The complete Gaussian kernel in the multi-factor Heston model: option pricing and implied volatility applications
Supplementary material

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1 Introduction

In this online supplementary material, we provide all the Theorems, Corollaries and Propositions introduced in the paper as well as some supplementary material. To make this online material self-consistent, we recall the main notation and quantities already introduced in the paper.

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- Section 7 Some details on asymptotic expansions
- Section 8 Further details on the expansions for option pricing and Greeks
- Section 9 Option price calibration

2 Detailed proofs of theoretical results in Section 2

To make this supplementary material self-consistent, we report some content contained in the associated paper by providing detailed proofs and some new results.

We consider the following stochastic volatility model

$$dx_t = \left(r(t) - \frac{1}{2}\sum_{j=1}^{n} v_{j,t} \right) dt + \sum_{j=1}^{n} \sqrt{v_{j,t}} dZ_{j,t}, \;\; t > 0, \tag{1}$$

$$dv_{j,t} = \chi_j (v_j^* - v_{j,t}) dt + \gamma_j \sqrt{v_{j,t}} dW_{j,t}, \;\; t > 0, \tag{2}$$

where x_t denotes the log-price variable and $v_{1,t}, \ldots, v_{n,t}$ the corresponding variances, while $r(t)$ is the instantaneous risk-free rate (assumed to be known in advance) and χ_j, v_j^*, γ_j are positive constants. $Z_{j,t}$, $W_{j,t}$, $j = 1, 2, \ldots, n$, are standard Wiener processes such that all correlations among the Wiener processes are zero except for $E(dZ_{j,t}, dW_{j,t}) = \rho_j \, dt$, with constant correlation coefficients $\rho_j \in (-1, 1)$, $j = 1, 2, \ldots, n$. Dividends are not included. The system of equations (1)-(2) is equipped with the following initial conditions:

$$x_0 = \log \widetilde{S}_0, \tag{3}$$

$$v_{j,0} = \widetilde{v}_{j,0}, \tag{4}$$

where $\widetilde{S}_0$ and $\widetilde{v}_{j,0}$, $j = 1, 2, \ldots, n$ are the initial spot price and variances respectively, which are assumed to be random variables concentrated at a point with probability one. Specifically, we use $\mathcal{G}_\Gamma$ to denote the Gaussian kernel with variance $\Gamma(t, t')$, $t < t'$, that is,

$$\mathcal{G}_\Gamma(y, t, t') = \frac{1}{\sqrt{2\pi \Gamma(t, t')}} e^{-\frac{1}{2\Gamma(t,t')} \left(y - \int_t^{t'} r(s) ds + \frac{1}{2}\Gamma(t,t') \right)^2} = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{\imath k \left[y - \int_t^{t'} r(s) ds + \frac{1}{2}\Gamma(t,t') \right] - \frac{1}{2}\Gamma(t,t') k^2} dk \, . \tag{5}$$

We consider two choices of Γ. The first is $\Gamma(t, t') = \Gamma_0(t, t')$:

$$\Gamma_0(t, t') = \sum_{j=1}^{n} \int_t^{t'} E(v_{j,s} \,|\, \mathcal{F}_t) ds, \tag{6}$$

where $\mathcal{F}_t$ is the information set, i.e., the continuous σ-algebra generated by the point-in-time volatility processes, while $E(v_{j,s} \,|\mathcal{F}_t)$ is the conditional mean of the point-in-time volatility given by

$$E(v_{j,t'} | \mathcal{F}_t) = v_{j,t} e^{-\chi_j (t'-t)} + v_j^* (1 - e^{-\chi_j (t'-t)}), \quad t < t' \, . \tag{7}$$

The second choice is $\Gamma(t, t') = \Gamma_2(t, t')$, where Γ_2 is defined as

$$\begin{aligned} \Gamma_2(t, t') &= \Gamma_0(t, t') - 2S_1(t, t') + 2S_2(t, t') \\ &= \sum_{j=1}^{n} \int_t^{t'} E(v_{j,s} \,|\, \mathcal{F}_t) \left[(1 - \rho_j^2) + \left(\frac{\gamma_j}{2\chi_j} \left(1 - e^{-\chi_j (t'-s)} \right) - \rho_j \right)^2 \right] ds, \end{aligned} \tag{8}$$

and S_1 and S_2 are given by

$$S_1(t,t') = \frac{1}{2}\sum_{j=1}^{n} \frac{\rho_j \gamma_j}{\chi_j} \int_t^{t'} E(v_{j,s}\,|\,\mathcal{F}_t)\left(1 - e^{-\chi_j(t'-s)}\right) ds, \tag{9}$$

$$S_2(t,t') = \sum_{j=1}^{n} \frac{\gamma_j^2}{8\chi_j^2} \int_t^{t'} E(v_{j,s}\,|\,\mathcal{F}_t)\left(1 - e^{-\chi_j(t'-s)}\right)^2 ds. \tag{10}$$

These kernels are related to the processes, $X_{t'}$ and $Y_{t'}$, associated with the multi-factor Heston model (1):

$$X_{t'} = \int_t^{t'} \sum_{j=1}^{n} \sqrt{v_{j,\tau}}\, dZ_{j,\tau}, \tag{11}$$

$$Y_{t'} = \int_t^{t'} \sum_{j=1}^{n} [v_{j,\tau} - E(v_{j,\tau}\,|\mathcal{F}_t)] d\tau\,. \tag{12}$$

According to Zhang et al. (2017), $X_{t'}$ measures the cumulative uncertainty of the asset return and $Y_{t'}$ the uncertainty of the integrated variance process over the time interval $[t,t']$. Note that Γ_0 is the conditional variance of $X_{t'}$, while Γ_2 is the conditional variance of the continuously compounded return $R_t^{t'}$ defined as

$$R_t^{t'} = x_{t'} - x_t = \int_t^{t'} \left[\left(r(\tau) - \frac{1}{2}\sum_{j=1}^{n} v_{j,\tau}\right) d\tau + \sum_{j=1}^{n} \sqrt{v_{j,\tau}}\, dZ_{j,\tau}\right],\ t < t', \tag{13}$$

with

$$E(R_t^{t'}\,|\,\mathcal{F}_t) = \int_t^{t'} \left[r(\tau) - \frac{1}{2}\sum_{j=1}^{n} E(v_{j,\tau}\,|\,\mathcal{F}_t)\right] d\tau,$$

which is related to the processes $X_{t'}$ and $Y_{t'}$ as follows

$$R_t^{t'} - E(R_t^{t'}\,|\,\mathcal{F}_t) = X_{t'} - \frac{1}{2} Y_{t'}. \tag{14}$$

Theorem 2.1, which provides two representation formulas for the above-mentioned marginal density, was already illustrated in the paper. Here we provide more details about the proof.

Theorem 2.1 *The marginal probability density of the log-price variable conditioned to $\underline{v}_t = \underline{v}$ is given by*

$$M(x,\underline{v},t,x',t') = \int_{\mathbb{R}^{n+}} p_f(x,\underline{v},t,x',\underline{v}',t') d\underline{v}'$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} \underbrace{e^{\imath k\left[(x'-x) - \int_t^{t'} r(s)ds + \frac{1}{2}\Gamma_0(t,t')\right] - \frac{1}{2}\Gamma_0(t,t')k^2}}_{Gaussian\,kernel\,Fourier\,transform} \underbrace{e^{\sum_{j=1}^{n}\int_t^{t'} E(v_{j,s}\,|\,\mathcal{F}_t)\left[\frac{\gamma_j^2}{2} B_j^2(k,s,t') + \imath k \rho_j \gamma_j B_j(k,s,t')\right] ds}}_{contribution\,from\,vols\,of\,vols} dk,$$

$$x,\, x' \in \mathbb{R},\ \underline{v} \in \mathbb{R}^{n^+},\ t, t' \geq 0, t' - t > 0, \tag{15}$$

where $\imath$ is the imaginary unit and $E(v_{j,s}\,|\,\mathcal{F}_t)$ is the conditional mean (7). Here, B_j is given by

$$B_j(k,t,t') = \frac{1}{2}(k^2 - \imath\, k)\frac{1 - e^{-2\zeta_j(t'-t)}}{(\zeta_j + \nu_j) + (\zeta_j - \nu_j)e^{-2\zeta_j(t'-t)}}, \tag{16}$$

where ζ_j and ν_j are the following quantities:

$$\zeta_j(k) = \frac{1}{2}\left(4\nu_j^2 + \gamma_j^2(k^2 - \imath\, k)\right)^{1/2}, \tag{17}$$

$$\nu_j(k) = \frac{1}{2}(\imath\, k\rho_j\gamma_j + \chi_j). \tag{18}$$

Furthermore, M can also be written as

$$M(x,\underline{v},t,x',t') = \int_{-\infty}^{+\infty} \mathcal{G}_{\Gamma_0}(x'-x-y,t,t')\mathcal{L}_{\underline{\gamma}}(y,t,t')dy, \tag{19}$$

where $\Gamma_0(t,t')$ is the integrated conditional variance given in (6), while $\mathcal{G}_{\Gamma_0}$ is the Gaussian kernel in (5) and $\mathcal{L}_{\underline{\gamma}}$ is the function that fully accounts for the effects of the vols of vols:

$$\mathcal{L}_{\underline{\gamma}}(y,t,t') = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{\imath\, k\, y} e^{\sum_{j=1}^{n}\int_t^{t'} E(v_{j,s}\,|\,\mathcal{F}_t)\left[\frac{\gamma_j^2}{2}B_j^2(k,s,t')+\imath\, k\rho_j\gamma_j B_j(k,s,t')\right]ds} dk\,. \tag{20}$$

Proof of Theorem 2.1 *We recall the backward Kolmogorov equation satisfied by the function M given in (15) as a function of the past log-price x and time t:*

$$-\frac{\partial M}{\partial t} = \frac{1}{2}\sum_{j=1}^{n} v_j\frac{\partial^2 M}{\partial x^2} + \frac{1}{2}\sum_{j=1}^{n}\gamma_j^2 v_j\frac{\partial^2 M}{\partial v_j^2} + \sum_{j=1}^{n}\gamma_j\rho_j v_j\frac{\partial^2 M}{\partial\, x\partial v_j} + \sum_{j=1}^{n}\chi_j(v_j^* - v_j)\frac{\partial M}{\partial v_j} + \left(r(t) - \frac{1}{2}\sum_{j=1}^{n} v_j\right)\frac{\partial M}{\partial x}, \tag{21}$$

with final condition

$$M(x,\underline{v},t',x',t') = \delta(x-x'), \tag{22}$$

where $\delta(\cdot)$ is the Dirac delta function. We look for M in the form

$$\begin{aligned} M(x,\underline{v},t,x',t') &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{\imath k(x'-x)-\imath\, k\int_t^{t'} r(s)ds + A(k,t,t') - \sum_{j=1}^{n} v_j B_j(k,t,t')}dk, \\ M(x,\underline{v},t,x',t') &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{\imath k(x'-x)) - \imath\, k\int_t^{t'} r(s)ds + Q(k,t,t',\underline{v};\underline{\Theta}_v)}dk, \\ & x,\, x' \in \mathbb{R},\ \underline{v}\in\mathbb{R}^{n^+},\ t,t'\geq 0, t'-t>0, \end{aligned} \tag{23}$$

where Q is defined as

$$Q(t'-t,\underline{v},k;\underline{\Theta}_v) = A(k,t,t') - \sum_{j=1}^{n} v_j B_j(k,t,t'). \tag{24}$$

Substituting Eq. (24) into Eq. (21), we obtain the Riccati equation satisfied by A and B_j (see Duffie et al. 2000; Fatone et al. 2009):

$$\frac{d}{dt}A = \sum_{j=1}^{n}\chi_j v_j^* B_j, \tag{25}$$

and for $j = 1, 2, \ldots, n$,

$$\frac{d}{dt}B_j = \chi_j B_j + \frac{1}{2}\gamma_j^2 B_j^2 + \imath\, k\,\rho_j\gamma_j B_j - \frac{k^2}{2} + \frac{\imath\, k}{2}, \tag{26}$$

with final conditions

$$A(k,t',t')=0,\;\; B_j(k,t',t')=0,\; j=1,2,\ldots,n. \tag{27}$$

We now rewrite Q in Eq. (24). Eqs. (25) and (27) give

$$A(k,t,t')=\sum_{j=1}^{n}A_j(k,t,t')=-\sum_{j=1}^{n}\chi_j v_j^*\int_t^{t'}B_j(k,\tau,t')d\tau\,, \tag{28}$$

where

$$A_j(k,t,t')=-\chi_j v_j^*\int_t^{t'}B_j(k,\tau,t')d\tau\,, \tag{29}$$

while Eqs. (26) and (27) give

$$\frac{d}{dt}\left(e^{-\chi_j t}B_j(k,t,t')\right)=e^{-\chi_j t}\left(\imath\, k\rho_j\gamma_j B_j(k,t,t')+\frac{1}{2}\gamma_j^2B_j^2(k,t,t')\right)-e^{-\chi_j t}\left(\frac{k^2}{2}-\imath\frac{k}{2}\right). \tag{30}$$

Integrating, we obtain

$$\int_t^{t'}\frac{d}{ds}\left(e^{-\chi_j s}B_j(k,s,t')\right)ds=\int_t^{t'}\left[e^{-\chi_j s}\left(\imath\, k\rho_j\gamma_j B_j(k,s,t')+\frac{1}{2}\gamma_j^2B_j^2(k,s,t')\right)-e^{-\chi_j s}\left(\frac{k^2}{2}-\imath\frac{k}{2}\right)\right]ds \tag{31}$$

and

$$e^{-\chi_j t'}B_j(k,t',t')-e^{-\chi_j t}B_j(k,t,t')=\int_t^{t'}\left[e^{-\chi_j s}\left(\imath\, k\rho_j\gamma_j B_j(k,s,t')+\frac{1}{2}\gamma_j^2B_j^2(k,s,t')\right)-e^{-\chi_j s}\left(\frac{k^2}{2}-\imath\frac{k}{2}\right)\right]ds. \tag{32}$$

Since $B_j(k,t',t')=0$ we have

$$\begin{aligned}-e^{-\chi_j t}B_j(k,t,t')&=\int_t^{t'}\left[e^{-\chi_j s}\left(\imath\, k\rho_j\gamma_j B_j(k,s,t')+\frac{1}{2}\gamma_j^2B_j^2(k,s,t')\right)-e^{-\chi_j s}\left(\frac{k^2}{2}-\imath\frac{k}{2}\right)\right]ds\,,\\ B_j(k,t,t')&=-\int_t^{t'}\left[e^{-\chi_j (s-t)}\left(\imath\, k\rho_j\gamma_j B_j(k,s,t')+\frac{1}{2}\gamma_j^2B_j^2(k,s,t')\right)-e^{-\chi_j (s-t)}\left(\frac{k^2}{2}-\imath\frac{k}{2}\right)\right]ds\,,\end{aligned} \tag{33}$$

so B_j can be written as

$$B_j(k,t,t')=-\int_t^{t'}e^{-\chi_j(s-t)}\left[\imath\, k\rho_j\gamma_j B_j(k,s,t')+\frac{1}{2}\gamma_j^2B_j^2(k,s,t')\right]ds-\left(-\frac{k^2}{2}+\imath\frac{k}{2}\right)\int_t^{t'}e^{-\chi_j(s-t)}ds. \tag{34}$$

From Eqs. (28) and (34) we have

$$\begin{aligned}A_j(k,t,t')&=-\chi_j v_j^*\int_t^{t'}B_j(k,\tau,t')d\tau\\ &=\chi_j v_j^*\int_t^{t'}\left[\int_\tau^{t'}\left[e^{-\chi_j(s-\tau)}\left(\imath\, k\rho_j\gamma_j B_j(k,s,t')+\frac{1}{2}\gamma_j^2B_j^2(k,s,t')\right)+e^{-\chi_j(s-\tau)}\left(-\frac{k^2}{2}+\imath\frac{k}{2}\right)\right]ds\right]d\tau\,.\end{aligned} \tag{35}$$

Inverting the integration order, we obtain

$$A_j(k,t,t') = \chi_j v_j^* \int_t^{t'} \left(\imath\, k\rho_j\gamma_j B_j(k,s,t') + \frac{1}{2}\gamma_j^2 B_j^2(k,s,t')\right)\left[\int_t^s e^{-\chi_j(s-\tau)}d\tau\right] ds$$
$$+\chi_j v_j^* \left(-\frac{k^2}{2}+\imath\frac{k}{2}\right)\int_t^{t'}\left[\int_t^s e^{-\chi_j(s-\tau)}d\tau\right] ds\,, \tag{36}$$

which reads

$$A_j(k,t,t') = \chi_j v_j^* \int_t^{t'} \left(\imath\, k\rho_j\gamma_j B_j(k,s,t') + \frac{1}{2}\gamma_j^2 B_j^2(k,s,t')\right)\left(\frac{1-e^{-\chi_j(s-t)}}{\chi_j}\right) ds$$
$$+\chi_j v_j^* \left(-\frac{k^2}{2}+\imath\frac{k}{2}\right)\int_t^{t'}\left(\frac{1-e^{-\chi_j(s-t)}}{\chi_j}\right) ds\,. \tag{37}$$

Using Eqs. (34) and (37), we obtain

$$A_j(k,t,t') - v_j B_j(k,t,t') = -\left(\frac{k^2}{2}-\imath\frac{k}{2}\right)\int_t^{t'}\left[v_j^*\left(1-e^{-\chi_j(s-t)}\right) + v_j e^{-\chi_j(s-t)}\right] ds$$
$$+\int_t^{t'}\left[\imath\, k\rho_j\gamma_j B_j(k,s,t') + \frac{1}{2}\gamma_j^2 B_j^2(k,s,t')\right]\left[v_j^*\left(1-e^{-\chi_j(s-t)}\right) + v_j e^{-\chi_j(s-t)}\right] ds\,. \tag{38}$$

Bearing in mind the conditional mean of the point-in-time volatility given in (7) and that v_j is the variance at time t, Eq. (38) becomes

$$A_j(k,t,t') - v_j B_j(k,t,t') = \int_t^{t'}\left[\imath\, k\rho_j\gamma_j B_j(k,s,t') + \frac{1}{2}\gamma_j^2 B_j^2(k,s,t') + \left(-\frac{k^2}{2}+\imath\frac{k}{2}\right)\right] E(v_{j,s}\,|\,\mathcal{F}_t) ds. \tag{39}$$

Eq. (39) implies

$$\sum_{j=1}^{n}\left(A_j(k,t,t') - v_j B_j(k,t,t')\right) =$$
$$= \quad -\frac{(k^2-\imath k)}{2}\Gamma_0(t,t') + \sum_{j=1}^{n}\int_t^{t'} E(v_{j,s}\,|\,\mathcal{F}_t)\left[\frac{1}{2}\gamma_j^2 B_j^2(k,s,t') + \imath\, k\rho_j\gamma_j B_j(k,s,t')\right] ds, \tag{40}$$

where Γ_0 is given in formula (6). This proves formula (15). Formula (19) follows if we apply the convolution theorem for the inverse Fourier transform to formula (15).

We now prove Eq. (16). First, we observe that B_j can be computed explicitly using a standard approach for the Riccati equations:

$$B_j = -\frac{2}{\gamma_j^2}\frac{\frac{d}{dt}C_j}{C_j}. \tag{41}$$

Substituting B_j into (26), we obtain

$$-\frac{2}{\gamma_j}\frac{\frac{d^2}{dt^2}C_j}{C_j} + \frac{2}{\gamma_j^2}\left(\frac{\frac{d}{dt}C_j}{C_j}\right)^2 = -\frac{2}{\gamma_j^2}(\chi_j + \imath\, k\rho_j\gamma_j)\frac{\frac{d}{dt}C_j}{C_j} + \frac{2}{\gamma_j^2}\left(\frac{\frac{d}{dt}C_j}{C_j}\right)^2 + \frac{1}{2}(-k^2+\imath\, k), \tag{42}$$

that is, C_j is the solution to the following initial value problem:

$$\frac{d^2}{dt^2}C_j - (\chi_j + \imath\, k\rho_j\gamma_j)\frac{d}{dt}C_j + \frac{\gamma_j^2}{4}(-k^2 + \imath\, k)C_j = 0, \tag{43}$$

with initial conditions

$$C_j(k, t', t') = 1, \quad \frac{d}{dt}C_j(k, t', t') = 0. \tag{44}$$

Solving problem (43), (44) we obtain

$$B_j(k, t, t') = \frac{1}{2}(k^2 - \imath\, k)\widetilde{B}_j(k, t, t'), \tag{45}$$

where

$$\widetilde{B}_j(k, t, t') = \frac{1 - e^{-2\zeta_j(t'-t)}}{(\zeta_j + \nu_j) + (\zeta_j - \nu_j)e^{-2\zeta_j(t'-t)}}, \tag{46}$$

in which ζ_j and ν_j are the quantities in Eqs. (17) and (18). Note that $\lambda_1 = \nu_j - \zeta_j$ and $\lambda_2 = \nu_j + \zeta_j$ are the complex roots of the characteristic equation associated with differential equation (43).
This concludes the proof. □

The following theorem provides a formula for the marginal density that captures the Gaussian kernel whose variance is the conditional variance of the continuously compounded return $R_t^{t'}$ in the multi-Heston model (see formula (81)). This Gaussian kernel is therefore the most relevant one hidden in the multi-factor Heston model.

The statement of this theorem differs from the statement in the associated paper since here we also provide the expansion in the third order of the Gaussian kernel (see formula (53)).

