate with an infinite rate. Given that the firm contracts for reserve capacity, the available production from reserve capacity is limited by the contracted amount. The contracted amount of the reserve capacity is typically specified in a contract between the CMO and the pharmaceutical company. Such contracts have been discussed in the case study by Samii and Van Wassenhove (2008).

Assumption 6 (A.6). As common in the disruption risk literature, we set customer demand to be deterministic with rate $\mu = 1$. For ease of exposition we set the lead time equal to zero. Our model, however, allows a deterministic, positive lead time. The use of a fixed positive lead time does not fundamentally chain any results (each stage would carry an additional fixed amount of inventory to cover the deterministic demand during the lead time), justifying the use of a lead time of 0.

Assumption 7 (A.7). This assumption is reasonable for the pharmaceutical industry because of the low likelihood of disruptions and the goal of quick recovery. It is reasonable to assume that the average time between disruptions is much longer than the average time to recovery in the event of a disruption.

Assumption 8 (A.8). We assume that at most one stage is disrupted at any given point in time. This assumption is reasonable when disruption rates are low relative to recovery rates (Assumption 7) and the number of stages is not too large (a reasonable assumption in practice where supply chains are unlikely to have more than ten stages).

These assumptions are quite common in the disruption risk management literature (compare for example Dong et al. (2018), Ang et al. (2017) and Schorpp et al. (2018)). The advantage of these assumptions is to somewhat simplify the analysis and allow us to focus our insights on the location and quantity of RMI and reserve capacity when dealing with disruptions.

Recall that the objective of our model is to determine optimal RMI and reserve capacity quantities at each stage by minimizing long-run expected cost per unit of time. In the following we provide an outline of the two-stage serial supply chain objective function (long-run expected cost per unit of time). The generalization of the objective function to n-stages is in the appendix. Given assumptions A.7 and A.8, we assume that after every disruption (whether upstream or downstream) the supply chain returns to the undisrupted stage before the next disruption occurs. This allows us to define a renewal cycle as a period of no disruption followed by a period with one disruption (either upstream or downstream). The duration of a renewal cycle is then defined as the expected time duration of not being disrupted plus the expected time duration of one disruption (see Figure 2). Given disruption rate downstream α_d and disruption rate upstream α_u, the expected time of not

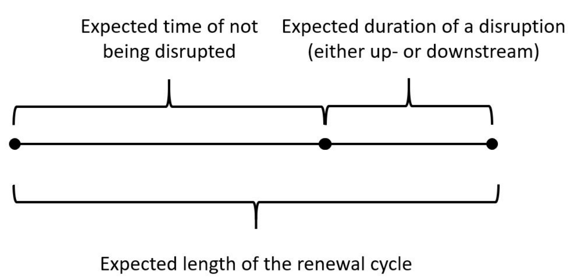

Figure 2 Expected length of the renewal cycle

being disrupted is $\frac{1}{\alpha_d+\alpha_u}$ (neglecting terms arising from simultaneous disruptions). The expected time duration of one disruption is $\frac{1}{\beta}$. Thus, the expected renewal cycle length is given by:

$$\frac{1}{\alpha_d+\alpha_u}+\frac{1}{\beta}.$$

Based on this definition of the renewal cycle we use the well-known renewal-reward theorem to calculate the long-run expected cost. Using the subscript u [d] to label the upstream [downstream] stage of the supply chain, the long-run expected cost $\mathbb{E}[C(I_u,I_d,a_u,a_d)]$ is the ratio of the expected cost per cycle and the expected renewal cycle length:

$$\mathbb{E}[C(I_u,I_d,a_u,a_d)]=\frac{\text{Expected cost per cycle}}{\text{Expected cycle length}}.$$

In order to determine the expected cost per cycle, let us introduce some definitions:

1. $\mathbb{E}[C_0(I_u,I_d,a_u,a_d)]$: Expected cost per unit of time when there is no disruption
2. $\mathbb{E}[C_d(I_u,I_d,a_u,a_d)]$: Expected cost per unit of time when there is a downstream disruption
3. $\mathbb{E}[C_u(I_u,I_d,a_u,a_d)]$: Expected cost per unit of time when there is an upstream disruption

The expected cost per cycle is then given by the expected time of not being disrupted times the expected cost per time when the supply chain is not disrupted plus the expected time of being disrupted times the expected cost per time when there is a disruption (either up- or downstream):

$$\frac{1}{\alpha_d+\alpha_u}\mathbb{E}[C_0(I_u,I_d,a_u,a_d)]+\frac{1}{\beta}(P(\text{disruption in downstream before upstream})\mathbb{E}[C_d(I_u,I_d,a_u,a_d)]$$
$$+P(\text{disruption in upstream before downstream})\mathbb{E}[C_u(I_u,I_d,a_u,a_d)]).$$

Using $P(\text{disruption in upstream before downstream})=\frac{\alpha_u}{\alpha_d+\alpha_u}$ and $P(\text{disruption in downstream before upstream})=\frac{\alpha_d}{\alpha_d+\alpha_u}$, we get the long-run expected cost:

$$\mathbb{E}[C(I_u,I_d,a_u,a_d)]=\frac{\frac{1}{\alpha_d+\alpha_u}\mathbb{E}[C_0(I_u,I_d,a_u,a_d)]+\frac{1}{\beta}(\frac{\alpha_d}{\alpha_d+\alpha_u}\mathbb{E}[C_d(I_u,I_d,a_u,a_d)]+\frac{\alpha_u}{\alpha_d+\alpha_u}\mathbb{E}[C_u(I_u,I_d,a_u,a_d)])}{\frac{1}{\alpha_d+\alpha_u}+\frac{1}{\beta}} \tag{1}$$

The expected cost per time when there is no disruption consists of costs for holding RMI up- and downstream as well as upfront costs for the reserve capacity up- and downstream:

$$\mathbb{E}[C_0(I_u,I_d,a_u,a_d)]=h_dI_d+h_uI_u+\hat{c}_ua_u+\hat{c}_da_d. \tag{2}$$

The expected cost per time when there is a downstream disruption consists of five components - a penalty cost per time for any unmet demand (if all demand cannot be met during the disruption), a holding cost per time for excess downstream RMI (if all downstream RMI is not used during the disruption), the cost of reserving downstream reserve capacity per time, a downstream reserve capacity production cost per time (if downstream reserve capacity is used during the disruption), and an upstream RMI holding cost per time (because the upstream RMI is not used during a downstream disruption).

First, the expected penalty cost per time is the unit penalty cost p times the expected unmet demand divided by the expected disruption time. Unmet demand arises when the disruption time exceeds the time $\frac{I_d}{1-a_d}$ required to deplete all RMI I_d given demand $\mu = 1$ and reserve capacity production rate a_d (resulting in RMI depletion rate $1-a_d$). Given a disruption of duration $k \geq \frac{I_d}{1-a_d}$, the unmet demand is given by $k(1-a_d)-I_d$. Thus, the expected unmet demand during a disruption is given by: $\int_{\frac{I_d}{1-a_d}}^{\infty} \pi(k)\big(k(1-a_d)-I_d\big)dk$. Given the expected disruption time $\frac{1}{\beta}$, the expected penalty cost per unit time is given by $\beta p \int_{\frac{I_d}{1-a_d}}^{\infty} \pi(k)\big(k(1-a_d)-I_d\big)dk$.

Second, we calculate the expected holding cost per time for excess downstream RMI. Excess downstream RMI arises when not all downstream RMI is used during the disruption (that is when the available downstream RMI I_d exceeds the demand during disruption time k. Thus, the expected holding cost for excess downstream RMI during a disruption equals $h_d \int_0^{I_d} \pi(k)(I_d-k)kdk$. Given the expected disruption time $\frac{1}{\beta}$, the expected holding cost per unit time for excess downstream RMI is given by $\beta h_d \int_0^{I_d} \pi(k)(I_d-k)kdk$.

Third, the expected downstream reserve capacity production cost per time is the unit production cost c_d times the expected number of units to be produced through the reserve capacity divided by the expected disruption time. The reserve capacity is used only when the disruption time demand (k) exceeds the downstream RMI (I_d). If $k \leq \frac{I_d}{1-a_d}$, the quantity produced using reserve capacity is $k-I_d$. If $k \geq \frac{I_d}{1-a_d}$, the quantity produced using reserve capacity is ka_d. Thus, given the expected disruption time $\frac{1}{\beta}$, the expected reserve capacity production quantity during a disruption is given by: $\beta c_d \int_{I_d}^{\frac{I_d}{1-a_d}} \pi(k)(k-I_d)dk + \beta c_d \int_{\frac{I_d}{1-a_d}}^{\infty} \pi(k)a_d kdk$.

Fourth, the expected cost per time for reserving downstream reserve capacity is given by $\hat{c}_u a_u + \hat{c}_d a_d$.

Fifth, the cost per time for holding unused RMI upstream is given by $h_u I_u$.

The expected cost per unit of time when there is a downstream disruption sums up to:

$$\mathbb{E}[C_d(I_u, I_d, a_u, a_d)] = \beta p \int_{\frac{I_d}{1-a_d}}^{\infty} \pi(k)\big((1-a_d)k - I_d\big)dk + \beta h_d \int_0^{I_d} \pi(k)(I_d - k)kdk \\ + \beta c_d \int_{I_d}^{\frac{I_d}{1-a_d}} \pi(k)(k - I_d)dk + \beta c_d \int_{\frac{I_d}{1-a_d}}^{\infty} \pi(k) a_d k dk + h_u I_u + \hat{c}_u a_u + \hat{c}_d a_d. \tag{3}$$

We calculate costs similarly when the supply chain is in the state of an upstream disruption. In this case, we incur a penalty cost per time for unmet demand or inventory holding costs per time for excess RMI up- and/or downstream as well as a reserve capacity production cost per time upstream. The only difference relative to a downstream disruption is that we now use the the entire RMI, $I_d + I_u$, before needing production from reserve capacity and thus only pay a holding cost for any unused RMI up or downstream. The expected cost per unit of time when there is an upstream disruption is (the derivation of the cost-terms is analogous to the case when a disruption occurs downstream):

$$\mathbb{E}[C_u(I_u, I_d, a_u, a_d)] = \beta p \int_{\frac{I_d+I_u}{1-a_u}}^{\infty} \pi(k)\big((1-a_u)k - I_d - I_u\big)dk + \beta h_u \int_{I_d}^{I_d+I_u} \pi(k)\big(I_d + I_u - k\big)kdk \\ + \beta \int_0^{I_d} \pi(k)\Big(h_d\big(I_d - k\big) + h_u I_u\Big)kdk + \beta c_u \int_{I_d+I_u}^{\frac{I_d+I_u}{1-a_u}} \pi(k)(k - I_d - I_u)dk \\ + \beta c_u \int_{\frac{I_d+I_u}{1-a_u}}^{\infty} \pi(k) a_u k dk + \hat{c}_u a_u + \hat{c}_d a_d. \tag{4}$$

The long-run expected cost can then be obtained using Eq. 1. We first consider the case where both RMI and reserve capacity are held at each stage at optimality. The case where some stages do not hold RMI or reserve capacity are a natural specialization of the more general case and are discussed in Appendix A.3. Applying the first-order conditions to the long-run expected cost, the optimal RMI and reserve capacity at each stage can be characterized as follows:

Proposition 1. *Let Assumptions 1-8 (A.1-A.8) hold. The optimal RMI and reserve capacity levels at the downstream stage satisfy:*

$$0 = \alpha_d\Big(\Pi(\frac{I_d}{1-a_d})(p-c_d) + \Pi(I_d)c_d + E^{I_d}h_d - p\Big) + \alpha_u(h_d - h_u)E^{I_d} + (h_d - h_u) - \frac{\alpha_d}{\beta}h_u, \quad (5)$$

$$E^{\frac{I_d}{1-a_d}} = \frac{1}{\beta} - \frac{\hat{c}_d(1+\frac{\alpha_d}{\beta}+\frac{\alpha_u}{\beta})}{\alpha_d(p-c_d)}. \quad (6)$$

The optimal RMI and reserve capacity levels at the upstream stage satisfy:

$$0 = \alpha_u\left(\Pi(\frac{I_d+I_u}{1-a_u})(p-c_u) + \Pi(I_d+I_u)c_u + E^{I_d+I_u}h_u - p\right) + \big(1+\frac{\alpha_d}{\beta}\big)h_u, \quad (7)$$

$$E^{\frac{I_d+I_u}{1-a_u}} = \frac{1}{\beta} - \frac{\hat{c}_u(1+\frac{\alpha_d}{\beta}+\frac{\alpha_u}{\beta})}{\alpha_u(p-c_u)} \quad (8)$$

with $E^{\frac{I_d+I_u}{1-a_u}} = \int_0^{\frac{I_d+I_u}{1-a_u}} \pi(k)k dk.$

Note that Eq. 5-8 can be solved in two steps. In the first step, Eq. 5 and 6 are solved simultaneously to determine $\frac{I_d}{1-a_d}$ and I_d. In the second step, Eq. 7 and 8 are solved simultaneously to determine $\frac{I_d+I_u}{1-a_u}$ and I_d+I_u. Observe that the optimal RMI levels are determined by balancing holding costs with penalty cost, while being modulated with the available reserve capacity.

Next, we introduce an approximation that allows for a simplification of the expressions governing the optimal RMI and optimal reserve capacities (Eq. 5 - 8). When a disruption takes place at any stage, observe that only RMI at the disrupted stage or further downstream can be used to meet disruption demand. All RMI upstream of the disrupted stage cannot be used to meet demand because the disrupted stage cannot transform upstream RMI. Because the disruption time is uncertain, there is some probability that not all of the available RMI at the disrupted stage or further downstream is used to meet disruption demand (for example when the disruption turns out to be short). In this case we incur inventory holding costs during the disruption time for this unused RMI at the disrupted stage or further downstream. Observe that in most instances we will have none or very little unused RMI at the disrupted stage or downstream in the event of a disruption. As a result, the holding cost will be negligible for this unused RMI. Under our suggested approximation, we ignore the inventory holding costs for any unused RMI during the disruption time that is carried at the disrupted stage or further downstream. Observe that we do not neglect the inventory holding costs for the unused RMI that is carried upstream of the disrupted stage because this will be a more significant quantity. This approximation is reasonable when disruption rates are low relative to recovery rates (see Assumption A.7). We refer to this approximation as the *low disruption/ high recovery rate* approximation. Under this approximation we can simplify Proposition 1 as follows:

PROPOSITION 2. *Let Assumptions 1-8 (A.1-A.8) hold. Under the low disruption/ high recovery rate approximation, the approximate RMI and reserve capacity levels at the downstream stage ($\tilde{I}_d$ and $\tilde{a}_d$ respectively) satisfy:*

$$0 = \alpha_d\Big((p-c_d)\Pi(\frac{\tilde{I}_d}{1-\tilde{a}_d}) - p + c_d\Pi(\tilde{I}_d)\Big) + h_d - h_u - \frac{\alpha_d}{\beta}h_u, \tag{9}$$

$$E^{\frac{\tilde{I}_d}{1-\tilde{a}_d}} = \frac{1}{\beta} - \frac{\hat{c}_d(1+\frac{\alpha_d}{\beta}+\frac{\alpha_u}{\beta})}{\alpha_d(p-c_d)}. \tag{10}$$

The approximate RMI and reserve capacity levels at the upstream stage ($\tilde{I}_u$ and $\tilde{a}_u$ respectively) satisfy:

$$0 = \alpha_u\left(\Pi(\frac{\tilde{I}_d+\tilde{I}_u}{1-\tilde{a}_u})(p-c_u) - p + c_u\Pi(\tilde{I}_d+\tilde{I}_u)\right) + h_u(1+\frac{\alpha_d}{\beta}h_u), \tag{11}$$

$$E^{\frac{\tilde{I}_d+\tilde{I}_u}{1-\tilde{a}_u}} = \frac{1}{\beta} - \frac{\hat{c}_u(1+\frac{\alpha_d}{\beta}+\frac{\alpha_u}{\beta})}{\alpha_u(p-c_u)}. \tag{12}$$

In Section 5, our numerical experiments show that this approximation is very close to optimal and provides robust insights. It is straight-forward to generalize Propositions 1 and 2 to n-stage supply chains.

