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Article

On the Market-Consistent Valuation of Participating Life Insurance Heterogeneous Contracts under Longevity Risk

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Abstract: The purpose of this paper is to conduct a market-consistent valuation of life insurance participating liabilities sold to a population of partially heterogeneous customers under the joint impact of biometric and financial risk. In particular, the heterogeneity between groups of policyholders stems from their offered minimum interest rate guarantees and contract maturities. We analyse the effects of these features on the company's insolvency while embracing the insurer's goal to achieve the same expected return for different cohorts of policyholders. Within our extensive numerical analyses, we determine the fair participation rates and other key figures, and discuss the implications for the stakeholders, taking account of various degrees of conservativeness of the insurer when pricing the contracts.

Keywords: Participating life insurance; Heterogeneous policyholders; Market-consistent valuation; Longevity risk; Fair contract analysis

JEL Classification: G13; G22

1. Introduction

This paper focuses on the market-consistent valuation of life insurance liabilities, a topic which has recently again attracted much attention both from academics and practitioners, see, for example, Sheldon and Smith (2004), Bauer et al. (2010), Broeders et al. (2013), Gambaro et al. (2019), Dorobantu et al. (2020) and Ghalehjooghi and Pelsser (2020). This growing interest mostly stems from the long-sought adoption of fair value based accounting standards in many countries, culminating with the full implementation of Solvency II in the European Union in 2016, see European Parliament and Council of the European Union (2009). According to these principles, assets and liabilities should be evaluated at the price, actual or hypothetical, they could be exchanged for in a liquid market. As in the last few decades financial markets have experienced a high volatility and permanently low (or even negative) interest rates, coupled with a steady increase in life expectancy, the introduction of these accounting standards has forced life insurers to deal with risks more carefully in valuation.

25 We propose a contingent claim model, along the lines of [Briys and de Varenne \(1994 1997\)](#), for the
26 valuation of the equity and the liabilities of a participating life insurance company. The pioneering
27 model by [Briys and de Varenne \(1994 1997\)](#) has been extended in several directions, e.g. by [Grosen and](#)
28 [Jørgensen \(2002\)](#), [Bernard et al. \(2005\)](#), [Chen and Suchanecski \(2007\)](#), [Cheng and Li \(2018\)](#), [Bacinello](#)
29 [et al. \(2018\)](#), [Hieber et al. \(2019\)](#) or [Orozco-Garcia and Schmeiser \(2019\)](#), just to quote a few. In most of
30 these papers, fair valuation is carried out for individual life insurance contracts with the exception of
31 [Hieber et al. \(2019\)](#), [Orozco-Garcia and Schmeiser \(2019\)](#) and [Bacinello et al. \(2018\)](#). [Hieber et al. \(2019\)](#)
32 show that individual insurance contracts with an annual guaranteed return can be incorporated into an
33 existing portfolio, resulting in no wealth transfer between two groups. [Orozco-Garcia and Schmeiser](#)
34 [\(2019\)](#) examine whether contracts can be priced such that all the generations pay fair premiums and
35 face the same level of default risk. Our paper extends [Bacinello et al. \(2018\)](#) who introduce biometric
36 risk for a population of completely homogeneous policyholders, and analyze how this risk can be split
37 into two components, namely diversifiable and systematic parts. We consider the valuation of life
38 insurance participating contracts sold to a population of partially *heterogeneous* customers under the
39 joint impact of biometric and financial risk. Here, the heterogeneity between groups of policyholders
40 stems from their offered minimum interest rate guarantees and contract maturities. In this respect,
41 this paper is related to [Hansen and Miltersen \(2002\)](#) and [Burkhart \(2018\)](#). [Hansen and Miltersen \(2002\)](#)
42 deal with the case of participating life insurance contracts sold to heterogeneous customers, but do not
43 take into account biometric and default risk.¹ [Burkhart \(2018\)](#) particularly addresses surrender risk in
44 the assessment of a heterogeneous insurance portfolio under Solvency II, and considers the interaction
45 between minimum interest rate guarantee, surplus participation and reserving requirement.

46 Analyzing customers with different minimum interest rate guarantees has some interesting
47 practical implications. As a reaction to new market conditions, in particular low interest rates, one
48 measure insurance companies have taken was to reduce the level of the minimum interest rate
49 guarantee drastically. For example, the German Federal Ministry of Finance has gradually decreased
50 the maximum technical interest rate for life insurance products over the past 26 years from 4% to 2.25%,
51 then to 1.75% and currently to 0.9%, see [Eling and Holder \(2013\)](#). However, these adjustments are
52 applied exclusively to new contracts, while older customers keep enjoying higher minimum interest
53 rate guarantees. A natural question that arises is whether new customers will then be penalized and
54 what measures could be taken to protect the policyholders. This problem has been widely discussed in
55 public, see, for instance, [Seibel \(2016\)](#).

56 Despite its stylized nature, this paper provides some useful insights into such and similar topics by
57 developing a rather comprehensive contingent claim model that explicitly considers financial, default
58 and longevity risk. We incorporate the heterogeneity of customers by dividing them into two groups.
59 We model the liabilities for these two groups by addressing the insurer's goal to protect both old and
60 new customers, usually endowed with different minimum interest rate guarantees, and provide them
61 with the same expected return. Alternatively, when the two groups have different contract maturities,
62 the payoff of the group with the earlier maturity is structured in such a way that the other group is also
63 adequately protected. We evaluate the outstanding liabilities in a market-consistent way and conduct
64 an analysis of fair contracts for both specifications of the heterogeneous groups. The subject of actuarial
65 fairness has been examined by several authors, see, for example, [Meyers and Hoyweghen \(2017\)](#)
66 for a very general discussion or [Knispel et al. \(2011\)](#). Based on the fair combinations of parameters,
67 we compute then the certainty equivalent returns, under the physical probability measure, for the

¹ Actually, [Hansen and Miltersen \(2002\)](#) introduce the diversifiable component of mortality.

68 heterogeneous policyholders. This helps answering two questions: what are the factors determining
 69 the relative magnitudes of the fair participation rates? How will the benefits of the two groups, as
 70 measured by their certainty equivalent returns, be impacted by considering both groups as a whole?

71 The main findings of the paper resulting from our numerical analysis can be summarized as
 72 follows: (i) The levels of the risk premium (or the degrees of prudence) arising from various longevity
 73 pricing assumptions play a substantial role in the magnitudes of the fair participation rates and of
 74 the certainty equivalent returns. (ii) Maintaining participation rates in the range 80 – 100% (often
 75 prescribed by law and used in practice) can severely affect the insurer’s balance sheet as some portfolio
 76 and parameter combinations actually require smaller participation rates to ensure the fairness of
 77 the contracts. Remarks (i) and (ii) are consistent with the findings in [Bacinello et al. \(2018\)](#). (iii)
 78 When the two groups differ in their minimum interest rate guarantees, the group with the lower rate
 79 receives a higher fair participation rate. This compensation is bound to rise if the insurance company
 80 does not explicitly aim at providing similar returns to all policyholders, as we propose to do instead
 81 by modifying the payoff structure of the group with the lower minimum interest rate guarantee.
 82 Consequently, the difference between the certainty equivalent returns would increase as well. Further,
 83 when there are few policyholders holding a lower individual guarantee, their participation in the
 84 surplus sharing must be very high to ensure fair contracts. (iv) If the two groups differ exclusively in
 85 the contract maturity, the fair participation rate for the group with the longer contract duration is much
 86 lower. As a consequence, the certainty equivalent return behaves similarly, although to a lesser extent.
 87 Further, the group with the longer contract duration receives a remarkably low fair participation rate if
 88 its size is much lower than the other group.

89 The remainder of this paper is structured as follows: Section 2 sets up the contract structure and
 90 describes the payoffs to the different customer groups and the modelling of the insurance and the
 91 financial risk. Section 3 focuses on the market valuation of the outstanding liabilities and explains
 92 how a fair contract analysis can be conducted in our framework. Section 4 is devoted to the numerical
 93 analysis and addresses the issue of fair pricing. Further, we find out which group benefits from
 94 specific portfolio compositions by comparing the certainty equivalent returns. Section 5 provides some
 95 concluding remarks and a short outlook on possible extensions.

