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On the analytics of infinite game theory problems

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On the analytics of infinite game theory problems

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Abstract: We consider zero-sum and a non zero-sum games of two players with generalized, not necessarily linear, utility functions and infinite, compact pure strategy spaces. Emphasis is given to comparisons with results obtained in mathematical theorems. The games chosen make specific points in relation to the conditions of the theorems. The idea of δ functions is exploited to construct mixed strategies. We interpret their significance in joining pure strategies and show the application in confirming NE. Uniqueness of NE is looked at. An issue is also how far an analogy can be drawn from the case of the finite matrix games. The usually discussed game theory problems are easy to analyze but they do not cover the whole range of possibilities.

Keywords and Phrases: Nash Equilibrium (NE), Infinite games, Compact set of strategies, Reaction functions, Dirac δ function, Mixed strategies, Quasi-concave utility function, Nash-von Neumman-Debreu-Fan-Glisberg theorems, multiple Nash equilibria, minimax theorem, saddle point, games with perfect recall, behavioural strategies.

JEL Classification Numbers: C61, C72

1 Introduction

The common theme of these notes is the application of fundamental results and ideas to types of games which are not commonly discussed. Mathematical results and developments have existed for a long time for infinite strategy games and games with imperfect recall. We analyze games in these areas. The complications in relation to finite matrix games and those with prefect recall are not insurmountable, at least as we move to a higher level.

We consider finite extensive, tree formulation games, both zero-sum and non zero-sum rather than in matrix form. The utility functions are always taken to be continuous and the two players have choices, each, [0, 1].

In non-cooperative normal games, simple examples usually described by finite matrices are used for both zero and non-zero games. They are in general easy to analyze the NE (Nash, 1950), using the intersection of reaction functions. What we have is the Nash result that equilibrium in mixed strategies exists. Of course NE exists in extensive form games. These are also in general straightforward to analyze.

For zero-sum matrix games we have a NE which is also a saddle point (Binmore p.228). Starting with zero-sum game in matrix form we can obtain a pair of ordinary dual linear programming problems. In effect, we can obtain also another pair where probabilities appear as an equality constraint. From this pair we can go to the ordinary pair through a change of variables (Morris, 1994). In the zero-sum games, there is a NE and the minimax property is confirmed.

However there exists also results for obtaining NE solutions in games with non linear payoff functions and generalized strategy spaces. They are in remarkable contributions (1952), by Debreu, Fan, and Glicksberg employ advanced topological and generalize the Kakutani fixed point theorem. The basic idea of the Brouwer-Kakutani theorem, that a continuous function from-to [0, 1] has to cross the 45^o line, is well known. This is really what we mean by self-fulfilling prophesies.

We analyse more general than the usually looked at problems. Through rather involved but basic calculus calculation we see how the NE results of the theoretical contribution emerge. It is more or less common place that simple finite game models can be used to understand the basic Nash theory. There is also a case for more advanced examples supporting the further theoretical development.

The utility functions of the two players are continuous (that is nice, measurable), on the product space of pure strategies. Our strategy sets are the infinite, and compact sets of [0, 1]. In the finite matrix games mixed strategies are ease to visualize. Infinite strategy games discussions involve more complicated generalized (Borel) probability sets.

In our case it is still easy to comprehend what is happening through δ functions, introduced by the mathematicians. These functions select a particular point in [0, 1] and attach to it probability 1. Mixed strategies are formed through convex combinations of such pure strategies. Finally, the games discussed are of perfect recall where a player does not forget what he once knew.

Game theory is a very active area with important developments. Among others, there are extensions of the Nash model to games with discontinuous utility functions, (Reny (2016)), and to "large games" with infinite players (a short introduction is in Glycopantis (2014)), but we are not concerned with these. We are concerned with explaining the application of fundamental results and ideas to types of games which are not widely discussed.

2 Results in the literature

A quick reminder of the idea of NE and a summary of results[1], simplified for our puposes: A pair of strategies (p_1^*, p_2^*) is NE if $u_1(p_1^*, p_2^*) \geq u_1(p_1, p_2^*)$ and $u_2(p_1^*, p_2^*) \geq u_2(p_1^*, p_2)$. That is the NE is a fixed point and can be thought of as a self-fulfilling property.

(i) T1: (The Nash theorem) Games with a finite number of players and finite sets of pure strategies have a NE in mixed strategies.

Also, statement by Nash: "In the two-person zero-sum case two equilibrium points lead to the same expectations for the players, but this need not occur in general." (1950).

(ii) T2: Consider a two-person, non-cooperative game with pure strategies $p_1 \in [0,\ 1]$ and $p_2 \in [0,\ 1]$. The utility functions are $u_1 = u_1(p_1,\ p_2)$, $u_2 = u_2(p_1,\ p_2)$ and they are continuous. If also u_i is quasi-concave in p_i, $i = 1,\ 2$, then the game has a pure strategy NE.

This is a simplified version of the Debreu, (1952), Fan (1952), Glicksberg (1952) theo-

[1] ‘′ετερος εξ ‘ετέρου σοφός τό τε πάλαι τό τε νυν, Bacchylides, 518-452 BC

rems, (see Fernando Vega-Redondo, 2003 pp. 52-53). The specification [0, 1] satisfies the technical conditions on the of pure strategies these theorems.

Of course these are sufficient conditions for the existence of a pure strategy NE, (PSNE). The question is what happens when in an explicit model the conditions of T1 are not satisfied and the reaction functions do not cross. The answer is in following,

(iii) T3: Consider a two-person, non-cooperative game with pure strategies $p_1 \in [0,\ 1]$ and $p_2 \in [0,\ 1]$ and utility functions $u_1 = u_1(p_1,\ p_2)$, $u_2 = u_2(p_1,\ p_2)$ assumed to be continuous. Then there exists a NE in mixed strategies, (MSNE). This is a simplified version of the Glicksberg (1952) theorem. The mixed strategies can go up to complicated Borel (probability) sets

(iv) T4: In a zero-sum game a strategy profile $(p_1^*,\ p_2^*)$ is a NE iff

(i) p_1^* is a maximin strategy for P1
(ii) p_2^* is a maximin strategy for P2
(iii) $\underset{p_1}{max}\ \underset{p_2}{min}\ u_1(p_1,\ p_2) = \underset{p_2}{min}\ \underset{p_1}{max}\ u_1(p_1,\ p_2)$
Also $u_1(p_1^*,\ p_2^*) = \underset{p_1}{max}\ \underset{p_2}{min}\ u_1(p_1,\ p_2)$.
In addition all NE give the same utility to P1 and to P2.

(v) Corollary 1: If for a zero-sum game $\underset{p_1}{max}\ \underset{p_2}{min}\ u_1(p_1,\ p_2) < \underset{p_2}{min}\ \underset{p_1}{max}\ u_1(p_1,\ p_2)$ then the game has no NE.

Theorem T4 and the Corollary are from Aliprantis - Chakrabarti, 2011, p. 436, p. 438.

(vi) (T5): Let a two-person, zero-sum game with utility functions $u_1(p_1,\ p_2)$, $u_2(p_1,\ p_2)$. The pair $(p_1^*,\ p_2^*)$ is NE if and only if it is saddle point, that is

$$u_1(p_1,\ p_2^*) \leq u_1(p_1^*,\ p_2^*) \leq u_1(p_1^*,\ p_2).$$

See Aliprantis - Chakrabarti, 2011, p. 436, Binmore p.228.

The examples which follow confirm the statements in the theorem above. They are cases of zero-sum and non-zero-sum games, of imperfect information o looking for a NE. For every example a number of calculations are on the figures. We believe this is easier for the readers.

The relation between the various types of strategies is briefly discussed in the prosthema. The issues involved are straightforward.

2.1 A Technical note.

We know how to analyze a simple two person, finite matrix game. For example, consider the following, game of *"Chicken"*, payoff matrix:

		P2: t_1	t_2
P1:	s_1	(3, 3)	(1, 4)
	s_2	(4, 1)	(0, 0)

Player P1 has a finite set of pure strategies s_1 and s_2 and P2 has t_1 and t_2. P1 can play his strategies with probabilities $(s_1,\ 1-p)$ and $(s_2,\ p)$ and P2 can choose independently $(t_1,\ 1-q)$ and $(t_2,\ q)$. For each player we have a finite set of pure strategies and it is perfectly acceptable and easy to comprehend that a player can choose a strategies with probability one.

It is straightforward to calculate three NE, $(p^*,\ q^*)$, derived from the intersection of the reaction functions $R1$ and $R2$. They are $(1,\ 0)$, $(0,\ 1)$, and $(\frac{1}{2},\frac{1}{2})$ with corresponding payoff pairs $(E_1,\ E_2)$ equal to $(4,\ 1)$, $(1,\ 4)$ and $(2,\ 2)$. By the way, we note that in this non zero-sum example two equilibrium points lead to different expectations for the players (Nash, 1950).

For the discussion of NE in the case of compact strategy spaces, as [0, 1], we have distributions of strategies. The idea of a (Dirac) δ function has a prominent role. These are (genelalized) distributions which pick out only one point in the space of strategies and they are used to specify a player's unique choice.[2]

That is from all distributions on $[0, 1]$ we choose the one with support 1 on (p). I.e. all the weight is attached on this point, giving it an atomic probability measure of 1. We can do this through what is known in probability theory as a "Dirac delta function". This attains the value 0 at all points except at a specific one where it is infinity, and integrates around it to the value 1.

A δ_{t-T^*} function gives 0 for $t \neq T^*$ and ∞ for $t = T^*$. The infinity is of a special nature in that δ_{t-T^*} times a function all under an integral sign will select the value of the function at T^*. These are 'linear functionals' on the underlying function space and examples of what are called generalized functions. A δ function is a generalized distribution referring to a point mass.

So δ_t gives 0 for $t \neq 0$ and ∞ for $t = 0$. The infinity is of a special nature in that δ_t times a function all under an integral sign will select the value of the function at 0.

We assume here that [0, 1] gives that set of choices (actions), p_i, of a player. A distribution (a density) function describes the probability of choosing actions by the player. For a density function $f(p_i)$ the probability that the choice p_i belongs to [a, b] is giving by $\int_a^b f(p_i)dp_1$. Through the distributions we obtain mixed strategies on [0, 1]. We can also attach [0, 1] itself to its distributions by considering the mapping of p_i to δ_{p_i}.

A δ function can be understood that it elevates an element of [0, 1] to a distribution, (mixed) strategy, concentrating completely on this particular action. It acts on a utility function through the process described above. For example $\delta_{p_1}(\frac{1}{2})$ on $u_1(p_1,\ p_2)$ will result to $u_1(\frac{1}{2},\ p_2)$. The recognition of this is simple to remember and executing a substitution step allows to proceed in the analysis.

Of course convex combination of δ functions (of integrals) can also be formed and applied to $u_i(p_1,\ p_2)$. We denote mixed δ strategies for Player i by π_i. In Example 1 we work initially using explicitly δ functions and then we resort to the appropriate substitution explained above.