4. Optimal location and quantity of RMI in multi-stage supply chains

In this section we study the optimal location and quantity of RMI in serial n-stage supply chains in the absence of reserve capacity. The optimal use of both RMI and reserve capacity in multi-stage supply chains is studied separately in Section 6.

Our main finding in this section is that in a serial supply chain where product is transformed at each stage, it often pays to push more RMI downstream even though RMI is cheaper to hold upstream (we refer to this phenomenon as downstream commitment to RMI). This finding is driven primarily by the observation that in a serial supply chain with product transformation at each stage, downstream RMI can help buffer against disruptions at all upstream stages. In other words, even though downstream RMI is more expensive to hold, it may compensate for the higher holding cost by being useful no matter whether the disruption is downstream or upstream. We also discuss conditions when there is no downstream commitment to RMI. In general, the cheaper upstream holding costs become, the less likely we are to observe downstream commitment at optimality.

All insights in this section are based on assumptions A.1-A.8 and the low disruption / high recovery rate approximation (Proposition 2). In Section 5, we validate our approximation and the key findings of this section numerically. We denote the *effective* inventory holding cost at stage m as $\bar{h}_m = h_m(1+\sum_{j=1}^{m-1}\frac{\alpha_j}{\beta})$. Further, we assume that the penalty cost is high enough (there exists a stage m such that $\bar{h}_m < p\alpha_m$) to ensure that we hold RMI at at least one stage.

We define a *gap* in the supply chain as a set of consecutive stages where it is optimal to not hold any RMI.

Definition 1. Consider a n-stage supply chain. We say that stage $m \in \{1,..,n-1\}$ is followed by a gap upstream of size $g \in \{1,...,n-m\}$ if it is optimal to hold RMI at stage m, i.e., $I_m > 0$, while holding no RMI at stages $\{e \mid m < e \leq m+g\}$, i.e., $I_e = 0$ for $\{e \mid m < e \leq m+g\}$. A stage m is followed upstream by a gap of size $g = 0$ if $I_m > 0$ and $I_{m+1} > 0$.

In Figure 3 we illustrate a gap of size 1 upstream of stage 1.

We find that the occurrence of gaps in the supply chain is primarily driven by the ratio of inventory holding costs between adjacent stages.

Lemma 1. *Let Assumption 1-8 (A.1-A.8) hold. Let $m \in \{1,2,...,n-2\}$ be any stage where it is optimal to hold RMI, $\tilde{I}_m > 0$. First consider the case where m is not the most upstream stage carrying RMI, i.e., there exists some stage $\bar{m} > m+1$ that also carries RMI. Then, under the low disruption/ high recovery rate approximation, stage m is followed by a gap upstream of size $0 < g \leq n-m-1$ (i.e., stages $m+1$, $m+2$,..., $m+g$ carry no RMI, but stage $m+g+1$ carries*

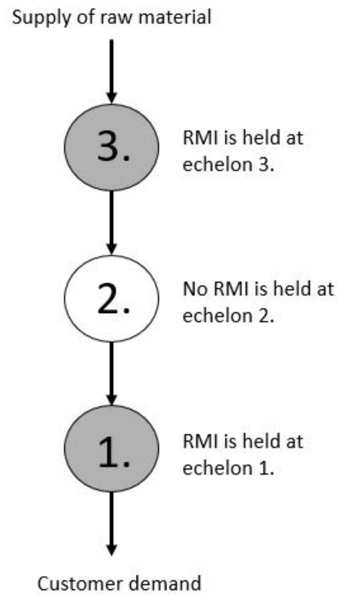

Figure 3 **Stage 1 is followed by a gap of size 1.**

RMI, with $g>0$) and stage $m+g+1$ is followed by a gap upstream of size $\hat{g}$, where $\hat{g}$ could be 0 (i.e., stage $m+g+1$ may or may not be followed upstream by a gap) iff

$$\frac{\bar{h}_m}{\bar{h}_{m+q}} \leq \frac{\sum_{e=m}^{m+g} \alpha_e - \frac{\bar{h}_{m+g+1}}{\bar{h}_{m+q}} \sum_{e=m}^{m+q-1} \alpha_e}{\sum_{e=m+q}^{m+g} \alpha_e} \quad \forall q \in \{1,2,...,g\} \tag{13}$$

$$\text{and} \quad \frac{\bar{h}_m}{\bar{h}_{m+g+1}} > \frac{\sum_{e=m}^{m+g+\hat{g}+1} \alpha_e - \frac{\bar{h}_{m+g+\hat{g}+2}}{\bar{h}_{m+g+1}} \sum_{e=m}^{m+g} \alpha_e}{\sum_{e=m+g+1}^{m+g+\hat{g}+1} \alpha_e} \tag{14}$$

where $\bar{h}_i = 0$ if $i > n$. If $m \in \{1,2,...,n-1\}$ is the most upstream stage carrying RMI, i.e., $m+g=n$, then, under the low disruption/ high recovery rate approximation, stage m is followed by a gap upstream of size g iff condition Eq. 13 holds.

The intuition behind the lemma is that gaps occur when effective inventory holding costs do not decrease much across adjacent stages as we move upstream. If there is only a small difference in effective inventory holding costs between adjacent stages (the fraction $\frac{\bar{h}_m}{\bar{h}_{m+q}}$ is close to 1), it is better to hold RMI only downstream rather than both upstream and downstream (resulting in the occurrence of a gap) because the downstream RMI can be used in the event of an up- or downstream disruption whereas the upstream RMI can only be used in the event of an upstream disruption. Note that Eq. 13 and 14 are equivalent to $\frac{\bar{h}_m}{\bar{h}_{m+q}} \leq \frac{g+1}{g+1-q}$ $\forall q \in \{1, 2, ..., g\}$ if the last stage of the gap is also the last stage of the supply chain ($\bar{h}_{m+g+1} = 0$) and if disruption rates are identical across stages. For example, for a two-stage supply chain with identical disruption rates up- and downstream, it is optimal to not hold RMI upstream when the effective inventory holding cost upstream is at least half as expensive as downstream. Even when downstream RMI is almost twice as expensive to hold, we hold no upstream RMI (holding all RMI downstream) because the downstream RMI helps buffer against a disruption at either stage. The additional value it creates by buffering against either disruption more than compensates for the higher holding cost.

The following theorem shows that if the inventory holding cost at the most upstream stage is not too low, the optimal quantity of RMI at the most downstream stage that carries RMI is larger than the sum of the RMI quantities held at all other stages. We refer to this finding as *global downstream commitment* to RMI.

Theorem 1. *Let Assumption 1-8 (A.1-A.8) hold. Let stage $m^* \in \{1, 2, ..., n-1\}$ be the most downstream stage that carries RMI, i.e., no RMI is held at stages 1,2,...,$m^* - 1$. Let stage m^* be followed by a gap upstream of size g, where g could be 0 (i.e., stage m^* may or may not be followed by a gap upstream). Then, under the low disruption/ high recovery rate approximation, RMI held at stage m^* is larger than the sum of RMI held at all stages upstream of stage m^* ($\tilde{I}_{m^*} > \sum_{j=m^*+1}^{n} \tilde{I}_j$) iff the effective inventory holding cost at the most upstream stage n is not too low, i.e., $\frac{(\bar{h}_{m^*} - \bar{h}_{m^*+g+1})^2 \alpha_n}{p(\sum_{e=m^*}^{m^*+g} \alpha_e)^2} < \bar{h}_n$ (where $\bar{h}_{m^*+g+1} = 0$ if $m^* + g = n$).*

The logic for global downstream commitment can be explained as follows. The RMI at the most downstream stage holding RMI can be used in the event of a disruption at any stage in the supply chain. Thus, it makes sense to hold more RMI downstream because it contributes more to improving RMI service level in the event of a disruption. However, this rationale breaks down when the upstream inventory holding cost at stage n is below a certain threshold ($\frac{(\bar{h}_1 - \bar{h}_{2+g})^2 \alpha_n}{p(\sum_{e=1}^{1+g} \alpha_e)^2} > \bar{h}_n$).

Theorem 1 translates into the following managerial insight:

Insight I: Holding the majority of RMI at the most downstream stage is the most effective way of dealing with disruptions if the inventory holding cost at the most upstream stage is not too low.

In the pharmaceutical industry, there are many settings where the inventory holding cost at the most upstream stage is not too small. In particular, this holds whenever the active ingredient (which tends to be the most expensive part of most drugs) is produced at the most upstream stage.

Clearly, the condition that the inventory holding cost of the most upstream stage is not too low is not always satisfied. Consider for example a manufacturer who holds large quantities of cheap raw-material upstream. In such cases a global downstream commitment to RMI may not be optimal anymore. However, it turns out that on a local scale a downstream commitment to RMI may still be optimal in a somewhat stylized setting.

In order to describe this local downstream commitment to RMI, recall that in general it is not necessarily optimal to hold RMI at each stage of the supply chain (occurrences of gaps). Instead, it may be optimal to hold RMI only at a subset of stages in the supply chain. We refer to a band as a set of stages that are adjacent to each other and all hold RMI.

Let us compare the optimal RMI quantities held at adjacent stages within a band. We find that as long as the difference in holding costs across adjacent stages is not too large, it is optimal to hold more RMI at a downstream stage than at the adjacent upstream stage- referred to as *local downstream commitment* to RMI.

THEOREM 2. *Let Assumption 1-8 (A.1-A.8) hold. Let* $m \in \{2,3,...,n-1\}$ *and* $m+1$ *be any two adjacent stages where it is optimal to hold RMI,* $\tilde{I}_m > 0$ *and* $\tilde{I}_{m+1} > 0$*. Let stage* $m+1$ *be followed by a gap upstream of size* g*, where* g *could be 0 (i.e., stage* $m+1$ *may or may not be followed by a gap upstream). Let stage* m *be preceded downstream by a gap of size* $\hat{g}$*, where* $\hat{g}$ *could be 0 (i.e., stage* m *may or may not be preceded downstream by a gap). Let* $\Delta_{m,g} = \frac{\bar{h}_m - \bar{h}_{m+g+1}}{\sum_{e=m}^{m+g} \alpha_e}$ *(where* $\bar{h}_i = 0$ *if* $i > n$*) satisfy* $\Delta^2_{m,0} < \Delta_{m+1,g}\Delta_{m-\hat{g}-1,\hat{g}}$*, i.e., the difference in holding costs between stages* m *and* $m+1$ *is not too large. Then, under the low disruption/ high recovery rate approximation, RMI held at stage* m*,* $\tilde{I}_m$*, is larger than RMI held at the adjacent upstream stage* $m+1$*,* $\tilde{I}_{m+1}$*, i.e.,* $\tilde{I}_m > \tilde{I}_{m+1}$.

The theorem translate into the following managerial insight:

Insight II: In multi-stage supply chains it makes sense to group adjacent stages together where the inventory holding costs do not deviate too much from each other (but are still increasing as we move downstream). Within such a group of stages, it is optimal to hold more RMI at any stage compared to its adjacent upstream stage.

Let us illustrate Theorem 1 and 2 with a numerical example.

EXAMPLE 1. We provide a numerical example illustrating both local and global downstream commitment to RMI for a four-stage supply chain (see Figure 4).

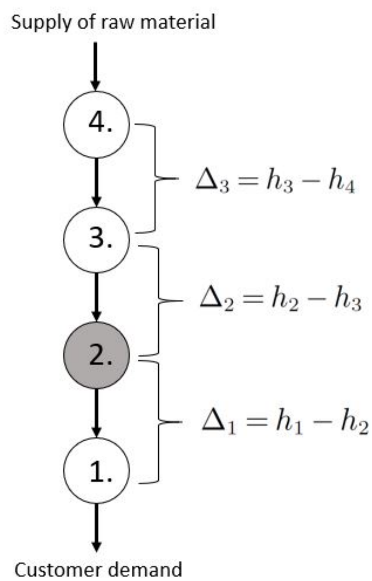

Figure 4 When does downstream commitment occur at stage 2?

The optimal RMI levels are (see proof of Theorem 2 for details of mathematical derivation):

$$\tilde{I}_1^* = -\frac{1}{\beta}\ln\left(\frac{\bar{h}_1 - \bar{h}_2}{p\alpha_1}\right)$$
$$\tilde{I}_2^* = -\frac{1}{\beta}\ln\left(\frac{\bar{h}_2 - \bar{h}_3}{p\alpha_2}\right) - I_1^*$$
$$\tilde{I}_3^* = -\frac{1}{\beta}\ln\left(\frac{\bar{h}_3 - \bar{h}_4}{p\alpha_3}\right) - I_1^* - I_2^*$$
$$\tilde{I}_4^* = -\frac{1}{\beta}\ln\left(\frac{\bar{h}_4}{p\alpha_4}\right) - I_1^* - I_2^* - I_3^*$$

Using the numerical values $p = 50,\ \beta = 0.1,\ \alpha_1 = 0.010,\ \alpha_2 = 0.012,\ \alpha_3 = 0.014,\ \alpha_4 = 0.016\ h_1 = 1.00,\ h_2 = 0.45,\ h_3 = 0.25,\ h_4 = 0.11$, we calculate: $\tilde{I}_1^* = 0,\ \tilde{I}_2^* = 11.6,\ \tilde{I}_3^* = 3.6,\ \tilde{I}_4^* = 1.7$. Clearly, we have a band consisting of stages 2, 3 and 4. Within this band we observe local downstream commitment to RMI at stages 2 and 3 ($\tilde{I}_2^* > \tilde{I}_3^*$ and $\tilde{I}_3^* > \tilde{I}_4^*$) even though holding costs decrease as we move upstream. At stage 3 we hold more RMI than at stage 4 (even though holding cost at stage 4 is lower) because the RMI at stage 3 can be used in the event of a disruption at either stage 3 or 4, whereas the RMI at stage 4 can only be used in the event of a disruption at stage 4. A similar argument holds for local downstream commitment at stage 2.

In addition to local downstream commitment we observe a global downstream commitment to RMI ($\tilde{I}_2 > \tilde{I}_3 + \tilde{I}_4$) because the upstream inventory holding cost ($h_4 = 0.12$) is larger than $\frac{(\bar{h}_2 - \bar{h}_3)^2 \alpha_4}{p\alpha_2^2} = 0.08$ (see Theorem 1). Even though h_4 is smaller than h_3 (and h_2), it is relatively large enough to justify global downstream commitment to RMI at optimality.

5. Numerical validation of the low disruption/ high recovery rate approximation

In this section we numerically evaluate the performance of the low disruption/ high recovery rate approximation relative to the optimal solution.

