Welfare theorems for random assignments with priorities*

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Abstract

We introduce new notions of priority-constrained efficiency and provide priority-constrained versions of the ordinal efficiency welfare theorem for school choice lotteries. Moreover, we show that a priority-constrained version of a cardinal second welfare theorem fails to hold, but can be restored for a relaxed notion of equilibrium with priority-specific prices.
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1 Introduction

The assignment of students to schools (Abdulkadiroglu and Sönmez, 2003) is one of the major applications of matching theory. A school choice mechanism assigns students to schools taking into account the preferences of students and priorities of the students at different schools. Coarse priorities are a generic feature in school choice. In practice, students are prioritized according to coarse criteria (e.g. based on catchment areas or having a sibling in the school) such that many students have the same priority for a seat at a school. Thus, one can sometimes not avoid treating students differently ex-post even though they have the same priorities and preferences. However, ex-ante, some form of fairness can be restored by the use of lotteries. This motivates the study of school choice lotteries.

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Respecting priorities in school choice mechanisms can lead to efficiency losses (Abdulkadiroglu and Sönmez, 2003). This motivates the question under which conditions priority-constrained efficiency can be achieved. We contribute to this discussion by studying different notions of constrained efficiency for school choice lotteries, and welfare in pseudo-market mechanisms with priorities.

Throughout the paper we are interested in three different classes of efficiency-related concepts and how they relate to each other: priority-constrained efficiency, priority-respecting improvement cycles,[1] and pseudo-market equilibria with priority-specific prices. We introduce a weaker and a stronger version of each of the three concepts.

Constrained efficiency requires that a random assignment cannot be welfare improvement upon by a random assignment that respects priorities as least as much as the original random assignment. Our two versions of the notion use different criteria to compare priority-respect of different random assignments: The first criterion compares priority-respect by first-order stochastic dominance according to schools' priorities. We call the induced efficiency notion "priority-constrained efficiency". The second (weaker) criterion compares priority-respect by cut-offs, i.e., the lowest priority that is sufficient to be admitted to a school with positive probability.[2] We call the induced (stronger) efficiency notion "cut-off-constrained efficiency".[3] We consider ordinal as well as cardinal versions of the two efficiency notions.

Correspondingly, we introduce two notions of an improvement cycle. Priority-respecting improvement cycle are improvements cycles whose execution does not impair priorities. Equal-priority improvement cycle are improvement cyles in which two consecutive agents have the same priority at the school under consideration.

Finally, we consider two notions of an equilibrium with priority-specific prices: In a pseudo-market (Hylland and Zeckhauser, 1979; He et al., 2018), a random assignment is generated by a market for probability shares. Each agent has a budget of tokens and can "buy" probability shares at the different schools. Agents face different prices depending on their priority. Our first equilibrium notion requires that prices are priority-specific and (weakly) decreasing with priority. The second and stronger equilibrium notion

[1]An improvement cycle is a sequence of agents $i_0, i_1, \dots, i_K$ and schools $j_0, j_1, \dots, j_K$ such that agent i_0 is matched to school j_0 with positive probability, but prefers j_1 to j_0, agent i_1 is matched to school j_1 with positive probability, but prefers j_2 to j_1, and so on taking indices modulo $K + 1$.

[2]If priorities are, for example, derived from exam scores or grade point averages, then cut-offs are specified by minimum scores that grant admission at the different schools, where randomization can be used to ration seats among applicants who achieve exactly the minimum score. These kinds of priorities occur frequently in centralized college admission (e.g. in China, and several European countries, see `http://www.matching-in-practice.eu/higher-education/`). In practice, different methods are used to ration seats among equal score students, such as lotteries, the use of additional tie-breaking criteria, or (in the case of Hungary) leaving seats unassigned.

[3]This notion is particularly natural for ex-ante stable random assignments: ex-ante stability requires that a student only can obtain a seat at a school with positive probability if there is no higher priority student for that school who obtains a less desirable school with positive probability (Kesten and Ünver, 2015). For deterministic matchings under strict priorities and preferences, priority cut-offs uniquely determine stable matchings (Azevedo and Leshno, 2016). Similarly, in the probabilistic setting with coarse priorities and strict preferences, ex-ante stable random assignments are uniquely determined by priority cut-offs and by a probabilistic rationing rule for admission at the cut-off. Cut-off constrained efficiency precludes inefficient rationing for given priority cut-offs.

(due to He et al., 2018) uses "cut-off pricing": At each school, there is a specific cut-off priority class, such that agents within the priority class face the same finite price, agents ranked strictly above the cut-off can obtain a seat in the school for free, and agents ranked strictly below the cut-off face an infinite price for a seat in the school. These equilibria are ex-ante stable by construction.

Our first main result (Theorem 1) establishes that priority-constrained ordinal efficiency is equivalent to the absence of priority-respecting improvement cycles, and moreover, equivalent to the existence of a cardinal utility profile consistent with the ordinal preferences such that social-welfare is maximized subject to priority constraints. Thus, we generalize the ordinal efficiency welfare theorem (McLennan, 2002; Manea, 2008) to the school-choice context. If the random assignment under consideration is ex-ante stable, we obtain a stronger version (Corollary 1) of the result: in this case, constrained ordinal efficiency according to the cut-off criterion is equivalent to the absence of equal-priority improvement cycles, and moreover, equivalent to the existence of a cardinal utility profile consistent with the ordinal preferences such that social-welfare is maximized subject to a set of priority cut-offs.

Next, we relate the different equilibrium notions to the efficiency notions: We show that equilibria in the sense of He et al. (2018) are cut-off constrained efficient (Corollary 2). Moreover, we prove (Theorem 2) a stronger version of Corollary 1, where for ex-ante stable assignments without equal-priority improvement cycles, supporting prices and budgets can be constructed along the utility profile, such that the random assignment under consideration is an equilibrium assignment in the sense of He et al. (2018) for the economy defined by the constructed utility profiles and budgets. This naturally leads to the question whether the result can be strengthened in such a way that for any utility profile and priority-constrained efficient random assignment, there exist corresponding budgets such that the random assignment is an equilibrium assignment in the sense of He et al. (2018). In other words, does a constrained second welfare theorem hold? We show using a counterexample that this is not the case. However, for the relaxed notion of equilibrium, we obtain (Theorem 3) a second welfare theorem.

1.1 Related literature

Ordinal efficiency welfare theorems for probabilistic assignments have been studied for object allocation without priorities (McLennan, 2002; Manea, 2008) and for marriage markets (Doğan and Yıldız, 2016). An ordinal efficiency welfare theorem establishes that ordinal efficiency (with welfare evaluated by first-order stochastic dominance) for a random assignment is equivalent to the existence of a cardinal utility profile consistent with the ordinal preferences under which the random assignment maximizes social welfare when lotteries are evaluated according to expected utility. The original ordinal welfare theorem is due to McLennan (2002), answering a question raised by Bogomolnaia and Moulin (2001). Manea (2008) provides a constructive proof.

Kesten and Ünver (2015) initiate the study of ex-ante stable school choice lotteries. For the classical marriage model, the condition was first considered by Roth et al. (1993). Kesten and Ünver (2015) consider mechanisms that implement ex-ante stable lotteries and satisfy constrained ordinal efficiency properties that are generally more stringent than the ones we consider. Han (2016) considers a refinement of ex-ante stability and mechanisms that implement such allocations. He et al. (2018) define an appealing class of mechanisms that implement ex-ante stable lotteries. These mechanisms generalize the pseudo-market mechanisms of Hylland and Zeckhauser (1979) by allowing for priority-specific prices (agents with different priorities are offered different prices). Cardinal welfare theorems for random assignments are counterparts to the classical welfare theorems for exchange economies. For cardinal welfare theorems, utility profiles are the primitive of the model. Hylland and Zeckhauser (1979) show that the equilibria of their pseudo markets are Pareto efficient and hence establish a cardinal first welfare theorem. Miralles and Pycia (2020) establish a cardinal second welfare theorem that demonstrates that each Pareto efficient random assignment can be decentralized as a pseudo-market equilibrium by appropriately choosing budgets and prices. We show that their result only generalizes to pseudo markets with priorities, only if we use the weaker notion of an equilibrium with priority-specific prices.

2 Model

There is a set of n agents N and a set of m schools M. A generic agent is denoted by i and a generic school by j. For each school j, there is a finite number of seats $q_j \in \mathbb{N}$. We assume that there are as many school seats as agents, $\sum_{j \in M} q_j = n$. The assumption is for ease of exposition and, more generally, we can modify our definitions and results to the case of $\sum_{j \in M} q_j \geq n$. We discuss the extension to excess capacity in Appendix E.[4] A lottery over schools is a probability distribution over M. We denote the set of all lotteries over schools by $\Delta(M)$.

Agents have preferences over lotteries over schools. Preferences of agents can be modeled in two different ways: In the first version, each agent i has a **preference relation** R_i over different schools. We call $R = (R_i)_{i \in N}$ a **preference profile**. We write $j \, P_i \, j'$ if $j \, R_i \, j'$ but not $j' \, R_i \, j$, and $j \, I_i \, j'$ if $j \, R_i \, j'$ and $j' \, R_i \, j$. The preferences can be extended to a partial preference order over lotteries using the stochastic dominance criterion: A lottery π' **weakly first-order stochastically dominates** lottery π with respect to preferences R_i, if for each $j \in M$ we have

$$\sum_{j' \in M : j' \, R_i \, j} \pi'_{j'} \geq \sum_{j' \in M : j' \, R_i \, j} \pi_{j'}.$$

In this case, we write $\pi' \, R_i^{sd} \, \pi$. We write $\pi \, I_i^{sd} \, \pi'$ if all of the above weak inequalities hold with equality, and $\pi' \, P_i^{sd} \pi$ if at least one of the inequalities is strict. In the latter case, we say that π' **strictly first-order stochastically dominates** lottery π.

In the second version, each agent i has a **von-Neumann-Morgenstern (vNM) utility** vector $U_i = (u_{ij})_{j \in M} \in \mathbb{R}_+^M$. We call $U = (U_i)_{i \in N}$ a **utility profile**. Lotteries are evaluated according to expected utility. Thus, agent i prefers lottery π' to lottery π if

$$\sum_{j \in M} u_{ij} \pi'_j > \sum_{j \in M} u_{ij} \pi_j.$$

[4] The case of excess aggregate demand $n > \sum_{j \in M} q_j$ and the case of outside options can be dealt with by adding a dummy school with a large enough capacity.

A utility vector contains more information than a preference relation. In addition to ranking the schools, the vNM-utilities express the rates with which agents substitute probabilities of obtaining seats at the different schools. Utility vector U_i is **consistent** with preferences R_i, if for each pair of schools $j, j' \in M$ we have $jR_ij' \Leftrightarrow u_{ij} \geq u_{ij'}$. Each utility vector U_i is consistent with one preference relation R_i that we call the preference relation **induced** by U_i. It is a standard result (see e.g., Proposition 6.D.1 in Mas-Colell et al., 1995), that if lottery π' strictly first-order stochastically dominates lottery π according to preferences R_i, then lottery π' yields higher expected utility than π according to any vNM-utilities U_i consistent with R_i.

Each school j has a weak (reflexive, complete and transitive) priority order $\succsim_j$ of the agents. We let $i \sim_j i'$ if and only if $i \succsim_j i'$ and $i' \succsim_j i$. We let $i \succ_j i'$ if and only if $i \succsim_j i'$ but not $i' \succsim_j i$. The priorities $\succsim_j$ of a school j partition N in equivalence classes of equal priority agents, i.e. in equivalence classes with respect to $\sim_j$. We call these equivalence classes **priority classes** and denote them by $N_j^1, N_j^2 \ldots, N_j^{\ell(j)}$ with indices decreasing with priority. Thus, for $\ell < \ell'$, $i \in N_j^\ell$ and $i' \in N_j^{\ell'}$ we have $i \succ_j i'$. In that case, we also write $N_j^\ell \succ_j N_j^{\ell'}$. We use the notation $i \succsim_j N_j^\ell$ to indicate that i has higher or equally high priority at j than the agents in the priority class N_j^ℓ. For $i \in N$ and $j \in M$ we denote the index of i's priority class at j by $\ell(i, j)$, i.e. $\ell(i, j) = \ell$ for the unique $1 \leq \ell \leq \ell(j)$ with $i \in N_j^\ell$.

