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Violations of locality and free choice are equivalent resources in Bell experiments

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Bell inequalities rest on three fundamental assumptions: realism, locality and free choice, which lead to nontrivial constraints on correlations in very simple experiments. If we retain realism, then violation of the inequalities implies that at least one of the remaining two assumptions must fail, which can have a profound consequences for the causal explanation of the experiment. In this paper, we investigate the extent to which a given assumption needs to be relaxed for the other to hold at all costs. Our discussion is based on the observation that an assumption violation need not occur on every experimental trial, even when describing some correlations violating Bell inequalities. How often this needs to be the case determines the degree of, respectively, locality or free choice in the observed experimental behaviour. Despite their disparate character and interpretation, we show that both assumptions are equally costly. Namely, the resources required to explain the experimental statistics (measured by the frequency of causal interventions of either sort) are exactly the same. Furthermore, we compute such defined measures of locality and free choice for any non-signalling statistics in a Bell experiment with binary settings, showing that it is directly related to the amount of violation of the so called Clauser-Horne-Shimony-Holt inequalities. This result is theory-independent as it refers directly to the experimental statistics (with quantum predictions being just one example). Additionally, we show how the local fraction results for quantum-mechanical frameworks with infinite number of settings translate into analogous statements for the measure of free choice introduced in the present paper. Such a parallel treatment of both assumptions demonstrates that, as far as the statistics is concerned, causal explanations resorting either to violation of locality or free choice are fully interchangeable.

locality | free choice | causal models | non-local correlations | Bell inequalities | measure of locality | measure of free choice

"I would rather discover one true cause than gain the kingdom of Persia."

– Democritus (c. 460-370 BC)

The study of experimental correlations provides a window into the underlying causal mechanisms, even when their exact nature remains obscured. In his seminal works (1–5), John Bell showed that seemingly innocuous assumptions about the structure of causal relationships leave a mark on the observed statistics. The first assumption, called *realism* (or counterfactual definiteness), presents the worldview in which physical objects and their properties exist, whether they are observed or not. Note that realism allows a standard notion of *causality* (6, 7), which in turn provides us with the language to express the remaining two assumptions. The *locality* assumption is a statement that physical (or causal) influences propagate in accord with the spatio-temporal structure of events (i.e., neither backward in time nor instantaneous causation).

The *free choice* assumption asserts that the choice of measurement settings can be made independently from anything in the (causal) past. These three assumptions are enough to derive testable constraints on correlations called Bell inequalities.

Surprisingly, nature violates Bell inequalities (8–15) which means that if the standard causal (or *realist*) picture is to be maintained at least one of the remaining two assumptions, that is *locality* or *free choice*, has to fail. It turns out that rejecting just one of those two assumptions is always enough to explain the observed correlations, while maintaining consistency with the causal structure imposed by the other. Either option poses a challenge to deep-rooted intuitions about reality, with a full range of viable positions open to serious philosophical dispute (16–18). Notably, quantum theory in its operational formulation does not provide any clue regarding the causal structure at work, leaving such questions to the domain of interpretation. It is therefore interesting to ask about the extent to which a given assumption needs to be relaxed, if we insist on upholding the other one (while always maintaining *realism*). In this paper, we seek to compare the cost of *locality* and *free choice* on an equal footing, without any preconceived conceptual biases. As a basis for comparison we choose to measure the weight of a given assumption in terms of the following question:

How often a given assumption, i.e. locality or free choice, can be retained, while safeguarding the other assumption, in order to fully reproduce some given experimental statistics within a standard causal (or realist) approach?

This question presumes that a Bell experiment is performed trial-by-trial and the observed statistics can be explained in

Significance Statement

Faced with a violation of Bell inequalities, a committed realist might pursue an explanation of the observed correlations on the basis of violations of the *locality* or *free choice* (sometimes called *measurement independence*) assumptions. The question of whether it is better to abandon (partially or completely) *locality* or *free choice* has been strongly debated since the inception of Bell inequalities, with committed supporters on either side. For the first time, we offer a comprehensive treatment that allows a comparison of both assumptions on an equal footing. This both advances the foundational debate and provides quantitative answers regarding the weight of each assumption for causal (or realist) explanation of observed correlations.

PB developed the mathematical ideas and initial versions of the manuscript. EMP, JMY, CG, and EB contributed equally in refining and polishing this work.

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the standard causal model (or hidden variable) framework (1–7, 19–21), which subsumes *realism*. It means that the remaining two assumptions of *locality* and *free choice* translate into conditional independence between certain variables in the model, whose causal structure is determined by their spatio-temporal relations (6, 22) (for some alternative approach endorsing indefinite causal structures see e.g. (23, 24), or (25) for discussion of retrocausality). Modelling of the experiment implies that in each run of the experiment all variables (including unobserved or hidden ones) always take definite values and the statistics accumulates over many trials. This leaves open the possibility that the violation of the assumptions do not have to occur on each run of the experiment to explain the given statistics. We can thus put flesh on the bones of the above question and seek the maximal proportion of trials in which a given assumption can be retained, while safeguarding the other assumption, so as to fully reproduce some given statistics. In the following, we shall denote so defined *measure of locality* (safeguarding freedom of choice) as μ_L and *measure of free choice* (safeguarding locality) as μ_F . Also, without stating this in every instance, we note that in all subsequent discussion *realism* is assumed.*

There has been some previous research on this theme. A measure of locality analogous to μ_L was first proposed by Elitzur, Popescu and Rohrlich (27) to quantify non-locality in a singlet state. Note, it seems that the original idea of a bound for such a locality measure was expressed earlier, in (28), but a bound was not worked out. In any case, Elitzur, Popescu, and Rohrlich’s measure was dubbed *local fraction* (or *content*) and shown (with improvements in (29–31)) to vanish in the limit of an infinite number of measurement settings. A substantial step was made in (32) where the local fraction is explicitly calculated for any pure two-qubit state for an arbitrary choice of settings. We note that those results concern measure μ_L only for the specific case of quantum-mechanical predictions. In this paper we go beyond this framework and consider the case of general experimental statistics (see (33) for extension to the idea of contextuality). To avoid confusion, the term *local fraction* for measure μ_L will be only used in relation to the quantum case. Furthermore, we propose a similar treatment of the free choice assumption quantified by measure μ_F . Natural as it may seem, this approach has not been pursued in the literature, with some other measures proposed to this effect (34–42) (all retaining locality as a principle, but departing from the original notion of free choice introduced by Bell (6, 22)).

We aim to comprehensively consider the extent to which a given assumption, i.e. *locality* or *free choice*, can be preserved through partial violation of the other assumption. To accomplish this, we provide similar definitions and discuss on an equal footing both measures of locality μ_L and free choice μ_F . Then, we derive the following results. First, we prove a general structural theorem about causal models explaining any given experimental statistics in a Bell experiment (for any number of settings) showing that such defined measures are necessarily equal, $\mu_L = \mu_F$. This result consolidates those two disparate concepts demonstrating their deep interchangeability. Second, we explicitly compute both measures for any *non-signalling*

*As noted, realism is subsumed in the standard notion of causality, which is implicit in the definition of locality and free choice (1–6). So, henceforth, referring to the standard causal framework implies the realist approach. We also remark that, although, “realism” goes under different guises in the literature (e.g. “counterfactual definiteness”, “local causality”, “hidden causes”, etc.), for our purposes those distinctions are irrelevant and the underlying mathematics remains the same, i.e. it boils down to the hidden variable framework (which beyond physics is frequently referred to as the structural causal models (7)). See (3, 26) for some discussion.

