# Hermitian versus non-Hermitian representations for minimal length uncertainty relations

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ABSTRACT: We investigate four different types of representations of deformed canonical variables leading to generalized versions of Heisenberg's uncertainty relations resulting from noncommutative spacetime structures. We demonstrate explicitly how the representations are related to each other and study three characteristically different solvable models on these spaces, the harmonic oscillator, the manifestly non-Hermitian Swanson model and an intrinsically noncommutative model with Pöschl-Teller type potential. We provide an analytical expression for the metric in terms of quantities specific to the generic solution procedure and show that when it is appropriately implemented expectation values are independent of the particular representation. A recently proposed inequivalent representation resulting from Jordan twists is shown to lead to unphysical models. We suggest an anti-$\mathcal{PT}$-symmetric modification to overcome this shortcoming.

## 1. Introduction

Generalized versions of Heisenberg's uncertainty relations for deformed canonical variables have attracted a considerable amount of attention [1, 2, 3, 4, 5, 6, 7, 8] since they lead to the interesting feature of minimal lengths and minimal momenta. In three dimensions it was explicitly shown [6] that the deformed canonical variables are related to noncommutative spacetime structures and the corresponding analogues of the creation and annihilation operators satisfy $q$-deformed oscillator algebras [9, 10]. We will focus here on a one dimensional version of a noncommutative space which results as a decoupled direction from a three dimensional version as shown in [6]

$$[X, P] = i\hbar \left(1 + \check{\tau} P^2\right). \tag{1.1}$$

Here $\check{\tau} := \tau/(m\omega\hbar) > 0$ has the dimension of an inverse squared momentum and $\tau$ is therefore dimensionless. Our intention is here to investigate different types of models for different representations for the operators obeying these relations. We will compare four representations, denoted as $\Pi_{(i)}$ with $i \in \{1,2,3,4\}$, for $X$ and $P$ in relation (1.1) expressed in terms of the standard canonical variables $x$ and $p$ satisfying $[x,p] = i\hbar$

$$X_{(1)} = (1+\check{\tau}p^2)x, \; P_{(1)} = p, \qquad X_{(2)} = (1+\check{\tau}p^2)^{1/2}x(1+\check{\tau}p^2)^{1/2}, \; P_{(2)} = p, \tag{1.2}$$

$$X_{(3)} = x, \; P_{(3)} = \frac{1}{\sqrt{\check{\tau}}}\tan\left(\sqrt{\check{\tau}}p\right), \quad X_{(4)} = ix(1+\check{\tau}p^2)^{1/2}, \; P_{(4)} = -ip(1+\check{\tau}p^2)^{-1/2}. \tag{1.3}$$

Representation $\Pi_{(1)}$ is most obvious and most commonly used, but manifestly non-Hermitian with regard to the standard inner product. This is adjusted in the Hermitian representation $\Pi_{(2)}$ obtained from $\Pi_{(1)}$ by an obvious similarity transformation, i.e. $\Pi_{(2)} = (1+\check{\tau}p^2)^{-1/2}\Pi_{(1)}(1+\check{\tau}p^2)^{1/2}$.

Representation $\Pi_{(3)}$ is Hermitian in the standard sense, albeit less evident. Apart from an additional term in $X_{(3)}$ commuting with $P_{(3)}$, it appeared already in [2] where it was found to be a representation acting on the quasiposition wave function. Below we demonstrate that for some concrete models it is also related in a non-obvious way to $\Pi_{(1)}$ by the transformations to be outlined in section 2.

We have also a particular interest in representation $\Pi_{(4)}$ as it can be constructed systematically from Jordan twists accompanied by an additional rotation. In [11] a closely related version of this representation, which we denote by $\Pi_{(4')}$, occurred without the additional factors $i$ and $-i$ in $X_{(4)}$ and $P_{(4)}$, respectively. However, it is easily checked that this is incorrect and does not produce the commutation relations (1.1), as instead this variant produces a minus sign on the right hand side in front of the $\check{\tau}P^2$-term. One might consider that version of a noncommutative space, which will, however, lead immediately to more severe problems such as a pole in the metric etc. We will argue here further that the construction provided in [11] results in unphysical models and requires the proposed adjustments.

We also note that representation $\Pi_{(4)}$ respects a different kind of $\mathcal{PT}$-symmetry. Whereas the $\mathcal{PT}$-symmetry $x \to -x$, $p \to p$, $i \to -i$ of the standard canonical variables is inherited in a one-to-one fashion by the deformed variables in representations $\Pi_{(i)}$ for $i = 1,2,3$, i.e. $X_{(i)} \to -X_{(i)}$, $P_{(i)} \to P_{(i)}$, $i \to -i$, it becomes an anti-$\mathcal{PT}$-symmetry for $\Pi_{(4)}$, that is $X_{(i)} \to X_{(i)}$, $P_{(i)} \to -P_{(i)}$, $i \to -i$. Both versions are of course symmetries of the commutation relation (1.1) and since both of them are antilinear involutions, they may equally well be employed to ensure the reality of spectra for operators respecting the symmetry [12, 13].

We expect that in concrete models the physics, such as the expectation values for observables, are independent of the representation. We will argue here that this is indeed the case.

Our manuscript is organized as follows: In section 2 we provide a general construction procedure which can be used to solve non-Hermitian models. In section 3, 4 and 5 we discuss the harmonic oscillator, the Swanson model and a model with Pöschl-Teller type potential, respectively, in terms of the aforementioned representations. Our conclusions are stated in section 6.

## 2. A general construction procedure for solvable non-Hermitian potentials

Once a Hamiltonian for a potential system is formulated on a noncommutative space it usually ceases to be of potential type. Our aim here is to find exact solutions for the corresponding Schrödinger equation. Let us first explain the general method we are going to employ. It consists of four main steps: In the first we convert the system to a potential one, in the second we construct the explicit solution to that system as a function of the energy eigenvalues $E$, in the third step we employ a quantization condition by means of the choice of appropriate boundary conditions and in the final step we have to construct an appropriate metric due to the fact that the Hamiltonian might not be Hermitian.

