A Model of Investment under Uncertainty with Time to Build, Market Incompleteness and Risk Aversion

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Abstract

In this paper I develop a theoretical framework of irreversible investment under uncertainty in which there is time to build (TTB). The novel aspects of this framework, compared with TTB models in the extant literature, are that the market is incomplete in that not all the uncertainty associated with investing can be diversified away by trading an appropriate spanning asset, and the decision-maker, who acts in the interest of an organisation, is risk averse. I show that models of investment under uncertainty with a TTB, and models of investment under uncertainty with market incompleteness and risk aversion, ought not to be mutually exclusive as they have been in research to date because the recognised results of market incompleteness and risk aversion on the optimal investment strategy are challenged when we incorporate a TTB feature. Conversely, there are also implications on the effect of a TTB when we incorporate market incompleteness and risk aversion. The framework I develop in this paper provides a robust and parsimonious means of facilitating this.

Keywords: Finance, Investment under Uncertainty, Time to Build, Market Incompleteness, Risk Aversion.

1 Introduction

Over the past few decades, the literature on irreversible investment under uncertainty has been useful in helping decision-makers acting in the interest of an organisation to determine the optimal time to invest in a project in which there is uncertainty over the future value of its payoffs.[1] However, there are many projects for which the total cost of investing in the project carries more uncertainty than its payoffs. These are projects for which the irreversible cost is paid sequentially over time rather than in one fixed lump sum. In this sense, there is a "time to build" (TTB) which spans the period between the first and final investment payout, and the literature on TTB focuses on the relationship between the decision to invest (and possibly suspend investment) in the project and the selling price of the completed product. Examples of such projects include the development of an infrastructure project, the manufacturing of items such as aircraft, vehicles, or machinary, and pharmaceutical research and

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[1] See, for example, Dixit and Pindyck [6], Brennan and Schwartz [3] and McDonald and Siegel [17] to name but a few.

development. These investment projects are typically characterised by the following features: (i) cost outlays occur sequentially over time; i.e., there is a TTB, (ii) there is uncertainty over the total cost that the investment will incur, (iii) there is uncertainty over the length of time it will take to develop the product fully and (iv) the payoffs only accrue once the development of the product has been fully completed.

The most salient characteristic of investment projects with a TTB is that two different types of operational uncertainty arise: one being internal to the organisation and the other being external. *Internal uncertainties* lie within the organisation and can be related to the physical difficulty of completing a project; i.e., how much time, effort, and materials will ultimately be required, the quality of the raw materials bring used, machine failures, etc. This type of operational uncertainty is reminiscent of the uncertainties described as 'technical cost uncertainty' by Pindyck [24], 'performance variability' by Huchzermeier and Loch [13] and 'internal uncertainty' by Grote [9]. Typically, uncertainties that lie within the organisation are largely independent from the overall economic environment and, hence, there is idiosyncratic risk associated with waiting to invest in such a project which cannot be diversified away through hedging.

On the other hand, *external uncertainties* arise from the external environment of the organisation and may relate to, for example, fluctuating economic conditions (cf. 'input cost uncertainty' of Pindyck [23]), changing customer demands/demographic changes or emerging competitors/substitute products ('situational uncertainty' of Grote [9] and 'market payoff variability' of Huchzermeier and Loch [13]). This source of uncertainty is typically correlated with the economic environment and, hence, is at least partly diversifiable.

The TTB feature has been incorporated into models of investment under uncertainty in a number of ways, both in the finance literature and the operations research literature. Pindyck [24] models the expected remaining capital expenditure to the completion (RCEC) of the project as a state variable in which he distinguishes between technical and input cost uncertainty. Pennings and Sereno [22] apply their approach to the valuation of a pharmaceutical firm's R&D investment projects. However, they model the technical cost uncertainty as a Poisson-type jump process, in which if a drug candidate fails for any reason (e.g. has harmful side-effects), investment in the project is abandoned. Majd and Pindyck [16] also model the RCEC as a state variable, but do not distinguish between the types of cost uncertainty. Instead, they let the future payoff evolve stochastically over time. Milne and Whalley [20] apply their model to the problem of understanding the aggregate dynamics of "work in progress" held by manufacturing firms of, for example, ships, vehicles and machinery. Schwartz and Moon [27] apply the RCEC as a state variable approach to R&D projects in the pharmaceutical industry and Hsu and Schwartz [12] to the design of research incentives. Thijssen [30], on the other hand, considers a model in which the RCEC is not a state variable but, instead, the decision to begin investing is based on the current prediction of revenues upon completion at an unknown future date, while Kort et al. [15] examine the tradeoff between completing an investment project in one stage at a chosen time or in stages that can be completed at optimally chosen points in time.

These studies highlight the extent and relevance of irreversible and uncertain investment projects in which there is a TTB and, moreover, the importance of using robust modelling techniques which incorporate TTB uncertainty into the valuation of such projects for improving managerial decision making.

However, they base their analyses under the assumptions of risk neutral decision-makers and complete capital markets, where all uncertainty can be perfectly hedged away. These assumptions are not very relevant to most real-world investments, and especially to investments with a TTB, because the internal uncertainty this feature generates cannot be easily diversified away through hedging. Moreover, unless the organisation is large and very well diversified, the decision-makers acting in the interests of the shareholders are typically risk-averse. According to recent statistics, small businesses accounted for 99.3% of all private sector businesses in the UK at the start of 2017, and 99.9% were small or medium-sized (SMEs). SMEs account for at least 99.5% of the businesses in every main industry sector, and nearly a fifth of all SMEs operate in construction compared with less than 1% in the Mining, Quarrying and Utilities sector.[2] Therefore, the majority of businesses in the UK operate in incomplete markets with high levels of idiosyncratic risk associated with their investments.

To address this issue, the main contribution of my paper is that I develop a theoretical framework for a TTB model of investment under uncertainty in an incomplete market in which the decision-maker of an organisation cannot offset all of the uncertainty associated with the investment through hedging. As such, he is risk averse and requires compensation for both diversifiable and non-diversifiable uncertainty. Similar to Pindyck [24], the RCEC is a state variable which distinguishes between internal and external cost uncertainty, and costs are paid sequentially over time. To offset some of the uncertainty associated with the investment, the decision-maker also invests in a hedge portfolio that is partially correlated with the RCEC of the project. Furthermore, investment can costlessly and temporarily be suspended during this period should the expected RCEC rise significantly above its payoff. When the development of the project has been fully completed, the organisation (in whose interest the decision-maker is acting) receives a one-off lump-sum payment.[3] The overall objective of the decision-maker is to find the optimal time to invest in the project, as well as the optimal rate at which to invest and an optimal hedge position, so that his discounted expected utility from investing in the project and the hedge portfolio is maximised. Hence, a combined optimal control and stopping problem is solved for and I refer to this problem set-up as the General Model, hereafter.

I also derive a complete markets (C-M) TTB model in which I show, via a change of measure reminiscent of the equivalent martingale measure from the asset pricing literature (cf. Shreve [28]), that it is a special case of the General Model when the market is complete and the decision-maker is risk neutral. Hence, the C-M model can be used as a benchmark against which to compare the General Model in order to understand the impacts of market incompleteness and risk aversion on investments that take TTB.

There are novel results generated by the model from a bidirectional perspective. The results provide a relevant contribution to the finance literature on irreversible investment under uncertainty, as well as complementing some existing results on operational flexibility and hedging in the operations research literature. This is discussed in detail in the section on related literature.

First, I show that market incompleteness and risk aversion impose less stringent investment criteria (equivalently, greater operational flexibility) in a TTB setting than the investment criteria imposed by an identical model under the assumptions of market completeness and risk neutrality. The reason for this is that the concave utility function for the risk averse decision-maker gives an optimal rate of investment that evolves with the RCEC, but the optimal investment rate for the risk neutral decision-maker with a linear utility function is fixed at the maximum rate at which he can productively invest.

[2] The data was obtained from the Department for Business, Energy and Industrial Strategy's 2017 business population estimates for the UK, available at *https://www.fsb.org.uk/media-centre/small-business-statistics*

[3] For example, a real estate entrepreneur who owns some land will decide when to develop that land by building, say, a shopping centre complex on it. He can choose to sell the complex or continue to manage it once it's complete. Assuming a lump-sum payment implies he takes the (exogeneous) decision to sell the complex once complete.

In this way, the risk averse decision-maker has less stringent investment criteria because he can choose to exacerbate or alleviate his investment depending on the RCEC at each point in time.

From the contrary perspective, the TTB feature implies that my model falls between two extreme versions of models of investment under uncertainty with market incompleteness and risk aversion. On one hand, Henderson [10] and Hugonnier and Morellec [14] develop such a model in which they assume that once the investment option is exercised, the payoff is received as a fixed lump sum and the investment problem immediately ends. On the other hand, Miao and Wang [19] assume the payoff is received as a flow over an infinite horizon once the investment option is exercised. However, in my model, contrary to the lump sum case, the investment problem does not end after the investment option is exercised since the cost is paid sequentially over time, but once all the cost has been paid after a finite time, the payoff becomes a fixed lump sum. As such, I find that the effects of market incompleteness and risk aversion on the optimal time to invest in irreversible and uncertain projects also fall between the effects they have in these two extreme versions of such models. Similar to the flow case of Miao and Wang [19], risk aversion always delays optimal investment in my model. Moreover, I find that the optimal investment threshold is concave in the degree of market incompleteness. However, in the flow payoff case of Miao and Wang [19], market incompleteness always delays optimal investment, but in the fixed payoff versions of Henderson [10] and Hugonnier and Morellec [14], market incompleteness always expedites it. In this way, this paper contributes to the literature on irreversible investment under uncertainty by providing a novel perspective, via the TTB feature, on the effects of market incompleteness and risk aversion on the optimal investment strategy for such projects. The relationship between my model and those in the existing literature is discussed in greater detail in Section 2.

To summarise, this paper highlights the fact that models of investment under uncertainty with a TTB and models of investment under uncertainty with market incompleteness and risk aversion ought not to be mutually exclusive as they have been in research to date for the following reasons. (i) The assumptions of market completeness and risk neutrality are not very relevant to real world investments with a TTB and (ii) the recognised effects of market completeness and risk neutrality are challenged when we incorporate the TTB feature, and also vice versa: (a) the effect of market incompleteness and risk aversion on the optimal investment strategy for a flow payoff and for a lump sum payoff are demonstrated by Miao and Wang [19] and Henderson [10], respectively. However, by adding the TTB, the effects are neither that of a flow or a fixed lump-sum, but somewhere in between. (b) On the other hand, risk aversion implies the optimal investment rate is variable over time rather than bang-bang and, as such, market incompleteness and risk aversion impose less stringent criteria than the TTB model without such features. Therefore, effort should be made to account for risk aversion and market incompleteness in models where there is a TTB, and vice versa. The theoretical framework I develop in this paper provides a robust and parsimonious method for facilitating this.