A **deterministic assignment** is a mapping $\mu : N \to M$ such that for each $j \in M$ we have $|\mu^{-1}(j)| = q_j$. A **random assignment** is a matrix $x = (x_{ij}) \in \mathbb{R}^{N\times M}$ with

$$0 \leq x_{ij} \leq 1, \quad \sum_{j\in M} x_{ij} = 1, \quad \sum_{i\in N} x_{ij} = q_j,$$

where x_{ij} is the probability that agent i is matched to school j. By the Birkhoff-von Neumann Theorem, each random assignment corresponds to a lottery over deterministic assignments and, vice versa, each such lottery corresponds to a random assignment (see Kojima and Manea (2010) for a proof in the set-up that we consider). For each $i \in N$ we write $x_i = (x_{ij})_{j\in M}$ and for each $j \in M$ we write $x_j = (x_{ij})_{i\in N}$.

A random assignment x is **ex-ante blocked** by agent i and school j if there is some agent $i' \neq i$ with $x_{i'j} > 0$ and $i \succ_j i'$ and some school $j' \neq j$ with $x_{ij'} > 0$ and $j\, P_i\, j'$. In this case, we say that i has **justified envy** at school j. A random assignment is **ex-ante stable** or **ex-ante priority respecting** if it is not blocked by any agent-school pair. The definition extends to the case where agents have vNM-utilities, by considering the preference profile induced by the utility profile.

2.1 Constrained efficiency

2.1.1 Priority-constrained efficiency

Next, we introduce priority-constrained efficiency notions. Constrained efficiency requires that there is not a more efficient random assignment that respects priorities as much as the original random assignment. There are different ways of comparing random assignments by how much they respect priorities. In the following, we will introduce two criteria, a stronger and a weaker one that will induce weaker and stronger notions of constrained efficiency.

The stronger criterion is by stochastic dominance according to priorities: For school j and random assignments x and y we let $y_j \succsim_j^{sd} x_j$ if for each $1 \leq \ell \leq \ell(j)$ we have

$$\sum_{i \succsim_j N_j^\ell} y_{ij} \geq \sum_{i \succsim_j N_j^\ell} x_{ij}.$$

We write $y_j \succ_j^{sd} x_j$ if at least one of the inequalities is strict. We introduce ordinal and cardinal versions of constrained efficiency:

Random assignment y **first-order stochastically dominates** (sd-dominates) random assignment x if for each $i \in N$ we have $y_i \, R_i^{sd} \, x_i$ and for at least one $i \in N$ we have $y_i \, P_i^{sd} \, x_i$. Random assignment x is **priority-constrained sd-efficient** if for each random assignment y that sd-dominates x there is a school j such that $y_j \not\succsim_j^{sd} x_j$.

Random assignment y **Pareto dominates** random assignment x with respect to vNM-utility profile U if for each $i \in N$ we have

$$\sum_{j \in M} u_{ij} y_{ij} \geq \sum_{j \in M} u_{ij} x_{ij},$$

and the inequality is strict for at least one agent. Random assignment x is **priority-constrained efficient** if for each random assignment y that Pareto dominates x there is a school j such that $y_j \not\succsim_j^{sd} x_j$. A random assignment y **dominates** random assignment x **in social welfare terms** with respect to profile U if

$$\sum_{i,j} u_{ij} y_{ij} > \sum_{i,j} u_{ij} x_{ij}.$$

Random assignment x **maximizes social welfare subject to priority constraints** if for each assignment y that dominates x in social welfare terms with respect to U there is a school j such that $y_j \not\succsim_j^{sd} x_j$.

He et al. (2018) consider a slightly stronger version of priority-constrained efficiency, called two-sided efficiency, in which also priority-improvements that leave agents' welfare unchanged are ruled out. Formally, a random assignment y makes a **welfare-indifferent priority improvement** on random assignment x if for each $i \in N$ we have $\sum_{j \in M} u_{ij} y_{ij} = \sum_{j \in M} u_{ij} x_{ij}$, for each $j \in M$ we have $y_j \succsim_j^{sd} x_j$ and for at least one $j \in M$ we have $y_j \succ_j^{sd} x_j$. A random assignment x is **two-sided efficient** if it is priority-constrained efficient and does not admit a welfare-indifferent priority improvement. Analogously, we can define **two-sided sd-efficiency** as the combination of priority-constrained sd-efficiency and the absence of ordinal welfare-indifferent priority improvements.

2.1.2 Cut-off constrained efficiency

As a second priority comparison criterion we consider lower bounds on priorities: The **cut-off** $C_j(x)$ for school j under random assignment x is the lowest priority class containing an agent that obtaining a seat in school j under x with positive probability,

$$C_j(x) := N_j^{\max\{\ell : \exists i \in N_j^\ell, x_{ij} > 0\}}.$$

A school j uses a **more lenient admission policy** under random assignment y than under random assignment x, if either the school has a lower cut-off in y than in x or it has the same cut-off, but admits a bigger fraction of the students in the cut-off class,

$$C_j(x) \succ_j C_j(y) \text{ or } [C_j(x) = C_j(y) \text{ and } \sum_{i \in C_j(x) = C_j(y)} y_{ij} > \sum_{i \in C_j(x) = C_j(y)} x_{ij}].$$

A random assignment x is **cut-off-constrained sd-efficient** if for each random assignment y that sd-dominates it, there exists a school that uses a more lenient admission policy under y than under x. Similarly, a random assignment x is **cut-off-constrained efficient** if for each random assignment y that Pareto dominates it, there exists a school that uses a more lenient admission policy under y than under x. A random assignment x **maximizes social welfare subject to priority cut-offs** if for each random assignment y that dominates x in social welfare terms there is a school j that uses a more lenient admission policy under y.

Remark 1. By definition, first-order stochastic dominance according to priorities implies a higher or the same cut-off, i.e., for $j \in M$ with $y_j \succsim_j^{sd} x_j$ we have $C_j(y) \succsim_j C_j(x)$. Moreover, $y_j \succsim_j^{sd} x_j$ implies that $\sum_{i \succ_j C_j(x)} y_{ij} \geq \sum_{i \succ_j C_j(x)} x_{ij}$ so that in the case that $C_j(x) = C_j(y)$ we have

$$\sum_{i \in C_j(x)} x_{ij} = 1 - \sum_{i \succ_j C_j(x)} x_{ij} \geq 1 - \sum_{i \succ_j C_j(x)} y_{ij} = \sum_{i \in C_j(x)} y_{ij}.$$

Hence $y_j \succsim_j^{sd} x_j$ implies that j uses a more lenient or the same admission policy under x than under y. Thus, cut-off-constrained (sd)-efficiency implies priority-constrained (sd)-efficiency. We will later show (see Example 1) that cut-off constrained efficiency is generally a stronger notion than priority-constrained efficiency.

On the other hand, cut-off-constrained (sd)-efficiency can be understood as priority-constrained (sd)-efficiency according to a "coarser" priority profile $\succsim'$ under which agents are partitioned into three priority classes at each school: Those that are ranked above the cut-off priority class, those that are in the cut-off class, and those ranked below the cut-off class. Formally, for $j \in M$ and $i, i' \in N$ define

$$i \succsim_j' i' \Leftrightarrow [i \succ_j C_j(x) \text{ or } (i \sim_j C_j(x) \succsim_j i') \text{ or } (C_j(x) \succ_j i \text{ and } C_j(x) \succ_j i')].$$

A random assignment is cut-off-constrained (sd)-efficient for priorities $\succsim$ if and only if it is priority-constrained (sd)-efficient for priorities $\succsim'$. □

2.2 Improvement cycles

Without priorities, sd-efficiency is equivalent to the absence of welfare-improving cycles of trades (see Lemma 3 in Bogomolnaia and Moulin, 2001). Formally, a **(stochastic) improvement cycle** for a random assignment x is a sequence of agents $i_0, i_1, \ldots, i_K$ and schools $j_0, j_1, \ldots, j_K$ such that the following holds:

1. $x_{i_k, j_k} > 0$ for each $0 \leq k \leq K$,
2. $j_K \, R_{i_{K-1}} \, j_{K-1} \, R_{i_{K-2}} \, j_{K-2} \ldots j_1 \, R_{i_0} \, j_0 \, P_{i_K} \, j_K$.

Next, we define priority-respecting versions of this notion. The first requires that reallocating probability shares along the cycle should not lead to a less priority-respecting (in the stochastic dominance sense) assignment. A **priority-respecting improvement cycle** is an improvement cycle such that $i_0 \succsim_{j_1} i_1 \succsim_{j_2} \dots i_K \succsim_{j_0} i_0$. The second and stronger notion leaves the allotment of probability shares to each priority class invariant; an **equal-priority improvement cycle** is an improvement cycle such that $i_0 \sim_{j_1} i_1 \sim_{j_2} i_2 \dots i_K \sim_{j_0} i_0$.

Remark 2. For the case that preferences R are strict, and x is ex-ante stable, Kesten and Ünver (2015) consider the notion of a stable improvement cycle that generalizes a notion introduced by Erdil and Ergin (2008) for deterministic stable matchings to the probabilistic set-up. A stable improvement cycle is an improvement cycle such that for $0 \leq k \leq K$, i_k is one of the highest priority agents at school j_{k+1} that prefers j_{k+1} to some of the schools that he is matched to with positive probability. Here we take indices modulo $K+1$.

The notion of a stable improvement cycle relates to the previous notions as follows: suppose R is a profile of strict preferences, and x is an ex-ante stable random assignment. Then each priority-respecting improvement cycle for x is a stable improvement cycle, as otherwise for each k if there would be a higher priority agent $i \succ_{j_{k+1}} i_k \succsim_{j_{k+1}} i_{k+1}$ such that i prefers j_{k+1} to a school that he is matched to with positive probability, then i and j_{k+1} ex-ante block x contradicting the ex-ante stability of x. Thus, for ex-ante stable random assignment under strict preferences, priority-respecting improvement cycles are a special case of stable improvement cycles. □

Analogous to the notion of a welfare-indifferent priority improvement, we can define a **welfare-indifferent priority-improvement cycle** to be a sequence of agents $i_0, i_1, \dots, i_K$ and schools $j_0, j_1, \dots, j_K$ such that $x_{i_k, j_k} > 0$ for each $0 \leq k \leq K$, $j_K \, I_{i_{K-1}} \, j_{K-1} \, I_{i_{K-2}} \, j_{K-2} \dots j_1 \, I_{i_0} \, j_0 \, I_{i_K} \, j_K$ and $i_0 \succsim_{j_1} i_1 \succsim_{j_2} \dots i_{K-1} \succsim_{j_K} i_K \succ_{j_0} i_0$.

2.3 Equilibrium with priority-specific prices

In an equilibrium with priority-specific prices, a pseudo-market of probability shares generates a random assignment. Each agent has a budget of tokens and can "buy" probability shares at the different schools. Agents face different prices depending on their priority.

Formally, a **pseudo-market** is a triple $(U, b, \succsim)$ consisting of a vNM-utility profile U, a vector of budgets $b \in \mathbb{R}^M_+$, and priorities $\succsim$. We consider two equilibrium notions for pseudo-markets. The first relaxes the equilibrium notion of He et al. (2018): An **equilibrium with priority-specific prices** for the pseudo market $(U, b, \succsim)$ is a pair (x, p) consisting of a random assignment x, and prices $p = (p_{j,\ell})_{j \in M, 1 \leq \ell \leq \ell(j)}$ that are (weakly) decreasing with priority, for each $j \in M$ we have $0 \leq p_{j,1} \leq p_{j,2} \leq \dots \leq p_{j,\ell(j)} \leq \infty$, such that for each $i \in N$ the lottery x_i is an optimum for the problem

$$\begin{aligned} \max_{\pi \in \Delta(M)} \quad & \sum_{j \in M} u_{ij} \pi_j \\ \text{subject to} & \sum_{j \in M} p_{j,\ell(i,j)} \pi_j \leq b_i. \end{aligned}$$

We denote the set of equilibria with priority-specific prices for $(U, b, \succsim)$ by $\mathcal{E}(U, b, \succsim)$.