We exploit the fact that for a large class of one dimensional models on noncommutative spaces the Schrödinger equation involving a Hamiltonian $H(p)$ in momentum space acquires the general form

$$H(p)\psi(p) = E\psi(p) \qquad \Leftrightarrow \qquad -f(p)\psi''(p) + g(p)\psi'(p) + h(p)\psi(p) = E\psi(p), \tag{2.1}$$

with $f(p)$, $g(p)$, $h(p)$ being some model specific functions and $E$ denoting the energy eigenvalue. This version of the equation may be converted to a potential system, see for instance [14, 15],

$$\tilde{H}(q)\psi(q) = E\psi(q) \qquad \Leftrightarrow \qquad -\phi''(q) + V(q)\phi(q) = E\phi(q), \tag{2.2}$$

when transforming simultaneously the wavefunction and the momentum,

$$\psi(p) = e^{\chi(p)}\phi(p), \quad \chi(p) = \int \frac{f'(p) + 2g(p)}{4f(p)}dp, \qquad \text{and} \qquad q = \int f^{-1/2}(p)dp, \tag{2.3}$$

respectively. In terms of the original functions $f(p)$, $g(p)$ and $h(p)$, as defined by equation (2.1), the potential is of the form

$$V(q) = \left.\frac{4g^2 + 3\left(f'\right)^2 + 8gf'}{16f} - \frac{f''}{4} - \frac{g'}{2} + h\right|_q . \tag{2.4}$$

At this stage one could simply compare with the literature on solvable potentials in order to extract an explicit solution. However, as the literature contains conflicting statements and ambiguous notations, we will present here a simple and transparent construction method for the solutions adopted from [14, 16].Furthermore, the quantities constructed in the next step occur explicitly in the expression for the metric. For the purpose of constructing a solvable potential we factorize the wavefunction $\phi(q)$ in (2.2) further into

$$\phi(q) = v(q)F[w(q)] \tag{2.5}$$

with as yet unknown functions $v(q)$, $w(q)$ and $F(w)$. This Ansatz converts the potential equation back into a second order equation of the type (2.1), albeit for the function $F(w)$,

$$F''(w) + Q(w)F'(w) + R(w)F(w) = 0, \tag{2.6}$$

where

$$Q(w) := \frac{2v'}{vw'} + \frac{w''}{(w')^2} \qquad \text{and} \qquad R(w) := \frac{E - V(q)}{(w')^2} + \frac{v''}{v\,(w')^2}. \tag{2.7}$$

Using the first relation in (2.7) we can express $v$ entirely in terms of $w$ and $Q$

$$v(q) = \left(w'\right)^{-1/2} \exp\left[\frac{1}{2}\int^{w(q)} Q(\tilde{w})d\tilde{w}\right]. \tag{2.8}$$

With the help of this expression we eliminate $v$ from the second relation in (2.7) and express the difference between the energy eigenvalue and the potential as

$$E - V(q) = \frac{w'''}{2w'} - \frac{3}{4}\left(\frac{w''}{w'}\right)^2 + \left(w'\right)^2 R(w) - \frac{(w')^2\, Q'(w)}{2} - \frac{(w')^2\, Q^2(w)}{4}. \tag{2.9}$$

Assuming now that $F$ as introduced in (2.5) is a particular special function satisfying the second order differential equation (2.6) with known $Q(w)$ and $R(w)$, the only unknown quantity left on the right hand side of (2.9) is $w(q)$. In the general pursuit of constructing solvable potentials one then selects terms on the right hand side of (2.9) to match the constant $E$ which in turn fixes the function $w$. The remaining terms on the right hand side must then compute to a meaningful potential. For the case at hand it has to equal $V(q)$ as computed in (2.4). Assembling everything one has therefore obtained an explicit form for $\phi(q)$ in (2.5) and hence $\psi(p)$, as given in (2.3), together with the energy eigenvalues $E$.

In the next step we need to implement the appropriate boundary conditions and quantize $\psi(p)$ to a well-defined $L^2(\mathbb{R})$-function $\psi_n(p)$ for discrete eigenvalues $E_n$.

What is left is to construct an appropriate metric, since some of our Hamiltonians $H$ are non-Hermitian with regard to the standard inner product, either resulting from the fact that we use a non-Hermitian representation or from the Hamiltonian being manifestly non-Hermitian in the first place, or a combination of both. In any of those two cases we have to re-define the metric $\rho$ on our Hilbert space involving the new inner product

$$\left\langle \tilde{\psi} \middle| \psi \right\rangle_\rho := \left\langle \tilde{\psi} \middle| \rho\psi \right\rangle, \tag{2.10}$$

with $\tilde{\psi}$ and $\psi$ being eigenstates of $H$. The metric operator $\rho$ can be constructed in various ways as outlined in the recent literature on non-Hermitian systems [17, 13, 18, 19, 20, 21]. When $\rho$ factorizes into a new operator $\eta$ we can use it to construct an isospectral Hermitian $h$ counterpart for $H$. The following relations

$$h = \eta H \eta^{-1} = h^\dagger \qquad \Leftrightarrow \qquad H^\dagger = \rho H \rho^{-1} \quad \text{with } \rho = \eta^\dagger \eta \tag{2.11}$$

hold. In other words assuming the existence of an inverse for $\rho$ and its factorization in form of (2.11) we can derive a Hermitian counterpart $h$ for $H$ and vice versa.

In general, it is difficult to construct $\eta$ or $\rho$ and in most concrete cases only perturbative solutions are known. However, for the scenario outlined in this section we can present a closed analytical formula. Assuming at this stage further that the function $F$, as introduced in (2.5), is an orthonormal function we have

$$\delta_{n,m} = \int \varrho(w) F_n\left(w\right) F_m\left(w\right)^* dw = \int \varrho(p) e^{-2\,\mathrm{Re}\,\chi(p)} \left|v(p)\right|^{-2} \frac{dw}{dp} \psi_n(p) \psi_n^*(p) dp, \tag{2.12}$$

such that the metric is read off as

$$\rho(p) = \varrho(p) e^{-2\,\mathrm{Re}\,\chi(p)} \left| v(p) \right|^{-2} \frac{dw}{dp}. \tag{2.13}$$

In the first integral we might need the additional metric $\varrho(w)$ in case the special function $F(w)$ is not taken to be orthonormal. All quantities on the right hand side are explicitly known at this point of the construction allowing us to compute $\rho(p)$ directly. We note that the positivity of the metric in entirely governed by $dw/dp$.