To perform the numerical calculation we first rewrite the conditions for the optimal RMI levels (as characterized by Proposition 1) using the Lambert W function $W_{-1}(\cdot)$ (see Corless et al. (1996) for further details). From the calculation in Example 2 (see Appendix B) we can deduce the optimal RMI levels in closed form:

$$I_d = -\frac{1}{\beta} W_{-1}\Big(\frac{-(h_d - h_u)\beta + \alpha_d h_u - \alpha_d h_d - \alpha_u (h_d - h_u)}{\Big(\alpha_d h_d + \alpha_u (h_d - h_u)\Big) \exp\Big(1 + \frac{\beta p \alpha_d}{\alpha_d h_d + \alpha_u (h_d - h_u)}\Big)} \Big) - \frac{1}{\beta} - \frac{p\alpha_d}{\alpha_d h_d + \alpha_u (h_d - h_u)} \quad (15)$$

and

$$I_d + I_u = -\frac{1}{\beta} W_{-1}\Big(- \frac{\big(1 + \frac{\alpha_d}{\beta}\big) h_u + \frac{\alpha_u h_u}{\beta}}{\frac{h_u \alpha_u}{\beta} \exp\big(1 + \frac{\beta p}{h_u}\big)} \Big) - \frac{1}{\beta} - \frac{p}{h_u}. \quad (16)$$

We then calculate the optimal RMI levels with MATLAB using the command *lambertw*.

First, we calculate the long-run expected cost per time for both the optimal solution and our approximation: (i) $\mathbb{E}[C(I_u, I_d)]$ (with optimal I_d and I_u given by Eq. 15 and Eq. 16) and (ii) $\mathbb{E}[C(\tilde{I}_u, \tilde{I}_d)]$ (with approximate $\tilde{I}_d$ and $\tilde{I}_u$ given by Proposition 2). In Figure 5.a we plot the relative

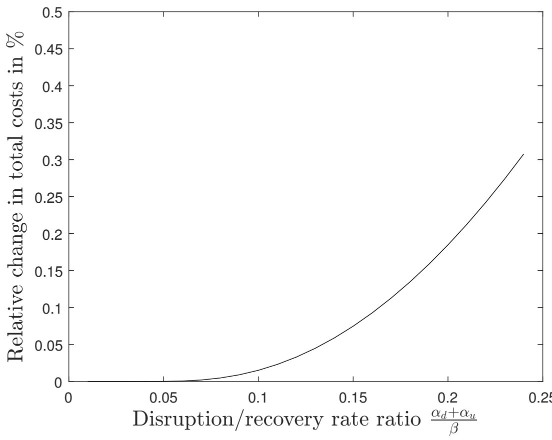

Figure 5.a **Change in total cost as a function of $\frac{\alpha_d+\alpha_u}{\beta}$**

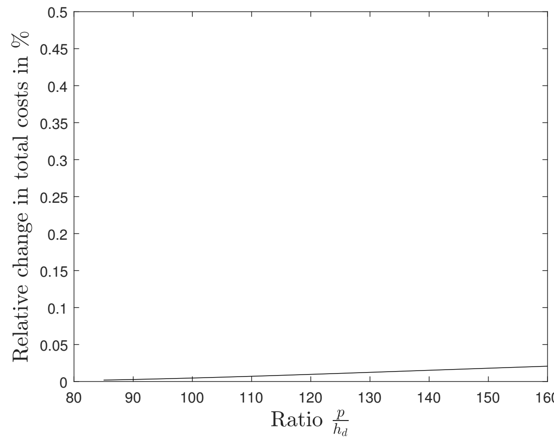

Figure 5.b **Change in total cost as a function of $\frac{p}{h_d}$**

change in expected total cost, $\frac{\mathbb{E}[C(\tilde{I}_u,\tilde{I}_d)]-\mathbb{E}[C(I_u,I_d)]}{\mathbb{E}[C(I_u,I_d)]}\times 100$, as a function of $\frac{\alpha_d+\alpha_u}{\beta}$ (keeping $\beta=0.2$ and $\alpha_u=\alpha_d$). Further parameters are: $h_d=1, h_u=0.4, p=140$. In Figure 5.b we plot the relative change in expected total cost, $\frac{\mathbb{E}[C(\tilde{I}_u,\tilde{I}_d)]-\mathbb{E}[C(I_u,I_d)]}{\mathbb{E}[C(I_u,I_d)]}\times 100$, as a function of $\frac{p}{h_d}$ (keeping $h_d=1$). Further parameters are: $\beta=0.2, \alpha_d=\alpha_u=0.01, h_u=0.4$.

Figures 5.a and 5.b show that the expected cost from our approximation is very close to the optimal expected cost. Observe that in both figures the relative difference in the long-run expected cost per time is less than 0.4%. We have conducted exhaustive numerical experiments by varying all parameter combinations and find consistent results. Our numerical experiments indicate that the approximation performs very well even when disruption rates are relatively high compared to recovery rates.

Second, we want to evaluate if the commitment to downstream RMI (which we proved using the low disruption/ high recovery rate approximation) is preserved at the true optimal solution. In Figure 6.a we plot the downstream RMI as a fraction of the total RMI for the true optimum ($\frac{I_d}{I_d+I_u}$) and the approximate optimum ($\frac{\tilde{I}_d}{\tilde{I}_d+\tilde{I}_u}$), as a function of $\frac{\alpha_d+\alpha_u}{\beta}$ (keeping $\beta=0.2$ and $\alpha_u=\alpha_d$). Further parameters are: $h_d=1, h_u=0.4, p=140$. The unbroken line corresponds to the true optimum and the broken line to the approximate optimum. In Figure 6.a we observe that for $\frac{\alpha_d+\alpha_u}{\beta}\geq 0.06$ it is optimal to hold more RMI downstream than upstream ($\frac{I_d}{I_d+I_u}>0.5$).

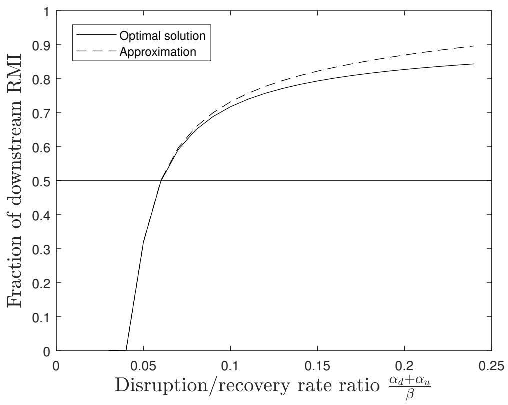

Figure 6.a **Fraction of downstream RMI**

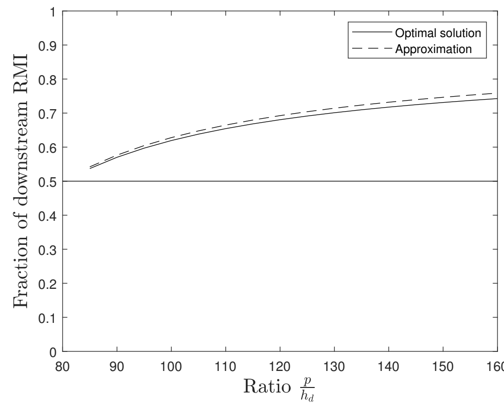

Figure 6.b **Fraction of downstream RMI**

In Figure 6.b we plot the downstream RMI as a fraction of the total RMI for the true optimum ($\frac{I_d}{I_d+I_u}$) and the approximate optimum ($\frac{\tilde{I}_d}{\tilde{I}_d+\tilde{I}_u}$), as a function of $\frac{p}{h_d}$ (keeping $h_d = 1$). Further parameters are: $\beta = 0.2, \alpha_d = \alpha_u = 0.01, h_u = 0.4$. The unbroken line corresponds to the true optimum and the broken line to the approximate optimum. Once again, we focus on the range $\frac{p}{h_d} \geq 85$ where it is optimal to hold RMI both up- and downstream. Observe that RMI held downstream is larger than RMI held upstream ($\frac{I_d}{I_d+I_u} > 0.5$).

Observe that the fraction of downstream RMI using our approximation is fairly close to the optimal fraction. Our approximation, however, slightly overestimates the fraction of downstream RMI. Fundamentally, the overestimation is driven by the fact that our approximation underestimates downstream inventory holdings costs (by ignoring holding costs downstream in the event of a disruption). Observe, that the overestimation of downstream RMI is low for reasonable penalty costs and disruption/recovery rate ratios, but becomes somewhat more pronounced at very high values of p and $\frac{\alpha_d+\alpha_u}{\beta}$. Our findings regarding the commitment to downstream RMI, however, continue to hold.

6. Optimal location and quantity of reserve capacity and RMI in multi-stage supply chains

In this section we study the joint use of RMI and reserve capacity in serial supply chains. We follow the same set of assumptions as in Section 4 (A.1-A.8).

Similar to downstream commitment to RMI when upstream holding costs are not too small, we find that it is optimal to hold more reserve capacity downstream than upstream as long as the reserve capacity costs upstream are not much cheaper than downstream and as long as disruption rates upstream are not higher than downstream. Fundamentally, this finding is driven by the fact that when some RMI is held upstream, downstream reserve capacity is used more often than upstream reserve capacity. In the event of a disruption at the downstream stage, we first use the downstream RMI (to save on holding cost) and then the downstream reserve capacity to meet customer demand. In the event of an upstream disruption, we first use both down- and upstream RMI (to save on holding cost) and then the upstream reserve capacity to meet demand. Thus, the presence of upstream RMI reduces the need for upstream reserve capacity while maintaining the need for downstream reserve capacity. Thus, more downstream reserve capacity is likely to be used than upstream reserve capacity. In Appendix B we provide a formal proof of this statement for the two-stage supply chain.

Tomlin (2006) has shown that in one-stage supply chains, RMI and reserve capacity always behave as substitutes. A goal in this section is to understand how RMI and reserve capacity interact in multi-stage supply chains with transformation.

Define *echelon* RMI at stage $m = \sum_{j=1}^{m} I_j$, as the sum of all downstream RMI up to stage m. We generalize the result in Tomlin (2006) and find that echelon RMI and reserve capacity at any stage of a multi-stage supply chain behave like substitutes.

THEOREM 3. *Let Assumption 1-8 (A.1-A.8) hold. Echelon RMI held at any stage and the reserve capacity held at the same stage are substitutes, i.e., an increase in echelon RMI held at any stage reduces the marginal value of reserve capacity at the same stage.*

As a result, if we increase (decrease) echelon inventory at any stage, we should decrease (increase) the reserve capacity at that stage.

Next, we show that across adjacent stages, RMI and reserve capacity behave as complements.

THEOREM 4. *Let Assumption 1-8 (A.1-A.8) hold. RMI held at any stage m is complementary to the reserve capacity held at the adjacent downstream stage $m-1$, i.e., an increase in RMI held at any stage increases the marginal value of reserve capacity at the adjacent downstream stage.*

An increase in RMI at any stage m reduces the need for echelon RMI at the adjacent downstream stage $m-1$ because less echelon RMI is needed at $m-1$ to mitigate upstream disruptions. By Theorem 3, the reduction of echelon RMI downstream at $m-1$ increases the need for reserve capacity at stage $m-1$. As a result, an increase in RMI at any stage results in an increase in reserve capacity at the adjacent downstream stage. Likewise, an increase in reserve capacity at any stage m reduces the need for echelon RMI at the same stage (by Theorem 3). The reduction of echelon RMI at stage m increases the need for RMI at the upstream stage $m+1$ because the downstream echelon RMI would have been helpful in the event of an upstream disruption. As a result, an increase in reserve capacity at any stage m results in an increase in RMI at the adjacent upstream stage $m+1$.

7. Conclusion

In this paper, we have analyzed the location and quantity of RMI and reserve capacity held in serial multi-stage supply chains with product transformation at each stage. We find that under reasonable conditions it is better to hold more RMI downstream than upstream even when the upstream holding costs are lower. We also find that it is often optimal to hold more reserve capacity downstream than upstream. A key contribution of this research is to provide an economic rationale for the occurrence of this commitment to downstream RMI and downstream reserve capacity. We further find that echelon RMI and reserve capacity at a stage behave as substitutes in multi-stage supply chains. In contrast, the RMI at any stage and the reserve capacity at the adjacent downstream stage behave as complements.

The research arose out of a collaboration with a leading pharmaceutical company. Based on our research, a key take-away for the company is to focus their risk mitigation effort on the downstream stages. Even though RMI and reserve capacity are cheaper upstream, holding more RMI and reserve capacity downstream is more cost effective in the long term.

References

B. Tomlin. On the value of mitigation and contingency strategies for managing supply chain disruption risks. *Management Science*, 52:639–657, 2006.

E. Ang, D.A. Iancu, and R. Swinney. Disruption risk and optimal sourcing in multitier supply networks. *Management Science*, 63 (8):2397–2419, 2017.

A.J. Schmitt and M. Singh. A quantitative analysis of disruption risk in a multi-echelon supply chain. *International Journal of Production Economics*, 139:22–32, 2012.

A.J. Clark and H. Scarf. Optimal policies for a multi-echelon inventory problem. *Management Science*, 5: 475–490, 1960.

A. Federgruen and P. Zipkin. Approximations of dynamic, multilocation production and inventory problems. *Management Science*, 30 (1):69–84, 1984.

F. Chen and YS. Zheng. Lower bounds for multi-echelon stochastic inventory systems. *Management Science*, 40 (11):1426–1443, 1994.

S.C. Graves. A multi-echelon inventory model for a repairable item with one-for-one replenishment. *Management Science*, 31:1247–1256, 1985.

K. Rosling. Optimal inventory policies for assembly systems under random demands. *Operations Research*, 37 (4):565–579, 1989.

C.P. Schmidt and S. Nahmias. Optimal policy for a two-stage assembly system under random demand. *Operations Research*, 33 (5):1130–1145, 1985.

G.A. DeCroix. Inventory management for an assembly system subject to supply disruptions. *Management Science*, 59 (9):2079–2092, 2013.

S. Chopra, G. Reinhardt, and U. Mohan. The importance of decoupling recurrent and disruption risks in a supply chain. *Naval Research Logistics*, 54 (5):544–555, 2007.

L. Qi. A continuous-review inventory model with random disruptions at the primary supplier. *European Journal of Operational Research*, 225:59–74, 2013.

L. Qi and K. Lee. Supply chain risk mitigations with expedited shipping. *Omega*, 57:98–113, 2015.

G. Schorpp, F. Erhun, and H.L. Lee. Multi-tiered supply chain risk management. *Working Paper*, 2018.

W.J. Hopp and Z. Yin. Protecting supply chain networks against catastrophic failures. *Working paper Wisconsin School of Business*, 2010.

L. Dong, S.Y. Tang, and B. Tomlin. Production chain disruptions: Inventory, preparedness, and insurance. *Production and Operations Management, forthcoming*, 2018.

M.K. Lim, A. Bassamboo, and S. Chopra. Facility location decisions with random disruptions and imperfect estimation. *Manufacturing & Service Operations Management*, 15 (2):239–249, 2013.

L.V. Snyder, Z. Atan, P. Peng, Y. Rong, A.J. Schmitt, and B. Sinsoysal. Or/ms models for supply chain disruptions: A review. *IIE Transactions*, 48(2):89–109, 2016.

ICH. Good manufacturing practice guide for active pharmaceutical ingredients. *https://www.ich.org/fileadmin/Public_Web_Site/ICH_Products/Guidelines/Quality/Q7/Step4/Q7_Guideline.pdf*, 2000.