Theorem 2.2 *The marginal probability density of the log-price variable conditioned to $\underline{v}_t = \underline{v}$ is given by*

$$\begin{aligned} M(x, \underline{v}, t, x', t') &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{\imath k\left[(x'-x) - \int_t^{t'} r(s)ds + \frac{1}{2}\Gamma_2(t,t')\right] - \frac{1}{2}\Gamma_2(t,t')k^2}\, e^{\sum_{j=1}^{n}\int_t^{t'} E(v_{j,s}\,|\,\mathcal{F}_t)H_j(k,s,t')ds} dk \\ &= \int_{-\infty}^{+\infty} \mathcal{G}_{\Gamma_2}(x' - x - y, t, t')\mathcal{L}^*_{\underline{\gamma}}(y, t, t')dy, \end{aligned} \tag{47}$$

where $\mathcal{L}^*_{\underline{\gamma}}$ is the function

$$\mathcal{L}^*_{\underline{\gamma}}(y, t, t') = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{\imath\, k\, y} e^{\sum_{j=1}^{n}\int_t^{t'} E(v_{j,s}\,|\,\mathcal{F}_t)H_j(k,s,t')ds} dk\,. \tag{48}$$

Here Γ_2 is defined by

$$\begin{aligned} \Gamma_2(t, t') &= \Gamma_0(t, t') - 2S_1(t, t') + 2S_2(t, t') \\ &= \sum_{j=1}^{n}\int_t^{t'} E(v_{j,s}\,|\,\mathcal{F}_t)\left[(1 - \rho_j^2) + \left(\frac{\gamma_j}{2\chi_j}\left(1 - e^{-\chi_j(t'-s)}\right) - \rho_j\right)^2\right] ds \end{aligned} \tag{49}$$

and S_1 and S_2 are given by

$$S_1(t, t') = \frac{1}{2}\sum_{j=1}^{n}\frac{\rho_j\gamma_j}{\chi_j}\int_t^{t'} E(v_{j,s}\,|\,\mathcal{F}_t)\left(1 - e^{-\chi_j(t'-s)}\right) ds, \tag{50}$$

$$S_2(t,t') = \sum_{j=1}^{n} \frac{\gamma_j^2}{8\chi_j^2} \int_t^{t'} E(v_{j,s}\,|\,\mathcal{F}_t) \left(1 - e^{-\chi_j(t'-s)}\right)^2 ds, \tag{51}$$

while H_j is given by:

$$H_j(k,t,t') = \frac{\gamma_j^2}{2} B_j^2(k,t,t') + \imath\, k\rho_j\gamma_j B_j(k,t,t') + \frac{1}{2}(k^2 - \imath k) \left[-\frac{\rho_j\gamma_j}{\chi_j}(1 - e^{-\chi_j(t'-t)}) + \frac{1}{4}\frac{\gamma_j^2}{\chi_j^2}\left(1 - e^{-\chi_j(t'-t)}\right)^2\right]. \tag{52}$$

Furthermore, the following expansion holds:

$$\mathcal{L}^*_{\underline{\gamma}}(y,t,t') = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{\imath\, k\, y} e^{S_1(t,t')(\imath\, k^3 + \imath\, k) + S_2(t,t')(k^4 - 2\imath k^3 - \imath k) + S_{2c}(t,t')(k^4 - \imath\, k^3) - \imath k(k^2 - \imath k)^2 S_{3c}(t,t') - \imath\, k(k^4 - \imath\, k^3) S_{3d}(t,t') + o(\|\underline{\gamma}\|^3)} dk\,, \quad \|\underline{\gamma}\| \to 0^+. \tag{53}$$

S_1 and S_2 are given in Eqs. (50) and (51), while S_{2c}, S_{3c} and S_{3d} are

$$S_{2c}(t,t') = \sum_{j=1}^{n} \frac{\gamma_j^2\rho_j^2}{2\chi_j} \int_t^{t'} E(v_{j,s}\,|\,\mathcal{F}_t) e^{-\chi_j(t'-s)} \int_s^{t'} \left(e^{\chi_j(t'-\tau)} - 1\right) d\tau ds, \tag{54}$$

$$S_{3c}(t,t') = \sum_{j=1}^{n} \frac{\gamma_j^3\rho_j}{8\chi_j} \int_t^{t'} E(v_{j,s}\,|\,\mathcal{F}_t) \left\{\psi_j^2(s,t') + 2\frac{(t'-s)}{\chi_j}\left(e^{-2\chi_j(t'-s)} - 2e^{-\chi_j(t'-t)}\right) + 2\frac{\psi_j(s,t')}{\chi_j}\right\} ds, \tag{55}$$

and

$$S_{3d}(t,t') = \sum_{j=1}^{n} \frac{\gamma_j^3\rho_j^3}{2\chi_j} \int_t^{t'} E(v_{j,s}\,|\,\mathcal{F}_t) \left[\frac{\psi_j(s,t')}{\chi_j} - \frac{(t'-s)}{\chi_j} e^{-\chi_j(t'-s)} - \frac{(t'-s)^2}{2} e^{-\chi_j(t'-s)}\right], \tag{56}$$

where ψ_j is given by

$$\psi_j(t,t') = \frac{(1 - e^{-\chi_j(t'-t)})}{\chi_j}, \;\; t < t'. \tag{57}$$

We remark that the functions $\mathcal{L}_{\underline{\gamma}}$ in (20) and $\mathcal{L}^*_{\underline{\gamma}}$ in (53) satisfy the following equation:

$$\widehat{\mathcal{L}}_{\underline{\gamma}} = e^{-(k^2 - \imath k)(S_2(t,t') - S_1(t,t'))} \widehat{\mathcal{L}}^*_{\underline{\gamma}}, \tag{58}$$

where $\widehat{\mathcal{L}}_{\underline{\gamma}}$ and $\widehat{\mathcal{L}}^*_{\underline{\gamma}}$ are the Fourier transforms of the functions $\mathcal{L}_{\underline{\gamma}}$ and $\mathcal{L}^*_{\underline{\gamma}}$ with respect to the log-price, respectively.

Proof of Theorem 2.2 *We start by proving Eqs. (47) and (53). Eqs. (47) and (52) follow from Eq. (15) by adding and subtracting the quantity $\left(k^2 - \imath k\right)(-S_1(t,t') + S_2(t,t'))$, where S_1 and S_2 are given in Eqs. (50) and (51), and applying the convolution theorem for the inverse Fourier transform to the inverse Fourier product of the Fourier transform of $\mathcal{G}_{\Gamma_2}$ and $\mathcal{L}^*_{\underline{\gamma}}$.*

We now prove the expansion of $\mathcal{L}^*_{\underline{\gamma}}$ in Eq. (53). To this end, we prove the following expansion for B_j (16):

$$B_j(k,t,t') = B_{j,0}(k,t,t') + \gamma_j B_{j,1}(k,t,t') + O(\gamma_j^2), \;\; \gamma_j \to 0^+, \;\; t < t'. \tag{59}$$

Substituting Eq. (59) in (26) and equating the coefficients of the same powers of γ_j, we see that the zero and first-order term $B_{j,0}$ and $B_{j,1}$ solve the following equations:

$$\frac{dB_{j,0}}{dt}(k,t,t') - \chi_j\, B_{j,0}(k,t,t') = -\frac{k^2}{2} + \frac{\imath\, k}{2}\,, \tag{60}$$

$$\frac{dB_{j,1}}{dt}(k,t,t') - \chi_j\, B_{j,1}(k,t,t') = \imath\, k\, \rho_j\, B_{j,0}(k,t,t'), \tag{61}$$

with final conditions

$$B_{j,0}(k,t',t') = 0, \qquad B_{1,1}(k,t,t') = 0. \tag{62}$$

The solution $B_{j,0}$ reads

$$B_{j,0}(k,t,t') = \frac{1}{2}\left(k^2 - \imath\, k\right)\frac{(1 - e^{-\chi_j(t'-t)})}{\chi_j} = \frac{1}{2}\left(k^2 - \imath\, k\right)\psi_j(t,t'), \tag{63}$$

where ψ_j is given in (86), while $B_{j,1}$ is

$$B_{j,1}(k,t,t') \quad = \quad -\frac{\imath\, k\, \rho_j}{2\chi_j}\left(k^2 - \imath\, k\right) f_j(t,t')\,, \tag{64}$$

where f_j is

$$f_j(t,t') = \left(\psi_j(t'-t) - (t'-t)e^{-\chi_j(t'-t)}\right) = e^{-\chi_j(t'-t)}\int_t^{t'}(e^{\chi_j(t'-s)} - 1)ds \quad t < t'. \tag{65}$$

Using Eq. (59) in Eq. (52), we have

$$\begin{aligned} H_j(k,s,t') &= \frac{\gamma_j^2}{2}\left(B_0^2(k,s,t') + \frac{(k^2 - \imath\, k)}{4}\psi_j^2(s,t')\right) \\ &+\imath\, k\rho_j\gamma_j\left(B_{j,0}(k,s,t') + \gamma_j B_{j,1}(k,s,t') + \frac{(\imath k + 1)}{2}\psi_j(s,t')\right) + o(\gamma_j^2)\,, \gamma_j \to 0^+, \end{aligned} \tag{66}$$

which also reads as follows:

$$\begin{aligned} H_j(k,s,t') &= \frac{\gamma_j^2}{2}\left(\frac{\psi_j^2(s,t')}{4}(k^2 - \imath k)^2 + \frac{(k^2 - \imath\, k)}{4}\psi_j^2(s,t')\right) \\ &+\imath\, k\rho_j\gamma_j\left(\frac{\psi_j(s,t')}{2}(k^2 - \imath\, k) + \frac{\gamma_j\rho_j}{2\chi_j} f_j(s,t')(-\imath k^3 - k^2) + \frac{(\imath k + 1)}{2}\psi_j(s,t')\right) + o(\gamma_j^2)\,, \gamma_j \to 0^+. \end{aligned} \tag{67}$$

Using expansion (67) in (20) and (47), we obtain

$$\begin{aligned} M(x,\underline{v},t,x',t') = \frac{1}{2\pi}\int_{-\infty}^{+\infty} & \underbrace{e^{\imath k\left[(x'-x) - \int_t^{t'} r(s)ds + \frac{1}{2}\Gamma_2(t,t')\right] - \frac{1}{2}\Gamma_2(t,t')k^2}}_{Fourier\,tranform\,Gaussian\,kernel} \times \\ & \underbrace{e^{\sum_{j=1}^n \int_t^{t'} E(v_{j,s}\,|\,\mathcal{F}_t)\left[\frac{\gamma_j^2}{2}(k^4 - 2\imath k^3 - \imath k)\frac{\psi_j^2(s,t')}{4} + \frac{\gamma_j^2\rho_j^2}{2\chi_j}(k^4 - \imath\, k^3) f_j(s,t') + \frac{\rho_j\,\gamma_j}{2}(\imath\, k^3 + \imath\, k)\psi_j(s,t') + o(\gamma_j^2)\right]ds}}_{function\ \widehat{\mathcal{L}}_{\underline{\gamma}},\, i.e.\ Fourier\,transform\,of\,\mathcal{L}_{\underline{\gamma}}} dk. \end{aligned} \tag{68}$$

Now we compute the third-order term in the expansion in volatilities of volatilities. To this end, we compute the third-order derivative of the function H_j in Eq. (52) with respect to γ_j. We drop the argument of the functions to simplify the demonstration.

$$\frac{\partial H_j}{\partial \gamma_j} = \gamma_j B_j^2 + \gamma_j^2 \frac{\partial B_j}{\partial \gamma_j} B_j + \imath k \rho_j B_j + \imath k \rho_j \gamma_j \frac{\partial B_j}{\partial \gamma_j} + \frac{1}{2}(k^2 - \imath k)(-\rho_j \psi_j + \frac{\gamma_j}{2}\psi^2), \tag{69}$$

$$\frac{\partial^2 H_j}{\partial \gamma_j^2} = B_j^2 + 4\gamma_j \frac{\partial B_j}{\partial \gamma_j} B_j + \gamma_j^2 \frac{\partial^2 B_j}{\partial \gamma_j^2} B_j + \gamma_j^2 \left(\frac{\partial B_j}{\partial \gamma_j}\right)^2 + 2\imath k \rho_j \frac{\partial B_j}{\partial \gamma_j} + \imath k \rho_j \gamma_j \frac{\partial^2 B_j}{\partial \gamma_j^2} + \frac{1}{4}(k^2 - \imath k)\psi^2, \tag{70}$$

$$\frac{\partial^3 H_j}{\partial \gamma_j^3} = 6\frac{\partial B_j}{\partial \gamma_j} B_j + 6\gamma_j \left(\frac{\partial B_j}{\partial \gamma_j}\right)^2 + 6\gamma_j \frac{\partial^2 B_j}{\partial \gamma_j^2} B_j + 3\gamma_j^2 \frac{\partial^2 B_j}{\partial \gamma_j^2}\frac{\partial B_j}{\partial \gamma_j} + 3\imath k \rho_j \frac{\partial^2 B_j}{\partial \gamma_j^2} + \imath k \rho_j \gamma_j \frac{\partial^3 B_j}{\partial \gamma_j^3} + \gamma_j^2 \frac{\partial^3 B_j}{\partial \gamma_j^3} B_j. \tag{71}$$

We now compute all the derivatives at $\gamma_j = 0$, which are

$$\frac{\partial H_j}{\partial \gamma_j} = \imath k \rho_j B_j - \frac{1}{2}(k^2 - \imath k)\rho_j \psi_j, \tag{72}$$

$$\frac{\partial^2 H_j}{\partial \gamma_j^2} = B_j^2 + 2\imath k \rho_j \frac{\partial B_j}{\partial \gamma_j} + \frac{1}{4}(k^2 - \imath k)\psi^2\,, \tag{73}$$

and

$$\frac{\partial^3 H_j}{\partial \gamma_j^3} = 6\frac{\partial B_j}{\partial \gamma_j} B_j + 3\imath k \rho_j \frac{\partial^2 B_j}{\partial \gamma_j^2}. \tag{74}$$

The function B_j evaluated at $\gamma_j = 0$ and its first-order derivative with respect to γ_j are given by the terms of the expansion in powers of γ_j as $\gamma_j \to 0^+$, that is, $B_{j,0}$ (see Eq. (63)) and $B_{j,1}$ (see Eq. (64)), respectively. Now we compute the second-order derivatives. The second-order derivative at $\gamma_j = 0$, which we denote as B_j'', is the solution to the following initial value problem:

$$\frac{d}{dt} B_j'' = \chi_J B'' + B_{j,0}^2 + 2\imath\, k \rho_j B_{j,1}, \tag{75}$$

with final condition

$$B_j''(k, t', t') = 0.$$

To solve Eq. (75), it is sufficient to solve

$$\frac{d}{dt}\left(e^{-\chi_j (t-t')} B_j''\right) = \frac{(k^2 - \imath k)^2}{4}\psi_j^2(t,t') e^{-\chi_j (t-t')} + \frac{\rho_j^2}{\chi_j} k^2 (k^2 - \imath k)\left[\psi_j e^{-\chi_j (t-t')} - (t'-t)\right]. \tag{76}$$

Integrating Eq. (76) with the final condition, we obtain

$$B_j''(k,t,t') = \frac{(k^2 - \imath k)^2}{4}\left[\frac{\psi_j^2(t,t')}{\chi_j} + 2\frac{(t'-t)e^{-\chi_j (t'-t)}}{\chi_j^2} - \frac{2}{\chi_j^2}\psi_j(t,t')\right]$$
$$-\frac{\rho_j^2}{\chi_j}(k^4 - \imath k^3)\left[\frac{\psi_j(t,t')}{\chi_j} - \frac{(t'-t)}{\chi_j} e^{-\chi_j (t'-t)} - \frac{(t'-t)^2}{2} e^{-\chi_j (t'-t)}\right]. \tag{77}$$

From Eqs. (74) and (77), we have

$$\sum_{j=1}^{n} \frac{\gamma_j^3}{6} \int_t^{t'} \left. \frac{\partial^3 H_j}{\partial \gamma_j^3} \right|_{\gamma_j=0} E_t(v_{j,s}) ds =$$

$$-\imath\, k(k^2 - \imath k)^2 \sum_{j=1}^{n} \frac{\gamma_j^3 \rho_j}{\chi_j} \int_t^{t'} E_t(v_{j,s}) \left[\frac{1}{8}\psi_j^2(s,t') + \frac{1}{4} \frac{(t'-s)(e^{-2\chi_j(t'-s)} - 2e^{-\chi_j(t'-s)})}{\chi_j} + \frac{\psi_j(s,t')}{4\chi_j} \right] ds$$

$$-\frac{\imath\, k}{2}(k^4 - \imath\, k^3) \sum_{j=1}^{n} \frac{\gamma_j^3 \rho_j^3}{\chi_j} \int_t^{t'} E_t(v_{j,s}) \left[\frac{\psi_j(s,t')}{\chi_j} - \frac{(t'-s)}{\chi_j} e^{-\chi_j(t'-s)} - \frac{(t'-s)^2}{2} e^{-\chi_j(t'-s)} \right] ds. \quad (78)$$

This concludes the proof. □

The following result, partially illustrated in the paper, is proved in more detail.

Proposition 2.3 *Let $t < t'$ and $X_{t'}$, $Y_{t'}$, $R_t^{t'}$ be the processes in (11), (12), and (13). We have the following expression for the conditional moment of $X_{t'}$ and $Y_{t'}$:*

$$E(X_{t'}^2 \,|\, \mathcal{F}_t) = \Gamma_0(t,T), \quad E(X_{t'}^3 \,|\, \mathcal{F}_t) = 6S_1(t,t'), \quad E(Y_{t'}^2 \,|\, \mathcal{F}_t) = 8S_2(t,t'), \quad E(X_{t'}Y_{t'} \,|\, \mathcal{F}_t) = 2S_1(t,t'), \quad (79)$$

$$E(X_{t'}^2 Y_{t'} \,|\, \mathcal{F}_t) = 4S_{2c}(t,t') + 8S_2(t,t'), \quad E(X_{t'} Y_{t'}^2 \,|\, \mathcal{F}_t) = 8S_{3c}(t,t'), \quad (80)$$

where Γ_0, S_1, S_2, S_{2c}, S_{3c} are given in (6), (50), (51), (54) and (55), respectively. Finally, in the multi-factor Heston model (1), the conditional variance of the continuously compounded return $R_t^{t'}$ and the price skewness formula as defined in Das and Sundaram (1999) are

$$var(R_t^{t'} \,|\, \mathcal{F}_t) = E\left((R_t^{t'} - E_t(R_t^{t'}))^2 \,|\, \mathcal{F}_t \right) = \Gamma_2(t,t') \quad (81)$$

and

$$\text{Skewness}_{DS} = \frac{E(X_{t'}^3 \,|\, \mathcal{F}_t)}{\left[E(X_{t'}^2 \,|\, \mathcal{F}_t)\right]^{3/2}} = 6\frac{S_1(t,t')}{\Gamma_0(t,t')^{3/2}}, \quad (82)$$

where Γ_2 is given in (49). The third conditional moments of the process $Y_{t'}$ constitute a homogenous fourth-degree polynomial in the vols of vols:

$$E(Y_{t'}^3 \,|\, \mathcal{F}_t) = 3\sum_{j=1}^{n} \gamma_j^4 \int_t^{t'} E_t(v_{j,s}) \psi_j(s,t') \int_s^{t'} \psi_j(\tau,t')^2 e^{-\chi_j(\tau - s)} d\tau\, ds\,. \quad (83)$$

Proof of Proposition 2.3 *This proof generalizes the results of Zhang et al. (2017) to the multi-factor Heston model (1) following the approach therein. Let $X_{t'}$, $Y_{t'}$, and $R_t^{t'}$ be the processes defined in (11), (12), and (13). We start by proving Eq. (13), that is,*

$$R_t^{t'} - E_t(R_t^{t'}) = X_{t'} - \frac{1}{2}Y_{t'}\,. \quad (84)$$

Here, $E_t(\cdot) = E(\cdot \,|\, \mathcal{F}_t)$ to keep the notation simple. Bearing in mind that Eq. (1) implies

$$d(e^{\chi_j(\tau - t)} v_{j,\tau}) = \chi_j v_j^* e^{-\chi_j(\tau-t)} + \gamma_j e^{-\chi_j(\tau-t)} \sqrt{v_{j,\tau}} dW_{j,\tau},\ t < \tau,\ j = 1, 2, \dots, n,$$

we integrate over the interval $[t, s]$, $t < s$, *thereby obtaining*

$$v_{j,s} = v_j^* + (v_{j,t} - v_j^*)e^{-\chi_j(s-t)} + \gamma_j \int_t^s e^{-\chi_j(s-\tau)}\sqrt{v_{j,\tau}}dW_{j,\tau} = E_t(v_{j,s}) + \gamma_j \int_t^s e^{-\chi_j(s-\tau)}\sqrt{v_{j,\tau}}dW_{j,\tau} \, ,$$

which implies

$$\sum_{j=1}^{n}[v_{j,s} - E_t(v_{j,s})] = \sum_{j=1}^{n}\gamma_j \int_t^s e^{-\chi_j(s-\tau)}\sqrt{v_{j,\tau}}dW_{j,\tau} \, .$$

Hence, we can rewrite the process $Y_{t'}$ *as follows:*

$$Y_{t'} = \sum_{j=1}^{n}\gamma_j \int_t^{t'}\int_t^s e^{-\chi_j(s-\tau)}\sqrt{v_{j,\tau}}dW_{j,\tau}ds = \sum_{j=1}^{n}\gamma_j \int_t^{t'}\sqrt{v_{j,\tau}}dW_{j,\tau}\int_\tau^{t'} e^{-\chi_j(s-\tau)}ds = \sum_{j=1}^{n}\gamma_j \int_t^{t'}\psi_j(\tau, t')\sqrt{v_{j,\tau}}dW_{j,\tau}, \tag{85}$$

where ψ_j *is given by*

$$\psi_j(t, t') = \frac{(1 - e^{-\chi_j(t'-t)})}{\chi_j}, \;\; t < t'. \tag{86}$$

The variance of the continuously compounded return R_t^T *is*

$$E_t\left(\left(R_t^{t'} - E_t(R_t^{t'})\right)^2\right) = E_t\left(\left(X_{t'} - \frac{1}{2}Y_{t'}\right)^2\right) = E_t(X_{t'}^2) - E_t(X_{t'}Y_{t'}) + \frac{1}{4}E_t(Y_{t'}^2). \tag{87}$$

Thus, the proof of Eq. (81) follows from Eq. (79).