Finally, it is worth noting that the seminal TTB models of Pindyck [24] and Majd and Pindyck [16] have typically advanced in the following way. Bar-Ilan and Strange [2] model a sequential investment in two stages and where each stage takes TTB. Similarly, Adkins and Paxson [1] model an investment opportunity as a set of distinct, ordered investments that have to be made before the project is complete and its benefits realised. Each *stage* takes TTB. In contrast to these models, investment in my model proceeds continuously at an (endogenous) optimal rate until the project has been completed, unless suspended in the meantime. The TTB involves the time from beginning to end of the project as opposed to from the beginning to the end of each stage. In that sense, compared to these models, my model is essentially a one-stage simplistic version of these models. The distinct stages approach is very useful in highlighting issues such as whether exploratory investment is beneficial for certain projects. It is not necessary to order stages in this way when exploring the impact of incorporating market incompleteness and risk aversion into a TTB model of irreversible investment under uncertainty. However, given the important impact market incompleteness and risk aversion are shown to have, these features ought to be incorporated into models involving multi-stage investments in which each stage takes TTB, and is a worthwhile extension for further research.

The remainder of the paper is organised as follows. In the next section I discuss how my model contributes and relates to those in the existing literature. In Section 3 I present the General Model (G-M) and outline its solution, while in Section 4 I do the same for the complete markets (C-M) case. In Section 5 I present the results, and in Section 6 I provide a discussion on the economic implications that are generated by the model. Section 7 concludes. All proofs are placed in the Appendix.

2 Related Literature

In this section I discuss the contribution of the methodology I develop in the paper in relation to previous research from two perspectives. In particular, I point out that by incorporating a TTB feature, there are implications for the recognised results of the effects of market incompleteness and risk aversion on investment, but also vice versa.

First, I discuss the effect of incorporating market incompleteness and risk aversion into a TTB model. Second, I discuss the impact of the TTB feature on existing results on the effects of market incompleteness and risk aversion on the optimal time to irreversibly invest in projects with uncertain payoffs in which there is no TTB.

The models most closely related to mine from the TTB perspective are those of Pindyck [24] and Majd and Pindyck [16]. However, the crucial difference is that in their models, all uncertainty over the RCEC can be diversified away by trading an appropriate spanning asset and, as such, the decision-maker is risk neutral. They find that the optimal rate of capital investment is of the bang-bang type; invest at some (constant) maximum intensity or not at all, which agrees with the result from my C-M model. However, by incorporating risk aversion, I find that the optimal rate of capital investment is not of the bang-bang type and at each point in time depends on the RCEC. This difference arises from the linear versus concave utility functions describing the decision-maker's tolerance for risk, and has significant implications for the optimal timing strategy. In particular, as I discuss in Section 3.3, the concave utility function ensures that all non-zero optimal rates of investment are such that the implicit dividend loss from suspending the investment is sufficiently high to make suspending suboptimal, whereas this is not always the case for a risk neutral decision-maker with a linear utility function. Thus, incorporating the features of market incompleteness and risk aversion yield less stringent investment criteria than an identical TTB model without these features.

In terms of risk aversion and market incompleteness, my model is most closely related to those of Henderson [10] and Hugonnier and Morellec [14] who find that higher levels of risk aversion and market incompleteness make early investment optimal. However, they do not consider the TTB feature in their models. Hence, the crucial difference that arises between those models and mine is that in theirs, once the investment is made, the lump sum payoff is received immediately and the investment problem ends. In my model, however, the investment problem is ongoing even when the investment has been initiated. This implies that, unlike Henderson [10] and Hugonnier and Morellec [14], investment in my model does not eliminate the non-diversifiable uncertainty faced by the organisation. Miao and Wang [19] do consider the impact of market incompleteness and risk aversion on the optimal time to make an irreversible investment but such that the payoff is received as a flow over time after the investment option has been exercised. This gives rise to opposite results on the effects of market incompleteness and risk aversion to Henderson [10] and Hugonnier and Morellec [14] because, similar to my model, investing in the flow case does not eliminate non-diversifiable uncertainty completely. However, the flow payoff in Miao and Wang [19] is defined over an infinite horizon, whereas the flow payoff in my TTB model is finite and ends once the development is complete. Hence, my TTB model has a flow payoff post initial investment, but for a limited time, after which the payoff is a fixed lump sum. In this way, it contrasts with the existing models of investment under uncertainty which examine the effects of market incompleteness and risk aversion. Importantly, my results also fall between those of the flow payoff models and the fixed payoff models. Specifically, similar to Miao and Wang [19], I find that risk aversion always delays investment and that market incompleteness delays investment *but only for high levels of market incompleteness.* However, for *lower levels of market incompleteness, market incompleteness speeds up investment* similar to the result of Henderson [10] and Hugonnier and Morellec [14].

3 The General Model

Consider an organisation whose decision-maker has the option to irreversibly invest in some project that takes TTB. The project could, for example, be the production of some fixed size output to be sold on an open market. This implies that the cost of production is paid sequentially over time. At the time of investing, the decision-maker is uncertain about the total costs the project will end up incurring. The uncertainty arises as two types: (i) internal cost uncertainty which cannot be diversified away through hedging and (ii) external cost uncertainty which is, at least, partly diversifiable. Note that throughout the paper, I use the terms internal cost uncertainty and idiosyncratic risk interchangeably, and the terms external cost uncertainty and systematic risk interchangeably. To offset the uncertainty that can be diversified away, he invests in a hedging portfolio comprised of a risk free bond and a risky asset that is partially correlated with the remaining capital expenditure to completion (RCEC). Furthermore, during this period of investing, production can be temporarily suspended at no cost should the expected remaining capital expenditure to completion (RCEC) of the investment rise substantially. When the development of the product has been fully completed, the output is sold and the organisation receives a one-off lump-sum payment, V, where V is known with certainty.[4] Note that the objectives of the decision-maker and the organisation's shareholders are assumed to be perfectly aligned, so there are no issues of agency costs in this model. Hence, the terms 'organisation' and 'decision-maker' can be interpreted as one and the same throughout the paper.

The objective of the decision-maker is to find the optimal time to invest in the product, the optimal rate at which to invest at each point in time during the production process, as well the optimal amount to invest in the risky asset, also at each point in time during the production process, so that the organisation's discounted expected utility from investment in the project is maximised.

[4] I make this assumption for simplicity and clarity in order to focus on effects of market completeness and risk aversion.

3.1 Cost Dynamics

Let time be continuous and indexed by $t \geq 0$. Uncertainty over the RCEC is represented by a probability space $(\Omega, \mathcal{F}, P)$, endowed with a filtration $(\mathcal{F}_t)_{t\geq 0}$. The RCEC, denoted by $(K_t)_{t\geq 0}$ is adapted to the filtration and follows a controlled diffusion process under P of the form

$$dK = -Idt + \rho\widetilde{h}(I,K)dW^M + \sqrt{1-\rho^2}\widehat{h}(I,K)dZ. \tag{1}$$

where $I_t > 0$ denotes the rate of investment at time t and, as such, K_t/I_t is the time it takes to complete the project at time t, for all t. $(\rho dW^M + \sqrt{1-\rho^2}dZ)$ is the increment of a standard P-Brownian motion correlated with $(W_t^M)_{t\geq 0}$ with correlation coefficient $\rho \in [-1, 1]$ and such that $(Z_t)_{t\geq 0}$, independent of $(W_t^M)_{t\geq 0}$. Therefore, the uncertainty associated with K is expressed in terms of the diversifiable and non-diversifiable uncertainty.

The uncertainty that is, at least, partly diversifiable, W^M, is not entirely independent of the economic environment. It is termed *external cost uncertainty* and relates to changes that are exogenous to the organisation; for example, fluctuating economic conditions, emerging competitors etc. Another feature of this uncertainty is that if stochastic changes in K are due to pure external cost uncertainty, then even if investment is not actively taking place, the dynamics of the RCEC are still affected by this source of uncertainty. Therefore, we require that $\widetilde{h}(0,K) > 0$ (Pindyck [24]).

On the other hand, the non-diversifiable uncertainty, Z, termed *internal cost uncertainty*, arises from the possible physical endogenous changes that may occur during the period of investing, such as, for example, changes in the amount of time, materials, etc that may ultimately be required or machine failures. This type of uncertainty can only affect the cost dynamics when investment is actively taking place and, hence, if stochastic changes in K are due to pure internal cost uncertainty, $\widehat{h}(0,K) = 0$.

In my model, I want the stochastic changes in the RCEC to be attributable to both sources of uncertainty, with pure internal cost and pure external cost uncertainty as extreme cases. Hence, I need to provide a structure on the uncertainty parameter to ensure the conditions I describe are met. A further condition that must be satisfied is that instantaneous variance of dK should be increasing K and bounded for all $K < \infty$ (cf. Pindyck [24]). This implies that as the project nears completion and the RCEC tends to zero, so too should the instantaneous variance. I adopt the specification of Pindyck [24] and let $\widehat{h}(I,K) := \theta K\,(I/K)^{\widehat{\nu}}$ and $\widetilde{h}(I,K) := \theta K\,(I/K)^{\widetilde{\nu}}$ for $\widehat{\nu}, \widetilde{\nu} \geq 0$ and $\theta \in [0,1]$ a constant. He shows that the instantaneous variance for such a structure on the uncertainty parameter is given by $V(K) = 2K^2\left(\frac{\theta^2}{2-\theta^2}\right)$ and, hence, we can only be certain that the variance increases in K for $\widehat{\nu} < 1/2$ and $\widetilde{\nu} < 1/2$. Hence, I restrict $\widehat{\nu}$ and $\widetilde{\nu}$ to the interval $[0, 1/2]$.

Now, for $\widetilde{h}(I,K) = \theta K\,(I/K)^{\widetilde{\nu}}$, the condition that pure external cost uncertainty can only arise if $\widetilde{h}(0,K) > 0$ will be satisfied iff $\widetilde{\nu} = 0$. This implies that the opposing extreme case of pure internal cost uncertainty should be satisfied for $\widehat{\nu} = 1/2$.

Allowing for both types of uncertainty to be represented, I let the RCEC dynamics under P be represented by the following equation

$$dK = -Idt + \theta\rho K dW^M + \theta\sqrt{IK(1-\rho^2)}dZ, \tag{2}$$

where W^M and Z correspond to external cost and internal cost uncertainty, respectively.

