RESEARCH ARTICLE

The Feedback Invariant Measures of distance to Uncontrollability and Unobservability

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Abstract
The selection of systems of inputs and outputs (input and output structure) forms part of the early system design and this is important since it preconditions the potential for control design. Existing methodologies for input, output structure selection rely on criteria expressing distance to uncontrollability, unobservability. Although controllability is invariant under state feedback, its corresponding degrees expressing distance to uncontrollability is not. The paper introduces new criteria for distance to uncontrollability which is invariant under feedback transformations. The approach uses the restricted input-state, state-output matrix pencils developed for the matrix pencil characterisation of invariant spaces of the geometric theory and then deploys exterior algebra to define the invariant input and output decoupling polynomials. This reduces the overall problem of distance to uncontrollability to two optimisation problems: the distance from the Grassmann variety and distance of a set of polynomials from non-coprimeness that relates to the notion of approximate Greatest Common Divisor. Results on the distance of Sylvester Resultants from singularity provide the new measures. By duality, the results also apply to the problem of invariant distance to unobservability related to the selection of the output structure.

KEYWORDS
Distance to uncontrollability, distance to unobservability, feedback invariant measures, decomposability of multivectors, Sylvester resultant, “approximate” GCD.

1. Introduction

The selection of sets of inputs and outputs in a system is a fundamental problem that has appeared in many areas of design such as aerospace (Müller & Weber, 1972), noise suppression (Dunn & Garcia, 1999), process systems (Kumar & Seinfeld, 1978) etc. This problem is independent from control system design, but the selection affects the resulting model and has a significant effect on resulting structure and system properties, which determine the potential for control design (Karcanias, 2008). The selection of inputs and outputs has been based so far on controllability, observability criteria for a given system (fixed input and output structure) (Müller & Weber, 1972), (Dunn & Garcia, 1999), (Kumar & Seinfeld, 1978) (and references therein). Such measures have been used for the design of the input and output maps (Müller & Weber,

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1972), (Dunn & Garcia, 1999), (Kumar & Seinfeld, 1978), (Georges, 1995), (Xia, Yin, & Zou, 2018). It is known that such measures of distance are clearly affected by the selection of coordinate transformations, as this is evident by the theory of balanced realisations (Moore, 1981).

The existing methods for selection do not take into account the fact that although controllability is invariant under state feedback and observability is invariant under output injection (Kailath, 1980), (Karcanias, 1979), their corresponding degrees (measures of distance to uncontrollability, unobservability) are not invariant under compensation (state feedback, output injection) as this can be demonstrated by simple examples. Given that selecting input and output structures fix the system structure and that different feedback schemes will be used, it is essential to derive selection procedures for input and output structures, based on criteria which are invariant under feedback. The paper considers state space descriptions based on physical state variables and deals with the development of criteria measuring distance of a given system from uncontrollability, unobservability, which are invariant under state feedback. We focus on the problem of distance to uncontrollability, whereas the case of distance to unobservability is treated along similar lines by using duality arguments.

The approach used is based on the matrix pencil characterisation of controllability using the controllability pencil $C(s)$ (Kailath, 1980), (Rosenbrock, 1970) (similarly $K(s)$ matrix pencil for observability), which has a feedback invariant version provided by the restricted input state pencil $R(s)$ (restricted output $Q(s)$) (Karcanias, 1979), (Jaffe & Karcanias, 1981). The restricted input-state pencil has been used in matrix pencil approach (Karcanias, 1979), (Karcanias & Kouvaritakis, 1979), (Jaffe & Karcanias, 1981) to geometric system theory (Wonham, 1979). The characterisation of uncontrollability is based on the presence of zeros (input decoupling zeros). In the effort to develop distance criteria from uncontrollability, it is natural to consider the distance to singularity (Gu, 2000) of these pencils, which is equivalent to the characterisation of almost input decoupling zeros (Karcanias, Giannakopoulos, & Hubbard, 1983), (Karcanias & Halikias, 2013).

We use the state feedback invariant pencil $R(s)$ $(Q(s))$ and by taking the exterior product (Marcus & Minc, 1964) of the rows of $R(s)$ $(Q(s))$ we define a polynomial vector $\underline{r}'(s)^t(\underline{q}'(s))$ the Greatest Common Divisor (GCD) which determine the input decoupling zeros of the system (Rosenbrock, 1970). Thus, controllability can be assessed by the coprimeness of the polynomials in $\underline{r}'(s)^t(\underline{q}'(s))$. This leads naturally to the characterisation of the distance to uncontrollability (unobservability) that is invariant under feedback to the distance of a set of polynomials from lack of coprimeness, or the distance to the GCD variety (Karcanias, Fatouros, Mitrouli, & Halikias, 2006). The latter problem is related to the notion of "almost zeros" (Karcanias et al., 1983) and the computation of "approximate" GCD of a set of polynomials (Mitrouli & Karcanias, 1993), (Karcanias et al., 2006). The distance for the set of corresponding polynomials from non-coprimeness are based on results expressing the distance of Sylvester Resultants from singularity. Given that the invariant input decoupling zero polynomial vector $\underline{r}(t)$ is obtained as exterior product of the rows of $R(s)$ implies that the coefficient vector of this has to belong to the Grassmann variety of the corresponding projective space. The overall problem of distance to non-coprimeness is thus reduced to two optimisation problems, that is the distance to the Grassmann and the GCD varieties. We use recent results (Karcanias & Leventides, 2016) allowing the computation of a vector $\underline{r}'(t)$ of the best approximation of a given vector $\underline{r}(t)$ to the Grassmann variety, and then use this for the computation of the distance to the GCD variety.

The paper is organised as follows: Section 2 deals with background definitions, provides a motivation for the study of the problem and introduces the matrix pencil descriptions. Section 3 deals with the restricted input-state and state output pencil descriptions and the feedback invariance property of the restricted input-state pencil under state feedback, and the restricted state-output pencil under output injection. The role of these pencils in defining invariant distance measures is investigated. Section 4 introduces the input-decoupling zero polynomial vector which provides the means for developing feedback invariant characterisations of distance to uncontrollability, by linking this distance to the computation of distance to non-coprimeness. Section 5 summarises recet results (Karcanias & Leventides, 2016) on multivector decomposability and defines an optimisation problem allowing the computation of the best approximation of a vector $\underline{r}(t)$ in a projective space by a vector $\underline{r}'(t)$ of the corresponding Grassmann variety. Section 6 reviews the basics on coprimeness and distance to non-coprimeness of a set of polynomials. Section 7 develops criteria for invariant distance to uncontrollability, where the approximate vector $\underline{r}'(t)$ is used in general. The development of distance to unobservability follows by using the results on the distance to uncontrollability and deploying duality.