We now derive explicit expressions for the conditional moments in Eq. (79). A key ingredient is that all correlations between the Wiener processes in (1) are zero except for $E(dZ_{j,t}, dW_{j,t}) = \rho_j dt$, $j = 1, 2, \ldots, n$. *We start with* $E_t(X_{t'}^2)$. *Bearing in mind* $dX_\tau = \sum_{j=1}^{n}\sqrt{v_{j,\tau}}dZ_{j,\tau}$, *the correlations between the Wiener processes* $Z_{j,\tau}$, $j = 1, 2, \ldots, n$, *and Itô's lemma, we obtain*

$$dE_t(X_\tau^2) = \sum_{j=1}^{n}E_t(v_{j,\tau})d\tau \, ,$$

which implies

$$E_t(X_{t'}^2) = \sum_{j=1}^{n}\int_t^{t'}E_t(v_{j,\tau})d\tau = \Gamma_0(t, t'). \tag{88}$$

Now we derive $E_t(X_{t'}Y_{t'})$ *and* $E_t(Y_{t'}^2)$. *To this end, we use the shadow process,* Y_s^* *introduced by Zhang et al. (2017), which is defined as*

$$Y_s^* = \sum_{j=1}^{n}\gamma_j \int_t^s \psi(\tau, t')\sqrt{v_{j,\tau}}dW_{j,\tau}, \tag{89}$$

which is an Itô's process (martingale) such that $Y_{t'}^* = Y_{t'}$.
Since we have $Y_{t'} = Y_{t'}^*$ *and* $dY_s^* = \sum_{j=1}^{n}\gamma_j\psi_j(s, t')\sqrt{v_{j,s}}dW_{j,s}$ *applying Itô's lemma and bearing in mind the correlation structure, we obtain*

$$\begin{aligned} E_t(X_{t'}Y_{t'}) &= E_t(X_{t'}Y_{t'}^*) = E_t\int_t^{t'}(Y_\tau^* dX_\tau + X_\tau dY_\tau^* + dX_\tau dY_\tau^*) \\ &= E_t\int_t^{t'}dX_\tau dY_\tau^* = \sum_{j=1}^{n}\rho_j\gamma_j\int_t^{t'}\psi_j(\tau, t')E_t(v_{j,\tau})d\tau = 2S_1(t, t'), \end{aligned} \tag{90}$$

$$
\begin{aligned}
E_t(Y_{t'}^2) &= E_t\left(Y_{t'}^{*^2}\right) = E_t\int_t^{t'} d\left(Y_\tau^{*^2}\right) = E_t\int_t^{t'}\left(2Y_\tau^* dY_\tau^* + (dY_\tau^*)^2\right) \\
&= E_t\int_t^{t'}(dY_\tau^*)^2 = \sum_{j=1}^n \gamma_j^2\int_t^{t'}\psi_j(s,t')^2 E_t(v_{j,s})ds = 8S_2(t,t'). \qquad (91)
\end{aligned}
$$

Now we prove the relationships of S_{2c} and the processes (11) and (12) in Eq (80). It is easy to see that S_{2c} can be rewritten as

$$
S_{2c}(t,t') = \sum_{j=1}^n \frac{\gamma_j^2\rho_j^2}{2}\int_t^{t'} E(v_{j,s}\,|\,\mathcal{F}_t)\int_s^{t'} e^{-\chi_j(\tau-s)}\frac{\left(1-e^{-\chi_j(t'-\tau)}\right)}{\chi_j}d\tau ds. \qquad (92)
$$

Due to the assumption about correlations in (1) and the definitions of the processes $X_{t'}$, $Y_{t'}$, and $Y_{t'}^*$ given in (11), (12), and (89), we have

$$
E_t(X_{t'}^2 Y_{t'}) = E_t(X_{t'}^2 Y_{t'}^*) = E_t\int_t^{t'}\left[2X_\tau Y_\tau^* dX_\tau + X_\tau^2 dY_\tau^* + Y_\tau^*(dX_\tau)^2 + 2X_\tau dX_\tau dY_\tau^*\right],
$$

$$
E_t(X_{t'}^2 Y_{t'}) = \sum_{j=1}^n\int_t^{t'} E_t(Y_\tau^* v_{j,\tau})d\tau + 2\sum_{j=1}^n \gamma_j\rho_j\int_t^{t'}\psi_j(\tau,t')E_t(X_\tau v_{j,\tau})d\tau\,,
$$

where ψ_j is given in (86) and $E_t(\cdot) = E(\cdot\,|\,\mathcal{F}_t)$. To complete the proof, we compute $E_t(Y_s^* v_{j,s})$ and $E_t(X_s v_{j,s})$, $s > t > 0$, following Zhang et al. (2017) Proposition 2. Bearing in mind again the assumption about the correlations in (1), we have

$$
\begin{aligned}
&E_t(Y_s^*\, v_{j,s}) = E_t\int_t^s d(Y_\tau^* v_{j,\tau}) = E_t\int_t^s\left(v_{j,\tau}dY_\tau^* + Y_\tau^* dv_{j,\tau} + dY_\tau^* dv_{j,\tau}\right) = \\
&E_t\left(\int_t^s\left(Y_\tau^*\chi_j(v_j^* - v_{j,\tau})\right)d\tau + \int_t^s \gamma_j^2\psi_j(\tau,t')v_{j,\tau}d\tau\right) = \\
&-\chi_j\int_t^s E_t(Y_\tau^* v_{j,\tau})d\tau \;+\; \gamma_j^2\int_t^s\psi_j(\tau,t')E_t(v_{j,\tau})d\tau\,. \qquad (93)
\end{aligned}
$$

We solve the ordinary differential equation (93) and obtain

$$
\begin{aligned}
&\int_t^{t'} E_t(Y_\tau^* v_{j,\tau})d\tau = \gamma_j^2\int_t^{t'}\int_t^\tau e^{-\chi_j(\tau-s)}\psi_j(s,t')E_t(v_{j,s})dsd\tau = \\
&\gamma_j^2\int_t^{t'} E_t(v_{j,s})\psi_j(s,t')\left[\int_s^{t'} e^{-\chi_j(\tau-s)}d\tau\right]ds = \gamma_j^2\int_t^{t'} E_t(v_{j,s})\psi_j(s,t')^2 ds\,. \qquad (94)
\end{aligned}
$$

Differentiating, we have

$$
E_t(Y_s^* v_{j,s}) = \gamma_j^2\int_t^s e^{-\chi_j(s-\tau)}\psi_j(\tau,t')E_t(v_{j,\tau})d\tau\,. \qquad (95)
$$

Now we compute $E_t(X_s v_{j,s})$. Using Itô's lemma and the martingale property of X_s, we have

$$
\begin{aligned}
E_t(X_s v_{j,s}) &= E_t\int_t^s d(X_\tau v_{j,\tau}) = E_t\int_t^s\left[v_\tau dX_\tau + X_\tau dv_{j,\tau} + dX_\tau dv_{j,\tau}\right] \\
&= E_t\int_t^s\left[X_\tau\chi_j(v_j^* - v_{j,\tau})d\tau + \rho_j\gamma_j v_{j,\tau}d\tau\right] = -\chi_j E_t(X_\tau v_{j,\tau})d\tau - \rho_j\gamma_j\int_t^s E_t(v_{j,\tau})d\tau\,. \; (96)
\end{aligned}
$$

We solve the ordinary differential equation (96) and obtain

$$\int_t^s E_t(X_\tau v_{j,\tau})d\tau = \rho_j\gamma_j \int_t^s e^{-\chi_j(s-u)} \int_t^u E_t(v_{j,\tau})d\tau du = \rho_j\gamma_j \int_t^s \psi_j(\tau,s)E_t(v_{j,\tau})d\tau\,. \tag{97}$$

Deriving Eq. (97) with respect to time, we obtain

$$E_t(X_s v_{j,s}) = \rho_j\gamma_j \int_t^s e^{-\chi_j(s-\tau)}E_t(v_{j,\tau})d\tau. \tag{98}$$

Using Eq. (94) and the following:

$$\int_t^{t'} \psi_j(\tau,t')E_t(X_\tau v_{j,\tau})d\tau = \rho_j\gamma_j \int_t^{t'}\int_t^\tau e^{-\chi_j(\tau-s)}E_t(v_{j,s})dsd\tau = \rho_j\gamma_j \int_t^{t'} E_t(v_{j,s})\psi_j(s,t')\left[\int_s^{t'} e^{-\chi_j(\tau-s)}d\tau\right]ds$$
$$= \rho_j\gamma_j \int_t^{t'} E_t(v_{j,s})\left[\int_s^{t'} e^{-\chi_j(\tau-s)}\psi_j(\tau,t')d\tau\right]ds, \tag{99}$$

we show that $E_t(X_{t'}^2 Y_{t'}) = 4S_{2c}(t,t') + 8S_2(t,t')$.

We now compute $E_t(X_{t'}Y_{t'}^2)$ *using the assumption about the correlations. We have*

$$E_t(X_{t'}Y_{t'}^2) = E_t(X_{t'}(Y_{t'}^{*2}) = E_t\int_t^{t'} d(X_sY_s^{*2}) = E_t\int_t^{t'}\left[Y_s^{*2}dX_s + 2X_sY_s^*dY_s^* + 2Y_s^*dX_sdY_s^* + X_s(dY_s^*)^2\right]$$
$$= \sum_{j=1}^n\left[2\rho_j\gamma_j\int_t^{t'}\psi_j(s,t')E_t(Y_s^*v_{j,s})ds + \gamma_j^2\int_t^{t'}\psi_j(t,s)^2E_t(X_sv_{j,s})ds\right]. \tag{100}$$

Substituting Eqs. (95) and (99) into Eq. (100), we obtain

$$E_t(X_{t'}Y_{t'}^2) = \sum_{j=1}^n 2\rho_j\gamma_j^3\int_t^{t'}\psi_j(s,t')\int_t^s e^{-\chi_j(s-\tau)}\psi_j(\tau,t')E_t(v_{j,\tau})d\tau\,ds$$
$$+\sum_{j=1}^n\gamma_j^3\rho_j\int_t^{t'}\psi_j(t,s)^2\int_t^s e^{-\chi_j(s-\tau)}E_t(v_{j,\tau})d\tau\,ds\,. \tag{101}$$

The thesis follows by computing the elementary integrals appearing in Eq. (101).

Now we derive is (82). To this end, we need to compute $E_t(X_{t'}^3)$*. Arguing as in the proof of Proposition 2 in Zhang et al. (2017), we have*

$$E_t(X_{t'}^3) = E_t\int_t^{t'} dX_s^3 = E_t\int_t^{t'}\left[3X_s^2dX_s + 3X_s(dX_s)^2\right] = 3\sum_{j=1}^n\int_t^{t'}E_t(X_sv_{j,s})ds. \tag{102}$$

Hence we have

$$E_t(X_{t'}^3) = 3\sum_{j=1}^n\int_t^{t'}E_t(X_sv_{j,s})ds = 3\sum_{j=1}^n\rho_j\gamma_j\int_t^{t'}\psi_j(s,t')E_t(v_{j,s})ds = 6S_1(t,t')\,.$$

Finally, we compute $E_t(Y_{t'}^3)$*. Using Itô's lemma and the martingale property of* Y_s^**, we have*

$$E_t(Y_{t'}^3) = E_t(Y_{t'}^{*^3}) = E_t\int_t^{t'} d(Y_{t'})^3 = E_t\int_t^{t'}\left[3Y_s^{*^2}dY_s^* + 3Y_s^*(dY_s^*)^2\right] = 3\sum_{j=1}^n\gamma_j^2\int_t^{t'}\psi_j(s,t')^2E_t(Y_s^*v_{j,s})ds. \tag{103}$$

Substituting Eq. (95) into Eq. (103) we have

$$E_t(Y_{t'}^3) = E_t(Y_{t'}^{*^3}) = 3\sum_{j=1}^{n}\gamma_j^4\int_t^{t'}\psi_j(s,t')^2\int_t^s e^{-\chi_j(s-\tau)}\psi_j(\tau,t')E_t(v_{j,\tau})d\tau\,ds. \tag{104}$$

This concludes the proof. $\square$

Corollary 2.4 *The following expansion of the conditional marginal M in (15) in powers of $\underline{\gamma}$ as $\|\underline{\gamma}\|\to 0$ holds:*

$$M(x,\underline{v},t,x',t') = \mathcal{G}_{\Gamma_2}(x'-x,t,t') + \mathcal{M}_1(x,\underline{v},t,x',t') + \mathcal{M}_2(x,\underline{v},t,x',t') + \mathcal{M}_3(x,\underline{v},t,x',t') + o\left(\|\underline{\gamma}\|^3\right), \quad \|\underline{\gamma}\|\to 0, \tag{105}$$

where $\mathcal{G}_{\Gamma_2}$ is the Gaussian kernel defined in (5).

$\mathcal{M}_1$ is given by

$$\mathcal{M}_1(x,\underline{v},t,x',t') = S_1(t,t')\left[-\frac{d^3\mathcal{G}_{\Gamma_2}}{dx'^3}(x'-x,t,t') + \frac{d\mathcal{G}_{\Gamma_2}}{dx'}(x'-x,t,t')\right], \tag{106}$$

$\mathcal{M}_2$ is given by

$$\begin{aligned}\mathcal{M}_2(x,\underline{v},t,x',t') &= S_2(t,t')\left[\frac{d^4\mathcal{G}_{\Gamma_2}}{dx'^4}(x'-x,t,t') + 2\frac{d^3\mathcal{G}_{\Gamma_2}}{dx'^3}(x'-x,t,t') - \frac{d\mathcal{G}_{\Gamma_2}}{dx'}(x'-x,t,t')\right] + \\ &S_{2c}(t,t')\left[\frac{d^4\mathcal{G}_{\Gamma_2}}{dx'^4}(x'-x,t,t') + \frac{d^3\mathcal{G}_{\Gamma_2}}{dx'^3}(x'-x,t,t')\right] \\ &+\frac{1}{2}S_1^2(t,t')\left[\frac{d^6\mathcal{G}_{\Gamma_2}}{dx'^6}(x'-x,t,t') - 2\frac{d^4\mathcal{G}_{\Gamma_2}}{dx'^4}(x'-x,t,t') + \frac{d^2\mathcal{G}_{\Gamma_2}}{dx'^2}(x'-x,t,t')\right],\end{aligned} \tag{107}$$

and $\mathcal{M}_3$ is given by

$$\begin{aligned}\mathcal{M}_3(x,\underline{v},t,x',t') \quad = \quad & S_{3c}\left[-\frac{d^3}{dx'^3}\mathcal{G}_{\Gamma_2} - 2\frac{d^4}{dx'^4}\mathcal{G}_{\Gamma_2} - \frac{d^5}{dx'^5}\mathcal{G}_{\Gamma_2}\right] + S_{3d}\left[-\frac{d^4}{dx'^4}\mathcal{G}_{\Gamma_2} - \frac{d^5}{dx'^5}\mathcal{G}_{\Gamma_2}\right] \\ &+\frac{1}{6}S_1^3\left[-\frac{d^9}{dx'^9}\mathcal{G}_{\Gamma_2} + 3\frac{d^7}{dx'^7}\mathcal{G}_{\Gamma_2} - 3\frac{d^5}{dx'^5}\mathcal{G}_{\Gamma_2} + \frac{d^3}{dx'^3}\mathcal{G}_{\Gamma_2}\right] \\ &+S_1S_2\left[-\frac{d^7}{dx'^7}\mathcal{G}_{\Gamma_2} - 2\frac{d^6}{dx'^6}\mathcal{G}_{\Gamma_2} + \frac{d^5}{dx'^5}\mathcal{G}_{\Gamma_2} + 3\frac{d^4}{dx'^4}\mathcal{G}_{\Gamma_2} - \frac{d^2}{dx'^2}\mathcal{G}_{\Gamma_2}\right] \\ &+S_1S_{2c}\left[-\frac{d^7}{dx'^7}\mathcal{G}_{\Gamma_2} - \frac{d^6}{dx'^6}\mathcal{G}_{\Gamma_2} + \frac{d^5}{dx'^5}\mathcal{G}_{\Gamma_2} + \frac{d^4}{dx'^4}\mathcal{G}_{\Gamma_2}\right].\end{aligned} \tag{108}$$

In Eq. (108) we have dropped the arguments on the right side to keep the notation simple. Here, S_1 is given by (50) and Γ_2, S_2, S_{2c}, S_{3c}, and S_{3d} are given in (49), (51), (54), (55), and (56), respectively.

Proof of Corollary 2.4 *The proof is based on the expansion in powers of vols of vols of the function $\mathcal{L}_{\underline{\gamma}}$. From Theorem 2.2 and the expansion in formula (53), we have*

$$\begin{aligned}M(x,\underline{v},t,x',t') = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{\imath k\left[(x'-x)-\int_t^{t'}r(s)ds+\frac{1}{2}\Gamma_2(t,t')\right]-\frac{1}{2}\Gamma_2(t,t')k^2}\times \\ e^{S_1(t,t')(\imath k^3+\imath k)+S_2(t,t')(k^4-2\imath k^3-\imath k)+S_{2c}(t,t')(k^4-\imath k^3)-S_{3c}(t,t')\imath k(k^2-\imath k)^2-S_{3d}(t,t')\imath k(k^4-\imath k^3)+o(\|\underline{\gamma}\|^3)}dk,\end{aligned} \tag{109}$$

where S_1 is a linearly homogeneous function of the vols of vols, while S_2 and S_{2c} are second-degree homogeneous functions and S_{3c} and S_{3d} are third-degree homogeneous functions. We compute the first four terms of the expansion in powers of the vols of vols of the function bearing in mind that the mixed partial derivative of the functions S_1, S_2, S_{2c}, S_{3c}, and S_{3d} as a functions of $\underline{\gamma}$ are equal to zero. We therefore have

$$\mathcal{E}(\underline{\gamma}, k, t, t') = e^{S_1(t,t')(\imath k^3+\imath k)+S_2(t,t')(k^4-2\imath k^3-\imath k)+S_{2c}(t,t')(k^4-\imath k^3)+S_{3c}(\imath k^3-2k^4-\imath k^5)+S_{3d}(-k^4-\imath k^5)}. \tag{110}$$

We derive the third-order expansion by using $e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + o(x^3)$, where x is the argument of the function $\mathcal{E}$ in Eq. (110) and we drop all the terms that go to zero faster than the third power. Thus we have

$$\begin{aligned}
&\mathcal{E}(\underline{\gamma}) = \\
&1 + \left[S_1(t,t')(\imath k^3 + \imath k) + S_2(t,t')(k^4 - 2\imath k^3 - \imath k) + S_{2c}(t,t')(k^4 - \imath k^3) + S_{3c}(\imath k^3 - 2k^4 - \imath k^5) + S_{3d}(-k^4 - \imath k^5)\right] \\
&+\frac{1}{2}\left[S_1(t,t')(\imath k^3 + \imath k) + S_2(t,t')(k^4 - 2\imath k^3 - \imath k) + S_{2c}(t,t')(k^4 - \imath k^3) + S_{3c}(\imath k^3 - 2k^4 - \imath k^5) + S_{3d}(-k^4 - \imath k^5)\right]^2 \\
&+\frac{1}{6}\left[S_1(t,t')(\imath k^3 + \imath k) + S_2(t,t')(k^4 - 2\imath k^3 - \imath k) + S_{2c}(t,t')(k^4 - \imath k^3) + S_{3c}(\imath k^3 - 2k^4 - \imath k^5) + S_{3d}(-k^4 - \imath k^5)\right]^3 \\
&+o\left(\left[S_1(t,t')(\imath k^3 + \imath k) + S_2(t,t')(k^4 - 2\imath k^3 - \imath k) + S_{2c}(t,t')(k^4 - \imath k^3) + S_{3c}(\imath k^3 - 2k^4 - \imath k^5) + S_{3d}(-k^4 - \imath k^5)\right]^3\right), \\
&\|\underline{\gamma}\| \to 0.
\end{aligned} \tag{111}$$

Now selecting only the terms with monomials of degree smaller than or equal to 3, we have

$$\begin{aligned}
\mathcal{E}(\underline{\gamma}) \approx\ & \underbrace{1 + \left[S_1(t,t')(\imath k^3 + \imath k) + S_2(t,t')(k^4 - 2\imath k^3 - \imath k) + S_{2c}(t,t')(k^4 - \imath k^3) + \frac{1}{2}S_1(t,t')^2(\imath k^3 + \imath k)^2\right]}_{contribution\ up\ to\ second\ order} \\
& \underbrace{+S_{3c}(\imath k^3 - 2k^4 - \imath k^5) + S_{3d}(-k^4 - \imath k^5) + \frac{1}{6}S_1(t,t')^3(\imath k^3 + \imath k)^3}_{third-order\ contribution} \\
& \underbrace{+S_1(t,t')S_2(t,t')(\imath k^3 + \imath k)(k^4 - 2\imath k^3 - \imath k) + S_1(t,t')S_{2c}(t,t')(\imath k^3 + \imath k)(k^4 - \imath k^3)}_{third-order\ contribution}.
\end{aligned} \tag{112}$$

We now order the powers of k, separating the first-, second- and third-order contributions. Bearing in mind that we have

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty} (\imath k)^m e^{\imath k y} dk = \frac{d^m}{dy^m}\delta(y),\ \ m = 0, 1, 2, \dots,$$

and that from the convolution theorem we have

$$M(x, \underline{v}, t, x', t') \approx \int_{-\infty}^{+\infty} \mathcal{G}_{\Gamma_2}(x' - x - y, t, t') \left[\frac{1}{2\pi} e^{\imath k y}\mathcal{E}(\underline{\gamma})(k, t, t') dk\right] dy, \tag{113}$$

we compute

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty}\mathcal{E}(\underline{\gamma})e^{\imath k y}dk \approx \delta(y)+S_1(t,t')\left[-\frac{d^3\delta(y)}{dy^3}+\frac{d\delta(y)}{dy}\right]$$
$$+S_2(t,t')\left[\frac{d^4\delta(y)}{dy^4}+2\frac{d^3\delta(y)}{dy^3}-\frac{d\delta(y)}{dy}\right]+S_{2c}(t,t')\left[\frac{d^4\delta(y)}{dy^4}+\frac{d^3\delta(y)}{dy^3}\right]$$
$$+\frac{1}{2}S_1(t,t')^2\left[-\frac{d^6\delta(y)}{dy^6}+\frac{d^2\delta(y)}{dy^2}-2\frac{d^4\delta(y)}{dy^4}\right]$$
$$+S_{3c}\left[-\frac{d^3\delta(y)}{dy^3}-2\frac{d^4\delta(y)}{dy^4}-\frac{d^5\delta(y)}{dy^5}\right]+S_{3d}\left[-\frac{d^4\delta(y)}{dy^4}-\frac{d^5\delta(y)}{dy^5}\right]$$
$$+\frac{1}{6}S_1(t,t')^3\left[-\frac{d^9\delta(y)}{dy^9}+3\,\frac{d^7\delta(y)}{dy^7}-3\frac{d^5\delta(y)}{dy^5}+\frac{d^3\delta(y)}{dy^3}\right]$$
$$+S_1(t,t')S_2(t,t')\left[-\frac{d^7\delta(y)}{dy^7}-2\frac{d^6\delta(y)}{dy^6}+\frac{d^5\delta(y)}{dy^5}+3\frac{d^4\delta(y)}{dy^4}-\frac{d^2\delta(y)}{dy^2}\right]$$
$$+S_1(t,t')S_{2c}(t,t')\left[-\frac{d^7\delta(y)}{dy^7}-\frac{d^6\delta(y)}{dy^6}+\frac{d^5\delta(y)}{dy^5}+\frac{d^4\delta(y)}{dy^4}\right]. \tag{114}$$

Using Eq. (114) in (113), the Dirac delta function property:

$$\int_{-\infty}^{+\infty}\mathcal{G}_{\Gamma_2}(x'-x-y,t,t')\frac{d^m\delta(y)}{dy^m}dy=(-1)^m\,\frac{d^m}{dy^m}\mathcal{G}_{\Gamma_2}(x'-x-y,t,t')\bigg|_{y=0}, \tag{115}$$

and the fact that

$$\frac{d^m}{dy^m}\mathcal{G}_{\Gamma_2}=(-1)^m\frac{d^m}{dx'^m}\mathcal{G}_{\Gamma_2}, \tag{116}$$

we have

$$M(x,\underline{v},t,x',t')\approx\mathcal{G}_{\Gamma_2}(x'-x,t,t')+\mathcal{M}_1(x,\underline{v},t,x',t')+\mathcal{M}_2(x,\underline{v},t,x',t')+\mathcal{M}_3(x,\underline{v},t,x',t'), \tag{117}$$

where $\mathcal{M}_1$, $\mathcal{M}_2$, and $\mathcal{M}_3$ are given in (106), (107), and (108), respectively. This concludes the proof. □

We denote the approximations of the marginal density up to the third order as

$$\begin{aligned}
&M_0(x,\underline{v},t,x',t')=\mathcal{G}_{\Gamma_2}(x'-x,t,t'),\\
&M_1(x,\underline{v},t,x',t')=\mathcal{G}_{\Gamma_2}(x'-x,t,t')+\mathcal{M}_1(x,\underline{v},t,x',t'),\\
&M_2(x,\underline{v},t,x',t')=\mathcal{G}_{\Gamma_2}(x'-x,t,t')+\mathcal{M}_1(x,\underline{v},t,x',t')+\mathcal{M}_2(x,\underline{v},t,x',t'),\\
&M_3(x,\underline{v},t,x',t')=\mathcal{G}_{\Gamma_2}(x'-x,t,t')+\mathcal{M}_1(x,\underline{v},t,x',t')+\mathcal{M}_2(x,\underline{v},t,x',t')+\mathcal{M}_3(x,\underline{v},t,x',t').
\end{aligned} \tag{118}$$

Proposition 2.5 below shows that the zero-, second-order, and third-order approximations of the marginal density in Eqs. (118) satisfy the conditions that guarantee mass conservation, the martingale property (i.e., the asset price should be a martingale in the multi-factor Heston model), and the so-called "symmetry condition'.' These conditions avoid norm-defecting and martingale-defecting pdfs as discussed in Lewis (2000) Chapter 2.