Note that prior to investment, the RCEC is a martingale. This is because when there is no active investment, the rate of investment I is zero. In that case, $dK = \theta\rho K dW^M$ implying that the dynamics are entirely governed by the external uncertainty such as, for example, changing economic conditions, cost of labour and materials, and so on. Hence, K will change irrespective of the progress of completing the project (Pindyck [24], pp.8).

3.2 Wealth Dynamics

Market incompleteness arises because not all of the uncertainty associated with the RCEC can be diversified away by trading an appropriate spanning asset. However, during the investment process, it can be partially hedged by trading (each period) in a correlated asset, hereafter referred to as the market asset and denoted by $(M_t)_{t\geq 0}$, and a riskless bond with constant interest rate $r > 0$, both of which are adapted to $(\mathcal{F}_t)_{t\geq 0}$.

Let ω_t denote the amount invested in the market asset at time t, and let $(X_t)_{t\geq 0}$ denote the organisation's *discounted* wealth from investment in the market asset and the riskless bond, discounted by the bond.

The dynamics of the wealth process under P are therefore given by

$$dX = \omega_t \frac{dM}{M} \tag{3}$$

such that

$$\frac{dM}{M} = \alpha_M dt + \sigma_M dW^M, \tag{4}$$

where α_M is the drift rate, σ_M the volatility of the market asset, and $(W_t^M)_{t\geq 0}$ a standard P-Brownian motion. Let $\lambda := \alpha_M/\sigma_M$ represent the instantaneous Sharpe ratio of the market asset.

3.3 The Investment Problem

There are three decisions to be made by the decision-maker: (i) how much to invest in the market asset at each period during the investment interval, (ii) when to exercise the investment option, and (iii) when the option has been exercised, the rate at which to invest at every period until the product is complete, so that his utility from investing in the project is maximised. Importantly, during the investment process, the wealth dynamics are given by $dX_t = \omega_t \frac{dM_t}{M_t} - I_t dt$, such that dM_t/M_t is given by Eq. (4). The problem for the decision-maker is thus to solve the following combined control and stopping problem:

$$F(X, K) = \sup_{0\leq\tau} \sup_{(\omega_s)} \sup_{(I_s)} \mathbb{E}_0^P \Big[\int_\tau^{\widetilde{\tau}} e^{-\zeta s} U_s \left(-I_s\right) ds + e^{-\zeta \widetilde{\tau}} B_{\widetilde{\tau}}(X_{\widetilde{\tau}} + V) | X_0 = W, K_0 = K \Big] \tag{5}$$

where τ denotes the time the investment option is first exercised and $\widetilde{\tau} := \inf\{t \geq \tau | K_t = 0\}$ denotes the time the production is complete and all costs have been paid. When all costs have been paid, the investor's wealth is comprised of his wealth from investing in the risky asset plus the lump sum payoff from the project; i.e., $X_{\widetilde{\tau}} + V$. The term $B_{\widetilde{\tau}}(X_{\widetilde{\tau}} + V)$ is the 'bequest function', which I assume is also concave in his wealth. Preferences are represented by the exponential utility function $U(x) = -1/\eta e^{-\eta x}$ $(\eta > 0)$, and $\zeta := -\frac{1}{2}\lambda^2$ denotes the decision-maker's subjective discount rate.

Before continuing, a brief explanation of the form of the objective function (5) and choice of subjective discount rate $\zeta = -\lambda^2/2$ is deserved. Prior to the completion of the product at $\widetilde{\tau}$ (i.e., at each time t during the investment interval $[\tau, \widetilde{\tau}]$), the decision-maker invests in the hedging portfolio to partially offset some of the uncertainty associated with the total investment outlay required for the project to reach completion. After the project is complete, the decision-maker then solves a portfolio choice problem to invest his organisation's total wealth over an infinite horizon in the style of Merton [18]. Therefore, as explained in Henderson [10] and Henderson and Hobson [11], the problem must be formulated in such a way that the portfolio choice problem does not influence the decision-maker's optimal choices of exercise time and investment rate in this specific project. In other words, the discount rate must be such that the DM's choices would be consistent with his optimal choices of exercise time and rate in the absence of the portfolio choice problem; i.e., in a complete market, risk neutral set up. This becomes apparent in Proposition 3 later on. While it is demonstrated in their papers that a choice of subjective discount rate for exponential utility of $\zeta = -\lambda^2/2$ ensures such consistency, for completeness I derive it again for my specific problem in Appendix A. Note that the risk-free rate r does not appear explicitly as a function of ζ because the dynamics of his hedge portfolio, described by Eq. (3) is given in discounted terms. If the choice of discount factor was chosen such that the dynamics of the portfolio were not given in discounted terms, then it would be $\zeta = -r - \lambda^2/2$.

Finally, I remark on the specification of the utility function. To remain as close as possible to the two related papers by Henderson [10] and Miao and Wang [19], I assume that the decision-maker's preferences are represented by an exponential utility function exhibiting constant absolute risk aversion (CARA). This specification eliminates the effect of wealth from the investment problem reducing it from a two-dimensional to a one-dimensional problem which is much easier to solve analytically. While other concave specifications could be used to represent preferences, it would be at the expense of increased dimensionality and the likelihood of obtaining an analytical solution is slim. Nonetheless, it is highly unlikely that the main conclusions of my paper (in terms of the qualitative impact of market incompleteness and risk aversion) would be altered by assuming a different specification.

3.4 The Optimal Investment Solution for the General Model

Theorem 1. *Let* $\beta_1 = 1 + 2\frac{\lambda}{\theta\rho}$.

1. *If* $\beta_1 < 0$, *the optimal investment time* τ^* *is given by*

$$\tau^* = \inf\{t | K_t \leq K^*\},$$

where K^* *denotes the optimal investment trigger and is given by*

$$K^* = \left(\frac{\beta_1(1-\beta_1)}{3\beta_1 - 2}\right) V \left[1 + \frac{\beta_1}{2}\theta^2(1-\rho^2)\left(1 - \eta\left(\frac{(1-\beta_1)}{3\beta_1 - 2}\right)V\right)\right]. \tag{6}$$

The optimal rate of investment is given by

$$I^* = \left[\frac{1}{\eta}\ln\left(\frac{1}{2}\theta^2(1-\rho^2)KF_{KK}(K) - F_K(K) - F_X(K)\right)\right]. \tag{7}$$

Finally let $F(X,K)$ *denote the value function for the investment problem and this is given by*

$$F(X,K) = \begin{cases} -\frac{1}{\eta}e^{-\eta\left(X+V-\left(\frac{\beta_1}{3\beta_1-2}\right)V\left(\frac{K}{K^*}\right)^{1-\beta_1}\right)} & \text{for } K \leq K^* \\ -\frac{1}{\eta}e^{-\eta\left(X+2\left(\frac{\beta_1-1}{3\beta_1-2}\right)V\left(\frac{K}{K^*}\right)^{\beta_1}\right)} & \text{otherwise.} \end{cases} \tag{8}$$

2. *If* $\beta_1 \geq 0$, *investment will be postponed indefinitely so that the optimal rate of investment is zero*

for all $K \geq 0$, and the associated value function for the investment problem is given by

$$F(X,K) = -\frac{1}{\eta}e^{-\eta X}, \ \forall K \geq 0. \tag{9}$$

Proof. See Appendix B. ■

We see from the theorem that there are some parameter values for which it is never optimal to invest (i.e. for parameter values such that $\beta_1 \geq 0$) and the option is kept alive indefinitely, and others for which there is a finite threshold below which it is optimal to invest (i.e., for all parameter values such that $\beta_1 < 0$). Hence, the investment option will be exercised if, and only if, $\lambda\rho < -\frac{1}{2}\theta\rho^2$, and not otherwise.

To interpret this condition more clearly and to facilitate a comparison with the corresponding result of Henderson [10], I re-write $\beta_1 = 1 - \frac{2(0-\lambda\rho)}{\theta\rho^2}$. Note that 0 corresponds to the Sharpe ratio of the RCEC when the investment is not actively taking place. Therefore, the implicit proportional dividend yield from investing, denoted by δ^Y, is simply $\delta^Y = \lambda\rho$. Thus, investment will only ever be optimal if $\delta^Y < -\frac{1}{2}\theta\rho^2$, which corresponds to a similar condition in Henderson [10]. The reversal in the inequality sign arises because the option to invest in my paper is analogous to an American put, whereas it is analogous to an American call in her paper. The intuition is that the option to invest should be exercised when the risky hedging asset offers a better opportunity to achieve higher payoffs in the future than that of the project. For this condition to hold, we let $\rho < 0$ owing to the implausibility of the Sharpe ratio λ being negative. Thus, hereafter, the market is complete for $\rho = -1$.

Note, moreover, that the investment condition in the standard real options model with no TTB, market completeness, and risk neutral decision-makers is for the dividend yield to be positive when the option is a call (correspondingly negative for a put) (cf. Dixit and Pindyck [6], for example). The condition in my model is more stringent than this, but the corresponding condition in Henderson [10] is less stringent. The explanation for this is given in Section 5.

The above intuition is based on the assumption that the investment is inactive. However, if the investment is actively taking place, then the decision-maker can suspend development if the RCEC gets too high or, equivalently, if the implicit proportional dividend *loss* from suspension is low. The implicit proportional dividend loss, denoted by δ^L, is represented by $\delta^L = -(I^* + \lambda)\rho$. This loss is small when I^* is low and suspension is more valuable than investment in the long-run. Once the investment is suspended, the decision-maker re-acquires his option to wait and, therefore, from the perspective of an inactive decision-maker, if the value of the option to suspend exceeds the value of investing, the investment option will never be exercised. Thus, the condition on whether investment will ever be optimal is dependent on $\delta^Y = \lambda\rho$ in both active and inactive investment regions.

Finally, the condition is independent of I^* in both regions, which is not the case in the complete, risk-neutral C-M case. The reasoning here is that I^* is dependent on K (see Eq. (7)) and, therefore, for any level of the RCEC, the optimal rate will always be such that the implicit dividend loss from suspending will be sufficiently high (so that active investment will be optimal for $I^* > 0$). Hence, the condition on investment does not need to be given in terms of this optimal rate for the General model. However, in the C-M model, the condition on investment (cf. Section 4) is not independent of the optimal investment rate k. The reason for the dependence on k in the C-M case is that the optimality condition on the rate at which to invest is different from that in the General model. In particular, the utility function in the former model is optimised for investment at the maximum rate at which it can productively occur k, or not at all. However, as discussed, the optimal rate of investment is dependent on the RCEC in the General model and adapts for every level of K. This difference arises from the linear versus concave utility functions representing the preferences of the risk neutral and risk averse decision-maker, respectively. The implications of this are explored in more detail in Section 5 below.