96 2. Model Setup

97 We consider a life insurance company which consists of equity holders and two heterogeneous
 98 groups of policyholders. All policyholders take out their contracts at time 0. The policyholders’
 99 heterogeneity stems from either the offered minimum interest rate guarantees or the contract maturities.
 100 In group i , $i = 1, 2$, there are $N_i(0)$ homogeneous policyholders, meaning that they have the same
 101 age, make the same initial contribution $l(0)$ and the contracts they hold are identical. At time 0, the
 102 insurer’s stylized balance sheet is:

Assets	Liabilities
$W(0)$	$E(0) = (1 - \alpha_1 - \alpha_2)W(0)$ $L_1(0) = \alpha_1 W(0)$ $L_2(0) = \alpha_2 W(0)$
$W(0)$	$W(0)$

103 The initial assets of the insurance company can be split up into three
 104 components: $W(0) = E(0) + L_1(0) + L_2(0)$, where the premiums $L_1(0) = l(0)N_1(0)$
 105 and $L_2(0) = l(0)N_2(0)$ are contributed by the first and the second group of policyholders,
 106 respectively, and the remainder $E(0)$ by the equity holders. We denote the fraction of initial assets
 107 contributed by group i , $i = 1, 2$, by $\alpha_i = N_i(0)l(0)/W(0) = L_i(0)/W(0) \in (0, 1)$, and the share of
 108 initial assets contributed by the equity holders by $e(0) := 1 - \alpha_1 - \alpha_2 \in (0, 1)$.² Note that, as we focus
 109 on the impact of different minimum interest rate guarantees and contract maturities, the individual
 110 contribution $l(0)$ is assumed to be the same for all policyholders. The benefits for each group will be
 111 determined according to the initial contribution and contract provisions.

112 2.1. Contract Structure

113 We assume that every policyholder buys from the life insurer a participating pure endowment
 114 contract whose payoff is contingent on the event that the policyholder survives the contract maturity.
 115 We discuss two different ways the heterogeneity of groups 1 and 2 could be specified. The two groups
 116 differ exclusively by

117 **Case 1:** the minimum interest rate $g_i \geq 0$, $i = 1, 2$, promised by the insurer;

118 **Case 2:** the contract maturity date T_i , $i = 1, 2$.

119 In both cases, the insurer provides the policyholders with some guaranteed payoff. The aggregated
 120 guaranteed payment to group i becomes due at the contract maturity T_i and is defined by

$$G_i(T_i) := N_i(T_i)l(0)e^{g_i T_i}, \quad i = 1, 2, \quad (1)$$

121 where $N_i(t)$ is the (random) number of surviving policyholders at time $t \geq 0$ in group i .

122 In the subsequent paragraphs, we specify the outstanding liabilities $L_i(T_j)$, $i = 1, 2$, $j = 1, 2$, for
 123 Cases 1 and 2. We note that, if there are no surviving policyholders in a group, then the insurer is free
 124 from any duty of payments with respect to that group. Therefore, we will set in Case 1, involving a
 125 common maturity $T := T_1 = T_2$ for the two groups,

$$L_i(T) = \begin{cases} \Psi_i & \text{if } N_i(T) > 0 \\ 0 & \text{if } N_i(T) = 0 \end{cases} = \Psi_i \mathbb{1}_{\{N_i(T) > 0\}}, \quad i = 1, 2, \quad (2)$$

126 where $\mathbb{1}_{\{\cdot\}}$ is the indicator function. Here, Ψ_i represents the global payment made at maturity to
 127 surviving policyholders in group i , $i = 1, 2$. Case 2 involves two contract maturities and will be treated
 128 separately. The amounts Ψ_i , $i = 1, 2$, depend on the guaranteed payments $G_i(T_i)$, the value of the
 129 assets of the life insurance company, which is denoted by $W(t)$ at time $t \geq 0$, and on the participation
 130 rates $\delta_i \in [0, 1]$, according to which a share of the assets exceeding the guaranteed payoff is paid to
 131 surviving policyholders as a bonus.³

132 **Case 1.** We assess the impact of different minimum interest rate guarantees. To set up this case
 133 recall that $T_1 = T_2 = T$ and, without loss of generality, assume that $g_1 > g_2$. As a result, the payoff
 134 promised to the policyholders of group i , $i = 1, 2$, at maturity T is $G_i(T) = N_i(T)l(0)e^{g_i T}$. Such instance
 135 is common to many life insurers since older products still in force often have significantly higher

² We assume that the insurance company issues no further debt, raises no capital and pays no dividends to the equity holders within the time frame of interest.

³ Note that we allow for different participation rates for the two groups as the insurance company's goal is to set these rates so as to achieve fairness for both groups, see Section 3.

136 guaranteed rates than those sold more recently which were penalized by the ongoing low interest
 137 environment. Although in this stylized model all contracts are issued at the same date, our findings
 138 provide some guidance on establishing some contractual parameters, in particular the participation
 139 coefficients for which there is usually some discretion on the insurer's side.

140 Since several large insurance companies issuing participating contracts aim to provide the same
 141 (expected) rate of return to customers endowed with different minimum interest rate guarantees, we
 142 adjust the definition of the outstanding liabilities in order to achieve this desirable goal. Inspired by
 143 the model in Briys and de Varenne (1994 1997), the global payment Ψ_1 in (2), relative to policyholders
 144 of the group with the higher guarantee rate g_1 , is defined by

$$\Psi_1 = \begin{cases} \frac{G_1(T)}{G(T)} W(T) & \text{if } W(T) < G(T) \\ G_1(T) & \text{if } G(T) \leq W(T) \leq \frac{G(T)}{\alpha_1 + \alpha_2} \\ \zeta_1 & \text{if } \frac{G(T)}{\alpha_1 + \alpha_2} < W(T) \end{cases}, \quad (3)$$

145 where $G(T) = G_1(T) + G_2(T)$ is the total guaranteed payment. The rationale behind (3) is as follows:
 146 if the insurance company becomes insolvent, i.e. the value of the assets at maturity $W(T)$ is insufficient
 147 to cover the total guaranteed payoff $G(T)$, the available assets are shared among the surviving
 148 policyholders according to a proportional splitting rule.⁴ This implies that the company has limited
 149 liability towards its customers and, in case of insolvency, nothing is left to the equity holders. If
 150 the assets perform moderately, i.e. $G(T) \leq W(T) \leq G(T)/(\alpha_1 + \alpha_2)$, the surviving policyholders
 151 of group 1 receive their guaranteed payoff $G_1(T)$, but no bonus. If the assets perform well,
 152 i.e. $G(T)/(\alpha_1 + \alpha_2) < W(T)$, then policyholders of group 1 are entitled to a share of the assets $W(T)$
 153 on top of their guaranteed payments. The corresponding payoff ζ_1 is specified further below.

154 The global payment Ψ_2 in (2) made to surviving policyholders of the group with the lower
 155 guarantee rate g_2 is defined by

$$\Psi_2 = \begin{cases} \frac{G_2(T)}{G(T)} W(T) & \text{if } W(T) < G(T) \\ G_2(T) e^{\min\{(g_1 - g_2)T, \ln\left(\frac{W(T) - G_1(T)}{G_2(T)}\right)\}} & \text{if } G(T) \leq W(T) \leq \frac{G(T)}{\alpha_1 + \alpha_2} \\ \zeta_2 & \text{if } \frac{G(T)}{\alpha_1 + \alpha_2} < W(T) \end{cases}. \quad (4)$$

156 The rationale behind the expression in the middle row in (4) is as follows: after serving the minimum
 157 interest rate of the first group, the insurance company aims at endowing the second group with an
 158 identical minimum interest rate g_1 provided there is enough capital left for this. When this is not the
 159 case, the second group obtains the remainder $W(T) - G_1(T) (\geq G_2(T))$. On the other hand, as long
 160 as the company's assets are sufficient to distribute some bonus (third row), group 2 will likely be
 161 compensated for its lower guaranteed payoff by receiving a higher participation coefficient.⁵

162 In order to specify the payoffs ζ_i , $i = 1, 2$, we first define the quantity

$$D(T) := W(T) - \sum_{i=1}^2 \zeta_i^+, \quad (5)$$

⁴ An alternative rule uses the weights $\alpha_i / (\alpha_1 + \alpha_2)$, $i = 1, 2$, so that the splitting rule is decided by the groups' initial contributions. Choosing this alternative could, if only one group survives until time T , result in the equity holders receiving the remaining assets after the insurer has served the group still existent.