2. Background Results

Consider a linear system described as follows

$$S(A,B,C,D): \ \dot{\underline{x}} = A\underline{x} + B\underline{u}, \quad A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times p} \tag{1}$$

$$\underline{y} = C\underline{x} + D\underline{u}, \quad C \in \mathbb{R}^{m\times n}, D \in \mathbb{R}^{m\times p} \tag{2}$$

where (A,B) is controllable, (A,C) is observable and $\text{rank}(B) = p,\ \text{rank}(C) = m$. With the pairs $(A,B),(A,C)$ we associate the controllability, observability pencils (Karcanias & Vafiadis, 2002), (Rosenbrock, 1970) respectively.

Definition 2.1 ((Rosenbrock, 1970))**.** Let $S(A,B,C)$ be a system description and consider the matrix pencils

$$C(s) = (sI - A, \ \ -B), \ \ K(s) = \begin{pmatrix} sI - A \\ -C \end{pmatrix} \tag{3}$$

Then, system's controllability and observability properties are characterised by the absence of finite (or Smith) zeros in (3), meaning that the finite Smith form has no polynomial entries.

We assume that the state space description is made from physical variables. For such descriptions performance conditions and constraints may be introduced on the states. Preserving the physical state space description implies that input, output coordinate transformations and state, output feedback and output injection may be used.

Remark 1. Note that matrix pencils $C(s), K(s)$ are modified under state feedback $L \in \mathbb{R}^{p\times n}$ and output injection $K \in \mathbb{R}^{n\times m}$, respectively, which is a generally known result and hence is omitted. Although the rank of the two pencils is invariant under these two transformations, the numerical rank may be considerably different.

A number of criteria for measuring the degrees of those two fundamental system properties have already been given in the literature (Xia et al., 2018), (Moore, 1981). However these open loop criteria suffer from the fact that they are affected by state feedback (output injection respectively) as this is demonstrated in the following example.

Example 1. Consider the state space description of a linearised two-mass model of the wind turbine, where the states are physical variables and the model parameters are similar to the physical parameters for the wind turbine model as in (Xia et al., 2018). The state and input matrices for this model are:

$$A = \begin{pmatrix} -0.0467 & 0 & -9.104 \times 10^{-7} \\ 0 & 0 & 3.3673 \times 10^{-4} \\ 8.0215 \times 10^{6} & -1.8583 \times 10^{5} & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ -0.014535 \\ 0 \end{pmatrix} \tag{4}$$

Let the distance of the open-loop system to uncontrollability be denoted by

$$d_o(A,B) = \min_{s \in \mathbb{C}} \sigma_{min}([sI - A, \ \ B])$$

Then for the closed-loop system (with the LQR controller denoted by $K = -kR^{-1}B'V$, where R is a positive-definite matrix, V is the unique positive definite solution of the corresponding Riccati equation and k is a gain parameter) the distance becomes $d_c(\hat{A}, B) = \min_{s\in\mathbb{C}} \sigma_{min}([sI - (A + BK), \ \ B])$. The optimal solution of the Riccati equation V verified in (Xia et al., 2018) is

$$V = \begin{pmatrix} 0.7439 & 0.0017 & -2.2412 \times 10^{-8} \\ 0.0017 & 1.0230 \times 10^{-5} & -5.9486 \times 10^{-11} \\ -2.2412 \times 10^{-8} & -5.9486 \times 10^{-11} & 1.0973 \times 10^{-14} \end{pmatrix} \tag{5}$$

With $k = 1$ the closed loop state matrix is computed as

$$\hat{A} = \begin{pmatrix} -0.0467 & 0 & -9.1040^{-7} \\ -3.5915 \times 10^{-7} & -2.1613 \times 10^{-9} & 3.3673 \times 10^{-4} \\ 8.0215 \times 10^{6} & -1.8583 \times 10^{5} & 0 \end{pmatrix} \tag{6}$$

Using Gu's algorithm along the straight line (Gu, 2000) for estimation of the distance to uncontrollability we start increasing the value of the gain k in order to observe differences in the distance. With $k = 10^8$ the closed-loop distance to uncontrollability of the system (which is already close to uncontrollability in open-loop) demonstrates a rapid decrease. The results can be evaluated with given tolerance $tol = 10^{-8}$ as follows

$$d_o(A,B) = 2.8997 \times 10^{-7}, \ \ d_c(\hat{A}, B) = 1.9118 \times 10^{-8} \tag{7}$$

It should be noted that the higher the value of the gain, the smaller the value of $d_c(\hat{A}, B)$ (for this particular example).

This example clearly demonstrates the property that although controllability is invariant under the state feedback, the corresponding degree of the property, that is equivalent to the distance to uncontrollability, varies under the feedback. This suggests that the measures of the distance to uncontrollability that can be used for the selection of the input matrix has to be based on the distance criteria which is invariant under state feedback. This provides the motivation for the search for invariant measures of the distances.

3. Feedback Invariant System Descriptions

Consider the linear system described as in 1. If N is a left annihilator of B, that is, $NB = 0$, $N \in \mathbb{R}^{(n-p)\times n}$, $\mathrm{rank}(N) = n - p$, $B^\dagger$ is a $(p \times n)$ left inverse of B $(B^\dagger B = I_n)$, M is a right annihilator of C, that is, $CM = 0$, $M \in \mathbb{R}^{n\times(n-m)}$, $\mathrm{rank}(M) = n-m$ and $C^\dagger$ is a $(n \times m)$ right inverse of C $(CC^\dagger = I_m)$. Then we define matrices Z, W

$$Z = \begin{pmatrix} N \\ B^\dagger \end{pmatrix} \in \mathbb{R}^{n\times n}, |Z| \neq 0, W = \begin{pmatrix} M & C^\dagger \end{pmatrix} \in \mathbb{R}^{n\times n}, |W| \neq 0 \tag{8}$$

The matrix pencils

$$R(s) = sN - NA, \;\; Q(s) = sM - AM \tag{9}$$

are known as the restricted state-input pencil and restricted state-output pencil respectively (Karcanias & Kouvaritakis, 1979), (Jaffe & Karcanias, 1981), (Rosenbrock, 1970). The following results are known:

Theorem 3.1 ((Rosenbrock, 1970))**.** *The system $S(A, B, C)$ is controllable if and only if the pencil $C(s)$ has no finite elementary divisors and it is observable if and only if the pencil $K(s)$ has no finite elementary divisors.*

Clearly, the case where $C(s)$ and $K(s)$ have zeros then we have uncontrollability, unobservability respectively and the corresponding elementary divisors define the sets of input and output decoupling zeros respectively (Rosenbrock, 1970).

Notation 1 ((Marcus & Minc, 1964))**.** Let $Q_{k,n}$ denote the set of lexicographically ordered, strictly increasing sequences of k integers from $\tilde{n} = \{1, 2, \ldots, n\}$. If $\{\underline{x}_{i_1}, \ldots, \underline{x}_{i_k}\}$ is a set of vectors of $\mathcal{V}, \omega = (i_1, \ldots, i_k) \in Q_{k,n}$, then $\underline{x}_{i_1} \wedge \cdots \wedge \underline{x}_{i_k} = \underline{x}_\omega \wedge$ denotes the exterior product and by $\wedge^r \mathcal{V}$ we denote the r-th exterior power of $\mathcal{V}$. If $H \in \mathcal{F}^{m\times n}$ and $r \leq \min(m, n)$, then by $C_r(H)$ we denote the r-th compound matrix of H.