		Locality	Free choice	
Any statistics	any no. settings	$\mu_L = \mu_F$		Theorem 1
	non-signalling two settings	$\mu_L = \frac{1}{2}(4 - S_{max})$	$\mu_F = \frac{1}{2}(4 - S_{max})$	Theorem 2
Quantum statistics	Bell state infinite no. settings	$\mu_L \xrightarrow{M \rightarrow \infty} 0^{(*)}$	$\mu_F \xrightarrow{M \rightarrow \infty} 0$	Theorem 3
	two-qubit state any no. settings	$\mu_L = \cos \theta^{(*)}$	$\mu_F = \cos \theta$	Theorem 4

Fig. 1. Summary of the results. The main **Theorem 1** is the backbone of the paper, consolidating both measures of locality μ_L and free choice μ_F . **Theorem 2** is a theory-independent result about both measures μ_L and μ_F . It offers a concrete interpretation for the amount of violation of the CHSH inequalities. **Theorems 3 and 4** are specific to the quantum-mechanical statistics stated here for measure μ_F . They are translations of some remarkable local fraction results μ_L in the literature (marked with an $(*)$, cf. (29–32)).

statistics in a two-setting and two-outcome Bell scenario. This enables a direct interpretation to the amount of violation of the Clauser-Horne-Shimony-Holt (CHSH) inequalities (43). Third, we consider the special case of the quantum statistics with infinite number of settings, utilising existing results for the local fraction μ_L , which thus translate on the newly developed concept of the measure of free choice μ_F . Fig. 1 summarises the results in the paper.

Results

Bell experiment and Fine’s theorem. Let us consider the usual Bell-type scenario with two parties, called Alice and Bob, playing the roles of agents conducting experiments on two separated systems (whose nature is irrelevant for the argument). We assume that on each side there are two possible outcomes labelled respectively $a, b = \pm 1$ and M possible measurement settings labelled respectively $x, y \in \mathfrak{M}$ where $\mathfrak{M} \equiv \{1, 2, \dots, M\}$. A Bell experiment consists of a series of trials in which Alice and Bob each choose a setting and make a measurement registering the outcome. After many repetitions, they compare their results described by the set of $M \times M$ distributions $\{P_{ab|xy}\}_{xy}$, where $P_{ab|xy}$ denotes the probability of obtaining outcomes a, b , given measurements x, y were made on Alice and Bob’s side respectively. For conciseness, following the terminology in (5), we will call $\{P_{ab|xy}\}_{xy}$ a “behaviour”. Note that without assuming anything about the causal structure underlying the experiment any behaviour is admissible (as long as the distributions are normalised, i.e. $\sum_{a,b} P_{ab|xy} = 1$ for each $x, y \in \mathfrak{M}$). In particular, quantum theory gives a prescription for calculating the experimental statistics $P_{ab|xy}$ for each choice of settings $x, y \in \mathfrak{M}$ based on the formalism of Hilbert spaces.

It is instructive to recall the special case of two measurement settings on each side $x, y \in \mathfrak{M} = \{0, 1\}$ for which Bell derived his seminal result. Briefly, this can be expressed by saying that any *local* hidden variable model with *free choice* has to satisfy the following four CHSH inequalities (43)

$$|S_i| \leq 2 \quad \text{for } i = 1, \dots, 4, \quad [1]$$

137 where

$$138 \quad S_1 = \langle ab \rangle_{00} + \langle ab \rangle_{01} + \langle ab \rangle_{10} - \langle ab \rangle_{11}, \quad [2]$$

$$139 \quad S_2 = \langle ab \rangle_{00} + \langle ab \rangle_{01} - \langle ab \rangle_{10} + \langle ab \rangle_{11}, \quad [3]$$

$$140 \quad S_3 = \langle ab \rangle_{00} - \langle ab \rangle_{01} + \langle ab \rangle_{10} + \langle ab \rangle_{11}, \quad [4]$$

$$141 \quad S_4 = -\langle ab \rangle_{00} + \langle ab \rangle_{01} + \langle ab \rangle_{10} + \langle ab \rangle_{11}, \quad [5]$$

142 with $\langle ab \rangle_{xy} = \sum_{a,b} ab P_{ab|xy}$ being correlation coefficients for a
 143 given choice of settings x, y . Interestingly, by virtue of Fine's
 144 theorem (44, 45), this is also a sufficient condition for a *non-*
 145 *signalling* behaviour $\{P_{ab|xy}\}_{xy}$ to be explained by a *local* hidden
 146 variable model with *freedom of choice* (for non-signalling see
 147 Eqs. (16) and (17)).

148 It is crucial to observe that, although locality and freedom
 149 of choice are two disparate concepts with different ramifications
 150 for our understanding of the experiment, they are in
 151 a certain sense interchangeable. If locality is dropped with
 152 Alice and Bob freely choosing their settings, then the boxes, by
 153 influencing one another, can produce any behaviour $\{P_{ab|xy}\}_{xy}$.
 154 Similarly, a violation of the free choice assumption can be
 155 used to reproduce any behaviour $\{P_{ab|xy}\}_{xy}$ without giving up
 156 locality. It is straightforward to see how this might work if one
 157 of the two assumptions fails on *every* experimental trial.[†]

158 However, such a complete renouncement of assumptions
 159 so central to our view of nature may seem excessive, espe-
 160 cially when the CHSH inequalities are violated only by a little
 161 amount (less than the maximal algebraic bound of $|S_i| \leq 4$),
 162 leaving room for a possible explanation of the experimental
 163 statistics by rejecting one of the assumptions *sometimes* only.
 164 Here we assess the cost of such a partial violation by asking
 165 how often a given assumption can be retained in order to
 166 account for a behaviour $\{P_{ab|xy}\}_{xy}$. We will investigate both
 167 cases in parallel: (♠) full freedom of choice with *occasional* non-
 168 locality (communication), and (♣) the possibility of retaining
 169 full locality at a price of compromising freedom of choice (by
 170 controlling or rigging measurement settings) on *some* of the
 171 trials. We shall use the least frequency of violation, required
 172 to model some statistics with a hypothetical simulation, as a
 173 natural figure of merit, guided by the principle that the less
 174 the violation the better. Notably, such simulations should not
 175 restrict possible distributions of measurement settings P_{xy} . In
 176 other words, we define a *measure of locality* μ_L as

177 *the maximal fraction of trials in which Alice and Bob do
 do not need to communicate trying to simulate a given be-
 haviour $\{P_{ab|xy}\}_{xy}$, optimised over all conceivable strategies
 with freely chosen settings.* [♠]

178 Similarly, we define a *measure of free choice* μ_F as

179 *the maximal fraction of trials in which Alice and Bob can
 grant free choice of settings in trying to simulate a given
 behaviour $\{P_{ab|xy}\}_{xy}$, optimised over all conceivable local
 strategies.* [♣]

180 In the quantum-mechanical context the measure μ_L is called
 181 a *local fraction* (27–32). By analogy, when considering the

182 quantum-mechanical statistics the measure μ_F might be called
 183 a *free fraction*. This provides an equal basis for comparing
 184 the two assumptions within the standard causal (or realist)
 185 approach, which we formalise in the following section.

186 **Causal models, locality and free choice.** The appropriate
 187 framework for the discussion of locality and free choice is
 188 provided by hidden variable models (1–5). First, a hidden
 189 variables model allows a formal statement of the *realism* as-
 190 sumption, understood to mean that properties of a physical
 191 system exist irrespective of an act of measurement (counterfac-
 192 tual definiteness). Second, hidden variable models provide the
 193 causal language in which the locality and free choice assump-
 194 tions are expressed (6, 7). The *locality* assumption conveys the
 195 requirement that the propagation of physical (or causal) influ-
 196 ences have to follow the spatio-temporal structure of events
 197 (i.e., preserve the arrow of time and respect that actions at a
 198 distance require time). The *free choice* assumption concerns
 199 the choice of measurement settings which are deemed causally
 200 unaffected by anything in the past (and thus it is sometimes
 201 called *measurement independence*).[‡] Both assumptions take the
 202 form of conditional independencies between certain variables
 203 in a hidden variables model.