M. Parlar and D. Perry. Inventory models of future supply uncertaintiy with single and multiple suppliers. *Naval Research Logistics*, 43:191–210, 1996.

R. Samii and L. N. Van Wassenhove. Fighting the flu. tamiflu stockpiling: A pandemic preparedness policy. *INSEAD Case Study*, 2008.

R. M. Corless, G. H. Gonnet, D. J. Hare, D. E. G.and Jeffrey, and D. E. Knuth. On the lambertw function. *Advances in Computational Mathematics*, 5:329–359, 1996.

Appendix A: Proofs

A.1. Proof of Proposition 1 for the two-stage supply chain

As described in Section 3, the long-run expected cost per cycle, $\mathbb{E}[C(I_u, I_d, a_u, a_d)]$, is given by (combining Eq. 1- 4):

$$\begin{aligned}
\mathbb{E}[C(I_u, I_d, a_u, a_d)] = \frac{\frac{1}{\alpha_d+\alpha_u}}{\frac{1}{\alpha_d+\alpha_u}+\frac{1}{\beta}}\Bigg[& h_d I_d + h_u I_u + \hat{c}_u a_u + \hat{c}_d a_d \\
&+\alpha_d\Bigg(p\int_{\frac{I_d}{1-a_d}}^{\infty}\pi(k)\big((1-a_d)k - I_d\big)dk + h_d\int_0^{I_d}\pi(k)(I_d-k)kdk \\
&+c_d\int_{I_d}^{\frac{I_d}{1-a_d}}\pi(k)(k-I_d)dk + c_d\int_{\frac{I_d}{1-a_d}}^{\infty}\pi(k)a_d kdk + \frac{1}{\beta}\big(h_u I_u + \hat{c}_u a_u + \hat{c}_d a_d\big)\Bigg) \\
&+\alpha_u\Bigg(p\int_{\frac{I_d+I_u}{1-a_u}}^{\infty}\pi(k)\big((1-a_u)k - I_d - I_u\big)dk + h_u\int_{I_d}^{I_d+I_u}\pi(k)\big(I_d+I_u-k\big)kdk \\
&+\int_0^{I_d}\pi(k)\Big(h_d\big(I_d-k\big)+h_u I_u\Big)kdk + c_u\int_{I_d+I_u}^{\frac{I_d+I_u}{1-a_u}}\pi(k)(k-I_d-I_u)dk \\
&+c_u\int_{\frac{I_d+I_u}{1-a_u}}^{\infty}\pi(k)a_u kdk + \frac{1}{\beta}\big(\hat{c}_u a_u + \hat{c}_d a_d\big)\Bigg)\Bigg]
\end{aligned} \tag{17}$$

The first order conditions with respect to α_d, α_u, I_d and I_u are:

$$\begin{aligned}
0 &= \frac{\partial\mathbb{E}[C(I_u, I_d, a_u, a_d)]}{\partial a_d} \quad \Leftrightarrow \\
0 &= \alpha_d\Big(-p\int_{\frac{I_d}{1-a_d}}^{\infty}\pi(k)kdk + c_d\int_{\frac{I_d}{1-a_d}}^{\infty}\pi(k)kdk\Big) + \hat{c}_d(1+\frac{\alpha_d}{\beta}+\frac{\alpha_u}{\beta}) \quad \Leftrightarrow \\
E^{\frac{I_d}{1-a_d}} &= \frac{1}{\beta} - \frac{\hat{c}_d(1+\frac{\alpha_d}{\beta}+\frac{\alpha_u}{\beta})}{\alpha_d(p-c_d)},
\end{aligned}$$

which is equivalent to Eq. 6 in Proposition 1.

$$\begin{aligned}
0 &= \frac{\partial\mathbb{E}[C(I_u, I_d, a_u, a_d)]}{\partial a_u} \quad \Leftrightarrow \\
0 &= \alpha_u\Big(-p\int_{\frac{I_d+I_u}{1-a_u}}^{\infty}\pi(k)kdk + c_u\int_{\frac{I_d+I_u}{1-a_u}}^{\infty}\pi(k)kdk\Big) + \hat{c}_u(1+\frac{\alpha_d}{\beta}+\frac{\alpha_u}{\beta}) \quad \Leftrightarrow \\
E^{\frac{I_d+I_u}{1-a_u}} &= \frac{1}{\beta} - \frac{\hat{c}_u(1+\frac{\alpha_d}{\beta}+\frac{\alpha_u}{\beta})}{\alpha_u(p-c_u)},
\end{aligned}$$

which is equivalent to Eq. 8 in Proposition 1.

$$\begin{aligned}
0 &= \frac{\partial\mathbb{E}[C(I_u, I_d, a_u, a_d)]}{\partial I_d} \quad \Leftrightarrow \\
0 &= h_d + \alpha_d\Bigg(-p\int_{\frac{I_d}{1-a_d}}^{\infty}\pi(k)dk + h_d\int_0^{I_d}\pi(k)kdk - c_d\int_{I_d}^{\frac{I_d}{1-a_d}}\pi(k)dk\Bigg) \\
&\quad +\alpha_u\Bigg(-p\int_{\frac{I_d+I_u}{1-a_u}}^{\infty}\pi(k)dk + h_u\int_{I_d}^{I_d+I_u}\pi(k)kdk + h_d\int_0^{I_d}\pi(k)kdk - c_u\int_{I_d+I_u}^{\frac{I_d+I_u}{1-a_u}}\pi(k)dk\Bigg).
\end{aligned} \tag{18}$$

$$
\begin{aligned}
0 &= \frac{\partial \mathbb{E}[C(I_u, I_d, a_u, a_d)]}{\partial I_u} \qquad \Leftrightarrow \\
0 &= \alpha_u \Bigg(-p \int_{\frac{I_d+I_u}{1-a_u}}^{\infty} \pi(k)dk + h_u \int_{I_d}^{I_d+I_u} \pi(k)kdk + h_u \int_0^{I_d} \pi(k)kdk - c_u \int_{I_d+I_u}^{\frac{I_d+I_u}{1-a_u}} \pi(k)dk \Bigg) + \big(1 + \frac{\alpha_d}{\beta}\big)h_u \quad \Leftrightarrow \\
0 &= \alpha_u \Bigg(-p \int_{\frac{I_d+I_u}{1-a_u}}^{\infty} \pi(k)dk + h_u \int_0^{I_d+I_u} \pi(k)kdk - c_u \int_{I_d+I_u}^{\frac{I_d+I_u}{1-a_u}} \pi(k)dk \Bigg) + \big(1 + \frac{\alpha_d}{\beta}\big)h_u \qquad \Leftrightarrow \\
0 &= \alpha_u \Bigg(\Pi(\frac{I_d+I_u}{1-a_u})(p-c_u) + \Pi(I_d+I_u)c_u + E^{I_d+I_u}h_u - p \Bigg) + \big(1 + \frac{\alpha_d}{\beta}\big)h_u,
\end{aligned}
$$

which is equivalent to Eq. 7 in Proposition 1. From Eq. 18 and Eq. 7 we get:

$$
0 = \alpha_d \Big((p-c_d)\Pi(\frac{I_d}{1-a_d}) - p + h_d E^{I_d} + c_d \Pi(I_d) \Big) + \alpha_u (h_d - h_u) E^{I_d} + (h_d - h_u) - \frac{\alpha_d}{\beta} h_u,
$$

which is equivalent to Eq. 5 in Proposition 1. Note that the second order condition is satisfied. The result thus follows.

A.2. Proof of Proposition 2 for the two-stage supply chain

Using Eq. 17 we rewrite the long-run expected cost per cycle, $\mathbb{E}[C(I_u, I_d, a_u, a_d)]$, as follows:

$$
\begin{aligned}
\mathbb{E}[C(I_u, I_d, a_u, a_d)] = \frac{\frac{1}{\alpha_d+\alpha_u}}{\frac{1}{\alpha_d+\alpha_u} + \frac{1}{\beta}} \Bigg[& h_d I_d + h_u I_u + \hat{c}_u a_u + \hat{c}_d a_d \\
& + \alpha_d \Bigg(p \int_{\frac{I_d}{1-a_d}}^{\infty} \pi(k)\big((1-a_d)k - I_d\big)dk + c_d \int_{I_d}^{\frac{I_d}{1-a_d}} \pi(k)(k-I_d)dk + c_d \int_{\frac{I_d}{1-a_d}}^{\infty} \pi(k) a_d k dk \\
& + \frac{1}{\beta}(\hat{c}_d a_d + \hat{c}_u a_u + h_u I_u) \Bigg) + \alpha_u \Bigg(p \int_{\frac{I_d+I_u}{1-a_u}}^{\infty} \pi(k)\big((1-a_u)k - I_d - I_u\big)dk \\
& + c_u \int_{I_d+I_u}^{\frac{I_d+I_u}{1-a_u}} \pi(k)(k - I_d - I_u)dk + c_u \int_{\frac{I_d+I_u}{1-a_u}}^{\infty} \pi(k) a_u k dk + \frac{1}{\beta}(\hat{c}_d a_d + \hat{c}_u a_u) \Bigg) + \Delta \Bigg]
\end{aligned}
$$

with

$$
\Delta = \alpha_d \Big(h_d \int_0^{I_d} \pi(k)(I_d - k)kdk \Big) + \alpha_u \Big(h_u \int_{I_d}^{I_d+I_u} \pi(k)\big(I_d + I_u - k\big)kdk + \int_0^{I_d} \pi(k) \Big(h_d \big(I_d - k\big) + h_u I_u \Big) kdk \Big).
$$

Observe that Δ corresponds to the inventory holding costs during the disruption time for the unused RMI at the disrupted stage or further downstream. Under the low disruption/ high recovery rate assumption we set $\Delta = 0$. Using this approximation, the approximate RMI and reserve capacity levels at the downstream stage ($\tilde{I}_d$ and $\tilde{a}_d$ respectively) follow from a similar calculation as in Proposition 1:

$$
\begin{aligned}
0 &= \alpha_d \Big((p-c_d)\Pi(\frac{\tilde{I}_d}{1-\tilde{a}_d}) - p + c_d \Pi(\tilde{I}_d) \Big) + h_d - h_u - \frac{\alpha_d}{\beta} h_u, \\
E^{\frac{\tilde{I}_d}{1-\tilde{a}_d}} &= \frac{1}{\beta} - \frac{\hat{c}_d (1 + \frac{\alpha_d}{\beta} + \frac{\alpha_u}{\beta})}{\alpha_d (p - c_d)},
\end{aligned}
$$

which is equivalent to Eq. 9 and 10 in Proposition 2. The approximate RMI and reserve capacity levels at the upstream stage ($\tilde{I}_u$ and $\tilde{a}_u$ respectively) satisfy:

$$0 = \alpha_u\left(\Pi(\frac{\tilde{I}_d+\tilde{I}_u}{1-\tilde{a}_u})(p-c_u) - p + c_u\Pi(\tilde{I}_d+\tilde{I}_u)\right) + h_u(1+\frac{\alpha_d}{\beta}h_u),$$
$$E^{\frac{\tilde{I}_d+\tilde{I}_u}{1-\tilde{a}_u}} = \frac{1}{\beta} - \frac{\hat{c}_u(1+\frac{\alpha_d}{\beta}+\frac{\alpha_u}{\beta})}{\alpha_u(p-c_u)},$$

which is equivalent to Eq. 11 and 12 in Proposition 2. Note that the second order condition is satisfied. The result thus follows.

A.3. Proof of Proposition 1 for the n-stage supply chain

Recall that the long-run expected cost for the two-stage supply chain is given by Eq. 1:

$$\mathbb{E}[C(I_u,I_d,a_u,a_d)] = \frac{\frac{1}{\alpha_d+\alpha_u}\mathbb{E}[C_0(I_u,I_d,a_u,a_d)] + \frac{1}{\beta}(\frac{\alpha_d}{\alpha_d+\alpha_u}\mathbb{E}[C_d(I_u,I_d,a_u,a_d)] + \frac{\alpha_u}{\alpha_d+\alpha_u}\mathbb{E}[C_u(I_u,I_d,a_u,a_d)])}{\frac{1}{\alpha_d+\alpha_u}+\frac{1}{\beta}}$$

The long-run expected cost is generalized to the n-stage supply chain as follows:

$$\mathbb{E}[C] = \frac{\frac{1}{\sum_{i=1}^n\alpha_i}\mathbb{E}[C_0] + \frac{1}{\beta}(\sum_{e=1}^n\frac{\alpha_e}{\sum_{i=1}^n\alpha_i}\mathbb{E}[C_e])}{\frac{1}{\sum_{i=1}^n\alpha_i}+\frac{1}{\beta}}$$

where $\mathbb{E}[C_0]$ is the expected cost per unit of time when there is no disruption and $\mathbb{E}[C_e]$ is the expected cost per unit of time when there is a disruption at stage e. We have: $\mathbb{E}[C_0] = \sum_{e=1}^n(h_eI_e + \hat{c}_ea_e)$. The expected cost per unit of time when there is a disruption at stage e, $\mathbb{E}[C_e]$, is given by the sum of the following four terms:

1. $\beta p\int_{\frac{\sum_{j=1}^e I_j}{1-a_e}}^{\infty}\pi(k)\big((1-a_e)k - \sum_{j=1}^e I_j\big)dk$, represents the penalty cost per unit of time when disruption demand is larger than $\frac{\sum_{j=1}^e I_j}{1-a_e}$.
2. $\beta\sum_{l=1}^e\int_{\sum_{j=1}^{l-1}I_j}^{\sum_{j=1}^{l}I_j}\pi(k)\Big(h_l(\sum_{j=1}^l I_j - k) + \sum_{o=l+1}^e h_oI_o\Big)kdk$, represents the inventory holding costs per unit of time for unused RMI at any of the stages $l = 1,2,...,e$.
3. $\beta c_e\int_{\sum_{j=1}^e I_j}^{\frac{\sum_{j=1}^e I_j}{1-a_e}}\pi(k)(k - \sum_{j=1}^e I_j)dk + \beta c_e\int_{\frac{\sum_{j=1}^e I_j}{1-a_e}}^{\infty}\pi(k)a_ekdk$, represents the cost for using the reserve capacity per unit of time.
4. $\big(\sum_{o=1}^n\hat{c}_oa_o + \sum_{o=e+1}^n h_oI_0\big)$, represents the costs per unit of time for reserving the reserve capacity as well as inventory holding costs for unused RMI at the stages $e+1,...,n$.

Thus, we obtain:

$$\mathbb{E}[C] = \frac{\frac{1}{\alpha_d+\alpha_u}}{\frac{1}{\alpha_d+\alpha_u}+\frac{1}{\beta}}\left[\sum_{e=1}^n\alpha_e\left(p\int_{\frac{\sum_{j=1}^e I_j}{1-a_e}}^{\infty}\pi(k)\big((1-a_e)k - \sum_{j=1}^e I_j\big)dk\right.\right.$$

$$+\sum_{l=1}^{e}\int_{\sum_{j=1}^{l-1}I_j}^{\sum_{j=1}^{l}I_j}\pi(k)\Big(h_l(\sum_{j=1}^{l}I_j-k)+\sum_{o=l+1}^{e}h_oI_o\Big)kdk+c_e\int_{\sum_{j=1}^{e}I_j}^{\frac{\sum_{j=1}^{e}I_j}{1-a_e}}\pi(k)(k-\sum_{j=1}^{e}I_j)dk$$
$$+c_e\int_{\frac{\sum_{j=1}^{e}I_j}{1-a_e}}^{\infty}\pi(k)a_ekdk+\frac{1}{\beta}\Big(\sum_{o=1}^{n}\hat{c}_oa_o+\sum_{o=e+1}^{n}h_oI_o\Big)\Big)+\sum_{e=1}^{n}(h_eI_e+\hat{c}_ea_e)\Big]$$

Observe that for $n=2$ we get the long-run expected cost for 2 stages (Eq. 17).