If $\underline{c}_i(s)^t, \underline{k}_i(s)^t, i \in \tilde{n}$, denote the rows of $C(s) \in \mathbb{R}^{n\times(p+n)}[s]$, columns of $K(s) \in \mathbb{R}^{(n+m)\times n}[s]$ and $C_r(X)$ denotes the r-th compound matrix of $X \in \mathbb{R}^{t\times w}[s], r \leq \min(t, w)$ (Marcus & Minc, 1964) then we may define the polynomial vectors

$$\begin{aligned} C_n(C(s)) &= \underline{c}_1(s)^t \wedge \cdots \wedge \underline{c}_n(s)^t = \underline{c}(s)^t \wedge^n \in \mathbb{R}^{1\times\rho}[s] \\ &= \tilde{\underline{c}}(s)^t, \;\; \rho = \begin{pmatrix} n+p \\ n \end{pmatrix} \end{aligned} \tag{10}$$

$$\begin{aligned} C_n(K(s)) &= \underline{k}_1(s) \wedge \cdots \wedge \underline{k}_n(s) = \underline{k}(s) \wedge^n \in \mathbb{R}^{\rho'}[s] \\ &= \tilde{\underline{k}}(s), \;\; \rho' = \begin{pmatrix} n+m \\ n \end{pmatrix} \end{aligned} \tag{11}$$

as the input decoupling zero and output decoupling zero polynomials of the system respectively.

Remark 2. The system is controllable, if the polynomials of $\underline{\tilde{c}}(s)$ are coprime and it is observable if the polynomials in $\underline{\tilde{k}}(s)$ are coprime. If the polynomials in $\underline{\tilde{c}}(s)$ and $\underline{\tilde{k}}(s)$ are coprime, then their distance to the corresponding GCD variety (Karcanias & Vafiadis, 2002) defines the distances to uncontrollability, $d(A, B)$, unobservability, $d(A, C)$ of the $S(A, B), S(A, C)$ systems respectively.

Note that the distances $d(A, B), d(A, C)$ may vary when state feedback, respectively output injection is applied. In fact, if $L \in \mathbb{R}^{p \times n}$ is a state feedback, then the controllability pencil of the closed loop system becomes

$$C'(s) = (sI - A - BL, \ \ -B) = (sI - A, \ \ -B) \begin{pmatrix} I_n & 0 \\ L & I_p \end{pmatrix} \tag{12}$$

and the corresponding input decoupling polynomial is then defined by

$$\begin{aligned} \underline{\tilde{c}}'(s)^t &= C_n(C'(s)) = C_n([sI - A - BL, \ \ -B]) \\ &= C_n([sI - A, \ \ -B]) C_n\left(\begin{bmatrix} I_n & 0 \\ L & I_p \end{bmatrix}\right) = \underline{\tilde{c}}(s)^t T(L) \end{aligned} \tag{13}$$

where $T(L) \in \mathbb{R}^{\sigma \times \sigma}$.

Remark 3. Condition (13) clearly demonstrates that although $\underline{\tilde{c}}'(s)^t$ and $\underline{\tilde{c}}(s)^t$ have the same GCD their distance to the GCD is affected by the choice of L.

Similar results may be stated for unobservability and the effect of output injection on the distance to unobservability when $C_n(K(s)) = \underline{\tilde{k}}(s)$, is considered. The above raises the question of defining measures of the distance to uncontrollability, unobservability which are feedback invariant.

4. The Invariant Distance Problem

The study of the invariant distance to uncontrollability and unobservability uses early results for characterising controllability and observability based on the restriction pencils introduced above (Karcanias, 1979), (Karcanias & Kouvaritakis, 1979), (Jaffe & Karcanias, 1981). Note that

$$ZC(s) = \begin{pmatrix} N \\ B^{\dagger} \end{pmatrix} (sI - A, \ \ -B) = \begin{pmatrix} sN - NA & 0 \\ sB^{\dagger} - B^{\dagger}A & I_p \end{pmatrix} = C^*(s) \tag{14}$$

and

$$K(s)W = \begin{pmatrix} sI - A \\ C \end{pmatrix} \begin{pmatrix} M & C^{\dagger} \end{pmatrix} = \begin{pmatrix} sM - AM & sC^{\dagger} - AC^{\dagger} \\ 0 & I_m \end{pmatrix} = K^*(s) \tag{15}$$

Clearly, $C(s)$ and $C^*(s)$, as well as $K(s)$ and $K^*(s)$ are strict equivalent (Gantmakher, 1998) and both $C(s)$ and $K(s)$ do not have zeros at infinity. Thus their finite zeros, if any, are given by their corresponding Smith forms defined by

$$C^*(s) = \begin{pmatrix} sN - NA & 0 \\ sB^\dagger - B^\dagger A & I_p \end{pmatrix} \lhd \begin{pmatrix} sN - NA & 0 \\ 0 & I_p \end{pmatrix} = C(s) \tag{16}$$

$$\begin{aligned} K^*(s) = &\begin{pmatrix} sM - AM & sC^\dagger - AC^\dagger \\ 0 & I_m \end{pmatrix} \\ &\lhd \begin{pmatrix} sM - AM & 0 \\ 0 & I_m \end{pmatrix} = K(s) \end{aligned} \tag{17}$$

where $\lhd$ denotes $R[s]$-equivalence. The above lead to the following result (Karcanias, 1979):

Theorem 4.1. *The system $S(A, B, C)$ is:*

(1) Controllable if and only if the pencil $R(s)$ has no finite elementary divisors
(2) Observable if and only if the pencil $Q(s)$ has no finite elementary divisors.

Corollary 4.2. *The input restriction pencil $R(s) = sN - NA$ is invariant under state feedback and the output restriction pencil $Q(s) = sM - AM$ is invariant under output injection.*

The above is rather obvious from the fact that if $L \in \mathbb{R}^{p\times n}$ is a state feedback, then $R'(s) = sN - N(A + BL) = sN - NA = R(s)$. Similarly, if $F \in \mathbb{R}^{n\times m}$ is an output injection, then $Q'(s) = sM - (A + FC)M = sM - AM = Q(s)$. The invariance of $R(s)$ and $Q(s)$ leads to the following definition.

Definition 4.3. (1) For the pencil $R(s) = sN - NA$ we define by $\underline{r}_i(s)^t, i = 1, \dots, n - p$ the rows of $R(s)$. The polynomial defined as

$$\begin{aligned} C_{n-p}(R(s)) &= \underline{r}_1(s)^t \wedge \cdots \wedge \underline{r}_{n-p}(s)^t \\ &= \underline{r}(s)^t \wedge^{n-p} \in \mathbb{R}^{1\times\sigma'}[s] \\ &= \underline{\tilde{r}}(s)^t, \;\; \sigma' = \begin{pmatrix} n \\ n-p \end{pmatrix} \end{aligned} \tag{18}$$

will be called the invariant controllability polynomial (ICP) of the system.