204 To make this idea more concrete, let us consider a given
 205 set of probability distributions (behaviour) $\{P_{ab|xy}\}_{xy}$ which
 206 describes the statistics in a Bell experiment. Without loss of
 207 generality, by conditioning on λ in some *a priori* unknown
 208 hidden variable space Λ , one can always write (4, 5, 7)

$$209 \quad P_{ab|xy} = \sum_{\lambda \in \Lambda} P_{ab|xy\lambda} \cdot P_{\lambda|xy}, \quad [6]$$

210 where $P_{\lambda|xy}$ and $P_{ab|xy\lambda}$ are valid (i.e. normalised) conditional
 211 probability distributions. The role of the hidden variable (cause
 212 in the past) $\lambda \in \Lambda$, distributed according to some P_{λ} , is to
 213 provide an explanation of the observed experimental statistics.
 214 This means that at each run of the experiment the outcomes
 215 are described by the distribution $P_{ab|xy\lambda}$ with $\lambda \in \Lambda$ fixed in
 216 a given trial, so that the accumulated experimental statistics
 217 $P_{ab|xy}$ obtains by sampling from some distribution $P_{\lambda|xy}$ over the
 218 whole hidden variable space Λ . It is customary to say that

219 *the choice of space Λ and probability distribution P_{λ}
 along with conditional distributions $P_{ab|xy\lambda}$ and $P_{\lambda|xy}$
 satisfying Eq. (6) specify a hidden variable (HV) model
 of a given behaviour $\{P_{ab|xy}\}_{xy}$.* [★]

220 Note that such a model implicitly describes the distribution of
 221 settings chosen by Alice and Bob through the standard formula

$$222 \quad P_{xy} = \sum_{\lambda \in \Lambda} P_{xy\lambda} \cdot P_{\lambda}. \quad [7]$$

223 So far the framework is general enough to accommodate *any*
 224 causal explanation of the statistics observed in the experiment.
 225 The assumptions of locality and free choice take the form of
 226 constraints on conditional distributions in (★). For a *local hidden*
 227 *variable* (LHV) model, we require the following factorisation[§]

$$228 \quad P_{ab|xy\lambda} = P_{a|x\lambda} \cdot P_{b|y\lambda}, \quad [8]$$

[†]As noted, the *free choice* assumption is sometimes called *measurement independence*. Instead
 of on the agent, measurement independence is focussed on the measurement devices and possi-
 ble correlations between their settings, which can affect the observed statistics. Regardless of
 interpretation, the mathematics remains the same, with the source of correlations traced to some
 common factor (in the causal past).

[§]Locality can be seen as a conjunction of two conditions: *parameter independence* $P_{a|xy\lambda} = P_{a|x\lambda}$
 & $P_{b|xy\lambda} = P_{b|y\lambda}$, and *outcome independence* $P_{a|bxy\lambda} = P_{a|xy\lambda}$ & $P_{b|axy\lambda} = P_{b|xy\lambda}$. One can
 show that such defined locality entails the factorisation condition $P_{ab|xy\lambda} = P_{a|x\lambda} \cdot P_{b|y\lambda}$ (46).

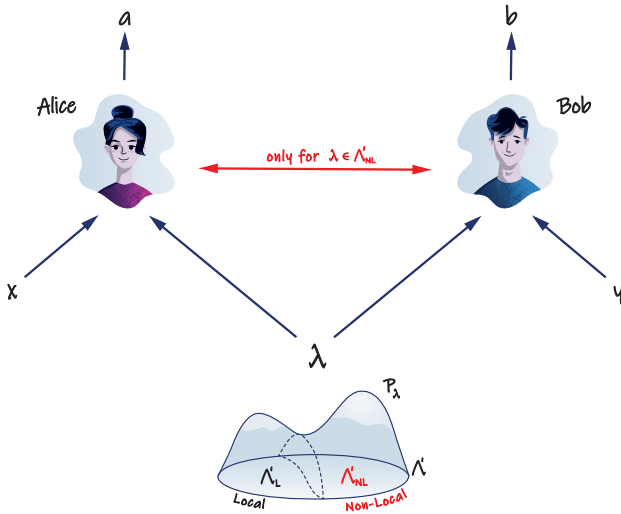


Fig. 2. Causal model with some non-locality (communication). In a Bell scenario, with free choice of settings, correlations between Alice and Bob's outcomes have two possible explanations: common cause in the past or causal influence between the parties. In any causal model the space of hidden variables (representing common causes) splits into two disjoint parts $\Lambda' = \Lambda'_L \cup \Lambda'_{NL}$ distinguished by whether, for a given $\lambda \in \Lambda'$, causal influence occurs or not, Eq. (10). Then, *locality* is measured by the proportion of events when locality is maintained, which is equal to the probability accumulated over subset Λ'_L , i.e. $Prob(\lambda \in \Lambda'_L) \equiv \sum_{\lambda \in \Lambda'_L} P_\lambda$.

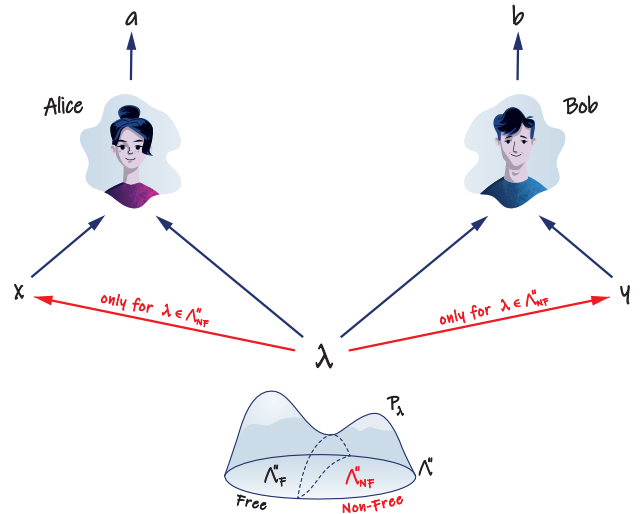


Fig. 3. Causal model with some freedom of choice (rigging). In a Bell scenario, with locality assumption, correlations between the outcomes on Alice and Bob's side can be explained by a common cause affecting choice or not (the latter implies freedom of choice). In any causal model the space of hidden variables (representing common causes) splits into two disjoint parts $\Lambda'' = \Lambda''_F \cup \Lambda''_{NF}$ distinguished by whether, for a given $\lambda \in \Lambda''$, the choice is free or not, Eqs. (11). Then, the parties enjoy *freedom of choice* only on the trials when $\lambda \in \Lambda''_F$, which happens with a frequency equal to the probability accumulated over subset Λ''_F , i.e. $Prob(\lambda \in \Lambda''_F) \equiv \sum_{\lambda \in \Lambda''_F} P_\lambda$.

229 for each $x, y \in \mathfrak{M}$ and all $\lambda \in \Lambda$. The *freedom of choice* assumption consists of requiring that λ does not contain any
 230 information about variables x, y representing Alice and Bob's
 231 choice of measurement settings. This boils down to the independence
 232 condition (6, 22)
 233

$$234 \quad P_{\lambda|xy} = P_\lambda \quad (\text{or equivalently } P_{xy|\lambda} = P_{xy}), \quad [9]$$

235 holding for $x, y \in \mathfrak{M}$ and all $\lambda \in \Lambda$. In the following, we will
 236 abbreviate a *hidden variable model with freedom of choice* as FHV
 237 model.