Proposition 2.5 *Let M_0, M_1, M_2, and M_3 be given in (118). The following equations then hold:*

$$\int_{-\infty}^{+\infty}M_l(x,\underline{v},t,x',t')dx'\;=\;1,\;\;l=0,1,2,3, \tag{119}$$

$$\int_{-\infty}^{+\infty} e^{x'} M_l(x, \underline{v}, t, x', t') dx' \;=\; e^{x} e^{\int_t^{t'} r(s)ds}, \;\; l = 0, 1, 2, 3, \tag{120}$$

and

$$\int_{-\infty}^{+\infty} \left(x' - x - \int_t^{t'} r(s)ds + \frac{1}{2}\Gamma_2(t, t') \right) M_l(x, \underline{v}, t, x', t') dx' \;=\; 0, \;\; l = 0, 1, 2, 3, \tag{121}$$

that is, mass conservation (119), the martingale property (120), and the symmetry condition (121). These properties also hold for the marginal density M in (15).

Proof of Proposition 2.5 *We prove Eq. (119) for the marginal probability density M in Eq. (15). Integrating Eq. (15) with respect to x', changing the integration order, and bearing in mind that $\delta(k) = (1/2\pi)\int e^{iky}dy$, we have*

$$\begin{aligned}
&\int_{-\infty}^{+\infty} M(x, \underline{v}, t, x', t') dx' = \\
&\frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{\imath k\left[-\int_t^{t'} r(s)ds + \frac{1}{2}\Gamma_0(t,t')\right] - \frac{1}{2}\Gamma_0(t,t')k^2} e^{\sum_{j=1}^n \int_t^{t'} E(v_{j,s}\,|\,\mathcal{F}_t)\left[\frac{\gamma_j^2}{2} B_j^2(k,s,t') + \imath\, k\rho_j\gamma_j B_j(k,s,t')\right]ds} \int_{-\infty}^{+\infty} e^{ik(x'-x)} dx' dk \\
&= \int_{-\infty}^{+\infty} \delta(k) e^{\imath k\left[-\int_t^{t'} r(s)ds + \frac{1}{2}\Gamma_0(t,t')\right] - \frac{1}{2}\Gamma_0(t,t')k^2} e^{\sum_{j=1}^n \int_t^{t'} E(v_{j,s}\,|\,\mathcal{F}_t)\left[\frac{\gamma_j^2}{2} B_j^2(k,s,t') + \imath\, k\rho_j\gamma_j B_j(k,s,t')\right]ds} dk = 1.
\end{aligned} \tag{122}$$

To prove the martingale property, we use the following result regarding the Dirac delta function with a complex argument:

$$\mathcal{I}(k) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{\imath\, y(k-\imath)} dy = \delta(k - \imath). \tag{123}$$

We have

$$\begin{aligned}
&\int_{-\infty}^{+\infty} e^{x'} M(x, \underline{v}, t, x', t') dx' = \\
&\frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{\imath k\left[-\int_t^{t'} r(s)ds + \frac{1}{2}\Gamma_0(t,t')\right] - \frac{1}{2}\Gamma_0(t,t')k^2} e^{\sum_{j=1}^n \int_t^{t'} E(v_{j,s}\,|\,\mathcal{F}_t)\left[\frac{\gamma_j^2}{2} B_j^2(k,s,t') + \imath\, k\rho_j\gamma_j B_j(k,s,t')\right]ds} \int_{-\infty}^{+\infty} e^{ik(x'-x)+x'} dx' dk \\
&= e^{x}\int_{-\infty}^{+\infty} \delta(k) e^{\imath k\left[-\int_t^{t'} r(s)ds + \frac{1}{2}\Gamma_0(t,t')\right] - \frac{1}{2}\Gamma_0(t,t')k^2} e^{\sum_{j=1}^n \int_t^{t'} E(v_{j,s}\,|\,\mathcal{F}_t)\left[\frac{\gamma_j^2}{2} B_j^2(k,s,t') + \imath\, k\rho_j\gamma_j B_j(k,s,t')\right]ds} \mathcal{I}(k) dk.
\end{aligned} \tag{124}$$

Formula (120) follows from Eq. (123) and the fact that the exponent in (127) reduces to $\int_t^{t'} r(s)ds$ when $k = \imath$ since $B_j(\imath, s, t') = 0$, $\forall s < t'$. Alternatively, the mass conservation and martingale property can be proved using the fact that $\mathcal{L}_{\underline{\gamma}}$ satisfies the following equations:

$$\int_{-\infty}^{+\infty} dy \mathcal{L}_{\underline{\gamma}}(y, t, t') = \int_{-\infty}^{+\infty} e^{\sum_{j=1}^n \int_t^{t'} E(v_{j,s}\,|\,\mathcal{F}_t)\left[\frac{\gamma_j^2}{2} B_j^2(k,s,t') + \imath\, k\rho_j\gamma_j B_j(k,s,t')\right]ds} \underbrace{\frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{\imath\, k\, y} dy}_{\delta(k)} dk, \tag{125}$$

$$\int_{-\infty}^{+\infty} dy\, e^{y}\, \mathcal{L}_{\underline{\gamma}}(y, t, t') = \int_{-\infty}^{+\infty} e^{\sum_{j=1}^n \int_t^{t'} E(v_{j,s}\,|\,\mathcal{F}_t)\left[\frac{\gamma_j^2}{2} B_j^2(k,s,t') + \imath\, k\rho_j\gamma_j B_j(k,s,t')\right]ds} \underbrace{\frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{\imath\, y(k-\imath)} dy}_{\delta(k-\imath)} dk, \tag{126}$$

since $B_j(k,t,t')=0$ at $k=0$ and $k=\imath$ for any $t<t'$. In fact, Eq. (126) holds since $B_j(k,t,t')=0$ when $k=\imath$, $j=1,2,\ldots,n$, as already stressed by Lewis (2000) Ch. 2, where conditions to avoid norm-defecting and martingale-defecting pdfs are discussed.

Now we prove (121) for M, bearing in mind that

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty} ye^{\imath ky}dy = -\imath\delta'(k),$$

where $\delta'(k)$ denotes the derivative of the Dirac delta function with respect to k. We have

$$\int_{-\infty}^{+\infty}\left(x'-x-\int_t^{t'} r(s)ds-\frac{1}{2}\Gamma_0(t,t')\right)M(x,\underline{v},t,x',t')dx' =$$

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-\frac{1}{2}\Gamma_0(t,t')k^2} e^{\sum_{j=1}^{n}\int_t^{t'} E(v_{j,s}\,|\,\mathcal{F}_t)\left[\frac{\gamma_j^2}{2}B_j^2(k,s,t')+\imath\,k\rho_j\gamma_jB_j(k,s,t')\right]ds}\times$$

$$\int_{-\infty}^{+\infty} e^{\imath k\left[(x'-x-\int_t^{t'} r(s)ds+\frac{1}{2}\Gamma_0(t,t')\right]}\left(x'-x-\int_t^{t'} r(s)ds-\frac{1}{2}\Gamma_0(t,t')\right)dx'dk,$$

$$=\int_{-\infty}^{+\infty} e^{-\frac{1}{2}\Gamma_0(t,t')k^2} e^{\sum_{j=1}^{n}\int_t^{t'} E(v_{j,s}\,|\,\mathcal{F}_t)\left[\frac{\gamma_j^2}{2}B_j^2(k,s,t')+\imath\,k\rho_j\gamma_jB_j(k,s,t')\right]ds}\int_{-\infty}^{+\infty} ye^{\imath y}dydk=0, \qquad (127)$$

since the derivative of the exponent with respect to k is zero because $B_j(0,s,t')=0$, $s<t'$. The proof of Eqs. (119)–(121) for M_0, M_1, and M_2 can through a similar argument, bearing in mind the Fourier transform of the Gaussian kernel and its derivatives, i.e., using Eqs. (118), (106), (107), (108), (115), and (116). This concludes the proof. □

3 Proofs of the results in Section 3

In this section we derive explicit formulas for European vanilla call and put options by using the third-order approximation, M_3, for the multi-factor Heston conditional marginal probability density, M. In the following, we use $C_{MH}(S_0, T, E)$ and $P_{MH}(S_0, T, E)$ to denote the price of European vanilla call and put options, respectively, in the multi-factor Heston model, with spot price S_0, maturity T, strike price E, and discount factor $B(T)$, which is given by

$$B(T) = e^{-\int_0^T r(s)ds}. \tag{128}$$

Specifically, C_{MH} and P_{MH} read as

$$C_{MH}(S_0, T, E) = B(T) \int_{\log E}^{+\infty} (e^{x'} - E) M(\log S_0, \underline{v}_0, 0, x', T) dx' \,, \tag{129}$$

and

$$P_{MH}(S_0, T, E) = B(T) \int_{-\infty}^{\log E} (E - e^{x'}) M(\log S_0, \underline{v}_0, 0, x', T) dx' \,, \tag{130}$$

where $\underline{v}_0$ is the vector of the variances at time $t = 0$. Furthermore, we use $C_{BS}\left(S_0, T, E, \sqrt{\frac{\Gamma}{T}}\right)$ and $P_{BS}\left(S_0, T, E, \sqrt{\frac{\Gamma}{T}}\right)$ to denote the classical Black-Scholes formulas for call and put vanilla options, where $\Gamma = \Gamma(0, T) > 0$ is the integrated variance over the time interval $[0, T]$, that is,

$$C_{BS}\left(S_0, T, E, \sqrt{\frac{\Gamma}{T}}\right) = S_0 N(d_1(\Gamma)) - E e^{-\int_0^T r(s)ds} N(d_2(\Gamma)), \tag{131}$$

and

$$P_{BS}\left(S_0, T, E, \sqrt{\frac{\Gamma}{T}}\right) = -S_0 N(-d_1(\Gamma)) + E e^{-\int_0^T r(s)ds} N(-d_2(\Gamma)), \tag{132}$$

where $N(x)$ ia given by

$$N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-y^2/2} dy, \tag{133}$$

and $d_1(\Gamma)$ and $d_2(\Gamma)$ are given by

$$d_1(\Gamma) = \frac{\log\left(\frac{S_0}{E}\right) + \int_0^T r(s)ds + \frac{1}{2}\Gamma}{\sqrt{\Gamma}} \,, \tag{134}$$

$$d_2(\Gamma) = d_1(\Gamma) - \sqrt{\Gamma} = \frac{\log\left(\frac{S_0}{E}\right) + \int_0^T r(s)ds - \frac{1}{2}\Gamma}{\sqrt{\Gamma}} \,. \tag{135}$$

Proposition 3.1 *Let $C(S_0, T, E)$, $P(S_0, T, E)$ be the prices of European call and put options with spot price S_0, maturity T, strike price E, and discount factor $B(T)$ as given in Eqs. (129)–(130). We have*

$$C(S_0, T, E) = C_{BS}\left(S_0, T, E, \sqrt{\frac{\Gamma_2}{T}}\right) + \mathcal{R}_1(S_0, T, E) + \mathcal{R}_2(S_0, T, E) + \mathcal{R}_3(S_0, T, E) + o\left(\|\underline{\gamma}\|^3\right), \ \|\underline{\gamma}\| \to 0, \tag{136}$$

and

$$P(S_0,T,E) = P_{BS}\left(S_0,T,E,\sqrt{\frac{\Gamma_2}{T}}\right) + \mathcal{R}_1(S_0,T,E) \quad + \quad \mathcal{R}_2(S_0,T,E) + \mathcal{R}_3(S_0,T,E)$$
$$+o\left(\|\underline{\gamma}\|^3\right), \ \ \|\underline{\gamma}\| \to 0. \tag{137}$$

Here, $\Gamma_2(0,T)$ is given by (49), C_{BS} and P_{BS} denote the classical Black-Scholes formulas as in (131) and (132), and $\mathcal{R}_1$, $\mathcal{R}_2$, and $\mathcal{R}_3$ are the corrections to the standard Black-Scholes formula due to the contribution of the first-, second-, and third-order correction terms of the expansion in powers of vols of vols of the marginal density function:

$$\mathcal{R}_1(S_0,T,E) = B(T)E\frac{S_1(0,T)}{\Gamma_2(0,T)}\left(+\log\left(\frac{E}{S_0 e^{\int_0^T r(s)ds}}\right) + \frac{3}{2}\Gamma_2(0,T)\right)\mathcal{G}_{\Gamma_2}(\log(E/S_0),0,T), \tag{138}$$

$$\begin{aligned}\mathcal{R}_2(S_0,T,E) = {} & S_2(0,T)B(T)E\left[\frac{d^2\mathcal{G}_{\Gamma_2}}{dx'^2} + \frac{d\mathcal{G}_{\Gamma_2}}{dx'} - \mathcal{G}_{\Gamma_2}\right](\log(E/S_0),0,T) \\ & + S_{2c}(0,T)B(T)E\frac{d^2\mathcal{G}_{\Gamma_2}}{dx'^2}(\log(E/S_0),0,T) \\ & + \frac{1}{2}S_1^2(0,T)B(T)E\left[\frac{d^4\mathcal{G}_{\Gamma_2}}{dx'^4} - \frac{d^3\mathcal{G}_{\Gamma_2}}{dx'^3} - \frac{d^2\mathcal{G}_{\Gamma_2}}{dx'^2} + \frac{d\mathcal{G}_{\Gamma_2}}{dx'}\right](\log(E/S_0),0,T),\end{aligned} \tag{139}$$

and

$$\begin{aligned}\mathcal{R}_3(S_0,T,E) = {} & B(T)E\,S_{3c}(0,T)\left[-\frac{d^3\mathcal{G}_{\Gamma_2}}{dx'^3} - \frac{d^2\mathcal{G}_{\Gamma_2}}{dx'^2}\right](\log(E/S_0),0,T) \\ & -B(T)ES_{3d}(0,T)\frac{d^3\mathcal{G}_{\Gamma_2}}{dx'^3}(\log(E/S_0),0,T) + B(T)E\,S_1(0,T)S_{2c}(0,T)\left[-\frac{d^5\mathcal{G}_{\Gamma_2}}{dx'^5} + \frac{d^3\mathcal{G}_{\Gamma_2}}{dx'^3}\right](\log(E/S_0),0,T) \\ & +\frac{1}{6}B(T)E\,S_1^3(0,T)\left[-\frac{d^7\mathcal{G}_{\Gamma_2}}{dx'^7} + \frac{d^6\mathcal{G}_{\Gamma_2}}{dx'^6} + 2\frac{d^5\mathcal{G}_{\Gamma_2}}{dx'^5} - 2\frac{d^4\mathcal{G}_{\Gamma_2}}{dx'^4} - \frac{d^3\mathcal{G}_{\Gamma_2}}{dx'^3} + \frac{d^2\mathcal{G}_{\Gamma_2}}{dx'^2}\right](\log(E/S_0),0,T) \\ & +B(T)E\,S_1(0,T)S_2(0,T)\left[-\frac{d^5\mathcal{G}_{\Gamma_2}}{dx'^5} - \frac{d^4\mathcal{G}_{\Gamma_2}}{dx'^4} + 2\frac{d^3\mathcal{G}_{\Gamma_2}}{dx'^3} + \frac{d^2\mathcal{G}_{\Gamma_2}}{dx'^2} - \frac{d\mathcal{G}_{\Gamma_2}}{dx'}\right](\log(E/S_0),0,T) \\ & +B(T)E\,S_1(0,T)S_{2c}(0,T)\left[-\frac{d^5\mathcal{G}_{\Gamma_2}}{dx'^5} + \frac{d^3\mathcal{G}_{\Gamma_2}}{dx'^3}\right](\log(E/S_0),0,T),\end{aligned} \tag{140}$$

where S_1, S_2, S_{2c}, S_{3c}, and S_{3d} are given in (50), (51) (54), (55), and (56), respectively. The notation $[\cdot](\cdot,\cdot,\cdot)$ in Eq. (139) and Eq. (140) means that the function in the square brackets is evaluated at the argument $(\cdot,\cdot,\cdot)$.

Note that for $\gamma = 0$, $\mathcal{G}_{\Gamma_2}$ coincides with $\mathcal{G}_{\Gamma_0}$, the correction terms $\mathcal{R}_1$, $\mathcal{R}_2$, and $\mathcal{R}_3$ become zero, and the option prices become the classical Black and Scholes price for options with time-dependent but deterministic volatilities. Dropping the arguments of Γ_2, S_1, S_2, and S_{2c}, Eqs. (138), (139) also reads as

$$\mathcal{R}_1(S_0,T,E) = \frac{Vega(\Gamma_2)}{\sqrt{T}\Gamma_2^{3/2}}S_1\left(m_E + \frac{3}{2}\Gamma_2\right), \tag{141}$$

$$
\begin{aligned}
\mathcal{R}_2(S_0,T,E) &= +S_{2c}\frac{Vega(\Gamma_2)}{\sqrt{T}\,\Gamma_2^{3/2}}\left[\frac{(m_E+\frac{1}{2}\Gamma_2)^2}{\Gamma_2}-1\right]+S_2\frac{Vega(\Gamma_2)}{\sqrt{T}\,\Gamma_2^{3/2}}\left[\frac{(m_E+\frac{1}{2}\Gamma_2)^2}{\Gamma_2}-(m_E+\frac{1}{2}\Gamma_2)-1-\Gamma_2\right]\\
&+\frac{1}{2}S_1^2\frac{Vega(\Gamma_2)}{\sqrt{T}\,\Gamma_2^{3/2}}\left[\frac{(m_E+\frac{1}{2}\Gamma_2)^4}{\Gamma_2^3}+\frac{(m_E+\frac{1}{2}\Gamma_2)^3}{\Gamma_2^2}-\frac{(m_E+\frac{1}{2}\Gamma_2)^2}{\Gamma_2}\left(1+\frac{6}{\Gamma_2}\right)\right]\\
&+\frac{1}{2}S_1^2\frac{Vega(\Gamma_2)}{\sqrt{T}\,\Gamma_2^{3/2}}\left[-(m_E+\frac{1}{2}\Gamma_2)\left(1+\frac{3}{\Gamma_2}\right)+\left(1+\frac{3}{\Gamma_2}\right)\right],
\end{aligned}
\tag{142}
$$

where m_E is the log-moneyness associated with the forward price defined as

$$
m_E=\log\left(\frac{E}{S_0e^{\int_0^T r(s)ds}}\right),
\tag{143}
$$

and the Black-Scholes Vega is $Vega(\Gamma_2)=\sqrt{T}Ee^{-\int_0^T r(s)ds}N'(d_2(\Gamma_2))$ with $d_2(\Gamma_2)=-\left(m_E+\frac{1}{2}\Gamma_2\right)/\sqrt{\Gamma_2}$.

Proof of Proposition 3.1 *The price of a European vanilla call option with maturity T, spot price S_0, and strike price E discounted by a deterministic factor $B(T)$ is given in Eq. (129). Thus, using formula (105) for M in (129), we have*

$$
\begin{aligned}
C(S_0,T,E) &= B(T)\int_{\log E}^{+\infty}(e^{x'}-E)\mathcal{G}_{\Gamma_2}(x'-\log S_0,0,T)dx'\\
&+B(T)S_1(0,T)\int_{\log E}^{+\infty}(e^{x'}-E)\left[-\frac{d^3\mathcal{G}_{\Gamma_2}}{dx'^3}+\frac{d\mathcal{G}_{\Gamma_2}}{dx'}\right](x'-\log S_0,0,T)dx'\\
&+B(T)S_2(0,T)\int_{\log E}^{+\infty}(e^{x'}-E)\left[\frac{d^4\mathcal{G}_{\Gamma_2}}{dx'^4}+2\frac{d^3\mathcal{G}_{\Gamma_2}}{dx'^3}-\frac{d\mathcal{G}_{\Gamma_2}}{dx'}\right](x'-\log S_0,0,T)dx'\\
&+B(T)S_{2c}(0,T)\int_{\log E}^{+\infty}(e^{x'}-E)\left[\frac{d^4\mathcal{G}_{\Gamma_2}}{dx'^4}+\frac{d^3\mathcal{G}_{\Gamma_2}}{dx'^3}\right](x'-\log S_0,0,T)dx'\\
&+B(T)\frac{1}{2}S_1^2(0,T)\int_{\log E}^{+\infty}(e^{x'}-E)\left[\frac{d^6\mathcal{G}_{\Gamma_2}}{dx'^6}-2\frac{d^4\mathcal{G}_{\Gamma_2}}{dx'^4}+\frac{d^2\mathcal{G}_{\Gamma_2}}{dx'^2}\right](x'-\log S_0,0,T)dx\\
&+B(T)S_{3c}(0,T)\int_{\log E}^{+\infty}(e^{x'}-E)\left[-\frac{d^3}{dx'^3}\mathcal{G}_{\Gamma_2}-2\frac{d^4}{dx'^4}\mathcal{G}_{\Gamma_2}-\frac{d^5}{dx'^5}\mathcal{G}_{\Gamma_2}\right](x'-\log S_0,0,T)dx'\\
&+B(T)S_{3d}(0,T)\int_{\log E}^{+\infty}(e^{x'}-E)\left[-\frac{d^4}{dx'^4}\mathcal{G}_{\Gamma_2}-\frac{d^5}{dx'^5}\mathcal{G}_{\Gamma_2}\right](x'-\log S_0,0,T)dx'\\
&+\frac{1}{6}B(T)S_1^3(0,T)\int_{\log E}^{+\infty}(e^{x'}-E)\left[-\frac{d^9}{dx'^9}\mathcal{G}_{\Gamma_2}+3\frac{d^7}{dx'^7}\mathcal{G}_{\Gamma_2}-3\frac{d^5}{dx'^5}\mathcal{G}_{\Gamma_2}+\frac{d^3}{dx'^3}\mathcal{G}_{\Gamma_2}\right](x'-\log S_0,0,T)dx'\\
&+B(T)S_1(0,T)S_2(0,T)\int_{\log E}^{+\infty}(e^{x'}-E)\left[-\frac{d^7}{dx'^7}\mathcal{G}_{\Gamma_2}-2\frac{d^6}{dx'^6}\mathcal{G}_{\Gamma_2}+\frac{d^5}{dx'^5}\mathcal{G}_{\Gamma_2}+3\frac{d^4}{dx'^4}\mathcal{G}_{\Gamma_2}-\frac{d^2}{dx'^2}\mathcal{G}_{\Gamma_2}\right](x'-\log S_0,0,T)dx'\\
&+B(T)S_1(0,T)S_{2c}(0,T)\int_{\log E}^{+\infty}(e^{x'}-E)\left[-\frac{d^7}{dx'^7}\mathcal{G}_{\Gamma_2}-\frac{d^6}{dx'^6}\mathcal{G}_{\Gamma_2}+\frac{d^5}{dx'^5}\mathcal{G}_{\Gamma_2}+\frac{d^4}{dx'^4}\mathcal{G}_{\Gamma_2}\right](x'-\log S_0,0,T)dx'.
\end{aligned}
\tag{144}
$$