(2) For the restriction pencil $Q(s) = sM - AM$ we define by $\underline{q}_i(s), i = 1, \dots, n - m$ the rows of $Q(s)$. The polynomial defined as

$$\begin{aligned} C_{n-m}(Q(s)) &= \underline{q}_1(s) \wedge \cdots \wedge \underline{q}_{n-m}(s) \\ &= \underline{q}(s) \wedge^{n-m} \in \mathbb{R}^{\sigma'}[s] \\ &= \underline{\tilde{q}}(s), \;\; \sigma' = \begin{pmatrix} n \\ n-m \end{pmatrix} \end{aligned} \tag{19}$$

will be called the invariant observability polynomial (IOP) of the system.

The invariance under feedback of $R(s)$ and $Q(s)$ implies the invariance of $\underline{\tilde{r}}(s)^t, \underline{\tilde{q}}(s)$ and these will be used for the study of feedback invariant distances to uncontrollabity, respectively unobservability of the system. Note that $\deg\{\underline{\tilde{r}}(s)^t\} = n - p$ and that

$\deg\{\tilde{\underline{q}}(s)\} = n - m$, where $\deg(\cdot)$ denotes the highest order degree of polynomials in the corresponding multivector. Given that the GCD of $\tilde{\underline{r}}(s)^t$, provides a state feedback invariant characterisation of input decoupling zeros and $\tilde{\underline{q}}(s)$ provides an output injection invariant characterisation of output decoupling zeros we are led to the following result:

Corollary 4.4. *The distances of the set of polynomials of $\tilde{\underline{r}}(s)^t, \tilde{\underline{q}}(s)^t$ from noncopriness define respectively the invariant distance to uncontrollability, unobservability.*

Thus such distance problems may be studied using the results on the "approximate" GCD of a set of polynomials (Karcanias et al., 1983), (Fatouros & Karcanias, 2003), (Karcanias et al., 2006) and express the distance of the corresponding sets of polynomials from their respective GCD variety (Karcanias et al., 2006). The existing results on the approximate GCD assume that the set of polynomials is arbitrarily defined. However, this is not the case for the polynomials of $\tilde{\underline{r}}(s)^t, \tilde{\underline{q}}(s)$ since these are defined as exterior products of rows, columns of matrix pencils and the study of such distances has to take into account these properties. We summarise next some of the basics from exterior algebra (Marcus, 1973).

Definition 4.5 ((Marcus, 1973)). Let $\tau = \begin{pmatrix} v \\ r \end{pmatrix}, r \leq v$ and let $\underline{k} \in \mathbb{R}^\tau$. The vector $\underline{k}$ is called decomposable, if there exists a set of vectors $\{\underline{h}_i, i = 1, \ldots, r, \underline{h}_i \in \mathbb{R}^v\}$ such that

$$\underline{h}_1 \wedge \cdots \wedge \underline{h}_r = \underline{h}\wedge^r = \underline{k} \tag{20}$$

The matrix $H = [\underline{h}_1, \ldots, \underline{h}_r] \in \mathbb{R}^{v \times r}$ defines a basis for subspace $\mathcal{H} = \text{span}\{\underline{h}_1, \ldots, \underline{h}_r\}$ which has dimension r and may be referred to as the parent space of the decomposable vector $\underline{k}$.

The set of r-dimensional subspaces $\mathcal{H}$ of $\mathbb{R}^v$ is referred to as the r-Grassmaniann, it is denoted by $\mathcal{G}(r, \mathbb{R}^v)$ and the column space of H, defines a basis for such subspaces. The mapping of each r-dimensional subspace $\mathcal{H}$ such as $\underline{h}_1 \wedge \cdots \wedge \underline{h}_r = \underline{h}\wedge^r = \underline{k}$ (where $\underline{h}_i$ are the column vectors of H) is a vector $\underline{k} \in \mathbb{R}^\tau$ defining a point in the projective space $\mathcal{P}^{\tau-1}(\mathbb{R})$; the points of $\mathcal{P}^{r-1}$ which satisfy for some $H \in \mathbb{R}^{r \times v}$ (20) are those which belong to the Grassmann variety $\Omega(r, v)$ of the projective space $\mathcal{P}^{\tau-1}(\mathbb{R})$ (Hodge, Hodge, & Pedoe, 1994). Let $\underline{k} \in \mathbb{R}^\sigma$ with coordinates $k_\omega, \omega = (i_1, \ldots, i_r) \in Q_{r,v}$.These are referred to as the Plücker coordinates of $\underline{k}$ and the mapping of $\mathcal{H}$ through $\wedge^r$ is known as the Plücker Embedding of the r-Grassmaniann $\mathcal{G}(r, \mathbb{R}^v)$ into the projective space $\mathcal{P}^{\tau-1}(\mathbb{R})$ (Hodge et al., 1994). The variety $\Omega(r, v)$ is characterised by the result (Marcus, 1973):

Theorem 4.6 ((Karcanias & Giannakopoulos, 1984)). *(1) Let $\underline{k} \in \mathbb{R}^\tau, \tau = \begin{pmatrix} v \\ r \end{pmatrix}$.*

Necessary and sufficient condition for a matrix H to exist, where $H \in \mathbb{R}^{r \times v}, H = [\underline{h}_1, \ldots, \underline{h}_r]^t$ such that

$$\underline{h}\wedge^r = \underline{h}_1 \wedge \cdots \wedge \underline{h}_r = \underline{k} = [\ldots, k_\omega, \ldots]^t \tag{21}$$

is that the coordinates k_ω satisfy the following quadratic relations

$$\sum_{k=1}^{r+1}(-1)^{v-1}k_{i_1,\ldots,i_{r-1},j_v^k j_1,\ldots,j_{v-1},j_{v+1},j_{r+1}} = 0 \tag{22}$$

where $k_{i_1,\ldots,i_r}$ are the coordinates of $\underline{k}$, $1 \leq i_1 < i_2 < \cdots < i_{r-1} \leq v$ and $1 \leq j_1 < j_2 < \cdots < j_{r+1} \leq v$.

(2) If the conditions (22) are satisfied, there exists a uniquely defined space $\mathcal{H}$ that corresponds to $\underline{k} \in \mathbb{R}^\tau$ that satisfies equation (21).

The vectors $\underline{k}$ which satisfy (21) are called decomposable and the set of quadratics defined by (22) (Marcus, 1973), (Hodge et al., 1994), as Quadratic Plücker Relations (QPR) and they define the Grassmann variety of $\mathcal{P}^{\tau-1}(\mathbb{R})$. Alternative conditions for decomposability are given in (Karcanias & Leventides, 2016), in terms of the Grassmann matrix; the latter criteria also provide the means for the reconstruction of the space $\mathcal{H}$ as thus its basis matrix H.