4 Complete Markets Case

The complete markets (C-M) case against which I compare the General Model (G-M) is that of a classic risk neutral investment problem with TTB in which the uncertainty associated with the RCEC can be perfectly diversified away by trading in the market asset. Thus, since $\rho < 0$ in the G-M case, I let the market be complete for $\rho = -1$ and the investor is risk neutral for $\eta = 0$. The associated optimal stopping problem is given by

$$\widetilde{F}^*(K) = \sup_{0 \leq \tau} \sup_{(I_u)_{\tau \leq u \leq \widetilde{\tau}}} \mathbb{E}_0^P \left[\widetilde{\Lambda}_{\widetilde{\tau}} V - \int_{\tau}^{\widetilde{\tau}} \widetilde{\Lambda}_{s-\tau} I_s ds \Big| K_0 = K \right], \tag{10}$$

where τ and $\widetilde{\tau}$ are the investment and completion times respectively, and $(\widetilde{\Lambda}_t)_{t \geq 0}$ denotes the decision-maker's subjective discount factor which is adapted to the filtration $(\mathcal{F}_t)_{t \geq 0}$, and which follows the diffusion

$$\frac{d\widetilde{\Lambda}}{\widetilde{\Lambda}} = -\widetilde{\alpha}_\Lambda dt - \widetilde{\sigma}_\Lambda W^M$$

under P. $\widetilde{\Lambda}_0 = 1$.

Proposition 1. *There exists a probability measure Q, equivalent to P, such that*

$$\frac{dQ}{dP} := e^{-\frac{1}{2}\lambda^2 t - \lambda W_t^M} \tag{11}$$

such that under Q, the dynamics of $(K_t)_{t \geq 0}$ are given by

$$dK = -\left(I - \theta \lambda K\right) dt - \theta K d\widetilde{W}^M,$$

where $(\widetilde{W}_t^M)_{t \geq 0}$ is a Brownian motion under Q and λ is, as previously defined, the Sharpe ratio of the market asset.

Furthermore, the discount factor in the Complete Markets model is given by

$$\widetilde{\Lambda}_t = e^{-rt} \frac{dQ}{dP}. \tag{12}$$

Proof. See Appendix C. ■

Let $k > 0$ denote the maximum rate at which the decision-maker can productively invest. The solution to the optimal stopping problem (10) is stated in the following proposition.

Proposition 2. *Let $\widetilde{\beta}_1 = 1 - 2\frac{\lambda}{\theta} + 2\frac{k}{\theta^2 K}$.*

1. *If $0 < \widetilde{\beta}_1 < \frac{2k}{\theta^2 K}$, the optimal investment time τ^* is given by*

$$\tau^* = \inf\{t | K_t \leq \widetilde{K}^*\},$$

where $\widetilde{K}^*$ *denotes the optimal investment trigger in the C-M case and such that*

$$\widetilde{K}^* = \left(1 - \frac{\theta}{2\lambda}\right)\left(\frac{1}{V} - \frac{\theta}{4\lambda k}(\theta - 2\lambda)^2\right)^{-1}. \tag{13}$$

The optimal rate of investment is given by

$$I^* = \begin{cases} k & \text{for } K \le K^* \\ 0 & \text{otherwise,} \end{cases} \tag{14}$$

and the associated value function for the investment problem is given by

$$\widetilde{F}^*(K) = \begin{cases} \left(\left(\frac{\theta^2 K^*}{2k}\widetilde{\beta}_1 - 1\right)V - \frac{\lambda\theta K^*}{k-\lambda\theta K^*}\widetilde{K}^*\right)\left(\frac{K}{\widetilde{K}^*}\right)^{\widetilde{\beta}_1} + V - \frac{kK}{k-\lambda\theta K} & \text{for } K \le \widetilde{K}^* \\ \left(\frac{\theta^2}{2k}\widetilde{\beta}_1 V - 1\right)K^*\left(\frac{K}{\widetilde{K}^*}\right)^{\widetilde{\beta}_1 - 2\frac{k}{\theta^2 K}} & \text{otherwise.} \end{cases} \tag{15}$$

Note that $\left(\left(\frac{\theta^2 K^*}{2k}\widetilde{\beta}_1 - 1\right)V - \frac{\lambda\theta K^*}{k-\lambda\theta K^*}\widetilde{K}^*\right)\left(\frac{K}{\widetilde{K}^*}\right)^{\widetilde{\beta}_1}$ *represents the value of the option to suspend production for the active investor.*

2. *If* $\widetilde{\beta}_1 \le 0$ *or if* $\widetilde{\beta}_1 \ge \frac{2k}{\theta^2 K}$*, investment will be postponed indefinitely such that* $I^* = 0$ *for all* $K \ge 0$*, and the associated value function for the investment problem is given by*

$$\widetilde{F}^*(K) = 0, \ \forall K \ge 0.$$

Proof. See Appendix D. ■

We see here that, as in the G-M, there is a range of parameter values for which it may be optimal to exercise the investment option, and another range for which exercise will never be optimal. In particular, the investment option may be exercised if, and only if, $\lambda\theta > \frac{1}{2}\theta^2$, where $\lambda\theta$ is the implicit proportional dividend yield from investing and is denoted by $\widetilde{\delta}^Y$, and if, and only if $\widetilde{\delta}^L < \frac{1}{2}\theta^2$, where $\widetilde{\delta}^L := -\left(\frac{k}{K} - \lambda\theta\right)$ and represents the implicit dividend loss from suspension. Overall, therefore, investment is optimal if, and only if $\frac{1}{2}\theta^2 < \lambda\theta < \frac{1}{2}\theta^2 + \frac{k}{K}$.

5 Results

5.1 Comparison with the Complete Markets Model

From Theorem 1, active investment may be optimal in the General Model if, and only if, $\lambda > -\frac{1}{2}\theta\rho$, where $\rho < 0$. However, according to Proposition 2, active investment may be optimal in the C-M Model if, and only if, $\frac{\theta}{2} < \lambda < \frac{k}{\theta K} + \frac{\theta}{2}$. Therefore, there are two additional regions in which active investment may be optimal when we account for market incompleteness and risk aversion, but in the C-M Model which does not account for such features, active investment would never be optimal. The additional regions are (i) $-\frac{\theta\rho}{2} \le \lambda \le \frac{\theta}{2}$ and (ii) $\lambda \ge \frac{k}{\theta K} + \frac{\theta}{2}$. Therefore, assuming market completeness and risk neutrality in a TTB model imposes more stringent conditions on investment than a TTB model with risk aversion and market incompleteness. For parameter values such that $\frac{\theta}{2} < \lambda < \frac{k}{\theta K} + \frac{\theta}{2}$, active investment may be optimal in both models, and when $\lambda \le -\frac{\theta\rho}{2}$, active investment will never be optimal in either model. I show these observations graphically in Fig. 1 below.

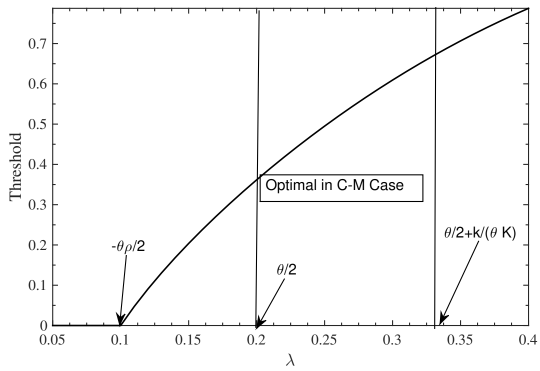

Figure 1: The effect of λ on K^* and $\widetilde{K}^*$ for $\rho = -0.5$ and $\eta = 1.5$.

Unless otherwise specified, the fixed parameter values chosen for all figures in the paper are as follows: $V = 1$; $\eta = 1.5$; $\rho = -0.5$; $\theta = 0.4$; $\lambda = 0.3$; $k = 0.05$. The results are robust to a wide choice of parameter values, but I depict the results for these values because they correspond with the values chosen in the comparable papers of Henderson [10] and Pindyck [24].

For this choice of parameter values, active investment will be optimal in the C-M model for $\lambda \in (0.2, 0.325)$. In the plot, this region is between the two vertical lines. For $\rho = -0.5$, if $\lambda < 0.1$ active investment is never optimal in both models, but if $\lambda \in (0.1, 0.2)$ or $\lambda > 0.325$, active investment will be optimal in the General model when it is not optimal in the C-M model. Hence, it is clear from the figure that the investment criteria is less stringent than in the complete markets, risk neutral case.

Furthermore, for the complete market, risk-neutral C-M case, the condition for investment to occur is that $\widetilde{\delta}^Y > \frac{1}{2}\theta^2$ but, according to the standard real options model of Dixit and Pindyck [6], a put option of this type should be exercised for $\widetilde{\delta}^Y > 0$. The difference between that model and the C-M is that there is a TTB. Therefore, the investment criteria is more stringent than in the standard model because the TTB feature implies that exercising the option will not eliminate the risk associated with investing. Hence, the decision-maker requires a greater compensation in terms of a higher dividend yield to encourage him to invest. This higher dividend offsets some of the effect of the risk he will continue to face after he has exercised the option and throughout the investment period.

Finally, it is important to note that the region in which $\lambda < -\frac{1}{2}\theta\rho$ corresponds with $\delta^{(\rho)} > -\frac{\eta}{2}$ in Henderson [10]. In her model, in which the market is incomplete, the decision-maker is risk averse, but there is no TTB, the investment criterion is to invest for all $\delta^{(\rho)} > -\frac{\eta}{2}$; in other words, investment is optimal in this region. The intuition in her case is that "the impact of idiosyncratic risk dominates the incentive to wait because of negative dividends, and the option is exercised". In this sense, her criterion on investment is less stringent than in the standard Dixit and Pindyck [6] type model. However, in the General model (and C-M model), it is not optimal to invest in this region. This is because investing in a project in which there is a TTB does not eliminate the idiosyncratic risk associated with the project and, as discussed, the dividend yield needs to compensate for this.

However, the following result implies that the C-M Model is a useful benchmark against which to assess the impact of incorporating the effects of market incompleteness and risk aversion into a TTB model of investment under uncertainty.

Proposition 3. *The value of the investment programme before and during active investment in the Complete Markets Model can be recovered from the corresponding values in the General Model when $\rho = -1$ and $\eta = 0$ by a change of measure from P to Q, where $\frac{dP}{dQ}$ is defined by Eq. (11).*

Proof. See Appendix E. ■

5.2 Comparative Static Results

5.2.1 The impact of risk aversion

It is readily obtained via standard calculus that

Proposition 4. *The optimal investment threshold decreases in the level of risk aversion η.*

This result arises from the fact that the rate of investment depends on the RCEC when the decision maker is risk averse and his utility function is concave. As discussed in Section 3.4, the rate adapts depending on K. However, for a risk neutral investor, production only occurs at a fixed maximum rate k or not at all. Therefore, the expected TTB for the risk neutral decision-maker will be no longer than it is for a risk averse decision-maker for all levels of RCEC. A long TTB generates greater uncertainty for a decision-maker over the true RCEC because $V(K)$ decreases in I^* (cf. Eq. (F.3)) and, therefore, it is optimal for the risk averse decision-maker to wait longer to invest until the RCEC is low. This result is depicted in Fig. 2 below.