⁵ It may happen in (4) that the payoff in the third row is smaller than that in the middle one, corresponding to a lower assets' value. However, this fairly rare event does not result in a contradiction since it is down to the insurer to decide to what extent the goal of achieving equal rates of return shall be pursued.

163 where

$$\zeta_i^+ = G_i(T) + \delta_i \max \{ \alpha_i W(T) - G_i(T), 0 \}, \quad (6)$$

164 is the guaranteed payment plus the *regular bonus* the insurance company aims at delivering
 165 to the group of policyholders i . The amount $D(T)$ represents then the excess (deficit) of the
 166 assets above (below) the target payments to both groups. If $W(T) \geq \max \{ G_1(T)/\alpha_1, G_2(T)/\alpha_2 \}$,
 167 then $D(T) \geq 0$. If $\min \{ G_1(T)/\alpha_1, G_2(T)/\alpha_2 \} < W(T) < \max \{ G_1(T)/\alpha_1, G_2(T)/\alpha_2 \}$, the sign
 168 of $D(T)$ is indeterminate. Moreover, we define

$$\zeta_i^- := \begin{cases} G_i(T) & \text{if } \frac{G_i(T)}{\alpha_i} = \max \left\{ \frac{G_1(T)}{\alpha_1}, \frac{G_2(T)}{\alpha_2} \right\}, \\ W(T) - G_j(T) & \text{otherwise} \end{cases}, \quad i = 1, 2, j \in \{1, 2\} - \{i\}, \quad (7)$$

169 and let

$$\zeta_i = \zeta_i^+ \mathbf{1}_{\{D(T) \geq 0\}} + \zeta_i^- \mathbf{1}_{\{D(T) < 0\}}. \quad (8)$$

170 As long as $D(T) \geq 0$, the firm can serve both groups with their regular bonuses. If, however, $D(T) < 0$,
 171 the assets of the insurer are insufficient to generate such payments. We assume then that the group
 172 with the higher value of $G_i(T)/\alpha_i$ obtains only the guaranteed payment, while the other group seizes
 173 the remaining assets' value which is larger than the guaranteed payment, but smaller than the payoff
 174 including the regular bonus.⁶

175 **Case 2.** We assume that the policies from the first and second group expire at time T_1 , respectively
 176 time T_2 , and, without loss of generality, let $T_2 > T_1$. The minimum guaranteed rates are set to be equal,
 177 i.e. $g_1 = g_2 =: g$. The guaranteed payoffs are now $G_i(T_i) = N_i(T_i)l(0)e^{gT_i}$, $i = 1, 2$.

178 In order to define the outstanding liabilities $L_i(T_j)$, $i = 1, 2$, $j = 1, 2$, we need the market
 179 value $V_2(T_1)$ at the earliest maturity T_1 of the guaranteed payoff paid at T_2 to the second group.
 180 The exact specification of $V_2(T_1)$ will be given in Section 4. Define then $V(T_1) := G_1(T_1) + V_2(T_1)$ the
 181 market value at T_1 of the insurer's liabilities, including the immediate guaranteed payment to the
 182 first group and the market value of future liabilities for the second group. A premature default event
 183 may occur at time T_1 when $W(T_1) < V(T_1)$ and $N_1(T_1) > 0$. In this case, we assume that group 1
 184 will accept less than $G_1(T_1)$ as insurance benefit, so that an equal treatment of both groups can be
 185 achieved. In other words, the two groups of policyholders have the same claiming priority in the case
 186 of bankruptcy. More specifically, the global payment made to surviving policyholders of the first group
 187 at T_1 is defined by

⁶ Through this way of modelling of the outstanding liabilities, it could happen that the payments to the equity holders decrease or even vanish, although the assets' value increases at the same time. The rationale behind this circumstance is that when the assets pertaining to the policyholders as a whole create some surplus over the minimum guarantees, the insurer's primary goal is to provide them with their regular bonuses, if possible. Some alternative modelling methods apply if the insurance company wants to calculate the possible bonus payments to the different stakeholders based on their initial contributions. In this case, the definition of $\zeta_i(T)$, $i = 1, 2$, needs to take into account that only $(\alpha_1 + \alpha_2)W(0)$ is provided by the policyholders of the two groups at time 0 leading to a modification of (5) and (7). However, for the sake of brevity, we examine only the case described before.

$$\Psi_1 = \begin{cases} \frac{G_1(T_1)}{V(T_1)} W(T_1) & \text{if } W(T_1) < V(T_1) \\ G_1(T_1) & \text{if } V(T_1) \leq W(T_1) \leq \frac{V(T_1)}{\alpha_1 + \alpha_2} \\ \min \{ \zeta_1^+, W(T_1) - V_2(T_1) \} & \text{if } \frac{V(T_1)}{\alpha_1 + \alpha_2} < W(T_1) \end{cases} \quad (9)$$

188 Here, the target regular payment to the first group ζ_1^+ , defined as in (6) with T replaced by T_1 , may be
 189 so high that the second group could not be served with the notional guaranteed amount $V_2(T_1)$. In
 190 this case, only the amount $W(T_1) - V_2(T_1)$ ($> G_1(T_1)$) is available to the policyholders in group 1, so
 191 that the remaining assets match the market value of the guaranteed payoff to the second group.

192 To define the outstanding liability for group 2, we need to distinguish whether there is default at
 193 time T_1 or not. If default occurs, the second group obtains a rebate payment at T_1 amounting to

$$L_2(T_1) = \frac{V_2(T_1)}{V(T_1)} W(T_1) \mathbb{1}_{\{N_1(T_1) > 0, N_2(T_1) > 0\}} \mathbb{1}_{\{W(T_1) < V(T_1)\}}. \quad (10)$$

194 If there is no default at T_1 , i.e. when $W(T_1) \geq V(T_1)$ or when $N_1(T_1) = 0$, the contract payoff for the
 195 second group becomes due at time T_2 and is given by

$$L_2(T_2) = \Psi_2 \mathbb{1}_{\{N_2(T_2) > 0\}} \mathbb{1}_{\{N_1(T_1) = 0\} \cup \{W(T_1) \geq V(T_1)\}}, \quad (11)$$

196 with

$$\Psi_2 = \begin{cases} W'(T_2) & \text{if } W'(T_2) < G_2(T_2) \\ G_2(T_2) & \text{if } G_2(T_2) \leq W'(T_2) \leq \frac{G_2(T_2)}{\alpha_2'} \\ G_2(T_2) + \delta_2 (\alpha_2' W'(T_2) - G_2(T_2)) & \text{if } \frac{G_2(T_2)}{\alpha_2'} < W'(T_2) \end{cases}, \quad (12)$$

197 where $\alpha_2' = \alpha_2 / (1 - \alpha_1)$ and $W'(T_2)$ is the assets' value at T_2 taking into account the previous outflows
 198 to the policyholders of the first group. The exact specification of $W'(T_2)$ will be given in Section 4.
 199 Note also that the events in the indicators in (10) and (11) are disjoint, so only one benefit will be paid
 200 to the second group either at T_1 or T_2 .

201 2.2. Modelling Insurance and Financial Risk

202 In this section, we follow and adapt [Bacinello et al. \(2018\)](#) to our situation.

203 The insurance risk is made up of the hedgeable (also called diversifiable or unsystematic) part,
 204 which is likely to tail off once the portfolio of all policyholders of the heterogeneous groups is
 205 sufficiently large, and the systematic (or unhedgeable) part. The latter, called longevity risk, cannot be
 206 diversified away through pooling and affects all contracts equally by entailing the misestimation of
 207 the decline in mortality rates. An introduction to longevity risk is given in e.g. [Barrieu et al. \(2012\)](#);
 208 [Biffis \(2005\)](#) and [Biffis et al. \(2010\)](#) cover the stochastic modelling of longevity risk.

209 We select a risk neutral probability measure Q for pricing purposes.⁷ Furthermore, we let τ_i^j
 210 be the residual lifetime of the j -th policyholder, $j = 1, \dots, N_i(0)$, of group i , $i = 1, 2$. The following
 211 assumptions will remain valid throughout the paper.