We also recall that a function $u(x)$, defined on convex set X, is quasi-concave if the set $x \in X : u(x) \geq r$ is convex for every real number r. This is a important property of functions for results, for example, in utility theory. Quasi-concavity will be looked at below in the context of the specific functions used.

[2]The author D.G. has profited from discussions with A. Muir.

2.2 A comparison with the usual matrix model.

We compare results here with those of the usual matrix games.

In the finite matrix game we form expected payoffs through mixed strategies, i.e. convex combinations of pure strategies. Each player chooses independently the probability of playing his pure strategies. In the case of the two-person game we determine easily the NE through the intersections of the two reaction functions R1 and R2. In a number of cases R1 and R2 contain flats parallel to an axis. They correspond to mix strategies. The linearity of expected payoff functions guarantees that choices of a player on a flat are optimal, given the (mixed) choice of the opponent.

The issues is the following. Also in compact sets of infinite strategies, as [0, 1], there are mixed strategies. As we have just seen, we can form mixed strategies, π_i by taking the convex combinations of δ functions of a finite number of pure strategies. The mixed π_i's will be applied to the corresponding utility function u_i.

The objective is again to consider the NE. Of course we can appeal to existence theorem, given above. But attempting to construct the NE can we mimic the idea employed in the finite matrix games? This an important point.

If we have a game and manage to find the NE, this a sense could be the end. The graphs with reaction functions might help us to find the NE. Of course one thing is the mathematical solution and the graphical representation is something else. However if we attach probabilities to δ functions and then the mixed strategies form a NE, it would still be interesting to obtain a graphical representation and understanding of the equilibrium.

In the graphs there are substantial differences between finite matrix problems and those with [0, 1] strategies. Consider two player models. In the case of a finite game the flat in the reaction functions combines with probabilities the payoffs of two pure strategies. We have probabilities on the axes and the intersection of two reaction functions solves the problem. There is nothing else to consider.

In the case of [0, 1] though things are different. We suppose that the conditions of T3 are satisfied. We have strategies on the axes and this is a completely different world. Using δ functions, we can still go to "reaction functions" find their intersection but then we have to go from strategies to probabilities which are needed for an MSNE.

If for example R1 and R2 both break up at $p_i = 0$ and $p_i = 1$, $i = 1, 2$, then $\pi_i = (k)\delta_{p_i}(0) + (1-k)\delta_{p_i}(1)$ gives the same constant value,for a fixed p_j. We have formed two "reaction functions" and suppose they intersect at $p_1 = p_2 = \frac{1}{5}$.

We now have to attach probabilities, that is significance, weight, to $p_i = 0$ and $p_i = 1$ in forming $p_i = \frac{1}{5}$. For this we calculate $0k + (1-k)1 = \frac{1}{5}$ and get probability (significance, weight) $\frac{4}{5}$ for 0 and probability (significance, weight) $\frac{1}{5}$ for 1. Now there is a possibility that the mixed strategies π_1 and π_2 form a MSNE. Thus we check the requirements for a MSNE.

However there is no guarantee that these will be automatically satisfied. For example in Figure 6(ii), where only R1 breaks up at $p_1 = 0$ and $p_1 = 1$, the strategies $\pi_2 = \delta_{p_2}(\frac{1}{2})$ and $\pi_1 = (k)\delta_{p_1}(0) + (1-k)\delta_{p_1}(1)$ are self fulfilling only for $k = \frac{1}{2}$.

If we have a MSNE then $\pi_1 = (k)\delta_{p_1}(0) + (1-k)\delta_{p_1}(1)$ on u_1 gives the same constant value, for a fixed $p_2 = \frac{1}{2}$. The value of u_1 at $p_1 = k(0) + (1-k)(1)$ for a MSNE will lie below the value at the mixed strategy. Graphs this relation are shown in Figures 4 and 5.

To sum up, suppose we know from the theory that for a case with [0, 1] a NE exists. Let the reaction function in pure strategies break at a particular p_i as it does in examples below.

A rule of thumb for proceeding might be to believe that what happens in the finite case can, only in sense, be mimicked. So we connect the edges where pure strategies break with straight lines, using δ functions. This process creates mixed strategies by going to probabilities. Then we consider the intersection of such "reaction functions" calculate and hope that we have a MSNE. This is not the always the case. Going through the rigorous mathematical calculations of a NE has surprises.

Infinite strategies, as [0, 1], can lead to different equilibrium resolutions. A variety of cases can be identified, as we shall see in the examples below:

(i) The intersection of the "reaction functions" obtained through δ functions can determine MSNE. This is shown in Figure 1(ii,a), Figure 4(ii) and Figure 7(i). Characteristic of Figure 4 is that the breaking point of R1 has to be argued.

(ii) As we explain below cases (a) and (b) in Figure 1 have in effect two different graphs looking identical to the one in Figure 1(ii).

In part (a) the interpretation of the flat section of R1 refers to the choices of k$u_1(0,\ \frac{1}{2})$+(1-k)$u_1(1,\ \frac{1}{2})$, where $0 \leq k \leq 1$. In part (b) the flat part of R1 is now made up of pure strategies p_1. The two cases give different utility payoff pairs.

(iii) In Figure 5(i) the non-linearity of the utility functions allows for pure strategies reactions parallel to the axis. This gives a NE with $0 < p_i < 0$. In Figure 5(ii) the game is again of non-zero-sum. There are pure and masses of mixed strategies NE, giving varying pairs if utility values. In Figure 5(iii) all NE give the same pair of utility values.

(iv) Figure (6)(ii) and (iii), Example 4, make the significant point that, in contrast to Figure 6(i), the intersection of the solid R2 with the "reaction function" of P1 does not imply a NE. We return to the significance of u_1 for this. Figure 6(iv) gives a NE away from intersection of "reaction functions".

In the finite matrix case this would not be possible. The resolution through mixed strategies is the only one available.

To capture the fact that we give weight to u_1 at only $p_1 = 0$ or $p_1 = 1$ we borrow from the mathematics δ functions and form linear combinations. This captures the significance of p_i's in terms of probabilities on the end points.

Intuitively, if we want to use (0, 1/3) in u_1 then we give it weight 1 to (0, 1/3) and 0 to (1, 1/3). If we want to use only (1, 1/3) we give it weight 1 to (1, 1/3) and 0 to (0, 1/3). If we want to implement the point which closer to 0 than to 1 then we must attach more weight to δ of 0. The significance of the δ functions must be reflected on the significance of a point. It is like forming convex combinations of 0 and 1, that is for the point $(\frac{2}{9},\ 0)$ $k0 + (1-k)1 = \frac{2}{9}$ which implies $k = \frac{7}{9}$.

3 The various games

Example 1. A non zero-sum game

Let $u_1 = (p_2 - \frac{1}{2})p_1$ and $u_2 = (p_1 - p_2)^2$; P1 and P2 chooses their strategies in [0, 1]. The reaction functions R1 and R2 of the two players are shown in Figure 1. There is no NE in pure strategies.

If the utility functions were quasi-concave in a player's own strategy, then a PSNE would exist, applying T2. However $u_2 = (p_2 - \frac{1}{2})^2$ is not quasi-concave. It attains its minimum, 0, at $p_2 = \frac{1}{2}$ and for $u_2 > \epsilon$, positive small, the quasi-concavity conditions is not satisfied. The set of p_2's with $u_2 \geq u_2'$ is not convex. This can easily be seen on a graph.

However from Glicksberg theorem, here T3, we know that there is a mixed strategy equilibrium, as the utility functions are continuous on [0, 1]. We show here the possibility of more than one NE in mixed strategies.

As we have said above δ distributions applied to an integral of a function allocates a specific value to a variable. Of course convex combination of δ distribution can also be formed.

The recognition of these facts allows to proceed in the analysis. In this example we work initially using explicitly δ functions and then we resort to the appropriate substitution of variables by the δ functions. We check that a pair satisfies the usual condition that given the specified strategy of one player the other one maximizes his utility at his strategy in the pair.

We show that the NE is not unique. Suppose we take $(\pi_1^*,\ \pi_2^*) = (\dfrac{1}{2}(\delta_{p_1}(0)+\delta_{p_1}(1)),\ \dfrac{1}{2}(\delta_{p_2}(0)+\delta_{p_2}(1))$. Putting π_2^* into u_1 we can choose $\pi_1^* = \dfrac{1}{2}(\delta_{p_1}(0)+\delta_{p_1}(1)$. Inserting this π_1^* into u_2 we obtain the normalized $z = p_2^2 + (1-p_2)^2$. This is a maximum for the mixed strategy 0 and 1, with any convex combination of these values. Therefore we can choose π_2^*.

Next we want to prove $(\pi_1^*,\ \pi_2^*) = (\delta_{p_1}(\frac{1}{2}),\ \dfrac{1}{2}(\delta_{p_2}(0)+\delta_{p_2}(1))$ is also a mixed strategies NE.

Inserting π_1^* into u_2 we obtain $\int_0^1 (p_2 - p_1)^2 \pi_1^* = (p_2 - \frac{1}{2})^2$. This is a maximum for the mixed strategy 0 and 1, with any convex combination of the δ functions. Therefore we can choose π_2^*.

Inserting now π_2^* into u_1 we obtain
$\int_0^1 (p_2 - \frac{1}{2}) p_1 \pi_2^* = (\frac{1}{2})(-\frac{1}{2})p_1 + (\frac{1}{2})(1-\frac{1}{2})p_1 = \dfrac{1}{4}(-p_1 + p_1) = 0$. Therefore any pure strategy will do and we can choose $p_1 = \dfrac{1}{2}$ with probability 1.

Next we note whether we could mimic the case of the finite number of pure strategies. This works here, in the sense discussed above, when we considered $(\pi_1^*,\ \pi_2^*) = (\dfrac{1}{2}(\delta_{p_1}(0)+\delta_{p_1}(1)),\ \dfrac{1}{2}\delta_{p_2}(0)+\delta_{p_2}(1))$. This works because of the special function $u_2 = (\frac{1}{2} - p_2)^2$, It attains its minimum at $p_2 = \frac{1}{2}$ and its maximum at 0, 1 and any convex combination.

Also this works for $(\pi_1^*,\ \pi_2^*) = (\delta_{p_1}(\frac{1}{2}),\ \dfrac{1}{2}(\delta_{p_2}(0)+\delta_{p_2}(1))$, but as we see below, the interpretation of the graph is different.

Next we look again briefly at the two MSNE, and compare the implied payoff pairs. We use directly the replacement of variables by exact values implied by the δ functions. We have $u_1 = (p_2 - \frac{1}{2})p_1$ and $u_2 = (p_1 - p_2)^2$ and we indicate now through the π^*'s the values that the variables will take and with what probability.