238 The crucial point is the distinction between *local vs non-local*
 239 as well as *free vs non-free* situations in the individual runs of
 240 the experiment modelled by Eq. (6). This means that each
 241 condition Eq. (8) and Eq. (9) should be considered separately
 242 for each $\lambda \in \Lambda$, i.e. whenever the respective condition does
 243 not hold for a given λ the assumption fails on the corresponding
 244 experimental trials. Such a distinction leads to a natural
 245 splitting of the underlying HV space into two *unique* partitions
 246 $\Lambda = \Lambda_L \cup \Lambda_{NL}$ and $\Lambda = \Lambda_F \cup \Lambda_{NF}$. The first one divides Λ by
 247 the locality property

$$248 \quad \begin{aligned} \lambda \in \Lambda_L &\Leftrightarrow \text{Eq. (8) holds for all } x, y \in \mathfrak{M}, \\ \lambda \in \Lambda_{NL} &\Leftrightarrow \text{Eq. (8) fails for some } x, y \in \mathfrak{M}, \end{aligned} \quad [10]$$

249 while the second one divides Λ with by the free choice property
 250

$$251 \quad \begin{aligned} \lambda \in \Lambda_F &\Leftrightarrow \text{Eq. (9) holds for all } x, y \in \mathfrak{M}, \\ \lambda \in \Lambda_{NF} &\Leftrightarrow \text{Eq. (9) fails for some } x, y \in \mathfrak{M}. \end{aligned} \quad [11]$$

252 Figs. 2 and 3 illustrate the causal structures for two extreme
 253 cases: FHV and LHV models (in general built on different
 254 HV spaces Λ' and Λ''). The first one grants full freedom of
 255 choice ($\Lambda' = \Lambda'_F$) while allowing for partial violation of locality
 256 ($\Lambda' \supset \Lambda'_L$). The second one retains full locality ($\Lambda'' = \Lambda''_L$)
 257 while admitting some violation of free choice ($\Lambda'' \supset \Lambda''_F$).

258 Thus, for a given experimental trial (with $\lambda \in \Lambda$ fixed)
 259 the constraints in Eqs. (10) and (11) indicate, respectively,
 260 whether some non-local influence between the parties takes
 261 place ($\lambda \in \Lambda_{NL}$) and whether some influence from the past on
 262 the measurement settings occurs ($\lambda \in \Lambda_{NF}$). In other words,
 263 in a hypothetical simulation scenario these possibilities correspond
 264 to, respectively, communication or rigging measurement
 265 settings. How often this has to happen depends on the distribution
 266 P_λ . This picture lends itself to quantifying the degree
 267 of locality and freedom choice in a given HV model.

Remark 1. For a given HV model (\star) *locality* is measured by
 268 $Prob(\lambda \in \Lambda_L) \equiv \sum_{\lambda \in \Lambda_L} P_\lambda$, and similarly *freedom of choice* is
 269 measured by $Prob(\lambda \in \Lambda_F) \equiv \sum_{\lambda \in \Lambda_F} P_\lambda$.
 270

271 This remark captures the intuition of measuring locality and
 272 freedom of choice by considering the proportion of trials when
 273 the respective property is maintained across the whole experimental
 274 ensemble. We note that this quantity is model-dependent, since it is a
 275 property of a particular HV model adopted to explain some given
 276 experimental statistics $\{P_{ab|xy}\}_{xy}$ (including the distribution of
 277 measurement settings P_{xy} , cf. Eq. (7)).
 278

279 The concepts just introduced allow a precise expression for the
 280 informal definitions (♣) and (♠) given above.

Definition 1. For a given behaviour $\{P_{ab|xy}\}_{xy}$ the measure of locality
 281 μ_L and freedom of choice μ_F are defined as
 282

$$283 \quad \mu_L := \min_{P_{xy}} \max_{FHV} \sum_{\lambda \in \Lambda_L} P_\lambda, \quad [12]$$

$$284 \quad \mu_F := \min_{P_{xy}} \max_{LHV} \sum_{\lambda \in \Lambda_F} P_\lambda, \quad [13]$$

285 where the maxima are taken respectively over all hidden variable
 286 models with freedom of choice (FHV) or all local hidden variable
 287 models (LHV) simulating given behaviour $\{P_{ab|xy}\}_{xy}$, with a fixed
 288 distribution of settings P_{xy} , minimized over any choice of the latter.

This definition follows the intuition of, respectively, locality or free choice as properties that can be relaxed only to the extent that is required to maintain the other assumption in every experimental situation (i.e., for any distribution of measurement settings P_{xy}). Formally, the measures μ_L and μ_F count the maximal frequency of, respectively, local or free choice events optimised over all protocols simulating $\{P_{ab|xy}\}_{xy}$ without violating of the other assumption, cf. **Remark 1**. The minimum over all P_{xy} amounts to the worst case scenario, which takes into account the possibility that P_{xy} is *a priori* unspecified (i.e., this amount of freedom is enough to simulate an experiment with any arbitrary choice of distribution P_{xy} in compliance with Eq. (7)).

At first glance, even if conceptually appropriate, such a definition might seem too general to provide a manageable notion, due to the range of experimental scenarios that need to be taken into account (i.e. arbitrariness of P_{xy}). However, the situation considerably simplifies because of the following lemma (see **Methods** section for further discussion and proof). This lemma also provides additional support for **Definition 1**.

Lemma 1. *In both Eqs. (12) and (13) in Definition 1 the first minimum can be omitted, i.e. we have*

$$\mu_L = \max_{FHV} \sum_{\lambda \in \Lambda_L} P_\lambda, \quad [14]$$

$$\mu_F = \max_{LHV} \sum_{\lambda \in \Lambda_F} P_\lambda, \quad [15]$$

where the respective maxima are taken for some fixed nontrivial distribution P_{xy} (i.e., the expression is insensitive to this choice provided all settings are probed, $P_{xy} \neq 0$ for all x, y).

It is in this way that the present measure of locality μ_L extends the notion of *local fraction* (27–32) to arbitrary experimental behaviour $\{P_{ab|xy}\}_{xy}$. Remarkably, the twin concept, which is the measure of free choice μ_F has not been considered at all. Perhaps the reason for this omission is the issue of arbitrariness of the distribution P_{xy} , for which there are non-trivial constraints when freedom of choice is violated (note that for the measure μ_L this problem does not occur). Those concerns can be dismissed only after the proper treatment in **Lemma 1**. This allows a so defined measure of freedom μ_F on a par with the more familiar measure of locality μ_L .

So far the concepts of violation of locality and freedom of choice, and the corresponding measures μ_L and μ_F , have been kept separate. This is expected given their disparate character. First, each concept plays a different role in the description of an experiment and hence offers a different explanation for any observed correlations, this is, direct influence (communication during the experiment) vs measurement dependence (employing common past for rigging measurement settings). Second, on the level of causal modelling those assumptions are expressed differently, Eq. (8) vs Eq. (9). Third, violating free choice gives rise to subtle issues regarding constraints on the distribution of settings P_{xy} (as noted, these concerns are addressed in **Lemma 1**).

Having brought all those issues to the spotlight, it is surprising that the assumption of locality and free choice are intrinsically connected. We now present the key result in this paper showing the exchangeability of both concepts, while maintaining the same degree of locality and freedom of choice so defined. It holds for any number of settings $x, y \in \mathfrak{M} = \{0, 1, \dots, M\}$ (see **Methods** for the proof).

Theorem 1. *For a given behaviour $\{P_{ab|xy}\}_{xy}$ the degree of locality and freedom of choice are the same, i.e. both measures in Definition 1 coincide $\mu_L = \mu_F$.*

This is a general structural theorem about causal modelling of a given behaviour $\{P_{ab|xy}\}_{xy}$. It means that the resources measured by the frequency of causal interventions of either sort, required to explain an experimental statistics, are equally costly. Thus, as far as the statistics is concerned, causal explanations resorting either to violation of locality or free choice (or measurement dependence) should be kept on an equal footing. Preference should be guided by a better understanding of a particular situation (design of the experiment as well as ontological commitments in its description).