A stronger notion of constrained equilibrium was introduced by He et al. (2018). A **cut-off-constrained equilibrium** is a pair $(x, \bar{p})$ of a random assignment x and prices $\bar{p} \in \mathbb{R}_+^M$ such that $(x, p) \in \mathcal{E}(U, b, \succsim)$ for

$$p_{j,\ell} := \begin{cases} 0, & \text{for } N_j^\ell \succ_j C_j(x), \\ \bar{p}_j, & \text{for } N_j^\ell = C_j(x), \\ \infty, & \text{for } C_j(x) \succ_j N_j^\ell, \end{cases}$$

We denote the set of cut-off-constrained equilibria for $(U, b, \succsim)$ by $\bar{\mathcal{E}}(U, b, \succsim)$. Requiring zero prices above the cut-offs guarantees that cut-off-constrained equilibrium assignments are ex-ante stable by construction: If an agent i and school j ex-ante block the random assignment x, then $i \succ_j C_j(x)$ and i can obtain j for free. Thus, i could afford a better lottery where he substitutes probability shares at a worse school for probability shares at school j.[5]

3 Results

3.1 Constrained ordinal efficiency welfare theorems

We now relate the concepts of priority-respecting improvement cycles, priority-constrained ordinal efficiency, and priority-constrained cardinal efficiency to each other by providing an ordinal efficiency welfare theorem for random assignments with priorities.

Theorem 1 (Constrained Ordinal Efficiency Welfare Theorem)**.** *For a random assignment x, preferences R and priorities $\succsim$ the following statements are equivalent:*

(1) x has no priority-respecting improvement cycle,

(2) x is priority-constrained sd-efficient for R and $\succsim$,

(3) there exists a vNM-utility profile U consistent with R, such that x maximizes social welfare subject to priority constraints for U and $\succsim$.

Proof. To show that (3) $\Rightarrow$ (2), note that if a random assignment y sd-dominates x under the preferences R induced by U, then y Pareto dominates x with respect to U and in particular y yields higher social welfare than x with respect to U. To show that (2) $\Rightarrow$ (1) note that if there is a priority-respecting improvement cycle, we can construct an assignment y that sd-dominates x by reallocating probabilities across the cycle: Let $i_0, \ldots, i_K$ and $j_0, \ldots, j_K$ be the agents and schools in the cycle. Choose

[5] Zero prices above the cut-off are essential for this argument. Otherwise, equilibria with priority-specific prices can fail to be ex-ante or even ex-post stable: Consider two agents i_1, i_2, two schools j_1, j_2 each with one seat and priorities $i_1 \succ_{j_1} i_2$ and $i_1 \succ_{j_2} i_2$. Suppose both agents prefer the first school to the second one: $u_{11} > u_{12} > 0$ and $u_{21} > u_{22} > 0$. Let $b_1 = 0$ and $b_2 = 1$. Consider the deterministic assignment defined by $\mu(i_1) = j_2$ and $\mu(i_2) = j_1$. Agent i_1 and school j_1 block the assignment. However, with prices $p_{1,2} = 1 = p_{1,1}$ and $p_{2,2} = p_{2,1} = 0$ the degenerate lottery that chooses μ for sure is an equilibrium.

$\epsilon > 0$ such that $\epsilon < \min\{x_{i_k,j_k}, 1 - x_{i_k,j_{k+1}}\}$ for each $0 \leq k \leq K$ (taking indices modulo $K+1$), and define $y_{i_kj_k} = x_{i_k,j_k} - \epsilon$ and $y_{i_kj_{k+1}} = x_{i_k,j_{k+1}} + \epsilon$ for each $0 \leq k \leq K$ (taking indices modulo $K+1$). Leave the assignment otherwise un-changed. By construction $y_j \succsim_j^{sd} x_j$ for each $j \in M$, $y_i \, R_i^{sd} \, x_i$ for each $i \in N$ and $y_{i_K} \, P_{i_K}^{sd} \, x_{i_K}$. Thus, (2) $\Rightarrow$ (1). To show (1) $\Rightarrow$ (3), we use the following result due to Manea (2008):

Lemma 1 (Proposition 2 in Manea, 2008)**.** *Let $\rhd$ and $\bowtie$ two disjoint binary relations over a finite set $\mathcal{O}$. Let $\unrhd := \rhd \cup \bowtie$. If $\unrhd$ is acyclic, i.e. if there exists no sequence $o_0, o_1, \ldots, o_k$ with $o_k \unrhd o_{k-1} \unrhd \ldots \unrhd o_0 \rhd o_k$, then there exist a mapping $v : \mathcal{O} \to \mathbb{R}$ such that for all $o, o' \in \mathcal{O}$ we have*

$$o \rhd o' \Rightarrow v(o) > v(o'),$$
$$o \bowtie o' \Rightarrow v(o) \geq v(o').$$

We choose $\mathcal{O}$ to be the set of all priority classes, i.e. $\mathcal{O} := \{N_j^\ell : j \in M, 1 \leq \ell \leq \ell(j)\}$ and define $\unrhd$ such that it reflects a common component of the agents' preferences: if an agent is contained in two priority classes (at different schools), consumes from the first one with positive probability, but (weakly) prefers consuming from the second one, then that second priority class is (weakly) preferred to the first one according to $\unrhd$. Formally, for $j \neq j'$, $1 \leq \ell \leq \ell(j)$ and $1 \leq \ell' \leq \ell(j')$ we let

$$N_j^\ell \rhd N_{j'}^{\ell'} \Leftrightarrow (\exists i \in N_j^\ell \cap N_{j'}^{\ell'}, x_{ij'} > 0, j \, P_i \, j'),$$
$$N_j^\ell \bowtie N_{j'}^{\ell'} \Leftrightarrow (N_j^\ell \not\rhd N_{j'}^{\ell'} \text{ and } \exists i \in N_j^\ell \cap N_{j'}^{\ell'}, x_{ij'} > 0, j \, I_i \, j').$$

We rank priority classes at the same school such that a lower priority is weakly preferred to a higher priority, i.e. for $j \in M$ and $1 \leq \ell, \ell' \leq \ell(j)$ we let

$$N_j^{\ell'} \bowtie N_j^\ell \Leftrightarrow \ell < \ell'.$$

The absence of priority-respecting improvement cycles implies that $\unrhd$ is acyclic. Indeed suppose there is a cycle

$$N_{j_K}^{\ell_K} \unrhd N_{j_{K-1}}^{\ell_{K-1}} \unrhd \ldots \unrhd N_{j_0}^{\ell_0} \rhd N_{j_K}^{\ell_K}.$$

In Figure 1 we illustrate an example of a cycle. Note that consecutive priority classes in the cycle could be at the same school: Let $k_0 = 0$, let $k_0 < k_1 \leq K$ be the smallest index such that $j_{k_1} \neq j_{k_0} = j_0$, let $k_1 < k_2 \leq K$ be the smallest index such that $j_{k_2} \neq j_{k_1}$ etc. Let $K' \leq K$ be the number of resulting subindices. By definition of $\unrhd$, we can find for $r = 0, \ldots, K'-1$ a $i_r \in N_{j_{k_r}}^{\ell_{k_{r+1}}-1} \cap N_{j_{k_{r+1}}}^{\ell_{k_{r+1}}}$ with $x_{i_rj_{k_r}} > 0$ and $j_{k_{r+1}} \, R_{i_r} \, j_{k_r}$, and a $i_{K'} \in N_{j_{k_0}}^{\ell_0} \cap N_{j_{k_{K'}}}^{\ell_{k_{K'}}}$ with $x_{i_{K'}j_{k_{K'}}} > 0$ and $j_{k_0} \, P_{i_{K'}} \, j_{k_{K'}}$. Moreover, by the definition of $\bowtie$, for each $r = 0, \ldots, K'$ we have $N_{j_{k_r}}^{\ell_{k_{r+1}}-1} \bowtie N_{j_{k_r}}^{\ell_{k_{r+1}}-2} \bowtie \ldots \bowtie N_{j_{k_r}}^{\ell_{k_r}}$ and therefore $i_r \succsim_{j_{k_r}} i_{r-1}$ taking indices modulo $K'+1$. The sequence of agents $i_0, i_1, \ldots, i_{K'}$ and schools $j_{k_0}, j_{k_1}, \ldots, j_{k_{K'}}$ form a priority respecting improvement cycle contradicting the absence of such cycles.

We use the lemma to define the utility profile U. Consider a mapping $v : \mathcal{O} \to \mathbb{R}$ as in Lemma 1. Since $\mathcal{O}$ is finite, we may choose v such that it takes on strictly positive

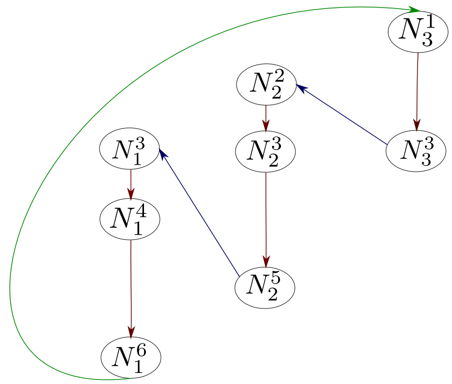

Figure 1: Illustrative example of a cycle. Brown edges depict the $\bowtie$ relation. Blue edges depict the $\trianglerighteq$ relation, while the green edge depicts the $\triangleright$ relation. Here $j_0 = j_1 = j_{k_0} = 3$, $j_2 = j_3 = j_4 = j_{k_1} = 2$ and $j_5 = j_6 = j_7 = j_{k_2} = 1$. Moreover, $K = 7$ and $\ell_0 = 1$, $\ell_1 = 3$, $\ell_2 = 2, \ell_3 = 3, \ell_4 = 5$, $\ell_5 = 3$, $\ell_6 = 4$, $\ell_7 = 6$.

values everywhere. We define utilities such that for each priority class $N_j^\ell \in \mathcal{O}$ all agents in N_j^ℓ that are matched to j with positive probability have the same utility $v(N_j^\ell)$ for attending j. Moreover, this utility is the maximal utility that any agent in N_j^ℓ has for attending j. Thus, we require for each $i \in N$ and $N_j^\ell \in \mathcal{O}$ that

$$x_{ij} > 0, i \in N_j^\ell \Rightarrow u_{ij} = v(N_j^\ell), \tag{1}$$

$$x_{ij} = 0, i \in N_j^\ell \Rightarrow u_{ij} \leq v(N_j^\ell). \tag{2}$$

We show that we can construct U consistent with R such that Conditions (1) and (2) hold. For $i \in N$ order the schools that i obtains with positive probability consistently with R_i, i.e., let $\{j_0, \ldots, j_K\} = \{j \in M : x_{ij} > 0\}$ with $j_K \, R_i \, j_{K-1} \, R_i \, \ldots \, R_i \, j_0$. For $0 \leq k \leq K$, let $\ell_k := \ell(i, j_k)$ and let for each $j \in M$:

$$j \, I_i \, j_k \Rightarrow u_{ij} = v(N_{j_k}^{\ell_k}).$$

Define U_i otherwise consistent with R_i such that (2) holds, by requiring for $0 \leq k \leq K-1$ that

$$j_{k+1} \, P_i \, j \, P_i \, j_k \Rightarrow v(N_{j_k}^{\ell_k}) < u_{ij} < \min_{j' \in M : j_{k+1} \, R_i \, j' \, P_i \, j_k} v(N_{j'}^{\ell(i,j')}),$$

and that

$$j \, P_i \, j_K \Rightarrow v(N_{j_K}^{\ell_K}) < u_{ij} < \min_{j' \in M : j' \, P_i \, j_K} v(N_{j'}^{\ell(i,j')}),$$

$$j_0 \, P_i \, j \Rightarrow 0 \leq u_{ij} < \min_{j' \in M : j_0 \, R_i \, j'} v(N_{j'}^{\ell(i,j')}).$$

By the construction of our ordering this is possible and yields Conditions (1) and (2), as for each $0 \leq k \leq K$ and $j \in M$: if $j \, P_i \, j_k$, then $N_j^{\ell(i,j)} \triangleright N_{j_k}^{\ell_k}$ and therefore $v(N_j^{\ell(i,j)}) > v(N_{j_k}^{\ell_k})$, and if $j \, I_i \, j_k$, then $N_j^{\ell(i,j)} \bowtie N_{j_k}^{\ell_k}$ and therefore $v(N_j^{\ell(i,j)}) \geq v(N_{j_k}^{\ell_k}) = u_{ij}$.