(a) $\pi_1^*:\ (\frac{1}{2}(0),\ \frac{1}{2}(1))$ and $\pi_2^*:\ (\frac{1}{2}(0),\ \frac{1}{2}(1))$

π_2^* into u_1 gives, as above, $u_1 = ((\frac{1}{2})(-\frac{1}{2})p_1 + (\frac{1}{2})(\frac{1}{2}))p_1 = 0$ so any p_1 will do.

π_1^* into u_2 gives $u_2 = (\frac{1}{2})p_1^2 + (\frac{1}{2})(p_1 - 1)^2 = p_1^2 + \frac{1}{2} - p_1$, given π_2^*, max u_2 at $p_1 = 0,\ 1$ which confirms π_1^*.

(b) $\pi_1^* :\ (1)(\frac{1}{2})$ and $\pi_2^* :\ (\frac{1}{2})(0),\ (\frac{1}{2})(1)$

π_2^* into u_1 gives $u_1 = ((\frac{1}{2})(-\frac{1}{2})p_1 + (\frac{1}{2})(\frac{1}{2}))p_1 = 0$ so any p_1 will do.

π_1^* into u_2 gives $u_2 = (\frac{1}{2} - p_2)^2$ implies π_2^*.

In order to calculate the values of u_1 and u_2 at the NE then we insert in the utility functions the NE strategies. In (a) we have $u_1 = 0$, $u_2 = \frac{1}{2}$ and in (b) $u_1 = 0$, $u_2 = (\frac{1}{2})^2$. Therefore the utility payoffs are not the same under (a) and (b). If P1 and P2 could get together they could achieve a Pareto improvement.

This is also an example of two-person, non zero-sum game in which two equilibrium points lead to different expectations for the players, Nash (1950).

The two MSNE can be obtained purely mathematically without the aid of graphs. However an explanation of the two different MSNE can given by carefully appealing to their help.

In Figure 1(ii) for cases (a) and (b) we have two different graphs. They look identical to the one in Figure 1(ii) but they have completely different interpretations. In part (a) the interpretation of the flat section of R1 refers to choices of mixed strategies on the basis of u_1 calculated at $p_1 = 0$ and $p_1 = 1$ while P2 plays $\frac{1}{2}$. The choices of k$u_1(0,\ \frac{1}{2})$+(1-k)$u_1(0,\ \frac{1}{2})$, where $0 \leq k \leq 1$, cover the whole flat section and $u_1 = 0$, $u_2 = \frac{1}{2}$. Using mixed strategies P1 makes no distinction between utility and expected utility. Then we solve the problem and get $u_1 = 0$ and $u_2 = \frac{1}{2}$.

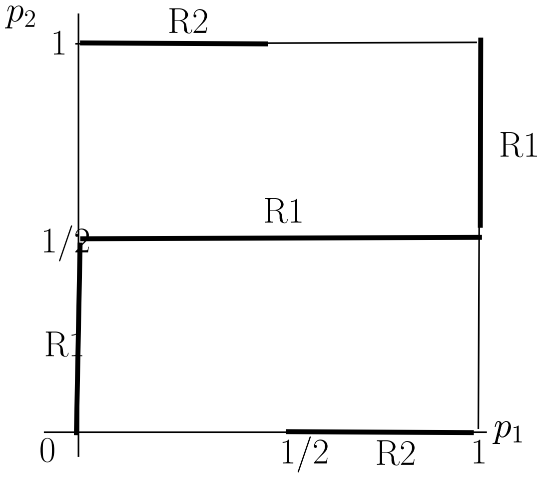
In part (b) the flat part of R1 is now made up of pure strategies p_1. Given that P2 plays $\frac{1}{2}$ then P1 can play a pure strategy p_1 to secure the maximum u_1 which is zero. Any pure strategy will do and the whole flat is covered. Then we have an intersection of R1 and R2, right in the middle, which gives $u_1 = 0$ and $u_2 = \frac{1}{4}$. In solution (a) player P1 gains nothing in relation to (b) but P2 has an advantage. He has more flexibility in the new circumstances. This is due to the strict convexity of $u_2 = (p_1 - p_2)^2$ with min $u_2 = \frac{1}{4}$ for $p_1 = \frac{1}{2}$.

The discussion extends into extensive form game trees. We keep the infinite choices of the players. A quick summary of the relevant concepts is in a short appendix (prosthema). The definitions and ideas have a wide circulation. They are needed in the tree formulation, where players take turns in choosing their strategies.

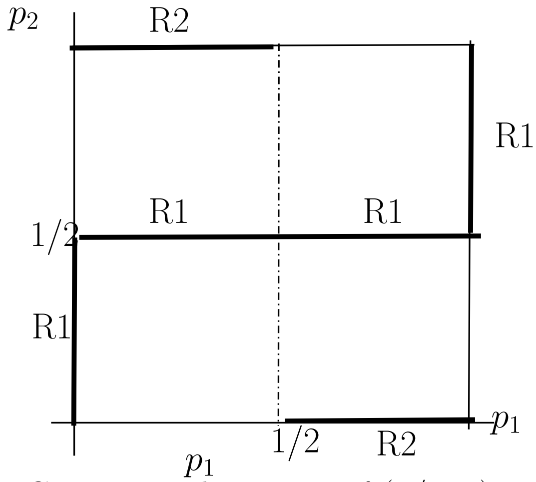
The games are of perfect recall, where the players never forget what they once knew, with perfect or imperfect information. In the case of imperfect information the players are constrained to make the same move from every node in an information set.

In Figure 2 we consider strategy profiles for the results under perfect information. All the NE can again be obtained. The novel result is that a NE might not be Pareto optimal which is harder to examine in a simultaneous move game.

In Figure 3 we consider the games under imperfect information and the previous MSNE are re-established. Behavioural strategies and Bayesian equilibria cast the discussion in current language. As explained in the prosthema, players act on the basis of beliefs attached to the nodes of an information set. For one of the MSNE calculations are on the graph.

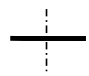


The reaction functions for
$u_1 = (p_2 - 1/2)p_1$ and $u_2 = (p_1 - p_2)^2$
For $p_2 = 1/2$ max u_1 at any p_1.
There is no NE in pure strategies.
(i)



Convex combinations of (1/2, 0) and (1/2, 1)

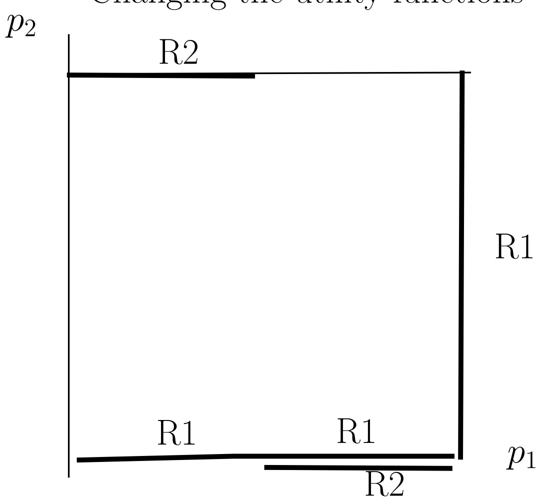
For $p_1 = 1/2$, at all convex com. $u_2 = (1/2)^2$.
On the flat of R1 $u_1 = 0$.
(a) A NE in mixed strat. at the intersection:



$u_1 = 0$
$u_2 = 1/2$

$\pi_1^* : (1/2)(0),\ \ (1/2)(1); \pi_2^* : (1/2)(0),\ \ (1/2)(1),$
Setting π_1^* into u_2 confirms π_2^*,
setting π_2^* into u_1 confirms π_1^*; ($u_1 = 0p$)
(b) Also a NE in mixed strategies:
$\pi_1^* : (1)(1/2)$ and $\pi_2^* : (1/2)(0),\ (1/2)(1)),$
π_2^* into u_1 confirms π_1^*;
π_1^* gives $u_2 = ((1/2) - p_2)^2$ max at π_2.
For $p_1 = 1/2$, at all convex com. $u_2 = (1/2)^2$
$> u_2(1/2,\ 1/2) = 0$
$u_1 = (1/2)((1/2)(-1/2) + (1/2)(1/2)) = 0$
(ii)

Changing the utility functions



$u_1 = (p_1 - 1/2)p_2$ and $u_2 = (p_2 - p_1)^2$
Masses of pure strategies NE:
$1/2 \leq p_1 \leq 1;\ p_2 = 0$
(iii)

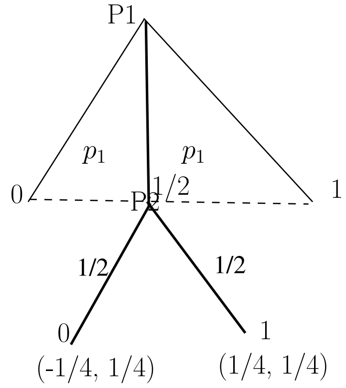
Figure 1

For $u_1 = (p_2 - 1/2)p_1$ and $u_2 = (p_1 - p_2)^2$
We saw in Figure 1:
-There is no NE in pure strategies.
- Also two mixed strategy NE (a) and (b).

In this Figure we look in terms of game trees at (b), in (ii) and (iii), and at (a) in (iv).

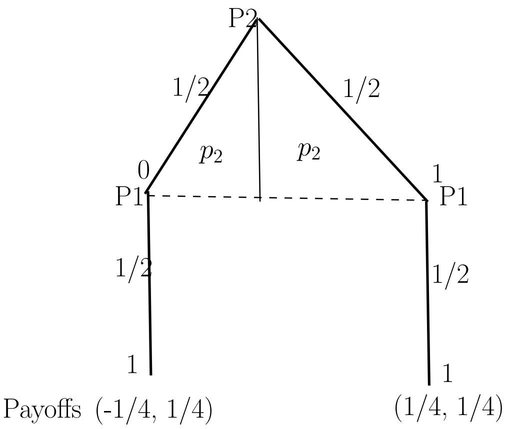
(i)

A two step problem looks at Pareto optimality.



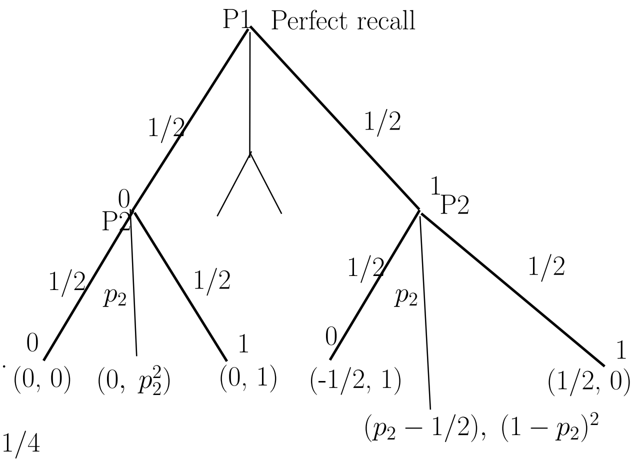
In (b): $u_1 = 0$ and $u_2 = (1/2)^2$
Now P1 plays 1/2.
If P2 plays (1/2)(0) and (1/2)(1).
Now if P1 plays $(1/2) - \epsilon$ with prob. 1 then P2 can play 1. (Ass. ϵ small.) Both players gain:
$u_1 = (1/2)^2 - (1/2)\epsilon > 0$, $u_2 = (1/2 + \epsilon)^2 > 1/4$
The NE was not Pareto optimal.