As mentioned above, the notation $[\cdot](\cdot,\cdot,\cdot)$ means that the function in the square parentheses is evaluated at $(\cdot,\cdot,\cdot)$. The put option price formula can be obtained from formula (144) by replacing $\int_{\log E}^{+\infty}(e^{x'}-E)$ with $\int_{-\infty}^{\log E}(E-e^{x'})$. Hence, the key ingredient to evaluate option prices is the computation of the integrals

$$
I_m^C=\int_{\log E}^{+\infty}e^{x'}\frac{d^m\mathcal{G}_{\Gamma_2}}{dx'^m}dx',\ m=1,2,\ldots,
\tag{145}
$$

and

$$I_m^P = \int_{-\infty}^{\log E} e^{x'} \frac{d^m \mathcal{G}_{\Gamma_2}}{dx'^m} dx', \;\; m = 1, 2, \ldots . \tag{146}$$

From Eqs. (145) and (146) we have

$$I_m^C = -e^{x'} \left. \frac{d^{m-1}\mathcal{G}_{\Gamma_2}}{dx'^{m-1}} \right|_{x'=\log E} + I_{m-1}^C \,, \tag{147}$$

$$I_m^P = e^{x'} \left. \frac{d^{m-1}\mathcal{G}_{\Gamma_2}}{dx'^{m-1}} \right|_{x'=\log E} - I_{m-1}^P \,, \tag{148}$$

so by recursion we have

$$I_m^C = \sum_{j=1}^{m} (-1)^j \, e^{x'} \left. \frac{d^{m-j}\mathcal{G}_{\Gamma_2}}{dx'^{m-j}} \right|_{x'=\log E} + (-1)^m I_0^c = I_m + (-1)^m \, I_0^C \,, \tag{149}$$

$$I_m^P = -\sum_{j=1}^{m} (-1)^j \, e^{x'} \left. \frac{d^{m-j}\mathcal{G}_{\Gamma_2}}{dx'^{m-j}} \right|_{x'=\log E} + (-1)^m I_0^p = -I_m + (-1)^m \, I_0^P \,, \tag{150}$$

where I_m is given in (151), while I_0^C and I_0^P are

$$I_m = \sum_{j=1}^{m} (-1)^j e^{x'} \frac{d^{m-j}}{dx'^{m-j}} \mathcal{G}_{\Gamma_2}(\log E - \log S_0, 0, T) \,, \tag{151}$$

$$I_0^c = \int_{\log E}^{+\infty} e^{x'} \mathcal{G}_{\Gamma_2}(x' - \log S_0, 0, T) dx' = S_0 e^{\int_0^T r(s)ds} N\left(d_1(\Gamma_2)\right) = S_0 B(T)^{-1} N\left(d_1(\Gamma_2)\right) \,, \tag{152}$$

$$I_0^p = \int_{-\infty}^{\log E} e^{x'} \mathcal{G}_{\Gamma_2}(x' - \log S_0, 0, T) dx' = S_0 e^{\int_0^T r(s)ds} N\left(-d_1(\Gamma_2)\right) = S_0 B(T)^{-1} N\left(-d_1(\Gamma_2)\right) \,, \tag{153}$$

with

$$d_1(\Gamma_2) = \frac{\log(S_0/E) + \int_0^T r(s)ds + \frac{1}{2}\Gamma_2}{\sqrt{\Gamma_2}}.$$

Furthermore, we have

$$\int_{\log E}^{+\infty} \mathcal{G}_{\Gamma_2}(x' - \log S_0, 0, T) dx' = N\left(d_2(\Gamma_2)\right) \,, \tag{154}$$

$$\int_{-\infty}^{\log E} \mathcal{G}_{\Gamma_2}(x' - \log S_0, 0, T) dx' = N\left(-d_2(\Gamma_2)\right) \,, \tag{155}$$

$$\tag{156}$$

where

$$d_2(\Gamma_2) = \frac{\log(S_0/E) + \int_0^T r(s)ds - \frac{1}{2}\Gamma_2}{\sqrt{\Gamma_2}}.$$

Integrating by parts yields

$$\begin{aligned}
&C(S_0,T,E) = B(T)\int_{\log E}^{+\infty}(e^{x'}-E)\mathcal{G}_{\Gamma_2}(x'-\log S_0,0,T)dx' \\
&+B(T)S_1(0,T)\int_{\log E}^{+\infty}e^{x'}\left[\frac{d^2\mathcal{G}_{\Gamma_2}}{dx'^2}-\mathcal{G}_{\Gamma_2}\right](x'-\log S_0,0,T)dx' \\
&-B(T)S_2(0,T)\int_{\log E}^{+\infty}e^{x'}\left[\frac{d^3\mathcal{G}_{\Gamma_2}}{dx'^3}+2\frac{d^2\mathcal{G}_{\Gamma_2}}{dx'^2}-\mathcal{G}_{\Gamma_2}\right](x'-\log S_0,0,T)dx' \\
&-B(T)S_{2c}(0,T)\int_{\log E}^{+\infty}e^{x'}\left[\frac{d^3\mathcal{G}_{\Gamma_2}}{dx'^3}+\frac{d^2\mathcal{G}_{\Gamma_2}}{dx'^2}\right](x'-\log S_0,0,T)dx' \\
&-B(T)\frac{1}{2}S_1^2(0,T)\int_{\log E}^{+\infty}e^{x'}\left[\frac{d^5\mathcal{G}_{\Gamma_2}}{dx'^5}-2\frac{d^3\mathcal{G}_{\Gamma_2}}{dx'^3}+\frac{d\mathcal{G}_{\Gamma_2}}{dx'}\right](x'-\log S_0,0,T)dx \\
&-B(T)S_{3c}(0,T)\int_{\log E}^{+\infty}e^{x'}\left[-\frac{d^2}{dx'^2}\mathcal{G}_{\Gamma_2}-2\frac{d^3}{dx'^3}\mathcal{G}_{\Gamma_2}-\frac{d^4}{dx'^4}\mathcal{G}_{\Gamma_2}\right]dx' \\
&-B(T)S_{3d}(0,T)\int_{\log E}^{+\infty}e^{x'}\left[-\frac{d^3}{dx'^3}\mathcal{G}_{\Gamma_2}-\frac{d^4}{dx'^4}\mathcal{G}_{\Gamma_2}\right]dx' \\
&-\frac{1}{6}B(T)S_1^3(0,T)\int_{\log E}^{+\infty}e^{x'}\left[-\frac{d^8}{dx'^8}\mathcal{G}_{\Gamma_2}+3\frac{d^6}{dx'^6}\mathcal{G}_{\Gamma_2}-3\frac{d^4}{dx'^4}\mathcal{G}_{\Gamma_2}+\frac{d^2}{dx'^2}\mathcal{G}_{\Gamma_2}\right]dx' \\
&-B(T)S_1(0,T)S_2(0,T)\int_{\log E}^{+\infty}e^{x'}\left[-\frac{d^6}{dx'^6}\mathcal{G}_{\Gamma_2}-2\frac{d^5}{dx'^5}\mathcal{G}_{\Gamma_2}+\frac{d^4}{dx'^4}\mathcal{G}_{\Gamma_2}+3\frac{d^3}{dx'^3}\mathcal{G}_{\Gamma_2}-\frac{d}{dx'}\mathcal{G}_{\Gamma_2}\right]dx' \\
&-B(T)S_1(0,T)S_{2c}(0,T)\int_{\log E}^{+\infty}e^{x'}\left[-\frac{d^6}{dx'^6}\mathcal{G}_{\Gamma_2}-\frac{d^5}{dx'^5}\mathcal{G}_{\Gamma_2}+\frac{d^4}{dx'^4}\mathcal{G}_{\Gamma_2}+\frac{d^3}{dx'^3}\mathcal{G}_{\Gamma_2}\right]dx'.
\end{aligned} \tag{157}$$

Now we can insert formulas (149), (151), (152), and (154) in formula (157) to obtain

$$\begin{aligned}
&C(S_0,T,E) = S_0N(d_1(\Gamma_2)) - B(T)E\,N(d_2(\Gamma_2)) \\
&+B(T)S_1(0,T)\left[I_2+I_0^c-I_0^c\right]-B(T)S_2(0,T)\left[I_3-I_0^c+2I_2+2I_0^c-I_0^c\right] \\
&-B(T)S_{2c}(0,T)\left[I_3-I_0^c+I_2+I_0^c\right]-B(T)\frac{1}{2}S_1^2(0,T)\left[I_5-I_0^c-2(I_3-I_0^c)+I_1-I_0^c\right] \\
&-B(T)S_{3c}(0,T)\left[-I_2-I_0^c-2(I_3-I_0^c)-I_4-I_0^c\right]-B(T)S_{3d}(0,T)\left[-I_3+I_0^c-I_4-I_0^c\right] \\
&-\frac{1}{6}B(T)S_1^3(0,T)\left[-I_8-I_0^c+3I_6+3I_0^c-3(I_4+I_0^c)+I_2+I_0^c\right] \\
&-B(T)S_1(0,T)S_2(0,T)\left[-(I_6+I_0^c)-2(I_5-I_0^c)+I_4+I_0^c+3(I_3-I_0^c)-(I_1-I_0^c)\right] \\
&-B(T)S_1(0,T)S_{2c}(0,T)\left[-(I_6+I_0^c)-(I_5-I_0^c)+I_4+I_0^c+I_3-I_0^c\right]
\end{aligned} \tag{158}$$

$$\begin{aligned}
&C(S_0,T,E) = S_0N(d_1(\Gamma_2)) - B(T)E\,N(d_2(\Gamma_2)) \\
&+B(T)S_1(0,T)\,I_2-B(T)S_2(0,T)\left[I_3+2I_2\right]-B(T)S_{2c}(0,T)\left[I_3+I_2\right]-B(T)\frac{1}{2}S_1^2(0,T)\left[I_5-2I_3+I_1\right] \\
&-B(T)S_{3c}(0,T)\left[-I_2-2I_3-I_4\right]-B(T)S_{3d}(0,T)\left[-I_3-I_4\right] \\
&-\frac{1}{6}B(T)S_1^3(0,T)\left[-I_8+3I_6-3I_4+I_2\right]-B(T)S_1(0,T)S_2(0,T)\left[-I_6-2I_5+I_4+3I_3-I_1\right] \\
&-B(T)S_1(0,T)S_{2c}(0,T)\left[-I_6-I_5+I_4+I_3\right].
\end{aligned} \tag{159}$$

Formula $\mathcal{R}_2$ follows bearing in mind that

$$I_2 = -\frac{d\mathcal{G}_{\Gamma_2}}{dx'} + \mathcal{G}_{\Gamma_2},$$

$$I_3 + 2I_2 = -E\left[\frac{d^2\mathcal{G}_{\Gamma_2}}{dx'^2} + \frac{d\mathcal{G}_{\Gamma_2}}{dx'} - \mathcal{G}_{\Gamma_2}\right],$$

$$I_5 - 2I_3 + I_1 = -E\left[\frac{d^4\mathcal{G}_{\Gamma_2}}{dx'^4} - \frac{d^3\mathcal{G}_{\Gamma_2}}{dx'^3} - \frac{d^2\mathcal{G}_{\Gamma_2}}{dx'^2} + \frac{d\mathcal{G}_{\Gamma_2}}{dx'}\right],$$

and

$$I_3 + I_2 = -E\frac{d^2\mathcal{G}_{\Gamma_2}}{dx'^2}.$$

The latter equation is more general since we have

$$I_m = -\frac{d^{m-1}\mathcal{G}_{\Gamma_2}}{dx'^{(m-1)}} - I_{m-1}, \tag{160}$$

where $I_0 = 0$, $I_1 = -\mathcal{G}_{\Gamma_2}$. Using (160) and Eq. (159), we have

$$\begin{aligned}
\mathcal{R}_3(S_0, T, E) &= -B(T)S_{3c}(0,T)\left[E\frac{d^3\mathcal{G}_{\Gamma_2}}{dx'^3} + E\frac{d^2\mathcal{G}_{\Gamma_2}}{dx'^2}\right] - B(T)S_{3d}(0,T)\left[E\frac{d^3\mathcal{G}_{\Gamma_2}}{dx'^3}\right] \\
&-\frac{1}{6}B(T)S_1^3(0,T)\left[E\frac{d^7\mathcal{G}_{\Gamma_2}}{dx'^7} - E\frac{d^6\mathcal{G}_{\Gamma_2}}{dx'^6} - 2E\frac{d^5\mathcal{G}_{\Gamma_2}}{dx'^5} + 2E\frac{d^4\mathcal{G}_{\Gamma_2}}{dx'^4} + E\frac{d^3\mathcal{G}_{\Gamma_2}}{dx'^3} - E\frac{d^2\mathcal{G}_{\Gamma_2}}{dx'^2}\right] \\
&-B(T)S_1(0,T)S_2(0,T)\left[E\frac{d^5\mathcal{G}_{\Gamma_2}}{dx'^5} + E\frac{d^4\mathcal{G}_{\Gamma_2}}{dx'^4} - 2E\frac{d^3\mathcal{G}_{\Gamma_2}}{dx'^3} - E\frac{d^2\mathcal{G}_{\Gamma_2}}{dx'^2} + E\frac{d\mathcal{G}_{\Gamma_2}}{dx'}\right] \\
&-B(T)S_1(0,T)S_{2c}(0,T)\left[E\frac{d^5\mathcal{G}_{\Gamma_2}}{dx'^5} - E\frac{d^3\mathcal{G}_{\Gamma_2}}{dx'^3}\right].
\end{aligned} \tag{161}$$

Proceeding in a similar manner, we obtain the approximation for the put option in Eq. (137). As mentioned above, the corrections $\mathcal{R}_1$, $\mathcal{R}_2$, and $\mathcal{R}_3$ for the call option are the same as the put correction since there are two changes of sign: one due to the payoff function and the other due to integration by parts over the interval $(-\infty, \log E)$ rather than $(\log E, +\infty)$.
This concludes the proof. □

The implied volatility Σ in the multi-factor Heston model is defined as the quantity such that the following equality holds:

$$C_{BS}\left(S_0, T, E, \sqrt{\frac{\Sigma^2}{T}}\right) = C(S_0, T, E). \tag{162}$$

We derive the first and second-order approximations of Σ, as a function of the vols of vols (i.e., $\Sigma = \Sigma(\underline{\gamma})$) by solving

$$C_{BS}\left(S_0, T, E, \sqrt{\frac{\Sigma_m^2(\underline{\gamma})}{T}}\right) = C_{BS}\left(S_0, T, E, \sqrt{\frac{\Gamma_2}{T}}\right) + \mathcal{R}_m(S_0, T, E), \ m = 1, 2. \tag{163}$$

Proposition 3.2 *The first-order approximation, Σ_1, and the second-order one, Σ_2 are given by:*

$$\Sigma_1(\underline{\gamma}) = \sqrt{\Gamma_0} + \frac{S_1}{\Gamma_0\sqrt{\Gamma_0}}\left(m_E + \frac{1}{2}\Gamma_0\right), \tag{164}$$

$$\Sigma_2(\underline{\gamma}) = \sqrt{\Gamma_0} + \sqrt{\Gamma_0}\left[a_0(T,\underline{\gamma}) + a_1(T,\underline{\gamma})\left(m_E + \frac{1}{2}\Gamma_0\right) + a_2(T,\underline{\gamma})\left(m_E + \frac{1}{2}\Gamma_0\right)^2\right]. \tag{165}$$

Here m_E is the log-moneyness associated with the forward price (see Eq. (143)), Γ_0 and S_1 are defined in (6), and $a_0(T,\underline{\gamma})$, $a_1(T,\underline{\gamma})$, and $a_2(T,\underline{\gamma})$ are given by

$$a_0(T,\underline{\gamma}) = \frac{3}{2}\frac{1}{\Gamma_0^3}S_1^2 - \frac{(S_2 + S_{2c})}{\Gamma_0^2}, \tag{166}$$

$$a_1(T,\underline{\gamma}) = \frac{(S_1 - S_2)}{\Gamma_0^2} + \frac{3}{2}\frac{1}{\Gamma_0^3}S_1^2, \tag{167}$$

and

$$a_2(T,\underline{\gamma}) = \frac{1}{\Gamma_0^3}\left(S_2 + S_{2c} - \frac{3}{\Gamma_0}S_1^2\right). \tag{168}$$

S_2 and S_{2c} are given in (51), (54), respectively. Here, we have dropped the arguments $(0,T)$ of the functions Γ_0, S_1, S_2, and S_{2c}.

Proof of Proposition 3.2 *Let us prove formula (164).*
When $\underline{\gamma} = \underline{0}$ (i.e., all vols of vols equal zero), we have $\Gamma_2(0,T)$ equal to $\Gamma_0(0,T)$ and the correction terms $\mathcal{R}_1$, $\mathcal{R}_2$ equal to zero, which implies

$$\Sigma_1(\underline{0}) = \sqrt{\Gamma_0(0,T)}. \tag{169}$$

We compute the partial derivative of both sides of equation (163) with respect to γ_j, $j = 1,2,\ldots,n$ and we evaluate the derivatives at $\underline{\gamma} = \underline{0}$. Using the Black-Scholes Vega (i.e., $\frac{\partial C_{BS}}{\partial\sigma}|_{\underline{\gamma}=\underline{0}} = S_0 N'(d_1(\Gamma_0))\sqrt{T}$), and the derivatives of Γ_2 and S_1 with respect to γ_j, we have

$$\left.\frac{\partial C_{BS}}{\partial\sigma}\right|_{\underline{\gamma}=\underline{0}}\frac{\partial}{\partial\gamma_j}\Sigma_1(\underline{0}) = \frac{\rho_j\,\mathcal{T}_j(0,T)}{2\chi_j}S_0 N'(d_1(\Gamma_0))\left[-\frac{1}{\sqrt{\Gamma_0(0,T)}} - d_2(\Gamma_0) + \frac{1}{\sqrt{\Gamma_0(0,T)}}\right], \tag{170}$$

where

$$\mathcal{T}_j(t,t') = \int_t^{t'}\left(1 - e^{-\chi_j(t'-s)}\right)E(v_{j,s}\,|\,\mathcal{F}_t)ds, \tag{171}$$

while d_1 and d_2 are given in Eqs. (134) and (135).

Eq. (170) and the expression for the Black-Scholes Vega, $\frac{\partial C_{BS}}{\partial\sigma}|_{\underline{\gamma}=\underline{0}} = S_0 N'(d_1(\Gamma_0))\sqrt{T}$, yield the derivative $\frac{\partial}{\partial\gamma_j}\Sigma_1$ at $\underline{\gamma} = \underline{0}$:

$$\left.\frac{\partial}{\partial\gamma_j}\Sigma_1\right|_{\underline{\gamma}=\underline{0}} = \frac{\rho_j\,\mathcal{T}_j(0,T)}{2\chi_j}\frac{1}{\sqrt{\Gamma_0(0,T)}}\left(+\frac{1}{2} - \frac{(\ln(S_0/E) + \int_0^T r(s)ds)}{\Gamma_0(0,T)}\right), \tag{172}$$

thus implying

$$\Sigma_1(\underline{\gamma}) = \sqrt{\Gamma_0(0,T)} - \frac{1}{\sqrt{\Gamma_0(0,T)}}\left(\frac{(\ln(S_0/E) + \int_0^T r(s)ds)}{\Gamma_0(0,T)} - \frac{1}{2}\right)\sum_{j=1}^n\frac{\gamma_j\rho_j}{2\chi_j}\mathcal{T}_j(0,T). \tag{173}$$

We now prove Eq. (165).

We need to compute the second-order derivatives of Σ with respect to the vols of vols. We have

$$\frac{1}{T}\left.\frac{\partial^2 C_{BS}}{\partial\sigma^2}\right|_{\underline{\gamma}=\underline{0}}\left.\left(\frac{\partial\Sigma_2}{\partial\gamma_j}\right)^2\right|_{\underline{\gamma}=\underline{0}}+\frac{1}{\sqrt{T}}\left.\frac{\partial C_{BS}}{\partial\sigma}\right|_{\underline{\gamma}=\underline{0}}\frac{\partial^2}{\partial\gamma_j^2}\Sigma_2(\underline{0})=\frac{1}{T}\left.\frac{\partial^2 C_{BS}}{\partial\sigma^2}\right|_{\underline{\gamma}=\underline{0}}\left(\frac{\partial S_1}{\partial\gamma_j}\right)^2\frac{1}{\Gamma_0}$$
$$+\frac{1}{\sqrt{T}}\left.\frac{\partial C_{BS}}{\partial\sigma}\right|_{\underline{\gamma}=\underline{0}}\frac{\partial^2 S_2}{\partial\gamma_j^2}\frac{1}{\sqrt{\Gamma_0}}\left[\frac{(m_E+\frac{1}{2}\Gamma_0)^2}{\Gamma_0^2}-\frac{(m_E+\frac{1}{2}\Gamma_0)}{\Gamma_0}-\frac{1}{\Gamma_0}\right]$$
$$+\frac{1}{\sqrt{T}}\left.\frac{\partial C_{BS}}{\partial\sigma}\right|_{\underline{\gamma}=\underline{0}}\frac{\partial^2 S_{2c}}{\partial\gamma_j^2}\frac{1}{\sqrt{\Gamma_0}}\left[\frac{(m_E+\frac{1}{2}\Gamma_0)^2}{\Gamma_0^2}-\frac{1}{\Gamma_0}\right]$$
$$+\frac{1}{\sqrt{T}}\left.\frac{\partial C_{BS}}{\partial\sigma}\right|_{\underline{\gamma}=\underline{0}}\frac{1}{\sqrt{\Gamma_0}}\left(\frac{\partial S_1}{\partial\gamma_j}\right)^2\left[+\frac{(m_E+\frac{1}{2}\Gamma_0)^4}{\Gamma_0^4}-\frac{(m_E+\frac{1}{2}\Gamma_0)^3}{\Gamma_0^3}-\left(1+\frac{6}{\Gamma_0}\right)\frac{(m_E+\frac{1}{2}\Gamma_0)^2}{\Gamma_0^2}\right]$$
$$+\frac{1}{\sqrt{T}}\left.\frac{\partial C_{BS}}{\partial\sigma}\right|_{\underline{\gamma}=\underline{0}}\frac{1}{\sqrt{\Gamma_0}}\left(\frac{\partial S_1}{\partial\gamma_j}\right)^2\left[\left(1+\frac{3}{\Gamma_0}\right)\frac{(m_E+\frac{1}{2}\Gamma_0)}{\Gamma_0}+\frac{3}{\Gamma_0^2}\right]. \tag{174}$$