Having defined U, we can show that x maximizes social welfare subject to priority constraints with respect to U and $\succsim$. Suppose for random assignment y we have

$$\sum_{i\in N}\sum_{j\in M} u_{ij}y_{ij} > \sum_{i\in N}\sum_{j\in M} u_{ij}x_{ij}.$$

By Condition (1), we have

$$\sum_{j\in M}\sum_{i\in N} u_{ij}x_{ij} = \sum_{j\in M}\sum_{\ell=1}^{\ell(j)}\sum_{i\in N_j^\ell} u_{ij}x_{ij} = \sum_{j\in M}\sum_{\ell=1}^{\ell(j)} v(N_j^\ell)(\sum_{i\in N_j^\ell} x_{ij}).$$

By Conditions (1) and (2), we have

$$\sum_{j\in M}\sum_{i\in N} u_{ij}y_{ij} = \sum_{j\in M}\sum_{\ell=1}^{\ell(j)}\sum_{i\in N_j^\ell} u_{ij}y_{ij} \leq \sum_{j\in M}\sum_{\ell=1}^{\ell(j)} v(N_j^\ell)(\sum_{i\in N_j^\ell} y_{ij}).$$

Thus,

$$\sum_{j\in M}\sum_{\ell=1}^{\ell(j)} v(N_j^\ell)(\sum_{i\in N_j^\ell} y_{ij}) > \sum_{j\in M}\sum_{\ell=1}^{\ell(j)} v(N_j^\ell)(\sum_{i\in N_j^\ell} x_{ij}).$$

Rearranging the terms and noting that $\sum_{i\succsim_j N_j^{\ell(j)}} y_{ij} = \sum_{i\in N} y_{ij} = q_j = \sum_{i\in N} x_{ij} = \sum_{i\succsim_j N_j^{\ell(j)}} x_{ij}$ we have

$$0 < \sum_{j\in M}\sum_{\ell=1}^{\ell(j)} v(N_j^\ell)(\sum_{i\in N_j^\ell}(y_{ij}-x_{ij})) = \sum_{j\in M}\sum_{\ell=1}^{\ell(j)-1}(v(N_j^\ell)-v(N_j^{\ell+1}))(\sum_{i\succsim_j N_j^\ell}(y_{ij}-x_{ij})).$$

By construction, $v(N_j^\ell) \leq v(N_j^{\ell'})$ for $\ell < \ell'$ and therefore $v(N_j^\ell) - v(N_j^{\ell+1}) \leq 0$ for $\ell = 1, \dots, \ell(j)-1$ for each $j \in M$. Thus, there is a $j \in M$ and $1 \leq \ell \leq \ell(j)$ with $\sum_{i\succsim_j N_j^\ell}(y_{ij}-x_{ij}) < 0$ and therefore $y_j \not\succsim_j^{sd} x_j$. □

Remark 3. An analogous ordinal efficiency theorem can be obtained for the two-sided efficiency notions. The absence of priority-respecting improvement cycles *and* welfare-indifferent priority-improvement cycles is equivalent to two-sided sd-efficiency and to the existence of a vNM-utility profile under which the assignment under consideration is two-sided efficient. We sketch the argument in the appendix. □

Immediately from the theorem follows a stronger version of the ordinal efficiency welfare theorem for the case that the random assignment in question is ex-ante stable. In this case, we can replace the notion of a priority-respecting improvement cycle by the stronger notion of an equal-priority improvement cycle so that the absence of a cycle is now a weaker efficiency notion. More specifically, it suffices to consider equal-priority improvements cycles in which all involved agents are in a cut-off priority class. Moreover, priority-constrained (sd)-efficiency can be replaced by cut-off-constrained (sd)-efficiency. More generally, the equivalence between cut-off-constrained sd-efficiency and cut-off-constrained efficiency under a consistent vNM-utility profile does not require ex-ante stability.

We have to make one additional assumption on random assignments for the result. A random assignment x satisfies **no indifferences between assigned cut-off schools and safe schools**[6] if for $i \in N$ $j, j' \in M$ with $i \in C_j(x)$ and $i \succ_{j'} C_{j'}(x)$, we have $j\, I_i\, j' \Rightarrow x_{ij} = 0$. The condition is, for example, redundant if preferences are strict.

Corollary 1 (Ex-ante Stable Ordinal Efficiency Welfare Theorem)**.** *For an ex-ante stable random assignment x that satisfies no indifferences between assigned cut-off schools and safe schools under preferences R and priorities $\succsim$ the following statements are equivalent:*

(1) x has no equal-priority improvement cycle according to R and $\succsim$,

(2) x is cut-off-constrained sd-efficient for R and $\succsim$,

(3) there exists a vNM-utility profile U consistent with R, such that x maximizes social welfare subject to priority cut-offs for U and $\succsim$.

Proof. Consider the auxiliary priority profile $\succsim'$ as defined in Remark 1. The equivalence between (2) and (3) follows by Theorem 1 applied to $\succsim'$. To show the equivalence between (1) and (2), we show that if x is ex-ante stable and satisfies no indifferences between assigned cut-off schools and safe schools, then an improvement cycle under R is priority-respecting under $\succsim'$ if and only if it is an equal-priority cycle under $\succsim$. The equivalence then follows from Theorem 1 applied to $\succsim'$. Let agents $i_0, i_1, \ldots, i_K$ and schools $j_0, j_1, \ldots, j_K$ form an improvement cycle for x. Suppose that $i_0 \succsim'_{j_1} i_1 \succsim'_{j_2} \ldots i_K \succsim'_{j_0} i_0$. By the assumption of no indifferences between assigned cut-off schools and safe schools, we have $i_0 \sim'_{j_1} i_1 \sim'_{j_2} \ldots i_K \sim'_{j_0} i_0$. If $i_k, i_{k-1} \in C_{j_k}(x)$ for $k = 0, \ldots K$ (taking indices modulo $K+1$), then $i_0 \sim_{j_1} i_1 \sim_{j_2} \ldots i_K \sim_{j_0} i_0$ and the cycle is an equal-priority improvement cycle according to $\succsim$. If $i_k, i_{k-1} \succ'_{j_k} C_{j_k}(x)$ for $k = 0, \ldots K$ (taking indices modulo $K+1$), then $i_K \succ'_{j_0} C_{j_0}(x)$ and thus $i_K \succ_{j_0} C_{j_0}(x)$. In that case i_K and j_0 form an ex-ante blocking pair according to $\succsim$, contradicting the ex-ante stability of x. Immediately from the definition of $\succsim'$, it follows that equal-priority improvement cycles according to $\succsim$ are priority-respecting under $\succsim'$. □

Remark 4. Note that in the proof, we have used ex-ante stability and the assumption of no indifferences between assigned cut-off schools and safe schools only for the equivalence between (1) and (2), and between (1) and (3). Thus, statements (2) and (3) are equivalent for arbitrary not necessarily ex-ante stable assignments. □

The following example demonstrates that, even under ex-ante stability (and even under no indifferences between assigned cut-off schools and safe schools), cut-off-constrained efficiency is strictly stronger than priority-constrained efficiency.[7]

[6] We use the term "safe school" for schools where the agent is ranked strictly above the cut-offs, since in cut-off constrained equilibrium allocations those are the schools where the agent faces a zero price and thus is always guaranteed a seat.

[7] The two notions are, however, equivalent for the case that priorities and preferences are strict and the

Example 1. Consider three agents, three schools, each with a single seat ($q_j = 1$ for each j), the following utilities and priorities,

U_{i_1}	U_{i_2}	U_{i_3}	$\succsim_{j_1}$	$\succsim_{j_2}$	$\succsim_{j_3}$
2	3	5	i_1	$\underline{i_1, i_2, i_3}$	$\underline{i_1, i_2, i_3}$
3	4	4	i_2		
4	9	3	$\underline{i_3}$		

The underlined entries in the priorities are priority cut-offs for the following assignment (here and in the following rows correspond to agents and columns to schools)

$$x = \begin{pmatrix} 0.6 & 0 & 0.4 \\ 0.2 & 0.2 & 0.6 \\ 0.2 & 0.8 & 0 \end{pmatrix}.$$

Random assignment x is ex-ante stable, since the only potential blocking pairs are agent i_1 with school j_1 and agent i_2 with school j_1. However, for both agents, j_1 is the worst school. We show in the appendix that x is two-sided efficient. However, x is not cut-off-constrained efficient, since the following assignment without more lenient schools dominates it:

$$y = \begin{pmatrix} 0.5 & 0.2 & 0.3 \\ 0.3 & 0 & 0.7 \\ 0.2 & 0.8 & 0 \end{pmatrix}$$

□

3.2 Equilibria with Priority-Specific Prices

3.2.1 First Welfare Theorems

He et al. (2018) show that cut-off-constrained equilibrium assignments are priority-constrained efficient if the following tie-breaking assumption is made: Whenever multiple lotteries are optimal for an agent, he chooses a cheapest one. Their argument immediately generalizes to equilibrium random assignments under priority-specific prices. For completeness, the appendix contains a proof.

Proposition 1 (Constrained First Welfare Theorem)**.** *For each pseudo-market $(U, b, \succsim)$, each equilibrium random assignment for priority-specific prices such that each agent chooses a cheapest lottery whenever multiple lotteries are optimal, is priority-constrained efficient with respect to U and $\succsim$.*

The result can be strengthened to two-sided efficiency if prices in the cut-offs are positive. However, two-sided efficiency can fail to hold if some prices in the cut-off are zero:

random assignment is ex-ante stable: In that case, Schlegel (2018) proves that in each ex-ante stable random assignment, at each school at most two agents have a positive probability of obtaining a seat at the school which implies the equivalence between the two efficiency notions.

Example 2. Consider two agents i_1, i_2, two schools j_1, j_2 each with one seat, arbitrary budgets b_1, b_2, and priorities $i_1 \succ_{j_1} i_2$ and $i_2 \succ_{j_2} i_1$. Suppose $u_{11} = u_{12} > 0$ and $u_{21} = u_{22} > 0$. Consider the random assignment x with $x_{ij} = \frac{1}{2}$ for all i and j. Since both agents are indifferent between both schools, but agent i_1 is ranked higher at j_1 and agent i_2 is ranked higher at j_2, the assignment that assigns i_1 to j_1 for sure and i_2 to j_2 for sure yields a welfare-indifferent priority improvement over x. Thus, x is not two-sided efficient. However, it can be decentralized as a cut-off constrained equilibrium with prices $\bar{p}_{j_1} = \bar{p}_{j_2} = 0$ and the second priority as cut-off. □

An immediate consequence of the proposition and our definition of cut-off-constrained efficiency is that cut-off-constrained equilibrium assignments are cut-off-constrained efficient.

Corollary 2. *For each pseudo market $(U, b, \succsim)$, each cut-off-constrained equilibrium assignment such that each agent chooses a cheapest lottery whenever multiple lotteries are optimal is cut-off-constrained efficient.*

Proof. Consider the auxiliary priority profile $\succsim'$ as in Remark 1. By definition, $\bar{\mathcal{E}}(U, b, \succsim) = \bar{\mathcal{E}}(U, b, \succsim') \subseteq \mathcal{E}(U, b, \succsim')$. Thus, by the proposition, each cut-off-constrained equilibrium assignment x in $(U, b, \succsim)$ is priority-constrained efficient in $(U, b, \succsim')$ and hence, by Remark 1, cut-off-constrained efficient in $(U, b, \succsim)$. □

3.2.2 Decentralizing random assignments as equilibrium

Next, we provide a strengthening of the ex-ante stable version of the ordinal efficiency welfare theorem (Corollary 1). Instead of constructing a utility profile such that the random assignment is cut-off constrained efficient, we now construct prices and budgets along with the utility profile such that the random assignment can be decentralized as a cut-off-constrained equilibrium.

Theorem 2 (Ordinal Efficiency Welfare Theorem with Prices)**.** *For a random assignment x, preferences R and priorities $\succsim$, the following statements are equivalent:*

(1) x is ex-ante stable, satisfies no indifferences between assigned cut-off schools and safe schools, and has no equal-priority improvement cycle according to R and $\succsim$.

(2) There exists a vNM-utility profile U consistent with R, a budget vector $b \in \mathbb{R}^N_+$, and strictly positive prices $\bar{p} = (\bar{p}_j)_{j \in M} \in \mathbb{R}^M_{++}$ such that $(x, \bar{p})$ is a cut-off-constrained equilibrium under U, b and $\succsim$ in which each agent chooses a cheapest lottery whenever multiple lotteries are optimal.