## 3. The harmonic oscillator in different representations

At first we will consider the harmonic oscillator in different representations

$$H_{(i)} = \frac{P_{(i)}^2}{2m} + \frac{m\omega^2}{2} X_{(i)}^2, \qquad \text{for } i = 1, 2, 3, 4, \tag{3.1}$$

with a particular focus on $\Pi_{(4)}$, which has not been dealt with so far. In principle the solutions for $\Pi_{(1)}$ are known, but it is instructive to consider here briefly how they emerge in the above scheme. In terms of the standard canonical variables the Hamiltonian reads

$$H_{(1)}(p) = \frac{p^2}{2m} + \frac{m\omega^2}{2}\left(x^2 + \check{\tau} p^2 x^2 + \check{\tau} x p^2 x + \check{\tau}^2 p^2 x p^2 x\right), \qquad \text{for } p \in \mathbb{R}. \tag{3.2}$$

With $x = i\hbar\partial_p$, the corresponding Schrödinger equation in momentum space acquires the general form (2.1), where we identify

$$f(p) = \frac{m\omega^2\hbar^2}{2}(1 + \check{\tau} p^2)^2, \quad g(p) = -\tau\hbar\omega p(1 + \check{\tau} p^2), \quad \text{and} \quad h(p) = \frac{p^2}{2m}. \tag{3.3}$$

Then the equations (2.3) and (2.4) convert this into an equation for a potential system with Hamiltonian $\tilde{H}_{(1)}(q)$

$$\psi(p) = \phi(p), \quad q = \sqrt{\frac{2}{\tau\omega\hbar}} \arctan\left(\sqrt{\check{\tau}} p\right), \quad \text{and} \quad V(q) = \frac{\hbar\omega}{2\tau} \tan^2\left(\sqrt{\frac{\tau\omega\hbar}{2}} q\right). \tag{3.4}$$

The $\tan^2$-potential is well known to be solvable, which is explicitly seen as follows. Assuming that $F(w)$ is an associated Legendre polynomial $P_\nu^\mu(w)$ we identify from the defining differential equation for these functions, see e.g. [22], the coefficient functions in (2.6) as

$$Q(w) = \frac{2w}{w^2 - 1} \qquad \text{and} \qquad R(w) = \frac{\nu(\nu+1)}{1 - w^2} - \frac{\mu^2}{(1 - w^2)^2}. \tag{3.5}$$

Then equation (2.9) acquires the form

$$E - \frac{\hbar\omega}{2\tau} \tan^2\left(\sqrt{\frac{\tau\omega\hbar}{2}} q\right) = \left(w'\right)^2 \left(\frac{\nu^2 + \nu + 1}{1 - w^2} + \frac{w^2 - \mu^2}{(1 - w^2)^2}\right) - \frac{3\left(w''\right)^2}{4\left(w'\right)^2} + \frac{w'''}{2w'} \tag{3.6}$$

for the unknown function $w(q)$ and constant $E$. Assuming that the first term on the right hand side gives rise to a constant, i.e. $(w')^2/(1-w^2) = c \in \mathbb{R}^+$, we obtain $w(q) = \sin(\sqrt{c}q)$ as solution of the latter equation. This function solves (3.6) with the identifications

$$E = \frac{\tau\omega\hbar}{8}(1+2\nu)^2 - \frac{\hbar\omega}{2\tau}, \qquad c = \frac{\tau\omega\hbar}{2}, \qquad \text{and} \qquad \mu = \mu_\pm = \pm\frac{1}{\tau}\sqrt{1+\frac{\tau^2}{4}}. \tag{3.7}$$

It remains to compute $v(q)$, which results from (2.8), such that all quantities assembled yield $\phi(q)$ in (2.5) as

$$\phi(q) = \sqrt{\cos(\sqrt{\tau\omega\hbar/2}q)}P_\nu^{\mu_\pm}\left[\sin(\sqrt{\tau\omega\hbar/2}q)\right]. \tag{3.8}$$

Hence with (2.3) we obtain finally a solution to the Schrödinger equation in momentum space involving the Hamiltonian $H_{(1)}(x,p)$

$$\psi(p) = \frac{1}{(1+\check{\tau}p^2)^{1/4}}P_\nu^{\mu_\pm}\left(\frac{\sqrt{\check{\tau}}p}{\sqrt{1+\check{\tau}p^2}}\right). \tag{3.9}$$

At this stage the constant $\nu$ is still unspecified. Implementing now the final step, the boundary conditions $\lim_{p\to\pm\infty}\psi(p) = 0$ yields the quantization condition for the energy. Using the property $\lim_{z\to\pm1}P_{n-m}^{m}(z) = 0$ for $n \in \mathbb{N}$, $m < 0$ we need to chose $\mu_-$ in (3.9), such that $\nu = n + 1/\tau\sqrt{1+\tau^2/4}$. Therefore the asymptotically vanishing eigenfunctions become

$$\psi_n(p) = \frac{1}{\sqrt{N_n}}\frac{1}{(1+\check{\tau}p^2)^{1/4}}P_{n-\mu_-}^{\mu_-}\left(\frac{\sqrt{\check{\tau}}p}{\sqrt{1+\check{\tau}p^2}}\right), \tag{3.10}$$

with corresponding energy eigenvalues

$$E_n = \omega\hbar\left(\frac{1}{2}+n\right)\sqrt{1+\frac{\tau^2}{4}} + \frac{\tau\omega\hbar}{4}(1+2n+2n^2), \tag{3.11}$$

and normalization constant $N_n$. The expression for $E_n$ agrees precisely with the one previously obtained in [2, 6] by different means. The corresponding eigenfunctions $\psi_n(p)$ are clearly $L^2(\mathbb{R})$-function, but since $H_{(1)}$ is non-Hermitian we do not expect them to be orthonormal. Noting that

$$\delta_{n,m} = \frac{1}{\sqrt{N_n N_m^*}}\int_{-1}^{1}P_{n-\mu_-}^{\mu_-}(w)\left[P_{m-\mu_-}^{\mu_-}(w)\right]^* dw, \tag{3.12}$$

with normalization constant $N_n := \int_{-1}^{1}\left|P_{n-\mu_-}^{\mu_-}(z)\right|^2 dz$, we use $w = \sqrt{\check{\tau}}p/\sqrt{1+\check{\tau}p^2}$ to compute the metric from (2.13). We obtain $\rho(p) = \sqrt{\check{\tau}}\left(1+\check{\tau}p^2\right)^{-1}$, which apart from an irrelevant overall factor $\sqrt{\check{\tau}}$ is the same as the operator obtained from solving the relations $\rho H\rho^{-1} = H^\dagger$ as previously reported in [3, 6].