(ii)



In (b): $u_1 = 0$ and $u_2 = (1/2)^2$ shown above.
If P2 plays 1 then P1 can play $1/2 - \epsilon$; prob.= 1.
Both players gain:
$u_1 = (1/2)^2 - (1/2)\epsilon > 0$ and $u_2 = (1/2 + \epsilon)^2 > 1/4$
The NE was not Pareto optimal.

(iii)



with expected values$u_1 = 0$, $u_2 = 2/4$.
for the strat. above with prob. 1/2.
Of course the same values obtain if we reverse the order of play between P1 and P2.

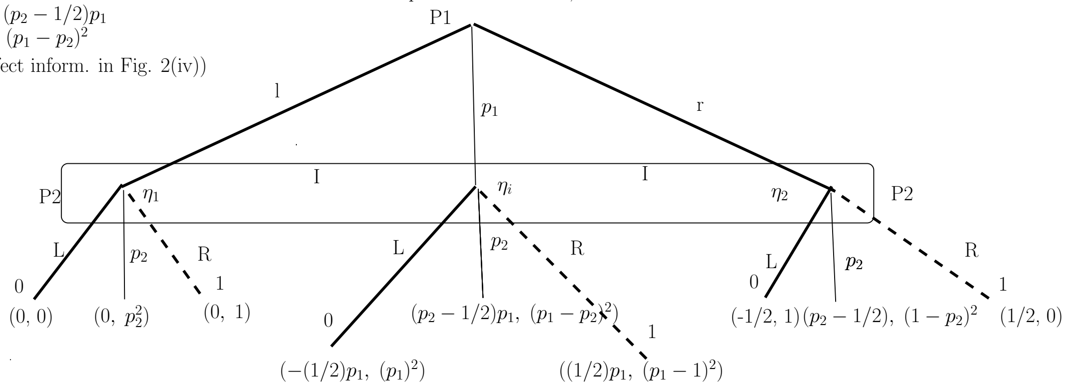
(iv)

Note. NE in mixed strategies does not mean that the idividual branches are Nash.
For example in (ii) P1 plays $p_1 = 1/2$ and P2 plays $p_2 = 0$ with $u_1 = -1/4$.
However if P1 switches to $p_1 = 0$ then he becomes better off with $u_1 = 0 > -1/4$.

Figure 2

Imperfect Information, Perfect Recall

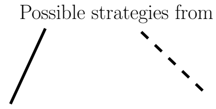
$u_1 = (p_2 - 1/2)p_1$
$u_2 = (p_1 - p_2)^2$
(perfect inform. in Fig. 2(iv))



Some not NE strategies: $(0 < p_2 < 1)$
(l, L) then P2 can move ∨
(l, p_2) P2 can move ∨
(l, R) P1 can move ∨
(r, L) then P2 can move ∨
(r, p_2) P2 can move ∨
(r. R) then P1 can move ∨

MSNE: (as for perfect infor.)
Case (i)
π_1^*: (1)(1/2), π_2^* : (1/2)(0), (1/2)(1)
Case (ii)
π_1^*: (1/2)(0), (1/2)(1), π_2^* : (1/2)(0), (1/2)(1)

Possible strategies from I

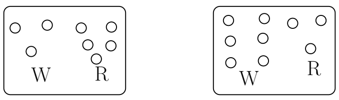


(Pure strategies are also behavioural strategies)

(i)

For a PBE we need to attach beliefs to nodes η_1 and η_2
Given that P1 plays 0 and 1 with prob 1/2, (Case (ii)),
for consistency P2 must attach beliefs 1/2 to each of the nodes.

Example of Bayesian calculation
W; white balls
R; black balls



U_1 1/2 U_2 1/2

$Pr\{U_1/W\} = A/B = \frac{3/16}{(3/16)+(6/18)}$
$A = Pr\{W/U_1\}Pr\{U_1\} = (3/8)(1/2)$
$B = Pr\{W/U_1\}Pr\{U_1\} + Pr\{W/U_2\}Pr\{U_2\}$
$= (3/8)(1/2) + (6/9)(1/2)$

The Bayesian calculations here for MSNE Case (ii)
$A = Pr\{\eta 1\} = Pr\{l/\eta_1\}Pr\{\eta_1\} = 1 \times 1/2)$
$B = Pr\{l/\eta_1\}Pr\{\eta_1\} + \Sigma_i Pr\{i/\eta_i\}Pr\{_i\} + Pr\{l/\eta_2\}Pr\{\eta_2\} = 1/2$
$Pr\{\eta_1/l\} = A/B = \frac{A}{B} = 1$
Similarly $Pr\{\eta_2\} = 1/2$.
So beliefs of P2 to support Case ii:
$Pr(\eta_1) = 1/2$ and $Pr(\eta_2 = 1/2)$.
We have a perfect Bayesian equilibrium (PBE).

(ii)

Figure 3

Example 2. A zero-sum game.

The utility functions of the players are $u_1 = p_1^2 + p_2^2 - 3p_1p_2$ and $u_2 = -p_1^2 - p_2^2 + 3p_1p_2$. Player Pi chooses strategies $p_i \in [0,\ 1]$.

First, it is seen that there are no PSNE. Then we obtain a mixed strategies NE, and confirm the minimax property, that is theorems T4 and T5.

The function u_1 is not quasi-concave because if we insert $p_2 = 1/3$ we get $u_1 = p_1^2 + 1/9 - p_1$ which is strictly convex with a minimum at $p_1 = \frac{1}{2}$ less than zero. On a simple graph we see clearly that u_1 is not quasi-concave.

So in this example the quasi-concavity condition is not satisfied. Theorem T2 gives sufficient condition for the existence of a PSNE. Since this is not a necessary requirement it is still useful to investigate the possibility of a PSNE.

The relevant graphs are in Figure 4. We now determine the reaction function R2, $p_2 = (\frac{3}{2})p_1$, of P1, which goes up to its limit $p_2 = 1$. The flat part of R2 is justifies by the fact u_2 increases in p_2 for $p_1 \geq \frac{2}{3}$.

The intersection of R1, including the interrupted line segment, with R2 will define a MSNE. Let us see what is happening. The intersection is at the point $(2/9,\ 1/3)$. In terms of p_1 the distance of the pure strategy $\frac{2}{9}$ to 0 is $\frac{2}{9}$ and from 1 is $\frac{7}{9}$. These are distances in the pure strategy space of P1. However the very idea of mixed strategy is to attach probabilities to $p_1 = 0$ and $p_1 = 1$ to capture the significance of $p_1 = 2/9$ in u_1. This is achieved by reversing the distances to get the probabilities. They correspond to the coefficients of the linear combinations of 0 and 1 which give $\frac{2}{9}$.

For R1 we have that $du_1/dp_1 = 2p_2 - 3p_2 = 0$, which means that u_1 is at such a point a minimum. So we must go away towards the two vertical lines $p_1 = 0$ and $p_1 = 1$. Here one has to be careful. We have to compare the values $u_1(0,\ x)$ and $u_1(1,\ x)$.

For $p_2 = x = 1/3$ we have $u_1(0,\ 1/3)$=$u_1(1,\ 1/3)$. This is a good starting point. We now have to compare the values $0 + p_2^2$ and $1 + p_2^2 - 3p_2$. So $p_2^2 > 1 + p_2^2 - 3p_2$ for $p_2 > 1/3$ and $p_2^2 < 1 + p_2^2 - 3p_2$ for $p_2 < 1/3$. This justifies completly the shape of R1.

R1 and R2 in Figure 4(ii) do not intersect and there is no PSNE.

We can also show through a direct argument that there is no PSNE. We examine the graph in Figure 4(i). The straight lines in the box, $p_2 = (2/3)p_1$ and $p_2 = (3/2)p_1$ correspond to $du_1/dp_1 = 0$ and $du_2/dp_2 = 0$ respectively. The areas corresponding to inequalities are then easy to establish.

No allocation, away from $p_1 = 1$, in the area $du_1/dp_1 < 0$ can be a NE because P1 can increase p_1, with p_2 fixed, and become better off. Also, no allocation away from $p_2 = 1$, in the area $du_2/p_2 > 0$ can be a NE because P2 can increase p_2, with p_1 fixed, and become better off.

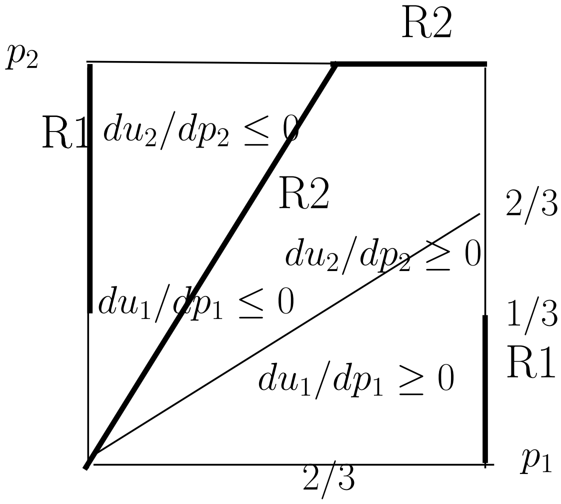
This argument leaves out the allocations $(0, 0)$ and $(1, 1)$. Then for the first P1 can choose along the horizontal axis and become better off and for $(1,\ 1)$ P1 can switch to $p_1 = 0$. Hence, the non-linear infinite mode has no PSNE.