Let us emphasise two features of **Theorem 1**. First, this is a *theory-independent* result in the sense that it applies directly to experimental statistics irrespective of the design or theoretical framework behind the experiment (with the quantum predictions being just one example). Second, the connection between those two seemingly disparate quantities μ_L and μ_F has a practical advantage: knowledge of one suffices to compute the other. Both features are illustrated by the following results.

Non-signalling behaviour with binary settings. Consider the case of Bell’s experiment with only two measurement settings on each side $x, y \in \mathfrak{M} = \{0, 1\}$. Let us recall that *non-signalling* of some given behaviour $\{P_{ab|xy}\}_{xy}$ means that Alice *cannot* infer Bob’s measurement setting (whether it is $y = 0$ or 1) from the statistics on her side alone, i.e.

$$P_{a|x0} = \sum_b P_{ab|x0} = \sum_b P_{ab|x1} = P_{a|x1} \quad \text{for all } a, x, \quad [16]$$

and similarly on Bob’s side (whether Alice chooses $x = 0$ or 1), i.e.

$$P_{b|0y} = \sum_a P_{ab|0y} = \sum_a P_{ab|1y} = P_{b|1y} \quad \text{for all } b, y. \quad [17]$$

Now we can state another result which explicitly computes both measures μ_L and μ_F in a surprisingly simple form (see **Methods** for the proof).

Theorem 2. *For a given non-signalling behaviour $\{P_{ab|xy}\}_{xy}$ with binary settings $x, y \in \mathfrak{M} = \{0, 1\}$ both measures of locality μ_L and free choice μ_F from Definition 1 are equal to*

$$\mu_L = \mu_F = \begin{cases} \frac{1}{2}(4 - S_{\max}), & \text{if } S_{\max} > 2, \\ 1, & \text{otherwise,} \end{cases} \quad [18]$$

where $S_{\max} = \max\{|S_i| : i = 1, \dots, 4\}$ is the maximum absolute value of the four CHSH expressions in Eqs. (2)–(5).

We thus obtain a systematic method for assessing the degree of locality and free choice directly from the observed statistics $\{P_{ab|xy}\}_{xy}$ without reference to the specifics of the experiment (the only requirement is non-signalling of the observed distributions). In this sense, this is a general *theory-independent* statement.

Overall, **Theorem 2** allows an interpretation of the amount of violation of the CHSH inequalities in Bell-type experiments as a fraction of trials violating locality (granted freedom of choice) or equivalently trials without freedom of choice (given locality).

The quantum case: Binary settings and beyond. Let us restrict our attention to the special case of the quantum statistics. Notably, various aspects of non-locality have been extensively researched in relation to the quantum-mechanical predictions, see (4, 5) for a review. This includes the notion of *local fraction* (27–32), which is the same as measure μ_L here defined for a general behaviour $\{P_{ab|xy}\}_{xy}$. As noted, it may be thus surprising that the equally natural measure of freedom μ_F has not been explored. **Theorem 1** bridges the gap between those two seemingly disparate notions: there is no actual need for separate study. We next review some crucial results for the *local fraction* in the quantum-mechanical framework, which allows us to make similar statements for the measure of freedom μ_F .

We first observe that **Theorem 2** can be readily applied to the quantum-mechanical statistics (where non-signalling holds). In a Bell experiment, quantum probabilities obtain through the standard formula $P_{ab|xy} = \text{Tr}[\rho \mathbb{P}_x^a \otimes \mathbb{P}_y^b]$ where ρ is a (bipartite) mixed state with two PVMs $\{\mathbb{P}_x^{a=\pm 1}\}$ and $\{\mathbb{P}_y^{b=\pm 1}\}$ representing Alice and Bob’s choice of measurement settings $x, y \in \mathfrak{M} = \{0, 1\}$. Calculating the CHSH expressions Eqs. (2)–(3) in each particular case is straightforward, which gives explicitly the expression for both measures μ_L and μ_F via Eq. (18). The result of special significance concerns the famous Tsirelson bound $S_{\max}^{\text{QM}} = 2\sqrt{2}$ for the maximal violation of the CHSH inequalities in quantum mechanics (47). By virtue of **Theorem 2**, this means that in order to locally recover the quantum predictions in a Bell experiment with two settings, Alice and Bob can enjoy freedom of choice in the worst case, at most, with a fraction $\mu_F = 2 - \sqrt{2} \approx 0.59$ of all trials (corresponding to the choice of measurements on a maximally entangled state that saturate the Tsirelson bound). Clearly, the same applies to local fraction μ_L in a two-setting scenario.

Interestingly, relaxing the constraint on the number of settings for Alice and Bob’s measurements $x, y \in \mathfrak{M} = \{1, 2, 3, \dots, M\}$ the quantum statistics forces us to further constrain, respectively, locality or free choice. The case of local fraction μ_L with arbitrary number of settings $M \rightarrow \infty$ has been thoroughly investigated for statistics generated by quantum states. Let us refer to two interesting results in the literature on local fraction μ_L which readily translate via **Theorem 1** to the measure of freedom μ_F . The first one concerns the statistics of a maximally entangled state, cf. (27, 29) (see **SI Appendix** for a direct proof).

Theorem 3. For every local hidden variable (LHV) model that explains the statistics of a Bell experiment for a maximally entangled state the amount of free choice tends to zero with increasing number of measurement settings M , i.e. $\mu_F \xrightarrow{M \rightarrow \infty} 0$.

Apparently, for less entangled states the amount of freedom increases, reaching the maximal value $\mu_F = 1$ for separable states. This is a consequence of the result in (32), which explicitly computes the local fraction μ_L for all pure two-qubit states. Stated for measure μ_F this takes the following form.

Theorem 4. For a pure two-qubit state, which by appropriate choice of the basis can always be written in the form $|\psi\rangle = \cos\frac{\theta}{2}|00\rangle + \sin\frac{\theta}{2}|11\rangle$ with $\theta \in [0, \frac{\pi}{2}]$, the amount of freedom is equal $\mu_F = \cos\theta$, whatever the choice and number of settings on Alice and Bob’s side.

Note that both **Theorem 3** and **Theorem 4** assume a specific form of behaviour $\{P_{ab|xy}\}_{xy}$ as obtained by the rules of

quantum theory. The theorems should be contrasted with **Theorem 2** which is a *theory-independent* statement, not limited to a particular theoretical framework.

Discussion

The ingenuity of Bell’s theorem lies in the fundamental nature of the premises from which the result is derived. Within the standard causal (or *realist*) approach, it is hard to assume less about two agents than having *free choice* and their systems being *localised* in space. Yet in some experiments nature refutes the possibility that both assumptions are concurrently true (8–15). It is not easy to reject either one of them without carefully rethinking the role of observers and how cause-and-effect manifests in the world.[‡] Our objective in this paper is this: *instead of pondering the question of how this could be possible, we ask about the extent to which a given assumption has to be relaxed in order to maintain the other*. Expressed more colloquially, it is natural for a realist to ask what is the cost of trading one concept for the other: *Is it possible to save free choice by giving up on some locality? Or, maybe is it better to forego a modicum of free choice in exchange for locality?* These questions can be compared on equal footing by computing a proportion of trials across the whole experimental ensemble in which a given assumption must fail, when the other holds at all times. Surprisingly, the answer can be obtained by looking at the observed statistics alone (avoiding the specifics of the experimental setup). The first question was formulated in the quantum-mechanical context by Elitzur, Popescu and Rohrlich (27) who introduced the notion of *local fraction* further elaborated in (29–32) (see (28) for an early indication of these ideas). Here, we generalise this notion to arbitrary experimental statistics (see also (33)). Furthermore, we answer the second question by adopting a similar approach to measuring the amount of free choice (which by analogy may be called *free fraction*). The first main result, **Theorem 1**, compares such defined measures in the general case (arbitrary statistics with any number of settings), showing that both assumptions are *equally costly*. This demonstrates a deeper symmetry between locality and free choice, which may come as a surprise, given our intuition of a profound difference in the role these concepts play in the description of an experiment.