Since by (1.2) it follows immediately that $H_{(2)} = \rho^{1/2}H_{(1)}\rho^{-1/2}$, the solutions for the Hermitian Hamiltonian $H_{(2)}$ are easily obtained from those for $H_{(1)}$ as $\rho^{-1/2}\psi_n$ with identical energy eigenvalues (3.11).

For the representation $\Pi_{(3)}$ we notice that the associated Hamiltonian $H_{(3)}(p)$ is just a rescaled version of the Hamiltonian $\tilde{H}_{(1)}(q)$, i.e. $H_{(3)}(p) = \tilde{H}_{(1)}(q = p\sqrt{2/m}/\hbar\omega)$ with $-\pi/2\sqrt{\check{\tau}} \leq p \leq \pi/2\sqrt{\check{\tau}}$. Thus the solution for the corresponding Schrödinger equation is simply $\phi(q = p\sqrt{2/m}/\hbar\omega)$. The metric results to be simply an overall constant factor $\rho(p) = \sqrt{\check{\tau}}$, which is consistent with the fact that $\Pi_{(3)}$ is Hermitian with regard to the standard inner product.

Leaving the aforementioned problems for $\Pi_{(4')}$ aside, we may still consider whether it might yield a physically meaningful Hamiltonian. In terms of the standard canonical variables we obtain

$$H_{(4')}(p) = \frac{p^2}{2m(1+\check{\tau}p^2)} + \frac{m\omega^2}{2}\left(x^2 + \check{\tau}x^2p^2 - i\hbar\check{\tau}xp\right). \tag{3.13}$$

In momentum space the corresponding Schrödinger equation is of the form (2.1) with

$$f(p) = \frac{m\hbar^2\omega^2}{2}(1+\check{\tau}p^2), \quad g(p) = -\frac{3}{2}\tau\hbar\omega p, \quad \text{and} \quad h(p) = \frac{p^2}{2m}(1+\check{\tau}p^2)^{-1} - \frac{\tau\hbar\omega}{2}. \tag{3.14}$$

Then equations (2.3) and (2.4) yield

$$\psi(p) = (1+\check{\tau}p^2)^{-1/2}\phi(p),\ q = \sqrt{\frac{2}{\tau\omega\hbar}}\operatorname{arcsinh}\left(\sqrt{\check{\tau}}p\right),\ V(q) = \frac{\hbar\omega}{2\tau}\tanh^2\left(\sqrt{\frac{\tau\omega\hbar}{2}}q\right). \tag{3.15}$$

With the same assumption on $F(w)$ as made previously we obtain again the relation (3.6) with the difference that the $\tan^2$-potential on the left hand side is replaced by a $\tanh^2$-potential. We may produce the latter potential by assuming $(w')^2/(1-w^2) = -c$, for $c \in \mathbb{R}^+$, which is solved by $w(q) = i\sinh(\sqrt{c}q)$. However, the resulting energy eigenvalues $E = \hbar\omega/2\tau - c/4(1+2\nu)^2$ are not bounded from below, which renders the Hamiltonian $H_{(4')}$ as unphysical.

Using instead $H_{(4)}$ yields the same version of the Schrödinger equation, but all functions in (3.14) are all replaced with an overall minus sign. The corresponding quantities in (3.15) are to be replaced by $\psi(p) = (1+\check{\tau}p^2)^{-1/2}P^{\mu_-}_{m-\mu_-}\left(\sqrt{\check{\tau}}p\right)$ with $-i/\sqrt{\check{\tau}} \leq p \leq i/\sqrt{\check{\tau}}$, the parameter $q$ needs to be multiplied by $-i$ and in the potential the $\tanh^2$ becomes a $\tan^2$. Then the energy spectrum becomes physically meaningful, being identical to (3.11). The metric results to $\rho(p) = -i\sqrt{\check{\tau}}\left(1+\check{\tau}p^2\right)^{1/2}$ in this case. We may now simply carry out a rotation $p \to ip$, $x \to ix$ on expressions for the expectation values to ensure the Hermiticity of the relevant operators involved.

With the explicit solutions we may now verify that the expectation values are indeed the same for all representations. For an arbitrary function $F\left(P_{(i)}, X_{(i)}\right)$ we compute a universal expression

$$\left\langle \psi_{(i)} \middle| F\left(P_{(i)}, X_{(i)}\right)\psi_{(i)} \right\rangle_{\rho_{(i)}} = \frac{1}{N}\int_{-1}^{1} F\left[\frac{z}{\sqrt{\check{\tau}(1-z^2)}}, i\hbar\sqrt{\check{\tau}(1-z^2)}\partial_z\right]\left|P^{\mu_-}_{m-\mu_-}(z)\right|^2 dz, \tag{3.16}$$

for $i = 1,2,3,4$. In particular we have $\left\langle \psi_{(i)} \middle| H_{(i)}\,\psi_{(i)} \right\rangle_{\rho_{(i)}} = E_n$, $\left\langle \psi_{(i)} \middle| P_{(i)}\,\psi_{(i)} \right\rangle_{\rho_{(i)}} = 0$.