Proof. To show that (2) ⇒ (1), first note that, as all prices are strictly positive and each agent chooses a cheapest lottery whenever multiple lotteries are optimal, x satisfies no indifferences between assigned cut-off schools and safe schools. As observed in Section 2.3, cut-off-constrained equilibrium assignments are ex-ante stable. By Corollary 2, x is cut-off constrained efficient with respect to U and $\succsim$. Since x is ex-ante stable, satisfies no indifferences between assigned cut-off schools and safe schools with respect to R and $\succsim$, and there exists a vNM profile U consistent with R, such that x

is cut-off constrained efficient with respect to U and $\succsim$, Corollary 1 implies that there is no equal-priority improvement cycle according to R and $\succsim$.

To show that (1) $\Rightarrow$ (2), define $C_j := C_j(x)$ for each $j \in M$ and reconsider the ordering $\trianglerighteq$ defined in the proof of Theorem 1. Observe that by the absence of equal-priority improvement cycles, $\trianglerighteq$ is acyclic on the set $\mathcal{O}' \subseteq \mathcal{O}$ of all cut-off classes $\mathcal{O}' := \{C_j : j \in M\}$. Thus, by Lemma 1, we find a mapping $v : \mathcal{O}' \to \mathbb{R}$ such that $C_j \triangleright C_{j'} \Rightarrow v(C_j) > v(C_{j'})$ and $C_j \bowtie C_{j'} \Rightarrow v(C_j) \geq v(C_{j'})$. Since $\mathcal{O}'$ is finite, we may choose v such that it takes on strictly positive values everywhere. For each $j \in M$ we define $\bar{p}_j := v(C_j)$. For each $i \in N$ we let $b_i := \sum_{j:i\in C_j} \bar{p}_j x_{ij}$. Moreover, we choose a number $0 \leq \bar{u} < \min_{j\in M} v(C_j)$ and define U as follows: For each $i \in N$ we choose U_i consistent with R_i such that

$$x_{ij} > 0, i \succ_j C_j \Rightarrow u_{ij} = \bar{u}, \tag{3}$$
$$x_{ij} = 0, i \succ_j C_j \Rightarrow u_{ij} \leq \bar{u}, \tag{4}$$
$$x_{ij} > 0, i \in C_j \Rightarrow u_{ij} = \bar{p}_j + \bar{u}, \tag{5}$$
$$x_{ij} = 0, i \in C_j \Rightarrow u_{ij} \leq \bar{p}_j + \bar{u}. \tag{6}$$

By construction of $\trianglerighteq$ and ex-ante stability this is possible: By ex-ante stability, for each $i \in N$, if $i \succ_j C_j$, then $j' \, R_i \, j$ for each $j' \in M$ with $x_{ij'} > 0$ and by the assumption of no indifference between cut-off schools and safe schools, we have $j' \, P_i \, j$ if additionally $i \in C_{j'}$. Thus, (3) and (4) can be satisfied. For $j, j' \in M$ with $i \in C_j \cap C_{j'}$ and $x_{ij} > 0$ we have that $j' \, R_i \, j$ implies $\bar{p}_{j'} \geq \bar{p}_j$ and that $j' \, P_i \, j$ implies $\bar{p}_{j'} > \bar{p}_j$. Thus, (5) and (6) can be satisfied.

We show that for each $i \in N$ lottery x_i is optimal given prices and his budget. It suffices to show that x_i is an optimum for the problem:

$$\begin{aligned} \max_{\pi} \quad & \sum_{j:i\succsim_j C_j} u_{ij}\pi_j \\ \text{subject to} \quad & \sum_{j\in M:i\in C_j} \bar{p}_j\pi_j \leq b_i, \\ & \sum_{j\in M} \pi_j \leq 1, \\ & \pi_j \geq 0, \quad \forall j \in M. \end{aligned}$$

The dual problem is

$$\begin{aligned} \min_{\lambda,\mu} \quad & \lambda b_i + \mu \\ \text{subject to} \quad & \bar{p}_j\lambda + \mu \geq u_{ij}, \quad \forall j : i \in C_j, \\ & \mu \geq u_{ij}, \quad \forall j : i \succ_j C_j, \\ & \lambda, \mu \geq 0. \end{aligned}$$

The choice of $\lambda = 1$ and $\mu = \bar{u}$ is feasible for the dual, as, by Conditions (5) and (6), for each j with $i \in C_j$ we have

$$\bar{p}_j\lambda + \mu = \bar{p}_j + \bar{u} \geq u_{ij},$$

and by Conditions (3) and (4), for each j with $i \succ_j C_j$ we have

$$\mu = \bar{u} \geq u_{ij}.$$

By Conditions (3) and (5) we have

$$\sum_{j \in M} u_{ij} x_{ij} = \sum_{j: i \in C_j} (\bar{p}_j + \bar{u}) x_{ij} + \sum_{j: i \succ_j C_j} \bar{u} x_{ij} = \sum_{j: i \in C_j} \bar{p}_j x_{ij} + \bar{u} = b_i + \bar{u} = \lambda b_i + \mu.$$

By linear programming duality, this shows that x_i is an optimal solution to the agent's maximization problem (and $(\lambda = 1, \mu = \bar{u})$ is optimal for the dual). Next, we show that there is no cheaper bundle that maximizes utility: If the budget constraint does not bind, then in the corresponding dual solution $\lambda = 0$, and therefore $\mu \geq \max_{j \in M: i \succsim_j C_j} u_{ij}$. But then by linear programming duality $\sum_{j \in M} u_{ij} x_{ij} = \mu \geq \max_{j \in M: i \succsim_j C_j} u_{ij}$. Thus, $u_{ij} = \mu = \max_{j \in M: i \succsim_j C_j} u_{ij}$ for each $j \in M$ with $x_{ij} > 0$. If there is a $j \in M$ with $i \succ_j C_j$ such that $x_{ij} > 0$, then by the assumption of no indifferences between assigned cut-off schools and safe schools, we have $x_{ij'} = 0$ for each $j' \in M$ with $i \in C_{j'}$, and therefore $b_i = 0$. Otherwise note that for each $j, j' \in M$ with $x_{ij} > 0$, $i \in C_j \cap C_{j'}$ and $u_{ij'} = \max_{j \in M: i \succsim_j C_j} u_{ij} = u_{ij}$ we have $j' \bowtie j$ and therefore $\bar{p}_{j'} \geq \bar{p}_j$. In either case, x_i is a cheapest utility maximizing lottery. □

The following example demonstrates that the assumption of "no indifferences between assigned cut-off schools and safe schools" is necessary for the construction of a utility profile and prices such that the random assignment is decentralized as a cut-off-constrained equilibrium.[8]

Example 3. Consider three agents, three schools, each with a single seat ($q_j = 1$ for each j), and the following preferences and priorities.

R_{i_1}	R_{i_2}	R_{i_3}	$\succsim_{j_1}$	$\succsim_{j_2}$	$\succsim_{j_3}$
j_1, j_2	j_1	j_2	i_1	i_2	$\underline{i_3}$
j_3	j_2	j_3	$\underline{i_2}$	$\underline{i_1, i_3}$	i_1, i_2
	j_3	j_1	i_3		

The underlined entries in the priorities are the cut-offs for the following assignment

$$x = \begin{pmatrix} 0.5 & 0.5 & 0 \\ 0.5 & 0.5 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

There exists no blocking pair. Moreover, one can show that the random assignment has no equal-priority improvement cycle and, more generally, is cut-off-constrained sd-efficient.

Now suppose there exist a utility profile U consistent with R and decentralizing cut-off prices p and budgets b. Observe that i_1 is indifferent between j_1 and j_2 and is

[8]In the other direction, it is necessary that cut-off prices are strictly positive to imply that "no indifferences between assigned cut-off schools and safe schools" is satisfied by the equilibrium assignment: The random assignment in Example 2 violates "no indifferences between assigned cut-off schools and safe schools", but can be decentralized with zero prices.

strictly above the cut-off at j_1. Thus, it must be the case that $p_2 = 0$, since otherwise i_1 could obtain probability shares at school j_1 instead to obtain a cheaper bundle. More generally, the assignment cannot be decentralized even if we allow for zero prices in the cut-off. It needs to be the case that $p_2 > p_3$ since otherwise agent i_3 could substitute shares at j_3 by shares at j_2 which he prefers. But then $0 = p_2 > p_3$, a contradiction. □

In Theorem 2 we started with ordinal preferences and constructed vNM-utilities, budgets, cut-offs, and prices to decentralize an ex-ante stable random assignment that is cut-off-constrained sd-efficient as a cut-off-constrained equilibrium. It is a natural question, whether the result can be strengthened in the following way: Start with a profile of vNM-utilities and show that each ex-ante stable random assignment that is cut-off-constrained efficient can be decentralized as a cut-off-constrained equilibrium. We demonstrate by means of a counterexample that this is not possible, and a cardinal second welfare theorem does not hold. In the example, each agent obtains different utility from the different schools. Thus, the induced ordinal preferences are strict (in particular, there is no indifference between assigned cut-off schools and safe schools), and the example does not rely on indifferences interfering with the construction of decentralizing prices.

Example 4. Consider three agents, three schools, each with a single seat ($q_j = 1$ for each j), and the following utilities and priorities

U_{i_1}	U_{i_2}	U_{i_3}		$\succsim_{j_1}$	$\succsim_{j_2}$	$\succsim_{j_3}$
1	3	4		i_1	i_2	i_3
2	1	2		$\underline{i_2, i_3}$	$\underline{i_1, i_3}$	$\underline{i_1, i_2}$
5	6	1				

The underlined entries in the priorities are the cut-offs for the following assignment

$$x = \begin{pmatrix} 0.1 & 0.6 & 0.3 \\ 0.8 & 0.2 & 0 \\ 0.1 & 0.2 & 0.7 \end{pmatrix}.$$

There exist no blocking pairs in this example since for all agents, their worst school is the only school where they are ranked strictly above the cut-off. Therefore, x is ex-ante stable. We show in the appendix that x is two-sided efficient (which, as there are only two priority classes at each school and utilities are strict, is equivalent to cut-off-constrained efficiency).

Next, we show that x cannot be decentralized as a cut-off-constrained equilibrium. Suppose there are budgets $b \in \mathbb{R}_+^N$ and prices $\bar{p} = (\bar{p}_1, \bar{p}_2, \bar{p}_3) \in \mathbb{R}_+^M$ such that $(x, \bar{p})$ is a cut-off-constrained equilibrium.

Agent i_1's optimization problem is

$$\max_{\pi \in \Delta(M)} \quad \pi_1 + 2\pi_2 + 5\pi_3$$
$$\text{subject to } \bar{p}_2\pi_2 + \bar{p}_3\pi_3 \leq b_1.$$

Substituting $\pi_1 = 1 - \pi_2 - \pi_3$ and ignoring the constant in the objective function, we obtain the following equivalent optimization problem

$$\begin{aligned}\max_{\pi_1,\pi_2} \quad & \pi_2 + 4\pi_3 \\ \text{subject to } & \bar{p}_2\pi_2 + \bar{p}_3\pi_3 \leq b_1 \\ & \pi_2 + \pi_3 \leq 1 \\ & \pi_2, \pi_3 \geq 0.\end{aligned}$$

If $(x, \bar{p})$ is a cut-off-constrained equilibrium, then $(\pi_1, \pi_2, \pi_3) = (0.1, 0.6, 0.3)$ is a solution to the original problem and hence $(\pi_2, \pi_3) = (0.6, 0.3)$ is a solution to the equivalent problem. At $(0.6, 0.3)$ neither the non-negativity constraints $\pi_2, \pi_3 \geq 0$ nor the constraint $\pi_2 + \pi_3 \leq 1$ binds. Thus, it is not a corner of the LP polytope and therefore indifference curves are parallel to the first constraint $\bar{p}_2\pi_2 + \bar{p}_3\pi_3$. That is $\frac{\bar{p}_2}{\bar{p}_3} = \frac{1}{4}$.