In this paper, the notions of locality and free choice are understood in the usual sense required to derive Bell’s theorem (6, 22). They are expressed in the standard causal model framework (which subsumes realism) as unambiguous yes-no criteria for each experimental trial (i.e. when all past variables are fixed), determining whether there is a causal link between certain variables in a model (without pondering its exact nature). The measures μ_L and μ_F count the fraction of trials when such a connection needs to be established, breaking locality or free choice respectively, in order to explain the observed statistics. This problem is prior to a discussion of how this actually occurs, which is particularly relevant when the exact nature of the phenomenon under study is obscured. **Theorem 1** shows no intrinsic reason for a realist to favour one assumption vs the other. The minimal frequency of the required causal influences of either sort, measured by μ_L and μ_F , is exactly the

[‡]We note that the conventional understanding of causality and the language of counterfactuals has recently gained a solid mathematical basis; see e.g. the work of J. Pearl (7). However, in view of the apparent difficulties with embedding quantum mechanics in that framework, the standard approach to causality based on Reichenbach’s principle or claims regarding spatio-temporal structure of events might need reassessment; see e.g. indefinite causal structures (23, 24) or retrocausality (25).

514 same. Notably, this is a general result which holds for *any*
515 behaviour $\{P_{ab|xy}\}_{xy}$. What remains is explicit calculation of
516 those measures for a given experimental statistics.

517 The second main result, **Theorem 2**, evaluates both mea-
518 sures μ_L and μ_F for any *non-signalling* behaviour in a Bell
519 experiment with two outcomes and two settings. It provides a
520 direct interpretation to the amount of violation of the CHSH
521 inequalities (43). The key motivation behind this result is that
522 the degree by which the inequalities are violated has not been
523 given tangible interpretation so far, beyond its use as a binary
524 test of whether the inequalities are obeyed or not in study of
525 Bell non-locality. Furthermore, **Theorem 2** has the advantage
526 of being *theory-independent* in the sense of being applicable to
527 the experimental statistics regardless of its theoretical origin
528 (i.e., beyond the quantum-mechanical framework). This makes
529 it suitable for quantitative assessment of the degree of locality
530 and free choice across different experimental situations, with
531 prospective applications beyond physics, e.g. in neuroscience,
532 cognitive psychology, social sciences or finance (48–52).

533 We also state two results, **Theorem 3** and **Theorem 4**, for
534 the measure of free choice μ_F in the case of the quantum
535 statistics generated by the pure two-qubit states. Both are
536 direct translation, via **Theorem 1**, of the corresponding results
537 for the local fraction μ_L (27–32).

538 It is worth noting a related idea of quantifying non-locality
539 through the amount of information transmitted between the
540 parties that is required to reproduce quantum correlations (un-
541 der free choice assumption). Together with the development
542 of the specific models (53–57), this has led to various results re-
543 garding communication complexity in the quantum realm (58).
544 However, in this paper we take a different perspective on mea-
545 suring non-locality by changing the question from “*how much*”
546 to “*how often*” communication needs to be established between
547 the parties to simulate given correlations. **Theorem 2** gives
548 the exact bound in the case of non-signalling statistics in the
549 two-setting and two-outcome Bell experiment. In the quantum
550 case, such a simulation requires communication in at least 41 %
551 of trials (because of Tsirelson’s bound (47)) and for maximally
552 entangled states increases to 100 % of trials when the number
553 of settings is arbitrary (cf. **Theorems 3** and **4**).

554 Natural as it may seem, the idea of measuring freedom of
555 choice by measure μ_F has not been developed in the literature.
556 The reason for this omission can be traced to the conceptual
557 and technical issues with handling arbitrariness of the distribu-
558 tion of settings P_{xy} . Those concerns are properly addressed in
559 the present paper with **Lemma 1**, which considerably simpli-
560 fies and supports **Definition 1**. We note that various measures
561 have been developed as a means of quantifying freedom of
562 choice (or *measurement independence*, as it is sometimes called).
563 They include maximal distance between distributions (35, 37),
564 mutual information (38, 42) or measurement dependent loca-
565 lity (39–41). Furthermore, some explicit models simulating
566 correlations in a singlet state with various degrees of measure-
567 ment dependence have been proposed (34, 36) and analysed
568 (e.g. see (42) for comparison of causal vs retrocausal models).
569 However, these attempts depart from the original understand-
570 ing of the free choice as introduced by Bell (6, 22) (strict in-
571 dependence of choice from anything in the past) in favour of
572 more sophisticated information-theoretic accounts. Notably,
573 the proposed measure of free choice builds on the Bell’s origi-
574 nal framework assessing the maximal frequency with which

575 such a freedom *can* be retained in a model strictly consistent
576 with locality. It thus benefits from a direct interpretation within
577 the established causal framework of Bell inequalities and has
578 a clear-cut operational meaning.

579 Regarding **Theorem 3**, which rules out *any* freedom of
580 choice *so defined*, it is interesting to take an adversarial per-
581 spective on the problem of free choice in relation to quantum
582 cryptography and device independent certification (59, 60). In
583 this narrative an eavesdropper controls the devices trying to
584 simulate the quantum statistics of a Bell test, which is impossi-
585 ble as long as the parties enjoy freedom of choice. However,
586 any breach of the latter, i.e. control of measurement settings,
587 shifts the balance in favour of the eavesdropper in her mali-
588 cious task. Taking the view that any causal influence comes
589 with a cost or danger of being uncovered there are two di-
590 verging strategies that reduce the cost/risk to be considered:
591 (a) resort to the use of control of choice as seldom as possible
592 during the experiment, or (b) minimise the intensity of each act
593 of control. **Theorem 3** completely rules out the first possibil-
594 ity when simulating quantum statistics, i.e., the eavesdropper
595 needs to manipulate both settings on each trial in order to sim-
596 ulate the quantum statistics. The question about the intensity
597 of the control is left open in our discussion, but amenable to
598 information-theoretic methods (35–42). This gives additional
599 security criteria for quantum cryptography and device inde-
600 pendent certification by forcing the eavesdropper to a more
601 challenging sort of attack (not only can she not miss a trial,
602 but the control has to be subtle enough).

603 We remark that the main **Theorem 1** readily extends to the
604 case of larger number of parties and outcomes $\{P_{abc\dots|xyz\dots}\}_{xyz\dots}$.
605 This should be also possible for **Theorem 2** when characteri-
606 sation of the local polytope is known, cf. (61–67). Yet another
607 valuable avenue for research in that case consists of completing
608 the analysis to include signalling scenarios (68, 69). As for the
609 quantum case, we considered the simplest Bell-type scenario
610 with two parties involved in the experiment, but extensions
611 may prove even more surprising (see (5) for a technical review
612 of the vast field of Bell non-locality). In particular, in three-
613 party scenarios the methods discussed presently can be used to
614 eliminate freedom of choice already for two settings per party
615 sharing the GHZ state (cf. Mermin inequalities which saturate
616 in that case (70)). We should also mention an intriguing re-
617 sult (71) for a triangle quantum network in which non-locality
618 can be proved with all measurements fixed. Remarkably, there
619 is nothing to choose in that setup, but there is another assump-
620 tion of preparation independence which plays a crucial role in
621 the argument.

622 In this paper we are trying to remain impartial as to which
623 assumption — *locality* or *free choice* — is more important on
624 the fundamental level. This is certainly a strongly debated
625 subject in general, both among physicists and philosophers,
626 with strong supporters on each side (16–18). As just one
627 example depreciating the role of freedom of choice let us quote
628 Albert Einstein¹¹: “*Human beings, in their thinking, feeling and*
629 *acting are not free agents but are as causally bound as the stars*
630 *in their motion.*” As a counterbalance, it is hard to resist the
631 objection that was eloquently stated by Nicolas Gisin (72): “*But*
632 *for me, the situation is very clear: not only does free will exist, but*
633 *it is a prerequisite for science, philosophy, and our very ability to*
634 *think rationally in a meaningful way.*” Without entering into this

¹¹ Statement to the Spinoza Society of America. September 22, 1932. AEA 33-291.