## 4. The Swanson model in different representations

Let us next consider a model which is a widely studied [23] solvable prototype example to investigate non-Hermitian systems, the so-called Swanson model [24]. On a noncommutative space it reads

$$
\begin{aligned}
H_{(i)} &= \hbar\omega\left(A_{(i)}^{\dagger}A_{(i)}+\frac{1}{2}\right)+\alpha A_{(i)}A_{(i)}+\beta A_{(i)}^{\dagger}A_{(i)}^{\dagger} \qquad \text{for } i=1,2,3,4, && (4.1)\\
&= \frac{\hbar\omega(1-\tau)-\alpha-\beta}{2m\hbar\omega}P_{(i)}^{2}+\frac{\Omega m\omega}{2\hbar}X_{(i)}^{2}+i\left(\frac{\alpha-\beta}{2\hbar}\right)\left(X_{(i)}P_{(i)}+P_{(i)}X_{(i)}\right), && (4.2)
\end{aligned}
$$

with $A_{(j)}=\left(m\omega X_{(j)}+iP_{(j)}\right)/\sqrt{2m\hbar\omega}$, $A_{(j)}^{\dagger}=\left(m\omega X_{(j)}-iP_{(j)}\right)/\sqrt{2m\hbar\omega}$ and $\Omega:=\alpha+\beta+\hbar\omega$, $\alpha,\beta\in\mathbb{R}$ with dimension of energy. Evidently for the standard inner product we have in general $H_{(i)}\neq H_{(i)}^{\dagger}$ when $\alpha\neq\beta$; even for $\tau=0$. Let us now study this model for the different types of representations. Starting with $\Pi_{(1)}$, we obtain the Schrödinger equation in momentum space once again in the form of (2.1), with

$$
\begin{aligned}
f(p) &= \frac{m\hbar\omega\Omega}{2}(1+\check{\tau}p^{2})^{2}, \quad g(p)=(\beta-\alpha-\tau\Omega)p(1+\check{\tau}p^{2}), && (4.3)\\
h(p) &= \frac{\beta-\alpha}{2}-\frac{\tau(\alpha-\beta+\hbar\omega)+\alpha+\beta-\hbar\omega}{2hm\omega}p^{2}.
\end{aligned}
$$

Then equations (2.3) and (2.4) yield

$$
\psi(p) = (1+\check{\tau}p^{2})^{\frac{(\beta-\alpha)}{2\tau\Omega}}\phi(p), \quad q=\sqrt{\frac{2}{\tau\Omega}}\arctan\left(\sqrt{\check{\tau}}p\right), \tag{4.4}
$$

$$
V_{S\tan}(q) = \frac{(1-\tau)\hbar^{2}\omega^{2}-\tau\hbar\omega(\alpha+\beta)-4\alpha\beta}{2\tau\Omega}\tan^{2}\left(\sqrt{\frac{\tau\Omega}{2}}q\right). \tag{4.5}
$$

Notice that we obtain again a $\tan^2$-potential, albeit with different constants involved. Using therefore as in the previous subsection the assumption that $F(w)$ is an associated Legendre polynomial, we compute with (3.5) the equation (3.6) with the left hand side replaced by $E-V_{S\tan}(q)$. With the same assumption on the function $w$, namely $(w')^2/(1-w^2)=c\in\mathbb{R}^+$, we obtain $w(q)=\sin(\sqrt{c}q)$ albeit now with $q$ taken from (4.4). The equivalent to equation (3.6) then yields

$$
E = \frac{\tau\Omega}{8}(1+2\nu)^{2}+\frac{4\alpha\beta+\tau\hbar\omega(\alpha+\beta)+\hbar^{2}(\tau-1)\omega^{2}}{2\tau\Omega}, \quad c=\frac{\tau\Omega}{2}, \tag{4.6}
$$

$$
\mu_{\pm} = \pm\frac{\sqrt{4\left(\hbar^{2}\omega^{2}-4\alpha\beta\right)+\tau\Omega(\tau\Omega-4\hbar\omega)}}{2\tau\Omega}. \tag{4.7}
$$

Since $\phi(p)$ takes on the same form as in (2.3) we obtain

$$
\psi_{n}(p)=\frac{1}{\sqrt{N_{n}}}(1+\check{\tau}p^{2})^{\frac{\beta-\alpha}{2\tau\Omega}-\frac{1}{4}}P_{n-\mu_{-}}^{\mu_{-}}\left(\frac{\sqrt{\check{\tau}}p}{\sqrt{1+\check{\tau}p^{2}}}\right), \tag{4.8}
$$

as a solution to the Schrödinger equation in momentum space involving the Hamiltonian $H_{(1)}(p)$ for $p\in\mathbb{R}$. We have used the same condition for the asymptotics of the wavefunction as stated before (3.10), such that the energy eigenvalues become

$$E_n = \frac{1}{4}\left[(\tau + 2n\tau + 2n^2\tau)\Omega + (2n+1)\sqrt{4\left(\hbar^2\omega^2 - 4\alpha\beta\right) + \tau\Omega(\tau\Omega - 4\hbar\omega)}\right]. \tag{4.9}$$

Notice that in the commutative limit $\tau \to 0$ we recover the well-known [24, 23] expression for the energy $E_n = (n + 1/2)\sqrt{\hbar^2\omega^2 - 4\alpha\beta}$. However, we find a discrepancy with the results reported in [15] when taking the parameter $\gamma$ in there to zero. The authors do not state any quantization condition, but besides that we can also not verify that the reported expression indeed satisfies the relevant Schrödinger equation.

In figure 1 we depict the onsets of the exceptional points as a function of the parameters $\alpha$ and $\beta$ with the remaining parameters fixed. We notice that for small values of $\alpha$ the domain for which the energy is real is usually reduced, i.e. a model which still has real energy eigenvalues on the standard space might develop complex eigenvalues on the noncommutative space of the type (2.1), e.g. for $\alpha = 2$, $\beta = 0.1$ we read off from the figure that $E_n(\tau = 0) \in \mathbb{R}$ whereas $E_n(\tau = 0.5) \notin \mathbb{R}$. In contrast, for larger values of $\alpha$ complex eigenvalues might become real again once the model is put onto the space of the type (2.1), e.g. for $\alpha = 15$, $\beta = 0.1$ we find $E_n(\tau = 0) \notin \mathbb{R}$ and $E_n(\tau = 0.5) \in \mathbb{R}$. Notice that the condition $\left(\hbar^2\omega^2 - 4\alpha\beta\right) > \tau\Omega(\tau\Omega/4 - \hbar\omega)$ which governs the reality of the energy in (4.9) is the same which controls the $\mathcal{PT}$-symmetry of the wavefunction $\psi(p)$, which is broken once $\mu_- \notin \mathbb{R}$.