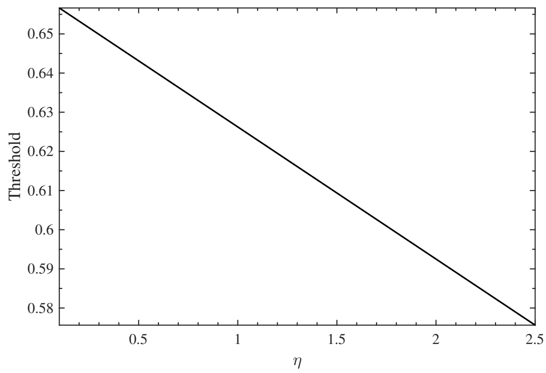

Figure 2: The effect of η on K^*.

5.2.2 The impact of market incompleteness and uncertainty

We see from Fig. 3 below that the optimal investment threshold is concave in the extent of market incompleteness. To understand this result, one must recognise that the impact of market completeness on the optimal investment strategy is analogous to the impacts of internal cost and external cost uncertainty; i.e., idiosyncratic and systematic risk, respectively. In particular, the more complete is the market, the lower is the idiosyncratic risk that cannot be hedged away and to which the decision-maker is risk averse, and the more the uncertainty surrounding the investment is attributable to external cost uncertainty. Therefore, to understand the impact of market completeness, I examine the variance of the RCEC, given in Appendix F for the two scenarios: (i) internal cost uncertainty dominates the impact of external cost uncertainty, and (ii) external cost uncertainty dominates the impact of internal cost uncertainty. The idiosyncratic risk dominates the impact of systematic risk when $\theta\sqrt{I^*(1-\rho^2)} > \theta\rho \Longleftrightarrow I^* > \frac{\rho^2}{1-\rho^2}$ (cf. Eq. (2)) or, equivalently, for $\rho \in \left(\left(\frac{I^*}{1+I^*}\right)^{1/2}, 0\right]$, and for $\rho \in \left[-1, \left(\frac{I^*}{1+I^*}\right)^{1/2}\right)$, the impact of external cost uncertainty is dominant.

Say ρ is such that the impact of idiosyncratic risk dominates the effect of systematic risk. If all of the risk associated with the project is attributable to idiosyncratic risk, then $\nu = 1/2$, $\rho = 0$, and the exposure to idiosyncratic risk is θ. The variance of the RCEC is then given by $V^{\rho=0}(K) = 2K^2 \frac{\theta^2}{2-\theta^2}$ (cf. Eq. (F.2)). However, for lower values of ρ (since $\rho < 0$), but such that the effect of idiosyncratic risk still dominates (i.e., such that $I^* > \frac{\rho^2}{1-\rho^2}$), then hedging reduces the exposure of the idiosyncratic risk to $\theta\sqrt{1-\rho^2}$ and the variance to $V(K) = 2K^2 \frac{\theta^2(1-\rho^2)}{2-\theta^2(1-\rho^2)} < V^{\rho=0}(K)$.[5]

The TTB feature implies that while the investment is being developed, the organisation is always exposed to random fluctuations in the RCEC. When the idiosyncratic risk is dominant, the uncertainty cannot be significantly hedged away by investing in the risky asset. The lower the correlation and the greater the extent of idiosyncratic risk, the worse is the hedge. We see from Fig. 3 that the more dominant is the idiosyncratic risk (i.e., the higher is ρ) the lower is the investment threshold. The ability to hedge, however, reduces the extent of idiosyncratic risk and the overall variance of the RCEC. Thus, K^* decreases in ρ in the region in which idiosyncratic risk is dominant.

Now, if ρ is such that $I^* < \frac{\rho^2}{1-\rho^2}$ and the impact of systematic external cost uncertainty dominates the impact of the idiosyncratic risk, then in the extreme case for $\rho = -1$ and $\nu = 0$, the variance of the RCEC is given by $V^{\rho=-1}(I) = 2K^2 \left(\frac{\theta^2 K}{2I^*(K)-\theta^2 K}\right)$ (cf. Eq. (F.3)) with $-\theta$ representing the exposure to systematic volatility. However, for higher values of ρ and such that the external cost uncertainty effect dominates, the exposure to external cost risk increases to $\theta\rho$ and the variance of the RCEC reduces to $2K^2 \frac{K\theta^2\rho^2}{2I^*-K\theta^2\rho^2} < V^{\rho=-1}(K)$. Thus, in the region $\rho \in \left[-1, \left(\frac{I^*}{1+I^*}\right)^{1/2}\right)$, $V(K)$ decreases in ρ (since $\rho < 0$) and in the extent of external cost uncertainty.

Furthermore, as seen in Fig. 3, the investment threshold increases in the extent of market completeness when the external cost uncertainty dominates, which is the opposite effect to the case in which idiosyncratic cost uncertainty dominates.

The reasoning is as follows. The pre-investment uncertainty associated with the project is all attributable to the systematic external cost risk. When the option is exercised, the uncertainty associated with external costs does not change because of the act of exercising the option, but only in the same way it would change even if the option was never exercised. However, the uncertainty associated with internal costs becomes non-zero once the option is exercised because, as mentioned previously, this source of uncertainty can only arise from the endogenous changes that occur through active investment. When the market is relatively incomplete, the internal cost uncertainty dominates and, therefore, the overall uncertainty associated with the RCEC increases when the option is exercised implying that investment will only be optimal at a low threshold. On the other hand, when external cost uncertainty dominates, actively exercising the option will not in itself change the overall variance of the RCEC but, as with standard real options models, waiting longer will be optimal when the volatility is high and, in my set up, volatility is high in the external-cost dominant region when the market is relatively complete. Overall, therefore, the (effective) variance of the RCEC is convex and the investment threshold is concave in the extent of market incompleteness.

[5] In principle, for $\rho \neq 0$, then $\nu < 1/2$ and the variance reduces to $V(K) = 2K^2 \frac{\theta^2(1-\rho^2)(I^*/K)^{2\nu-1}}{2-\theta^2(1-\rho^2)(I^*/K)^{2\nu-1}}$. However, without loss of generality, for ease of analysis I let $\nu = 1/2$ whenever internal cost uncertainty dominates external cost uncertainty.

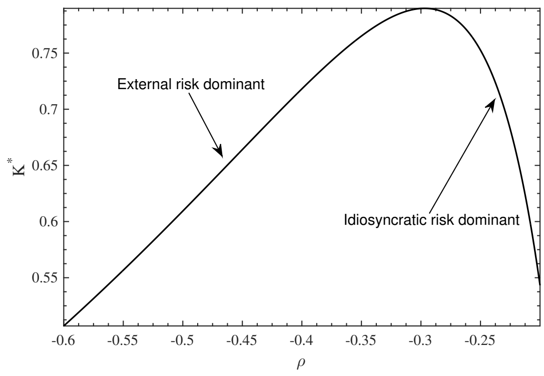

Figure 3: The impact of market incompleteness on the optimal investment threshold.

6 Implications of the Model

The General model is a new theoretical framework of investment under uncertainty with a TTB in that it incorporates the features of risk aversion and market incompleteness. In this sense, the model is comprised of important features of the TTB model of Pindyck [24] married with important features of the incomplete markets with a risk averse decision-maker model of Henderson [10]. In this section, I discuss some economic implications that are generated by the model.

My results have practical implications for decision-makers considering investing in projects with a TTB. However, the model predicts that the implications are different for organisations across industries, depending on the level of risk aversion and idiosyncratic risk associated with the investment projects for that industry. Typically large organisations hold well-diversified portfolios of investments making them mostly immune to idiosyncratic internal cost risk. Such organisations can represent those risk neural decision-makers operating in a complete market. However, small organisations are typically exposed to non-diversifiable investment uncertainty and, as such, these organisations represent those operating in an incomplete market with risk averse decision-makers.

As discussed in 5.1, the TTB in the risk neutral, complete markets case imposes a more stringent investment criterion than the standard real option model of Dixit and Pindyck [6] in which there is no TTB because actively investing does not eliminate the risk associated with investing in the former case. However, as I also point out, incorporating the features of risk aversion and market incompleteness alleviates some of the stringency. My model, therefore, predicts that the TTB feature has a negative impact on investment for large, well-diversified organisations, but less of an effect for smaller organisations.

This prediction is supported by Kort et al. [15] who examine the tradeoff between completing the project in one stage at a chosen time or in stages that can be completed at optimally chosen points in time in a complete market, risk neutral setting. The tradeoff in their paper arises from economies of scale in the sense that lumpy investment costs less, but proceeding stepwise gives the flexibility of selecting the optimal time to invest for subsequent stages. They show that high uncertainty favours the lumpy investment commitment relative to the stepwise investment. In my model, lumpy investment would also be expected to cost less overall because the expected RCEC decreases in the rate of investment (cf. Eq. (2)) and, moreover, it generates less uncertainty than investing in a product that takes a long time to develop (recall that the variance of the RCEC decreases in I^* and, thus, increases in the TTB). However, investing in a project with a TTB implies the risk associated with the investment project is not immediately eliminated and this dominates the effect of the lower and less uncertain RCEC on the optimal investment criteria. Thus, in essence, both models imply that for large well-diversified organisations (i.e., with risk neutral decision-makers), investment in projects with a long TTB is lower than in projects requiring lumpy investment.

Empirically, it has been shown by Pohl and Mihaljek [25], Flyvbjerg et al. [7], Flyvbjerg et al. [8] and Thijssen [30] that cost overruns are pronounced when the time it takes to develop large-scale transportation infrastructure projects is long and uncertain. Projects of this scale would typically be carried out by highly diversified organisations or government bodies and, indeed, their results are all based on the assumption of complete capital markets and risk neutral decision-makers. While my model does not lead to inferences about cost overruns as such, the fact that investment by well-diversified organisations in these type of large-scale projects with a TTB is predicted to be low does correspond with the effect large cost overruns are likely to have.

Testing these implications empirically is challenging because the investment threshold is dependent on the non-measuable parameter η - the degree of risk aversion. However, as cited by Sarkar and Zhang [26], "Bulan et al. [4] use a hazard model of time to development as a proxy for real-estate development trigger, and Moel and Tuffano [21] use a probit model of the operational state of a mine (i.e., open/closed) as a proxy for mine opening/closing triggers". The Bulan et al. [4] model could thus potentially be used as a proxy for the investment threshold in small organisations, whereas the Moel and Tuffano [21] model could be used as a proxy for large well-diversified organisations.