This lack of PSNE can also be confirmed by Corollary 1:

$$\max_{p_1} \min_{p_2} u_1(p_1, p_2) = 0 < \min_{p_2} \max_{p_1} u_1(p_1, p_2) = 1.$$

Next, we recall T2 above and calculate a mixed strategy equilibrium, shown in Figure 4(ii) which, is really one of the issues in this note. Then we shall prove the saddle point

The reaction functions R1 and R2



$u_1 = p_1^2 + p_2^2 - 3p_1p_2$

$du_1/dp_1 \geq 0$ for $2p_1 - 3p_2 \geq 0$

$du_1/dp_1 \leq 0$ for $2p_1 - 3p_2 \leq 0$

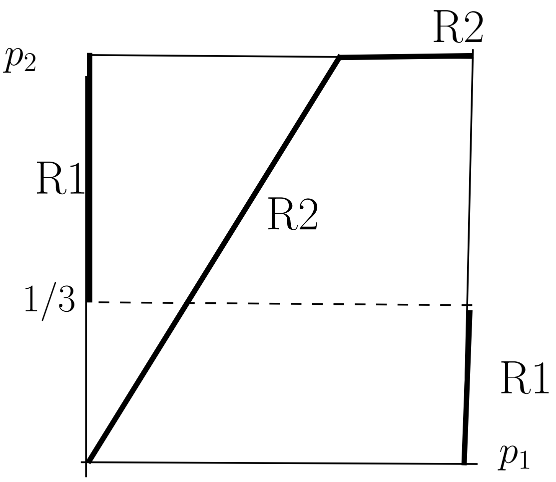
$u_1 = (1/3)^2$ at $p_1 = 0,\ 1$ for $p_2 = 1/3$

$u_2 = -p_1^2 - p_2^2 + 3p_1p_2$

$du_2/dp_2 \geq 0$ for $3p_1 - 2p_2 \geq 0$

$du_2/dp_2 \leq 0$ for $3p_1 - 2p_2 \leq 0$

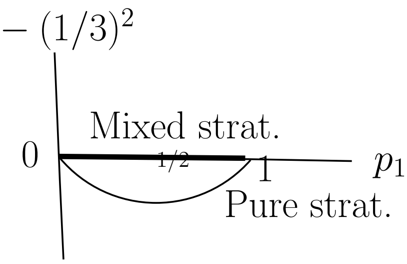
(i)



A MSNE: $\pi_2^* : (1)(1/3)$,
$\pi_1^* : (7/9)(0),\ \ (2/9)(1)$
π_1^* into u_2 gives:
$u_2 = -p_2^2 - (2/9) + (2/3)p_2$ max: $p_2 = 1/3$.
π_2^* into u_1:
$u_1 = p_1^2 + (1/3)^2 - p_1$
max at $p_1 = 0,\ 1$ and conv. com.
(2/9 from $du_2/dp_2 = 0$ at $p_2 = 1/3$).
$u_1 = 1/9$ and $u_2 = -1/9$.

(ii)

For $p_2 = 1/3$, $u_1 = p_1^2 + (1/3)^2 - p_1$:
Mixed strat. do better.



$u_1 < (1/3)^2$ for $p_1 \in (0,\ 1)$

(iii)

Figure 4

property. In Figure 4(iii) we mimic the finite matrix form ideas of finite game;

We claim that $\pi_1^* : (7/9)(0),\ (2/9)(1)$ and $\pi_2^* : (1)(1/3)$ is a MSNE. This is(at the intersection of R2 with the δ functions convex combinations of points p_1 at which R1 breaks.

We put π_2^* into u_1. We get $u_1 = p_1^2 + (1/3)^2 - p_1 = (1/3)^2 + p(p-1)$ which is maximized both at $p_1 = 0$ and $p_1 = 1$; of course $\pi_1^* : (7/9)(0),\ (1,\ 2/9)$ is a convex combination of 0 and 1 and $u_1 = \frac{1}{9}$.

Now we put $\pi_1^* : (7/9)(0),\ (2/9)(1)$ into $u_2 = -p_1^2 - p_2^2 + 3p_1p_2$.
We get $u_2 = -(7/9)p_2^2 + (2/9)(-p_2^2 - 1 + 3p_2 = -p_2^2 - (2/9) + 2/3p_2$ which is maximized at $p_2 = 1/3$ with $u_2 = -\frac{1}{9}$. Hence $\pi_1^* : (7/9)(0),\ (2/9)(1)$ and $\pi_2^* : (1)(1/3)$ is a MSNE.

The graph in the right-hand-side corner explains that the δ functions convex combinations, (expected choice), of 0 and 1 give for u_1 better values than the corresponding $p_1 = (7/9)(0) + (2/9)(1)$. Hence we can mimic the finite matrix approach.

Now imagine, Player 2 plays $p_2 = \frac{1}{3}$, Player 1 plays the linear combination of two strategies $p_1 = (\frac{7}{9})(0) + (\frac{2}{9})(1) = \frac{2}{9}$, instead of the linear combination of utilities.
$u_1((\frac{7}{9})(0) + (\frac{2}{9})(1), \frac{1}{3}) = (\frac{2}{9})^2 - \frac{2}{9} + \frac{1}{9} < u_1(\pi_1^*, \frac{1}{3}) = \frac{1}{9}$.

Hence mixing utilities is more advantageous to P1 than mixing strategies.

Now we want to check the saddle point property, (T4 and T5), on u_1, i.e.

$$u_1(\pi_1,\ \pi_2^*) \leq u_1(\pi_1^*,\ \pi_2^*) \leq u_1(\pi_1^*,\ \pi_2) \quad (**)$$

Now $u_1(\pi_1^*,\ \pi_2^*) = (1/3)^2 + (2/9) - 2/9 = 1/9$ and $u_1(p_1,\ \pi_2^*) = (1/3)^2 + p_1(p_1 - 1)$ is maximiz ed at $p_1 = 0$ and $p_1 = 1$, and hence at π_1^*, to $u_1 = \frac{1}{9}$. For all other combinations which involve different p_1 pure strategies $u_1(\pi_1,\ \pi_2^*) < \frac{1}{9}$. This is used to explain the left-hand-side inequality of (**).

We also have $u_1(\pi_1^*,\ p_2) = (7/9)p_2^2 + 2/9(p_2^2 + 1 - 3p_2) = 2/9 + p_2^2 - (2/3)p_2$ which is minimized at $p_2 = 1/3$ to $u_1 = \frac{1}{9}$. For all other combinations which involve different p_2 pure strategies $u_1(\pi_1^*,\ \pi_2) > \frac{1}{9}$. This is used to explain the right-hand-side inequality of (**).

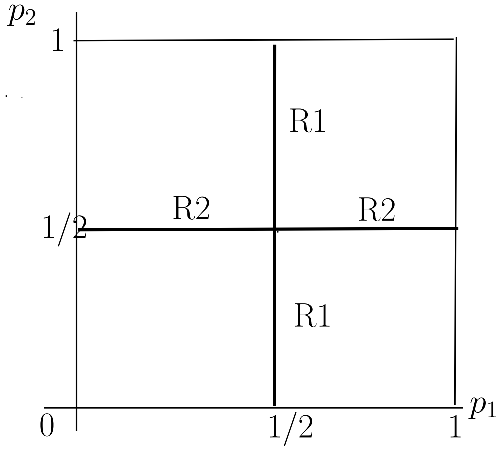
We have used pure strategies to show that the saddle property (**) holds for all further such choices. Our space is that of mixed strategies. However there is no problem because mixed strategies are convex combinations of pure strategies. For the left-hand-side we take the pure strategy in the mixture with the greatest value and for the right-hand-side the pure strategy with the lowest value. The same argument should hold even if we are in the space of complicated mixtures.

Example 3. Two non-zero and one zero-sum games.

(a) In part (i) of Figure 5 we consider the non-zero sum game with $u_1 = -p_1^2 + p_1 + p_2$ and $u_2 = -p_2^2 + p_2 + p_1$ where $p_i \in [0,\ 1]$.

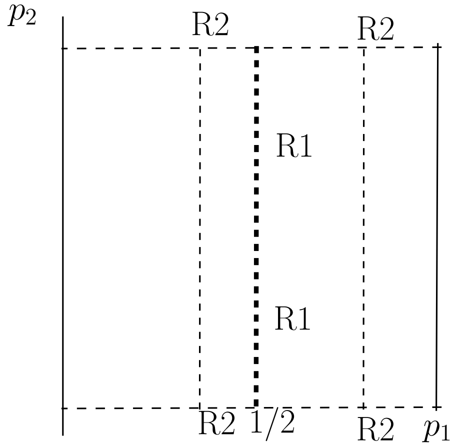
It is straightforward, and it can be read in Figure 5, that there is only one PSNE, $p_1 = p_2 = \frac{1}{2}$, with $u_1 = u_2 = 3/4$. The explanation is shown clearly. The two reaction functions, R1 and R2, are simple straight lines in $[0,\ 1]^2$. $R1$ is vertical at $p_1 = \frac{1}{2}$ and $R2$ horizontal at $p_2 = \frac{1}{2}$.

(b) In part (ii) We consider the non-zero sum game with $u_1 = -p_1^2 + p_1 + p_2$ and $u_2 = p_2^2 - p_2 - p_1$ where $p_i \in [0,\ 1]$. Things are more complicated now and there is a multiplicity



non-zero-sum
$u_1 = -p_1^2 + p_1 + p_2$ max $p_1 = 1/2$
$u_2 = -p_2^2 + p_2 + p_1$ max $p_2 = 1/2$
Intersection NE in pure strategies.
Only one optimal strategy, which is pure.
$u_1 = u_2 = 3/4$
(i)

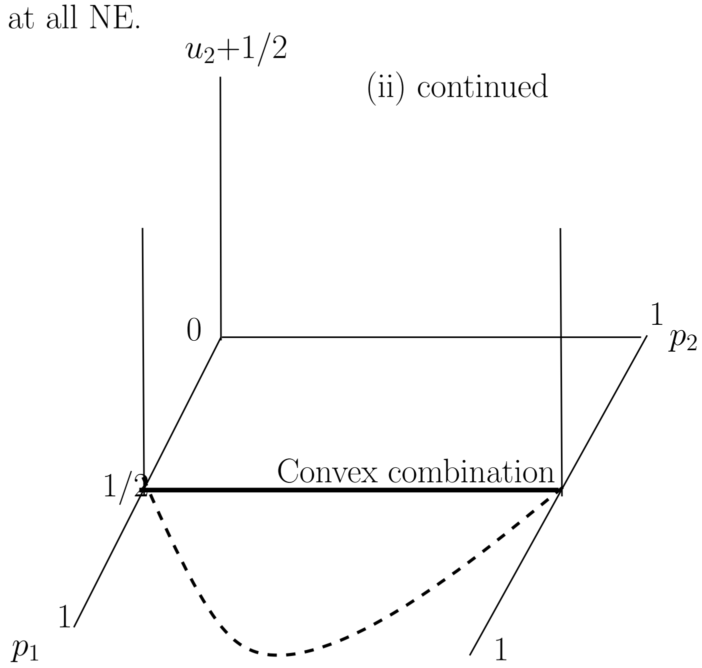
zero-sum
$u_1 = p_1^2 - p_1 - p_2$ max at $p_1 = 0,\ 1$
$u_2 = -p_1^2 + p_1 + p_2$ max at $p_2 = 1$
Two pure strat. NE: (0, 1) and (1, 1)
Mixed strategies NE: $\pi_2^* : (1)(1)$
$\pi_1^* : (1/2)(0),\ (1/2)\ (1)$
$u_1 = -1$ and $u_2 = 1$ at all NE.



non-zero-sum
$u_1 = -p_1^2 + p_1 + p_2$ max at $p_1 = 1/2$
$u_2 = p_2^2 - p_1 - p_2$ max $p_2 = 0,\ 1$
PSNE: (1/2, 0) and (1/2, 1), respectively
$u_1,\ u_2$: (1/4, -1/2) and (5/4, -1/2).
MSNE: $\pi_1^* : (1)(1/2)$
$\pi_2^* :$ (k)(0), (1-k)(1)
with $0 < k < 1$; $u_1 = (5/4) - k$ and $u_2 = -1/2$.