Agent i_3's optimization problem is

$$\begin{aligned}\max_{\pi\in\Delta(M)} \quad & 4\pi_1 + 2\pi_2 + \pi_3 \\ \text{subject to } & \bar{p}_1\pi_1 + \bar{p}_2\pi_2 \leq b_3\end{aligned}$$

Similarly to the previous case, we can substitute $\pi_3 = 1 - \pi_1 - \pi_2$ and ignore the constant in the objective function, to obtain an equivalent problem

$$\begin{aligned}\max_{\pi_1,\pi_2} \quad & 3\pi_1 + \pi_2 \\ \text{subject to } & \bar{p}_1\pi_1 + \bar{p}_2\pi_2 \leq b_3 \\ & \pi_1 + \pi_2 \leq 1 \\ & \pi_1, \pi_2 \geq 0\end{aligned}$$

If $(x, \bar{p})$ is a cut-off-constrained equilibrium, then $(\pi_1, \pi_2, \pi_3) = (0.1, 0.2, 0.7)$ is a solution to the original problem and hence $(\pi_1, \pi_2) = (0.1, 0.2)$ is a solution to the equivalent problem. At $(0.1, 0.2)$ neither the non-negativity constraints $\pi_1, \pi_2 \geq 0$ nor the constraint $\pi_1 + \pi_2 \leq 1$ binds. Thus, it is not a corner of the LP polytope and therefore indifference curves are parallel to the first constraint $\bar{p}_1\pi_1 + \bar{p}_2\pi_2$. That is $\frac{\bar{p}_1}{\bar{p}_2} = \frac{3}{1} = 3$.

Without loss of generality, we can assume that $\bar{p}_2 = 1$. Then $\bar{p}_1 = 3$ and $\bar{p}_3 = 4$. Then, agent i_2's optimization problem becomes:

$$\begin{aligned}\max_{\pi\in\Delta(M)} \quad & 3\pi_1 + \pi_2 + 6\pi_3 \\ \text{subject to } & 3\pi_1 + 4\pi_3 \leq b_2\end{aligned}$$

Similarly to the previous case, we can substitute $\pi_2 = 1 - \pi_1 - \pi_3$ and ignore the constant in the objective function to obtain an equivalent problem

$$\begin{aligned}\max_{\pi_1,\pi_3} \quad & 2\pi_1 + 5\pi_3 \\ \text{subject to } & 3\pi_1 + 4\pi_3 \leq b_2 \\ & \pi_1 + \pi_3 \leq 1 \\ & \pi_1, \pi_3 \geq 0\end{aligned}$$

If $(x, \bar{p})$ is a cut-off-constrained equilibrium, then $(\pi_1, \pi_2, \pi_3) = (0.8, 0.2, 0)$ is a solution to the original problem and hence $(\pi_1, \pi_3) = (0.8, 0)$ is a solution to the equivalent problem. Since the second constraint does not bind at $(0.8, 0)$, the first constraint binds. That is, $b_2 = \bar{p}_1\pi_1 + \bar{p}_3\pi_3 = 3 \times 0.8 = 2.4$. But if $b_2 = 2.4$ then $(\pi_1, \pi_3) = (0, 0.6)$ is feasible and gives a value $5 \times 0.6 = 3$, instead of $2 \times 0.8 = 1.6$. This contradicts the assumption that $(x, \bar{p})$ is a cut-off-constrained equilibrium. □

Note that in the example, the optimality of x_{i_1} for agent i_1 determines the relative cut-off price $\frac{\bar{p}_2}{\bar{p}_3}$ whereas the optimality of x_{i_3} for agent i_3 determines the relative cut-off price $\frac{\bar{p}_1}{\bar{p}_2}$. Thus, agents i_1 and i_2 jointly determine the relative cut-off prices for all pairs of schools. However, the induced relative cut-off price $\frac{\bar{p}_1}{\bar{p}_3}$ is too high to make the lottery x_{i_2} optimal for agent i_2. If we allow for positive prices above the cut-off at school j_3 we can decrease the relative cut-off price of schools j_1 and j_2 while maintaining optimality of x_{i_3}. Decreasing relative prices in such a way we can achieve optimality of x_{i_2} for agent i_2 while maintaining optimality for the other two agents (similarly, we could use a positive price above the cut-off at school j_1 to change the relative cut-off prices for schools j_2 and j_3 to achieve the same result). Thus, with positive prices above the cut-off we can restore the second welfare theorem:

Example 4 (cont.). Random assignment x can be decentralized as an equilibrium with priority-specific prices as follows: Prices are $p_{1,1} = 0 < p_{1,2} = 1.6$, $p_{2,1} = 0 < p_{2,2} = 1$, $p_{3,1} = 0.7 < p_{3,2} = 4$, and budgets are $b_1 = 1.8$, $b_2 = 1.28$ and $b_3 = 0.85$. Thus, for school j_3 there is a strictly positive price above the cut-off. One can check that for each agent $i \in N$, lottery x_i is a solution to the utility maximization problem subject to the budget constraint:

Substituting $\pi_1 = 1 - \pi_2 - \pi_3$ and ignoring the constant in the objective function as above, agent i_1's optimization problem is equivalent to the following problem

$$\begin{aligned} \max_{\pi_2, \pi_3} \quad & \pi_2 + 4\pi_3 \\ \text{subject to } & \pi_2 + 4\pi_3 \leq 1.8, \\ & \pi_2 + \pi_3 \leq 1, \\ & \pi_2, \pi_3 \geq 0. \end{aligned}$$

Note that indifference curves and the first constraint are parallel. Thus, any feasible (π_2, π_3) with $\pi_2 + 4\pi_3 = 1.8$, maximizes i_1's utility, in particular $(0.6, 0.3)$.

Substituting $\pi_2 = 1 - \pi_1 - \pi_3$ and ignoring the constant in the objective function as above, agent i_2's optimization problem is equivalent to the following problem

$$\begin{aligned} \max_{\pi_1, \pi_3} \quad & 2\pi_1 + 5\pi_3 \\ \text{subject to } & 1.6\pi_1 + 4\pi_3 \leq 1.28, \\ & \pi_1 + \pi_3 \leq 1, \\ & \pi_1, \pi_3 \geq 0. \end{aligned}$$

Note that indifference curves and the first constraint are parallel. Thus, any feasible (π_1, π_3) with $1.6\pi_1 + 4\pi_3 = 1.28$ maximizes agent i_2's utility, in particular $(0.8, 0)$.

Substituting $\pi_3 = 1 - \pi_1 - \pi_2$ and ignoring the constant in the objective function as above, agent i_3's optimization problem is equivalent to the following problem

$$\begin{aligned} \max_{\pi_1,\pi_2} \quad & 3\pi_1 + \pi_2 \\ \text{subject to } & 0.9\pi_1 + 0.3\pi_2 \leq 0.15, \\ & \pi_1 + \pi_2 \leq 1, \\ & \pi_1, \pi_2 \geq 0. \end{aligned}$$

Note that indifference curves and the first constraint are parallel. Thus, any feasible (π_1, π_2) with $0.9\pi_1 + 0.3\pi_2 = 0.15$ maximizes i_3's utility, in particular a vector $(0.1, 0.2)$. □

More generally, we can obtain a second welfare theorem for equilibria with priority-specific prices by using the second welfare theorem of Miralles and Pycia (2020) and treating priority classes as separate objects that have to be priced. The second welfare theorem for cut-off-constrained equilibria follows immediately as a corollary.

Theorem 3. *For each priority-constrained efficient random assignment x under U and $\succsim$, there exist priority-specific prices p and budgets b such that $(x, p) \in \mathcal{E}(U, b, \succsim)$.*

Proof. We rely on Theorem 3 of Miralles and Pycia (2020). Importantly, a careful inspection of their proof shows that their Theorem 3 does not hinge on the assumption that the supply of each object is an integer number. Miralles and Pycia prove a more general result for multi-unit demand, however, for our set-up, the following version of their theorem is sufficient.

Lemma 2 ((Adapted) Theorem 3 in Miralles and Pycia, 2020)**.** *Let N be a finite set of agents and let $\mathcal{O}$ be a finite set of objects where object $o \in \mathcal{O}$ is supplied in $\tilde{q}_o \in \mathbb{R}_+$ units such that $\sum_{o \in \mathcal{O}} \tilde{q}_o = |N|$. Suppose each agent $i \in N$ has a set of feasible objects, $B_i \subseteq \mathcal{O}$ and a utility function $\tilde{u}_i : B_i \to \mathbb{R}_+$. Let $\tilde{x} = ((\tilde{x}_{io})_{o \in B_i})_{i \in N} \in \times_{i \in N} \Delta(B_i)$ such that for each $o \in \mathcal{O}$ we have $\sum_{i \in N} \tilde{x}_{io} \leq \tilde{q}_o$ and there is no $\tilde{y} \in \times_{i \in N} \Delta(B_i)$ such that for each $o \in \mathcal{O}$ we have $\sum_{i \in N} \tilde{y}_{io} \leq \tilde{q}_o$ and $\tilde{y}$ Pareto dominates $\tilde{x}$ in the sense that for each $i \in N$ we have*

$$\sum_{o \in B_i} \tilde{u}_{io} \tilde{y}_{io} \geq \sum_{o \in B_i} \tilde{u}_{io} \tilde{x}_{io}$$

where the inequality is strict for at least one agent $i \in N$. Then there exist prices $p = (p_o)_{o \in \mathcal{O}} \in \mathbb{R}_+^{\mathcal{O}}$ and budgets $b = (b_i)_{i \in N} \in \mathbb{R}_+^N$ such that for each $i \in N$ we have $\sum_{o \in B_i} p_o \tilde{x}_{io} \leq b_i$ and for each $\pi \in \Delta(B_i)$ we have

$$\sum_{o \in B_i} \tilde{u}_{io} \pi_o > \sum_{o \in B_i} \tilde{u}_{io} \tilde{x}_{io} \Rightarrow \sum_{o \in B_i} p_o \pi_o > b_i.$$

We define an auxiliary market as in Lemma 2. We treat priority classes as objects which are supplied with the probability mass allotted to the priority class in the random assignment x. Agents are allowed to buy from a priority class, as long as they are ranked in or above that priority class at the school. Formally, let $\mathcal{O} := \{N_j^\ell : j \in M, \ell = 1, \ldots, \ell(j)\}$, let $\tilde{q}_{j,\ell} := \sum_{i \in N_j^\ell} x_{ij}$, for each $i \in N$ let $B_i = \{N_j^\ell \in \mathcal{O} : i \succsim_j N_j^\ell\}$

and let $\tilde{u}_{i,j,\ell} := u_{ij}$ for each $N_j^\ell \in B_i$. For each random assignment $\tilde{y} \in \times_{i\in N}\Delta(B_i)$, we can define a corresponding random assignment $y \in \times_{i\in N}\Delta(M)$ by

$$y_{ij} := \sum_{\ell=1}^{\ell(j)} \tilde{y}_{i,j,\ell}. \tag{7}$$

Note that by construction, for each $i \in N$ we have

$$\sum_{j\in M} u_{ij}y_{ij} = \sum_{j\in M}\sum_{\ell: i\succsim_j N_j^\ell} \tilde{u}_{i,j,\ell}\tilde{y}_{i,j,\ell}.$$

Moreover, $\tilde{y}_i \in \Delta(B_i)$ and therefore $\sum_{j\in M}\sum_{\ell:i\succsim_j N_j^\ell} \tilde{y}_{i,j,\ell} = 1$. As $\sum_{N_j^\ell\in\mathcal{O}} \tilde{q}_{j,\ell} = |N|$, this implies that $\sum_{i\in N} \tilde{y}_{i,j,\ell} = \tilde{q}_{j,\ell}$ for each $N_j^\ell \in \mathcal{O}$. Thus, for each $j \in M$ and $1 \leq \ell' \leq \ell(j)$ we have

$$\sum_{i:i\succsim_j N_j^{\ell'}} y_{ij} = \sum_{\ell=1}^{\ell'}\sum_{i\in N} \tilde{y}_{i,j,\ell} = \sum_{\ell=1}^{\ell'} \tilde{q}_{j,\ell} = \sum_{\ell=1}^{\ell'}\sum_{i\in N_j^\ell} x_{ij} = \sum_{i:i\succsim_j N_j^{\ell'}} x_{ij},$$

and therefore $y_j \succsim_j^{sd} x_j$. Similarly, we can derive a random assignment $\tilde{x} \in \times_{i\in N}\Delta(B_i)$ from $x \in \mathbb{R}^{N\times M}$ by

$$\tilde{x}_{i,j,\ell} = \begin{cases} x_{ij}, & \text{for } i \in N_j^\ell, \\ 0, & \text{else.} \end{cases}$$