212 **Assumption 1.** *There exists a positive random variable Δ , measurable with respect to the σ -algebra containing*
 213 *the information available to market participants at time T in Case 1, and T_1 in Case 2, such that*

⁷ By assuming that the markets are arbitrage-free, such a probability measure Q exists. As insurance markets are incomplete, the measure Q is chosen among infinitely many equivalent martingale measures.

$$\begin{aligned}
Q\left(\tau_1^1 > t_1^1, \dots, \tau_1^{N_1(0)} > t_1^{N_1(0)}, \tau_2^1 > t_2^1, \dots, \tau_2^{N_2(0)} > t_2^{N_2(0)} \mid \Delta\right) &= \prod_{i=1}^2 \prod_{j=1}^{N_i(0)} Q\left(\tau_i^j > t_i^j \mid \Delta\right) \\
&= \prod_{i=1}^2 \prod_{j=1}^{N_i(0)} e^{-\Delta \int_0^{t_i^j} m_i(v) dv},
\end{aligned} \tag{13}$$

for any $t_i^j \geq 0$, $i = 1, 2$, $j = 1, \dots, N_i(0)$, where m_i is a deterministic force of mortality depending on the initial age x_i of group i , non-negative, continuous, and satisfying $\int_0^{+\infty} m_i(v) dv = +\infty$.

Then, conditionally on Δ , the residual lifetimes τ_i^j , $i = 1, 2$, $j = 1, \dots, N_i(0)$, are independent. The random variable Δ can be thought of as a systematic risk factor whose effect is to rescale the deterministic forces of mortality m_i , $i = 1, 2$, by a random percentage. It is assumed to be the same for both groups of policyholders, so that all biometric differences between them stem from the deterministic forces of mortality, that are already equipped with safety margins, as will be specified in Section 4. Moreover, the fact that Δ is assumed to be part of the information available at the earliest maturity date means that its true value is unveiled within the (first) contract maturity, whereas today the market participants can merely anticipate the impact of the systematic risk since the rescaling amount is unknown at the valuation date 0. This greatly simplified circumstance is in some way acceptable by the fact that the insurer collects a vast quantity of demographic information from the examined and similar portfolios over the years, whose analysis can reveal the actual character of Δ . As we see in Section 4, we actually exploit this property only in Case 2.

The t -years survival probability for an individual belonging to group i , $i = 1, 2$, can be derived from Assumption 1 through

$${}_t p_{x_i} := Q\left(\tau_i^j > t\right) = E^Q\left[e^{-\Delta \int_0^t m_i(v) dv}\right], \tag{14}$$

for $t \geq 0$ and $j = 1, \dots, N_i(0)$. In the following, we also set

$${}_u p_{y_i}^* := e^{-\int_{y_i - x_i}^{u + y_i - x_i} m_i(v) dv}, \tag{15}$$

for $y_i \geq x_i$, $i = 1, 2$, and $u \geq 0$, so that in particular, for $0 \leq t \leq s$, we obtain a ‘deterministic’ survival probability ${}_{s-t} p_{x_i+t}^* = e^{-\int_t^s m_i(v) dv}$. Further, conditional on Δ , we have $N_i(t) \sim \text{Binomial}\left(N_i(0), \left({}_t p_{x_i}^*\right)^\Delta\right)$ for $i = 1, 2$ and $t \geq 0$.

Concerning the modelling of the financial risk, we presume that it is entirely driven by the randomness of the assets which is captured by the next important assumption.

Assumption 2. At time $t \geq 0$, the assets’ value $W(t)$ is defined by $W(t) = W(0)e^{R(0,t)}$, where $R(0, t)$ is the random assets’ log-return over $[0, t]$. This return is independent of Δ and the residual lifetimes τ_i^j , $i = 1, 2$, $j = 1, \dots, N_i(0)$.

The possibility of having an interest risk component, that can be implemented by introducing stochastic interest rates, is ignored in this paper whereby the market short rate, denoted by r , is supposed to be constant.

3. Valuation

We denote by V_i the initial market value of the outstanding liabilities for group i , $i = 1, 2$. The contracts issued by the insurer are fair if the following conditions hold:

$$V_1 = L_1(0), \quad V_2 = L_2(0). \quad (16)$$

245 In other words, the initial market value of the claim of group i , $i = 1, 2$, coincides with its initial
 246 investment $L_i(0) = \alpha_i W(0)$. Note that, if the equations in (16) hold, then fairness is guaranteed
 247 for equity holders as well. Economically, we can interpret these conditions as constraints on the
 248 participation coefficients δ_i , $i = 1, 2$. Nevertheless, it may happen that (16) implies participation
 249 coefficients that exceed 100% because there is a chance that the benefits of the customers are excessively
 250 low. Moreover, negative coefficients are also possible to compensate for too high benefits. We only
 251 consider fair contracts for which $\delta_i \in [0, 1]$, $i = 1, 2$.

252 Armed with the pricing measure Q , we can calculate V_i , $i = 1, 2$. For Case 1, entailing T as the
 253 maturity date, these quantities are specified by

$$V_i = E^Q \left[e^{-rT} L_i(T) \right], \quad i = 1, 2. \quad (17)$$

254 For Case 2 we have, for one thing,

$$V_1 = E^Q \left[e^{-rT_1} L_1(T_1) \right]. \quad (18)$$

255 Due to the possible premature deficit of the insurer at the earlier maturity T_1 , the outstanding payoff
 256 to group 2 is paid out either at T_1 or at the regular maturity T_2 . Then

$$V_2 = E^Q \left[e^{-rT_1} L_2(T_1) + e^{-rT_2} L_2(T_2) \right]. \quad (19)$$

257 4. Numerical Analysis

258 We conduct some numerical analyses to understand the relative size of participation rates for the
 259 heterogeneous customers and which group of policyholders is better or worse off when pooling them
 260 together. To serve this purpose, we compute for Cases 1 and 2

- 261 • firstly, the fair participation rates δ_i^* , $i = 1, 2$, under the pricing measure Q ;
- 262 • secondly, the annual certainty equivalent log-returns of the life insurance contracts under the
 263 real world measure P , henceforth just *certainty equivalent returns*, denoted by C_i , $i = 1, 2$, based
 264 on the fair participation rates δ_i^* .⁸

265 In Case 1, the certainty equivalent returns for group i , $i = 1, 2$ are given by

$$E^P \left[\frac{L_i(T)}{L_i(0)} \right] = e^{C_i T} \Leftrightarrow C_i = \frac{1}{T} \ln \left(E^P \left[\frac{L_i(T)}{L_i(0)} \right] \right). \quad (20)$$

266 Depending on different pooling schemes, we can figure out, based on C_i , $i = 1, 2$, how attractive our
 267 life insurance policies are against alternative investments. Through comparing C_1 and C_2 , we can also
 268 identify which group benefits more from a certain pooling scheme.

269 Concerning the modelling of the insurance risk under the pricing measure Q , we assume
 270 Gompertz's law, i.e. $m_i(t) = \lambda_i c_i^{x_i+t}$ for $i = 1, 2$ and $t \geq 0$, yielding ${}_{s-t}p_{x_i+t}^* = e^{-\lambda_i c_i^{x_i} (c_i^s - c_i^t) / \ln(c_i)}$
 271 for $0 \leq t \leq s$. Furthermore, we suppose that Δ follows a Gamma distribution $\Gamma(\beta, \theta)$

⁸ To calculate δ_i^* , $i = 1, 2$, we solve numerically the equations in (16) with V_i , $i = 1, 2$, given by (17) for Case 1, and (18) and (19) for Case 2. Due to the complicated structure of the outstanding liabilities, the computation of V_i , $i = 1, 2$, is based on a standard Monte Carlo simulation encompassing 100 000 draws. The calculation of C_i , $i = 1, 2$, is again based on the Monte Carlo method.

272 with $Var_Q(\Delta) = 0.1$ and $E^Q[\Delta] \in \{0.4, 0.8, 1\}$. The latter stipulation provides the possibility to
 273 integrate various degrees of the insurer's conservativeness into the pricing of the contracts. Specifically,
 274 lower values of $E^Q[\Delta]$ are associated with higher risk premiums or, put it another way, with increasing
 275 conservativeness of the company concerning longevity risk.

276 The financial risk merely depends on the stochastic log-return of the assets $R(0, t)$, that is assumed
 277 to be normally distributed under Q with mean $(r - \sigma^2/2)t$ and standard deviation $\sigma\sqrt{t}$, where σ is the
 278 assets' volatility.