Example 2 ((Karcanias & Giannakopoulos, 1984))**.** Let $v = 5, r = 3$ and let $(k_0, k_1, k_2, \ldots, k_8)$ be the coordinates of a vector defining a point in the projective space $\mathcal{P}^{\tau-1}(\mathbb{R}), \tau = 9$. The set of QPRs describing the Grassmann variety $\Omega(3,5)$ are given by

$$k_0k_5 - k_1k_4 + k_2k_3 = 0,\;\; k_0k_8 - k_1k_7 + k_2k_6 = 0,$$
$$k_0k_9 - k_3k_7 + k_4k_6 = 0 \tag{23}$$

$$k_1k_9 - k_3k_8 + k_5k_6 = 0,\;\; k_2k_9 - k_4k_8 + k_5k_7 = 0 \tag{24}$$

Note that the above set of equations are not minimal. It may be readily shown that the above set of equations is not minimal; in fact, the set (24) may be obtained from the set (23) and thus (23) is a minimal set of quadratics describing the Grassmann variety $\Omega(3,5)$.

5. The Distance Problem from the Grassmann Variety

In this section we examine the problem of distance from the Grassmann Variety which is an essential part in the effort to compute the invariant distance. We first review some standard results from exterior algebra and algebraic geometry.

Lemma 5.1 ((Marcus, 1973))**.** *Let $\mathcal{U}$ be a vector space over a field $\mathcal{F}$ with $dim\mathcal{U} = v$. Then any vector in $\wedge^{v-1}\mathcal{U}$ as well as $\wedge^1\mathcal{U}$ is decomposable.*

Note that the exterior algebra results on decomposability of real or complex vectors also apply for vectors of rational functions and thus polynomial vectors and they will be used in the following analysis. We consider next two cases: the first concerns with the case where the polynomial vector is always decomposable, and the second where the polynomial multivector has to satisfy the decomposability conditions.

Clearly, in this case, that is, when the polynomial vector is decomposable the Grassmann variety of $\mathcal{P}^{\tau-1}(\mathcal{F})$, $\left(\tau = \begin{pmatrix} v \\ r \end{pmatrix}, r = v-1 \text{ or } r = 1\right)$ coincides with the projective space $\mathcal{P}^{\tau-1}(\mathcal{F})$, or in other words there are no Quadratic Plücker Relations defining the variety. Thus the above study of the invariant distance to uncontrollability, unobservability is reduced to the following two cases:

Linear Invariant Distance Problem (LIDP):

(1) The invariant distance to uncontrollability is defined by the distance to the GCD variety of the polynomial $C_{n-p}(R(s)) = \underline{\tilde{r}}(s)^t$, where the resulting multivector $\underline{k} \in \mathbb{R}^p, \underline{\tilde{r}}(s)^t$ is free if $p = 1$ or $p = n-1$.
(2) The invariant distance to unobservability is defined by the distance to the GCD variety of the polynomial $C_{n-m}(Q(s)) = \underline{\tilde{q}}(s)$, where the result of compound matrix, the vector $\underline{k} \in \mathbb{R}^m, \underline{\tilde{q}}(s)$ is free if $m = 1$ or $m = n-1$.

Note that the general case $p \neq 1, n-1$ and $m \neq 1, n-1$ implies that the polynomials $\underline{\tilde{r}}(s)^t$ and $\underline{\tilde{q}}(s)$ are generated by vectors $\underline{k} \in \mathbb{R}^p, \underline{k} \in \mathbb{R}^m$ respectively, which are not any longer free, but they have to satisfy the corresponding QPRs. Note that in this case the general problem of invariant distance to uncontrollability, unobservability becomes a problem of distance between the GCD variety and the corresponding Grassmann variety. Thus we are led to the formulation of the following problem.

General Invariant Distance Problem (GIDP):

(1) The case where $p \neq 1, n-1$ the general invariant distance to uncontrollability is defined by the distance to the GCD variety of the polynomial $C_{n-p}(R(s)) = \underline{\tilde{r}}(s)^t$, where the resulting polynomials denoted as the vector $\underline{k} \in \mathbb{R}^p, \underline{\tilde{r}}(s)^t$ also satisfies the corresponding QPRs.
(2) The case where $m \neq 1, n-1$ the general invariant distance to unobservability is defined by the distance to the GCD variety of the polynomial $C_{n-m}(Q(s)) = \underline{\tilde{q}}(s)$, where vector $\underline{k} \in \mathbb{R}^m, \underline{\tilde{q}}(s)$ also satisfies the corresponding QPRs.

We consider next the non-linear case and this raises the important problem of defining the distance of a general polynomial multivector to the Grassmann Variety.

6. The Study of the General Invariant Distance Problem

The General Invariant Distance Problem is based on the study of the vector invariant zero polynomial $\underline{z}(s)$ ($\underline{\tilde{r}}(s)$ for input and $\underline{\tilde{q}}(s)$ for output decoupling polynomials respectively) and it is reduced to two optimisation problems:

(1) Define the distance of $\underline{z}(s)$ polynomial vector from the GCD variety.
(2) Define conditions for the decomposability of $\underline{z}(s)$ polynomial vector, that is its distance from the Grassmann variety of the corresponding projective space.

We may handle those two optimisation problems as follows:
First, we define the best approximation of $\underline{z}(s)$ from the Grassmann variety, and let us say that this polynomial vector $\underline{z}'(s) \in \Omega(m, n; \mathbb{R}[s])$. Then, define the distance of $\underline{z}(s)$ from the GCD variety and this shall be the solution to our problem.
Note that in the linear case, where $\underline{z}(s)$ is decomposable, then $\underline{z}'(s) = \underline{z}(s)$, whereas in the nonlinear case $\underline{z}(s)$ is approximated by $\underline{z}'(s)$. Throughout this analysis, the polynomial vectors $\underline{z}(s), \underline{z}'(s)$ have coordinates polynomials, which are the Plücker coordinates in the projective space of the corresponding vector. Assuming that $\underline{z}(s)$ is obtained as the exterior product of a matrix $Z(s) \in \mathbb{R}^{m \times n}[s], m < n$, where $\underline{z}_i(s), i \in \tilde{m}$, denote the rows of $Z(s)$ then we define

$$C_m(Z(s)) = \underline{z}_1(s)^t \wedge \cdots \wedge \underline{z}_m(s)^t = \underline{z}(s)^t \wedge^m \in \mathbb{R}^{l \times \rho}[s] = \tilde{\underline{z}}(s)^t, \;\; \rho = \begin{pmatrix} n \\ m \end{pmatrix} \tag{25}$$

and the Plücker coordinates are $\{z_\omega(s), \omega \in Q_{m,n}\}$

The standard results on decomposability of real, or complex vectors, carry over to the case of rational functions and polynomials (Gantmakher, 1998), (Marcus, 1973) and can be used here for the decomposability of polynomial multivectors. In this case we shall denote the set of QPRs associated with $\underline{z}(s)$ by

$$Q = \{q_i(s), i \in \tilde{\rho}, q_i(s) \in \mathbb{R}[s]\} \tag{26}$$

and they characterise the Grassmann variety $\Omega(m,n)$. Clearly, the set Q is defined by the set of Plücker coordinates $\{z_\omega(s), \omega \in Q_{m,n}\}$.