7 Conclusion

In this paper I develop a theoretical framework of investment under uncertainty with a TTB in which the market the organisation operates in is incomplete and its decision-maker is risk averse. I find that previous results on the effects of market incompleteness and risk aversion on the optimal time to invest in projects with uncertain payoffs are challenged by the addition of a TTB feature because they either focus on either a fixed lump sum payoff post investment, or a flow over an infinite horizon. The TTB feature implies the model falls between these two extremes by providing a flow payoff for a limited time post initial investment, after which the payoff is a fixed lump sum. On the other hand, I also find that market incompleteness and risk aversion have implications for TTB models in that TTB models in a complete market, risk neutral setting have more stringent investment criteria than those in my model. Thus, models of investment under uncertainty with a TTB, and models of investment under uncertainty with market incompleteness and risk aversion, ought not to be mutually exclusive as they have been in research to date. The robust and parsimonious methodology developed in the paper facilitates this.

Acknowledgements Thanks to Saqib Jafarey for helpful comments and suggestions.

Appendix

A Choice of Discount Factor

We want to write the mixed control/stopping problem at the time of the project completion

$$\sup_{\tau \geq 0} \sup_{(\omega_u)_{0\leq u\leq \widetilde{\tau}}} \sup_{((\alpha_I)_u)_{\tau\leq u\leq \widetilde{\tau}}} \mathbb{E}\left[-\frac{1}{\eta}e^{-\zeta\widetilde{\tau}-\eta(X_{\widetilde{\tau}}+Y_{\widetilde{\tau}})}\right], \tag{A.1}$$

where $Y_{\widetilde{\tau}}$ represents the value of the investment at the time of its completion, for an inter-temporal utility function $U(t,x)$. In other words, we want the decision-maker to have no in-built preferences or biases which would lead him to prefer one optimal time to invest over another and, as such, the subjective discount factor (and, by extension, $U(t,x)$) cannot be arbitrary (cf. Henderson and Hobson [11]).

Ignoring the option to invest temporarily, and considering the case where the terminal horizon of the portfolio choice problem is some fixed time T, the problem then becomes

$$M_t^{\omega} = \sup_{(\omega_u)_{0\le u\le T}} \mathbb{E}\left[-\frac{1}{\eta}e^{-\eta X_T-\zeta T}\Big| X_t = x\right]. \tag{A.2}$$

Then, by a standard application of the Bellman principle, we obtain

$$M_t^{\omega} = \sup_{\omega_t}\lim_{dt\downarrow 0} e^{-\zeta dt}\mathbb{E}[M_{t+dt}^{\omega}] = \sup_{\omega_t}\lim_{dt\downarrow 0}(1-\zeta dt + o(dt))\mathbb{E}\left[M_t^{\omega} + dM_t^{\omega}\right] \tag{A.3}$$

$$\begin{aligned} dM_t^{\omega} =& \frac{dM_t^{\omega}}{dt}dt + \frac{dM_t^{\omega}}{dX_t^{\omega}}dX_t^{\omega} + \frac{1}{2}\frac{d^2M_t^{\omega}}{d(X_t^{\omega})^2}(dX_t^{\omega})^2 \\ =& -\eta\omega_t\sigma_M M_t^{\omega}dW_t^M + \left(\frac{\eta^2}{2}\omega_t^2\sigma_M^2 - \eta\omega_t\alpha_M - \zeta\right)M_t^{\omega}dt \end{aligned} \tag{A.4}$$

Therefore, by substitution of dM_t^{ω} into the portfolio problem (A.3) above, the problem reduces to

$$\sup_{\omega_t}\left(\frac{\eta^2}{2}\omega_t^2\sigma_M^2 - \eta\omega_t\alpha_M\right) = 0. \tag{A.5}$$

This gives $\omega_t^* = \frac{\lambda}{\eta\sigma_M}$, and substituting for ω_t^* into Eq. (A.4) gives

$$\begin{aligned} \frac{dM_t^{\omega^*}}{M_t^{\omega^*}} &= \left(-\frac{1}{2}\lambda^2 - \zeta\right)dt - \lambda dW_t^M \\ \Longrightarrow M_t^{\omega^*} &= -\frac{1}{\eta}e^{-\eta x-\left(\lambda^2+\zeta\right)t-\lambda W_t^M}. \end{aligned} \tag{A.6}$$

Now, in order the mixed control/stopping problem (A.1) to have no in-build biases concerning the optimal time to invest, we need that the solution to the control problem (A.2) to be independent of the horizon $\widetilde{\tau}$; in other words, for $M_t^{\omega^*}$ to be a martingale.

Clearly, it is a martingale iff $\zeta = -\frac{1}{2}\lambda^2$. Hence, letting $\zeta = -\frac{1}{2}\lambda^2$ ensures that no such biases exist.

B Proof of Theorem 1

First I consider the problem where the decision-maker is actively investing in the project (i.e., $I_t \neq 0$ and $\tau \le t < \widetilde{\tau}$) and has to choose the rate at which to invest and the amount to invest in the market asset at each period so that his organisation's discounted expected utility is maximised. Letting $F(X,K)$ denote the value function for an active decision-maker, such that

$$dX_t = \omega_t\frac{dM_t}{M_t} - I_t dt,$$

where dM_t is defined by Eq. (4) and the dynamics of $(K_t)_{t\geq 0}$ by Eq. (2). By usual arguments, $F(X,K)$ satisfies the following HJB equation:

$$\begin{aligned}&\sup_{\omega}\{\frac{1}{2}(\omega)^2\sigma_M^2F_{XX}+\omega\alpha_M F_X+\theta\rho\sigma_M\omega KF_{XK}\}+\frac{1}{2}\theta^2\rho^2K^2F_{KK}\\&+\sup_{I}\{\frac{1}{2}\theta^2(1-\rho^2)IKF_{KK}-I(F_K+F_X)+U(-I)\}+\frac{1}{2}\lambda^2F=0,\end{aligned}\tag{B.1}$$

where $F_y=\partial F/\partial y$, $F_{yy}=\partial^2F/\partial y^2$, and $F_{yz}=\partial^2F/\partial y\partial z$.

Maximising over ω gives

$$\omega^*=\frac{-\lambda F_X-\theta\rho KF_{XK}}{\sigma_M F_{XX}}\tag{B.2}$$

and maximising over I gives

$$I^*=\frac{1}{\eta}\ln\left(\frac{1}{2}\theta^2(1-\rho^2)KF_{KK}-F_K-F_W\right)\tag{B.3}$$

since the utility function is exponential.

Replacing for the optimal values of ω and I, the HJB equation becomes

$$\begin{aligned}&\frac{1}{2}\theta^2\rho^2K^2F_{KK}-\frac{(\lambda F_X+\theta\rho KF_{XK})^2}{2F_{XX}}+\frac{1}{2}\lambda^2F\\&+\frac{1}{\eta}\left(\ln\left(\frac{1}{2}\theta^2(1-\rho^2)KF_{KK}-F_K-F_X\right)-1\right)\left(\frac{1}{2}\theta^2(1-\rho^2)KF_{KK}-F_K-F_X\right)=0,\end{aligned}\tag{B.4}$$

which is subject to the following boundary condition:

$$F(X,0)=B(X+V)\tag{B.5}$$

implying that when the product is fully developed and all costs have been paid (at time $t=\widetilde{\tau}$), his wealth is comprised of the lump sum payoff from investing V and the value of his hedging portfolio at $\widetilde{\tau}$.

I conjecture that a general solution to Eq. (B.4) takes the following form:

$$F(X,K)=-\frac{1}{\eta}e^{-\eta g(X+C+G(K))},$$

where C is some constant and $G(K)$ some function to be determined. Letting

$$g=\frac{1}{\eta G_K^2}\left(G_{KK}-\frac{2}{\theta^2(1-\rho^2)K}(G_K+1)\right)$$

ensures the logarithmic term in the PDE is eliminated completely. The PDE (B.4) then reduces to

$$\frac{1}{2}\theta^2\rho^2K^2G_{KK}-\lambda\theta\rho KG_K=0\tag{B.6}$$

which has a general solution of the form

$$G(K)=B_1K^{\beta_1}+B_2,$$

where B_1 and B_2 are constants to be determined, and β_1 is the non-zero root of the quadratic equation

$$\frac{1}{2}\beta(\beta-1)\theta^2\rho^2 - \lambda\theta\rho\beta = 0.$$

Thus $\beta_1 = 1 + 2\lambda/\theta\rho$.

In order for the boundary condition (B.5) to be satisfied, we must have $C = \frac{1}{g}\left[X + V - \frac{2}{\eta\beta_1}\left(\frac{\lambda}{\theta\rho} - \frac{1}{\theta^2(1-\rho^2)}\right)\right] - X - B_2$, $B(x) := U(x)$ and $\beta_1 < 0$. Thus, letting C and $B(\cdot)$ be defined as such, and for negative values of β_1, we have

$$F(X,K) = -\frac{1}{\eta}e^{-\eta\left(X+V-\frac{2}{\eta\beta_1^2\theta^2(1-\rho^2)B_1K^{\beta_1-1}}\right)}.$$

Next I derive the value function for an *inactive* decision-maker; i.e., $I_t = 0$ for all $t < \tau$. I let the value function in this region be denoted by $f(X,K)$ which satisfies the following HJB equation:

$$\frac{1}{2}\theta^2K^2 f_{KK} - \frac{(\lambda f_X + \theta\rho K f_{XK})^2}{2f_{XX}} + \frac{1}{2}\lambda^2 f = 0. \tag{B.7}$$

This equation is subject to the condition

$$f(X,\infty) = -\frac{1}{\eta}e^{-\eta X}, \tag{B.8}$$

which implies that when the RCEC is too high relative to the payoff V, the value of the investment project to the risk averse decision-maker is also zero and, hence, he will never invest.

Now I conjecture that $f(X,K) = -\frac{1}{\eta}e^{-\eta(X+D+s(K))}$ is a general solution to Eq. (B.7), such that D is a constant to be determined, $s(K)$ a function to be determined.

Substituting for $f(X,K)$ in Eq. (B.7) yields the following ODE for which $s(K)$ is a solution

$$\frac{1}{2}\theta^2\rho^2K^2 s_{KK} - \lambda\theta\rho K s_K = 0. \tag{B.9}$$

A general solution to this equation is of the form $s(K) = A_1K^{\gamma_1} + A_2$, where A_1 and A_2 are constant parameters to be determined, and γ_1 is the non-zero root of the quadratic equation

$$\frac{1}{2}\gamma(\gamma-1)\theta^2\rho^2 - \lambda\theta\rho\gamma = 0.$$

Thus $\gamma_1 = \beta_1 = 1 + 2\lambda/\theta\rho$.