279 Assumption 1 shall hold under P as well, with the same rescaling random variable Δ and
 280 with deterministic forces of mortality \tilde{m}_i , $i = 1, 2$. However, as the considered contracts are pure
 281 endowments, we have, for all $t \geq 0$,

$${}_t p_{x_i} = E^Q \left[e^{-\Delta \int_0^t m_i(v) dv} \right] > E^P \left[e^{-\Delta \int_0^t \tilde{m}_i(v) dv} \right] =: {}_t \tilde{p}_{x_i}, \quad i = 1, 2. \quad (21)$$

282 In other words, the risk neutral survival probability contains a safety loading. To achieve (21),
 283 we set $\tilde{m}_i = m_i/\gamma$, $i = 1, 2$, where $\gamma < 1$, and assume that the distribution of Δ under P
 284 is Gamma with the same variance as under Q , i.e. $Var_P(\Delta) = Var_Q(\Delta) = 0.1$, but with
 285 expectation $E^P[\Delta] = 1$. Under P , the distribution of the number of surviving policyholders at
 286 time $t \geq 0$ is $N_i(t)|\Delta \sim \text{Binomial} \left(N_i(0), \left({}_t \tilde{p}_{x_i}^* \right)^\Delta \right)$ with ${}_u \tilde{p}_{y_i}^* = e^{-\int_{y_i-x_i}^{u+y_i-x_i} \tilde{m}_i(v) dv}$ for $y_i \geq x_i$, $i = 1, 2$,
 287 and $u \geq 0$, and that of the assets' log-return is $R(0, t) \sim \mathcal{N} \left((\mu - \sigma^2/2)t, \sigma^2 t \right)$, where μ is the expected
 288 instantaneous rate of return of the assets. Then, Assumption 2 shall hold under P as well.

289 Subsequently, Table 1 summarizes the assumed values for the parameters that are not case-specific
 290 and valid in all of the following numerical analyses.

Symbol	Description	Value
$l(0)$	Initial contribution of a single policyholder	35
$e(0)$	Equity holders' share of initial assets	0.3
$x_1 (= x_2)$	Initial age of policyholders	40
$\lambda_1 (= \lambda_2)$	Age independent Gompertz parameter	$2.6743 \cdot 10^{-5}$
$c_1 (= c_2)$	Age dependent Gompertz parameter	1.098
β	Shape parameter of Δ under Q	$\{1.6, 6.4, 10\}$
θ	Scale parameter of Δ under Q	$\{0.25, 0.125, 0.1\}$
r	Risk free short rate	3%
σ	Assets' volatility	15%
μ	Assets' expected instantaneous rate of return	5%
γ	Adjustment factor to force of mortality	0.9

Table 1. Non case-specific parameters for numerical analyses.

291 The two values for the Gompertz parameters given in Table 1 were obtained by fitting the survival
 292 probabilities ${}_t p_{40}^*$ to the corresponding probabilities implied by the projected life table IPS55 in use in
 293 the Italian annuity market, see [Bacinello et al. \(2018\)](#). Further, combining pairwise the parameters of
 294 the Gamma distribution given in Table 1 will result in the values of $E^Q[\Delta]$ and $Var_Q(\Delta)$ considered.⁹

295 **Results for Case 1.** We set $T \in \{12, 25\}$ and consider reasonable choices for the minimum interest
 296 rate guarantees, $g_1 = 1.75\%$ and $g_2 = 1.25\%$. As previously outlined, a potentially desirable goal of
 297 an insurer is to provide all policyholders with the same (expected) rate of return regardless of the

⁹ The parameters of $\Delta \sim \Gamma(\beta, \theta)$ are calculated via $E^Q[\Delta] = \beta\theta$ and $Var_Q(\Delta) = \beta\theta^2$.

individual minimum interest rate guarantee. Hence, comparing the certainty equivalent returns C_1 and C_2 for the two groups in the present case seems particularly interesting. Tables 2 – 4 contain our findings for the fair participation rates and the annual certainty equivalent returns.

$T = 12$		$E^Q[\Delta] = 0.4$		$E^Q[\Delta] = 0.8$		$E^Q[\Delta] = 1$	
$N_1(0)$	$N_2(0)$	δ_1^*	δ_2^*	δ_1^*	δ_2^*	δ_1^*	δ_2^*
		C_1	C_2	C_1	C_2	C_1	C_2
1	1	72.42	78.01	76.38	81.60	78.37	83.41
		4.22	4.32	4.35	4.44	4.42	4.51
10	10	70.03	75.65	71.32	76.66	71.94	77.14
		4.30	4.41	4.35	4.44	4.37	4.46
100	100	70.05	75.68	71.34	76.69	71.96	77.17
		4.30	4.41	4.35	4.44	4.37	4.46
1000	1000	70.06	75.68	71.35	76.70	71.97	77.19
		4.30	4.41	4.35	4.45	4.37	4.46
100 000	100 000	70.06	75.68	71.35	76.70	71.97	77.18
		4.30	4.41	4.35	4.45	4.37	4.46

Table 2. Fair participation rates and certainty equivalent returns for Case 1 with $T = 12$, different portfolio sizes with $N_1(0) = N_2(0)$ and different values of $E^Q[\Delta]$. All results are in percentage.

From Table 2, we observe the following:

- (2.1) Group 2, endowed with a lower interest rate guarantee, is naturally provided with a higher fair participation rate δ_2^* and a perceptibly larger implied certainty equivalent return C_2 . In order to examine the goodness of the contract design in (4), middle row, to achieve similar rates of return for different groups of customers, we further carry out the analysis under the assumption that the same payout structure of group 1 is applied to group 2 (but still with different guarantees $g_1 > g_2$) and find that δ_2^* and C_2 are even higher. Therefore, if the insurance company aimed at treating both groups fairly when the payoff structures are identical, it should assign a much larger fair participation rate to the second group than to the first one, resulting in a greater difference between C_1 and C_2 . That is why our attempt at designing contracts that potentially provide the same rate of return to customers endowed with different minimum interest rate guarantees leads to more desirable results.
- (2.2) For any portfolio size, an increase in the longevity risk premium, i.e. lower values for $E^Q[\Delta]$, leads to smaller fair participation rates. This is because longevity improvements anticipated by the insurer increase the expected number of survivors, and consequently the value of the outstanding liabilities. To offset this effect and simultaneously ensure fairness, lower participation rates are offered. The same observation holds true for the certainty equivalent returns of the policyholders under the physical measure P . The reason for this is due to the smaller participation rates δ_i^* , $i = 1, 2$, since a greater degree of conservativeness harms the customers' benefits.
- (2.3) For the exceptional case where $N_1(0) = N_2(0) = 1$, the fair participation rates are considerably higher than those obtained with larger portfolio sizes due to the sizeable extinction probability of the groups and the fact that the equity holders seize all the assets pertaining to the extinct group(s). As a compensation, the policyholders need to be served with substantially larger participation rates.

- 325 (2.4) It seems that, with a very small portfolio size, e.g. $N_i(0) = 10$, $i = 1, 2$, the portfolio is already
 326 well-diversified, i.e. the expected impact of the systematic part of the biometric risk is the only
 327 component still playing a role, since similar results are achieved as, e.g. when $N_i(0) = 100\,000$.
 328 (2.5) It is notable that all certainty equivalent returns lie between the risk free rate of return and the
 329 expected rate of return of the assets under P , i.e. $C_i \in (r, \mu) = (0.03, 0.05)$, $i = 1, 2$. It is true that
 330 our life insurance policies cannot beat the pure investment into the assets due to the guaranteed
 331 interest rate, although the values obtained are much closer to μ than to r . Nevertheless, the
 332 included guarantees of the insurance products make them much less risky and are crucial for
 333 many potential customers when comparing different investment opportunities.

$T = 12$		$E^Q[\Delta] = 0.4$		$E^Q[\Delta] = 0.8$		$E^Q[\Delta] = 1$	
$N_1(0)$	$N_2(0)$	δ_1^*	δ_2^*	δ_1^*	δ_2^*	δ_1^*	δ_2^*
		C_1	C_2	C_1	C_2	C_1	C_2
10	1000	68.51	74.59	69.80	75.66	70.44	76.18
		4.26	4.38	4.31	4.42	4.33	4.44
100	100 000	68.55	74.57	69.91	75.64	70.56	76.15
		4.26	4.37	4.31	4.41	4.33	4.43
1000	100	71.34	76.75	72.57	77.71	73.16	78.18
		4.33	4.44	4.38	4.47	4.40	4.49
100 000	10	71.63	77.01	72.84	77.96	73.43	78.43
		4.34	4.44	4.38	4.48	4.40	4.50

Table 3. Fair participation rates and certainty equivalent returns for Case 1 with $T = 12$, different portfolio sizes with $N_1(0) \neq N_2(0)$ and different values of $E^Q[\Delta]$. All results are in percentage.