Lemma 6.1. *The multivector $\tilde{\underline{z}}(s)^t \in \mathbb{R}^{1\times\rho}[s]$ is decomposable, if and only if the QPRs are exactly zero, that is*

$$q_i(s) \equiv 0, \;\; \forall i \in \tilde{\rho}, q_i(s) \in \mathbb{R}[s] \tag{27}$$

Alternative decomposability conditions may be defined in terms of the Grassmann matrices (Karcanias & Leventides, 2016). The above set of conditions are polynomial equations and can be checked for each of the $q_i(s)$ defined by the set of Plücker coordinates $\{z_\omega(s), \omega \in Q_{m,n}\}$. In the case where at least one of the above conditions is not satisfied, then we address the problem of minimal distance of $\underline{z}(s)$ from the Grassmann variety $\Omega(m,n)$.

Note that we can always find vectors $\underline{k}(s) \in \wedge^m \mathbb{R}[s]^n$ such that $\underline{z}'(s) = \underline{z}(s) + \underline{k}(s)$, with $\underline{z}'(s) \in \Omega(m,n)$. This generates an approximate solution and can evaluate the derivation for the non-generic cases, where the exact problem does not have solution. Finding $\|\underline{k}(s)\|$ which is minimum expresses the distance problem of the given $\underline{z}(s)$ from the Grassmann variety (Leventides, Petroulakis, & Karcanias, 2014) defined as:

Definition 6.2. Define a vector $\underline{k}(s) \in \wedge^m \mathbb{R}[s]^n$, such that the vector $\underline{z}'(s) = \underline{z}(s) + \underline{k}(s), \underline{z}'(s) \in \Omega(m,n)$, where $\|\underline{k}(s)\|$ is the minimum and $\|\cdot\|$ corresponds to the spectral norm. The resulting $\underline{z}'(s)$ may be referred to as the *projection of $\underline{z}(s)$ on the Grassmann variety $\Omega(m,n)$*.

The above defines the *minimum distance problem* of $\underline{z}(s)$ from the Grassmann variety. If $\{z_\omega(s), \omega \in Q_{m,n}\}, \{k_\omega(s), \omega \in Q_{m,n}\}$ are the Plücker coordinates of $\underline{z}(s)$ and $\underline{k}(s)$ respectively, then the corresponding coordinates of $\underline{z}'(s)$ are $\{z'_\omega(s) = z_\omega(s) + k_\omega(s), \omega \in Q_{m,n}\}$. Using Lemma 6.1, the minimum distance problem is then defined as:

Problem 1 (Minimum Distance Problem)**.** Define the minimum solution $\{k_\omega(s), \omega \in Q_{m,n}\}$ for the set of quadratic equations

$$q_i(s) = 0, \forall i \in \tilde{\rho}, q_i(s) \in \mathbb{R}[s] \text{ defined for } \{z'_\omega(s) = z_\omega(s) + k_\omega(s), \omega \in Q_{m,n}\} \tag{28}$$

The above problem may be tackled using analytical, as well as numerical techniques.

The resulting $\underline{z}'(s)$ will then be used for the estimation of distance from the GCD variety.

Remark 4. Given that the decomposable set $\underline{z}'(s) = \{z_i'(s), i \in \tilde{\rho}\}$ is a constrained version of the free $\{z_i(s), i \in \tilde{\rho}\}$, we expect that the distance from singularity of the free set to define a lower bound for the distance from singularity of $\underline{z}'(s) = \{z_i'(s), i \in \tilde{\rho}\}$. Of course, whenever $\{z_i'(s), i \in \tilde{\rho}\}$ is decomposable, then the two distances are equivalent, such that the vectors coincide.

Characterisation of the distance from the GCD variety is considered next.

7. "Almost" non-coprimeness of a set of polynomials: Background Results

In this section the problem of defining the GCD of a set of polynomials is reviewed first which leads to the study of "almost GCD" of a set of polynomials and thus enables the computation of the distance to the GCD variety. This will be first described for the simple, unconstrained case where there are no QPRs and then the results will be extended to the general case where the set of QPRs is non-trivial. In the latter case we will use the polynomial vector defined as the best approximation that satisfies the QPRs.

Following this we will deal with the invariant distance to uncontrollability of the polynomial $C_{n-p}(R(s)) = \underline{\tilde{r}}(s)^t$ and the results for the invariant distance to unobservability based on $\underline{\tilde{q}}(s)$ follow by duality.

7.1. Background Results on the GCD: The Sylvester Resultant

Consider the set of σ polynomials in $\underline{\tilde{r}}(s)$, where v denotes the maximal degree, $v = n - p$. This set may be denoted by

$$\begin{aligned} P_{\sigma,v} &= \{a(s)\} \cup \{b_i(s) \in \mathbb{R}[s],\ \ i = 1, \ldots, \sigma - 1, v = \deg\{a(s)\}, \\ & v \geq \deg\{b_i(s)\}, i = 1, 2, \ldots, \sigma - 1\} \quad (29) \\ \delta &= \max\{\deg\{b_i(s)\}, i = 1, \ldots, \sigma - 1\} \quad (30) \end{aligned}$$

We represent the polynomials $a(s), b_i(s)$ with respect to the largest degrees (v, δ) as:

$$\begin{aligned} a(s) &= s^v + a_{v-1}s^{v-1} + \cdots + a_1 s + a_0 \\ b_i(s) &= b_{i,\sigma}s^{\delta} + \cdots + b_{i,1}s + b_{i,0},\ \ i = 1, 2, \ldots, \sigma - 1 \quad (31) \end{aligned}$$

$P_{\sigma,v}$ will be called an (v, σ) order polynomial set and whenever we denote the number of elements and the maximal degree of a polynomial set we shall use this notation, otherwise the set of polynomials will be abbreviated as P. The GCD of P will be denoted by $\phi(s)$.

The classical approaches for the study of coprimeness and determination of the GCD makes use of the Sylvester Resultant (Fatouros & Karcanias, 2003), (Barnett, 1971), (Vardulakis & Stoyle, 1978).

Definition 7.1. Consider the set $P_{\sigma,v}$ of (29). We can define $\delta \times (v + \delta)$ matrix associated with $a(s)$:

$$S_0 = \begin{pmatrix} 1 & a_{v-1} & a_{v-2} & \dots & a_1 & a_0 & 0 & \dots & 0 \\ 0 & 1 & a_{v-1} & \dots & \dots & a_1 & a_0 & \ddots & \vdots \\ \vdots & & \ddots & \ddots & & & \ddots & \ddots & 0 \\ 0 & \dots & 0 & 1 & a_{v-1} & \dots & \dots & a_1 & a_0 \end{pmatrix} \tag{32}$$

and $v \times (v+\delta)$ matrix associated with $b_i(s), i = 1, 2, \dots, \sigma - 1$ as:

$$S_i = \begin{pmatrix} b_{i,\delta} & b_{i,\delta-1} & \dots & b_{i,0} & 0 & \dots & \dots & 0 \\ 0 & b_{i,\delta} & b_{i,\delta-1} & \dots & b_{i,0} & 0 & \ddots & 0 \\ \vdots & \ddots & \ddots & & & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & & & \ddots & 0 \\ 0 & \dots & 0 & b_{i,\delta} & b_{i,\delta-1} & \dots & \dots & b_{i,0} \end{pmatrix} \tag{33}$$

for every $i = 1, 2, \dots, \sigma - 1$. An extended Sylvester matrix or a Sylvester Resultant for the set P is then defined by:

$$S_P = \begin{pmatrix} S_0 \\ S_1 \\ \vdots \\ S_{\sigma-1} \end{pmatrix} \in \mathbb{R}^{(\delta+\sigma v-v)\times(v+\delta)} \tag{34}$$

The resultant provides a criteria for the evaluation of GCD and this is summarised by the results (Fatouros & Karcanias, 2003),(Barnett, 1971), (Vardulakis & Stoyle, 1978).