Substituting for $s(K)$ into $f(X,K)$ gives

$$f(X,I) = -\frac{1}{\eta}e^{-\eta\left(X+D+A_1K^{\beta_1}+A_2\right)}.$$

The boundary condition (B.8) is satisfied for $A_2 = -D$, which I assume to be true, and for $\beta_1 < 0$. Hence, for $\beta_1 < 0$

$$f(X,K) = -\frac{1}{\eta}e^{-\eta\left(X+A_1K^{\beta_1}\right)}.$$

The decision-maker is indifferent between investing in the project and waiting at some critical level K^* which is the RCEC at time τ^*. Thus, it is optimal for the decision-maker to invest for all levels of $K \le K^*$, and to wait otherwise. Therefore, the following value matching condition must be satisfied at the boundary

$$F(X,K^*) = f(X,K^*). \tag{B.10}$$

Moreover, the value of the investment program must be continuous at the boundary so that

$$F_I(X, K^*) = f_I(X, K^*) \tag{B.11}$$

must also be satisfied.

Together these equations give

$$f(X, K) = -\frac{1}{\eta} e^{-\eta\left(X + 2\left(\frac{\beta_1 - 1}{3\beta_1 - 2}\right) V \left(\frac{K}{K^*}\right)^{\beta_1}\right)}$$

and

$$F(X, K) = -\frac{1}{\eta} e^{-\eta\left(X + V - \left(\frac{\beta_1}{3\beta_1 - 2}\right) V \left(\frac{K}{K^*}\right)^{1-\beta_1}\right)}. \tag{B.12}$$

Finally, we need to solve for K^*, and since we have already used the standard value-matching and smooth pasting conditions to solve for A_1 and B_1, we need another condition. The value matching condition implies that at K^*, $F(X, K^*) = f(X, K^*)$. However, from Eqs. (B.4) and (B.7), we see this is true if

$$\begin{aligned} &\left(\ln\left(\frac{1}{2}\theta^2(1-\rho^2)K^* F_{KK}(K^*) - F_K(K^*) - F_X(K^*)\right) - 1\right) \\ &\times \left(\frac{1}{2}\theta^2(1-\rho^2)K^* F_{KK}(K^*) - F_K(K^*) - F_X(K^*)\right) = 0. \end{aligned} \tag{B.13}$$

Thus, applying the condition at K^* using Eq. (B.12) gives the optimal investment timing threshold:

$$K^* = \left(\frac{\beta_1(1-\beta_1)}{3\beta_1 - 2}\right) V \left[1 + \frac{\beta_1}{2}\theta^2(1-\rho^2)\left(1 - \eta\left(\frac{(1-\beta_1)}{3\beta_1 - 2}\right) V\right)\right]. \tag{B.14}$$

Finally, for values of $\beta_1 \geq 0$, the boundary condition in $f(X, K)$ will hold iff $A_1 = 0$. In this case, the decision-maker would postpone investment indefinitely because the value of the option to invest is zero. Therefore, $F(X, K) = -1/\eta e^{-\eta X}$ for all values of K. Furthermore, the boundary condition in $F(X, K)$ will hold iff $B = 0$. In that case, it would only be optimal for the decision-maker to invest when the RCEC is zero; in other words, for him to invest at zero cost. But since that case can never arise, this case also implies that investment will be postponed indefinitely.

C Proof of Proposition 1

When the market is complete, $\rho = -1$ and the dynamics of the RCEC follow

$$dK = -Idt - \theta K dW^M.$$

In order to determine an appropriate discount factor, we assume that in the financial market, a risk-free asset and the market portfolio of risky assets are traded, with price processes $(B_t)_{t\geq 0}$ and $(e^{rt}M_t)_{t\geq 0}$, respectively, where $(M_t)_{t\geq 0}$ is given by Eq. (4). Note that in order to compare my model with Henderson [10], similar to her paper, the dynamics of the wealth process (4) are given in discounted terms, in which the discounting is by the bond (see Section 3.2). Thus, I consider the non-discounted process to determine the DF here. Moreover, $dB = rBdt$, $B_0 = 1$, and r is the riskless rate. However, since $\rho = -1$, the uncertainty associated with the RCEC can be perfectly diversified by trading in the market asset. Therefore, denoting the discount factor of the project in the C-M case by $(\widetilde{\Lambda}_t)_{t\geq 0}$, adapted to the filtration $(\mathcal{F}_t)_{t\geq 0}$, which, under P must have the following dynamics

$$\frac{d\widetilde{\Lambda}}{\widetilde{\Lambda}} = -\widetilde{\alpha}_\Lambda dt - \widetilde{\sigma}_\Lambda dW^M$$

and $\widetilde{\Lambda}_0 = 1$.

If we assume the absence of arbitrage, then decision-makers must discount uncertain payoffs according to $(\widetilde{\Lambda}_t)_{t\geq 0}$ such that

$$E^P[d\widetilde{\Lambda}B] = 0 \text{ and } E^P\left[d\widetilde{\Lambda}e^{rt}M\right] = 0 \tag{C.1}$$

(cf. Thijssen [29] and Cochrane [5] pp. 27). Applying Applying Ito's lemma[6], we get that $\alpha_\Lambda = r$ and $\sigma_\Lambda = \lambda$. Hence, the discount factor must take the specific form

$$\frac{d\widetilde{\Lambda}}{\widetilde{\Lambda}} = -rdt - \lambda dW^M,$$

where $\lambda = \alpha_M/\sigma_M$.

Thus, since $\widetilde{\Lambda}_0 = 1$,

$$\widetilde{\Lambda}_t = e^{-\left(r+\frac{1}{2}\lambda^2\right)t-\lambda W_t^M}.$$

To determine a probability measure Q which is equivalent to P, we must have that the discounted price process $(M_t)_{t\geq 0}$ is a martingale under Q. This is satisfied for Q when

$$\left.\frac{dQ}{dP}\right|_t = e^{-\frac{1}{2}\lambda^2 t-\lambda W_t^M}.$$

Therefore

$$\widetilde{\Lambda}_t = e^{-rt}\left.\frac{dQ}{dP}\right|_t$$

and by a straightforward application of Girsanov's theorem, the dynamics of the RCEC under Q are given by

$$dK = -\left(I - \theta\lambda K\right)dt - \theta K d\widetilde{W}^M, \tag{C.2}$$

where $(\widetilde{W}_t^M)_{t\geq}$ is the Q-Brownian motion.

D Proof of Proposition 2

Let $\widetilde{F}(K)$ denote the value function for an active decision-maker. Under Q, $\widetilde{F}(K)$ satisfies the following Bellman equation

$$\frac{1}{2}\theta^2 K^2 \widetilde{F}_{KK} + \theta\lambda K\widetilde{F}_K - \sup_I\{I\widetilde{F}_K + I\} = 0. \tag{D.1}$$

Since this equation is linear in I, the rate of investment that maximises $\widetilde{F}(K)$ is zero or the maximum rate at which the decision-maker can productively invest. I let this maximum rate be denoted by the

[6]For two geometric Brownian motions $(X_t)_{t\geq 0}$ and $(Y_t)_{t\geq 0}$

$$E^P[\frac{dXY}{XY}] = E^P\left[\frac{dX}{X}\right] + E^P\left[\frac{dY}{Y}\right] + E^P\left[\frac{dX}{X}\frac{dY}{Y}\right].$$

constant k. Therefore,

$$I^* = \begin{cases} k & \text{for } \widetilde{F}_K(K) + 1 \leq 0 \\ 0 & \text{otherwise.} \end{cases} \tag{D.2}$$

Hence, Eq. (D.1) has a free boundary at a point $\widetilde{K}^*$ such that $I^* = k$ for all $K \leq \widetilde{K^*}$ and zero otherwise.

Thus, when the decision-maker is investing, his value function solves

$$\frac{1}{2}\theta^2 K^2 \widetilde{F}_{KK} - (k - \theta\lambda K)\,\widetilde{F}_K - k = 0 \tag{D.3}$$

subject to the boundary condition

$$\widetilde{F}(0) = V. \tag{D.4}$$

A general solution, subject to the boundary condition, is given by

$$\widetilde{F}(K) = V + \widetilde{B}_1 K^{\widetilde{\beta}_1} - \frac{kK}{k - \theta\lambda K}, \tag{D.5}$$

where $\widetilde{B}_1$ is a constant to be determined, and

$$\widetilde{\beta}_1 = 1 - 2\frac{\lambda}{\theta} + 2\frac{k}{\theta^2 K}.$$

Note that $B_1 K^{\widetilde{\beta}_1}$ represents the value of the option to suspend production.

In order for the boundary condition to hold, we must have that $\widetilde{\beta}_1 > 0$.

Let $\widetilde{f}(K)$ denote the value function for an inactive decision-maker who has yet to invest. $\widetilde{f}(K)$ satisfies the Bellman equation

$$\frac{1}{2}\theta^2 K^2 \widetilde{f}_{KK} + \theta\lambda K \widetilde{f}_K = 0, \tag{D.6}$$

subject to the boundary condition

$$\lim_{K\to\infty} \widetilde{f}(K) = 0.$$

A general solution, subject to the boundary condition, is given by

$$\widetilde{f}(K) = \widetilde{A}_1 K^{\widetilde{\gamma}_1} + \widetilde{A}_2$$

where $\widetilde{A}_1$ and $\widetilde{A}_2$ are constants to be determined, and $\widetilde{\gamma}_1$ is the non-zero root of the quadratic equation

$$\frac{1}{2}\theta^2 \widetilde{\gamma}(\widetilde{\gamma} - 1) + \theta\lambda\widetilde{\gamma} = 0.$$

Thus, $\widetilde{\gamma}_1 = 1 - 2\lambda/\theta = \widetilde{\beta}_1 - 2\frac{k}{\theta^2 K}$. The boundary condition will hold iff $\widetilde{\gamma}_1 < 0$; i.e., iff $\widetilde{\beta}_1 < 2\frac{k}{\theta^2 K}$ and iff $\widetilde{A}_2 = 0$.