Bearing in mind that we have

$$\left.\frac{\partial C_{BS}}{\partial\sigma}\right|_{\underline{\gamma}=\underline{0}}=Vega(\Gamma_0), \tag{175}$$

$$\left.\frac{\partial^2 C_{BS}}{\partial\sigma^2}\right|_{\underline{\gamma}=\underline{0}}=Vomma(\Gamma_0)=Vega(\Gamma_0)\frac{\sqrt{T}}{\Gamma_0^{3/2}}(m_E+\frac{1}{2}\Gamma_0)(m_E-\frac{1}{2}\Gamma_0)$$
$$=Vega(\Gamma_0)\frac{\sqrt{T}}{\Gamma_0^{3/2}}\left[(m_E+\frac{1}{2}\Gamma_0)^2-\Gamma_0(m_E+\frac{1}{2}\Gamma_0)\right], \tag{176}$$

and

$$\left.\left(\frac{\partial\Sigma_2}{\partial\gamma_j}\right)\right|_{\underline{\gamma}=\underline{0}}=\frac{1}{\sqrt{\Gamma_0}}\left(\frac{\partial S_1}{\partial\gamma_j}\right)\frac{(m_E+\frac{1}{2}\Gamma_0)}{\Gamma_0}, \tag{177}$$

an easy but involved computation shows that the addenda containing powers of $(m_E+\frac{1}{2}\Gamma_0)$ higher than two are canceled by the addenda involving the Black-Scholes Vomma. In fact, we have

$$\frac{1}{\sqrt{T}}\left.\frac{\partial C_{BS}}{\partial\sigma}\right|_{\underline{\gamma}=\underline{0}}\frac{\partial^2}{\partial\gamma_j^2}\Sigma_2(\underline{0})=\frac{1}{T}\left.\frac{\partial^2 C_{BS}}{\partial\sigma^2}\right|_{\underline{\gamma}=\underline{0}}\left[\left(\frac{\partial S_1}{\partial\gamma_j}\right)^2\frac{1}{\Gamma_0}-\left.\left(\frac{\partial\Sigma_2}{\partial\gamma_j}\right)^2\right|_{\underline{\gamma}=\underline{0}}\right]$$
$$+\frac{1}{\sqrt{T}}\left.\frac{\partial C_{BS}}{\partial\sigma}\right|_{\underline{\gamma}=\underline{0}}\frac{\partial^2 S_2}{\partial\gamma_j^2}\frac{1}{\sqrt{\Gamma_0}}\left[\frac{(m_E+\frac{1}{2}\Gamma_0)^2}{\Gamma_0^2}-\frac{(m_E+\frac{1}{2}\Gamma_0)}{\Gamma_0}-\frac{1}{\Gamma_0}\right]$$
$$+\frac{1}{\sqrt{T}}\left.\frac{\partial C_{BS}}{\partial\sigma}\right|_{\underline{\gamma}=\underline{0}}\frac{\partial^2 S_{2c}}{\partial\gamma_j^2}\frac{1}{\sqrt{\Gamma_0}}\left[\frac{(m_E+\frac{1}{2}\Gamma_0)^2}{\Gamma_0^2}-\frac{1}{\Gamma_0}\right]$$
$$+\frac{1}{\sqrt{T}}\left.\frac{\partial C_{BS}}{\partial\sigma}\right|_{\underline{\gamma}=\underline{0}}\frac{1}{\sqrt{\Gamma_0}}\left(\frac{\partial S_1}{\partial\gamma_j}\right)^2\left[+\frac{(m_E+\frac{1}{2}\Gamma_0)^4}{\Gamma_0^4}-\frac{(m_E+\frac{1}{2}\Gamma_0)^3}{\Gamma_0^3}-\left(1+\frac{6}{\Gamma_0}\right)\frac{(m_E+\frac{1}{2}\Gamma_0)^2}{\Gamma_0^2}\right]$$
$$+\frac{1}{\sqrt{T}}\left.\frac{\partial C_{BS}}{\partial\sigma}\right|_{\underline{\gamma}=\underline{0}}\frac{1}{\sqrt{\Gamma_0}}\left(\frac{\partial S_1}{\partial\gamma_j}\right)^2\left[\left(1+\frac{3}{\Gamma_0}\right)\frac{(m_E+\frac{1}{2}\Gamma_0)}{\Gamma_0}+\frac{3}{\Gamma_0^2}\right]. \tag{178}$$

Using Eqs. (176) and (177) in Eq. (178) and simplifying, we have

$$\frac{\partial^2}{\partial\gamma_j^2}\Sigma_2(\underline{0}) = \frac{1}{\sqrt{\Gamma_0}}\left[+\frac{(m_E+\frac{1}{2}\Gamma_0)^2}{\Gamma_0^2} - \frac{(m_E+\frac{1}{2}\Gamma_0)}{\Gamma_0}\right]\left[1-\frac{(m_E+\frac{1}{2}\Gamma_0)^2}{\Gamma_0^2}\right]\left(\frac{\partial S_1}{\partial\gamma_j}\right)^2$$
$$+\frac{\partial^2 S_2}{\partial\gamma_j^2}\frac{1}{\sqrt{\Gamma_0}}\left[\frac{(m_E+\frac{1}{2}\Gamma_0)^2}{\Gamma_0^2} - \frac{(m_E+\frac{1}{2}\Gamma_0)}{\Gamma_0} - \frac{1}{\Gamma_0}\right] + \frac{\partial^2 S_{2c}}{\partial\gamma_j^2}\frac{1}{\sqrt{\Gamma_0}}\left[\frac{(m_E+\frac{1}{2}\Gamma_0)^2}{\Gamma_0^2} - \frac{1}{\Gamma_0}\right]$$
$$+\frac{1}{\sqrt{\Gamma_0}}\left(\frac{\partial S_1}{\partial\gamma_j}\right)^2\left[+\frac{(m_E+\frac{1}{2}\Gamma_0)^4}{\Gamma_0^4} - \frac{(m_E+\frac{1}{2}\Gamma_0)^3}{\Gamma_0^3} - \left(1+\frac{6}{\Gamma_0}\right)\frac{(m_E+\frac{1}{2}\Gamma_0)^2}{\Gamma_0^2}\right]$$
$$\frac{1}{\sqrt{\Gamma_0}}\left(\frac{\partial S_1}{\partial\gamma_j}\right)^2\left[\left(1+\frac{3}{\Gamma_0}\right)\frac{(m_E+\frac{1}{2}\Gamma_0)}{\Gamma_0} + \frac{3}{\Gamma_0^2}\right]. \tag{179}$$

An easy computation gives

$$\frac{\partial^2}{\partial\gamma_j^2}\Sigma_2(\underline{0}) = \frac{1}{\sqrt{\Gamma_0}}\left(\frac{\partial S_1}{\partial\gamma_j}\right)^2\left[-\frac{6}{\Gamma_0}\frac{(m_E+\frac{1}{2}\Gamma_0)^2}{\Gamma_0^2} + \frac{3}{\Gamma_0}\frac{(m_E+\frac{1}{2}\Gamma_0)}{\Gamma_0} + \frac{3}{\Gamma_0^2}\right]$$
$$+\frac{\partial^2 S_2}{\partial\gamma_j^2}\frac{1}{\sqrt{\Gamma_0}}\left[\frac{(m_E+\frac{1}{2}\Gamma_0)^2}{\Gamma_0^2} - \frac{(m_E+\frac{1}{2}\Gamma_0)}{\Gamma_0} - \frac{1}{\Gamma_0}\right] + \frac{\partial^2 S_{2c}}{\partial\gamma_j^2}\frac{1}{\sqrt{\Gamma_0}}\left[\frac{(m_E+\frac{1}{2}\Gamma_0)^2}{\Gamma_0^2} - \frac{1}{\Gamma_0}\right]. \tag{180}$$

Proceeding in a similar manner, we obtain mixed-order mixed derivatives:

$$\frac{\partial^2}{\partial\gamma_j\partial\gamma_k}\Sigma_2(\underline{0}) = \frac{1}{\sqrt{\Gamma_0}}\left(\frac{\partial S_1}{\partial\gamma_j}\right)\left(\frac{\partial S_1}{\partial\gamma_k}\right)\left[-\frac{6}{\Gamma_0}\frac{(m_E+\frac{1}{2}\Gamma_0)^2}{\Gamma_0^2} + \frac{3}{\Gamma_0}\frac{(m_E+\frac{1}{2}\Gamma_0)}{\Gamma_0} + \frac{3}{\Gamma_0^2}\right]. \tag{181}$$

The thesis follows since we have $S_2 = \frac{1}{2}\sum_{j=1}^n \gamma_j^2 \frac{\partial^2 S_2}{\partial\gamma_j^2}$, $S_{2c} = \frac{1}{2}\sum_{j=1}^n \gamma_j^2 \frac{\partial^2 S_{2c}}{\partial\gamma_j^2}$, *and* $S_1^2 = \sum_{j=1}^n\sum_{k=1}^n \gamma_j\gamma_k \frac{\partial S_1}{\partial\gamma_j}\frac{\partial S_1}{\partial\gamma_k}$. *This concludes the proof.* □

4 Estimated parameters in Christoffersen et al. (2009) for the Heston and Double Heston models

Tables 1 and 2 show the parameters used in the paper.

Table 1: Estimated parameters, one-factor stochastic volatility model (see Panel A, Table 3 in Christoffersen et al. (2009)).

year	χ	v^*	γ	ρ	$\frac{2\chi v^*}{\gamma^2}$
1990	1.9561	0.0593	0.8516	-0.6717	0.3198
1991	2.4240	0.0442	0.5834	-0.6957	0.6295
1992	2.5476	0.0375	0.5519	-0.6865	0.6272
1993	2.6846	0.0254	0.5105	-0.6703	0.5233
1994	4.4324	0.0233	0.4560	-0.8519	0.9933
1995	2.5070	0.0190	0.5597	-0.5061	0.3041
1996	3.1798	0.0298	0.5823	-0.5619	0.5589
1997	2.1672	0.0528	0.6018	-0.5666	0.6319
1998	1.8315	0.1029	0.8079	-0.7521	0.5774
1999	2.1310	0.1091	0.7552	-0.7404	0.8152
2000	2.5751	0.0678	0.6561	-0.6975	0.8111
2001	3.8191	0.0564	0.6489	-0.7410	1.0231
2002	3.3760	0.0532	0.5973	-0.7725	1.0068
2003	1.7201	0.0691	0.6837	-0.5939	0.5085
2004	1.6048	0.0464	0.3796	-0.7670	1.0335

Table 2: Estimated parameters, two-factor stochastic volatility model (see Panel B, Table 3 in Christoffersen et al. (2009)).

year	χ_1	v_1^*	γ_1	ρ_1	χ_2	v_2^*	γ_2	ρ_2	$\frac{2\chi_1 v_1^*}{\gamma_1^2}$	$\frac{2\chi_2 v_2^*}{\gamma_2^2}$
1990	0.2370	0.0227	1.0531	-0.7695	8.4983	0.0273	0.6827	-0.8417	0.0097	0.9955
1991	0.2966	0.0197	1.8157	-0.8575	4.4513	0.0319	0.3360	-0.6057	0.0035	2.5155
1992	0.2022	0.0051	6.2755	-0.9670	0.7424	0.0684	0.2740	-0.8040	0.0001	1.3527
1993	0.2000	0.0052	5.2500	-0.9666	0.6131	0.0569	0.2123	-0.8216	0.0001	1.5480
1994	0.1668	0.0050	9.4346	-0.9877	0.2098	0.1633	0.1706	-0.9364	0.0000	2.3543
1995	0.2061	0.0050	6.8941	-0.9206	1.4677	0.0242	0.2413	-0.7512	0.0000	1.2200
1996	0.2101	0.0052	2.0149	-0.9684	0.5561	0.0575	0.1868	-0.7978	0.0005	1.8327
1997	0.1397	0.0053	1.5423	-0.9914	0.1878	0.1648	0.1239	-0.8928	0.0006	4.0321
1998	0.1374	0.0051	2.1196	-0.9917	0.6247	0.1733	0.3965	-0.9117	0.0003	1.3772
1999	0.1388	0.0051	1.9895	-0.9917	0.7322	0.1736	0.3828	-0.9108	0.0003	1.7372
2000	0.1404	0.0052	1.9382	-0.9915	0.3542	0.1690	0.2292	-0.9024	0.0004	2.2789
2001	0.1433	0.0054	1.9115	-0.9911	0.2347	0.1655	0.2047	-0.8983	0.0004	1.8539
2002	0.1491	0.0058	1.9754	-0.9902	0.1855	0.1607	0.1715	-0.8896	0.0004	2.0270
2003	0.1638	0.0032	8.8078	-0.9838	0.4625	0.1198	0.3976	-0.6569	0.0000	0.7009
2004	0.1500	0.0059	1.9829	-0.9902	0.2335	0.1621	0.1971	-0.8918	0.0005	1.9486

5 Formulas derived from Recchioni and Sun (2016)

From the Recchioni and Sun (2016) approach* and a simple computation, we derived the following explicit formulas for option prices:

$$C_{MH}(S_0,T,E) = e^{(q-1)\int_0^T r(s)ds}\frac{S_0}{2\pi}\int_{-\infty}^{+\infty}\frac{\left(\frac{S_0}{E}\right)^{(q-1-\imath k)} e^{-\imath\, k\int_0^T r(s)ds} e^{Q_{v,q}(T,v_0,k;\underline{\Theta}_v)}}{-k^2-(2q-1)\imath k+q(q-1)}dk, \tag{182}$$

and

$$P_{MH}(S_0,T,E) = e^{(q-1)\int_0^T r(s)ds}\frac{S_0}{2\pi}\int_{-\infty}^{+\infty}\frac{\left(\frac{S_0}{E}\right)^{(q-1-\imath k)} e^{-\imath\, k\int_0^T r(s)ds} e^{Q_{v,q}(T,v_0,k;\underline{\Theta}_v)}}{-k^2-(2q-1)\imath k+q(q-1)}dk, \tag{183}$$

where, in the case of the Heston/double Heston model, $Q_{v,q}$ is the elementary function given by

$$\begin{aligned} Q_{v,q}(t'-t,v,k;\underline{\Theta}_v) = \sum_{j=1}^{n} -(2\chi_j v_j^*/\gamma_j^2)\ln(s_{q,v_j,b,}/(2\zeta_{q,v_j})) \\ -(2\chi v_j^*/\gamma_j^2)(\zeta_{q,v_j}+\mu_{q,v_j})(t'-t) - (2v_j/\gamma_j^2)(\zeta_{q,v_j}^2-\mu_{q,v_j}^2)s_{q,v_j,g}/s_{q,v_j,b}\,, \end{aligned} \tag{184}$$

with μ_{q,v_j}, ζ_{q,v_j}, $s_{q,v_j,g}$, and $s_{q,v_j,b}$ defined as follows:

$$\mu_{q,v_j} = -\frac{1}{2}(\chi_j+(\imath\, k-q)\,\gamma_j\,\rho_j), \qquad \zeta_{q,v_j} = \frac{1}{2}\left[4\mu_{q,v_j}^2+2\gamma_j^2\varphi_q(k)\right]^{1/2}, \tag{185}$$

$$s_{q,v_j,g} = 1-e^{-2\zeta_{q,v_j}(t'-t)}, \qquad s_{q,v_j,b} = (\zeta_{q,v_j}+\mu_{q,v_j})e^{-2\zeta_{q,v_j}(t'-t)}+(\zeta_{q,v_j}-\mu_{q,v_j}). \tag{186}$$

The quantity φ_q appearing in Eq. (185) is given by

$$\varphi_q(k) = \frac{k^2}{2}+\imath\frac{k}{2}(2q-1)-\frac{1}{2}(q^2-q),\ \ k\in\mathbb{R}. \tag{187}$$

Formulas (182) and (183) are obtained starting from the following formula for the marginal probability density M in Eq. (15):

$$M(x,\underline{v},t,x',t') = e^{-q(x'-x)}M_q(x,\underline{v},t,x',t'), \tag{188}$$

where q is a regularization parameter that allows for a suitable change of integration order in the option pricing.
The Fourier transforms of M and M_q, denoted by $\widehat{M}$ and $\widehat{M_q}$ satisfy the following equation:

$$\widehat{M_q}(k,\underline{v},t,t') = \widehat{M}(k+\imath q,\underline{v},t,t'). \tag{189}$$

The Fourier transform of M is in Eq. (15), so formula (182) for the call option follows by computing the integral

$$\int_{-\infty}^{+\infty}(e^{x'}-E)_+e^{-q(x'-x)}\frac{1}{-}2\pi\int_{-\infty}^{+\infty}e^{\imath k(x'-x)}\widehat{M}(k+\imath q,\underline{v},t,t')dkdx'$$

for $q>1$, while formula (183) follows for the put option:

$$\int_{-\infty}^{+\infty}(E-e^{x'})_+e^{-q(x'-x)}\left[\frac{1}{2\pi}\int_{-\infty}^{+\infty}e^{\imath k(x'-x)}\widehat{M}(k+\imath q,\underline{v},t,t')dk\right]dx'$$

for $q<0$, with $x=\log(S_0)$, $t=0$, $t'=T$. The double integrals reduce to one-dimensional integrals by changing the integration order.

*Note that formulas in Appendix C for option pricing in the multi-factor Heston model have been derived by applying the transformation suggested in Recchioni and Sun 2016 to the Kolmogorov backward equation associated with the multi-factor Heston model. It is a straightforward computation and this is why we do not stress these formulas.

6 Accuracy of the second- and third-order approximations in the Heston framework on a "reasonable" grid of parameters

In this section we assess the performance of the first- and second-order approximations $C_{2,MH}$, $P_{2,MH}$ and $C_{3,MH}$, $P_{3,MH}$ of the call and put option prices in the Heston framework. This analysis is done to show that these first- and second-order Black-Scholes approximations to option European vanilla option prices are of sufficient quality to be used for estimating the multi-factor Heston model parameters.

Figures 1 and 2 show the empirical distributions of the relative errors in call and put prices, $e_{C,m} = |C_H - C_{m,H}|/|C_H|$, $m = 2, 3$ (upper panels) and $e_{P,m} = |P_H - P_{m,H}|/|P_H|$, $m = 2, 3$ (lower panel), when $\gamma = 0.15$ (left panel), $\gamma = 0.25$ (middle panel), and $\gamma = 0.5$ (right panel) based on grid $\mathcal{M}$. We observe that the third-order approximations $C_{3,H}$ and $P_{3,H}$ of the call and put option prices, respectively, slightly outperform the second-order approximations $C_{2,H}$ and $P_{2,H}$ in terms of accuracy for vols of vols.

The exact Heston formula is obtained by imposing $n = 1$ in Eqs. (184) in Appendix C of the paper. As previously mentioned, Eqs. (182), (183) in Appendix B are equal except for the values of q, which are valid over different intervals. In the following, we choose $q = 1.05$ for a call option and $q = -0.05$ for a put option. Equations (182) and (183) are defined via convergent integrals that can be computed accurately using a simple composite rectangular rule with 2^{16} quadrature nodes. Obviously, depending on the choice of the model parameters and the time to maturity, the number of quadrature points could be reduced.

We evaluate the exact formulas C_H, P_H and the second-order Black-Scholes formulas $C_{m,H}$, $P_{m,H}$, for $m = 1, 2$ at the points in the following set:

$$\begin{aligned}\mathcal{M} = \{ & (S_0, E, T, \gamma, v_0, \chi, v^*, \rho, r) \; | S_0 = 100,\ E = 80 + 10(j-1),\ T = 2\,j/5,\ j = 1, 2, \ldots, 5, \\ & \gamma = 0.01, 0.05, 0.15, 0.25, 0.5, 0.8, 2\, v_0 = 2 + j/5, j = 1, 2, \ldots, 5,\ \chi = 1.5 + 1.5(j-1), j = 1, 2, \ldots, 5, \\ & v^* = j\,\gamma^2/(2\chi), \rho = -j/6, j = 1, 2, \ldots, 5, r = 0.01 \}\,. \end{aligned} \tag{190}$$

These values of model parameters in grid $\mathcal{M}$ include those estimated by Christoffersen et al. (2009) in Section 4.2 Table 3.

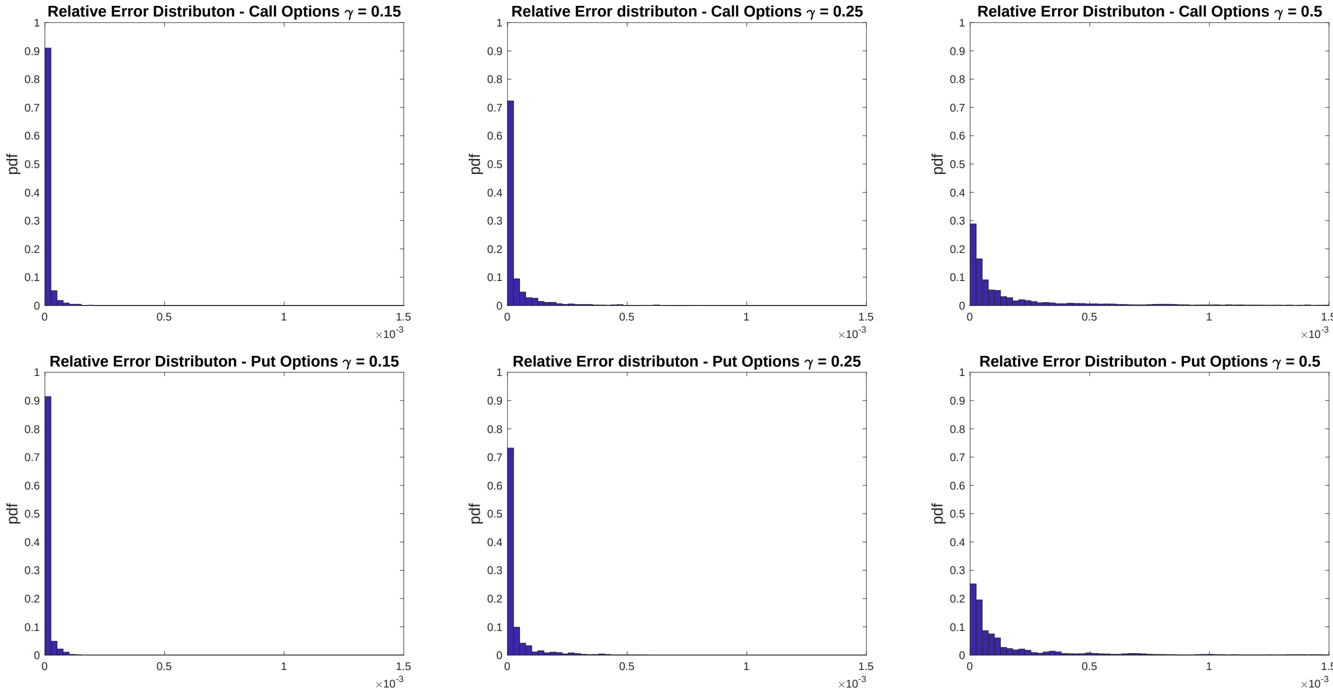

Figure 1: Relative error distributions $e_{C,2} = |C_H - C_{2,H}|/|C_H|$ (upper panels) and $e_{P,2} = |P_H - P_{2,H}|/|P_H|$ (lower panels) when $\gamma = 0.15$ (left panels), $\gamma = 0.25$ (middle panels), and $\gamma = 0.5$ (right panels) obtained with second-order approximations $C_{2,H}$, $P_{2,H}$.