Theorem 7.2 (Generalised Resultant Theorem)**.** *Given the set of polynomials of (29) with a generalised resultant S_P, the following properties hold true:*

(1) The necessary and sufficient condition for a set of polynomials to be coprime is that

$$rank(S_P) = v + \sigma \tag{35}$$

(2) Let $\phi(s)$ be the GCD of P. Then

$$rank(S_P) = v + \delta - deg\{\phi(s)\} \tag{36}$$

(3) If we reduce S_P, by using elementary row operations, to its row echelon form, the last non-vanishing row defines the coefficients of the GCD.

The Sylvester Resultant result stated above is central in establishing a number of important computational procedures for the GCD of many polynomials (Mitrouli & Karcanias, 1993). Clearly, the set of polynomials is not coprime if S_P is singular, which occurs when the polynomial vector $\underline{\tilde{r}}(s)^t$ is on the GCD variety defined as:

Definition 7.3. For a polynomial vector $\underline{\tilde{r}}(s)^t$ the GCD variety is defined by the set of maximal order minors of S_P which are all are equal to zero. This variety is defined by a polynomial in parameters defined by the coefficients of the polynomials of $\underline{\tilde{r}}(s)^t$ specified in (31).

An obvious Corollary of the above result is:

Corollary 7.4. *The distance of the polynomial vector $\underline{\tilde{r}}(s)^t$ from the GCD variety is defined by the distance of Sylvester matrix S_P from the set of singular matrices.*

In the following section we will provide estimates for this distance.

7.2. Distance to the GCD variety for unconstrained and constrained polynomials

Important properties of the Sylvester resultant matrix, summarised in Theorem 7.2, characterise the distance of a set of polynomials from the GCD variety. The well-known results (Meyer, 2000) correspond to the fact that the distance of S_P to singularity is denoted by the smallest singular value of the matrix.

Corollary 7.5. *For the set of unconstrained polynomials the distance to the GCD variety is exact and defined by the smallest singular value of the Sylvester matrix S_P.*

If additional constraints are involved to satisfy the QPRs, then the smallest singular value of S_P is the lower bound of the distance. The problem we address is equivalent to finding the nearest common root of the polynomials that characterises the distance of S_P to singularity. Generically any arbitrary set of polynomials is coprime, thus the distance to coprimeness is an important notion. This is linked to defining and computing the “approximate” GCD and the notion of “almost zeros” (Mitrouli & Karcanias, 1993), (Karcanias & Mitrouli, 1994), (Fatouros & Karcanias, 2003), (Karcanias et al., 2006), (Karcanias & Halikias, 2013), (Barnett, 1971).

Calculation of the “approximate” GCD can be defined from various perspectives. One of the common approaches (Fatouros & Karcanias, 2003) aims to approximate the calculation of the GCD by reducing the residual error that arises after matrix factorisation. The computational procedure determines the nearest polynomial set with the common divisor of defined degree d. Such a degree can be evaluated as the sensitivity measure of the set of smallest singular values of the corresponding Sylvester matrix.

An alternative approach for the calculation of the "approximate” GCD, recently evaluated in (Halikias, Galanis, Karcanias, & Milonidis, 2012), (Limantseva, Halikias, & Karcanias, 2019), defines the distance problem as the standard structured singular value problem, considering the minimal norm perturbation in the coefficients of the nominal polynomials to the set of the polynomials with a common root. Such an approach uses powerfully notions of the μ-value in order to tackle highly structured nature of the Sylvester resultant.

First, consider the representation in (Fatouros & Karcanias, 2003) that evaluates distance of the polynomial set P from the GCD variety Δ_d with a GCD of degree d based on the matrix factorisation.

Theorem 7.6 ((Fatouros & Karcanias, 2003))**.** *Let $P_{\sigma,v}$ be a set of polynomials as in (29) and (30) with the largest degrees (v, δ) respectively and define a corresponding*

Sylvester matrix as S_P*. If* $\phi(s)$ *is the GCD of degree* d *and* $\Phi_\phi \in \mathbb{R}^{(v+\delta)\times(v+\delta)}$*, then*

$$\hat{S}^{(k)}_{P^*} = S_P \Phi_\phi = [0_d|\hat{S}_{P^*}] \tag{37}$$

or

$$S_P = \hat{S}^{(d)}_{P^*} \hat{\Phi}_\phi = [0_d|\hat{S}_{P^*}]\hat{\Phi}_\phi \tag{38}$$

where $\hat{\Phi}_\phi$ *is a Toeplitz representation of* $\phi(s) = \phi_d s^d + \cdots + \phi_1 s + \phi_0$*, defined as*

$$\hat{\Phi}_\phi = \begin{pmatrix} \phi_0 & 0 & 0 & \dots & \dots & \dots & 0 \\ \phi_1 & \phi_0 & 0 & & & & \vdots \\ \vdots & \ddots & \ddots & \ddots & & & \vdots \\ \phi_d & & \ddots & \ddots & \ddots & & \vdots \\ 0 & \phi_d & & \phi_1 & \phi_0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & & \ddots & \ddots & 0 \\ 0 & \dots & 0 & \phi_d & \dots & \phi_1 & \phi_0 \end{pmatrix},$$

i.e. $\Phi_\phi = \hat{\Phi}_\phi^{-1}$ *and the expanded resultant of coprime polynomials defined as* $\hat{S}^{(d)}_{P^*}$ *is obtained by division of the initial matrix* S_P *by the GCD. The matrix factorisation implies the following formulation of the distance minimisation problem:*

$$d(P,\Delta) = \min_{\forall P^*,\phi} \|S_P - [0_d|\hat{S}_{P^*}]\hat{\Phi}_\phi\|_F, \tag{39}$$

where $S_P, \hat{S}_{P^*}$ *are the Sylvester resultant matrices of the initial and reduced polynomial sets respectively and* $\hat{\Phi}_\phi$ *is a Toeplitz representation of a polynomial nearest common divisor* $\phi(s)$*,* $deg\phi(s) = d, 1 \le d \le max(v,\delta)$*.*

It can be observed that the reduced set $\hat{S}_{P^*}$ and corresponding Toeplitz matrix $\hat{\Phi}_\phi$ in (39) form the non-exact factorisation that leads to the residual error matrix. The minimisation of (39) is equivalent to the solution of the nonlinear least-squares problem, which is computationally hard to obtain. In (Ruhe & Wedin, 1980), (G. H. Golub & Pereyra, 1973), (G. Golub & Pereyra, 2003) the nonlinear least-squares problem is studied as a separable problem of two sets of parameters. It is argued that such an approach develops a robust procedure and guarantees the convergence in fewer iterations than the traditional non-linear least-squares methods. Following this, the problem in (39) can be simplified as a separable problem of two linear least-squares projections, where iterating between the two the minimum of (39) can be achieved. Such an approach is of the particular interest and is going to be evaluated in future work.