The first order conditions with respect to a_m and I_m are given by:

$$\begin{aligned}0 &= \frac{\partial\mathbb{E}[C]}{\partial a_m} \quad\Leftrightarrow\\ 0 &= \alpha_m\Big(-p\int_{\frac{\sum_{j=1}^{m}I_j}{1-a_m}}^{\infty}\pi(k)kdk+c_m\int_{\frac{\sum_{j=1}^{m}I_j}{1-a_m}}^{\infty}\pi(k)kdk\Big)+\hat{c}_m(1+\sum_{o=1}^{n}\frac{\alpha_o}{\beta}) \quad\Leftrightarrow\\ E^{\frac{\sum_{j=1}^{m}I_j}{1-a_m}} &= \frac{1}{\beta}-\frac{\hat{c}_m(1+\sum_{o=1}^{n}\frac{\alpha_o}{\beta})}{\alpha_m(p-c_m)},\end{aligned} \tag{19}$$

which generalizes Eq. 6 and Eq. 8 in Proposition 1.

$$\begin{aligned}0 &= \frac{\partial\mathbb{E}[C]}{\partial I_m} \quad\Leftrightarrow\\ 0 &= \sum_{e=m}^{n}\alpha_e\Big(-p\int_{\frac{\sum_{j=1}^{e}I_j}{1-a_e}}^{\infty}\pi(k)dk+\sum_{l=m}^{e}h_l\int_{\sum_{j=1}^{l-1}I_j}^{\sum_{j=1}^{l}I_j}\pi(k)kdk+h_m\int_0^{\sum_{j=1}^{m-1}I_j}\pi(k)kdk\\ &\quad -c_e\int_{\sum_{j=1}^{e}I_j}^{\frac{\sum_{j=1}^{e}I_j}{1-a_e}}\pi(k)dk\Big)+h_m(1+\sum_{j=1}^{m-1}\frac{\alpha_j}{\beta}).\end{aligned}$$

Next, we rewrite this equation using the definition of the cumulative distribution function of the pdf $\pi(k)$, $\Pi(K)=\int_0^K\pi(k)dk$, and the definition $E^K=\int_0^K\pi(k)kdk$:

$$\begin{aligned}0 &= \sum_{e=m}^{n}\alpha_e\Big(-p\Big(1-\Pi(\frac{\sum_{j=1}^{e}I_j}{1-a_e})\Big)+\sum_{l=m}^{e}h_l(E^{\sum_{j=1}^{l}I_j}-E^{\sum_{j=1}^{l-1}I_j})+h_mE^{\sum_{j=1}^{m-1}I_j}\\ &\quad -c_e\Big(\Pi(\frac{\sum_{j=1}^{e}I_j}{1-a_e})-\Pi(\sum_{j=1}^{e}I_j)\Big)\Big)+h_m(1+\sum_{j=1}^{m-1}\frac{\alpha_j}{\beta}).\end{aligned}$$

Next, we use the identity $\sum_{l=m}^{e}h_l(E^{\sum_{j=1}^{l}I_j}-E^{\sum_{j=1}^{l-1}I_j})=-h_mE^{\sum_{j=1}^{m-1}I_j}+\sum_{l=m+1}^{e}E^{\sum_{j=1}^{l-1}I_j}\Big(h_{l-1}-h_l\Big)+h_eE^{\sum_{j=1}^{e}I_j}$. Thus:

$$\begin{aligned}0 &= \sum_{e=m}^{n}\alpha_e\Big(-p\Big(1-\Pi(\frac{\sum_{j=1}^{e}I_j}{1-a_e})\Big)-h_mE^{\sum_{j=1}^{m-1}I_j}+\sum_{l=m+1}^{e}E^{\sum_{j=1}^{l-1}I_j}\Big(h_{l-1}-h_l\Big)+h_eE^{\sum_{j=1}^{e}I_j}\\ &\quad +h_mE^{\sum_{j=1}^{m-1}I_j}-c_e\Big(\Pi(\frac{\sum_{j=1}^{e}I_j}{1-a_e})-\Pi(\sum_{j=1}^{e}I_j)\Big)\Big)+h_m(1+\sum_{j=1}^{m-1}\frac{\alpha_j}{\beta}).\end{aligned}$$

We rearrange some terms:

$$\begin{aligned}0 &= \sum_{e=m}^{n}\alpha_e\Big((p-c_e)\Pi(\frac{\sum_{j=1}^{e}I_j}{1-a_e})+c_e\Pi(\sum_{j=1}^{e}I_j)+h_eE^{\sum_{j=1}^{e}I_j}+\sum_{l=m+1}^{e}E^{\sum_{j=1}^{l-1}I_j}\big(h_{l-1}-h_l\big)\Big)\\ &\quad +h_m(1+\sum_{j=1}^{m-1}\frac{\alpha_j}{\beta})-p\sum_{e=m}^{n}\alpha_e.\end{aligned} \tag{20}$$

Without loss of generality we assume that there is a gap of size g upstream of stage m where g could be 0. Let us differentiate between the following two cases: First, consider the case where m is not the most upstream stage carrying RMI, i.e., stage $m+g+1 \leq n$ carries RMI, but all stages in between do not carry RMI. Second, consider the case where m is the most upstream stage carrying RMI, i.e., stages $m+1,...,n$ do not carry RMI. Note that in this case stage $m+g+1$ does not exist because $m+g+1 = n+1$. To facilitate our analysis, let us first focus on the first case. We can then rewrite Eq. 20 as follows: We split the sum $\sum_{e=m}^{n}$ into two sums, $\sum_{e=m}^{m+g}$ and $\sum_{e=m+g+1}^{n}$, and further split the sum $\sum_{l=m+1}^{e}$ into these two sums, $\sum_{l=m+1}^{m+g+1}$ and $\sum_{l=m+g+2}^{e}$. We have:

$$
\begin{aligned}
0 = & \sum_{e=m}^{m+g} \alpha_e \left((p-c_e)\Pi(\frac{\sum_{j=1}^{e} I_j}{1-a_e}) + c_e \Pi(\sum_{j=1}^{e} I_j) + h_e E^{\sum_{j=1}^{e} I_j} + \sum_{l=m+1}^{e} E^{\sum_{j=1}^{l-1} I_j}(h_{l-1}-h_l) \right) \\
& + \sum_{e=m+g+1}^{n} \alpha_e \left((p-c_e)\Pi(\frac{\sum_{j=1}^{e} I_j}{1-a_e}) + c_e \Pi(\sum_{j=1}^{e} I_j) + h_e E^{\sum_{j=1}^{e} I_j} + \sum_{l=m+1}^{m+g+1} E^{\sum_{j=1}^{l-1} I_j}(h_{l-1}-h_l) \right. \\
& \left. + \sum_{l=m+g+2}^{e} E^{\sum_{j=1}^{l-1} I_j}(h_{l-1}-h_l) \right) + h_m(1+\sum_{j=1}^{m-1}\frac{\alpha_j}{\beta}) - p\sum_{e=m}^{n}\alpha_e. \qquad (21)
\end{aligned}
$$

Next, we calculate the first order condition with respect to I_{m+g+1}:

$$
\begin{aligned}
0 = & \frac{\partial \mathbb{E}[C]}{\partial I_{m+g+1}} \quad \Leftrightarrow \\
0 = & \sum_{e=m+g+1}^{n} \alpha_e \left((p-c_e)\Pi(\frac{\sum_{j=1}^{e} I_j}{1-a_e}) + c_e \Pi(\sum_{j=1}^{e} I_j) + h_e E^{\sum_{j=1}^{e} I_j} + \sum_{l=m+g+2}^{e} E^{\sum_{j=1}^{l-1} I_j}(h_{l-1}-h_l) \right) \\
& + h_{m+g+1}(1+\sum_{j=1}^{m+g}\frac{\alpha_j}{\beta}) - p\sum_{e=m+g+1}^{n}\alpha_e. \qquad (22)
\end{aligned}
$$

Combining Eq. 21 and Eq. 22 leads to:

$$
\begin{aligned}
0 = & \sum_{e=m}^{m+g} \alpha_e \left((p-c_e)\Pi(\frac{\sum_{j=1}^{e} I_j}{1-a_e}) + c_e \Pi(\sum_{j=1}^{e} I_j) + h_e E^{\sum_{j=1}^{e} I_j} + \sum_{l=m+1}^{e} E^{\sum_{j=1}^{l-1} I_j}(h_{l-1}-h_l) \right) \\
& + \sum_{e=m+g+1}^{n} \alpha_e \left(\sum_{l=m+1}^{m+g+1} E^{\sum_{j=1}^{l-1} I_j}(h_{l-1}-h_l) \right) - h_{m+g+1}(1+\sum_{j=1}^{m+g}\frac{\alpha_j}{\beta}) \\
& + p\sum_{e=m+g+1}^{n}\alpha_e + h_m(1+\sum_{j=1}^{m-1}\frac{\alpha_j}{\beta}) - p\sum_{e=m}^{n}\alpha_e.
\end{aligned}
$$

Next, we observe that $\sum_{j=1}^{m} I_j = \sum_{j=1}^{m+1} I_j = ... = \sum_{j=1}^{m+g} I_j$ (by assumption there is a gap of size g upstream of stage m). Rearranging some terms leads then to:

$$
\begin{aligned}
0 = & \sum_{e=m}^{m+g} \alpha_e \left((p-c_e)\Pi(\frac{\sum_{j=1}^{m} I_j}{1-a_e}) + c_e \Pi(\sum_{j=1}^{m} I_j) + h_e E^{\sum_{j=1}^{m} I_j} + \sum_{l=m+1}^{e} E^{\sum_{j=1}^{m} I_j}(h_{l-1}-h_l) \right) \\
& + \sum_{e=m+g+1}^{n} \alpha_e \left(\sum_{l=m+1}^{m+g+1} E^{\sum_{j=1}^{m} I_j}(h_{l-1}-h_l) \right) - h_{m+g+1}(1+\sum_{j=1}^{m+g}\frac{\alpha_j}{\beta}) \\
& + h_m(1+\sum_{j=1}^{m-1}\frac{\alpha_j}{\beta}) - p\sum_{e=m}^{m+g}\alpha_e.
\end{aligned}
$$

Next we use following identities: $h_e + \sum_{l=m+1}^{e}(h_{l-1} - h_l) = h_m$ and likewise $\sum_{l=m+1}^{m+g+1}(h_{l-1} - h_l) = h_m - h_{m+g+1}$. Thus:

$$0 = \sum_{e=m}^{m+g} \alpha_e \left((p - c_e)\Pi(\frac{\sum_{j=1}^{m} I_j}{1 - a_e}) + c_e \Pi(\sum_{j=1}^{m} I_j) + h_m E^{\sum_{j=1}^{m} I_j} \right) \tag{23}$$
$$+ (h_m - h_{m+g+1}) E^{\sum_{j=1}^{m} I_j} \sum_{e=m+g+1}^{n} \alpha_e - h_{m+g+1}(1 + \sum_{j=1}^{m+g} \frac{\alpha_j}{\beta}) + h_m(1 + \sum_{j=1}^{m-1} \frac{\alpha_j}{\beta}) - p \sum_{e=m}^{m+g} \alpha_e.$$

Note that the second order condition is satisfied. Let us now return to the second case where stage m is the most upstream stage carrying RMI, i.e., $m + g = n$. An analogous calculation (see Eq. 20) then yields:

$$0 = \sum_{e=m}^{m+g} \alpha_e \left((p - c_e)\Pi(\frac{\sum_{j=1}^{m} I_j}{1 - a_e}) + c_e \Pi(\sum_{j=1}^{m} I_j) + h_m E^{\sum_{j=1}^{m} I_j} \right) + h_m(1 + \sum_{j=1}^{m-1} \frac{\alpha_j}{\beta}) - p \sum_{e=m}^{m+g} \alpha_e. \tag{24}$$

Observe that Eq. 23 and 24 are identical if (i) we set in Eq. 23 $m + g = n$ and (ii) we set in Eq. 23 $h_{m+g+1} = 0$. Technically, we would have to treat both cases as independently throughout the manuscript. However, to facilitate the analysis we abuse notation and simply refer to Eq. 23. We keep in mind that in the second case when $m + g = n$ we simply set $h_{m+g+1} = 0$.

Thus, Eq. 19 and Eq. 23 generalize Eq. 5- 8 in Proposition 1 to n-stages.

Let us generalize Proposition 2 to n-stages. Under the low disruption/high recovery rate assumption we ignore the inventory holding costs for the unused RMI during a disruption at the disrupted stage or further downstream. As a result, under the low disruption/high recovery rate approximation, Eq. 23 and Eq. 19 respectively become:

$$0 = \sum_{e=m}^{m+g} \alpha_e \left((p - c_e)\Pi(\frac{\sum_{j=1}^{m} \tilde{I}_j}{1 - \tilde{a}_e}) + c_e \Pi(\sum_{j=1}^{m} \tilde{I}_j) \right) - h_{m+g+1}(1 + \sum_{j=1}^{m+g} \frac{\alpha_j}{\beta}) + h_m(1 + \sum_{j=1}^{m-1} \frac{\alpha_j}{\beta}) - p \sum_{e=m}^{m+g} \alpha_e \tag{25}$$

and

$$E^{\frac{\sum_{j=1}^{m} \tilde{I}_j}{1 - \tilde{a}_m}} = \frac{1}{\beta} - \frac{\hat{c}_m(1 + \sum_{o=1}^{n} \frac{\alpha_o}{\beta})}{\alpha_m(p - c_m)}. \tag{26}$$

Thus, Eq. 25 and Eq. 26 generalize Eq. 9- 12 in Proposition 2 to n-stages.