334 Table 3 displays the effect of assuming different group sizes. Our findings are listed below:

- 335 (3.1) We observe that policyholders in the larger group obtain relatively higher fair participation
 336 rates δ_i^* , $i = 1, 2$, and consequently mostly also relatively higher certainty equivalent returns C_i
 337 on their investments when comparing them with the corresponding numbers from Table 2.
 338 (3.2) For the first group, an increase in $N_1(0)$ leads to an increase in δ_1^* and C_1 in general, independently
 339 of the size of the other group, while, for the second group, the opposite relations apply. We can
 340 conclude that, if there are only a few policyholders holding a lower individual guarantee than
 341 the rest, their participation in the surplus distribution must be very high to ensure fair contracts,
 342 especially if they represent a clear minority. Another related interesting fact is that a sudden
 343 spread within the values for δ_i^* and C_i , $i = 1, 2$, occurs as soon as the size ratio between the two
 344 groups shifts.

345 Figure 1 clearly illustrates the fact described in Remark (3.2) using the example where $E^Q[\Delta] = 0.8$. In
 346 the two plots, we can further detect that the corresponding numbers from Table 2 lie in-between the
 347 ones from Table 3.

348 Table 4 completes Case 1 by addressing the influence of the maturity date. We observe that a longer
 349 contract duration increases both the fair participation rates δ_i^* , $i = 1, 2$, and the certainty equivalent
 350 returns C_i compared to Table 3. Moving T from 12 to 25 years implies that less policyholders are
 351 expected to survive the maturity date, consequently a lower aggregated guaranteed payment needs to
 352 be provided by the insurer. Due to the fair contract principles, the insurer is then able to provide larger

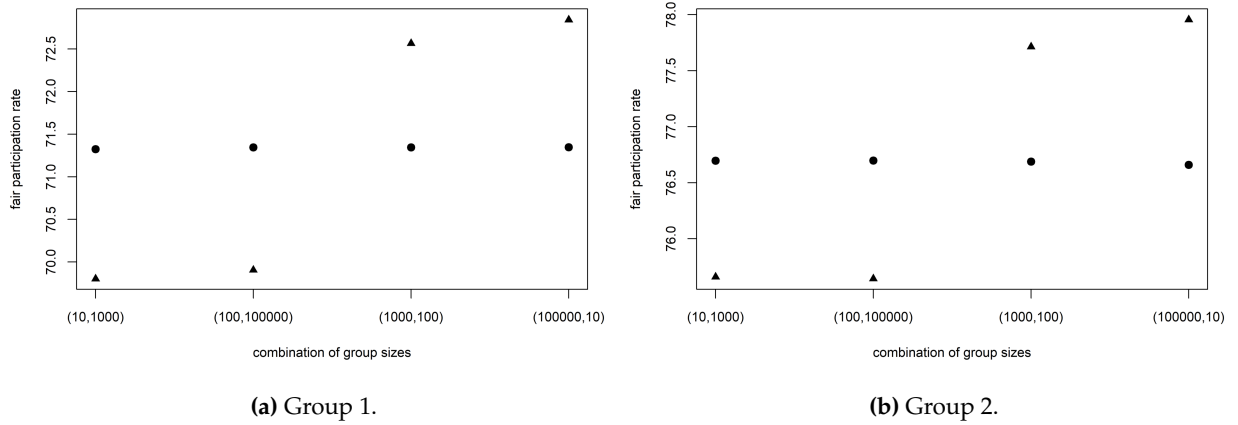


Figure 1. Case 1: fair participation rates in percentage when $E^Q[\Delta] = 0.8$. Triangles show rates from Table 3 for given combinations of group sizes. Circles additionally show rates from Table 2 when group sizes fulfil $N_2(0) = N_1(0) = 10, 100, 1000, 100\,000$ (left) and $N_1(0) = N_2(0) = 1000, 100\,000, 100, 10$ (right).

$T = 25$		$E^Q[\Delta] = 0.4$		$E^Q[\Delta] = 0.8$		$E^Q[\Delta] = 1$	
$N_1(0)$	$N_2(0)$	δ_1^*	δ_2^*	δ_1^*	δ_2^*	δ_1^*	δ_2^*
		C_1	C_2	C_1	C_2	C_1	C_2
10	1000	82.45	88.11	84.59	89.59	85.56	90.24
		4.61	4.72	4.67	4.77	4.69	4.79
100	100 000	82.56	88.09	84.80	89.57	85.81	90.22
		4.61	4.72	4.67	4.77	4.70	4.79
1000	100	85.80	90.45	87.58	91.58	88.39	92.09
		4.69	4.78	4.74	4.82	4.76	4.83
100 000	10	86.13	90.73	87.87	91.84	88.66	92.33
		4.69	4.79	4.74	4.82	4.76	4.84

Table 4. Fair participation rates and certainty equivalent returns for Case 1 with $T = 25$, different portfolio sizes with $N_1(0) \neq N_2(0)$ and different values of $E^Q[\Delta]$. All results are in percentage.

353 participation rates for a longer contract maturity. Again, a high participation in the surplus results in a
 354 high certainty equivalent return.

355 **Results for Case 2.** In this case, the maturity dates of the groups' policies differ. The quantity $V_2(T_1)$ in
 356 the definitions of the payoff functions (see Equations (9) – (12)) is given by

$$V_2(T_1) = E_{T_1}^Q \left[G_2(T_2) e^{-r(T_2-T_1)} \right], \quad (22)$$

357 where $E_{T_1}^Q[\cdot]$ denotes the expectation under Q conditional on information available at time T_1 .
 358 Since $G_2(T_2) = N_2(T_2)l(0)e^{gT_2}$ is proportional to $N_2(T_2)$, and by exploiting Assumptions 1 and 2
 359 of Section 2.2, we can write (22) as

$$V_2(T_1) = e^{-r(T_2-T_1)}l(0)e^{gT_2}N_2(T_1) \left(T_{2-T_1}p_{x_2+T_1}^* \right)^\Delta. \quad (23)$$

360 Furthermore, the remaining assets' value $W'(T_2)$ is defined as

$$W'(T_2) = (W(T_1) - L_1(T_1)) e^{R(T_1, T_2)}, \quad (24)$$

361 where $R(T_1, T_2)$ is the assets' log-return for the period from T_1 to T_2 , which is normally distributed
 362 with mean $(r - \sigma^2/2)(T_2 - T_1)$ and variance $\sigma^2(T_2 - T_1)$ under Q .¹⁰

363 As the first group is entirely dealt with at the earlier maturity T_1 , its certainty equivalent return is
 364 defined as in (20). On the contrary, the second group receives a payout either at the regular maturity T_2 ,
 365 or at T_1 if the insurance company is then insolvent. To be able to incorporate this contingency into
 366 the computation of C_2 , we assume that the possible payment $L_2(T_1)$ is invested into the riskless asset
 367 from T_1 to T_2 yielding an annual log-return of r , so that

$$C_2 = \frac{1}{T_2} \ln \left(E^P \left[\frac{L_2(T_1)e^{r(T_2-T_1)} + L_2(T_2)}{L_2(0)} \right] \right). \quad (25)$$

368 The number of surviving policyholders of the second group is now needed at both
 369 times T_1 and T_2 . Consequently, under Q and conditional on Δ and $N_2(T_1)$, we
 370 obtain $N_2(T_2) \sim \text{Binomial} \left(N_2(T_1), \left(T_{2-T_1}p_{x_2+T_1}^* \right)^\Delta \right)$.¹¹

371 Concerning the case-specific parameters, we stipulate that $(T_1, T_2) \in \{(10, 12), (12, 25)\}$
 372 and $g = 1.25\%$. Tables 5 – 7 show our values for δ_i^* and C_i , $i = 1, 2$, for this second case.