Since, the problem of "approximate" GCD is closely related to the coefficients of polynomials and the Sylvester matrix is a Toeplitz structure matrix, thus stronger criteria for the distance to singularity should be considered. Motivated by the recent results of the authors in (Halikias et al., 2012), (Limantseva et al., 2019) one can define the distance to non-coprimeness as an optimisation problem, where we seek to find the minimal magnitude norm perturbation in the coefficients.

Definition 7.7. Let $M \in \mathbb{R}^{n\times n}$ and consider the structured set of uncertainties as

$$\Delta = \{\text{diag}(\theta_1 I_{r_1}, \theta_2 I_{r_2}, \dots, \theta_s I_{r_s}) : \theta_i \in \mathbb{R}, i = 1, \dots, s\}, \tag{40}$$

where r_i are positive integers corresponding to the block-structure of Δ, i.e. $\sum_{i=1}^{s} r_i = n$. If there exists $\Delta \in \mathbf{\Delta}$, such that $\det(I_n - M\Delta) = 0$, then the structured singular value of M is:

$$\mu_\Delta(M) = \frac{1}{\min\{\|\Delta\| : \Delta \in \mathbf{\Delta},\ \det(I_n - M\Delta) = 0\}}, \tag{41}$$

If for all $\Delta \in \mathbf{\Delta},\ \det(I_n - M\Delta) \neq 0$ then $\mu_\Delta(M) = 0$ (Hinrichsen & Pritchard, 2005).

The simple case of a two-polynomial case, where Sylvester resultant is a square matrix, is presented in (Halikias et al., 2012). However, the structured singular value approach is considerably more difficult for an arbitrary number of polynomials. In the resent paper (Limantseva et al., 2019) an alternative way to approach the μ-problem of many polynomials is proposed. With the concept of structured singular values the set of perturbed polynomials P^*, such that

$$\begin{aligned} a(s) &= s^v + (a_{v-1} + \theta_{v-1})s^{v-1} + \dots + (a_0 + \theta_0), \\ b_i(s) &= (b_{i,\delta} + \epsilon_{i,\delta})s^\delta + \dots + (b_{i,0} + \epsilon_{i,0}),\ \ i = 1, \dots, \sigma - 1 \end{aligned} \tag{42}$$

form a reduced Sylvester matrix has the null space, such that $\text{null}(S_{P^*}) \geq 1$. Then formally define

$$\gamma = \max\{|\theta_0|, \dots, |\theta_{n-1}|, |\epsilon_{0,1}|, \dots, |\epsilon_{t-1,h}|\} \tag{43}$$

we seek to minimise γ so that the perturbed polynomials (42) have a common root. Equivalently the problem can be formulated as: $\inf\{\gamma : \text{null}(S_P) \geq d\}$ generalising the problem for the degree of d GCD.

8. Feedback Invariant Measures of Distance Uncontrollability and Unobservability

Combining characteristics of the input, output restriction matrix pencils $R(s) = sN - NA, Q(s) = sM - MA$ respectively and the notion of the distance to the GCD variety of a polynomial set, it is possible to specify invariant measures to uncontrollability and unobservability.

Corollary 8.1. *Let $P_{\sigma,v}$ be a set of polynomials, obtained from $C_{n-p}(R(s)) = \underline{\tilde{r}}(s)^t$. If $p = 1, n-1$, then the smallest singular value of a corresponding Sylvester Resultant $S_{P_{\sigma,v}}$ denotes the invariant distance to uncontrollability. If $p \neq 1, n-1$ and there exists such a multivector $\underline{k} \in \mathbb{R}^p$ that satisfies the QPRs, then the smallest singular value of a respective $S_{P_{\sigma,v}}$ defines the bound of the distance.*

Similar results can be defined for the distance to unobservability that follows by duality.

Corollary 8.2. *Let $P_{\sigma,v}$ be a set of polynomials, obtained from $C_{n-m}(Q(s)) = \underline{\tilde{q}}(s)$.*

For $m = 1, n-1$ the smallest singular value of $S_{P_{\sigma,v}}$ defines the invariant distance to uncontrollability.

Remark 5. For the case of $p \neq 1, n-1$ a Sylvester matrix $S_{P_{\sigma,v}}$ is structured from a corresponding multivector of the polynomials $\underline{k} \in \mathbb{R}^p$ that satisfy the QPRs, then the smallest singular value of $S_{P_{\sigma,v}}$ characterises the bound of the distance to uncontrollability, respectively unobservability.

It is obvious that constrained polynomials lead to a bounded result of the feedback invariant distance to uncontrollability or unobservability respectively. It can be observed that for an arbitrary system the problem of finding invariant measures of system properties can be narrowed to the analysis of the lower bound, where we have to satisfy the constraints of the Grassmann variety. Such observations form a basis of the future work and possible ways of finding the better bound of the distance to uncontrollability and unobservability for the design problem.

9. Conclusions

We have addressed an alternative approach for measuring fundamental properties of a system, namely controllability and observability, that are invariant under the state feedback and output injection respectively. The framework uses concepts of restricted input $R(s)$ and restricted output $Q(s)$ pencils in order to develop criteria for evaluating the distance to uncontrollability, unobservability. By studying properties of the invariant polynomials of a system, $\underline{\tilde{r}}(s)^t, \underline{\tilde{q}}(s)$, it is shown that the problem is equivalent to the distance of a set of polynomials $P_{\sigma,v}$ to the GCD variety that is reduced to the two special cases: LIDP, where polynomials are decomposable and the QPRs are satisfied, and GIDP that requires additional optimisation of the polynomials in order to satisfy the QPRs.

The invariant distance to uncontrollability (unobservability by the duality) is characterised by the smallest singular value of the corresponding Sylvester matrix $S_{P_{\sigma,v}}$, for the decomposable case and a lower bound for the general case. For the case where the invariant polynomial vector is not decomposable, we are using its best approximation on the Grassmann variety. In this case the decomposable set provides a lower bound for the distance to uncontrollability, unobservability. Working out efficient methods for computing the best approximation of an invariant zero polynomial vector to the Grassmann variety remains an open question which is a topic for further research.

Overall, proposed framework defines invariant criteria for measuring systems properties that can be used as an alternative measure prior control design. Hence, computation of the invariant distance to uncontrollability, unobservability can lead to the optimal control design and input/output structure selection that is the going to be studied in the future work.

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