At the boundary, the following value-matching and smooth pasting conditions must be satisfied:

$$\widetilde{F}(\widetilde{K}^*) = \widetilde{f}(\widetilde{K}^*)$$

and

$$\widetilde{F}_K(\widetilde{K}^*) = \widetilde{f}_K(\widetilde{K}^*)$$

which together give

$$\widetilde{F}(K) = \left(\left(\frac{\theta^2 K^*}{2k}\widetilde{\beta}_1 - 1\right)V - \frac{\lambda\theta K^*}{k - \lambda\theta K^*}\widetilde{K}^*\right)\left(\frac{K}{\widetilde{K}^*}\right)^{\widetilde{\beta}_1} + V - \frac{kK}{k - \lambda\theta K}$$

and

$$\widetilde{f}(K) = \left(\frac{\theta^2 K^*}{2k}\widetilde{\beta}_1 V - \widetilde{K}^*\right)\left(\frac{K}{\widetilde{K}^*}\right)^{\widetilde{\beta}_1 - 2\frac{k}{\theta^2 K}}.$$

Finally, to solve for $\widetilde{I}^*$, we use the condition that at the boundary

$$\widetilde{F}_K(\widetilde{K}^*) + 1 = 0.$$

Therefore,

$$\widetilde{K}^* = \left(1 - \frac{\theta}{2\lambda}\right)\left(\frac{1}{V} - \frac{\theta}{4\lambda k}(\theta - 2\lambda)^2\right)^{-1}.$$

However, if $\widetilde{\beta}_1 \geq 2\frac{k}{\theta^2 K}$, the boundary condition in $f(K)$ will only be satisfied if $\widetilde{A} = 0$; in other words, an inactive decision-maker will never invest. Therefore, his value function is $\widetilde{F}^*(K) = 0$ for all values of K.

Finally, if $\widetilde{\beta}_1 \leq 0$, then the boundary condition for $F(K)$ would only be satisfied for $\widetilde{B}_1 = 0$ and $\widetilde{B}_2 = V$. In that case, we have $\widetilde{F}(K) = V - \frac{kK}{k - \theta\lambda K}$. But, $\widetilde{\beta}_1 \leq 0$ iff $k - \theta\lambda K < 0$, where $k - \theta\lambda K$ represents the drift rate of the RCEC under the probability measure Q (cf. Eq. C.2), $\widetilde{\beta}_1 \leq 0$ implies the RCEC would increase with ongoing investment. Hence, investing would never be optimal in this scenario either.

E Proof of Proposition 3

Given the value function in Theorem 1, the value of the investment *option* can be obtained via the standard certainty equivalent argument. Letting $p^{(\rho,\eta)}(K)$ denote the certainty equivalent value of the option to invest (that is, for $K > K^*$), we evaluate it as the solution to $F(X, K) = F(X + p^{(\rho,\eta)}(K), \infty)$, where $F(X, K)$ is given by Eq. (8) (cf. Henderson [10]). In effect, we compare the value function in Theorem 1 with the value function when the option to invest is zero; i.e., when K is infinitely large, but with an adjusted initial wealth of $X + p^{(\rho,\eta)}(K)$ (cf. Henderson [10]).

Hence, The certainty-equivalent wealth associated with the investment opportunity is given by

$$p^{(\rho,\eta)}(K) = 2\left(\frac{\beta_1 - 1}{3\beta_1 - 2}\right)V\left(\frac{K}{K^*}\right)^{\beta_1},$$

where $\beta_1 = 1 + \frac{2\lambda}{\theta\rho} < 0$, as previously defined, and K^* is given by Eq. (6).

To compare his with the C-M model, we let $\rho = -1$ and $\eta = 0$. By letting $\rho = -1$, $\beta_1 = 1 - 2\lambda/\theta \equiv \widetilde{\gamma}_1$, where $\widetilde{\gamma}_1$ is defined in Appendix D and is the power on K in the value of the option to invest in the C-M model.

Now, it is a standard result (cf. Dixit and Pindyck [6] Ch. 9) that $\left(\frac{K}{K^*}\right)^{\beta_1}$ represents the discount factor. However, letting $\rho = -1$ and $\eta = 0$ implies we should use the discount factor in the risk-neutral C-M case. As such, we replace this General Model discount factor with the discount factor of the C-M model defined by Eq. (12). Hence, the certainty equivalent value of the option to invest, with

$\rho=-1$, $\eta=0$ and $\left(\frac{K}{K^*}\right)^{\beta_1}=\widetilde{\Lambda}_{\tau^*}$ can be written as

$$p^{(\rho,\eta)}(K)=E_0^P\left[2\left(\frac{\widetilde{\gamma}_1-1}{3\widetilde{\gamma}_1-2}\right)Ve^{-r\tau^*}\frac{dQ}{dP}\right]=E_0^Q\left[2\left(\frac{\widetilde{\gamma}_1-1}{3\beta\widetilde{\gamma}_1-2}\right)Ve^{-r\tau^*}\right].$$

Under Q, the dynamics of K for $K>K^*$ and $\rho=-1$ are given by

$$dK=\lambda\theta K dt-\theta K d\widetilde{W}^M,$$

which are exactly the dynamics in the C-M case for $K>K^*$.

Finally, by the standard approach to solving optimal stopping problems, the value of the option to invest is a martingale. As such, $E^Q[dp^{(\rho,\eta)}(K)]=0$. Now, since $e^{-r\tau^*}$ is a function of K (Dixit and Pindyck [6]),

$$E^Q[dp^{(\rho,\eta)}(K)]=0\Longleftrightarrow\frac{1}{2}\theta^2K^2p_{KK}^{(\rho,\eta)}+\lambda\theta Kp_K^{(\rho,\eta)}=0,$$

which is exactly the same Bellman equation for the option to invest as that in the C-M case (see Eq. (D.6)).

Therefore, when $\rho=-1$, by a change of measure from P to Q, we show that $p^{(\rho,\eta)}(K)=\widetilde{f}(K)$; i.e., the certainty equivalent value of the option to invest in the General Model is equal to the value of the option to invest in the C-M model.

Next we consider the situation when the development is in process; i.e., for $K\le K^*$. The certainty-equivalent wealth associated with investing is defined by $w^{(\rho,\eta)}(K)$ and solves $F(X,K)=F(X+w^{(\rho,\eta)}(K),0)$; i.e., we compare the value function in Theorem 1 from active investment with the value from having completed the project when all costs have been paid. Hence, for $\rho=-1$, $\eta=0$ and replacing the discount factor in this case (i.e., $(K/K^*)^{1-\beta_1}$) with that of the risk neutral C-M model; i.e., Eq. (12), by the same argument as previously, we get that

$$\begin{aligned}w^{(\rho,\eta)}(K)=&E^P\left[-\frac{\beta_1}{3\beta_1-2}V\left(\frac{K}{K^*}\right)^{1-\beta_1}\right]\\=&E^Q\left[-\frac{\widetilde{\gamma}_1}{3\widetilde{\gamma}_1-2}Ve^{-r\tau^*}\right],\end{aligned}\tag{E.1}$$

where $e^{-r\tau^*}$ is a function of K. Now, for active investment, when $K\le K^*$ and $\rho=-1$, under Q, the dynamics of the RCEC are

$$dK=-\left(I-\lambda\theta K\right)dt-\theta K d\widetilde{W}^M.$$

The certainty-equivalent value from active investment must also be a martingale (cf. Pindyck [24]); i.e., it must be the case that

$$\begin{aligned}&\mathbb{E}^Q\left[dw^{(\rho,\eta)}(K)\right]=0\\\Longleftrightarrow&\frac{1}{2}\theta^2K^2w_{KK}^{(\rho,\eta)}(K)+\lambda\theta Kw_K^{(\rho,\eta)}(K)-\sup_I\left[Iw_K^{(\rho,\eta)}(K)-U(-I)\right]=0.\end{aligned}\tag{E.2}$$

But, since we let $\eta=0$, by L'Hôpital's rule we get that $U(-I)=-I$. Thus,

$$\begin{aligned}&\mathbb{E}^Q\left[dw^{(\rho,\eta)}(K)\right]=0\\\Longleftrightarrow&\frac{1}{2}\theta^2K^2w_{KK}^{(\rho,\eta)}(K)+\lambda\theta Kw_K^{(\rho,\eta)}(K)-\sup_I\left[Iw_K^{(\rho,\eta)}(K)+I\right]=0,\end{aligned}\tag{E.3}$$

which is exactly the Bellman equation obtained for active investment in the C-M model (see Eq. (D.1)).

Thus, when $\rho = -1$ and $\eta = 0$, by a change of measure from P to Q, we find that $w^{(\rho,\eta)}(K) = \widetilde{F}(K)$; i.e., the certainty equivalent value from active investment in the General model is equal to the value from active investment in the C-M model also.

F Variance of the RCEC

Let $\widetilde{\tau}$ denote the time the product has been completed and the final instalment has been paid; i.e., $K_{\widetilde{\tau}} = 0$. Therefore, the variance of K during the period of investing is given by

$$V(K) = E^{t=0}\left[\int_0^{\widetilde{\tau}} I_s^* ds\right]^2 - \left(\mathbb{E}^{t=0}\left[\int_0^{\widetilde{\tau}} I_s^* ds\right]\right)^2.$$

Letting $M(K) := \mathbb{E}^{t=0}\left[\int_0^{\widetilde{\tau}} I_s^* ds\right]$, and for

$$dK = -I^* dt + \theta\rho K dW^M + \theta\sqrt{I^* K(1-\rho^2)}dZ$$

$M(K)$ must satisfy the following Kolmogorov backward equation corresponding to Eq. (1):

$$\frac{\theta^2}{2}\left(\rho^2 K + (1-\rho^2)I^*\right) K M_{KK} - I^* M_K + I^* = 0$$

subject to the boundary conditions $M(0) = 0$ and $M(\infty) = \infty$. Now, clearly, $M(K) = K$ is one such solution.

Now, let $J(K) := \mathbb{E}^{t=0}\left[\int_0^{\widetilde{\tau}} I_s^* ds\right]^2$ so that the variance becomes

$$V(K) = J(K) - K^2.$$

$J(K)$ must satisfy the Kolmogorov equation

$$\frac{\theta^2}{2}\left(\rho^2 K + (1-\rho^2)I^*\right) K J_{KK} - I^* J_K + 2I^* K = 0 \tag{F.1}$$

subject to $J(0) = 0$ and $J(\infty) = \infty$.

I solve for this for two extreme cases cases: (i) $\rho = 0$ and (ii) $|\rho| = 1$.

When $\rho = 0$, the solution to Eq. (F.1), subject to the boundary conditions is given by $J(K) = 2K^2/(2-\theta^2)$ so that

$$V^{\{\rho=0\}}(K) = 2\left(\frac{\theta^2}{2-\theta^2}\right)K^2 > 0. \tag{F.2}$$

When $|\rho| = 1$, A solution to this equation, subject to the boundary conditions is $J(K) = 2I^*K^2/(2I^* - \theta^2 K)$. Therefore,

$$V^{\{|\rho|=1\}}(I) = 2K^2\left(\frac{\theta^2 K}{2I^*(K) - \theta^2 K}\right). \tag{F.3}$$

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