373 Table 5 delivers the fair participation rates and certainty equivalent returns for the two
 374 heterogeneous and equal-sized groups of policyholders with $T_1 = 10$ and $T_2 = 12$. We can establish
 375 the following:

376 (5.1) One of the main questions is: why are fair participation coefficients for the second group so much
 377 smaller compared to group 1, even though in Case 1 it was seen that a longer duration leads
 378 to higher fair participation rates? A possible explanation is the fact that only a portion of the
 379 assets $\alpha_1 W(T_1)$ pertaining to the first group at T_1 , is paid out to its policyholders if the insurer is
 380 able to achieve a surplus. As a consequence, group 2 profits from the residual amount staying in

¹⁰ Under the physical measure P , $R(T_1, T_2) \sim \mathcal{N}((\mu - \sigma^2/2)(T_2 - T_1), \sigma^2(T_2 - T_1))$.

¹¹ Under the physical measure P , $N_2(T_2) | (\Delta, N_2(T_1)) \sim \text{Binomial} \left(N_2(T_1), \left(T_{2-T_1} \tilde{p}_{x_2+T_1}^* \right)^\Delta \right)$.

$(T_1, T_2) = (10, 12)$		$E^Q[\Delta] = 0.4$		$E^Q[\Delta] = 0.8$		$E^Q[\Delta] = 1$	
$N_1(0)$	$N_2(0)$	δ_1^*	δ_2^*	δ_1^*	δ_2^*	δ_1^*	δ_2^*
		C_1	C_2	C_1	C_2	C_1	C_2
1	1	75.52	49.27	78.36	52.07	79.90	53.47
		4.26	4.04	4.36	4.16	4.42	4.22
10	10	73.62	47.73	74.51	48.80	74.92	49.32
		4.34	4.13	4.37	4.18	4.39	4.20
100	100	73.63	47.74	74.53	48.82	74.95	49.34
		4.34	4.13	4.37	4.18	4.39	4.20
1000	1000	73.64	47.74	74.53	48.82	74.96	49.34
		4.34	4.13	4.37	4.18	4.39	4.20
100 000	100 000	73.63	47.74	74.53	48.82	74.96	49.35
		4.34	4.13	4.37	4.18	4.39	4.20

Table 5. Fair participation rates and certainty equivalent returns for Case 2 with $(T_1, T_2) = (10, 12)$, different portfolio sizes with $N_1(0) = N_2(0)$ and different values of $E^Q[\Delta]$. All results are in percentage.

381 the company which boosts the probability of gaining relatively high assets' values in the future.
 382 To maintain fairness, a lower δ_2^* is required.
 383 (5.2) The spread between C_1 and C_2 is always significantly positive, although it decreases as $E^Q[\Delta]$
 384 increases. Clearly, the substantially higher fair participation rates for group 1 play a major role
 385 here. Nevertheless, these differences are much less relevant than those between δ_1^* and δ_2^* .

386 Figure 2 illustrates the situation described in (5.1) and shows that obtaining fairness is associated with
 387 providing the second group with a much smaller participation coefficient.

$(T_1, T_2) = (10, 12)$		$E^Q[\Delta] = 0.4$		$E^Q[\Delta] = 0.8$		$E^Q[\Delta] = 1$	
$N_1(0)$	$N_2(0)$	δ_1^*	δ_2^*	δ_1^*	δ_2^*	δ_1^*	δ_2^*
		C_1	C_2	C_1	C_2	C_1	C_2
10	1000	72.82	53.45	73.65	54.45	74.04	54.94
		4.32	4.11	4.35	4.16	4.37	4.19
100	100 000	72.84	53.52	73.71	54.53	74.14	55.02
		4.32	4.11	4.35	4.16	4.37	4.19
1000	100	74.30	38.77	75.21	39.88	75.65	40.43
		4.36	4.15	4.39	4.20	4.41	4.22
100 000	10	74.45	35.60	75.36	36.68	75.80	37.21
		4.36	4.16	4.40	4.20	4.41	4.23

Table 6. Fair participation rates and certainty equivalent returns for Case 2 with $(T_1, T_2) = (10, 12)$, different portfolio sizes with $N_1(0) \neq N_2(0)$ and different values of $E^Q[\Delta]$. All results are in percentage.

388 As before, the case when $N_1(0) \neq N_2(0)$ holds is evaluated for Case 2 and the corresponding
 389 results are presented in Tables 6 and 7. As far as Table 6 is concerned, two striking features are:

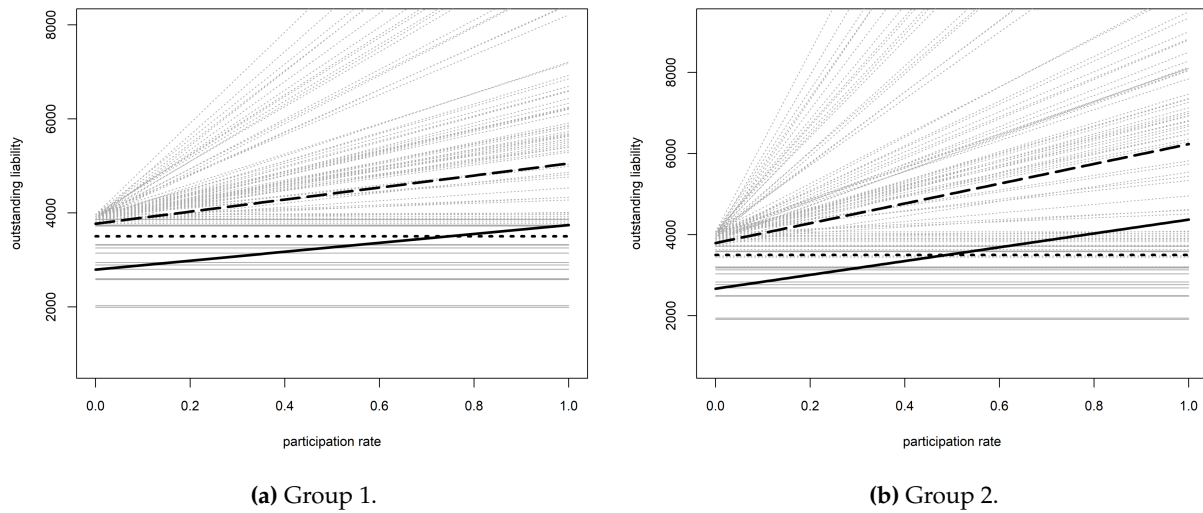


Figure 2. Case 2: 100 scenarios of the outstanding liability in terms of the participation rate when $E^Q[\Delta] = 0.8$ (grey dotted if no default at maturity T_1 , grey otherwise). Additionally, the mean (black dashed) and discounted mean (black) of the outstanding liability, and the premium (black dotted) are shown.

- 390 (6.1) The fair participation rates δ_i^* , $i = 1, 2$, grow with $N_i(0)$. Therefore, it is surprising that, unlike
 391 the certainty equivalent return C_1 of the first group, C_2 declines as the second group size $N_2(0)$
 392 increases (as in Table 3 where this also holds for δ_2^*). Yet, this finding reinforces the fact that low
 393 values of δ_2^* , when $N_2(0)$ is small, are necessary.
- 394 (6.2) The most remarkable feature is given by the variation in the values of δ_2^* for a given mortality
 395 pricing assumption when changing the composition of the portfolio (in particular, when
 396 comparing cases with $N_1(0) < N_2(0)$ to cases with $N_1(0) > N_2(0)$).

397 To get a better understanding of Remark (6.2), a key example is shown in Figure 3 where, on the
 398 left-hand side, δ_2^* is plotted against numbers of policyholders in group 1. An increase in $N_1(0)$ results
 399 in a remarkable decrease of the participation coefficient δ_2^* . The adjoining graph in Figure 3 displays,
 400 inter alia, the development of the assets' value $W'(T_2)$ at time T_2 , which is relevant for the contract
 401 payoff of the second group if the insurer is solvent at T_1 . The rise of $W'(T_2)$ stems from the fact
 402 that the two quantities $W(T_1)$ and $L_1(T_1)$ occurring in (24) evolve differently, i.e. $W(T_1)$ grows faster
 403 than $L_1(T_1)$. Thus, the second group would benefit from a bigger size of group 1 and consequently δ_2^*
 404 needs to be lowered.