A.4. Proof of Lemma 1

The generalization of Eq. 9 and Eq. 11 in Proposition 2 to n-stages is given by Eq. 25:

$$0 = \sum_{e=m}^{m+g} \alpha_e \left((p - c_e)\Pi(\frac{\sum_{j=1}^{m} \tilde{I}_j}{1 - \tilde{a}_e}) + c_e \Pi(\sum_{j=1}^{m} \tilde{I}_j) \right) - h_{m+g+1}(1 + \sum_{j=1}^{m+g} \frac{\alpha_j}{\beta}) + h_m(1 + \sum_{j=1}^{m-1} \frac{\alpha_j}{\beta}) - p \sum_{e=m}^{m+g} \alpha_e.$$

Let us first consider the case when $m + g < n$. In the absence of reserve capacity we have:

$$0 = \sum_{e=m}^{m+g} \alpha_e p \Pi(\sum_{j=1}^{m} \tilde{I}_j) - h_{m+g+1}(1 + \sum_{j=1}^{m+g} \frac{\alpha_j}{\beta}) + h_m(1 + \sum_{j=1}^{m-1} \frac{\alpha_j}{\beta}) - p \sum_{e=m}^{m+g} \alpha_e.$$

Using the definition of the effective inventory holding costs, $\bar{h}_m = h_m(1+\sum_{j=1}^{m-1}\frac{\alpha_j}{\beta})$, we have:

$$0 = \sum_{e=m}^{m+g}\alpha_e p\Pi(\sum_{j=1}^{m}\tilde{I}_j) - \bar{h}_{m+g+1} + \bar{h}_m - p\sum_{e=m}^{m+g}\alpha_e.$$

Thus,

$$\Pi(\sum_{j=1}^{m}\tilde{I}_j) = \frac{\bar{h}_{m+g+1} - \bar{h}_m + p\sum_{e=m}^{m+g}\alpha_e}{p\sum_{e=m}^{m+g}\alpha_e}.$$

This condition is equivalent to (using $\Pi(\sum_{j=1}^{m}\tilde{I}_j) = \int_0^{\sum_{j=1}^{m}\tilde{I}_j}\pi(k)dk = \int_0^{\sum_{j=1}^{m}\tilde{I}_j}\beta\exp\big(-\beta k\big)dk$):

$$\begin{aligned}
\int_0^{\sum_{j=1}^{m}\tilde{I}_j}\beta\exp\big(-\beta k\big)dk &= \frac{\bar{h}_{m+g+1} - \bar{h}_m + p\sum_{e=m}^{m+g}\alpha_e}{p\sum_{e=m}^{m+g}\alpha_e} \quad \Leftrightarrow \\
\exp\big(-\beta\sum_{j=1}^{m}\tilde{I}_j\big) &= 1 - \frac{\bar{h}_{m+g+1} - \bar{h}_m + p\sum_{e=m}^{m+g}\alpha_e}{p\sum_{e=m}^{m+g}\alpha_e} \quad \Leftrightarrow \\
\sum_{j=1}^{m}\tilde{I}_j &= -\frac{1}{\beta}\ln\Big(1 - \frac{\bar{h}_{m+g+1} - \bar{h}_m + p\sum_{e=m}^{m+g}\alpha_e}{p\sum_{e=m}^{m+g}\alpha_e}\Big). \qquad (27)
\end{aligned}$$

By assumption, there exists a stage m where it is optimal to hold some RMI, i.e., $\sum_{l=1}^{m}\tilde{I}_l > 0$. The condition that there is a gap of size $g > 0$ upstream of stage m and a gap of size $\hat{g}$ upstream of stage $m+g+1$ (where $\hat{g}$ could be 0) is then equivalent to:

$$\sum_{l=1}^{m}\tilde{I}_l \geq \sum_{l=1}^{m+q}\tilde{I}_l \quad \forall q \in \{1,2,...,g\} \quad \text{and} \quad \sum_{l=1}^{m}\tilde{I}_l < \sum_{l=1}^{m+g+1}\tilde{I}_l \qquad (28)$$

with (using Eq. 27)

$$\sum_{j=1}^{m}\tilde{I}_j = -\frac{1}{\beta}\ln\Big(1 - \frac{\bar{h}_{m+g+1} - \bar{h}_m + p\sum_{e=m}^{m+g}\alpha_e}{p\sum_{e=m}^{m+g}\alpha_e}\Big)$$

and

$$\sum_{j=1}^{m+q}\tilde{I}_j = -\frac{1}{\beta}\ln\Big(1 - \frac{\bar{h}_{m+g+1} - \bar{h}_{m+q} + p\sum_{e=m+q}^{m+g}\alpha_e}{p\sum_{e=m+q}^{m+g}\alpha_e}\Big)$$

and

$$\sum_{j=1}^{m+g+1}\tilde{I}_j = -\frac{1}{\beta}\ln\Big(1 - \frac{\bar{h}_{m+g+\hat{g}+2} - \bar{h}_{m+g+1} + p\sum_{e=m+g+1}^{m+g+1+\hat{g}}\alpha_e}{p\sum_{e=m+g+1}^{m+g+1+\hat{g}}\alpha_e}\Big).$$

Conditions 28 are equivalent to:

$$\begin{aligned}
&\frac{\bar{h}_{m+g+1} - \bar{h}_m + p\sum_{e=m}^{m+g}\alpha_e}{p\sum_{e=m}^{m+g}\alpha_e} \geq \frac{\bar{h}_{m+g+1} - \bar{h}_{m+q} + p\sum_{e=m+q}^{m+g}\alpha_e}{p\sum_{e=m+q}^{m+g}\alpha_e} \quad \forall q \in \{1,2,...,g\} \\
\text{and} \quad &\frac{\bar{h}_{m+g+1} - \bar{h}_m + p\sum_{e=m}^{m+g}\alpha_e}{p\sum_{e=m}^{m+g}\alpha_e} < \frac{\bar{h}_{m+g+\hat{g}+2} - \bar{h}_{m+g+1} + p\sum_{e=m+g+1}^{m+g+1+\hat{g}}\alpha_e}{p\sum_{e=m+g+1}^{m+g+1+\hat{g}}\alpha_e}.
\end{aligned}$$

These conditions are equivalent to:

$$\frac{\bar{h}_{m+g+1}-\bar{h}_m}{\sum_{e=m}^{m+g}\alpha_e} \geq \frac{\bar{h}_{m+g+1}-\bar{h}_{m+q}}{\sum_{e=m+q}^{m+g}\alpha_e} \quad \forall q\in\{1,2,...,g\}$$

$$\text{and} \quad \frac{\bar{h}_{m+g+1}-\bar{h}_m}{\sum_{e=m}^{m+g}\alpha_e} < \frac{\bar{h}_{m+g+\hat{g}+2}-\bar{h}_{m+g+1}}{\sum_{e=m+g+1}^{m+g+1+\hat{g}}\alpha_e}$$

These conditions are equivalent to:

$$\frac{\bar{h}_m}{\bar{h}_{m+q}} \leq \frac{\sum_{e=m}^{m+g}\alpha_e - \frac{\bar{h}_{m+g+1}}{\bar{h}_{m+q}}\sum_{e=m}^{m+q-1}\alpha_e}{\sum_{e=m+q}^{m+g}\alpha_e} \quad \forall q\in\{1,2,...,g\} \quad \text{and} \tag{29}$$

$$\frac{\bar{h}_m}{\bar{h}_{m+g+1}} > \frac{\sum_{e=m}^{m+g+\hat{g}+1}\alpha_e - \frac{\bar{h}_{m+g+\hat{g}+2}}{\bar{h}_{m+g+1}}\sum_{e=m}^{m+g}\alpha_e}{\sum_{e=m+g+1}^{m+g+\hat{g}+1}\alpha_e}. \tag{30}$$

Observe that the first condition (Eq. 29) arises from the fact that echelon RMI does not increase within the gap $m+1$ to $m+g$ while the second condition (Eq. 30) arises from the presence of an upstream stage $m+g+1$ at the end of the gap carrying positive RMI.

Let us now return to the second case where stage m is the most upstream stage carrying RMI, i.e., $m+g=n$. In this case the condition that there is a gap of size $g>0$ upstream of stage m is equivalent to (compare Eq. 28):

$$\sum_{l=1}^{m}\tilde{I}_l \geq \sum_{l=1}^{m+q}\tilde{I}_l \quad \forall q\in\{1,2,...,g\}$$

with

$$\sum_{j=1}^{m}\tilde{I}_j = -\frac{1}{\beta}\ln\Big(1-\frac{\bar{h}_{m+g+1}-\bar{h}_m+p\sum_{e=m}^{m+g}\alpha_e}{p\sum_{e=m}^{m+g}\alpha_e}\Big)$$

and

$$\sum_{j=1}^{m+q}\tilde{I}_j = -\frac{1}{\beta}\ln\Big(1-\frac{\bar{h}_{m+g+1}-\bar{h}_{m+q}+p\sum_{e=m+q}^{m+g}\alpha_e}{p\sum_{e=m+q}^{m+g}\alpha_e}\Big).$$

An analogous calculation shows that this condition is equivalent to Eq. 29:

$$\frac{\bar{h}_m}{\bar{h}_{m+q}} \leq \frac{\sum_{e=m}^{m+g}\alpha_e - \frac{\bar{h}_{m+g+1}}{\bar{h}_{m+q}}\sum_{e=m}^{m+q-1}\alpha_e}{\sum_{e=m+q}^{m+g}\alpha_e} \quad \forall q\in\{1,2,...,g\},$$

which simplifies to (because $h_{m+g+1}=0$ for $m+g=n$):

$$\frac{\bar{h}_m}{\bar{h}_{m+q}} \leq \frac{\sum_{e=m}^{m+g}\alpha_e}{\sum_{e=m+q}^{m+g}\alpha_e} \quad \forall q\in\{1,2,...,g\}.$$

The result thus follows.

A.5. Proof of Theorem 1

We provide the proof for the general case when $m+g<n$. In case $m+g=n$ all our arguments can be specialized by setting $h_{m+g+1}=0$ as introduced in Appendix A.3 (Proof of Proposition 1 for the n-stage supply chain).

The generalization of Eq. 9 and Eq. 11 in Proposition 2 to n-stages is given by Eq. 25:

$$0=\sum_{e=m}^{m+g}\alpha_e\left((p-c_e)\Pi(\frac{\sum_{j=1}^{m}\tilde{I}_j}{1-\tilde{a}_e})+c_e\Pi(\sum_{j=1}^{m}\tilde{I}_j)\right)-h_{m+g+1}(1+\sum_{j=1}^{m+g}\frac{\alpha_j}{\beta})+h_m(1+\sum_{j=1}^{m-1}\frac{\alpha_j}{\beta})-p\sum_{e=m}^{m+g}\alpha_e.$$

In the absence of reserve capacity we have:

$$0=\sum_{e=m}^{m+g}\alpha_e p\Pi(\sum_{j=1}^{m}\tilde{I}_j)-h_{m+g+1}(1+\sum_{j=1}^{m+g}\frac{\alpha_j}{\beta})+h_m(1+\sum_{j=1}^{m-1}\frac{\alpha_j}{\beta})-p\sum_{e=m}^{m+g}\alpha_e.$$

Using the definition of the effective inventory holding costs, $\bar{h}_m=h_m(1+\sum_{j=1}^{m-1}\frac{\alpha_j}{\beta})$, we have:

$$0=\sum_{e=m}^{m+g}\alpha_e p\Pi(\sum_{j=1}^{m}\tilde{I}_j)-\bar{h}_{m+g+1}+\bar{h}_m-p\sum_{e=m}^{m+g}\alpha_e.$$

Thus,

$$\Pi(\sum_{j=1}^{m}\tilde{I}_j)=\frac{\bar{h}_{m+g+1}-\bar{h}_m+p\sum_{e=m}^{m+g}\alpha_e}{p\sum_{e=m}^{m+g}\alpha_e}.$$

This condition is equivalent to (using $\Pi(\sum_{j=1}^{m}\tilde{I}_j)=\int_0^{\sum_{j=1}^{m}\tilde{I}_j}\pi(k)dk=\int_0^{\sum_{j=1}^{m}\tilde{I}_j}\beta\exp\big(-\beta k\big)dk$):

$$\begin{aligned}\int_0^{\sum_{j=1}^{m}\tilde{I}_j}\beta\exp\big(-\beta k\big)dk &= \frac{\bar{h}_{m+g+1}-\bar{h}_m+p\sum_{e=m}^{m+g}\alpha_e}{p\sum_{e=m}^{m+g}\alpha_e} \quad\Leftrightarrow\\ \exp\big(-\beta\sum_{j=1}^{m}\tilde{I}_j\big) &= 1-\frac{\bar{h}_{m+g+1}-\bar{h}_m+p\sum_{e=m}^{m+g}\alpha_e}{p\sum_{e=m}^{m+g}\alpha_e} \quad\Leftrightarrow\\ \sum_{j=1}^{m}\tilde{I}_j &= -\frac{1}{\beta}\ln\Big(1-\frac{\bar{h}_{m+g+1}-\bar{h}_m+p\sum_{e=m}^{m+g}\alpha_e}{p\sum_{e=m}^{m+g}\alpha_e}\Big).\end{aligned}$$

By assumption we know that there exists a most downstream stage m^* where $\tilde{I}_{m^*}>0$. Thus:

$$\tilde{I}_{m^*}=\sum_{j=1}^{m^*}\tilde{I}_j=-\frac{1}{\beta}\ln\Big(1-\frac{\bar{h}_{m^*+g+1}-\bar{h}_{m^*}+p\sum_{e=m^*}^{m^*+g}\alpha_e}{p\sum_{e=m^*}^{m^*+g}\alpha_e}\Big). \tag{31}$$

Likewise, we have:

$$\sum_{j=m^*}^{n}\tilde{I}_j=\sum_{j=1}^{n}\tilde{I}_j=-\frac{1}{\beta}\ln\left(1-\frac{p\alpha_n-\bar{h}_n}{p\alpha_n}\right). \tag{32}$$

The condition $\tilde{I}_{m^*}>\sum_{j=m^*+1}^{n}\tilde{I}_j$ is equivalent to (by adding $\tilde{I}_{m^*}$ to both sides of the inequality):

$$2\tilde{I}_{m^*}>\sum_{j=m^*}^{n}\tilde{I}_j.$$

This condition is equivalent to (using Eq. 31 and Eq. 32):

$$
\begin{aligned}
-\frac{2}{\beta}\ln\left(1-\frac{\bar{h}_{m^*+g+1}-\bar{h}_{m^*}+p\sum_{e=m^*}^{m^*+g}\alpha_e}{p\sum_{e=m^*}^{m^*+g}\alpha_e}\right) > -\frac{1}{\beta}\ln\left(1-\frac{p\alpha_n-\bar{h}_n}{p\alpha_n}\right) \quad &\Leftrightarrow \\
\left(1-\frac{\bar{h}_{m^*+g+1}-\bar{h}_{m^*}+p\sum_{e=m^*}^{m^*+g}\alpha_e}{p\sum_{e=m^*}^{m^*+g}\alpha_e}\right)^2 < \left(1-\frac{p\alpha_n-\bar{h}_n}{p\alpha_n}\right) \quad &\Leftrightarrow \\
\frac{(\bar{h}_{m^*}-\bar{h}_{m^*+g+1})^2\alpha_n}{p(\sum_{e=m^*}^{m^*+g}\alpha_e)^2} < \bar{h}_n &
\end{aligned}
$$

The result thus follows.

A.6. Proof of Theorem 2

We provide the proof for the general case where stage $m+1$ is not the most upstream stage carrying RMI, i.e., stage $m+1+g+1 \leq n$ carries RMI, but all stages in between do not carry RMI. In case $m+1+g=n$ all our arguments can be specialized by setting $h_{m+1+g+1}=h_{m+g+2}=0$ as introduced in Appendix A.3 (Proof of Proposition 1 for the n-stage supply chain).