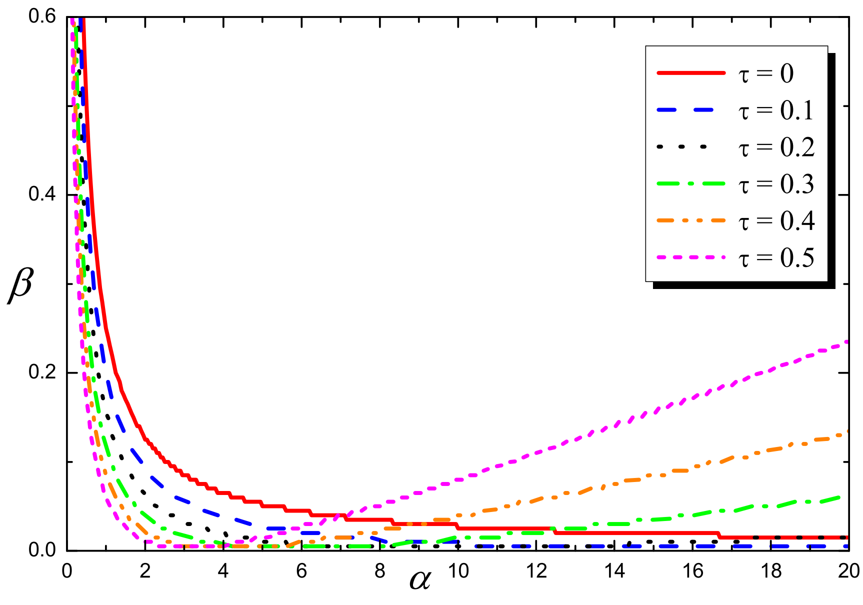


**Figure 1:** Domain of spontaneously broken (above the curve) and unbroken (below the curve) $\mathcal{PT}$-symmetry for the Swanson model with $\omega = 1$, $\hbar = 1$, $m = 1$ and different values of $\tau$.

With the help of (2.13) the metric is now computed to $\rho(p) = \sqrt{\check{\tau}}\left(1 + \check{\tau}p^2\right)^{\frac{\alpha-\beta-\tau\Omega}{\tau\Omega}}$.

Once again the solution for $H_{(2)}$ is $\rho^{-1/2}\psi_n$ with energy eigenvalues (4.9) due to $H_{(2)} = \rho^{1/2}H_{(1)}\rho^{-1/2}$.

Next we consider the representation $\Pi_{(3)}$. The Schrödinger equation in momentum space acquires the general form of (2.1), with

$$f(p) = \frac{m\hbar\omega\Omega}{2}, \quad g(p) = \frac{\beta-\alpha}{\sqrt{\check{\tau}}}\tan\left(\sqrt{\check{\tau}}p\right), \tag{4.10}$$
$$h(p) = \frac{\hbar\omega}{2} + \frac{\beta-\alpha-\hbar\omega}{2}\sec^2\left(\sqrt{\check{\tau}}p\right) + \frac{\hbar\omega-\alpha-\beta}{2\tau}\tan^2\left(\sqrt{\check{\tau}}p\right).$$

In this case the equations (2.3) and (2.4) yield

$$\psi(p) = \left[\cos\left(\sqrt{\check{\tau}}p\right)\right]^{\frac{\alpha-\beta}{\tau\Omega}}\phi(p), \quad q = \sqrt{\frac{2}{m\hbar\omega\Omega}}p, \quad V(q) = V_{S\tan}(q). \tag{4.11}$$

Notice that in the $q$-variables the potential obtained is exactly the same as the one previously computed (4.5) for representation $\Pi_{(3)}$. Thus we obtain the same equation (4.6) and (4.7) for the energy and the parameter $\mu$, respectively. However, the corresponding wavefunctions differ, resulting in this case to

$$\psi_n(p) = \frac{1}{\sqrt{N_n}}\left[\cos\left(\sqrt{\check{\tau}}p\right)\right]^{\frac{\alpha-\beta}{\tau\Omega}+\frac{1}{2}} P^{\mu_-}_{n-\mu_-}\left[\sin\left(\sqrt{\check{\tau}}p\right)\right], \tag{4.12}$$

for $-\pi/2\sqrt{\check{\tau}} \leq p \leq \pi/2\sqrt{\check{\tau}}$. We compute $\rho(p) = \sqrt{\check{\tau}}\left[\cos\left(\sqrt{\check{\tau}}p\right)\right]^{\frac{2(\beta-\alpha)}{\tau\Omega}}$ from (2.13) as relevant metric. Notice that $\rho(p)$ reduces to the standard metric for $\alpha = \beta$ reflecting the fact that $H_{(3)}$ is Hermitian for these values.

Since representation $\Pi_{(4')}$ was identified as being unphysical in the previous subsection, it is clear that this will also be the case for the Swanson model and we will therefore not treat it any further here.