635 debate, we remark that both assumptions are interchangeable
636 on a deeper level. Namely, for a given experimental statistics
637 $\{P_{ab|xy}\}_{xy}$ in a Bell-type experiment the measure of locality μ_L
638 and measure of free choice μ_F are exactly the same. This makes
639 an even stronger case regarding the inherent impossibility of
640 inferring causal structure from experimental statistics alone.

641 Materials and Methods

642 In order to facilitate the following discussion we begin with two
643 technical lemmas. See **SI Appendix** for the proofs.

644 The first one holds for a Bell experiment with arbitrary number of
645 settings $x, y \in \mathfrak{M} = \{1, 2, 3, \dots, M\}$.

646 **Lemma 2.** *For any behaviour $\{P_{ab|xy}\}_{xy}$ and distribution of settings P_{xy}
647 there exists a local hidden variable model (LHV) which fully violates the
648 freedom of choice assumption. [i.e. if $\tilde{\Lambda}$ is the relevant HV space, then we
649 have $\tilde{\Lambda} = \tilde{\Lambda}_L = \tilde{\Lambda}_{NF}$, cf. Eqs. (10) and (11)].*

650 The second one concerns a Bell scenario with binary settings $x, y \in$
651 $\mathfrak{M} = \{0, 1\}$.

652 **Lemma 3.** *Each non-signalling behaviour $\{P_{ab|xy}\}_{xy}$ with binary settings
653 $x, y \in \mathfrak{M} = \{0, 1\}$ can be decomposed as a convex mixture of a local
654 behaviour $\{\tilde{P}_{ab|xy}\}_{xy}$ and a PR-box $\{\tilde{P}_{ab|xy}\}_{xy}$ in the form*

$$655 P_{ab|xy} = p \cdot \tilde{P}_{ab|xy} + (1-p) \cdot \tilde{P}_{ab|xy}, \quad [19]$$

656 with $p = \frac{1}{2}(4 - S_{max})$ for all $x, y \in \{0, 1\}$.

657 Recall that a PR-box (73) is a non-signalling behaviour for which
658 one of the CHSH expressions in Eqs. (2)-(5) reaches the maximal
659 algebraic bound of $|S_i| = 4$. Here, local behaviour means existence of
660 a LHV+FHV model of $\{P_{ab|xy}\}_{xy}$ and $S_{max} = \max\{|S_i| : i = 1, \dots, 4\}$.

661 We are now ready to proceed with the proofs.

662 **Proof of Lemma 1.** Suppose we have a HV model (\star) of some behaviour
663 $\{P_{ab|xy}\}_{xy}$ for some nontrivial distribution of settings P_{xy} . The latter
664 obtains via Eq. (7) from the conditional probabilities $P_{xy|\lambda}$ which are
665 related to probabilities specified by the model, $P_{\lambda|xy}$ and P_λ , by the
666 usual Bayes' rule. The point at issue is whether a given HV model can
667 simulate any other distribution of settings \tilde{P}_{xy} via Eq. (7) by changing
668 $P_{xy|\lambda} \rightsquigarrow \tilde{P}_{xy|\lambda}$, while keeping the remaining components of the HV
669 model (\star) intact. This requires consistency with Bayes' rule, i.e.

$$670 \tilde{P}_{xy|\lambda} = \frac{P_{\lambda|xy} \cdot \tilde{P}_{xy}}{P_\lambda}, \quad [20]$$

671 which should be a well-defined probability distribution for each λ .
672 Since distributions $P_{\lambda|xy}$ and P_λ are fixed by the HV model (\star), then
673 the distribution of settings \tilde{P}_{xy} is arbitrary as long as the expression
674 in Eq. (20) is less than 1 for each $\lambda \in \Lambda$ (normalisation is trivially
675 fulfilled). Now, whenever freedom of choice from Eq. (9) holds, this
676 condition is always satisfied, and hence such a HV model can be
677 trivially adjusted for any distribution \tilde{P}_{xy} (by redefining $\tilde{P}_{xy|\lambda} := \tilde{P}_{xy}$
678 in compliance with Eq. (20), and keeping all the remaining components
679 of the HV model (\star) unchanged). Of course, for FHV models in the
680 definition of μ_L in Eq. (12) this is the case, which thus entails the
681 simpler expression for μ_L in Eq. (14).

682 Clearly, such a simple argument falls apart for models without
683 freedom of choice, like those in the definition of μ_F in Eq. (13), when
684 $P_{\lambda|xy}$ and P_λ do not cancel out and the probability in Eq. (20) may be
685 ill-defined. In that case, some deeper intervention into the model is
686 required as shown below.

687 Let us take some LHV model (\star) simulating a given behaviour
688 $\{P_{ab|xy}\}_{xy}$ with nontrivial distribution of settings P_{xy} . Then the related
689 HV space decomposes as $\Lambda = \Lambda_F \uplus \Lambda_{NF}$ and the degree of freedom
690 is measured by $p_F := \sum_{\lambda \in \Lambda_F} P_\lambda$, cf. **Remark 1**. Now, consider a
691 restriction of the model to the respective subspaces Λ_F and Λ_{NF} which
692 amounts to the following rescaling

$$693 P_\lambda^F := \frac{1}{p_F} P_\lambda, \quad P_{\lambda|xy}^F := \frac{1}{p_F} P_{\lambda|xy}, \quad P_{ab|xy\lambda}^F := P_{ab|xy\lambda}, \quad [21]$$

694 for $\lambda \in \Lambda_F$, and similarly

$$695 P_\lambda^{NF} := \frac{1}{1-p_F} P_\lambda, \quad P_{\lambda|xy}^{NF} := \frac{1}{1-p_F} P_{\lambda|xy}, \quad P_{ab|xy\lambda}^{NF} := P_{ab|xy\lambda}, \quad [22]$$

for $\lambda \in \Lambda_{NF}$. Both are LHV models with marginals

$$696 P_{ab|xy}^F = \sum_{\lambda \in \Lambda_F} P_{ab|xy\lambda}^F \cdot P_{\lambda|xy}^F, \quad [23] \quad 697$$

$$698 P_{ab|xy}^{NF} = \sum_{\lambda \in \Lambda_{NF}} P_{ab|xy\lambda}^{NF} \cdot P_{\lambda|xy}^{NF}, \quad [24] \quad 699$$

which provide a convex decomposition of the original behaviour
700 $\{P_{ab|xy}\}_{xy}$, i.e.

$$701 P_{ab|xy} = p_F \cdot P_{ab|xy}^F + (1-p_F) \cdot P_{ab|xy}^{NF}. \quad [25] \quad 702$$

The crucial point is a careful adjustment of these two models to
703 recover some arbitrary distribution of settings \tilde{P}_{xy} , while maintaining
704 the respective marginals Eqs. (23) and (24). For the first one (restriction
705 to Λ_F) the situation is trivial as explained above: since it is a FHV
706 model, then it suffice to redefine $\tilde{P}_{xy|\lambda}^F := \tilde{P}_{xy}$ (in compliance with
707 Eq. (20)) and leave all rest intact. As for the second one (restriction
708 to Λ_{NF}), we can use **Lemma 2** for constructing another HV space $\tilde{\Lambda}_{NF}$
709 with a LHV model without any free choice, that simulates behaviour
710 $\{P_{ab|xy}^{NF}\}_{xy}$ with the required distribution of settings \tilde{P}_{xy} . Then, such
711 modified models can be stitched back together on the compound
712 HV space $\tilde{\Lambda} := \Lambda_F \uplus \tilde{\Lambda}_{NF}$ with respective weights p_F and $1-p_F$.
713 This guarantees reconstruction of the original behaviour $\{P_{ab|xy}\}_{xy}$
714 (see Eq. (25)) with the new distribution of settings \tilde{P}_{xy} . The model
715 is local and has the same degree of freedom equal to p_F (the first
716 component has full freedom of choice, while in the second one it is
717 entirely missing).