The generalization of Eq. 9 and Eq. 11 in Proposition 2 to n-stages is given by Eq. 25:

$$
0=\sum_{e=m}^{m+g}\alpha_e\left((p-c_e)\Pi(\frac{\sum_{j=1}^{m}\tilde{I}_j}{1-\tilde{a}_e})+c_e\Pi(\sum_{j=1}^{m}\tilde{I}_j)\right)-h_{m+g+1}(1+\sum_{j=1}^{m+g}\frac{\alpha_j}{\beta})+h_m(1+\sum_{j=1}^{m-1}\frac{\alpha_j}{\beta})-p\sum_{e=m}^{m+g}\alpha_e.
$$

In the absence of reserve capacity we have:

$$
0=\sum_{e=m}^{m+g}\alpha_e p\Pi(\sum_{j=1}^{m}\tilde{I}_j)-h_{m+g+1}(1+\sum_{j=1}^{m+g}\frac{\alpha_j}{\beta})+h_m(1+\sum_{j=1}^{m-1}\frac{\alpha_j}{\beta})-p\sum_{e=m}^{m+g}\alpha_e.
$$

Using the definition of the effective inventory holding costs, $\bar{h}_m=h_m(1+\sum_{j=1}^{m-1}\frac{\alpha_j}{\beta})$, we have:

$$
0=\sum_{e=m}^{m+g}\alpha_e p\Pi(\sum_{j=1}^{m}\tilde{I}_j)-\bar{h}_{m+g+1}+\bar{h}_m-p\sum_{e=m}^{m+g}\alpha_e.
$$

Thus,

$$
\Pi(\sum_{j=1}^{m}\tilde{I}_j)=\frac{\bar{h}_{m+g+1}-\bar{h}_m+p\sum_{e=m}^{m+g}\alpha_e}{p\sum_{e=m}^{m+g}\alpha_e}.
$$

This condition is equivalent to (using $\Pi(\sum_{j=1}^{m}\tilde{I}_j)=\int_0^{\sum_{j=1}^{m}\tilde{I}_j}\pi(k)dk=\int_0^{\sum_{j=1}^{m}\tilde{I}_j}\beta\exp\big(-\beta k\big)dk$):

$$
\begin{aligned}
\int_0^{\sum_{j=1}^{m}\tilde{I}_j}\beta\exp\big(-\beta k\big)dk &= \frac{\bar{h}_{m+g+1}-\bar{h}_m+p\sum_{e=m}^{m+g}\alpha_e}{p\sum_{e=m}^{m+g}\alpha_e} \quad \Leftrightarrow \\
\exp\big(-\beta\sum_{j=1}^{m}\tilde{I}_j\big) &= 1-\frac{\bar{h}_{m+g+1}-\bar{h}_m+p\sum_{e=m}^{m+g}\alpha_e}{p\sum_{e=m}^{m+g}\alpha_e} \quad \Leftrightarrow \\
\sum_{j=1}^{m}\tilde{I}_j &= -\frac{1}{\beta}\ln\Big(1-\frac{\bar{h}_{m+g+1}-\bar{h}_m+p\sum_{e=m}^{m+g}\alpha_e}{p\sum_{e=m}^{m+g}\alpha_e}\Big).
\end{aligned}
$$

By assumption we have: $\tilde{I}_m > 0$ and $\tilde{I}_{m+1} > 0$, i.e., the gap upstream of stage m has size $g = 0$. Thus:

$$\sum_{j=1}^{m} \tilde{I}_j = -\frac{1}{\beta} \ln\Big(1 - \frac{\bar{h}_{m+1} - \bar{h}_m + p\alpha_m}{p\alpha_m}\Big). \tag{33}$$

By assumption stage $m+1$ is followed by a gap upstream of size g, where g could be 0 (i.e., stage $m+1$ may or may not be followed by a gap upstream). Thus,

$$\sum_{j=1}^{m+1} \tilde{I}_j = -\frac{1}{\beta} \ln\Big(1 - \frac{\bar{h}_{m+g+2} - \bar{h}_{m+1} + p\sum_{e=m+1}^{m+1+g} \alpha_e}{p\sum_{e=m+1}^{m+1+g} \alpha_e}\Big). \tag{34}$$

By assumption stage m is preceded downstream by a gap of size $\hat{g}$, where $\hat{g}$ could be 0 (i.e., stage m may or may not be preceded downstream by a gap). Thus,

$$\sum_{j=1}^{m-\hat{g}-1} \tilde{I}_j = -\frac{1}{\beta} \ln\Big(1 - \frac{\bar{h}_m - \bar{h}_{m-\hat{g}-1} + p\sum_{e=m-\hat{g}-1}^{m-1} \alpha_e}{p\sum_{e=m-\hat{g}-1}^{m-1} \alpha_e}\Big). \tag{35}$$

The condition $\tilde{I}_m > \tilde{I}_{m+1}$ is equivalent to (using $\tilde{I}_m = \sum_{l=1}^{m} \tilde{I}_l - \sum_{l=1}^{m-1} \tilde{I}_l$ and $\tilde{I}_{m+1} = \sum_{l=1}^{m+1} \tilde{I}_l - \sum_{l=1}^{m} \tilde{I}_l$):

$$2\sum_{l=1}^{m} \tilde{I}_l > \sum_{l=1}^{m+1} \tilde{I}_l + \sum_{l=1}^{m-1} \tilde{I}_l$$

This condition is equivalent to (using $\sum_{l=1}^{m-1} \tilde{I}_l = \sum_{l=1}^{m-\hat{g}-1} \tilde{I}_l$ because stage m is preceded downstream by a gap of size $\hat{g}$):

$$2\sum_{l=1}^{m} \tilde{I}_l > \sum_{l=1}^{m+1} \tilde{I}_l + \sum_{l=1}^{m-\hat{g}-1} \tilde{I}_l$$

This condition is equivalent to (using Eq. 33, Eq. 34 and Eq. 35):

$$-\frac{2}{\beta} \ln\Big(1 - \frac{\bar{h}_{m+1} - \bar{h}_m + p\alpha_m}{p\alpha_m}\Big) > -\frac{1}{\beta} \ln\Big(1 - \frac{\bar{h}_{m+g+2} - \bar{h}_{m+1} + p\sum_{e=m+1}^{m+1+g} \alpha_e}{p\sum_{e=m+1}^{m+1+g} \alpha_e}\Big) -\frac{1}{\beta} \ln\Big(1 - \frac{\bar{h}_m - \bar{h}_{m-\hat{g}-1} + p\sum_{e=m-\hat{g}-1}^{m-1} \alpha_e}{p\sum_{e=m-\hat{g}-1}^{m-1} \alpha_e}\Big)$$

This condition is equivalent to:

$$\Big(1 - \frac{\bar{h}_{m+1} - \bar{h}_m + p\alpha_m}{p\alpha_m}\Big)^2 < \Big(1 - \frac{\bar{h}_{m+g+2} - \bar{h}_{m+1} + p\sum_{e=m+1}^{m+1+g} \alpha_e}{p\sum_{e=m+1}^{m+1+g} \alpha_e}\Big)\Big(1 - \frac{\bar{h}_m - \bar{h}_{m-\hat{g}-1} + p\sum_{e=m-\hat{g}-1}^{m-1} \alpha_e}{p\sum_{e=m-\hat{g}-1}^{m-1} \alpha_e}\Big)$$

This condition is equivalent to:

$$\Big(\frac{\bar{h}_m - \bar{h}_{m+1}}{\alpha_m}\Big)^2 < \Big(\frac{\bar{h}_{m+1} - \bar{h}_{m+g+2}}{\sum_{e=m+1}^{m+1+g} \alpha_e}\Big)\Big(\frac{\bar{h}_{m-\hat{g}-1} - \bar{h}_m}{\sum_{e=m-\hat{g}-1}^{m-1} \alpha_e}\Big).$$

The result thus follows.

A.7. Proof of Theorem 3

The Theorem follows directly from the fact that at optimality, the first order conditions imply that at any stage m we have $\frac{\sum_{j=1}^{m} I_j}{1-a_m} = const$ (Eq. 19).

A.8. Proof of Theorem 4

From the first order optimality conditions in Eq. 23 we know that $\sum_{j=1}^{m} I_j = const$. As a result, RMI held at any stage reduces the marginal value of RMI at any adjacent stage.

From Eq. 19 we have that $\frac{\sum_{j=1}^{m} I_j}{1-a_m} = const$. As a result, RMI held at any stage reduces the marginal value of reserve capacity at the same stage.

Combining both statements completes the proof.

Appendix B: Commitment to downstream reserve capacity

Example 2. In this example we set the modeling parameters of a two-stage supply chain such that it is optimal to carry RMI and reserve capacity up- and downstream. We show that the downstream reserve capacity is larger than the upstream reserve capacity even when disruption rates are identical up- and downstream ($\alpha_d = \alpha_u =: \alpha$) and when reserve capacity costs are identical up- and downstream ($\hat{c}_d = \hat{c}_u =: \hat{c}$ and $c_d = c_u =: c$).

Let us first impose four conditions that ensure that RMI and reserve capacity is held up- and downstream. These conditions are expressed in terms of the Lambert W function, $W_{-1}(\cdot)$ (see Corless et al. (1996) for further details).

1. $\hat{c} < \frac{\alpha(p-c)}{\beta(1+\frac{2\alpha}{\beta})}$
2. $$-\frac{1}{\beta}W_{-1}\Big(-\frac{\exp(-1)\beta\hat{c}(1+\frac{2\alpha}{\beta})}{\alpha(p-c)}\Big) > -\frac{1}{\beta}W_{-1}\Big(-\frac{\alpha\Big(\Pi(\frac{I_d+I_u}{1-a_u})(p-c)-p\Big)+\big(1+\frac{2\alpha}{\beta}\big)h_u+\alpha c}{\frac{h_u\alpha}{\beta}\exp(1+\frac{\beta c}{h_u})}\Big)-\frac{c}{h_u}$$
3. $-W_{-1}\Big(-\frac{(\alpha\Big(\Pi(\frac{I_d}{1-a_d})(p-c)-p\Big)+(h_d-h_u)(1+\frac{2\alpha}{\beta})+\alpha c)\beta}{\alpha(2h_d-h_u)\exp\Big(1+\frac{\beta c}{2h_d-h_u}\Big)}\Big) > 1+\frac{\beta c}{2h_d-h_u}$
4. $$-\frac{1}{\beta}W_{-1}\Big(-\frac{(\alpha\Big(\Pi(\frac{I_d+I_u}{1-a_u})(p-c)-p\Big)+\big(1+\frac{2\alpha}{\beta}\big)h_u+\alpha c)\beta}{h_u\alpha\exp(1+\frac{\beta c}{h_u})}\Big)-\frac{c}{h_u}$$
$$> -\frac{1}{\beta}W_{-1}\Big(-\frac{(\alpha\Big(\Pi(\frac{I_d}{1-a_d})(p-c)-p\Big)+(h_d-h_u)(1+\frac{2\alpha}{\beta})+\alpha c)\beta}{\alpha(2h_d-h_u)\exp\Big(1+\frac{\beta c}{2h_d-h_u}\Big)}\Big)-\frac{c}{2h_d-h_u}$$

with $\frac{I_d}{1-a_d} = \frac{I_d+I_u}{1-a_u} = -\frac{1}{\beta}W_{-1}\Big(-\frac{\exp(-1)\beta\hat{c}(1+\frac{2\alpha}{\beta})}{\alpha(p-c)}\Big)-\frac{1}{\beta}$. Conditions 1 and 2 ensure that reserve capacity is held up- and downstream. In essence, we require the reservation cost of the reserve capacity to be sufficiently low relative to the penalty cost (condition 1) and the inventory holding cost upstream not to be too low (condition 2). Condition 3 ensures that RMI is held downstream. In essence, we require the downstream inventory holding cost to be sufficiently low relative to the emergency production cost of the reserve capacity. Condition 4 ensures that RMI is held upstream. In essence, we require the upstream inventory holding cost to be sufficiently low relative to the downstream inventory holding cost.

Below we formally proof that if conditions 1-4 are satisfied, we have $I_d > 0$, $I_u > 0$, $a_d > 0$ and $a_u > 0$. We further prove below that we then have: $a_d > a_u$. To convince the reader that the solution set is not empty, we provide one specific example. For $h_d = 1$, $h_u = 0.4$, $p = 100$, $\beta = 0.2$, $\alpha = 0.02$, $\hat{c} = 2$, $c = 40$, we have: $I_d = 3.2$, $I_u = 1.9$, $a_d = 0.7$, $a_u = 0.5$.

For ease of exposition, we do not generalize this example to multi-stage supply chains as it becomes tedious to express mathematical conditions when it is optimal to hold RMI at some upstream stage in a general setting.

Proof. We want to find conditions when $a_d > 0$, $a_u > 0$, $I_d > 0$ and $I_d + I_u > I_d$. We have:

$$
\begin{aligned}
E^{\frac{I_d+I_u}{1-a_u}} &= \frac{1}{\beta} - \frac{\hat{c}_u(1+\frac{\alpha_d}{\beta}+\frac{\alpha_u}{\beta})}{\alpha_u(p-c_u)} \\
\int_0^{\frac{I_d+I_u}{1-a_u}} \pi(k)k dk &= \frac{1}{\beta} - \frac{\hat{c}_u(1+\frac{\alpha_d}{\beta}+\frac{\alpha_u}{\beta})}{\alpha_u(p-c_u)} \\
\frac{1}{\beta} - (\exp\Big(-\beta\frac{I_d+I_u}{1-a_u}\Big)\frac{\beta\frac{I_d+I_u}{1-a_u}+1}{\beta} &= \frac{1}{\beta} - \frac{\hat{c}_u(1+\frac{\alpha_d}{\beta}+\frac{\alpha_u}{\beta})}{\alpha_u(p-c_u)}.
\end{aligned}
$$

Let us define $S_1 := -(\beta\frac{I_d+I_u}{1-a_u}+1)$. Then:

$$
\begin{aligned}
\exp\Big(S_1+1\Big)\frac{S_1}{\beta} &= -\frac{\hat{c}_u(1+\frac{\alpha_d}{\beta}+\frac{\alpha_u}{\beta})}{\alpha_u(p-c_u)} \\
\exp\Big(S_1\Big)S_1 &= -\frac{\exp(-1)\beta\hat{c}_u(1+\frac{\alpha_d}{\beta}+\frac{\alpha_u}{\beta})}{\alpha_u(p-c_u)}
\end{aligned}
$$

We can solve this equation for S_1 using the Lambert W function $W_{-1}(\cdot)$:

$$
\begin{aligned}
S_1 &= W_{-1}\Big(-\frac{\exp(-1)\beta\hat{c}_u(1+\frac{\alpha_d}{\beta}+\frac{\alpha_u}{\beta})}{\alpha_u(p-c_u)}\Big) \\
-\beta\frac{I_d+I_u}{1-a_u}-1 &= W_{-1}\Big(-\frac{\exp(-1)\beta\hat{c}_u(1+\frac{\alpha_d}{\beta}+\frac{\alpha_u}{\beta})}{\alpha_u(p-c_u)}\Big) \\
\frac{I_d+I_u}{1-a_u} &= -\frac{1}{\beta}W_{-1}\Big(-\frac{\exp(-1)\beta\hat{c}_u(1+\frac{\alpha_d}{\beta}+\frac{\alpha_u}{\beta})}{\alpha_u(p-c_u)}\Big)-\frac{1}{\beta} \qquad (36)
\end{aligned}
$$

Likewise, we have:

$$
\frac{I_d}{1-a_d} = -\frac{1}{\beta}W_{-1}\Big(-\frac{\exp(-1)\beta\hat{c}_d(1+\frac{\alpha_d}{\beta}+\frac{\alpha_u}{\beta})}{\alpha_d(p-c_d)}\Big)-\frac{1}{\beta} \qquad (37)
$$

We are left with determining the optimal RMI levels. We have:

$$
0 = \alpha_u\Bigg(\Pi(\frac{I_d+I_u}{1-a_u})(p-c_u)+\Pi(I_d+I_u)c_u+E^{I_d+I_u}h_u-p\Bigg)+\big(1+\frac{\alpha_d}{\beta}\big)h_u,
$$