(ii)

(iii)



P1 plays $p_1 = 1/2$
If also P2 plays convex comb. of 0 and 1, then $u_2 = -1/2$.
If instead P2 plays pure strat. $p_2 \in (0,\ 1)$ then
$u_2 = -3/4$ and $u_2 < -1/2 < 0$. No NE.
P2 has better switch to a convex comb.

Figure 5

of equilibria.

As above R1 is vertical at $p_1 = \frac{1}{2}$. On the other hand R2 consists of the horizontal lines $p_1 = 0$ and $p_1 = 1$ in the box. This follows from the fact the irrespective of p_1 the function u_2 attains its maximum at $p_2 = [0,\ 1]$. Hence up to this point we have obtained two PSNE: that is $(p_1,\ p_2) = (\frac{1}{2},\ 0)$ and $(p_1,\ p_2) = (\frac{1}{2},\ 1)$, shown in the two ends of the interrupted black line. The implied utility functions are $(u_1,\ u_2)$=$(1/4, -\frac{1}{2})$ and $(u_1,\ u_2) = (5/4, -\frac{1}{2})$ respectively.

The mixed strategies $\pi_1^* : (1)(\frac{1}{2})$ and the convex combination $\pi_2^* : (k)(0),\ (1-k)(1)$ are MSNE. This is also also easily seen. Inserting π_i^* into u_j and maximizing with respect to p_j we get π_j^* as an implication. The utility levels are $u_1 = (5/4) - k$ and $u_2 = -\frac{1}{2}$.

The continuation of part (ii) explains that in the MSNE the convex combination in p_2 does better than the corresponding pure strategy. If P1 plays $p_1 = \frac{1}{2}$ then it is better for P2 to play his mixed strategy which brings in $u_1 = -\frac{1}{2}$ rather the corresponding pure strategy $p_2 = (k)(0) + (1-k)(1) = 1-k$ which was available but brings in $u_1 = (1-k)^2 - \frac{1}{2} - (1-k) < -\frac{1}{2}$ for $k \in (0,\ 1)$.

(c) In part (iii) the utility functions are $u_1 = p_1^2 - p_1 - p_2$ and $u_2 = -p_1^2 + p_1 + p_2$ where $p_i \in [0,\ 1]$. The strategies $\pi_1^* : (\frac{1}{2})(0), (\frac{1}{2})(1)$ and $\pi_2^* : (1)(\frac{1}{2})$ form a MSNE. There are also two PSNE, $(0,1)$ and $(1,1)$.

The proof that $(0,1)$ and $(1,1)$ are NE is straightforward. With respect to the MSNE It is also easy to inserting π_i^* into u_j and maximizing with respect to p_j confirms p_j^*, where $i \neq j$. It is also routine to show that in all NE we have $u_1 = -1$ and $u_2 = 1$.

Finally, if P2 plays $p_2 = 1$ then it is better for P1 to play his mixed strategy which brings in $u_1 = -1$ rather the corresponding pure strategy $p_1 = (\frac{1}{2})(0) + (\frac{1}{2})(1) = \frac{1}{2}$ which was available but brings in only $u_1 = (\frac{1}{2})^2 - \frac{1}{2} - 1 < -1$.

Examples 4.

We consider various cases in Figure 6.

(i) we have $u_1 = (p_1 - p_2)^2$ and $u_2 = -(p_1 - p_2)^2$ with $p_1,\ p_2 \in [0,\ 1]$.

It is easy to see that the reaction functions, R1 and R2, shown in Figure 6(i), do not intersect and there is no PSNE. For example, when P1 chooses $p_1 = 0$, it is best response for P2 to choose $p_2 = p_1 = 0$. However, then P1 deviates to $p_1 = 1$.

We can also show here that Corollary 1 is confirmed:

$\underset{p_1}{max}\ \underset{p_2}{min}\ u_1(p_1, p_2) = 0 < \underset{p_2}{min}\ \underset{p_1}{max}\ u_1(p_1, p_2) = \frac{1}{4}$.

The interrupted line combining the branches of R1 gives the convex combinations of the strategies $(0,\ \frac{1}{2})$ and $(1,\ \frac{1}{2})$. Hence we have the intersection of two “reaction functions”.

We now show that $\pi_1^* : (\frac{1}{2})(0); (\frac{1}{2})(1)$ and $\pi_2^* : (1)(\frac{1}{2})$ is a MSNE.

Inserting π_1^* into u_2 player P2 chooses p_2 to maximize $u_2(\pi_1^*, p_2) = -\frac{1}{2}(0-p_2)^2 - \frac{1}{2}(1-p_2)^2 = -\frac{1}{2} + p_2 - p_2^2$. The function u_2 is a is maximized at $p_2 = \frac{1}{2}$. Thus, π_2^* is a best response to π_1^* and maximum $u_2 = -\frac{1}{4}$.

Inserting π_2^* into u_1 player P1 chooses π_1 to maximize $u_1(p_1, \pi_2^*) = (p_1 - \frac{1}{2})^2 = p_1^2 - p_1 + \frac{1}{4}$. The maximization is at the pure strategies p_1 =0, 1, and their convex combinations, with (expected value) $u_1 = \frac{1}{4}$.

Hence we have confirmed that $\pi_1^* : (\frac{1}{2})(0); (\frac{1}{2})(1)$ and $\pi_2^* : (1)(\frac{1}{2})$ is a MSNE which is at the intersection of the two "reaction functions".

We consider the behaviour of $u_1 = (p_1 - \frac{1}{2})^2$. It is strictly convex and attains its minimum at $p_1 = \frac{1}{2}$ and the expected value of the mixed strategy is $u_1 = 1/4$. On the other hand if P2 plays $p_2 = \frac{1}{2}$ the value at the pure strategy $p_1 = (\frac{1}{2})(0) + (\frac{1}{2})(1)$, which was always available, is to bring in only $u_1 = 0$. Hence the advantage in mixing is distinct.

(ii) We have the non-zero-sum game $u_1 = (p_1 - p_2)^2$ and $u_2 = -(\frac{1}{2})p_2^2 + p_1^3 p_2$ with $p_1,\ p_2 \in [0,\ 1]$.

R1 is the same as in Figure 6(i) and R2 is now $p_2 = p_1^3$ represented by the black convex curve in Figure 6(ii). R1 and R2 do not intersect and there is no PSNE. For example, if P2 chose $p_2 = p_1^3 = 1$ player P1 would choose $p_1 = 0$ and not $p_1 = 1$

We now show that $\pi_1^* : (\frac{1}{2})(0); (\frac{1}{2})(1)$ and $\pi_2^* : (1)(\frac{1}{2})$ is a MSNE.

Inserting $\pi_1^* : (\frac{1}{2})(0); (\frac{1}{2})(1)$ into u_2 player P2 will choose π_2 to maximize $u_2(\pi_1^*, p_2) = \frac{1}{2}(-\frac{1}{2}p_2^2 + 0) + \frac{1}{2}(-\frac{1}{2}p_2^2 + p_2) = -\frac{1}{2}p_2^2 + \frac{1}{2}p_2$. The maximum is at $p_2 = \frac{1}{2}$. Thus, $\pi_2^* : (1)(\frac{1}{2})$ is confirmed and $u_2 = \frac{1}{8}$.

When P2 plays $\pi_2^* : (1)(\frac{1}{2})$, P1 chooses π_1 to maximize $u_1(p_1, \pi_2^*) = (p_1 - \frac{1}{2})^2 = p_1^2 - p_1 + \frac{1}{4}$. The maximum is at the pure strategies p_1 =0, 1, and their convex combinations and π_1^* is confirmed and the (expected value) is $u_1 = \frac{1}{4}$.

This proves[3] that $\pi_1 : (\frac{1}{2})(0); (\frac{1}{2})(1)$ and $\pi_2 : (1)(\frac{1}{2})$ is a MSNE, with $u_1 = \frac{1}{4}$ and $u_2 = \frac{1}{8}$.

Now assume P2 plays $p_2 = \frac{1}{2}$ but instead of the linear combination of utilities, π_1, player P1 plays the corresponding linear combination of strategies $p_1 = \frac{1}{2}(0) + \frac{1}{2}(1) = \frac{1}{2}$ which is itself a pure strategy. In this case,

$u_1(\frac{1}{2}(0) + \frac{1}{2}(1), \pi_2) = (\frac{1}{2} - \frac{1}{2})^2 = 0 < u_1(\pi_1, \pi_2) = \frac{1}{2}(0 - \frac{1}{2})^2 + \frac{1}{2}(1 - \frac{1}{2})^2 = \frac{1}{4}$.

Thus, the available linear combination of strategies is not as good for P1 in terms of utility.

(iii) We now have $u_1 = (p_1 - p_2)^2$ and $u_2 = -(\frac{1}{2})p_2^2 + p_1^v p_2$ with $p_1,\ p_2 \in [0,\ 1]$ with $v < 1$. R1 is known and the reaction function R2 has equation $p_2 = p_1^v$ represented by the strictly concave curve corresponding to Figure (iii). The two reaction functions do not intersect and there is no PSNE.

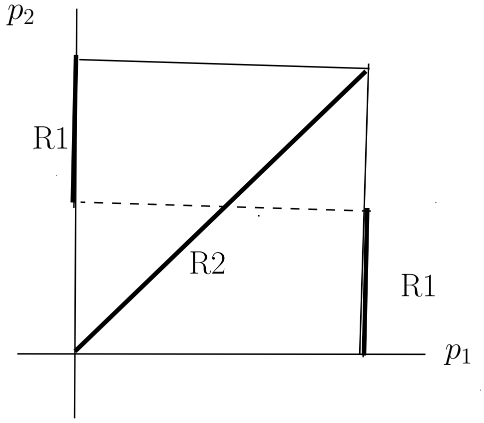
We can consider (iii) as we did for (ii). Again as in (i) the mixed strategies $\pi_1^* : (\frac{1}{2})(0),\ (1/2)(1)$ and $\pi_2^* : (1)(\frac{1}{2})$ is a MSNE, in spite of the substantial change in R2. Inserting π_i^* into u_j we get its maximum at π_j^*. The initial thought might have been that the NE, guaranteed by the continuity theorem, will be at the intersection of the reaction function of P2 and the interrupted extension of R1.

Of course the issue of existence of NE is settled by the conditions of theorems, presented here adjusted to the case the compact sets of pure strategy [0, 1]. Characteristic of investigating the type of NE is the existence of a utility function $(p_1 - p_2)^2$ for one of the players.