The above construction shows that for every LHV model of some
718 behaviour $\{P_{ab|xy}\}_{xy}$ there is always another one adjusted for any other
719 distribution of settings \tilde{P}_{xy} with the same degree of freedom. This
720 justifies the simpler expression for μ_F in Eq. (15) and hence concludes
721 the proof of **Lemma 1**. 722

Proof of Theorem 1. Note that **Lemma 1** Eqs. (14) and (15) can be taken
723 as a definition of measures μ_L and μ_F . This is very convenient, since it
724 allows a discussion free from any concerns about the distribution of
725 settings P_{xy} (this is particularly relevant in the case of μ_F as explained
726 above). 727

It is instructive to observe that the calculation of both measures μ_L
728 and μ_F can be succinctly formulated as a convex optimisation problem.
729 Suppose, we can decompose some given behaviour $\{P_{ab|xy}\}_{xy}$ as a
730 mixture 731

$$732 P_{ab|xy} = p_L \cdot P_{ab|xy}^L + (1-p_L) \cdot P_{ab|xy}^{NL}, \quad [26] \quad 733$$

where $\{P_{ab|xy}^L\}_{xy}$ is a local behaviour with full freedom of choice (i.e.,
734 has a LHV+FHV model), and $\{P_{ab|xy}^{NL}\}_{xy}$ is a free behaviour (i.e., has a
735 FHV model). And similarly, suppose that

$$736 P_{ab|xy} = p_F \cdot P_{ab|xy}^F + (1-p_F) \cdot P_{ab|xy}^{NF}, \quad [27] \quad 737$$

where $\{P_{ab|xy}^F\}_{xy}$ is a local behaviour with full freedom of choice (i.e.,
738 has a LHV+FHV model), and $\{P_{ab|xy}^{NF}\}_{xy}$ is a local behaviour (i.e., has a
739 LHV model). In both cases we assume that $0 \leq p_L, p_F \leq 1$, and both
740 Eq. (26) and Eq. (27) have to hold for all $a, b = \pm 1$ and $x, y \in \mathfrak{M}$. Then,
741 we have

Remark 2. *Measures μ_L and μ_F evaluate the maxima over all possible
742 decompositions in Eqs. (26) and (27) of behaviour $\{P_{ab|xy}\}_{xy}$, i.e.*

$$743 \mu_L = \max_{\text{decomp. (26)}} p_L, \quad [28] \quad 744$$

$$745 \mu_F = \max_{\text{decomp. (27)}} p_F. \quad [29] \quad 746$$

Proof. We will justify only Eq. (28), since the argument for Eq. (29) is
747 analogous. 748

Let us observe that every HV model (\star) of behaviour $\{P_{ab|xy}\}_{xy}$ as
749 described by Eq. (6) splits into two components (cf. Eq. (10))

$$750 P_{ab|xy} = \underbrace{\sum_{\lambda \in \Lambda_L} P_{ab|xy\lambda} \cdot P_\lambda}_{p_L \cdot P_{ab|xy}^L} + \underbrace{\sum_{\lambda \in \Lambda_{NL}} P_{ab|xy\lambda} \cdot P_\lambda}_{(1-p_L) \cdot P_{ab|xy}^{NL}}, \quad [30] \quad 751$$

which defines decomposition of the type in Eq. (26) with $p_L :=$
752 $\sum_{\lambda \in \Lambda_L} P_\lambda$. Therefore, by Eq. (14), we get $\mu_L \leq \max_{\text{decomp. (26)}} p_L$. 752

To see the reverse, we note that every decomposition of the type in Eq. (26) implies existence of a LHV+FHV model of behaviour $\{P_{ab|xy}^L\}_{xy}$ on some HV space $\tilde{\Lambda}_L$ and a FHV model of behaviour $\{P_{ab|xy}^{NL}\}_{xy}$ on some HV space $\tilde{\Lambda}_{NL}$. Those two models, when combined on a compound HV space $\Lambda := \tilde{\Lambda}_L \uplus \tilde{\Lambda}_{NL}$ with the respective weights p_L and $1 - p_L$, provide a HV model of behaviour $\{P_{ab|xy}\}_{xy}$. Since the local domain of such a model contains $\tilde{\Lambda}_L$, then from Eq. (14) we have $\mu_L \geq p_L$, which entails $\mu_L \geq \max_{\text{decomp. (26)}} p_L$. This concludes the proof of Eq. (28). \square

Now, in order to prove **Theorem 1** it is enough to show that for every decomposition of the type in Eq. (26) there exists a decomposition of the type in Eq. (27) with the same weight $p_L = p_F$, and vice versa. A closer look at both expressions reveals that behaviours $\{P_{ab|xy}^L\}_{xy}$ and $\{P_{ab|xy}^{NL}\}_{xy}$ are both local with full freedom of choice (i.e., share the same LHV+FHV model). Thus, the problem can be reduced to justifying that: (a) behaviour $\{P_{ab|xy}^{NL}\}_{xy}$ also has a LHV model (possibly a non-FHV model), and (b) behaviour $\{P_{ab|xy}^L\}_{xy}$ also has a FHV model (possibly a non-LHV model).

Ad. (a) This readily follows from **Lemma 2**.
Ad. (b) Here, a trivial model will suffice. Let us take $\Lambda := \{\lambda_0\}$ (a single-element set) with $P_{\lambda_0} \equiv P_{\lambda_0|xy} := 1$ and conditional distribution defined as $P_{ab|xy\lambda_0} := P_{ab|xy}^{NL}$. Clearly, it is a FHV model of behaviour $\{P_{ab|xy}^{NL}\}_{xy}$.

Thus, we have shown equivalence of both decompositions Eqs. (26) and (27), which, by virtue of **Remark 2**, proves **Theorem 1**.

Proof of Theorem 2. By virtue of **Theorem 1** it suffices to prove the result for one of the measures. Let it be measure μ_L evaluated by means of Eq. (28) in **Remark 2**.

Consider some arbitrary decomposition Eq. (26) of behaviour $\{P_{ab|xy}\}_{xy}$. Then, by linearity, the four CHSH expressions Eqs. (2)-(5) decompose as well, i.e. we get

$$S_i = p_L \cdot S_i^L + (1 - p_L) \cdot S_i^{NL}, \quad [31]$$

where S_i^L and S_i^{NL} are calculated for the respective behaviours $\{P_{ab|xy}^L\}_{xy}$ and $\{P_{ab|xy}^{NL}\}_{xy}$. Since the first one is a local behaviour with full freedom of choice (i.e. having a LHV+FHV model), then from the CHSH inequalities Eq. (1) we have $|S_i^L| \leq 2$. For the second one there is nothing interesting to be said other than noting the maximal algebraic bound $|S_i^{NL}| \leq 4$. As a consequence, the following inequality obtains

$$|S_i| \leq p_L \cdot 2 + (1 - p_L) \cdot 4 = 4 - 2p_L, \quad [32]$$

and we get $p_L \leq \frac{1}{2}(4 - |S_i|)$. Thus, by assumed arbitrariness of decomposition, Eq. (26) gives the upper bound on expression in Eq. (28)

$$\mu_L \leq \frac{1}{2}(4 - |S_i|), \quad [33]$$

where $S_{\max} = \max\{|S_i| : i = 1, \dots, 4\}$. By **Lemma 3** we conclude that the bound is tight, which ends the proof of **Theorem 2**.

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