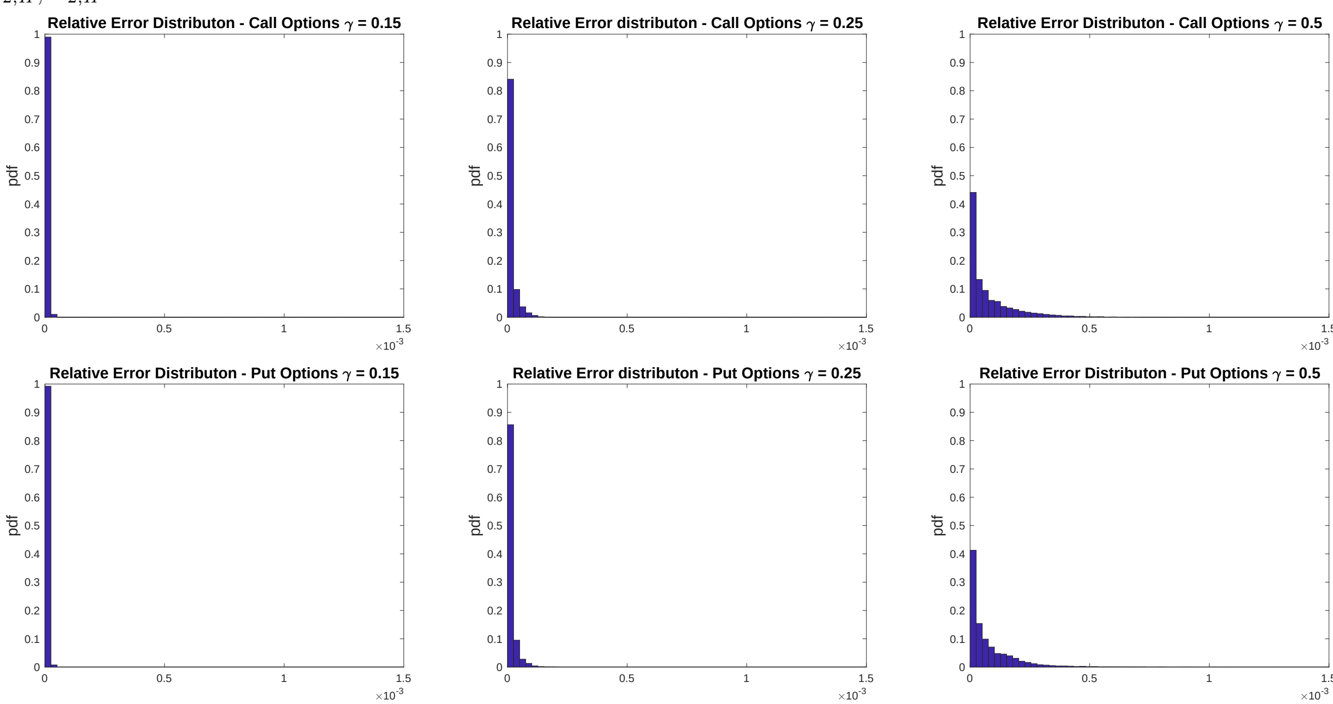

Figure 2: Relative error distributions $e_{C,3} = |C_H - C_{3,H}|/|C_H|$ (upper panels) and $e_{P,3} = |P_H - P_{3,H}|/|P_H|$ (lower panels) when $\gamma = 0.15$ (left panels), $\gamma = 0.25$ (middle panels), and $\gamma = 0.5$ (right panels) obtained with third-order approximations $(C_{3,H},\ P_{3,H})$.

7 Some details about asymptotic expansions

This section is related to Section 4.2 of the paper. It illustrates some details of the proposed expansion and specifically, we stress that the two expansions of $\mathcal{L}_{\underline{\gamma}}$ and $\mathcal{L}^*_{\underline{\gamma}}$ as a functions of the vols of vols are asymptotic power series expansions.
We recall that the function $f(z)$, $z \in \Omega \subset \mathbb{C}$ has an asymptotic representation (expansion) if

$$f(z) \sim f_N(z) = \sum_{j=0}^{N} a_n \phi_n(z) + o\left(\phi_N\right), \text{ as z} \to \text{z}_0, \tag{191}$$

for every $N = 0, 1, 2, \ldots$, where $\phi_n(z) = z^n$. Intuitively, an asymptotic expansion of a given function f is a finite sum which might diverge, yet it still provides an increasingly accurate description of the asymptotic behaviour of f.
There is a caveat here: for a divergent asymptotic expansion, there is an optimal $N_0 = N_0(z)$ for some z that gives a best approximation to f, i.e., adding more terms actually gives worse accuracy. When z tends to z_0, however, we have

$$|f(z) - f_N(z)| < \epsilon, \ \forall\, |z - z_0| < \delta, \ \ N > N_0. \tag{192}$$

Coming back to our two representation formulas for the marginal density functions and the corresponding asymptotic expansions of $\mathcal{L}_{\underline{\gamma}}$ and $\mathcal{L}^*_{\underline{\gamma}}$ as $\underline{\gamma} \to 0$, we stress that:
(i) the asymptotic expansion of $\mathcal{L}_{\underline{\gamma}}$ is convoluted with the Gaussian kernel $\mathcal{G}_{\Gamma_0}$, which is independent of the vols of vols, so the resulting asymptotic expansion for the marginal density function diverges for large value of vols of vols;
(ii) the asymptotic expansion of $\mathcal{L}^*_{\underline{\gamma}}$ is convoluted with the Gaussian kernel $\mathcal{G}_{\Gamma_2}$, which is dependent on the vols of vols and approaches zero exponentially for large vols of vols. This implies that the asymptotic expansion of the marginal density function does not make vols of vols that tend to $+\infty$ diverge. Figure 3 shows the average relative errors of the second-order approximations to the call options evaluated on the parameters estimated by Christoffersen et al. (2009) obtained using the expansion with kernel $\mathcal{G}_{\Gamma_0}$ (solid line) and with $\mathcal{G}_{\Gamma_2}$. Both panels show the natural logarithm of the average relative error as a function of vol of vol. The vol of vol in the left panel varies from 1 to 9.5, while in the right panel it varies from 1 to 95. We see that the expansion with base point $\mathcal{G}_{\Gamma_2}$ does not diverge.

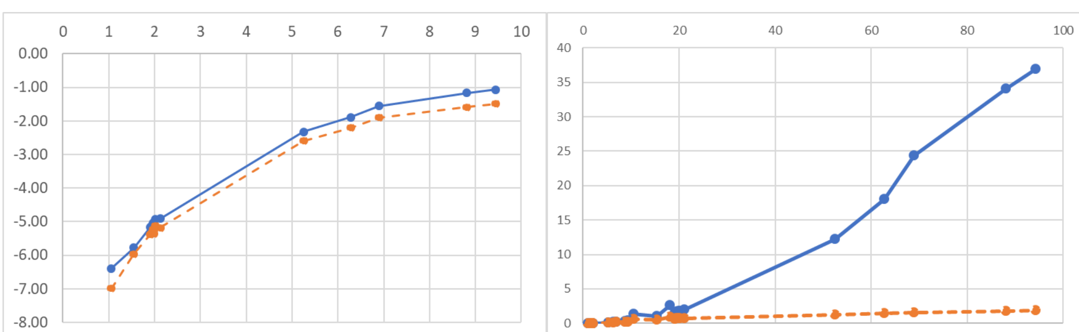

Figure 3: Average relative errors of the second-order approximation to the call option obtained using the kernel $\mathcal{G}_{\Gamma_2}$ (dashed line) and the kernel $\mathcal{G}_{\Gamma_0}$ (solid line). The y-axis is the natural logarithm of the average relative errors; the x-axis is the natural logarithm of the vol of vol.

We therefore focus on the expansion based on the kernel $\mathcal{G}_{\Gamma_2}$ and we investigate whether the first-order approximation outperforms the second- and third- order approximations for large vol of vols. We consider the double Heston model with the parameters estimated in 1990 (see Table 2 Year 1990) except for the vol of vol γ_1, which is chosen to be $\gamma_1^m = e^{-3+m/2}$, $m = 1, 2, \ldots, 30$, and $\gamma_2 = 0.007$.

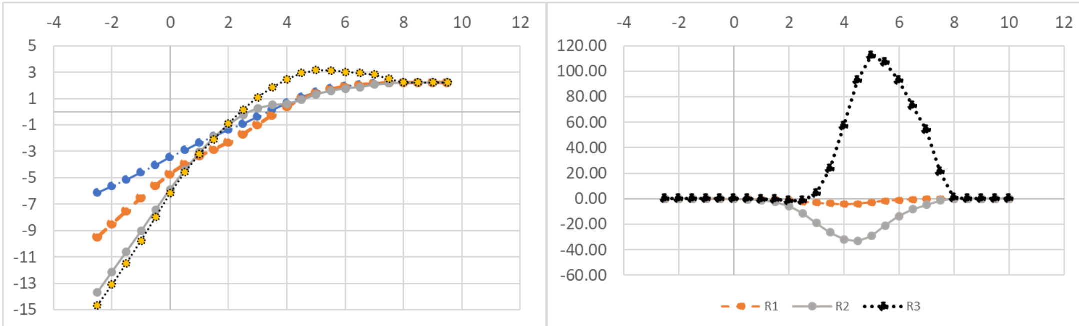

Figure 4: Left panel: relative errors of zero-, second-, and third-order approximations to the call options obtained using the kernel $\mathcal{G}_{\Gamma_2}$ in a log-log scale. Zero order: dash-dot line; first order: dashed line; second order: solid line; third order: dotted line. Right panel: correction terms $\mathcal{R}_1$ (dashed line), $\mathcal{R}_2$ (solid line), and $\mathcal{R}_3$ (dotted line) as a function of the log of vol of vol.

Figure 4 shows that the correction terms go to zero for very large values of vol of vol and that the second-order expansion provides a satisfactory approximation up to a vol of vol equal to three (remember that the x-axis is in log-scale). In fact, for values of the log of vol of vol less than 3, we observe that the first-order correction is larger than the second-order correction and the latter is larger than the third-order correction. This suggests that the second-order approximation can be used up to vol of vols of 250%.

8 Further details on the expansions for option pricing and Greeks

In this section we provide some details regarding the use of the representation formulas for the marginal probability density functions in Theorem 2.1 and Theorem 2.2. To keep the demonstration simple, we limit our attention to the second-order correction.

We recall the following proposition:

Proposition 8.1 *Let f be a function $f = f(y)$, $y \in \mathbb{R}$ with Fourier transform $\widehat{f}(k) = \int_{-\infty}^{+\infty} e^{-\imath\, k y} f(y)$, $k \in \mathbb{R}$:*

if $\widehat{f}(k=0) = 0$ we have $\int_{-\infty}^{+\infty} f(y) = 0$

if $\widehat{f}(k=\imath) = 0$ we have $\int_{-\infty}^{+\infty} e^y f(y) = 0$.

Proof of Proposition 8.1 *The proof follows by simply evaluating the Fourier transform at $k = 0$ and $k = \imath$.*

This proposition allows us to prove the following properties of the functions $\mathcal{L}_{\underline{\gamma}}$ as in Theorem 2.1 and $\mathcal{L}^*_{\underline{\gamma}}$ as in Theorem 2.2:

Proposition 8.2 *Let $\mathcal{L}_{\underline{\gamma}}$ be as in Theorem 2.1 and $\mathcal{L}^*_{\underline{\gamma}}$ as in Theorem 2.2. We have*

$$\int_{-\infty}^{+\infty} \left[\mathcal{L}_{\underline{\gamma}}(y,t,t') - \delta(y)\right] dy = 0, \qquad \int_{-\infty}^{+\infty} \left[\mathcal{L}^*_{\underline{\gamma}}(y,t,t') - \delta(y)\right] dy = 0, \tag{193}$$

and

$$\int_{-\infty}^{+\infty} e^y \left[\mathcal{L}_{\underline{\gamma}}(y,t,t') - \delta(y)\right] dy = 0, \qquad \int_{-\infty}^{+\infty} e^y \left[\mathcal{L}^*_{\underline{\gamma}}(y,t,t') - \delta(y)\right] dy = 0. \tag{194}$$

Proof of Proposition 8.2 *The proof follows, observing that the Fourier transforms of the functions $\mathcal{L}_{\underline{\gamma}}(y,t,t') - \delta(y)$ and $\mathcal{L}^*_{\underline{\gamma}}(y,t,t') - \delta(y)$ vanish at $k = 0$ and $k = \imath$ with Proposition 8.1.*

Now we focus on the expansion of function $\mathcal{L}^*_{\underline{\gamma}}$ up to the second-order terms in powers of vol of vol. We recall that formula (53) for $\mathcal{L}^*_{\underline{\gamma}}$ is

$$\mathcal{L}^*_{\underline{\gamma}}(y,t,t') = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{\imath\, k\, y} e^{S_1(t,t')(\imath\, k^3 + \imath\, k) + S_2(t,t')(k^4 - 2\imath k^3 - \imath k) + S_{2c}(t,t')(k^4 - \imath\, k^3) + o(\|\underline{\gamma}\|^2)} dk\,, \qquad \|\underline{\gamma}\| \to 0^+, \tag{195}$$

which implies the following expansion for the Fourier transform, $\widehat{\mathcal{L}}^*_{\underline{\gamma}}$ of $\mathcal{L}^*_{\underline{\gamma}}$:

$$\widehat{\mathcal{L}}^*_{\underline{\gamma}}(y,t,t') = e^{S_1(t,t')(\imath\, k^3 + \imath\, k) + S_2(t,t')(k^4 - 2\imath k^3 - \imath k) + S_{2c}(t,t')(k^4 - \imath\, k^3) + o(\|\underline{\gamma}\|^2)}, \quad \|\underline{\gamma}\| \to 0^+. \tag{196}$$

Eq. (196) allows for the following expansion of $\widehat{\mathcal{L}}^*_{\underline{\gamma}}$ for small vol of vols:

$$\widehat{\mathcal{L}}^*_{\underline{\gamma}}(y,t,t') = 1 + S_1(t,t')(\imath\, k^3 + \imath\, k) + S_2(t,t')(k^4 - 2\imath k^3 - \imath k) + S_{2c}(t,t')(k^4 - \imath\, k^3) + \frac{1}{2} S_1(t,t')^2 (\imath\, k^3 + \imath\, k)^2 + o(\|\underline{\gamma}\|^2), \quad \|\underline{\gamma}\| \to 0^+. \tag{197}$$

Eq. (197) can be rewritten as

$$\widehat{\mathcal{L}}^*_{\underline{\gamma}}(y,t,t') = 1 + S_1(t,t')\left[-(\imath\, k)^3 + (\imath\, k)\right] + S_2(t,t')\left[(\imath\, k)^4 + 2(\imath k)^3 - (\imath k)\right] + S_{2c}(t,t')\left[(\imath k)^4 + (\imath\, k)^3\right] + \frac{1}{2} S_1^2(t,t')\left[(\imath\, k)^6 - 2(\imath k)^4 + (\imath\, k)^2\right] + o(\|\underline{\gamma}\|^2), \quad \|\underline{\gamma}\| \to 0^+. \tag{198}$$

It is worth of noting that the functions S_1, S_2, S_{2c}, and S_1^2 multiply polynomial functions of k, which vanish at $k=0$ and $k=\imath$. This is a relevant aspect since the following proposition holds.

Bearing in mind that in the sense of distributions we have

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty}(\imath k)^m e^{\imath k y}dk = \frac{d^m}{dy^m}\delta(y), \tag{199}$$

we obtain the following expansion for $\mathcal{L}^*_{\underline{\gamma}}(y,t,t')$:

$$\mathcal{L}^*_{\underline{\gamma}}(y,t,t') = \delta(y) + S_1(t,t')\left[-\frac{d^3}{dy^3}\delta(y)+\frac{d}{dy}\delta(y)\right] + S_2(t,t')\left[\frac{d^4}{dy^4}\delta(y)-2\frac{d^3}{dy^3}\delta(y)-\frac{d}{dy}\delta(y)\right]$$
$$+S_{2c}(t,t')\left[\frac{d^4}{dy^4}\delta(y)+\frac{d^3}{dy^3}\delta(y)\right]+\frac{1}{2}S_1^2(t,t')\left[\frac{d^6}{dy^6}\delta(y)-2\frac{d^4}{dy^4}\delta(y)+\frac{d^2}{dy^2}\delta(y)\right]+o(\|\underline{\gamma}\|^2),\ \ \|\underline{\gamma}\|\to 0^+. \tag{200}$$

We now consider the integral representation formula in Theorem 2.2. Replacing Eq. 200 in Eq. (47) we have

$$\begin{aligned}
&M(x,\underline{v},t,x',t') = \int_{-\infty}^{+\infty}\mathcal{G}_{\Gamma_2}(x'-x-y,t,t')\mathcal{L}^*_{\underline{\gamma}}(y,t,t')dy\\
&\approx \int_{-\infty}^{+\infty}\mathcal{G}_{\Gamma_2}(x'-x-y,t,t')\delta(y)dy\\
&+S_1(t,t')\int_{-\infty}^{+\infty}\mathcal{G}_{\Gamma_2}(x'-x-y,t,t')\left[-\frac{d^3}{dy^3}\delta(y)+\frac{d}{dy}\delta(y)\right]dy\\
&+S_2(t,t')\int_{-\infty}^{+\infty}\mathcal{G}_{\Gamma_2}(x'-x-y,t,t')\left[\frac{d^4}{dy^4}\delta(y)-2\frac{d^3}{dy^3}\delta(y)-\frac{d}{dy}\delta(y)\right]dy\\
&+S_{2c}(t,t')\int_{-\infty}^{+\infty}\mathcal{G}_{\Gamma_2}(x'-x-y,t,t')\left[\frac{d^4}{dy^4}\delta(y)+\frac{d^3}{dy^3}\delta(y)\right]dy\\
&+\frac{1}{2}S_1^2(t,t')\int_{-\infty}^{+\infty}\mathcal{G}_{\Gamma_2}(x'-x-y,t,t')\left[\frac{d^6}{dy^6}\delta(y)-2\frac{d^4}{dy^4}\delta(y)+\frac{d^2}{dy^2}\delta(y)\right]dy\\
&= M_0(x,\underline{v},t,x',t')+\mathcal{M}_1(x,\underline{v},t,x',t')+\mathcal{M}_2(x,\underline{v},t,x',t'),
\end{aligned} \tag{201}$$

where M_0 is given by

$$M_0(x,\underline{v},t,x',t') = \int_{-\infty}^{+\infty}\mathcal{G}_{\Gamma_2}(x'-x-y,t,t')\delta(y)dy = \mathcal{G}_{\Gamma_2}(x'-x,t,t'), \tag{202}$$

and $\mathcal{M}_m$, $m=1,2$, are given by

$$\mathcal{M}_1(x,\underline{v},t,x',t') = +S_1(t,t')\int_{-\infty}^{+\infty}\mathcal{G}_{\Gamma_2}(x'-x-y,t,t')\left[-\frac{d^3}{dy^3}\delta(y)+\frac{d}{dy}\delta(y)\right]dy, \tag{203}$$

$$\begin{aligned}
&\mathcal{M}_2(x,\underline{v},t,x',t') = S_2(t,t')\int_{-\infty}^{+\infty}\mathcal{G}_{\Gamma_2}(x'-x-y,t,t')\left[\frac{d^4}{dy^4}\delta(y)-2\frac{d^3}{dy^3}\delta(y)-\frac{d}{dy}\delta(y)\right]dy\\
&+S_{2c}(t,t')\int_{-\infty}^{+\infty}\mathcal{G}_{\Gamma_2}(x'-x-y,t,t')\left[\frac{d^4}{dy^4}\delta(y)+\frac{d^3}{dy^3}\delta(y)\right]dy\\
&+\frac{1}{2}S_1^2(t,t')\int_{-\infty}^{+\infty}\mathcal{G}_{\Gamma_2}(x'-x-y,t,t')\left[\frac{d^6}{dy^6}\delta(y)-2\frac{d^4}{dy^4}\delta(y)+\frac{d^2}{dy^2}\delta(y)\right]dy\,.
\end{aligned} \tag{204}$$

Formulas (202), (203), and (204) imply formulas (105), (106), and (107).