405 As far as Table 7 is concerned, we further observe the following:

- 406 (7.1) Compared to Remark (6.2), the fair participation rate of the second group seems to smooth out
 407 over time within one longevity pricing assumption since the fluctuations between the varying
 408 pooling schemes subside. Specifically, δ_2^* goes down if $N_2(0)/N_1(0)$ is large (unlike Table 4 when
 409 compared to Table 3) and it goes up if $N_2(0)/N_1(0)$ is small (as in Table 4 when compared to
 410 Table 3). Looking at the values of δ_1^* , the same pattern is observed, i.e. a high $N_2(0)/N_1(0)$ results
 411 in lower fair participation coefficients and a low $N_2(0)/N_1(0)$ leads to (much) higher ones.
- 412 (7.2) Concerning the certainty equivalent returns, those of group 1 behave quite as expected, i.e. for
 413 the first two pooling schemes (low δ_1^*), smaller figures of C_1 are obtained and for the last two

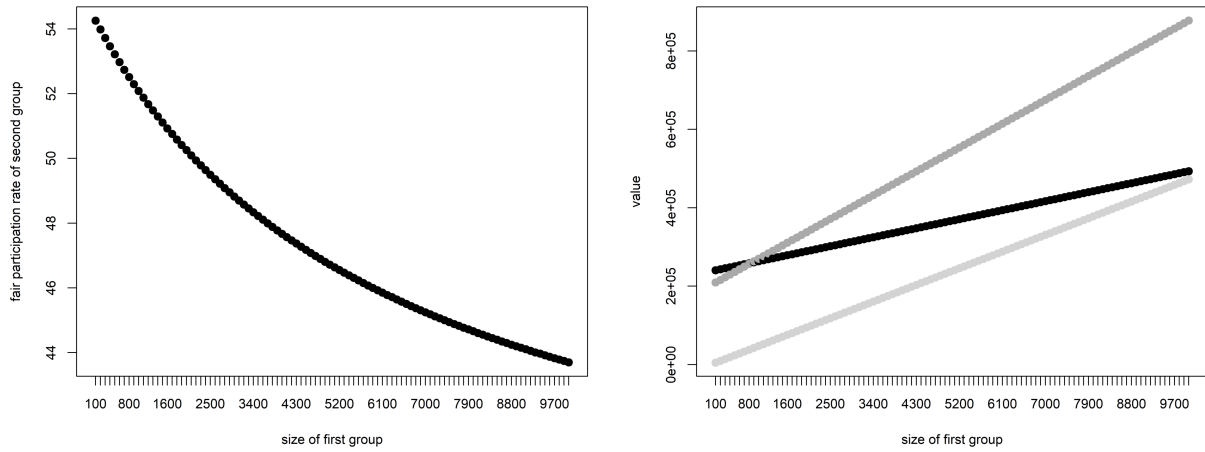


Figure 3. Case 2: fair participation rate δ_2^* in percentage (left), assets' and liability's values (right) in terms of the size of the first group when $N_2(0) = 3000$ and $E^Q[\Delta] = 0.8$. In the right-hand plot, total assets' value $W(T_1)$ at T_1 (dark grey), outstanding liability for the first group $L_1(T_1)$ at T_1 (light grey) and assets' value $W'(T_2)$ at T_2 (black) are shown.

$(T_1, T_2) = (12, 25)$		$E^Q[\Delta] = 0.4$		$E^Q[\Delta] = 0.8$		$E^Q[\Delta] = 1$	
$N_1(0)$	$N_2(0)$	δ_1^*	δ_2^*	δ_1^*	δ_2^*	δ_1^*	δ_2^*
		C_1	C_2	C_1	C_2	C_1	C_2
10	1000	70.61	51.90	71.29	53.75	71.66	54.62
		4.25	4.39	4.27	4.48	4.29	4.52
100	100 000	70.62	51.99	71.35	53.83	71.75	54.71
		4.25	4.39	4.28	4.48	4.29	4.52
1000	100	77.35	39.07	78.24	41.06	78.67	42.01
		4.44	4.33	4.47	4.43	4.48	4.48
100 000	10	78.19	36.59	79.13	38.59	79.58	39.52
		4.46	4.33	4.49	4.43	4.51	4.47

Table 7. Fair participation rates and certainty equivalent returns for Case 2 with $(T_1, T_2) = (12, 25)$, different portfolio sizes with $N_1(0) \neq N_2(0)$ and different values of $E^Q[\Delta]$. All results are in percentage.

414 combinations (high δ_1^*), larger values occur, compared to Table 6. By contrast, the relevant values
 415 of C_2 are always significantly higher than their counterparts in the previous table. A possible
 416 reason for that is our assumption made in the specific definition of this quantity given in (25),
 417 namely that the premature payoff to the second group conditioned by a default event at time T_1
 418 is invested into the riskless asset until T_2 .

419 5. Concluding Remarks

420 Participating life insurance products with minimum guarantees still represent a large portion
 421 of the contract portfolios of many life insurers. Due to the challenges these products have had to
 422 face in the recent past, such as the ongoing low interest rate environment, it is of special importance
 423 to adequately assess the financial standing of the firms. For this purpose, we seek to establish a
 424 model which strives to include several possible influencing factors. In addition to the introduction of
 425 a financial risk component, i.e. the uncertainty about future developments of the assets, and of the
 426 default risk, i.e. the chance of a distress of the company, we also integrate the longevity risk that is
 427 specifically crucial for life insurers. Especially in the light of the fact that people steadily get older on
 428 average and because of our focus on products cashing out only the claims of surviving policyholders,
 429 like pure endowments, it is reasonable to enhance our exploration by taking this risk into account. In
 430 this way, we can also study the effects of different longevity pricing assumptions made by the insurance
 431 company that reflect its degree of conservativeness. Furthermore, we aim to incorporate an often
 432 unconsidered circumstance, namely that customers and their contracts are (partially) heterogeneous.
 433 Therefore, we simplistically divide them into two homogeneous group. As a consequence, crucial
 434 issues, such as the impact of the usually high guarantees of old policyholders on the payout structures
 435 of new customers who are endowed with much lower guaranteed interest rates, can be surveyed.

436 After modelling the liabilities, we value them on a market-consistent basis leading to the feasibility
 437 of a fair contract analysis. With the aid of such an analysis, it is possible to determine appropriate
 438 policy parameters and in particular the participation rates. Building on the outcomes, we are also
 439 able to compute other interesting key figures, like the physical returns for the diverse insured persons.
 440 Eventually, we detect the effects of the different elements included in the model on the life insurer's
 441 and the policyholders' positions. Our main findings are listed in the following overview:

- 442 • If the insurer decides to heavily load risk premiums for the systematic part of the insurance risk,
 443 lower fair participation rates result. This in turn also hits the customers' returns, particularly if
 444 the presumptions on the longevity risk are very prudent.
- 445 • Maintaining usual practised participation rates of $\sim 80 - 100\%$ (often prescribed by law) can give
 446 rise to severe financial problems for the insurer, as certain portfolio and parameter combinations
 447 actually imply smaller participation coefficients ensuring the fairness of the contracts.
- 448 • If the two groups differ exclusively in the promised minimum interest rate guarantee provided
 449 by the insurer (Case 1), the group endowed with the lower minimum interest rate guarantee
 450 receives a larger fair participation rate. This increase is intensified if the insurance company does
 451 not explicitly aim to provide similar returns to all policyholders. Consequently, the difference
 452 between the actual returns widens as well. Thus, the proposed definition of the payoff structure
 453 for this case turns out to be an option the insurer can exploit in order to protect the customers
 454 and advance the desirable goal of achieving similar returns for everyone.
- 455 • In Case 1, another observation leads to the insight that, if there are only a few policyholders
 456 holding a lower individual guarantee than the rest, their participation in the surplus sharing
 457 must be really high to ensure fair contracts, especially if they represent a clear minority.

- 458 • If the two groups differ exclusively in the contract maturity date (Case 2), the fair participation
459 rate for the group with the longer contract duration is much lower, and so is the resulting actual
460 return, although on a considerably smaller scale.
- 461 • In Case 2, the group with the longer contract duration receives a remarkably low fair participation
462 rate if it outnumbers the members of the other group.

463 While the paper at hand is not able to capture every facet, in the given context, of a modern insurance
464 company acting in an open market economy, we think that our setup, paired with the wide numerical
465 analyses and related findings, helps to get a better understanding of the interaction between the several
466 influencing variables and to assess more thoroughly possibly occurring situations with their inherent
467 chances.

468 Potential aspects of future research can include, for instance, the study of alternative splitting
469 rules in the event of bankruptcy, the allowance for a stochastic short rate model, the combination of
470 different elements of heterogeneity, or the adoption of more sophisticated assumptions concerning the
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