Note that $\tilde{x}$ is Pareto efficient under $\tilde{U}$ among (in expectation) feasible random assignments for $\tilde{q}$, since otherwise if $\tilde{y} \in \times_{i\in N}\Delta(B_i)$ is feasible (in expectation) under $\tilde{q}$ and Pareto dominates $\tilde{x}$ according to $\tilde{U}$, the corresponding $y \in \times_{i\in N}\Delta(M)$, defined by Equation (7), Pareto dominates x according to U. As $y_j \succsim_j^{sd} x_j$ for each $j \in M$, this contradicts the priority-constrained efficiency of x. Since $\tilde{x}$ is Pareto efficient under $\tilde{U}$ among (in expectation) feasible random assignments for $\tilde{q}$, there exists, by Lemma 2, prices $\tilde{p} \in \mathbb{R}_+^{\mathcal{O}}$ and budgets $b \in \mathbb{R}_+^N$ such that $(\tilde{x}, \tilde{p})$ is an equilibrium for $(\tilde{U}, b)$. For each $j \in M$ and $1 \leq \ell \leq \ell(j)$ define

$$p_{j,\ell} := \min_{\ell\leq\ell'\leq\ell(j),\tilde{q}_{j,\ell'}>0} \tilde{p}_{j,\ell'},$$

with the usual convention that the minimum over an empty set is ∞. Note that by construction, for each $j \in M$ and $1 \leq \ell \leq \ell' \leq \ell(j)$ we have $p_{j,\ell} \leq p_{j,\ell'}$. We show that (x, p) is an equilibrium in $(U, b, \succsim)$. Let $i \in N$. First note that x_i is affordable under p since for $j \in M$ with $x_{ij} > 0$ we have $\tilde{q}_{j,\ell(i,j)} > 0$ and therefore $p_{j,\ell(i,j)} \leq \tilde{p}_{j,\ell(i,j)}$ and

$$\sum_{j\in M} p_{j,\ell(i,j)}x_{i,j} \leq \sum_{j\in M} \tilde{p}_{j,\ell(i,j)}\tilde{x}_{i,j,\ell(i,j)} = \sum_{j\in M}\sum_{\ell=\ell(i,j)}^{\ell(j)} \tilde{p}_{j,\ell}\tilde{x}_{i,j,\ell} \leq b_i.$$

Let $\pi_i \in \Delta(M)$ such that

$$\sum_{j\in M} p_{j,\ell(i,j)}\pi_{i,j} \leq b_i.$$

We show that x_i yields weakly higher expected utility than π_i under U_i. Define $\tilde{\pi}_i \in \Delta(B_i)$ as follows: for each $j \in M$ choose a $\ell(i,j) \leq \ell \leq \ell(j)$ such that $p_{j,\ell(i,j)} = \tilde{p}_{j,\ell}$ and let $\tilde{\pi}_{i,j,\ell} := \pi_{i,j}$ and $\pi_{i,j,\tilde{\ell}} := 0$ for $\tilde{\ell} \neq \ell$. By construction $\tilde{\pi}_i$, is affordable under $\tilde{p}$ since it costs the same as π_i under p. Since $(\tilde{x}, \tilde{p})$ is an equilibrium in $(\tilde{U}, b)$ and $\tilde{\pi}_i$ is affordable under $\tilde{p}$ this implies that $\tilde{\pi}_i$ does not yield higher expected utility than $\tilde{x}_i$ under $\tilde{U}$. By construction $\tilde{\pi}_i$ and π_i, yield the same expected utility

$$\sum_{j \in M} \sum_{\ell=\ell(i,j)}^{\ell(j)} \tilde{u}_{i,j,\ell} \tilde{\pi}_{j,\ell} = \sum_{j \in M} u_{i,j} \sum_{\ell=\ell(i,j)}^{\ell(j)} \tilde{\pi}_{j,\ell} = \sum_{j \in M} u_{i,j} \pi_{i,j},$$

and $\tilde{x}_i$ and x_i yield the same expected utility

$$\sum_{j \in M} \sum_{\ell=\ell(i,j)}^{\ell(j)} \tilde{u}_{i,j,\ell} \tilde{x}_{j,\ell} = \sum_{j \in M} \tilde{u}_{i,j,\ell(i,j)} \tilde{x}_{j,\ell(i,j)} = \sum_{j \in M} u_{i,j} x_{i,j}.$$

Thus, x_i yields weakly higher expected utility than π_i. □

For the case of cut-off constrained efficient random assignments, we can strengthen the result by having two prices for each school, a finite cut-off price, and a (weakly) lower price for agents ranked above the cut-off. The following corollary follows immediately from the previous theorem and Remark 1.

Corollary 3. *For each cut-off-constrained efficient random assignment x under U and R, there exist priority-specific prices p and budgets b such that $(x, p) \in \mathcal{E}(U, b, \succsim)$, where for each $j \in M$ there are two prices $\underline{p}_j \leq \bar{p}_j$ such that*

$$p_{j,\ell} = \begin{cases} \underline{p}_j \text{ for } N_j^\ell \succ_j C_j(x), \\ \bar{p}_j \text{ for } N_j^\ell = C_j(x), \\ \infty \text{ for } C_j(x) \succ_j N_j^\ell. \end{cases}$$

A Proof Sketch for Remark 3

A welfare-indifferent priority-improvement cycle can be used (in a similar way as a priority-respecting improvement cycle can be used to achieve a priority-constrained sd-improvement in the proof of Theorem 1) to generate a random assignment y such that $y_i \, I_i^{sd} x_i$ for each $i \in N$, $y_j \succsim_j^{sd} x_j$ for each $j \in M$, and $y_{j_0} \succsim_{j_0}^{sd} x_{j_0}$. Thus, if there is a welfare-indifferent priority-improvement cycle, the random assignment is not two-sided efficient. Moreover, each two-sided sd-efficient random assignment is two-sided efficient for any vNM utilities consistent with the ordinal preferences.

For the other direction, we adapt the proof of Theorem 1 as follows: We modify the ordering $\trianglerighteq$: For $j \neq j'$, $1 \leq \ell \leq \ell(j)$ and $1 \leq \ell' \leq \ell(j')$ we leave $\trianglerighteq$ unchanged. For $j \in M$ and $1 \leq \ell < \ell' \leq \ell(j)$ we now let $N_j^{\ell'} \triangleright N_j^\ell$. One can show that if there is no priority-respecting improvement cycle and no welfare-indifferent priority-improvement cycle, then the modified $\trianglerighteq$ is acyclic. We derive U as before from $\trianglerighteq$.

The same argument as before shows that x maximizes social welfare subject to priority constraints. To show that there is no welfare-indifferent priority improvement, observe that for y with

$$\sum_{j\in M}\sum_{i\in N} u_{ij}y_{ij} = \sum_{j\in M}\sum_{i\in N} u_{ij}x_{ij},$$

we can rearrange the terms as before to obtain

$$0 = \sum_{j\in M}\sum_{\ell=1}^{\ell(j)-1}(v(N_j^\ell) - v(N_j^{\ell+1}))(\sum_{i\succsim_j N_j^\ell}(y_{ij} - x_{ij})).$$

Now by the modified construction of $\trianglerighteq$, we have $v(N_j^\ell) < v(N_j^{\ell'})$ for $\ell < \ell'$ and therefore $v(N_j^\ell) - v(N_j^{\ell+1}) < 0$ for $\ell = 1, \ldots, \ell(j) - 1$ for each $j \in M$. Thus, for each $j \in M$ and $1 \leq \ell \leq \ell(j)$ we have $\sum_{i\succsim_j N_j^\ell} y_{ij} = \sum_{i\succsim_j N_j^\ell} x_{ij}$.

B Two-Sided Efficiency of x in Example 1

Let y be a random assignment such that

$$\begin{aligned}
&u_{11}y_{11} + u_{12}y_{12} + u_{13}y_{13} \geq 2\times 0.6 + 3\times 0 + 5\times 0.4 = 3.2,\\
&u_{21}y_{21} + u_{22}y_{22} + u_{23}y_{23} \geq 3\times 0.2 + 4\times 0.2 + 9\times 0.6 = 6.8,\\
&u_{31}y_{31} + u_{32}y_{32} + u_{33}y_{33} \geq 5\times 0.2 + 4\times 0.8 + 3\times 0 = 4.2,\\
&y_{11} \geq 0.6, \quad y_{11} + y_{21} \geq 0.8.
\end{aligned}$$

We show that $y = x$. First we show that agent i_1 obtains the same lottery, i.e. $y_{i_1} = x_{i_1} = (0.6, 0, 0.4)$. As $2y_{11} + 3y_{12} + 5y_{13} \geq 3.2$ and $y_{11} + y_{12} + y_{13} = 1$ we have $y_{12} + 3y_{13} \geq 3.2 - 2 = 1.2$. Since $y_{11} \geq 0.6$ we have $y_{12} + y_{13} \leq 0.4$ and therefore $1.2 \leq y_{12} + 3y_{13} \leq 0.4 + 2y_{13}$. As $y_{12} + y_{13} \leq 0.4$, this implies $y_{13} = 0.4$ and $y_{12} = 0$. Finally, since $y_{11} + y_{12} + y_{13} = 1$, we also have $y_{11} = 0.6$.

Next, we show that agent i_3 obtains the same lottery, i.e. $y_{i_3} = x_{i_3} = (0.2, 0.8, 0)$. As $5y_{31} + 4y_{32} + 3y_{33} \geq 4.2$ and $y_{31} + y_{32} + y_{33} = 1$ we have $2y_{31} + y_{32} \geq 1.2$. As $y_{11} + y_{21} \geq 0.8$ we have $y_{31} \leq 0.2$ and therefore, $0.4 + y_{32} \geq 1.2$ with the last inequality strict only if $y_{31} < 0.2$. Thus, $y_{32} = 0.8$ and $y_{31} = 0.2$. Finally, since $y_{31} + y_{32} + y_{33} = 1$, we also have $y_{33} = 0$.

As agents i_1 and i_3 obtain the same lottery also agent i_2 obtains the same lottery, i.e. $y_{i_2} = x_{i_2} = (0.2, 0.2, 0.6)$.

C Two-Sided Efficiency of x in Example 4

Let y be a random assignment such that

$$\begin{aligned}
&u_{11}y_{11} + u_{12}y_{12} + u_{13}y_{13} \geq 1\times 0.1 + 2\times 0.6 + 5\times 0.3 = 2.8,\\
&u_{21}y_{21} + u_{22}y_{22} + u_{23}y_{23} \geq 3\times 0.8 + 1\times 0.2 + 6\times 0 = 2.6,\\
&u_{31}y_{31} + u_{32}y_{32} + u_{33}y_{33} \geq 4\times 0.1 + 2\times 0.2 + 1\times 0.7 = 1.5,\\
&y_{11} \geq x_{11} = 0.1, \quad y_{22} \geq x_{22} = 0.2, \quad y_{33} \geq x_{33} = 0.7.
\end{aligned}$$

We show that $x = y$. First we show that agent i_1 obtains the same lottery, i.e. $y_{i_1} = x_{i_1} = (0.1, 0.6, 0.3)$. By $y_{11} + y_{12} + y_{13} = 1$ and $y_{11} + 2y_{12} + 5y_{13} \geq 2.8$ we have $y_{12} + 4y_{13} \geq 1.8$. As $y_{11} \geq 0.1$, we have $y_{12} + y_{13} \leq 0.9$ and therefore $1.8 \leq y_{12} + 4y_{13} \leq 0.9 + 3y_{13}$ with strict last inequality only if $y_{11} > 0.1$. Therefore $y_{13} \geq 0.3$ and, as $y_{33} \geq 0.7$, we have $y_{13} = 0.3$. Thus, the last inequality from before holds with equality and therefore $y_{11} = 0.1$. Since $y_{11} + y_{12} + y_{13} = 1$ this implies moreover $y_{12} = 0.6$.

Next, we show that agent i_2 obtains the same lottery, i.e. $y_{i_2} = x_{i_2} = (0.8, 0.2, 0)$. As $y_{13} = 0.3$ and $y_{33} \geq 0.7$, we have $y_{33} = 0.7$ and $y_{23} = 0$. Thus, $2.6 \leq 3y_{21} + y_{22} + 6y_{23} = 3y_{21} + y_{22}$. As $y_{22} \geq 0.2$ we have $y_{21} \leq 0.8$ and, as $y_{21} + y_{22} \leq 1$ the previous inequality can only hold for $y_{22} = 0.2$ and $y_{21} = 0.8$.

As agents i_1 and i_2 obtain the same lottery also agent i_3 obtains the same lottery, i.e. $y_{i_3} = x_{i_3} = (0.1, 0.2, 0.7)$.