Figure (6)(ii) and (iii) make the significant point that, in contrast to Figure 6(i), intersection of R2 with the "reaction function" of P1 does not imply a NE. Player P2 will play

[3] One can also argue as follows. Consider $\pi_1 : (x)(0),\ (1-x)(1)$. Inserting this into u_2 we get $u_2 = -\frac{1}{2}p_2^2 + (1-x)p_2$ which implies $p_2 = 1 - x$. If $x = \frac{1}{2}$ then we check as self-fulfilling pair $\pi_1 : (\frac{1}{2})(0),\ (\frac{1}{2})(1)$ and $\pi_2 : (1)(\frac{1}{2})$ which is confirmed. On the other hand $x < \frac{1}{2}$ will imply $p_2 > \frac{1}{2}$ which on R1 gives $p_1 = 0$; this gives on R2 $p_2 = 0$ different than $p_2 > 0$. — Analogously $x > \frac{1}{2}$ will imply $p_2 < \frac{1}{2}$ which on R1 gives $p_1 = 1$; this gives on R2 $p_2 = 1$ different than $p_2 < 0$. —

Zero Sum Game



$u_1 = (p_1 - p_2)^2$

$u_2 = -(p_1 - p_2)^2$

MSNE:

$\pi_1^* : (1/2)(0);\ (1/2)(1)$

$\pi_2^* : (1)(1/2).$

π_2^* gives max u_1 at conv. com. $p_1 = 0,\ 1$

π_1^* in u_2 gives

max $u_2 = -(1/2)(2p_2^2 + 1 - 2p_2)$ at $p_2 = 1/2$

Hence a NE in mixed strategies.

$u_1 = 1/4,\ u_2 = -1/4.$

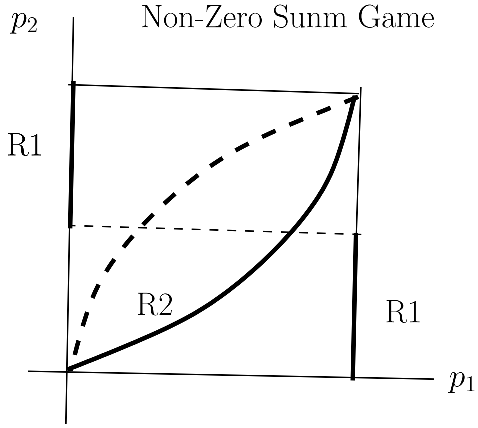
Considering only pure strategies:

$\max_{p_1}\min_{p_2} u_1 = 0 < \min_{p_2}\max_{p_1} u_1 = 1/4.$

Confirming Corollary 1.

(i)

Non-Zero Sunm Game



$u_1 = (p_1 - p_2)^2$

$u_2 = -(1/2)p_2^2 + p_1^3 p_2$

(ii)

Consider the mixed strategy NE:

$\pi_1^* :\ (1/2)(0),\ (1/2)(1)$

$\pi_2^* = (1)(1/2).$

π_2^* gives max u_1 at conv. com. $p_1 = 0,\ 1$

π_1^* in u_2 gives

$u_2 = (1/2)(-(1/2)p_2^2) + 1/2(-(1/2)p_2^2 + p_2)$, i.e.

Max $u_2 = -(1/2)p_2^2 + (1/2)p_2$, i.e. $p_2 = 1/2$

Hence a NE in mixed strategies.

Away from - - - - - -

$u_1 = 1/4,\ u_2 = 1/8.$

Alternative case:

$u_1 = (p_1 - p_2)^2$

$u_2 = -(1/2)p_2^2 + p_1^v p_2,$

$0 < v < 1$

(iii)

$\pi_1^* :\ (1/2)(0),\ (1/2)(1)$

$\pi_2^* = (1)(1/2)$ in u_1 as above.

π_1^* in u_2 gives

Max $u_2 = -(1/2)p_2^2 + (1/2)p_2$, i.e. $p_2 = 1/2$

Hence a NE in mixed strategies.

Away from - - - - - -

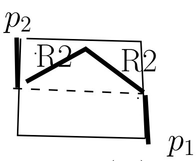
$u_1 = 1/4,\ u_2 = 1/8.$

Suppose $u_1 = (p_1 - p_2)^2$,

$u_2 = -(1/2)p_2^2 - p_2(p_1^2 - p_1) + (1/2)p_2$

from du_2/dp_2: $p_2 = -p_1^2 + p_1 + (1/2)$

hatzi1 Figure 6



(iv)

NE(a) $\pi_1^* :\ (1)(1)$
$\pi_2^* : (1)(1/2)$
(b) $\pi_1^* : (1)(0)$
$\pi_2^* : (1)(1/2)$
(c) $\pi_1^* : (1/2)(0);\ \ (1/2)(1)$
$\pi_2^* : (1)(1/2).$
$u_1 = 1/4,\ u_2 = 1/8.$

Figure 6

$p_2 = 1/2$. So in general even if there is no intersection of the "reaction functions" at a particular point a NE might still exist there.

In the finite matrix games the resolution would be at the intersection of the reactions functions. The flat section gives the only values to be considered. The issue in the infinite case is more involved. The expected value of u_1 at the intersection should also be compared with the value at the corresponding pure strategy. The function $u_1 = (p_1 - 1/2)^2$ is strictly convex with a minimum value at $p_1 = 1/2$. This means that the "corresponding" pure strategy gives a higher value to u_1 than the constant convex values.

Now as in (ii) P2 plays $p_2 = \frac{1}{2}$ but instead of the linear combination of utilities, π_1, player P1 plays the corresponding linear combination of strategies $p_1 = \frac{1}{2}(0) + \frac{1}{2}(1) = \frac{1}{2}$, itself a pure strategy. In this case,

$u_1(\frac{1}{2}(0) + \frac{1}{2}(1),\ 1/2) = (\frac{1}{2} - \frac{1}{2})^2 = 0 < u_1(\pi_1,\ \pi_2) = \frac{1}{2}(0 - \frac{1}{2})^2 + \frac{1}{2}(1 - \frac{1}{2})^2 = \frac{1}{4}$

Thus, the available linear combination of strategies is not as good for P1 in terms of utility.

More than that, in this particular game P1 has $u_1 = (p_1 - p_2)^2$, player P2 plays $p_2 = \frac{1}{2}$ and max $u_1 = \frac{1}{4}$. For all pure strategies, $p_1 \in (0,\ 1)$, $u_1 = (p_1 - (1/2))^2$ is below $\frac{1}{4}$, the constant value of convex combinations of utilities. There is no danger that the utility of a corresponding pure strategy could do better.

Figure 6(iv) presents a model with multiple NE. It gives a game with two pure strategies NE, and also a mixed NE, again, away from intersection of "reaction functions". The calculations are straightforward.

Let $u_1 = (p_1 - p_2)^2$, $u_2 = -\frac{1}{2}p_2^2 - p_2(p_1^2 - p_1) + \frac{1}{2}p_2$ where $p_1 \in [0, 1]$ and $p_2 \in [0, 1]$.

R1 has been established and from $du_2/dp_2 = 0$ the reaction function R2 with equation $p_2 = -p_1^2 + p_1 + \frac{1}{2}$ follows. It is a concave function with $p_2 = 1/2$ at $p_1 = 0,\ 1$ and a maximum value $\frac{3}{4}$ at $p_2 = \frac{1}{2}$.

There are two PSNE: $(0, \frac{1}{2})$ and $(1, \frac{1}{2})$. When P1 chooses $p_1 = 0$ player P2 will play p_2 to maximize $p_2 = -p_1^2 + p_1 + \frac{1}{2} = \frac{1}{2}$ which as a constant allows $p_2 = \frac{1}{2}$ can be chosen. When P2 chooses $p_2 = \frac{1}{2}$ then in order to maximize u_1 player P1 will choose $p_1 = 0$. Hence, $(0, \frac{1}{2})$ is a PSNE, with $u_1 = 1/4$ and $u_2 = \frac{1}{8}$. Analogously $(1, \frac{1}{2})$ is a PSNE with $u_1 = \frac{1}{4}$ and $u_2 = \frac{1}{8}$, as above.

There is also a MSNE, where $\pi_1^* : (\frac{1}{2})(0); (\frac{1}{2})(1)$ and $\pi_2^* : (1)(\frac{1}{2})$. When player P1 plays $\pi_1^* : (\frac{1}{2})(0); (\frac{1}{2})(1)$, player P2 chooses p_2 to maximize

$u_2(\pi_1, p_2) = \frac{1}{2}(-\frac{1}{2}p_2^2 - p_2(0^2 - 0) + \frac{1}{2}p_2) + \frac{1}{2}(-\frac{1}{2}p_2^2 - p_2(1^2 - 1) + \frac{1}{2}p_2) = -\frac{1}{2}p_2^2 + \frac{1}{2}p_2$

Thus, $\pi_2^* : (1)(\frac{1}{2})$ is a best response to $\pi_1^* : (\frac{1}{2})(0); (\frac{1}{2})(1)$.

When P2 plays $\pi_2^* : (1)(\frac{1}{2})$, player P1 chooses p_1 to maximize $u_1(p_1, \pi_2^*) = (p_1 - \frac{1}{2})^2 = p_1^2 - p_1 + \frac{1}{4}$. The function u_1 is maximized at $p_1 = 0$ and $p_2 = 1$ and takes the values $u_1 = \frac{1}{4}$ at both points. The linear combination of $u_1(0, \frac{1}{2})$ and $u_1(1, \frac{1}{2})$ are also equal to $\frac{1}{4}$. Thus, $\pi_1^* : (\frac{1}{2})(0); (\frac{1}{2})(1)$ is a best response to $\pi_2^* : (1)(\frac{1}{2})$. It follows that $\pi_1^* : (\frac{1}{2})(0); (\frac{1}{2})(1)$ and $\pi_2^* : (1)(\frac{1}{2})$ form a MSNE with $u_1 = \frac{1}{4}$ and $u_2 = \frac{1}{8}$.

Now, assume instead of the linear combination of utilities we consider the effect of P1 playing $p_1 = \frac{1}{2}(0) + (1 - \frac{1}{2})(1)$, i.e. the corresponding convex combination of strategies, while $p_2 = (\frac{1}{2})$. Then, $u_1(\frac{1}{2}(0) + (1 - \frac{1}{2})(1), \frac{1}{2}) = (\frac{1}{2} - \frac{1}{2})^2 = 0 < u_1(\pi_1, \frac{1}{2}) = \frac{1}{4}$.

This is as it should be, because we are in a model with an infinite set of pure strategies and $p_1 = \frac{1}{2}(0) + (1 - \frac{1}{2})(1)$ was also available as such but it was not chosen in the maximization of u_1, which selected instead $\pi_1 : (\frac{1}{2})(0); (\frac{1}{2})(1)$.

Examples 5.

Reny (2005) discusses the zero-sum game: $u_1 = |p_1+p_2|$, $u_2 = -|p_1+p_2|$ where P1 and P2 choose their strategies in $[-1,\ 1]$, (which does not affect the theorems), and explains that there is no PSNE. Here we use graphs and show also that a MSNE exists. The structure of the model is given in Figure 7. The discussion is interesting because of the absolute value functions involved.