For $\Pi_{(4)}$ the Schrödinger equation in momentum space is also of the form (2.1), with

$$f(p) = \frac{m\hbar\omega\Omega}{2}(1+\check{\tau}p^2), \quad g(p) = p\left(\beta-\alpha+\frac{3}{2}\tau\Omega\right), \tag{4.13}$$
$$h(p) = \frac{1}{2(1+\check{\tau}p^2)}\left\{(\beta-\alpha+\tau\Omega) + \frac{p^2}{m\hbar\omega}\left[\alpha+\beta-\hbar\omega+\tau(2\beta-2\alpha+\hbar\omega)+\tau^2\Omega\right]\right\}.$$

In this case the equations (2.3) and (2.4) yield

$$\psi(p) = (1+\check{\tau}p^2)^{\frac{\alpha-\beta}{2\tau\Omega}-\frac{1}{2}}\phi(p), \quad q = -i\sqrt{\frac{2}{\tau\Omega}}\operatorname{arcsinh}\left(\sqrt{\check{\tau}}p\right), \quad V(q) = V_{S\tan}(q). \tag{4.14}$$

Notice that in the $q$-variables the potential obtained is exactly the same as the one previously computed (4.5) for representation $\Pi_{(3)}$. Thus we obtain the same equations (4.6) and (4.7) for the energy eigenvalues and the parameter $\mu$, respectively. However, the final wavefunction differs, resulting, after imposing the boundary conditions, to

$$\psi_n(p) = \frac{1}{\sqrt{N_n}}(1-\check{\tau}p^2)^{\frac{\alpha-\beta}{2\tau\Omega}-\frac{1}{4}} P^{\mu_-}_{n-\mu_-}\left(\sqrt{\check{\tau}}p\right), \tag{4.15}$$

with $-1/\sqrt{\check{\tau}} \leq p \leq 1/\sqrt{\check{\tau}}$. Now we evaluate $\rho(p) = \sqrt{\check{\tau}}\left(1-\check{\tau}p^2\right)^{\frac{\beta-\alpha}{\tau\Omega}+\frac{1}{2}}$ as metric from our general formula (2.13). We have already performed a rotation of $\pi/2$ in $p$ for the reasons indicated in the previous section.

We compute again the expectation values for some arbitrary function $F\left(P_{(i)}, X_{(i)}\right)$ in all four representations

$$\left\langle \psi_{(i)} \right| F\left(P_{(i)}, X_{(i)}\right) \psi_{(i)} \Big\rangle_{\rho_{(i)}} = \frac{1}{N} \int_{-1}^{1} F\left[\frac{z}{\sqrt{\check{\tau}(1-z^2)}}, i\hbar\sqrt{\check{\tau}(1-z^2)}\partial_z\right] \left| P_{m-\mu_-}^{\mu_-}(z) \right|^2 dz, \tag{4.16}$$

which looks formally exactly the same as (3.16) with the difference that $\mu_-$ is given by the expression in (4.7).

## 5. A Pöschl-Teller potential in disguise

In the previous sections we observed that simple models on a noncommutative space may lead to more unexpected solvable potential systems when expressed in terms of the standard canonical variables and a subsequent transformation. We may also reverse the question and explore which type of model on a noncommutative space one obtains when we start from a well-known solvable potential in the standard canonical variables. For instance, we wish to construct the widely studied Pöschl-Teller potential [25]. Since the transformations are difficult to invert, we use trial and error and find that this indeed achieved when starting with the Hamiltonian

$$H_{(i)} = \frac{\beta}{2m} P_{(i)}^2 + \frac{\hbar\omega\alpha}{2\check{\tau}} P_{(i)}^{-2} + \frac{m\omega^2}{2} X_{(i)}^2 + \frac{\hbar\omega\alpha}{2} + \frac{\beta}{2m\check{\tau}} \quad \text{for } i = 1,2,3,4; \alpha, \beta \in \mathbb{R}. \tag{5.1}$$

We note that this Hamiltonian can not be viewed as a deformation of a model on a standard commutative space as it is intrinsically noncommutative, in the sense that it does not possess a trivial commutative limit $\tau \to 0$. Proceeding as in the previous subsections we find for the representation $\Pi_{(1)}$ that the Schrödinger equation in momentum space is once more of the general form of (2.1), with

$$f(p) = \frac{m\hbar^2\omega^2}{2}(1+\check{\tau}p^2)^2, \quad g(p) = -m\hbar^2\omega^2\check{\tau}p(1+\check{\tau}p^2), \quad h(p) = \frac{(1+\check{\tau}p^2)(\alpha m\hbar\omega + \beta p^2)}{2m\check{\tau}p^2}. \tag{5.2}$$

From equation (2.3) we obtain now

$$\psi(p) = \phi(p), \quad q = \sqrt{\frac{2}{\tau\hbar\omega}} \arctan\left(\sqrt{\check{\tau}}p\right), \tag{5.3}$$

and as anticipated we compute a Pöschl-Teller potential with the help of equation (2.4)

$$V_{PT}(q) = \frac{\hbar\omega\alpha}{2} \csc^2\left(q\sqrt{\frac{\hbar\omega\tau}{2}}\right) + \frac{\beta}{2m\check{\tau}} \sec^2\left(q\sqrt{\frac{\hbar\omega\tau}{2}}\right). \tag{5.4}$$

Assuming now that the special function $F(w)$ in (2.6) is a Jacobi polynomial $P_n^{(a,b)}(w)$, with $n \in \mathbb{N}_0$, $a, b \in \mathbb{R}$, we identify from its defining differential equation, see e.g. [22], the coefficient functions in (2.6) as

$$Q(w) = \frac{b-a-(2+a+b)w}{1-w^2} \qquad \text{and} \qquad R(w) = \frac{n(n+1+a+b)}{1-w^2}. \tag{5.5}$$

Then equation (2.9) is evaluated to

$$
\begin{aligned}
E - V_{PT}(q) &= \frac{n\,(w')^2\,(a+b+n+1)}{1-w^2} + \frac{(w')^2\left[w^2(a+b+2)+2w(a-b)+a+b+2\right]}{2\,(1-w^2)^2} \\
&\quad - \frac{(w')^2\,[b-a-w(a+b+2)]^2}{4\,(1-w^2)^2} - \frac{3\,(w'')^2}{4\,(w')^2} + \frac{w'''}{2w'}, \qquad (5.6)
\end{aligned}
$$

with as yet unknown function $w(q)$ and constant $E$. As in the previous section we assume again that the first term on the right hand side gives rise to a constant, i.e. $(w')^2/(1-w^2) = c \in \mathbb{R}^+$, but this time we choose the solution $w(q) = \cos(\sqrt{c}q)$, which solves (3.6) with the identifications

$$
E_n = \frac{\hbar\omega\tau}{2}(1+2n+a+b)^2, \quad c = 2\tau\omega\hbar, \quad a_\pm = \pm\frac{1}{2}\sqrt{1+\frac{4\alpha}{\tau}}, \quad b_\pm = \pm\frac{1}{2}\sqrt{1+\frac{4\beta}{\tau^2}}. \qquad (5.7)
$$