D Proof of Proposition 1

Proof. Let $(x, p) \in \mathcal{E}(U, b, \succsim)$ such that each agent chooses a cheapest lottery if multiple lotteries are optimal. Suppose random assignment y Pareto dominates x and $y \succsim^{sd} x$. Then for each $i \in N$,

$$\sum_{j \in M} u_{ij} y_{ij} \geq \sum_{j \in M} u_{ij} x_{ij},$$

where the inequality is strict for at least one agent. For an agent i, for which the inequality is strict, we have by revealed preferences

$$\sum_{j \in M} p_{ij} y_{ij} > b_i \geq \sum_{j \in M} p_{ij} x_{ij}.$$

For an agent i, for which equality holds, we have

$$\sum_{j \in M} p_{ij} y_{ij} \geq \sum_{j \in M} p_{ij} x_{ij},$$

since otherwise the tie-breaking rule that a cheapest lottery is chosen in case of multiple optimal lotteries would be violated. Summing the inequalities over all agents, we obtain

$$\sum_{i \in N} \sum_{j \in M} p_{ij} y_{ij} > \sum_{i \in N} \sum_{j \in M} p_{ij} x_{ij}.$$

We can rearrange the right-hand side of the inequality,

$$\sum_{i \in N} \sum_{j \in M} p_{ij} x_{ij} = \sum_{j \in M} \sum_{i \in N} p_{ij} x_{ij} = \sum_{j \in M} \sum_{\ell=1}^{\ell(j)} p_{j,\ell} \sum_{i \in N_j^\ell} x_{ij}.$$

Similarly, we can rearrange the left-hand side of the inequality,

$$\sum_{i \in N} \sum_{j \in M} p_{ij} y_{ij} = \sum_{j \in M} \sum_{i \in N} p_{ij} y_{ij} = \sum_{j \in M} \sum_{\ell=1}^{\ell(j)} p_{j,\ell} \sum_{i \in N_j^\ell} y_{ij}.$$

Thus,

$$\sum_{j\in M}\sum_{\ell=1}^{\ell(j)} p_{j,\ell}\sum_{i\in N_j^\ell} y_{ij} > \sum_{j\in M}\sum_{\ell=1}^{\ell(j)} p_{j,\ell}\sum_{i\in N_j^\ell} x_{ij}$$

Rearranging the terms:

$$0 < \sum_{j\in M}\sum_{\ell=1}^{\ell(j)} p_{j,\ell}(\sum_{i\in N_j^\ell}(y_{ij}-x_{ij})) = \sum_{j\in M}\sum_{\ell=1}^{\ell(j)-1}(p_{j,\ell}-p_{j,\ell+1})(\sum_{i\succsim_j N_j^{\ell+1}}(y_{ij}-x_{ij}))$$

As $p_{j,\ell}-p_{j,\ell+1}\leq 0$ for each $1\leq\ell\leq\ell(j)-1$ and $j\in M$, we thus have $\sum_{i\succsim_j N_j^{\ell+1}}(y_{ij}-x_{ij})<0$ for at least one $1\leq\ell\leq\ell(j)-1$ and $j\in M$, contradicting $y_j \succsim_j^{sd} x_j$. □

E Excess Capacity

Throughout the paper we have made the assumption that there is an equal number of school seats and agents. We made this assumption for ease of exposition, and our main results generalize to the case of excess capacity where $n\leq\sum_{j\in M} q_j$. We briefly describe how proofs and definitions have to be adapted in this case.

Now, a **random assignment** is a stochastic matrix x such that $\sum_{j\in M} x_{ij}=1$ for each $i\in N$ and $\sum_{i\in N} x_{ij}\leq q_j$ for each $j\in M$. We can incorporate the case of outside option by adding a dummy school with large capacity, i.e. by adding an additional school j_0 with $q_{j_0}>n$ to the problem.

With excess capacity, we distinguish between **ex-ante priority respect** which is the absence of ex-ante blocking pairs and **ex-ante stability** which is the combination of ex-ante priority respect and **non-wastefulness** which requires that no agent prefers a school with unfilled capacity to a school that he is matched to with positive probability, i.e. there is no $i\in N$ and $j,j'\in M$ with $j'\neq j$ such that $x_{ij'}>0$, $j\,P_i\,j'$ and $q_j>\sum_{i'\in N} x_{i'j}$. The notions of ex-ante stability and ex-ante priority respect coincide for the case of no excess capacity.

The notion of an **equilibrium with priority-specific prices** and a **cut-off constrained equilibrium** are modified as follows: We require additionally that for schools that have unfilled capacity prices are zero: thus, we restrict ourselves to equilibria $(x,p)\in\mathcal{E}(U,b,\succsim)$ such that if $q_j>\sum_{i\in N} x_{ij}$ then $p_{j,1}=p_{j,2}=\ldots=p_{j,\ell(j)}=0$, resp. $(x,\bar{p})\in\bar{\mathcal{E}}(U,b,\succsim)$ such that if $q_j>\sum_{i\in N} x_{ij}$ then $\bar{p}_j=0$ where we define the cut-off for schools with unfilled capacity as the lowest priority class, $C_j(x):=N_j^{\ell(j)}$, for $j\in M$ with $q_j>\sum_{i\in N} x_{ij}$. By construction, these equilibria are non-wasteful.

The definitions of stochastic dominance for schools, and hence the notion of priority constrained (sd)-efficiency remains unchanged, as do the notions of a priority-respecting and an equal-priority improvement cycle. We modify the notion of cut-off constrained (sd)-efficiency as follows: A random assignment x is **(priority)-cut-off-constrained sd-efficient** if for each random assignment y that sd-dominates it, there exists a school that fills its capacity under x, $q_j=\sum_{i\in N} x_{ij}$ but uses a more lenient admission policy under y than under x. **Cut-off-constrained efficiency** is defined analogously.

Our results remain valid in the more general set-up with excess capacity and proofs have to be adapted as follows:

Theorem 1

In the proof of Theorem 1 the assumption of no excess capacity is only used to prove the identity:

$$\sum_{j\in M}\sum_{\ell=1}^{\ell(j)} v(N_j^\ell)(\sum_{i\in N_j^\ell}(y_{ij}-x_{ij})) = \sum_{j\in M}\sum_{\ell=1}^{\ell(j)-1}(v(N_j^\ell)-v(N_j^{\ell+1}))(\sum_{i\succsim_j N_j^\ell}(y_{ij}-x_{ij})).$$

In the case of excess capacity, this identity becomes

$$\sum_{j\in M}\sum_{\ell=1}^{\ell(j)} v(N_j^\ell)(\sum_{i\in N_j^\ell}(y_{ij}-x_{ij})) = \sum_{j\in M}\sum_{\ell=1}^{\ell(j)-1}(v(N_j^\ell)-v(N_j^{\ell+1}))(\sum_{i\succsim_j N_j^\ell}(y_{ij}-x_{ij})) + v(N_j^{\ell(j)})\sum_{i\succsim_j N_j^{\ell(j)}}(y_{ij}-x_{ij})).$$

We can now modify the proof as follows: Instead of choosing a mapping v with strictly positive values, we choose a mapping v with strictly negative values. In this case, for each $j \in M$, the extra term

$$v(N_j^{\ell(j)})\sum_{i\succsim_j N_j^{\ell(j)}}(y_{ij}-x_{ij})$$

is positive only if $\sum_{i\succsim_j N_j^{\ell(j)}}(y_{ij}-x_{ij})) < 0$ and the proof continues as before to conclude that $y_j \not\succsim_j^{sd} x_j$ for some $j \in M$. To guarantee that the constructed vNM utility profile U takes on positive values, we can re-scale utilities by adding the same positive constant everywhere. Since, agents consume lotteries this does not change the social welfare ranking of random allocations.

Proposition 1 and Corollary 2

The proof of Proposition 1 remains unchanged. Note that in the case of no excess capacity we implicitly used that $\sum_{i\succsim_j N_j^{\ell(j)}} y_{ij} = \sum_{i\in N} y_{ij} = q_j = \sum_{i\in N} x_{ij} = \sum_{i\succsim_j N_j^{\ell(j)}} x_{ij}$ to obtain the equation

$$\sum_{j\in M}\sum_{\ell=1}^{\ell(j)} p_{j,\ell}(\sum_{i\in N_j^\ell}(y_{ij}-x_{ij})) = \sum_{j\in M}\sum_{\ell=1}^{\ell(j)-1}(p_{j,\ell}-p_{j,\ell+1})(\sum_{i\succsim_j N_j^{\ell+1}}(y_{ij}-x_{ij})),$$

whereas now we implicitly use the fact that $p_{j,\ell(j)} = 0$ for $j \in M$ with $q_j > \sum_{i\in N} x_{ij}$ to obtain the same equation. The proof of Corollary 2 follows along similar lines: Let $M' := \{j \in M : q_j = \sum_{i\in N} x_{ij}\}$ be the schools with filled capacity. By revealed preferences and the cheapest lottery tie-breaking rule, we find that for any y that Pareto dominates x, we have

$$\sum_{i\in N}\sum_{j\in M'} p_{ij}y_{ij} > \sum_{i\in N}\sum_{j\in M'} p_{ij}x_{ij}.$$

Thus, either there is a $i \in N$ and $j \in M'$ with $C_j(x) \succ_j i$ or

$$\sum_{i\in N}\sum_{j\in M'} p_{ij}y_{ij} = \sum_{j\in M'} \bar{p}_j(\sum_{i\in C_j(x)} y_{ij}) > \sum_{j\in M'} \bar{p}_j(\sum_{i\in C_j(x)} x_{ij}) = \sum_{i\in N}\sum_{j\in M'} p_{ij}x_{ij},$$

in which case we find a $j \in M'$ with $C_j(x) = C_j(y)$ and $\sum_{i\in C_j(x)} y_{ij} > \sum_{i\in C_j(x)} x_{ij}$.

Corollary 1 and Theorem 2

For Corollary 1 and Theorem 2, we modify the definition of **no indifferences between assigned cut-off schools and safe schools** such that also schools with unfilled seats count as safe schools: for $i \in N$ $j, j' \in M$ with $i \in C_j(x)$ and $q_j = \sum_{i\in N} x_{ij}$, if $i \succ_{j'} C_{j'}(x)$ or $q_{j'} > \sum_{i\in N} x_{ij'}$, then $j \, I_i \, j' \Rightarrow x_{ij} = 0$. In the statement (2) of Theorem 2, instead of requiring that prices are strictly positive, we now require that prices for schools with filled capacity are strictly positive and for schools with unfilled capacity are zero, i.e. $\bar{p} \in \mathbb{R}^M_+$ with $\bar{p}_j > 0$ for $q_j = \sum_{i\in M} x_{ij}$ and $\bar{p}_j = 0$ for $q_j > \sum_{i\in M} x_{ij}$.

The proof of (1) $\Rightarrow$ (2) in Theorem 2 can be modified as follows: For each $j \in M$ with $q_j > \sum_{i\in N} x_{ij}$ we let $\bar{p}_j = 0$. For each $j \in M$ with $q_j = \sum_{i\in N} x_{ij}$ we let $C_j := C_j(x)$ and define $\mathcal{O}' = \{C_j : j \in M, \sum_{i\in N} x_{ij} = q_j\}$. Lemma 1 applies as before to obtain a mapping $v : \mathcal{O}' \to \mathbb{R}_{++}$ that we can use to define prices $\bar{p}_j$ for $j \in M$ with $q_j = \sum_{i\in N} x_{ij}$. Budgets are defined as before. Utility values u_{ij} for $j \in M$ with $q_j = \sum_{i\in N} x_{ij}$ are defined as before, and for $j \in M$ with $q_j > \sum_{i\in N} x_{ij}$, we require that $x_{ij} = 0 \Rightarrow u_{ij} \leq \bar{u}$ and $x_{ij} > 0 \Rightarrow u_{ij} = \bar{u}$. With these modifications the proof continues as before. For the direction that (2) $\Rightarrow$ (1), ex-ante stability holds since equilibria are non-wasteful by construction, as prices for schools with unfilled capacity are zero.

To prove (1) $\Rightarrow$ (3) in Corollary 1, we can use the same construction of U as for the direction (1) $\Rightarrow$ (2) in Theorem 2. The implications (3) $\Rightarrow$ (2) $\Rightarrow$ (1) are straightforward to prove.

Theorem 3

The proof of Theorem 3 applies verbatim to the case of excess capacity. Note that in general, the constructed prices can be positive for schools with unfilled capacity. However, it is easy to see that if the assignment x is non-wasteful, we can set prices for schools with unfilled capacity to 0. By non-wastefulness, this will not change the optimality of x_i in agent i's utility maximization problem.

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