Proposition 8.3 *The functions $\mathcal{M}_1$, $\mathcal{M}_2$, and $\mathcal{M}_3$ in (106), (107), and (108) satisfy*

$$\int_{-\infty}^{+\infty} \mathcal{M}_m(x, \underline{v}, t, x', t')dx' = 0, \quad m = 1, 2, 3, \tag{205}$$

$$\int_{-\infty}^{+\infty} e^{x'}\mathcal{M}_m(x, \underline{v}, t, x', t')dx' = 0, \quad m = 1, 2, 3\,. \tag{206}$$

Proof of Proposition 8.3 *Eq. (205) is proved by integrating formulas (106), (107), and (108). The proof of Eq. (206) follows from Proposition 8.1 since the Fourier transform of (106), (107), and (108) with respect to x' is zero at $k = \imath$. Note that the Fourier transform is obtained using the properties*

$$\int_{-\infty}^{+\infty} e^{-\imath\, k\, x'} \frac{\partial^m}{\partial x'^m}\mathcal{G}_{\Gamma_2} dx' = (-\imath\, k)^m \widehat{\mathcal{G}}_{\Gamma_2}.$$

Proposition 8.3 is useful for understanding why the put-call parity is satisfied at each order of the expansion in powers of vols of vols. To illustrate this point, we consider the second-order approximation. Bearing in mind that the marginal probability density function is approximated by the following formula (see Theorem 2.2), we have

$$\begin{aligned}
M_2(x, \underline{v}, t, x', t') \;\; = \;\; & \int_{-\infty}^{+\infty} \mathcal{G}_{\Gamma_2}(x' - x - y, t, t')\mathcal{L}^*_{\underline{\gamma}}(y, t, t')dy \\
& \approx \int_{-\infty}^{+\infty} \mathcal{G}_{\Gamma_2}(x' - x - y, t, t')\delta(y)dy \\
& + S_1(t, t') \int_{-\infty}^{+\infty} \mathcal{G}_{\Gamma_2}(x' - x - y, t, t') \left[-\frac{d^3}{dy^3}\delta(y) + \frac{d}{dy}\delta(y) \right] dy \\
& + S_2(t, t') \int_{-\infty}^{+\infty} \mathcal{G}_{\Gamma_2}(x' - x - y, t, t') \left[\frac{d^4}{dy^4}\delta(y) - 2\frac{d^3}{dy^3}\delta(y) - \frac{d}{dy}\delta(y) \right] dy \\
& + S_{2c}(t, t') \int_{-\infty}^{+\infty} \mathcal{G}_{\Gamma_2}(x' - x - y, t, t') \left[\frac{d^4}{dy^4}\delta(y) + \frac{d^3}{dy^3}\delta(y) \right] dy \\
& + \frac{1}{2}S_1^2(t, t') \int_{-\infty}^{+\infty} \mathcal{G}_{\Gamma_2}(x' - x - y, t, t') \left[\frac{d^6}{dy^6}\delta(y) - 2\frac{d^4}{dy^4}\delta(y) + \frac{d^2}{dy^2}\delta(y) \right] dy \\
& = M_0 + \mathcal{M}_1 + \mathcal{M}_2.
\end{aligned} \tag{207}$$

We price vanilla call and put options under the multi-Heston model using the following formula:

$$\begin{aligned}
& C(S_0, T, E) = B(T) \int_{-\infty}^{+\infty} (e^{x'} - E)_+ M(\log S_0, \underline{v}_0, 0, x', T)dx' \\
& = B(T) \int_{-\infty}^{+\infty} (e^{x'} - E)_+ \int_{-\infty}^{+\infty} \mathcal{G}_{\Gamma_2}(x' - \log S_0 - y, 0, T)\mathcal{L}^*_{\underline{\gamma}}(y, 0, T)dydx' \\
& \approx B(T) \int_{-\infty}^{+\infty} (e^{x'} - E)_+ M_0(x, \underline{v}, t, x', t')dx' + B(T) \int_{-\infty}^{+\infty} (e^{x'} - E)_+ \left(\mathcal{M}_1(x, \underline{v}, t, x', t') + \mathcal{M}_2(x, \underline{v}, t, x', t') \right) dx',
\end{aligned} \tag{208}$$

$$\begin{aligned}
& P(S_0, T, E) = B(T) \int_{-\infty}^{+\infty} (E - e^{x'})_+ M(\log S_0, \underline{v}_0, 0, x', T)dx' \\
& = B(T) \int_{-\infty}^{+\infty} (E - e^{x'})_+ \int_{-\infty}^{+\infty} \mathcal{G}_{\Gamma_2}(x' - \log S_0 - y, 0, T)\mathcal{L}^*_{\underline{\gamma}}(y, 0, T)dydx' \\
& \approx B(T) \int_{-\infty}^{+\infty} (E - e^{x'})_+ M_0(x, \underline{v}, t, x', t')dx' + B(T) \int_{-\infty}^{+\infty} (E - e^{x'})_+ \left(\mathcal{M}_1(x, \underline{v}, t, x', t') + \mathcal{M}_2(x, \underline{v}, t, x', t') \right) dx'.
\end{aligned} \tag{209}$$

Thus, the Black-Scholes second-order approximations to call and put option prices are

$$C_2(S_0,T,E) = C_{BS}\left(S_0,E,T,\sqrt{\frac{\Gamma_2}{T}}\right) + B(T)\int_{-\infty}^{+\infty}(e^{x'}-E)_+\left(\mathcal{M}_1(x,\underline{v},t,x',t')+\mathcal{M}_2(x,\underline{v},t,x',t')\right)dx', \tag{210}$$

$$P_2(S_0,T,E) = P_{BS}\left(S_0,E,T,\sqrt{\frac{\Gamma_2}{T}}\right) + B(T)\int_{-\infty}^{+\infty}(E-e^{x'})_+\left(\mathcal{M}_1(x,\underline{v},t,x',t')+\mathcal{M}_2(x,\underline{v},t,x',t')\right)dx'. \tag{211}$$

Subtracting P_2 from C_2 while considering the put-call parity for the Black-Scholes option prices, we have

$$\begin{aligned}&C_2(S_0,T,E) - P_2(S_0,T,E) = S_0 - B(T)E\\&+B(T)\int_{-\infty}^{+\infty}\left[(e^{x'}-E)_+ - (E-e^{x})_+\right]\left(\mathcal{M}_1(\log S_0,\underline{v},0,x',T)+\mathcal{M}_2(\log S_0,\underline{v},0,x',T)\right)dx'.\end{aligned} \tag{212}$$

By virtue of Proposition 8.3, we obtain

$$\begin{aligned}&C_2(S_0,T,E) - P_2(S_0,T,E) = S_0 - B(T)E\\&+B(T)\int_{-\infty}^{+\infty}(e^{x'}-E)\left(M_1(\log S_0,\underline{v},0,x',T)+M_2(\log S_0,\underline{v},0,x',T)\right)dx' = S_0 - B(T)E\,.\end{aligned} \tag{213}$$

Finally, we underline that the correction terms appearing in the first-, second-, and third-order approximations can be interpreted as Greeks. Therefore, looking at the price of a vanilla call option under the multi-Heston model (see Theorem 2.2), we rewrite formula (208):

$$\begin{aligned}C(S_0,T,E) &= B(T)\int_{-\infty}^{+\infty}(e^{x'}-E)_+M(\log S_0,\underline{v}_0,0,x',T)dx'\\&= B(T)\int_{-\infty}^{+\infty}(e^{x'}-E)_+\int_{-\infty}^{+\infty}\mathcal{G}_{\Gamma_2}(x'-\log S_0-y,0,T)\mathcal{L}^*_{\underline{\gamma}}(y,0,T)dydx'\\&= B(T)\int_{-\infty}^{+\infty}\mathcal{L}^*_{\underline{\gamma}}(y,0,T)\left[\int_{-\infty}^{+\infty}(e^{x'}-E)_+\mathcal{G}_{\Gamma_2}(x'-\log S_0-y,0,T)\right]dx'dy\\&= \int_{-\infty}^{+\infty}\mathcal{L}^*_{\underline{\gamma}}(y,0,T)C_{BS}(S_0e^y,0,T)dy.\end{aligned} \tag{214}$$

Substituting (200) into (214), we obtain the following Black-Scholes second-order approximation for the call option price:

$$\begin{aligned}&C_2(S_0,T,E) = B(T)C_{BS}\left(S_0,T,E,\sqrt{\frac{\Gamma_2}{T}}\right)\\&+S_1(0,T)\int_{-\infty}^{+\infty}C_{BS}\left(S_0e^y,T,E,\sqrt{\frac{\Gamma_2}{T}}\right)\left[-\frac{d^3}{dy^3}\delta(y)+\frac{d}{dy}\delta(y)\right]dy\\&+S_2(0,T)\int_{-\infty}^{+\infty}C_{BS}\left(S_0e^y,T,E,\sqrt{\frac{\Gamma_2}{T}}\right)\left[\frac{d^4}{dy^4}\delta(y)-2\frac{d^3}{dy^3}\delta(y)-\frac{d}{dy}\delta(y)\right]dy\\&+S_{2c}(0,T)\int_{-\infty}^{+\infty}C_{BS}\left(S_0e^y,T,E,\sqrt{\frac{\Gamma_2}{T}}\right)\left[\frac{d^4}{dy^4}\delta(y)+\frac{d^3}{dy^3}\delta(y)\right]dy\\&\frac{1}{2}S_1^2(0,T)\int_{-\infty}^{+\infty}C_{BS}\left(S_0e^y,T,E,\sqrt{\frac{\Gamma_2}{T}}\right)\left[\frac{d^6}{dy^6}\delta(y)-2\frac{d^4}{dy^4}\delta(y)+\frac{d^2}{dy^2}\delta(y)\right]dy.\end{aligned} \tag{215}$$

Bearing in mind that

$$\int_{-\infty}^{+\infty} C_{BS}(S_0 e^y, 0, T) \frac{d^m}{dy^m} \delta(y) dy = (-1)^m \left. \frac{\partial^m}{\partial y^m} C_{BS}(S_0 e^y, 0, T) \right|_{y=0},$$

formula (215) approximates the call option as the Black-Scholes call options plus corrections expressed as derivatives of the Black-Scholes call option with respect to the log-price. These derivatives can be rewritten in terms of Greeks.

9 Option price calibration

In this section we assess the performance of the Heston second order approximation formulas (58)-(59) of the paper to reproduce and to forecast traded European call and put option prices on the US S&P 500 index. In this exercise, the U.S. three-month government bond index is used as a proxy for the interest rate r.

The availability of an explicit and very elementary formula for the implied volatility provides an advantage in terms of calibrating the model rather than estimating the parameters directly from the option prices. This is because it avoids biases caused by different magnitudes of option prices that are typically corrected introducing appropriate weights in the optimization algorithm (i.e., the inverse of option Vegas, see Christoffersen et al. 2009, or bid-ask spread, Date and Islyaev, 2015). Additionally, the simple link between implied volatility and model parameters allows for reliable estimates while reducing the time necessary to solve the optimization problem. Specifically, we provide empirical evidence that by using the second-order approximations for the implied volatility, $\Sigma_{2,H}$, via Eq. (165), we can obtain "consistent" estimates of the Heston model parameters from both the call and put options. Option prices are usually filtered to avoid inconsistency arising from the simultaneous use of call and put option prices (see Pacati et al. 2018). We do not filter out any observation not satisfying standard non-arbitrage conditions, and we investigate how this lack affects the model calibration.

Our dataset consists of 1200 call and put options with four strike prices (i.e., $n_E = 4$) and $n_T = 150$. Starting from the traded call option prices $C^o(S_i, T_i, E_j)$ with spot price S_i, time to maturity T_i and strike price E_j, and using the U.S. three-month government bond yield as risk-free interest rate, r, we compute the observed implied volatility, $\sigma^o_C(S_i, T_i, E_j)$, for $i = 1, 2, \ldots, n_T$, $j = 1, 2, \ldots, N_E$. This computation is done using the Matlab function *calcBSImpVol*, which uses Li's rational function approximator for the initial estimate (see, Li 2006; 2008), followed by Householder's root finder of the third order to improve the convergence rate of the Newton-Raphson method.

For any time $i = 1, 2, \ldots, n_T$, we then estimate the Heston model parameters $\underline{\Theta}_i = (\gamma_i, v^*_i, \chi_i, \rho_i, v^i_0) \in \mathbb{R}^5$, $i = 1, 2, \ldots, n_T$, solving the optimization problem:

$$\min_{\underline{\Theta}^C \in \mathcal{V}} \sum_{j=1}^{n_E} \left[\sigma^o_C(S_i, T_i, E_j) - \frac{\Sigma_{2,H}(S_i, T_i, E_j)}{\sqrt{T_i}} \right]^2, \tag{216}$$

where $\Sigma_{2,H}$ is given in formula (165) with n=1 and $\mathcal{V}$ is the following set of constraints:

$$\mathcal{V} = \left\{ \underline{\Theta} = (\gamma, v^*, \chi, \rho, v_0) \in \mathbb{R}^5 \mid \gamma, v^*, \chi, v_0 > 0, \ -1 < \rho < 1 \right\}. \tag{217}$$

To solve problem (216) we use a metric variable steepest descent algorithm (see, for example, Recchioni and Scoccia (2000), Fatone et al. (2013)). This is an iterative algorithm which generates a sequence of points, $\underline{\Theta}^k$, $k = 0, 1, \ldots$, belonging to the interior of the feasible region and moving opposite to the gradient vectors of the objective function computed in a suitable metric.

We then repeat the calibration procedure starting from the observed put prices $P^o(S_i, T_i, E_j)$, where P^o is the observed value of the put option, $i = 1, 2, \ldots, n_T$, and $j = 1, 2, \ldots, n_E$, and we solve the problem

$$\min_{\underline{\Theta}^P \in \mathcal{V}} \sum_{j=1}^{n_E} \left[\sigma_P^o(S_i, T_i, E_j) - \frac{\Sigma_{2,H}(S_i, T_i, E_j)}{\sqrt{T_i}} \right]^2, \tag{218}$$

In this way we obtain two optimal sets of model parameters, one starting from the call options, $\underline{\Theta}^C$, and the other starting from the put options, $\underline{\Theta}^P$.

Some descriptive statistics for the estimated model parameters, initial variance, Feller ratio, and observed implied volatility are given for the two sets in Table 3. We observe that the two sets of parameters are almost identical, with the exception of the estimate of the long-term mean parameter. We argue that the difference in the v^* parameter estimate from the call and put prices is due to market imperfections that lead to a spread between the implied volatility σ^o of call and put options. In fact, the absolute value of the implied volatility spread, $|\sigma_C^o(S_i, T_i, E_j) - \sigma_P^o(S_i, T_i, E_j)|$, derived from the call and put options, is 0.04 on average while the relative absolute spread (i.e., the ratio of the spread to implied volatility from the call) is 0.24. Interestingly, the absolute difference between the square root of the two long-term variance parameters is 0.05 and the ratio of this difference to square root of the call variances is 0.29, thus mirroring the implied volatility spread.

Table 3: Descriptive statistics for estimated values of the model parameters and observed implied volatility σ^o.

Call Set	χ	v^*	γ	ρ	v_0	$\frac{2\chi v^*}{\gamma^2}$	obj. func.	σ^o
mean	5.7999	0.014663	0.50098	-0.8502	0.08060	0.677512	8.35e-5	0.1581
median	5.7999	0.012726	0.50100	-0.8502	0.08200	0.588756	2.28e-5	0.1546
std	0.00057	0.007032	0.000303	0.000220	0.004912	0.324606	1.46e-4	0.020
Put Set	χ	v^*	γ	ρ	v_0	$\frac{2\chi v^*}{\gamma^2}$	obj. func.	σ^o
mean	5.7999	0.029102	0.5009	-0.8502	0.08384	1.34530	7.84e-5	0.1931
median	5.7999	0.029114	0.5009	-0.8502	0.08489	1.29907	2.02e-5	0.1923
std	0.000020	0.006205	0.00026	0.00018	0.004243	0.28708	2.87e-4	0.0168

To evaluate the model consistency, we compute the European call and put option prices using formulas $C_{2,H}$, $P_{2,H}$, with both sets of estimated parameters. Figures 5 and 6 show the observed and second-order (solid line and dotted line, respectively) call and put option prices. The approximations in Figure 5 are obtained using the model parameters estimated by the observed implied volatility from call options (i.e., Call Set) while those in Figure 6 are obtained using the model parameters estimated by the observed implied volatility from put options (i.e., Put Set). For each set we compute the mean and standard deviation of the relative errors for the Call as

$$E_{i,j}^{C,\underline{\Theta}^L} = |C^o(S_i, T_i, E_j) - C_{2,H}(S_i, T_i, E_j; \underline{\Theta}^L)|/C^o(S_i, T_i, E_j), \text{with } L = C, P,$$

and the equivalent errors for the Put. The average relative errors $E^{C,\underline{\Theta}^C}$ and $E^{P,\underline{\Theta}^P}$ (i.e., when the parameters are estimated starting from the corresponding option prices) are, respectively, 0.027 (i.e., 2.7%) and 0.031 (i.e., 3.1%). These errors are in line with those in Pacati et al. 2018, where a double Heston model with jumps is used. By contrast, using the model parameters of the Put set to estimate the Call prices and vice versa, the relative errors $E^{C,\underline{\Theta}^P}$ and $E^{P,\underline{\Theta}^C}$ are, on average, 0.21 (i.e., 21%) for the call and 0.22 (i.e., 22%) for the Put. Thus, while the cross estimates produce a clear bias, the error is of the same order as the relative error in the implied volatility (i.e., 24%), suggesting the bias is driven by market imperfections

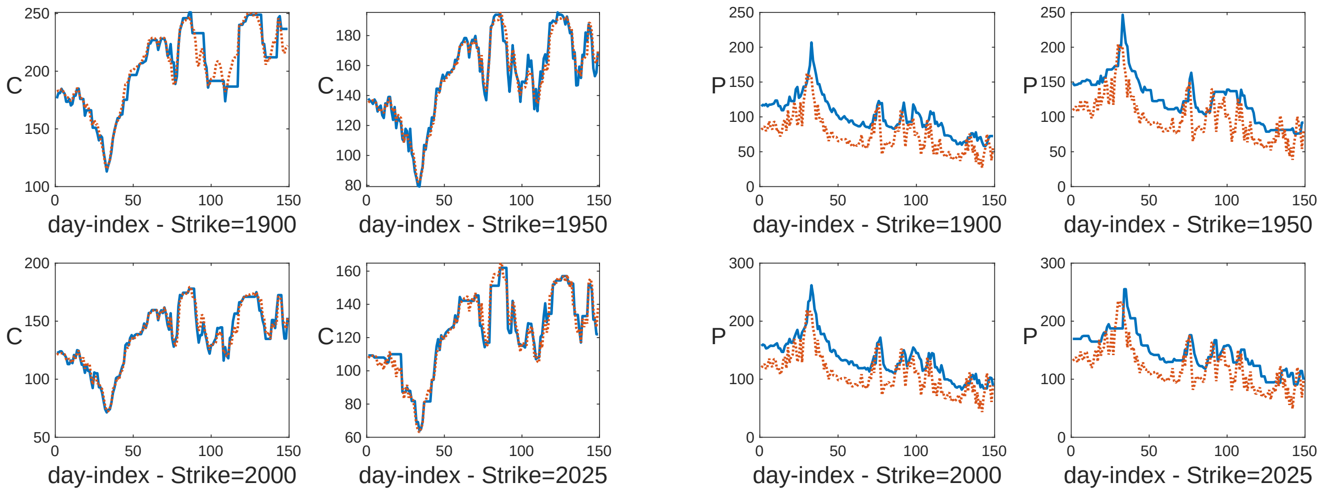

Figure 5: Left Panel: Observed call option prices (solid line) and the second-order approximations $C_{2,H}$ (dotted line) for four different strike prices $E_1 = 1900$, $E_2 = 1975$, $E_3 = 2000$, and $E_4 = 2025$ and expiry date $T =$ December 19, 2015, versus time (September 1, 2014 – March 30, 2015) obtained with the optimal parameters from the observed implied volatility of call options (i.e., Call Set). Right Panel: Observed put option prices (solid line) and Black-Scholes second-order approximations $P_{2,H}$ (dotted line) for four different strike prices $E_1 = 1900$, $E_2 = 1975$, $E_3 = 2000$, and $E_4 = 2025$ and expiry date $T =$ December 19, 2015, versus time (September 1, 2014 – March 30, 2015) obtained with the optimal parameters from the observed implied volatility of call options (i.e., Call Set).

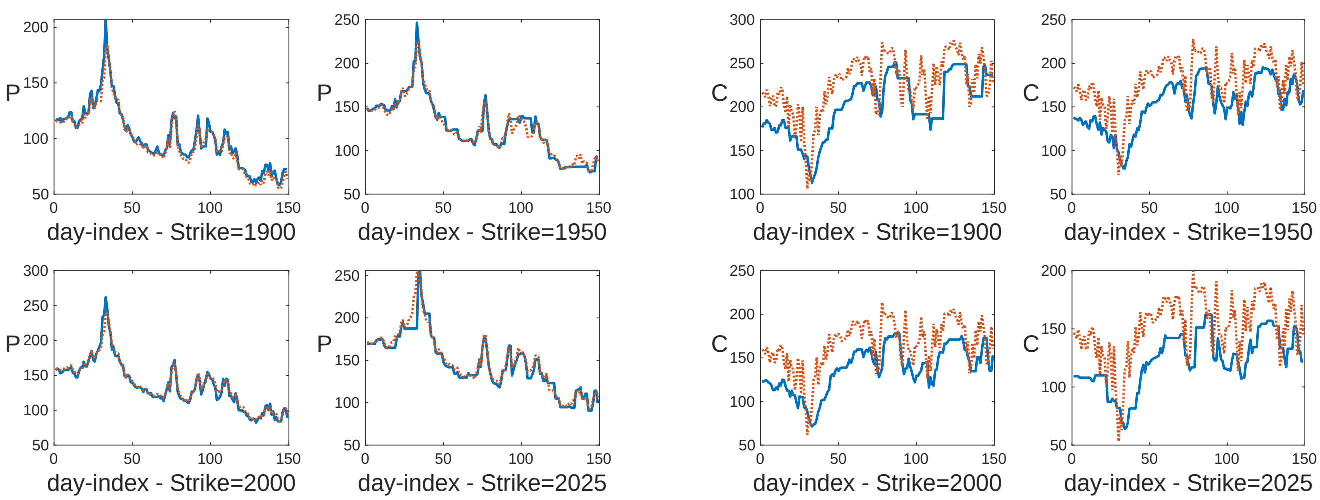

Figure 6: Left Panel: Observed put option prices (solid line) and second-order approximations $P_{2,H}$ (dotted line) for four different strike prices $E_1 = 1900$, $E_2 = 1975$, $E_3 = 2000$, and $E_4 = 2025$ and expiry date $T =$ December 19, 2015, versus time (September 1, 2014 – March 30, 2015) obtained with the optimal parameters from the observed implied volatility of put options (i.e., Put Set). Right Panel: Observed call option prices (solid line) and second-order approximations $C_{2,H}$ (dotted line) for four different strike prices $E_1 = 1900$, $E_2 = 1975$, $E_3 = 2000$, and $E_4 = 2025$ and expiry date $T =$ December 19, 2015, versus time (September 1, 2014 – March 30, 2015) obtained with the optimal parameters from the observed implied volatility of put options (i.e., Put Set).

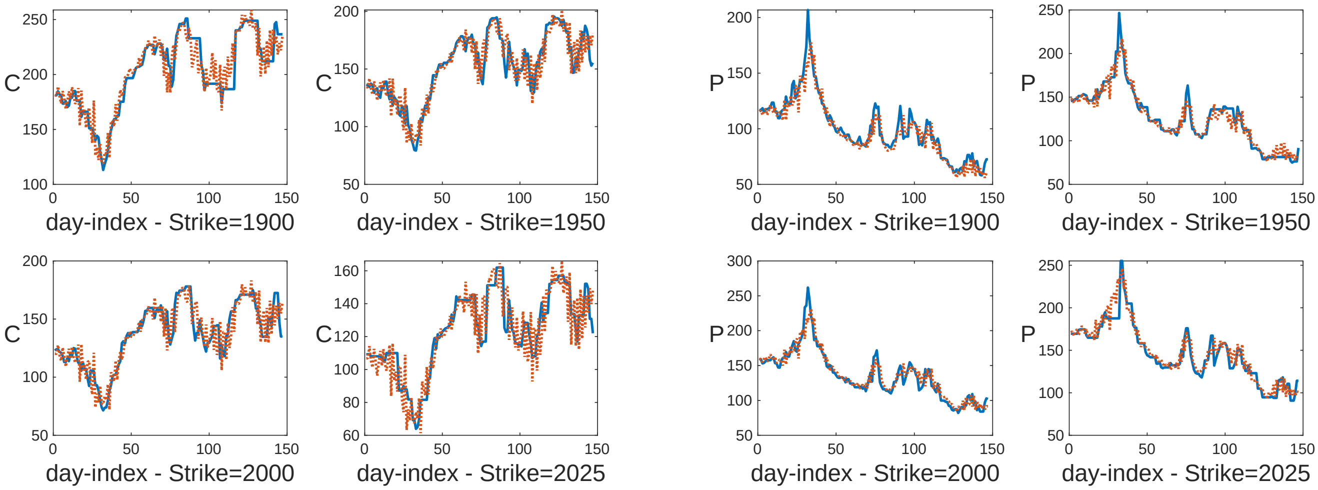

Figure 7: Observed option prices (solid line) and one-day ahead estimates computed using second-order approximation (dotted line) for four different strike prices $E_1 = 1900$, $E_2 = 1975$, $E_3 = 2000$, and $E_4 = 2025$ and with expiry date $T =$ December 19, 2015, versus time (September 1, 2014 – March 30, 2015). Call price one-day ahead estimates using the Call Set (left panel); Put price one-day ahead using the Put Set (on the right). Average relative errors of call and put options: 7.9% and 6.2% respectively.

rather than inconsistencies with the methodology.

We conclude this section by testing the potential of the calibrated parameters to forecast option prices one-day ahead. Figure 7 shows the one-day ahead estimates for call (left panel) and put (right panel) option prices. Specifically, the option estimates at time $t + 1$ is carried out using the value of the optimal parameters at time t. The call one-day ahead estimated prices are obtained using the model parameters $\underline{\Theta}_t^C$, while the put one-day ahead estimated prices use the parameters $\underline{\Theta}_t^P$. The relative errors of the one-day ahead estimates are, on average, 7.9% for the call options and 6.2% for the put options.