Computing $v(q)$ by means of (2.8) we assemble everything into the solution of the Schrödinger equation involving $H_{(1)}(x,p)$

$$
\psi_n(p) = \frac{1}{\sqrt{N_n}} p^{1/2+a_+}(1+\check{\tau}p^2)^{-(1+a_+ + b_+)/2} P_n^{(a_+,b_+)}\left(\frac{1-\check{\tau}p^2}{1+\check{\tau}p^2}\right). \qquad (5.8)
$$

We have selected here $a_+$ and $b_+$ in order to implement the appropriate boundary conditions $\lim_{p\to\pm\infty}\psi_n(p) = 0$ together with $\psi_n(0) = 0$. We note that the energy eigenvalues are real and bounded from below as long as $\alpha > -\tau/4$ and $\beta > -\tau^2/4$. The occurrence of exceptional points is due to the $\mathcal{PT}$-symmetry breaking of the wavefunction $\psi_n(p)$ when $a_+, b_+ \notin \mathbb{R}$.

Following the same procedure as in the previous subsections we find for the remaining representations

$$
\tilde{H}_{(1)}(q) = \tilde{H}_{(2)}(q) = \tilde{H}_{(3)}(q) = \tilde{H}_{(4)}(q), \qquad (5.9)
$$

where $q$ is related to $p$ differently in each case. Converting between the different variables and computing the relevant pre-factors as in the previous subsection we then find

$$
\begin{aligned}
\psi_{(2)}(p) &= \rho_{(1)}^{-1/2}\psi_{(1)}(p), \qquad (5.10)\\
\psi_{(3)}(p) &= \frac{1}{\sqrt{N_n}} \frac{\left[1-\cos(2p\sqrt{\check{\tau}})\right]^{\frac{1+a_+}{2}}\left[\cos(2p\sqrt{\check{\tau}})+1\right]^{\frac{1+b_+}{2}}}{\sqrt{\sin(2p\sqrt{\check{\tau}})}} P_n^{(a_+,b_+)}\left[\cos\left(2p\sqrt{\check{\tau}}\right)\right], \qquad (5.11)\\
\psi_{(4)}(p) &= \frac{i}{\sqrt{N_n}}(ip)^{a_+ + 1/2}(1-\check{\tau}p^2)^{\frac{2b_+ - 1}{4}} P_n^{(a_+,b_+)}\left(1-2\check{\tau}p^2\right), \qquad (5.12)
\end{aligned}
$$

for the energy eigenvalue (5.7) where $p > 0$ for $\Pi_{(2)}$, $-\pi/2\sqrt{\check{\tau}} \le p \le \pi/2\sqrt{\check{\tau}}$ for $\Pi_{(3)}$ and $-1/\sqrt{\check{\tau}} \le p \le 1/\sqrt{\check{\tau}}$ for $\Pi_{(4)}$. Using the the orthogonality relation for the Jacobi polynomial,[1] we compute the metrics from (2.13) to $\rho_{(1)}(p) = -2\sqrt{\check{\tau}}(1+\check{\tau}p^2)^{-1}$, $\rho_{(2)}(p) = 1$, $\rho_{(3)}(p) = -2\sqrt{\check{\tau}}$ and $\rho_{(4)}(p) = 2\sqrt{\check{\tau}}(1-\check{\tau}p^2)^{1/2}$.

[1]
$$
\int_{-1}^{1}(1-x)^a(1+x)^b P_n^{(a,b)}(x)\,P_m^{(a,b)}(x)\,dx = \delta_{n,m}N_n \qquad \text{for } \operatorname{Re} a, \operatorname{Re} b > -1
$$

with $N_n = \frac{2^{a+b+1}\Gamma(a+n+1)\Gamma(b+n+1)}{n!\Gamma(a+b+n+1)\Gamma(a+b+2n+1)}$.

We also note that for representation $\Pi_{(4')}$ we obtain the same potential (5.4) with $\csc^2 \to \operatorname{csch}^2$, $\sec^2 \to -\operatorname{sech}^2$ plus an overall constant, which is once again unphysical in the sense of leading to an unbounded spectrum from below.

Finally we compute the expectation values for some arbitrary function $F\left(P_{(i)}, X_{(i)}\right)$

$$\left\langle \psi_{(i)} \right| F\left(P_{(i)}, X_{(i)}\right) \psi_{(i)} \Big\rangle_{\rho_{(i)}} = \frac{1}{N}\int_{-1}^{1} F\left[\frac{z}{\sqrt{\check{\tau}(1-z^2)}}, i\hbar\sqrt{\check{\tau}(1-z^2)}\partial_z\right] \left| P_n^{(a_+,b_+)}(z)\right|^2 dz, \tag{5.13}$$

which is again the same for all four representations.

## 6. Conclusions

We have shown how different representations for the operators $X$ and $P$ obeying a generalized version of Heisenberg's uncertainty relation are related to each other by the transformations outlined in section. We have demonstrated their equivalence within the setting of three characteristically different types of solvable models, a Hermitian one, a non-Hermitian one and an intrisically noncommutative one. In all cases we showed that an appropriate metric can be found such that expectation values result to be representation independent. We provided an explicit formula for this metric, involving the quantities computed in the first two steps of the general procedure. The computations were carried out in momentum space, but naturally the method works equally well in standard $x$-space. In both cases the order of the differential equation imposes a limitation on the type of models which may be considered.

For representation $\Pi_{(4')}$ proposed in [11] we found that it does not lead to the uncertainty relations (1.1) and moreover that for the models investigated it always gives rise to unphysical spectra which are not bounded from below. This suggests that the general procedure of Jordan twists requires a mild modification as outlined in the manuscript.

Clearly it would be interesting to extent this analysis to different types of full three dimensional algebras for noncommutative spaces and investigate alternative representations, such as for instance for those already reported in [6].

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