We look at the reaction function R1 and R2. Player 2 chooses p_2 to maximize $u_2 = -|p_1 + p_2|$ for p_1 given. Hence he chooses $p_2 = -p_1$ and we obtains R2 in the graph. Player P1 chooses p_1 to maximize $u_1 = |p_1 + p_2|$ for given p_2. For all $p_2 \in [-1; 0]$, u_1 is maximized at $p_1 = -1$, and for all $p_2 \in [0; 1]$, u_1 is maximized at $p_1 = 1$. Hence for $p_2 = 0$ the function u_1 is maximized at both $p_1 = -1$ and $p_1 = 1$. Hence we obtain R1, in two vertical pieces in the graph of Figure 7(i). R1 and R2 do not intersect and there is no PSNE.

An explanation can be like this. For all $p_2 \in [-1; 0)$, the reaction for P1 is to choose $p_1 = -1$. However, P2's best response is $p_2 = 1$. Analogously for all $p_2 \in [0; 1)$, the reaction for P1 is to choose $p_1 = 1$. However, P2's best response is $p_2 = -1$. Also for $p_2 = 0$ there is no coincidence of optimal strategies. Hence, there is no PSNE.

Also, from Corollary 1, there is no PSNE, as $\underset{p_1}{max}\ \underset{p_2}{min}\ u_1 = 0 < \underset{p_2}{min}\ \underset{p_1}{max} u_1 = 1$.
P1 argues: no matter what p_1 that I play, P2 will minimize my utility by playing $p_2 = -p_1$. Hence the left-hand-side is zero. For the right-hand-side, P2 argues: no matter what p_2 that I play, P1 can secure $u_1 \geq 1$. In particular, I can play $p_2 = 0$ and P1 will play either -1 or 1.

Finally, Figure 7(ii) explain that the set of p_1's with $u_1 = |p_1 + 0| \geq u_1 = |a + 0|$
is not convex. Hence, $u_1 = |p_1 + p_2|$ is not quasi-concave. Hence the conditions of T2 are not satisfied.

On the other hand the continuity of the absolute vale functions implies by T3 that a MSNE exists.

Now, R1 is extended through δ functions and the two branches are united through the interrupted line. We show that an MSNE consists of: $\pi_1^* : (\frac{1}{2})(-1), (\frac{1}{2})(1)$ and $\pi_2^* : (1)(0)$. π_1^* in u_2 confirms π_2^* and π_2^* in u_1 confirms π_1^*.

When P1 plays $\pi_1^* : (\frac{1}{2})(-1), (\frac{1}{2})(1)$, player P2 chooses p_2 to maximize
$u_2(\pi_1, p_2) = -(\frac{1}{2})|-1+p_2|-(\frac{1}{2})|1+p_2| = -\frac{1}{2}(|p_2-1|+|p_2+1|) = -\frac{1}{2}(1-p_2+1+p_2) = -1.$
Hence $u_2 = -1$ irrespective of p_2 and we choose $p_2 = 0$ with probability 1.

Next, we check whether $\pi_1^* : (\frac{1}{2})(-1), (\frac{1}{2})(1)$ is a best response to $\pi_2^* : (1)(0)$. When P2 plays $\pi_2 : (1)(0)$, P1 chooses π_1 to maximize $u_1(p_1, \pi_2) = |p_1 + 0|$. This function takes the maximum value $u_1(-1, 0) = u_1(1, 0) = 1$ and $\pi_1^* : (\frac{1}{2})(-1), (\frac{1}{2})(1)$ is also a best response to $\pi_2^* : (1)(0)$. Hence $\pi_1 : (\frac{1}{2})(-1), (\frac{1}{2})(1)$ and $\pi_2 : (1)(0)$ is a MSNE.

Now, suppose that P2 plays $\pi_2 : (1)(0)$, and P1 chooses the pure strategy $p_1 = \frac{1}{2}(-1) + \frac{1}{2}(1) = 0$, instead of using on u_1 the mixed strategy $\pi_1 : (\frac{1}{2})(-1), (\frac{1}{2})(1)$. In this case,
$u_1(\frac{1}{2}(-1) + \frac{1}{2}(1), 0) = |0 + 0| = 0 < u_1(\pi_1, 0) = \frac{1}{2}| - 1 + 0| + \frac{1}{2}|1 + 0| = 1.$

Thus, the linear combination of utilities guarantees higher payoff for P1, than the corresponding linear combination of strategies.

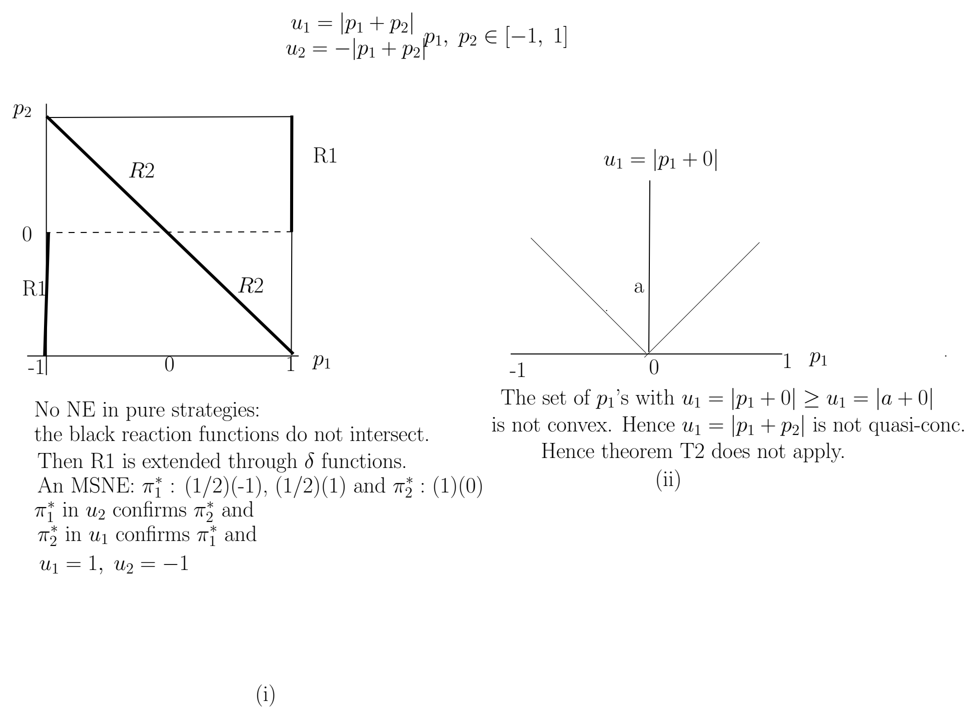


Figure 7

4 Concluding comments.

We have attempted to discuss MSNE in two-person games with an infinite compact sets of pure strategy [0, 1] and nonlinear utility functions. The use of such games is not very familiar and the examples chosen clarify the significance of the utility functions.

The issue is whether attempting to construct the NE we could use, (mimic), and in what sense, the idea employed in the finite matrix games of crossing reaction functions. This is an important point to consider. Another point is that there might be NE away from crossing points.

If the reaction function in pure strategies break up at a particular choice of the opponent a rule of thumb would be to connect the pure strategies with straight lines through δ functions and form mixed strategy "reaction functions".

Then we consider the intersection of the "reaction functions". And on the basis of a rigorous mathematical calculation proceed to see whether this is a MSNE. However for the mixed strategies we need to calculate the probabilities of the intersection point. In the finite model the intersection point is itself in probabilities. So when we are attempting to "mimic" the finite pure strategies model we mean in the limited sense that the intersection could again be an equilibrium point. We bear in mind that this in now in terms of strategies and we ask whether the implied probabilities lead to MSNE.

We look at the significance of the strictly convex utility function $u_1 = (p_1 - \frac{1}{2})^2$ in Figure 6(ii). It attains its minimum at $p_1 = \frac{1}{2}$.

It implies that reaction function of P1 beaks up at $p_2 = \frac{1}{2}$ and it is clear that there is no PSNE. On the other hand from the continuity of the utility functions we know that a

MSNE exists. The mixed strategy of P1 must now be of the form $\pi_1 : (k)(0),\ (1-k)(1)$ and this determines what happens next. We introduce this into the given u_2 and maximize with respect to p_2. This gives $p_2 = \frac{1}{2}$ and we have obtained a MSNE. One might have thought originally that the intersection point $(\frac{4}{5},\ \frac{1}{2})$ was a candidate for an equilibrium.

We also provider further justification for the existence of NE. In the case of Examples 2 and 5, the relevant utility functions are such that the value along the "reaction function" is above the value at the expected pure strategy. So there is no problem in believing that we can 'mimic' what is happening in the finite matrix games. There is a richness of cases.

Short Posthema.

In game theory, the extensive form is a way of describing a game using the idea of a tree. A finite tree is a directed graph whose nodes are positions in a game and whose edges are moves. A path can cross each information set at most once. A (directed) path connects in a unique manner the initial node of the tree with the terminal node. The payoffs are shown at the end of each branch.

The nodes form information sets which are assigned to the players. In a perfect information set the player know exactly his position. In the case of imperfect information the player does not know at which individual node he finds himself.

A *pure strategy*, (PS), of a player maps each of his information sets into the actions available at that set. *Mixed strategies*, (MS),)are defined to be probability distributions over pure strategies. *Behavioural strategies*, (BS), attach a probability distribution to the moves from each information set; these probabilities are attached independently. A pure strategy is also a behavioural strategy.

A behavioural strategy means that when a player is an information set he spins a wheel to decide which move he will make. The wheel is the same for all nodes in this set.

A *Perfect Bayesian Equilibrium*, PBE, consists of a set of players' optimal behavioural strategies, and consistent with these, a set of beliefs which attach a probability distribution to the nodes of each information set. Consistency requires that the decision from an information set is optimal given the particular player's beliefs about the nodes of this set and the strategies from all other sets, and that beliefs are formed from updating, using the available information. If the optimal play of the game enters an information set then updating of beliefs must be Bayesian. Otherwise appropriate beliefs are assigned arbitrarily to the nodes of the set.

Definitions. A player P is said to have *perfect recall* if he never forgets what he once knew. A game is said to be of *perfect recall* if every player has perfect recall.

If a player can recall all of his previous actions, he might still not be able to distinguish between the nodes of an information set by recalling how exactly he got to that set. All he will recall from any node of that set is the same sequence of previous information sets and the moves he made from them.

It may be that a player has an infinite number of possible actions to choose from at a particular decision node. The device used to represent this is an arc joining two edges protruding from the decision node in question. If the action space is a continuum between two numbers, the lower and upper delimiting numbers are placed at the bottom and top of the arc respectively, usually with a variable that is used to express the payoffs. The infinite number of decision nodes that could result are represented by a single node placed in the centre of the arc.

The strategic form (or normal form) is a way of describing a two-player game using a matrix. Different players are exhibited on a side of the matrix with their strategies or choices. The entries give corresponding payoffs. For the matrix game we know that a NE equilibrium exists and the reaction functions must intersect.

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