Let us define $H_u = \alpha_u\Big(\Pi(\frac{I_d+I_u}{1-a_u})(p-c_u)-p\Big)+\big(1+\frac{\alpha_d}{\beta}\big)h_u+\alpha_u c_u+\frac{\alpha_u h_u}{\beta}$. Then:

$$\begin{aligned}
0 &= \alpha_u\Bigg(\Pi(I_d+I_u)c_u+E^{I_d+I_u}h_u\Bigg)+H_u-\alpha_u c_u-\frac{\alpha_u h}{\beta},\\
0 &= \alpha_u\Bigg(\int_0^{I_d+I_u}\beta\exp\left(-\beta k\right)dk c_u+\int_0^{I_d+I_u}\pi(k)kdk h_u\Bigg)+H_u-\alpha_u c_u-\frac{\alpha_u h_u}{\beta},\\
0 &= \alpha_u\Bigg(\Big(1-\exp\left(-\beta(I_d+I_u)\right)\Big)c_u+\Bigg(\frac{1}{\beta}-(\exp\Big(-\beta(I_d+I_u)\Big)\frac{\beta(I_d+I_u)+1}{\beta}\Bigg)h_u\Bigg)+H_u-\alpha_u c_u-\frac{\alpha_u h_u}{\beta},
\end{aligned}$$

Let us define $S_2 = -\beta(I_d+I_u)-1-\frac{\beta c_u}{h_u}$. Then:

$$\begin{aligned}
0 &= \alpha_u\Bigg(\Big(1-\exp\left(S_2+1+\frac{\beta c_u}{h_u}\right)\Big)c_u+\Bigg(\frac{1}{\beta}+(\exp\Big(S_2+1+\frac{\beta c_u}{h_u}\Big)\frac{S_2+\frac{\beta c_u}{h_u}}{\beta}\Bigg)h_u\Bigg)+H_u-\alpha_u c_u-\frac{\alpha_u h_u}{\beta},\\
0 &= -\alpha_u c_u\exp\left(S_2+1+\frac{\beta c_u}{h_u}\right)+\frac{\alpha_u}{\beta}h_u\exp\Big(S_2+1+\frac{\beta c_u}{h_u}\Big)(S_2+\frac{\beta c_u}{h_u})+H_u,\\
0 &= -\alpha_u c_u\exp\left(S_2+1+\frac{\beta c_u}{h_u}\right)+\exp\Big(S_2+1+\frac{\beta c_u}{h_u}\Big)(S_2\frac{\alpha_u}{\beta}h_u+\alpha_u c_u)+H_u,\\
0 &= \exp\Big(S_2+1+\frac{\beta c_u}{h_u}\Big)S_2\frac{\alpha_u}{\beta}h_u+H_u,\\
0 &= \exp\left(S_2\right)S_2+\frac{H_u}{\frac{h_u\alpha_u}{\beta}\exp\left(1+\frac{\beta c_u}{h_u}\right)},
\end{aligned}$$

Using the Lambert W function $W_{-1}(\cdot)$ we have:

$$\begin{aligned}
S_2 &= W_{-1}\Big(-\frac{H_u}{\frac{h_u\alpha_u}{\beta}\exp\left(1+\frac{\beta c_u}{h_u}\right)}\Big)\\
-\beta(I_d+I_u)-1-\frac{\beta c_u}{h_u} &= W_{-1}\Big(-\frac{H_u}{\frac{h_u\alpha_u}{\beta}\exp\left(1+\frac{\beta c_u}{h_u}\right)}\Big)\\
I_d+I_u &= -\frac{1}{\beta}W_{-1}\Big(-\frac{H_u}{\frac{h_u\alpha_u}{\beta}\exp\left(1+\frac{\beta c_u}{h_u}\right)}\Big)-\frac{1}{\beta}-\frac{c_u}{h_u}
\end{aligned}$$

where we use Eq. 36 to determine H_u.

Regarding the downstream RMI we have:

$$0 = \alpha_d\Big(\Pi(\frac{I_d}{1-a_d})(p-c_d)+\Pi(I_d)c_d+E^{I_d}h_d-p\Big)+\alpha_u(h_d-h_u)E^{I_d}+(h_d-h_u)-\frac{\alpha_d}{\beta}h_u,$$

Let us define $H_d = \alpha_d\Big(\Pi(\frac{I_d}{1-a_d})(p-c_d)-p\Big)+(h_d-h_u)-\frac{\alpha_d}{\beta}h_u+\alpha_d c_d+\frac{\alpha_d h_d+\alpha_u(h_d-h_u)}{\beta}$. Then:

$$\begin{aligned}
0 &= \alpha_d\Big(\Pi(I_d)c_d+E^{I_d}h_d\Big)+\alpha_u(h_d-h_u)E^{I_d}+H_d-\alpha_d c_d-\frac{\alpha_d h_d+\alpha_u(h_d-h_u)}{\beta},\\
0 &= \alpha_d\Pi(I_d)c_d+E^{I_d}\Big(\alpha_d h_d+\alpha_u(h_d-h_u)\Big)+H_d-\alpha_d c_d-\frac{\alpha_d h_d+\alpha_u(h_d-h_u)}{\beta},\\
0 &= \alpha_d\int_0^{I_d}\beta\exp\left(-\beta k\right)dk c_d+\int_0^{I_d}\beta\exp\left(-\beta k\right)kdk\Big(\alpha_d h_d+\alpha_u(h_d-h_u)\Big)+H_d-\alpha_d c_d-\frac{\alpha_d h_d+\alpha_u(h_d-h_u)}{\beta},\\
0 &= \alpha_d\Big(1-\exp\left(-\beta I_d\right)\Big)c_d+\Bigg(\frac{1}{\beta}-(\exp\Big(-\beta I_d\Big)\frac{\beta I_d+1}{\beta}\Bigg)\Big(\alpha_d h_d+\alpha_u(h_d-h_u)\Big)+H_d-\alpha_d c_d-\frac{\alpha_d h_d+\alpha_u(h_d-h_u)}{\beta},
\end{aligned}$$

Let us define $S_3 = -\beta I_d - 1 - \frac{\beta c_d \alpha_d}{\alpha_d h_d + \alpha_u(h_d - h_u)}$. Then:

$$\begin{aligned}
0 &= -\alpha_d \exp\big(S_3 + 1 + \frac{\beta c_d \alpha_d}{\alpha_d h_d + \alpha_u(h_d - h_u)}\big) c_d \\
&\quad + \exp\Big(S_3 + 1 + \frac{\beta c_d \alpha_d}{\alpha_d h_d + \alpha_u(h_d - h_u)}\Big) \frac{S_3 + \frac{\beta c_d \alpha_d}{\alpha_d h_d + \alpha_u(h_d - h_u)}}{\beta} \Big(\alpha_d h_d + \alpha_u(h_d - h_u)\Big) + H_d, \\
0 &= \exp\Big(S_3 + 1 + \frac{\beta c_d \alpha_d}{\alpha_d h_d + \alpha_u(h_d - h_u)}\Big) \frac{S_3}{\beta} \Big(\alpha_d h_d + \alpha_u(h_d - h_u)\Big) + H_d, \\
0 &= \exp(S_3) S_3 + \frac{H_d \beta}{\Big(\alpha_d h_d + \alpha_u(h_d - h_u)\Big) \exp\Big(1 + \frac{\beta c_d \alpha_d}{\alpha_d h_d + \alpha_u(h_d - h_u)}\Big)},
\end{aligned}$$

Thus,

$$\begin{aligned}
S_3 &= W_{-1}\Big(- \frac{H_d \beta}{\Big(\alpha_d h_d + \alpha_u(h_d - h_u)\Big) \exp\Big(1 + \frac{\beta c_d \alpha_d}{\alpha_d h_d + \alpha_u(h_d - h_u)}\Big)}\Big) \\
-\beta I_d - 1 - \frac{\beta c_d \alpha_d}{\alpha_d h_d + \alpha_u(h_d - h_u)} &= W_{-1}\Big(- \frac{H_d \beta}{\Big(\alpha_d h_d + \alpha_u(h_d - h_u)\Big) \exp\Big(1 + \frac{\beta c_d \alpha_d}{\alpha_d h_d + \alpha_u(h_d - h_u)}\Big)}\Big)
\end{aligned}$$

Thus,

$$I_d = -\frac{1}{\beta} W_{-1}\Big(- \frac{H_d \beta}{\Big(\alpha_d h_d + \alpha_u(h_d - h_u)\Big) \exp\Big(1 + \frac{\beta c_d \alpha_d}{\alpha_d h_d + \alpha_u(h_d - h_u)}\Big)}\Big) - \frac{1}{\beta} - \frac{c_d \alpha_d}{\alpha_d h_d + \alpha_u(h_d - h_u)}$$

where we use Eq. 37 to determine H_d.

Recall that we want to find conditions when $a_d > 0$, $a_u > 0$, $I_d > 0$ and $I_d + I_u > I_d$. First, observe that $\frac{I_d + I_u}{1 - a_u} > 0$ if $\frac{1}{\beta} - \frac{\hat{c}_u (1 + \frac{\alpha_d}{\beta} + \frac{\alpha_u}{\beta})}{\alpha_u (p - c_u)} > 0$, which is equivalent to $\frac{\alpha(p-c)}{\beta(1 + \frac{2\alpha}{\beta})} > \hat{c}$ when disruption rates up- and downstream are equal ($\alpha_d = \alpha_u =: \alpha$) and when reserve capacity costs up- and downstream equal ($\hat{c}_d = \hat{c}_u =: \hat{c}$ and $c_d = c_u =: c$). Likewise $\frac{I_d}{1 - a_d} > 0$ if $\frac{\alpha(p-c)}{\beta(1 + \frac{2\alpha}{\beta})} > \hat{c}$. We have $a_d > 0$ and $a_u > 0$ if $\frac{I_d + I_u}{1 - a_u} > I_d + I_u$. This condition is equivalent to:

$$\begin{aligned}
-\frac{1}{\beta} W_{-1}\Big(- \frac{\exp(-1) \beta \hat{c} (1 + \frac{2\alpha}{\beta})}{\alpha(p - c)}\Big) - \frac{1}{\beta} &> -\frac{1}{\beta} W_{-1}\Big(- \frac{\alpha\Big(\Pi(\frac{I_d + I_u}{1 - a_u})(p - c) - p\Big) + \big(1 + \frac{\alpha}{\beta}\big) h_u + \alpha c + \frac{\alpha h_u}{\beta}}{\frac{h_u \alpha}{\beta} \exp(1 + \frac{\beta c}{h_u})}\Big) - \frac{1}{\beta} - \frac{c}{h_u} \\
-\frac{1}{\beta} W_{-1}\Big(- \frac{\exp(-1) \beta \hat{c} (1 + \frac{2\alpha}{\beta})}{\alpha(p - c)}\Big) &> -\frac{1}{\beta} W_{-1}\Big(- \frac{\alpha\Big(\Pi(\frac{I_d + I_u}{1 - a_u})(p - c) - p\Big) + \big(1 + \frac{2\alpha}{\beta}\big) h_u + \alpha c}{\frac{h_u \alpha}{\beta} \exp(1 + \frac{\beta c}{h_u})}\Big) - \frac{c}{h_u}
\end{aligned}$$

Next, we have $I_d > 0$ if

$$\begin{aligned}
-\frac{1}{\beta} W_{-1}\Big(- \frac{H_d \beta}{\alpha(2h_d - h_u) \exp\Big(1 + \frac{\beta c}{2h_d - h_u}\Big)}\Big) - \frac{1}{\beta} - \frac{c}{2h_d - h_u} &> 0 \\
-W_{-1}\Big(- \frac{H_d \beta}{\alpha(2h_d - h_u) \exp\Big(1 + \frac{\beta c}{2h_d - h_u}\Big)}\Big) &> 1 + \frac{\beta c}{2h_d - h_u} \\
-W_{-1}\Big(- \frac{H_d \beta}{\alpha(2h_d - h_u) \exp\Big(1 + \frac{\beta c}{2h_d - h_u}\Big)}\Big) &> 1 + \frac{\beta c}{2h_d - h_u}
\end{aligned}$$

Using $H_d = \alpha\Big(\Pi(\frac{I_d}{1-a_d})(p-c)-p\Big) + (h_d - h_u)(1+\frac{2\alpha}{\beta}) + \alpha c$ we get:

$$-W_{-1}\Big(-\frac{(\alpha\Big(\Pi(\frac{I_d}{1-a_d})(p-c)-p\Big) + (h_d-h_u)(1+\frac{2\alpha}{\beta}) + \alpha c)\beta}{\alpha(2h_d-h_u)\exp\Big(1+\frac{\beta c}{2h_d-h_u}\Big)}\Big) > 1 + \frac{\beta c}{2h_d - h_u}$$

where $\frac{I_d}{1-a_d}$ is determined using Eq. 37.

Next, we have $I_d + I_u > I_d$ if:

$$-\frac{1}{\beta}W_{-1}\Big(-\frac{H_u\beta}{h_u\alpha\exp(1+\frac{\beta c}{h_u})}\Big) - \frac{c}{h_u} > -\frac{1}{\beta}W_{-1}\Big(-\frac{H_d\beta}{\alpha(2h_d-h_u)\exp\Big(1+\frac{\beta c}{2h_d-h_u}\Big)}\Big) - \frac{c}{2h_d-h_u}.$$

Using $H_u = \alpha\Big(\Pi(\frac{I_d+I_u}{1-a_u})(p-c)-p\Big) + \big(1+\frac{2\alpha}{\beta}\big)h_u + \alpha c$ we get:

$$\begin{aligned}
&-\frac{1}{\beta}W_{-1}\Big(-\frac{(\alpha\Big(\Pi(\frac{I_d+I_u}{1-a_u})(p-c)-p\Big) + \big(1+\frac{2\alpha}{\beta}\big)h_u + \alpha c)\beta}{h_u\alpha\exp(1+\frac{\beta c}{h_u})}\Big) - \frac{c}{h_u} \\
&> -\frac{1}{\beta}W_{-1}\Big(-\frac{(\alpha\Big(\Pi(\frac{I_d}{1-a_d})(p-c)-p\Big) + (h_d-h_u)(1+\frac{2\alpha}{\beta}) + \alpha c)\beta}{\alpha(2h_d-h_u)\exp\Big(1+\frac{\beta c}{2h_d-h_u}\Big)}\Big) - \frac{c}{2h_d-h_u}
\end{aligned}$$

where $\frac{I_d}{1-a_d}$ is determined using Eq. 37 and $\frac{I_d+I_u}{1-a_u}$ is determined using Eq. 36.

Next, we show that $a_d > a_u$. We have (using Eq. 19):

$$\begin{aligned}
E^{\frac{I_1}{1-a_1}} &= \frac{1}{\beta} - \frac{\hat{c}(1+\frac{2\alpha}{\beta})}{\alpha(p-c)} \\
E^{\frac{I_1+I_2}{1-a_2}} &= \frac{1}{\beta} - \frac{\hat{c}(1+\frac{2\alpha}{\beta})}{\alpha(p-c)}
\end{aligned}$$

As a result, $\frac{I_1}{1-a_1} = \frac{I_1+I_2}{1-a_2}$. Because $I_1 + I_2 > I_1$, we have: $a_1 > a_2$.